ALGEBRAIC TOPOLOGY II LECTURE NOTES

HIDENORI SHINOHARA

Contents

1. Geometric Description of Cocycles

Example 1.1. Let X be a Δ-complex and $\phi \in C^k(X;\mathbb{Z})$. What does it mean that $\delta\phi = 0$? As a toy model, we will consider a surface. One way to construct such a $\phi \in C^1(X;\mathbb{Z})$ is to take an oriented closed curve γ transverse to 1-simplices. Then we define $\phi(\sigma)$ to be the number of intersections between σ and γ with signs. See the following example:

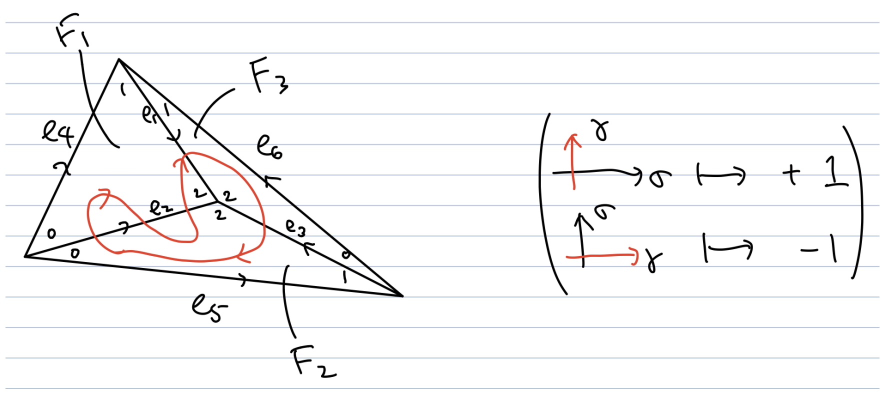

Figure 1. Oriented closed curve in a surface

γ gives us the following 1-cocycle ϕ which maps each e_i to an integer as following:

$$\begin{aligned} e_1 &\mapsto 1 \\ e_2 &\mapsto 1 - 1 + 1 = 1 \\ e_3 &\mapsto 1 \\ e_4 &\mapsto 0 \\ e_5 &\mapsto 0 \\ e_6 &\mapsto 0. \end{aligned}$$

Then $\delta\phi = 0$ because

$$\begin{aligned}
(\delta\phi)(F_1) &= \phi(\partial F_1) = \phi(e_1) - \phi(e_2) + \phi(e_4) = 0 \\
(\delta\phi)(F_2) &= \phi(\partial F_2) = \phi(e_3) - \phi(e_2) + \phi(e_5) = 0 \\
(\delta\phi)(F_3) &= \phi(\partial F_3) = \phi(e_1) - \phi(e_3) + \phi(e_6) = 0.
\end{aligned}$$

This is not a coincidence because $\phi(\partial\sigma)$ represents

(the number of times γ enters σ) - (the number of times γ exists σ)

which is always 0 for any 2-simplex σ and any traverse closed oriented curve γ. In this case, we call ϕ the **Poincare dual** to γ, or simply $\phi = \mathrm{PD}(\gamma)$, and this concept can be generalized further.

Definition 1.2. Let X be a topological manifold of dimension n. Let γ be an $(n-k)$-cycle in X transverse to the k-skeleton of X. Define $\phi \in C^k(X;\mathbb{Z})$ such that

$$\phi(\sigma) = \text{The number of intersections between } \sigma \text{ and } \gamma \text{ with signs.}$$

Then we call ϕ the **Poincare dual** of γ and denote it by $\phi = \mathrm{PD}(\gamma)$.

Example 1.3. We will look at a torus which is a slightly more complicated example. We obtain the following

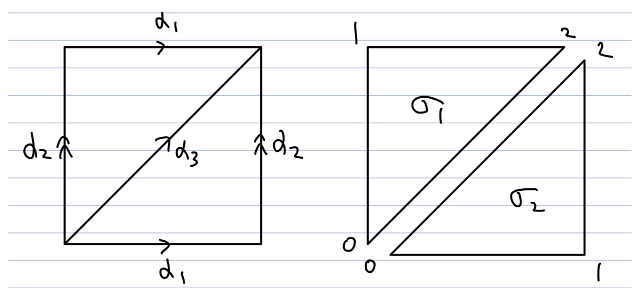

FIGURE 2. Torus

cellular chain complex from Figure 2:

$$\begin{array}{ccccc}
C_2 = \langle \sigma_1, \sigma_2 \rangle & \longrightarrow & C_1 = \langle \alpha_1, \alpha_2, \alpha_3 \rangle & \longrightarrow & C_0 = \langle v \rangle \\
\sigma_i & \longmapsto & \alpha_1 + \alpha_2 - \alpha_3 & & \\
 & & \alpha_i & \longmapsto & 0
\end{array}$$

Thus

$$H_i(T^2) = \begin{cases} \mathbb{Z} & (i = 0, 2) \\ \mathbb{Z}^2 & (i = 1). \end{cases}$$

Now, we will examine the cellular cochain complex.

$$\begin{aligned}
\delta(v^*)(\alpha_i) &= v^*(\partial\alpha_i) \\
&= v^*(0) \\
&= 0
\end{aligned}$$

for any $\alpha_i \in C_1$. Therefore, $\delta(v^*) = 0$.

$$\begin{aligned}\delta(\alpha_1^*)(\sigma_i) &= \alpha_1^*(\partial(\sigma_i)) \\ &= \alpha_1^*(\alpha_1 + \alpha_2 - \alpha_3) \\ &= 1\end{aligned}$$

for any $\sigma_i \in C_2$. By performing similar calculation on α_2^* and α_3^*, we obtain

$$\begin{array}{ccccc} C^2 = \langle \sigma_1^*, \sigma_2^* \rangle & \longleftarrow & C^1 = \langle \alpha_1^*, \alpha_2^*, \alpha_3^* \rangle & \longleftarrow & C^0 = \langle v^* \rangle \\ \\ \sigma_1^* + \sigma_2^* & \longleftarrow\!\!\shortmid & \alpha_1^*, \alpha_2^* & & \\ \\ -(\sigma_1^* + \sigma_2^*) & \longleftarrow\!\!\shortmid & \alpha_3^* & & \\ \\ & & 0 & \longleftarrow\!\!\shortmid & v^* \end{array}$$

Thus

$$H^i(T^2) = \begin{cases} \mathbb{Z} & (i = 0, 2) \\ \mathbb{Z}^2 & (i = 1). \end{cases}$$

- $H^0(T^2) = \langle v^* \rangle / 0$, so $H^0(T^2)$ is generated by $[v^*]$.
- $H^2(T^2)$ is generated by $[\sigma_2^*]$ because $H^2(T^2) = \langle \sigma_1^*, \sigma_2^* \rangle / \langle \sigma_1^* + \sigma_2^* \rangle$.

We can picture $H^1(T^2)$ by using Poincare duals. Let $\phi_1 = \mathrm{PD}(\gamma_1), \phi_2 = \mathrm{PD}(\gamma_2)$. Then $\phi_1(\alpha_1) = 0, \phi_1(\alpha_2) =$

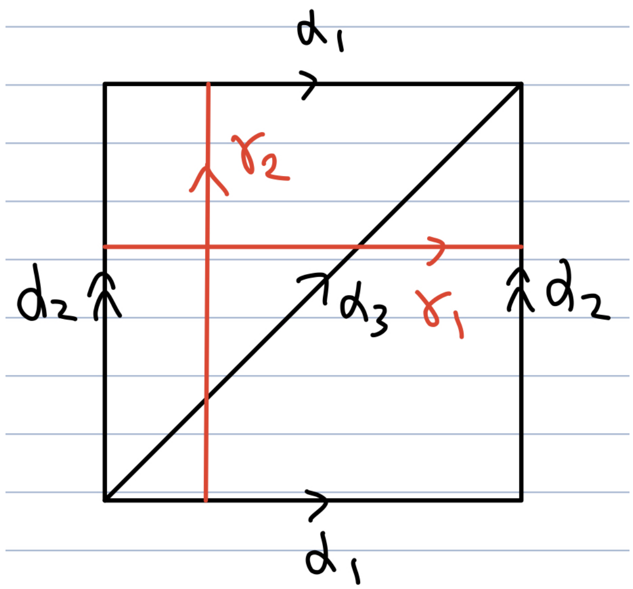

FIGURE 3. Torus Poincare Duals

$1, \phi_1(\alpha_3) = 1$, and $\phi_2(\alpha_1) = -1, \phi_2(\alpha_2) = 0, \phi_2(\alpha_3) = -1$. Moreover,

$$(\delta\phi_1)(\sigma_i) = \phi_1(\partial\sigma_i) = \phi_1(\alpha_1 + \alpha_2 - \alpha_3) = \phi_1(\alpha_1) + \phi_1(\alpha_2) - \phi_1(\alpha_3) = 0.$$
$$(\delta\phi_2)(\sigma_i) = \phi_2(\partial\sigma_i) = \phi_2(\alpha_1 + \alpha_2 - \alpha_3) = \phi_1(\alpha_1) + \phi_1(\alpha_2) - \phi_1(\alpha_3) = 0.$$

Therefore, the classes represented by ϕ_1, ϕ_2 are in $H^1(T^2;\mathbb{Z})$. Moreover, they clearly generate $H^1(T^2;\mathbb{Z})$, so every element in $H^1(T^2;\mathbb{Z})$ can be considered as a function with a fixed closed curve that counts the number of times the curve intersects a given 1-simplex.

Now, we will examine the cup product structure. By definition, we know that $[\phi_1] \smile [\phi_2] \in H^2(T^2;\mathbb{Z})$. Therefore, we will check what σ_1, σ_2 get sent to.

$$\begin{aligned}
(\phi_1 \smile \phi_2)(\sigma_1) &= \phi_1(\sigma_1|_{[0,1]})\phi_2(\sigma_1|_{[1,2]}) \\
&= \phi_1(\alpha_2)\phi_2(\alpha_1) \\
&= -1 \cdot 1 = -1. \\
(\phi_1 \smile \phi_2)(\sigma_2) &= \phi_1(\sigma_2|_{[0,1]})\phi_2(\sigma_2|_{[1,2]}) \\
&= \phi_1(\alpha_1)\phi_2(\alpha_2) \\
&= 0.
\end{aligned}$$

Recall that $[\sigma_1^* + \sigma_2^*] = 0$ because $\sigma_1^* + \sigma_2^*$ is in the kernel. Therefore, $[\phi_1 \smile \phi_2] = [-\sigma_1^*] = [\sigma_2^*]$. Similarly, we obtain $(\phi_2 \smile \phi_1)(\sigma_1) = 0$ and $(\phi_2 \smile \phi_1)(\sigma_2) = -1$. Thus $[\phi_2 \smile \phi_1] = -[\sigma_2^*]$.

I don't understand the alternative approach using the universal coefficient theorem.

Finally, we obtain the following multiplication table for $H^*(T^2;\mathbb{Z}) \cong \mathbb{Z}\langle 1, [\phi_1], [\phi_2], [\sigma_2^*]\rangle$:

$\smile$	1	$[\phi_1]$	$[\phi_2]$	$[\sigma_2^*]$
1	1	$[\phi_1]$	$[\phi_2]$	$[\sigma_2^*]$
$[\phi_1]$	$[\phi_1]$	0	$[\sigma_2^*]$	0
$[\phi_2]$	$[\phi_2]$	$-[\sigma_2^*]$	0	0
$[\sigma_2^*]$	$[\sigma_2^*]$	0	0	0

2. Poincare Duality

Since $H^*(T^n) \equiv \wedge_{\mathbb{Z}} M$ where $M = \langle v_1, \cdots, v_n\rangle$, we have

k	0	1	2	$\cdots$	$n-1$	n
$\operatorname{rank} H^k(T^n;\mathbb{Z})$	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\cdots$	$\binom{n}{n-1}$	$\binom{n}{n}$

This symmetry phenomenon is true in general and very useful. Another example: The cellular complex for $\mathbb{CP}^n$ is $\mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z} \to 0$. Thus

k	0	1	2	$\cdots$	$2n-1$	$2n$
$\operatorname{rank} H^k(\mathbb{CP}^n;\mathbb{Z})$	1	0	1	$\cdots$	0	1

2.1. **Orientations.**

Definition 2.1. Let M be a triangulable closed n-manifold. Let $\sigma_1, \cdots, \sigma_k$ be n-simplices such that $M = \sigma_1 \cup \cdots \cup \sigma_k$. Then $\sigma_i \in C_n(M)$ for each i. Suppose that the ordering of the vertices in σ_i and the signs $\pm$ can be chosen such that

$$\sum \pm \partial\sigma_i = 0 \in C_{n-1}(M).$$

Then M is said to be *orientable.*

Example 2.2. A tetrahedron and torus are examples of orientable manifolds.

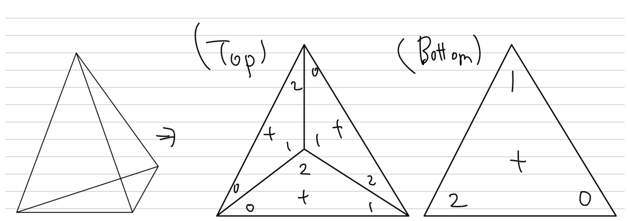

Figure 4. Orientation of a tetrahedron

Definition 2.3. Let M be an n-dimensional orientable manifold. Choose $\sigma_i \in C_n(M)$ and signs $\mathrm{sgn}_i \in \{-1, 1\}$ such that $M = \sigma_1 \cup \cdots \cup \sigma_k$ and $\sum \mathrm{sgn}_i \partial\sigma_i = 0$. The class represented by $\sum \mathrm{sgn}_i \sigma_i \in \ker(\partial)$ in $H_n(M)$ is called a fundamental class $[M]$.

Theorem 2.4. *If M connected, then $[M]$ is a generator of $H_n(M)$.*

Proof. By Poincare Duality(which will be discussed later (2.7)), $H_n(M) \cong H^0(M) = \mathbb{Z}$. Let $\sum c_i\sigma_i$ represent a generator of $H_n(M)$ where $c_i \in \mathbb{Z}$. Then $\sum \mathrm{sgn}_i \sigma_i = \lambda \sum c_i\sigma_i = \sum(\lambda c_i)\sigma_i$ for some $\lambda \in \mathbb{Z}$. Since each $\lambda c_i = \mathrm{sgn}_i \in \{-1, 1\}$, λ must be 1 or -1. Therefore, the class represented by $\sum \mathrm{sgn}_i \sigma_i$ is a generator of $H_n(M)$. ☐

Corollary 2.5. *There are two fundamental classes for any connected orientable manifold.*

Proof. By (2.4), a fundamental class $[M]$ is a generator of $H_n(M) = \mathbb{Z}$. Since $\mathbb{Z}$ has exactly two generators $1, -1$, M has exactly two fundamental classes. ☐

Definition 2.6. Let M be a connected, orientable manifold. Then a choice of a fundamental class is called an orientation of M.

2.2. **Poincare Duality(Version 1).**

Theorem 2.7. *If M is an orientable closed n-manifold, then*

$$H^k(M; G) \cong H_{n-k}(M; G)$$

for any integer $0 \le k \le n$ and an abelian group G.

Remark. *If M is not orientable, Poincare Duality holds when $G = \mathbb{Z}/2$. In other words,*

$$H^k(M; \mathbb{Z}/2) \cong H_{n-k}(M; \mathbb{Z}/2).$$

For instance,

$H_*(\mathbb{RP}^2; \mathbb{Z})$	*0*	$\mathbb{Z}/2$	$\mathbb{Z}$
$H^*(\mathbb{RP}^2; \mathbb{Z})$	$\mathbb{Z}/2$	*0*	$\mathbb{Z}$

so Poincare Duality does not hold in this case. However, with $\mathbb{Z}/2$,

$H_*(\mathbb{RP}^2; \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H^*(\mathbb{RP}^2; \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

so Poincare Duality holds in this case.

Definition 2.8. The kth Betti number of a manifold M is defined to be $b_k = \mathrm{rank}(H^k(M; \mathbb{Z}))$.

Theorem 2.9. *$b_k = b_{n-k}$ for all k if M is a closed orientable n-manifold.*

Proof. By the Universal Coefficient Theorem, $\mathrm{rank}(H^k(M)) = \mathrm{rank}(H_k(M))$. By Poincare Duality, $\mathrm{rank}(H^k(M)) = \mathrm{rank}(H_{n-k}(M))$. Therefore, $b_k = \mathrm{rank}(H^k(M)) = \mathrm{rank}(H_{n-k}(M)) = \mathrm{rank}(H^{n-k}(M)) = b_{n-k}$. ☐

2.3. **Cap Products.** There is a nice way to explicitly write down the isomorphism when G is a commutative ring.

Definition 2.10. Let a space X and a commutative ring R be given. For any $k \ge l$, define the cap product

$$\frown: \qquad C_k(X; R) \otimes C^l(X; R) \longrightarrow C_{k-l}(X; R)$$

$$\sigma \otimes \phi \longmapsto \phi(\sigma|_{[v_0, \cdots, v_l]})\sigma|_{[v_l, \cdots, v_k]}$$

Theorem 2.11. $\partial(\sigma \frown \phi) = (-1)^l((\partial\sigma) \frown \phi - \sigma \frown (\partial\phi))$.

Proof. Let $k \geq l, \sigma \in C_{k+1}(X;R), \phi \in C^l(X;R)$ be given.

$$\begin{aligned}
(\partial\sigma) \frown \phi &= (\sum_j (-1)^j \sigma|_{[v_0,\cdots,\hat{v}_j,\cdots,v_k]}) \frown \phi \\
&= \sum_j (-1)^j (\sigma|_{[v_0,\cdots,\hat{v}_j,\cdots,v_k]} \frown \phi) \\
&= \sum_{j=1}^{l} (-1)^j \phi(\sigma|_{[v_0,\cdots,\hat{v}_j,\cdots,v_{l+1}]})\sigma|_{[v_{l+1},\cdots,v_k]} + \sum_{j=l+1}^{k} (-1)^j \phi(\sigma|_{[v_0,\cdots,v_l]})\sigma|_{[v_l,\cdots,\hat{v}_j,\cdots,v_k]} \\
&= \sum_{j=1}^{l} (-1)^j \phi(\sigma|_{[v_0,\cdots,\hat{v}_j,\cdots,v_{l+1}]})\sigma|_{[v_{l+1},\cdots,v_k]} + (-1)^{l+1}\phi(\sigma|_{[v_0,\cdots,v_l,\hat{v}_{l+1}]})\sigma|_{[v_{l+1},\cdots,v_k]} \\
&+ (-1)^l \phi(\sigma|_{[v_0,\cdots,v_l]})\sigma|_{[\hat{v}_l,v_{l+1},\cdots,v_k]} + \sum_{j=l+1}^{k} (-1)^j \phi(\sigma|_{[v_0,\cdots,v_l]})\sigma|_{[v_l,\cdots,\hat{v}_j,\cdots,v_k]} \\
&= \sum_{j=1}^{l+1} (-1)^j \phi(\sigma|_{[v_0,\cdots,\hat{v}_j,\cdots,v_{l+1}]})\sigma|_{[v_{l+1},\cdots,v_k]} + \sum_{j=l}^{k} (-1)^j \phi(\sigma|_{[v_0,\cdots,v_l]})\sigma|_{[v_l,\cdots,\hat{v}_j,\cdots,v_k]}.
\end{aligned}$$

We will compute each summand.

$$\begin{aligned}
\sum_{j=1}^{l+1} (-1)^j \phi(\sigma|_{[v_0,\cdots,\hat{v}_j,\cdots,v_{l+1}]})\sigma|_{[v_{l+1},\cdots,v_k]} &= \phi(\sum_{j=1}^{l+1} (-1)^j \sigma|_{[v_0,\cdots,\hat{v}_j,\cdots,v_{l+1}]})\sigma|_{[v_{l+1},\cdots,v_k]} \\
&= \phi(\partial\sigma|_{[v_0,\cdots,v_{l+1}]})\sigma|_{[v_{l+1},\cdots,v_k]} \\
&= \sigma \frown \delta\phi.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{j=l}^{k} (-1)^j \phi(\sigma|_{[v_0,\cdots,v_l]})\sigma|_{[v_l,\cdots,\hat{v}_j,\cdots,v_k]} &= (-1)^l \sum_{j=0}^{k-l} (-1)^j \phi(\sigma|_{[v_0,\cdots,v_{l+j}]})\sigma|_{[v_l,\cdots,\hat{v}_{l+j},\cdots,v_k]} \\
&= (-1)^l \partial(\phi(\sigma|_{[v_0,\cdots,v_{l+j}]})\sigma|_{[v_l,\cdots,v_k]}) \\
&= (-1)^l \partial(\sigma \frown \phi).
\end{aligned}$$

Therefore, we obtain the desired result $\partial(\sigma \frown \phi) = (-1)^l((\partial\sigma) \frown \phi - \sigma \frown (\delta\phi))$. □

Theorem 2.12. *$\frown: C_k(X;R) \otimes C^l(X;R) \to C_{k-l}(X;R)$ induces a map $H_k(X;R) \otimes H^l(X;R) \to H_{k-l}(X;R)$.*

Proof. Let $[\sigma] \in H_k(X;R)$ and $[\phi] \in H^l(X;R)$ be given where $\sigma \in C_k(X;R)$ and $\phi \in C^l(X;R)$. Then $\partial(\sigma) = \delta(\phi) = 0$. By (2.11), $\partial(\sigma \frown \phi) = (-1)^l(0 - 0) = 0$. Therefore, $\sigma \frown \phi$ represents a class in $H_{k-l}(X;R)$. □

2.4. **Poincare Duality(Version 2).**

Theorem 2.13. *If M is an orientable closed n-manifold, let $[M] \in H_n(M)$ be a fundamental class for M, and let R be a commutative ring. View $[M] \in H_n(M;R)$. Then the map*

$$\begin{aligned}
H^l(M;R) &\longrightarrow H_{n-l}(M;R) \\
\phi &\longmapsto [M] \frown \phi
\end{aligned}$$

is an R-module isomorphism.

If M is not orientable, then $[M] \in H_n(M;\mathbb{Z}/2)$ still exists, and we have

$$H^l(M;\mathbb{Z}/2) \longrightarrow H_{n-l}(M;\mathbb{Z}/2)$$

$$\phi \longmapsto [M] \frown \phi$$

2.5. **Differential Forms on $\mathbb{R}^n$.**

Definition 2.14. Let $dx_1, \cdots, dx_n$ be formal indeterminates. Let V be the vector space over $\mathbb{R}$ generated by $dx_1, \cdots, dx_n$. Let Ω^* be the $\mathbb{R}$-algebra generated by $dx_1, \cdots, dx_n$ module relations $dx_i \wedge dx_i = 0$ and $dx_i \wedge dx_j = -dx_j \wedge dx_i$.

Lemma 2.15. *The dimension of Ω^* over $\mathbb{R}$ is 2^n.*

Proof. Ω^* is generated by the set of elements of the form $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ where $0 \leq k \leq n$ and $i_1 < \cdots < i_k$. This is because of the two relations of Ω^*. We denote it by dx_I where $I = \{i_1, \cdots, i_k\}$. Then there are exactly 2^n I's, so the dimension is 2^n. □

Definition 2.16. Let $U \subset \mathbb{R}^n$ be open. Then $\Omega^*(U) = C^\infty(U) \otimes \Omega^*$.

Lemma 2.17. *$\Omega^*(U)$ is a graded, sign-commutative $\mathbb{R}$-algebra.*

Proof. $\Omega^*(U)$ is a vector space over $\mathbb{R}$ because addition can be defined in a trivial way and for any $c \in \mathbb{R}$ and $f \otimes dx_I \in \Omega^*(U)$, $c(f \otimes dx_I) = (cf) \otimes dx_I$. Moreover, $(f_I dx_I) \wedge (f_J dx_J) = (f_I f_J)(dx_I \wedge dx_J)$ is a well-defined multiplication. It is easy to see that the multiplication, addition and scalar multiplication commute. Thus $\Omega^*(U)$ is a $\mathbb{R}$-algebra. Moreover, $\Omega^*(U) = \oplus_{k=1}^n \Omega^k(U)$ where $\Omega^k(U) = \{\sum_{|I|=k} f_I dx_I\}$. Finally, for any $\omega_1 \in \Omega^{k_1}(U), \omega_2 \in \Omega^{k_2}(U), \omega_1 \wedge \omega_2 = (-1)^{k_1 k_2} \omega_2 \wedge \omega_1$. □

Definition 2.18. The exterior derivative $d : \Omega^k(U) \to \Omega^{k+1}(U)$ is defined as following:

- For any $f \in \Omega^0(U)$, $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$.
- For any $\omega \in \sum f_I dx_I$, $d\omega = \sum df_I \wedge dx_I$.