

ALGEBRAIC TOPOLOGY II LECTURE NOTES

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1. GEOMETRIC DESCRIPTION OF COCYCLES

Example 1.1. Let X be a Δ -complex and $\phi \in C^k(X; \mathbb{Z})$. What does it mean that $\delta\phi = 0$? As a toy model, we will consider a surface. One way to construct such a $\phi \in C^1(X; \mathbb{Z})$ is to take an oriented closed curve γ transverse to 1-simplices. Then we define $\phi(\sigma)$ to be the number of intersections between σ and γ with signs. See the following example:

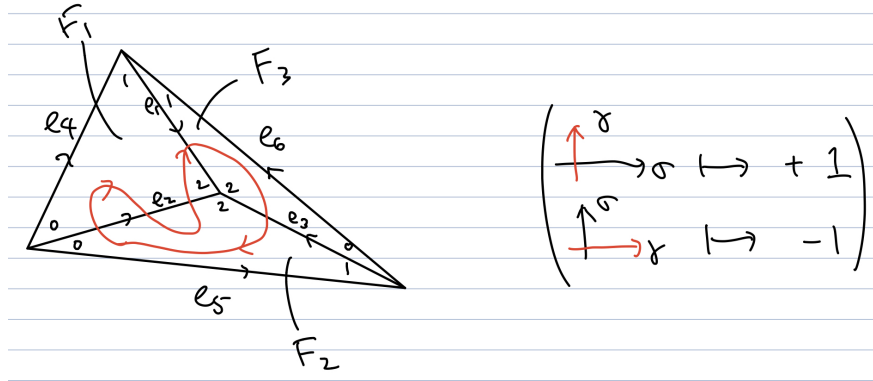


FIGURE 1. Oriented closed curve in a surface

γ gives us the following 1-cocycle ϕ which maps each e_i to an integer as following:

$$\begin{aligned}
 e_1 &\mapsto 1 \\
 e_2 &\mapsto 1 - 1 + 1 = 1 \\
 e_3 &\mapsto 1 \\
 e_4 &\mapsto 0 \\
 e_5 &\mapsto 0 \\
 e_6 &\mapsto 0.
 \end{aligned}$$

Then $\delta\phi = 0$ because

$$\begin{aligned}(\delta\phi)(F_1) &= \phi(\partial F_1) = \phi(e_1) - \phi(e_2) + \phi(e_4) = 0 \\(\delta\phi)(F_2) &= \phi(\partial F_2) = \phi(e_3) - \phi(e_2) + \phi(e_5) = 0 \\(\delta\phi)(F_3) &= \phi(\partial F_3) = \phi(e_1) - \phi(e_3) + \phi(e_6) = 0.\end{aligned}$$

This is not a coincidence because $\phi(\partial\sigma)$ represents

$$(\text{the number of times } \gamma \text{ enters } \sigma) - (\text{the number of times } \gamma \text{ exists } \sigma)$$

which is always 0 for any 2-simplex σ and any closed oriented curve γ . In this case, we call ϕ the **Poincare dual** to γ , or simply $\phi = \text{PD}(\gamma)$, and this concept can be generalized further.

Definition 1.2. Let X be a topological manifold of dimension n . Let γ be an $(n-k)$ -cycle in X transverse to the k -skeleton of X . Define $\phi \in C^k(X; \mathbb{Z})$ such that

$$\phi(\sigma) = \text{The number of intersections between } \sigma \text{ and } \gamma \text{ with signs.}$$

Then we call ϕ the **Poincare dual** of γ and denote it by $\phi = \text{PD}(\gamma)$.

2. POINCARÉ DUALITY

Since $H^*(T^n) \equiv \wedge_{\mathbb{Z}} M$ where $M = \langle v_1, \dots, v_n \rangle$, we have

k	0	1	2	\dots	$n-1$	n
$\text{rank } H^k(T^n; \mathbb{Z})$	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	\dots	$\binom{n}{n-1}$	$\binom{n}{n}$

This symmetry phenomenon is true in general and very useful. Another example: The cellular complex for \mathbb{CP}^n is $\mathbb{Z} \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0$. Thus

k	0	1	2	\dots	$2n-1$	$2n$
$\text{rank } H^k(\mathbb{CP}^n; \mathbb{Z})$	1	0	1	\dots	0	1

2.1. Orientations.

Definition 2.1. Let M be a triangulable closed n -manifold. Let $\sigma_1, \dots, \sigma_k$ be n -simplices such that $M = \sigma_1 \cup \dots \cup \sigma_k$. Then $\sigma_i \in C_n(M)$ for each i . Suppose that the ordering of the vertices in σ_i and the signs \pm can be chosen such that

$$\sum \pm \partial\sigma_i = 0 \in C_{n-1}(M).$$

Then M is said to be *orientable*.

Example 2.2. A tetrahedron and torus are examples of orientable manifolds.

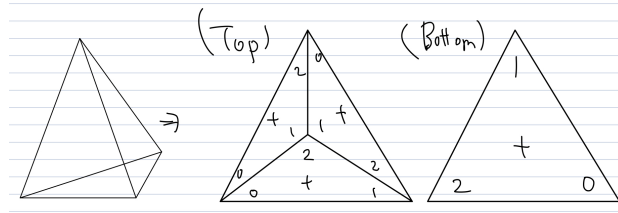


FIGURE 2. Orientation of a tetrahedron

Definition 2.3. Let M be an n -dimensional orientable manifold. Choose $\sigma_i \in C_n(M)$ and signs $\text{sgn}_i \in \{-1, 1\}$ such that $M = \sigma_1 \cup \dots \cup \sigma_k$ and $\sum \text{sgn}_i \partial\sigma_i = 0$. The class represented by $\sum \text{sgn}_i \sigma_i \in \ker(\partial)$ in $H_n(M)$ is called a fundamental class $[M]$.

Theorem 2.4. If M connected, then $[M]$ is a generator of $H_n(M)$.

Proof. By Poincare Duality (which will be discussed later (2.7)), $H_n(M) \cong H^0(M) = \mathbb{Z}$. Let $\sum c_i \sigma_i$ represent a generator of $H_n(M)$ where $c_i \in \mathbb{Z}$. Then $\sum \text{sgn}_i \sigma_i = \lambda \sum c_i \sigma_i = \sum (\lambda c_i) \sigma_i$ for some $\lambda \in \mathbb{Z}$. Since each $\lambda c_i = \text{sgn}_i \in \{-1, 1\}$, λ must be 1 or -1. Therefore, the class represented by $\sum \text{sgn}_i \sigma_i$ is a generator of $H_n(M)$. \square

Corollary 2.5. *There are two fundamental classes for any connected orientable manifold.*

Proof. By (2.4), a fundamental class $[M]$ is a generator of $H_n(M) = \mathbb{Z}$. Since \mathbb{Z} has exactly two generators 1, -1, M has exactly two fundamental classes. \square

Definition 2.6. Let M be a connected, orientable manifold. Then a choice of a fundamental class is called an orientation of M .

2.2. Poincare Duality (Version 1).

Theorem 2.7. *If M is an orientable closed n -manifold, then*

$$H^k(M; G) \cong H_{n-k}(M; G)$$

for any integer $0 \leq k \leq n$ and an abelian group G .

Remark. *If M is not orientable, Poincare Duality holds when $G = \mathbb{Z}/2$. In other words,*

$$H^k(M; \mathbb{Z}/2) \cong H_{n-k}(M; \mathbb{Z}/2).$$

For instance,

$H_*(\mathbb{RP}^2; \mathbb{Z})$	0	$\mathbb{Z}/2$	\mathbb{Z}
$H^*(\mathbb{RP}^2; \mathbb{Z})$	$\mathbb{Z}/2$	0	\mathbb{Z}

so Poincare Duality does not hold in this case. However, with $\mathbb{Z}/2$,

$H_*(\mathbb{RP}^2; \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H^*(\mathbb{RP}^2; \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

so Poincare Duality holds in this case.

Definition 2.8. The k th Betti number of a manifold M is defined to be $b_k = \text{rank}(H^k(M; \mathbb{Z}))$.

Theorem 2.9. $b_k = b_{n-k}$ for all k if M is a closed orientable n -manifold.

Proof. By the Universal Coefficient Theorem, $\text{rank}(H^k(M)) = \text{rank}(H_k(M))$. By Poincare Duality, $\text{rank}(H^k(M)) = \text{rank}(H_{n-k}(M))$. Therefore, $b_k = \text{rank}(H^k(M)) = \text{rank}(H_{n-k}(M)) = \text{rank}(H^{n-k}(M)) = b_{n-k}$. \square

2.3. Cap Products. There is a nice way to explicitly write down the isomorphism when G is a commutative ring.

Definition 2.10. Let a space X and a commutative ring R be given. For any $k \geq l$, define the cap product

$$\frown: C_k(X; R) \otimes C^l(X; R) \longrightarrow C_{k-l}(X; R)$$

$$\sigma \otimes \phi \longmapsto \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[v_l, \dots, v_k]}$$

Theorem 2.11. $\partial(\sigma \frown \phi) = (-1)^l((\partial\sigma) \frown \phi - \sigma \frown (\partial\phi))$.

Proof. Let $k \geq l, \sigma \in C_{k+1}(X; R), \phi \in C^l(X; R)$ be given.

$$\begin{aligned}
(\partial\sigma) \frown \phi &= \left(\sum_j (-1)^j \sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_k]} \right) \frown \phi \\
&= \sum_j (-1)^j (\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_k]} \frown \phi) \\
&= \sum_{j=1}^l (-1)^j \phi(\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} + \sum_{j=l+1}^k (-1)^j \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[v_l, \dots, \hat{v}_j, \dots, v_k]} \\
&= \sum_{j=1}^l (-1)^j \phi(\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} + (-1)^{l+1} \phi(\sigma|_{[v_0, \dots, v_l, \hat{v}_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} \\
&\quad + (-1)^l \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[\hat{v}_l, v_{l+1}, \dots, v_k]} + \sum_{j=l+1}^k (-1)^j \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[v_l, \dots, \hat{v}_j, \dots, v_k]} \\
&= \sum_{j=1}^{l+1} (-1)^j \phi(\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} + \sum_{j=l}^k (-1)^j \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[v_l, \dots, \hat{v}_j, \dots, v_k]}.
\end{aligned}$$

We will compute each summand.

$$\begin{aligned}
\sum_{j=1}^{l+1} (-1)^j \phi(\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} &= \phi \left(\sum_{j=1}^{l+1} (-1)^j \sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{l+1}]} \right) \sigma|_{[v_{l+1}, \dots, v_k]} \\
&= \phi(\partial\sigma|_{[v_0, \dots, v_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} \\
&= \sigma \frown \delta\phi.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{j=l}^k (-1)^j \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[v_l, \dots, \hat{v}_j, \dots, v_k]} &= (-1)^l \sum_{j=0}^{k-l} (-1)^j \phi(\sigma|_{[v_0, \dots, v_{l+j}]}) \sigma|_{[v_l, \dots, \hat{v}_{l+j}, \dots, v_k]} \\
&= (-1)^l \partial(\phi(\sigma|_{[v_0, \dots, v_{l+j}]}) \sigma|_{[v_l, \dots, v_k]}) \\
&= (-1)^l \partial(\sigma \frown \phi).
\end{aligned}$$

Therefore, we obtain the desired result $\partial(\sigma \frown \phi) = (-1)^l((\partial\sigma) \frown \phi - \sigma \frown (\delta\phi))$. \square

Theorem 2.12. $\frown: C_k(X; R) \otimes C^l(X; R) \rightarrow C_{k-l}(X; R)$ induces a map $H_k(X; R) \otimes H^l(X; R) \rightarrow H_{k-l}(X; R)$.

Proof. Let $[\sigma] \in H_k(X; R)$ and $[\phi] \in H^l(X; R)$ be given where $\sigma \in C_k(X; R)$ and $\phi \in C^l(X; R)$. Then $\partial(\sigma) = \delta(\phi) = 0$. By (2.11), $\partial(\sigma \frown \phi) = (-1)^l(0 - 0) = 0$. Therefore, $\sigma \frown \phi$ represents a class in $H_{k-l}(X; R)$. \square

2.4. Poincare Duality (Version 2).

Theorem 2.13. If M is an orientable closed n -manifold, let $[M] \in H_n(M)$ be a fundamental class for M , and let R be a commutative ring. View $[M] \in H_n(M; R)$. Then the map

$$H^l(M; R) \longrightarrow H_{n-l}(M; R)$$

$$\phi \longmapsto [M] \frown \phi$$

is an R -module isomorphism.

If M is not orientable, then $[M] \in H_n(M; \mathbb{Z}/2)$ still exists, and we have

$$H^l(M; \mathbb{Z}/2) \longrightarrow H_{n-l}(M; \mathbb{Z}/2)$$

$$\phi \longmapsto [M] \frown \phi$$