ALGEBRAIC TOPOLOGY II LECTURE NOTES

HIDENORI SHINOHARA

Contents

1. Poincare Duality

Since $H^*(T^n) \equiv \wedge_{\mathbb{Z}} M$ where $M = \langle v_1, \cdots, v_n \rangle$, we have

k	0	1	2	$\cdots$	$n-1$	n
rank $H^k(T^n;\mathbb{Z})$	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\cdots$	$\binom{n}{n-1}$	$\binom{n}{n}$

This symmetry phenomenon is true in general and very useful. Another example: The cellular complex for $\mathbb{CP}^n$ is $\mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z} \to 0$. Thus

k	0	1	2	$\cdots$	$2n-1$	$2n$
rank $H^k(\mathbb{CP}^n;\mathbb{Z})$	1	0	1	$\cdots$	0	1

1.1. **Orientations.**

Definition 1.1. Let M be a triangulable closed n-manifold. Let $\sigma_1, \cdots, \sigma_k$ be n-simplices such that $M = \sigma_1 \cup \cdots \cup \sigma_k$. Then $\sigma_i \in C_n(M)$ for each i. Suppose that the ordering of the vertices in σ_i and the signs $\pm$ can be chosen such that

$$\sum \pm \partial\sigma_i = 0 \in C_{n-1}(M).$$

Then M is said to be *orientable*.

Example 1.2. A tetrahedron and torus are examples of orientable manifolds.

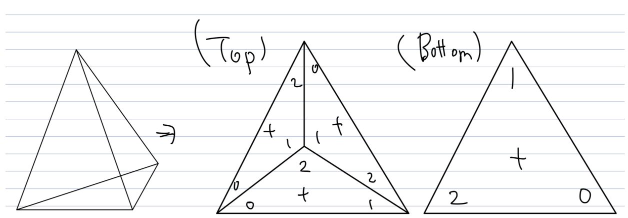

Figure 1. Orientation of a tetrahedron

Definition 1.3. Let M be an n-dimensional orientable manifold. Choose $\sigma_i \in C_n(M)$ and signs $\mathrm{sgn}_i \in \{-1, 1\}$ such that $M = \sigma_1 \cup \cdots \cup \sigma_k$ and $\sum \mathrm{sgn}_i \partial\sigma_i = 0$. The class represented by $\sum \mathrm{sgn}_i \sigma_i \in \ker(\partial)$ in $H_n(M)$ is called a fundamental class $[M]$.

Theorem 1.4. *If M connected, then $[M]$ is a generator of $H_n(M)$.*

Proof. By Poincare Duality(which will be discussed later (1.7)), $H_n(M) \cong H^0(M) = \mathbb{Z}$. Let $\sum c_i\sigma_i$ represent a generator of $H_n(M)$ where $c_i \in \mathbb{Z}$. Then $\sum \operatorname{sgn}_i \sigma_i = \lambda \sum c_i\sigma_i = \sum(\lambda c_i)\sigma_i$ for some $\lambda \in \mathbb{Z}$. Since each $\lambda c_i = \operatorname{sgn}_i \in \{-1, 1\}$, λ must be 1 or -1. Therefore, the class represented by $\sum \operatorname{sgn}_i \sigma_i$ is a generator of $H_n(M)$. □

Corollary 1.5. *There are two fundamental classes for any connected orientable manifold.*

Proof. By (1.4), a fundamental class $[M]$ is a generator of $H_n(M) = \mathbb{Z}$. Since $\mathbb{Z}$ has exactly two generators $1, -1$, M has exactly two fundamental classes. □

Definition 1.6. Let M be a connected, orientable manifold. Then a choice of a fundamental class is called an orientation of M.

1.2. **Poincare Duality(Version 1).**

Theorem 1.7. *If M is an orientable closed n-manifold, then*

$$H^k(M; G) \cong H_{n-k}(M; G)$$

for any integer $0 \leq k \leq n$ and an abelian group G.

Remark. *If M is not orientable, Poincare Duality holds when $G = \mathbb{Z}/2$. In other words,*

$$H^k(M; \mathbb{Z}/2) \cong H_{n-k}(M; \mathbb{Z}/2).$$

For instance,

$H_*(\mathbb{RP}^2; \mathbb{Z})$	*0*	$\mathbb{Z}/2$	$\mathbb{Z}$
$H^*(\mathbb{RP}^2; \mathbb{Z})$	$\mathbb{Z}/2$	*0*	$\mathbb{Z}$

so Poincare Duality does not hold in this case. However, with $\mathbb{Z}/2$,

$H_*(\mathbb{RP}^2; \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H^*(\mathbb{RP}^2; \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

so Poincare Duality holds in this case.

Definition 1.8. The kth Betti number of a manifold M is defined to be $b_k = \operatorname{rank}(H^k(M; \mathbb{Z}))$.

Theorem 1.9. *$b_k = b_{n-k}$ for all k if M is a closed orientable n-manifold.*

Proof. By the Universal Coefficient Theorem, $\operatorname{rank}(H^k(M)) = \operatorname{rank}(H_k(M))$. By Poincare Duality, $\operatorname{rank}(H^k(M)) = \operatorname{rank}(H_{n-k}(M))$. Therefore, $b_k = \operatorname{rank}(H^k(M)) = \operatorname{rank}(H_{n-k}(M)) = \operatorname{rank}(H^{n-k}(M)) = b_{n-k}$. □

1.3. **Cap Products.** There is a nice way to explicitly write down the isomorphism when G is a commutative ring.

Definition 1.10. Let a space X and a commutative ring R be given. For any $k \geq l$, define the cap product

$$\frown: \qquad C_k(X; R) \otimes C^l(X; R) \longrightarrow C_{k-l}(X; R)$$

$$\sigma \otimes \phi \longmapsto \phi(\sigma|_{[v_0, \cdots, v_l]})\sigma|_{[v_l, \cdots, v_k]}$$