ALGEBRAIC TOPOLOGY II LECTURE NOTES

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Contents

1. Poincare Duality

Since $H^*(T^n) \equiv \wedge_{\mathbb{Z}} M$ where $M = \langle v_1, \cdots, v_n \rangle$, we have

k	0	1	2	$\cdots$	$n-1$	n
rank $H^k(T^n; \mathbb{Z})$	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\cdots$	$\binom{n}{n-1}$	$\binom{n}{n}$

This symmetry phenomenon is true in general and very useful. Another example: The cellular complex for $\mathbb{CP}^n$ is $\mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z} \to 0$. Thus

k	0	1	2	$\cdots$	$2n-1$	$2n$
rank $H^k(\mathbb{CP}^n; \mathbb{Z})$	1	0	1	$\cdots$	0	1

1.1. **Orientations.**

Definition 1.1. Let M be a triangulable closed n-manifold. Let $\sigma_1, \cdots, \sigma_k$ be n-simplices such that $M = \sigma_1 \cup \cdots \cup \sigma_k$. Then $\sigma_i \in C_n(M)$ for each i. Suppose that the ordering of the vertices in σ_i and the signs $\pm$ can be chosen such that

$$\sum \pm \partial \sigma_i = 0 \in C_{n-1}(M).$$

Then M is said to be *orientable*.

Example 1.2. A tetrahedron and torus are examples of orientable manifolds.

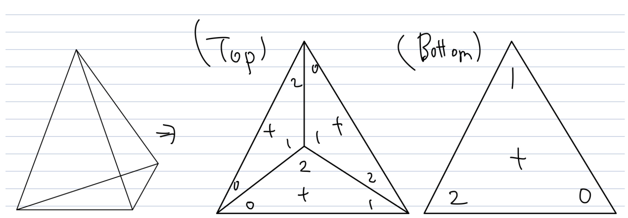

Figure 1. Orientation of a tetrahedron

Definition 1.3. Let M be an n-dimensional orientable manifold. Choose $\sigma_i \in C_n(M)$ and signs $\mathrm{sgn}_i \in \{-1, 1\}$ such that $M = \sigma_1 \cup \cdots \cup \sigma_k$ and $\sum \mathrm{sgn}_i \partial \sigma_i = 0$. The class represented by $\sum \mathrm{sgn}_i \sigma_i \in \ker(\partial)$ in $H_n(M)$ is called a fundamental class $[M]$.

Theorem 1.4. *If M connected, then $[M]$ is a generator of $H_n(M)$.*

Proof. By Poincare Duality(which will be discussed later (1.7)), $H_n(M) \cong H^0(M) = \mathbb{Z}$. Let $\sum c_i\sigma_i$ represent a generator of $H_n(M)$ where $c_i \in \mathbb{Z}$. Then $\sum \mathrm{sgn}_i\,\sigma_i = \lambda \sum c_i\sigma_i = \sum(\lambda c_i)\sigma_i$ for some $\lambda \in \mathbb{Z}$. Since each $\lambda c_i = \mathrm{sgn}_i \in \{-1, 1\}$, λ must be 1 or -1. Therefore, the class represented by $\sum \mathrm{sgn}_i\,\sigma_i$ is a generator of $H_n(M)$. ☐

Corollary 1.5. *There are two fundamental classes for any connected orientable manifold.*

Proof. By (1.4), a fundamental class $[M]$ is a generator of $H_n(M) = \mathbb{Z}$. Since $\mathbb{Z}$ has exactly two generators $1, -1$, M has exactly two fundamental classes. ☐

Definition 1.6. Let M be a connected, orientable manifold. Then a choice of a fundamental class is called an orientation of M.

1.2. **Poincare Duality(Version 1).**

Theorem 1.7. *If M is an orientable closed n-manifold, then*

$$H^k(M; G) \cong H_{n-k}(M; G)$$

for any integer $0 \leq k \leq n$ and an abelian group G.

Remark. *If M is not orientable, Poincare Duality holds when $G = \mathbb{Z}/2$. In other words,*

$$H^k(M; \mathbb{Z}/2) \cong H_{n-k}(M; \mathbb{Z}/2).$$

For instance,

$H_*(\mathbb{RP}^2; \mathbb{Z})$	*0*	$\mathbb{Z}/2$	$\mathbb{Z}$
$H^*(\mathbb{RP}^2; \mathbb{Z})$	$\mathbb{Z}/2$	*0*	$\mathbb{Z}$

so Poincare Duality does not hold in this case. However, with $\mathbb{Z}/2$,

$H_*(\mathbb{RP}^2; \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H^*(\mathbb{RP}^2; \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

so Poincare Duality holds in this case.

Definition 1.8. The kth Betti number of a manifold M is defined to be $b_k = \mathrm{rank}(H^k(M; \mathbb{Z}))$.

Theorem 1.9. *$b_k = b_{n-k}$ for all k if M is a closed orientable n-manifold.*

Proof. By the Universal Coefficient Theorem, $\mathrm{rank}(H^k(M)) = \mathrm{rank}(H_k(M))$. By Poincare Duality, $\mathrm{rank}(H^k(M)) = \mathrm{rank}(H_{n-k}(M))$. Therefore, $b_k = \mathrm{rank}(H^k(M)) = \mathrm{rank}(H_{n-k}(M)) = \mathrm{rank}(H^{n-k}(M)) = b_{n-k}$. ☐

1.3. **Cap Products.**

There is a nice way to explicitly write down the isomorphism when G is a commutative ring.

Definition 1.10. Let a space X and a commutative ring R be given. For any $k \geq l$, define the cap product

$$\frown: \qquad C_k(X; R) \otimes C^l(X; R) \longrightarrow C_{k-l}(X; R)$$

$$\sigma \otimes \phi \longmapsto \phi(\sigma|_{[v_0, \cdots, v_l]})\sigma|_{[v_l, \cdots, v_k]}$$

Theorem 1.11. $\partial(\sigma \frown \phi) = (-1)^l((\partial\sigma) \frown \phi - \sigma \frown (\partial\phi))$.

Proof. Let $k \geq l, \sigma \in C_{k+1}(X;R), \phi \in C^l(X;R)$ be given.

$$\begin{aligned}
(\partial\sigma) \frown \phi &= (\sum_j (-1)^j \sigma|_{[v_0,\cdots,\hat{v}_j,\cdots,v_k]}) \frown \phi \\
&= \sum_j (-1)^j (\sigma|_{[v_0,\cdots,\hat{v}_j,\cdots,v_k]} \frown \phi) \\
&= \sum_{j=1}^{l} (-1)^j \phi(\sigma|_{[v_0,\cdots,\hat{v}_j,\cdots,v_{l+1}]})\sigma|_{[v_{l+1},\cdots,v_k]} + \sum_{j=l+1}^{k} (-1)^j \phi(\sigma|_{[v_0,\cdots,v_l]})\sigma|_{[v_l,\cdots,\hat{v}_j,\cdots,v_k]} \\
&= \sum_{j=1}^{l} (-1)^j \phi(\sigma|_{[v_0,\cdots,\hat{v}_j,\cdots,v_{l+1}]})\sigma|_{[v_{l+1},\cdots,v_k]} + (-1)^{l+1}\phi(\sigma|_{[v_0,\cdots,v_l,\hat{v}_{l+1}]})\sigma|_{[v_{l+1},\cdots,v_k]} \\
&\quad + (-1)^l \phi(\sigma|_{[v_0,\cdots,v_l]})\sigma|_{[\hat{v}_l,v_{l+1},\cdots,v_k]} + \sum_{j=l+1}^{k} (-1)^j \phi(\sigma|_{[v_0,\cdots,v_l]})\sigma|_{[v_l,\cdots,\hat{v}_j,\cdots,v_k]} \\
&= \sum_{j=1}^{l+1} (-1)^j \phi(\sigma|_{[v_0,\cdots,\hat{v}_j,\cdots,v_{l+1}]})\sigma|_{[v_{l+1},\cdots,v_k]} + \sum_{j=l}^{k} (-1)^j \phi(\sigma|_{[v_0,\cdots,v_l]})\sigma|_{[v_l,\cdots,\hat{v}_j,\cdots,v_k]}.
\end{aligned}$$

We will compute each summand.

$$\begin{aligned}
\sum_{j=1}^{l+1} (-1)^j \phi(\sigma|_{[v_0,\cdots,\hat{v}_j,\cdots,v_{l+1}]})\sigma|_{[v_{l+1},\cdots,v_k]} &= \phi(\sum_{j=1}^{l+1} (-1)^j \sigma|_{[v_0,\cdots,\hat{v}_j,\cdots,v_{l+1}]})\sigma|_{[v_{l+1},\cdots,v_k]} \\
&= \phi(\partial\sigma|_{[v_0,\cdots,v_{l+1}]})\sigma|_{[v_{l+1},\cdots,v_k]} \\
&= \sigma \frown \delta\phi.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{j=l}^{k} (-1)^j \phi(\sigma|_{[v_0,\cdots,v_l]})\sigma|_{[v_l,\cdots,\hat{v}_j,\cdots,v_k]} &= (-1)^l \sum_{j=0}^{k-l} (-1)^j \phi(\sigma|_{[v_0,\cdots,v_{l+j}]})\sigma|_{[v_l,\cdots,\hat{v}_{l+j},\cdots,v_k]} \\
&= (-1)^l \partial(\phi(\sigma|_{[v_0,\cdots,v_{l+j}]})\sigma|_{[v_l,\cdots,v_k]}) \\
&= (-1)^l \partial(\sigma \frown \phi).
\end{aligned}$$

Therefore, we obtain the desired result $\partial(\sigma \frown \phi) = (-1)^l((\partial\sigma) \frown \phi - \sigma \frown (\delta\phi))$. □

Theorem 1.12. *$\frown: C_k(X;R) \otimes C^l(X;R) \to C_{k-l}(X;R)$ induces a map $H_k(X;R) \otimes H^l(X;R) \to H_{k-l}(X;R)$.*

Proof. Let $[\sigma] \in H_k(X;R)$ and $[\phi] \in H^l(X;R)$ be given where $\sigma \in C_k(X;R)$ and $\phi \in C^l(X;R)$. Then $\partial(\sigma) = \delta(\phi) = 0$. By (1.11), $\partial(\sigma \frown \phi) = (-1)^l(0-0) = 0$. Therefore, $\sigma \frown \phi$ represents a class in $H_{k-l}(X;R)$. □