

# ALGEBRAIC TOPOLOGY II LECTURE NOTES

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## 1. GEOMETRIC DESCRIPTION OF COCYCLES

**Example 1.1.** Let  $X$  be a  $\Delta$ -complex and  $\phi \in C^k(X; \mathbb{Z})$ . What does it mean that  $\delta\phi = 0$ ? As a toy model, we will consider a surface. One way to construct such a  $\phi \in C^1(X; \mathbb{Z})$  is to take an oriented closed curve  $\gamma$  transverse to 1-simplices. Then we define  $\phi(\sigma)$  to be the number of intersections between  $\sigma$  and  $\gamma$  with signs. See the following example:

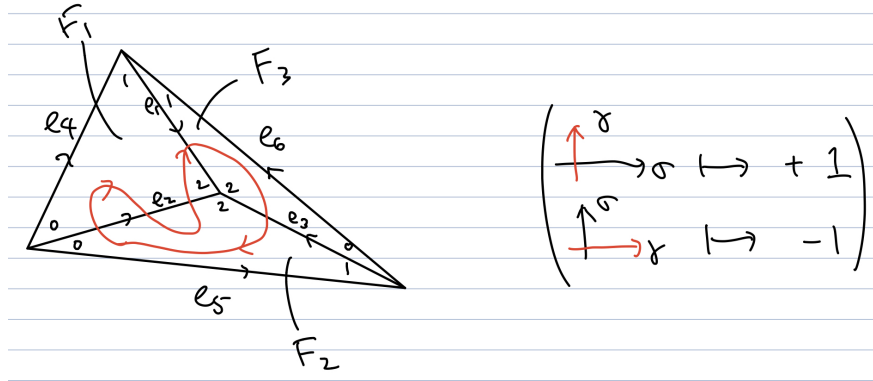


FIGURE 1. Oriented closed curve in a surface

$\gamma$  gives us the following 1-cocycle  $\phi$  which maps each  $e_i$  to an integer as following:

$$\begin{aligned}
 e_1 &\mapsto 1 \\
 e_2 &\mapsto 1 - 1 + 1 = 1 \\
 e_3 &\mapsto 1 \\
 e_4 &\mapsto 0 \\
 e_5 &\mapsto 0 \\
 e_6 &\mapsto 0.
 \end{aligned}$$

Then  $\delta\phi = 0$  because

$$\begin{aligned}(\delta\phi)(F_1) &= \phi(\partial F_1) = \phi(e_1) - \phi(e_2) + \phi(e_4) = 0 \\(\delta\phi)(F_2) &= \phi(\partial F_2) = \phi(e_3) - \phi(e_2) + \phi(e_5) = 0 \\(\delta\phi)(F_3) &= \phi(\partial F_3) = \phi(e_1) - \phi(e_3) + \phi(e_6) = 0.\end{aligned}$$

This is not a coincidence because  $\phi(\partial\sigma)$  represents

$$(\text{the number of times } \gamma \text{ enters } \sigma) - (\text{the number of times } \gamma \text{ exits } \sigma)$$

which is always 0 for any 2-simplex  $\sigma$  and any traverse closed oriented curve  $\gamma$ . In this case, we call  $\phi$  the **Poincare dual** to  $\gamma$ , or simply  $\phi = \text{PD}(\gamma)$ , and this concept can be generalized further.

**Definition 1.2.** Let  $X$  be a topological manifold of dimension  $n$ . Let  $\gamma$  be an  $(n-k)$ -cycle in  $X$  transverse to the  $k$ -skeleton of  $X$ . Define  $\phi \in C^k(X; \mathbb{Z})$  such that

$$\phi(\sigma) = \text{The number of intersections between } \sigma \text{ and } \gamma \text{ with signs.}$$

Then we call  $\phi$  the **Poincare dual** of  $\gamma$  and denote it by  $\phi = \text{PD}(\gamma)$ .

**Example 1.3.** We will look at a torus which is a slightly more complicated example. We obtain the following

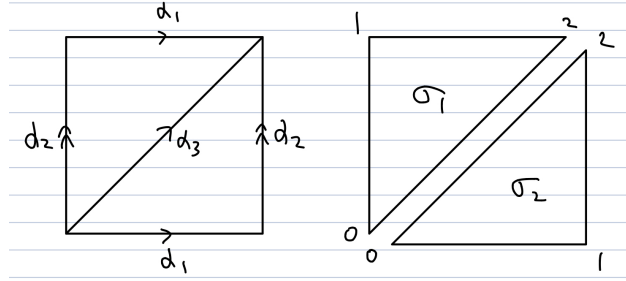


FIGURE 2. Torus

cellular chain complex from Figure 2:

$$C_2 = \langle \sigma_1, \sigma_2 \rangle \longrightarrow C_1 = \langle \alpha_1, \alpha_2, \alpha_3 \rangle \longrightarrow C_0 = \langle v \rangle$$

$$\sigma_i \longmapsto \alpha_1 + \alpha_2 - \alpha_3$$

$$\alpha_i \longmapsto 0$$

Thus

$$H_i(T^2) = \begin{cases} \mathbb{Z} & (i = 0, 2) \\ \mathbb{Z}^2 & (i = 1). \end{cases}$$

Now, we will examine the cellular cochain complex.

$$\begin{aligned}\delta(v^*)(\alpha_i) &= v^*(\partial\alpha_i) \\ &= v^*(0) \\ &= 0\end{aligned}$$

for any  $\alpha_i \in C_1$ . Therefore,  $\delta(v^*) = 0$ .

$$\begin{aligned}\delta(\alpha_1^*)(\sigma_i) &= \alpha_1^*(\partial(\sigma_i)) \\ &= \alpha_1^*(\alpha_1 + \alpha_2 - \alpha_3) \\ &= 1\end{aligned}$$

for any  $\sigma_i \in C_2$ . By performing similar calculation on  $\alpha_2^*$  and  $\alpha_3^*$ , we obtain

$$C^2 = \langle \sigma_1^*, \sigma_2^* \rangle \longleftarrow C^1 = \langle \alpha_1^*, \alpha_2^*, \alpha_3^* \rangle \longleftarrow C^0 = \langle v^* \rangle$$

$$\sigma_1^* + \sigma_2^* \longleftarrow \alpha_1^*, \alpha_2^*$$

$$-(\sigma_1^* + \sigma_2^*) \longleftarrow \alpha_3^*$$

$$0 \longleftarrow v^*$$

Thus

$$H^i(T^2) = \begin{cases} \mathbb{Z} & (i = 0, 2) \\ \mathbb{Z}^2 & (i = 1). \end{cases}$$

- $H^0(T^2) = \langle v^* \rangle / 0$ , so  $H^0(T^2)$  is generated by  $[v^*]$ .
- $H^2(T^2)$  is generated by  $[\sigma_2^*]$  because  $H^2(T^2) = \langle \sigma_1^*, \sigma_2^* \rangle / \langle \sigma_1^* + \sigma_2^* \rangle$ .

We can picture  $H^1(T^2)$  by using Poincare duals. Let  $\phi_1 = \text{PD}(\gamma_1), \phi_2 = \text{PD}(\gamma_2)$ . Then  $\phi_1(\alpha_1) = 0, \phi_1(\alpha_2) =$

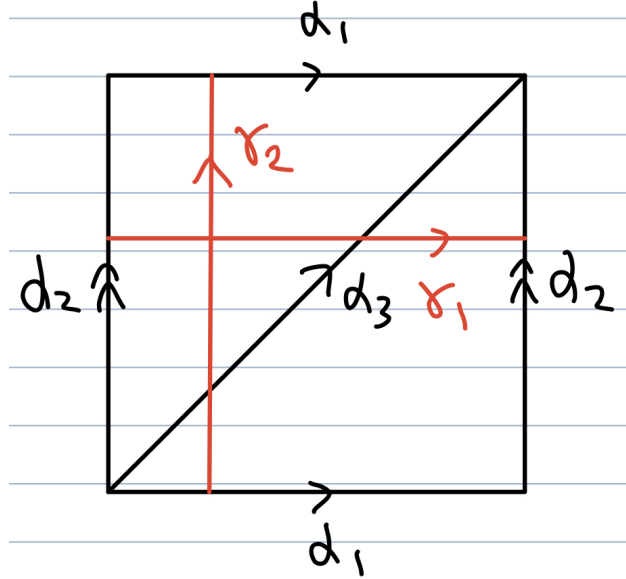


FIGURE 3. Torus Poincare Duals

1,  $\phi_1(\alpha_3) = 1$ , and  $\phi_2(\alpha_1) = -1, \phi_2(\alpha_2) = 0, \phi_2(\alpha_3) = -1$ . Moreover,

$$(\delta\phi_1)(\sigma_i) = \phi_1(\partial\sigma_i) = \phi_1(\alpha_1 + \alpha_2 - \alpha_3) = \phi_1(\alpha_1) + \phi_1(\alpha_2) - \phi_1(\alpha_3) = 0.$$

$$(\delta\phi_2)(\sigma_i) = \phi_2(\partial\sigma_i) = \phi_2(\alpha_1 + \alpha_2 - \alpha_3) = \phi_2(\alpha_1) + \phi_2(\alpha_2) - \phi_2(\alpha_3) = 0.$$

Therefore, the classes represented by  $\phi_1, \phi_2$  are in  $H^1(T^2; \mathbb{Z})$ . Moreover, they clearly generate  $H^1(T^2; \mathbb{Z})$ , so every element in  $H^1(T^2; \mathbb{Z})$  can be considered as a function with a fixed closed curve that counts the number of times the curve intersects a given 1-simplex.

Now, we will examine the cup product structure. By definition, we know that  $[\phi_1] \smile [\phi_2] \in H^2(T^2; \mathbb{Z})$ . Therefore, we will check what  $\sigma_1, \sigma_2$  get sent to.

$$\begin{aligned} (\phi_1 \smile \phi_2)(\sigma_1) &= \phi_1(\sigma_1|_{[0,1]})\phi_2(\sigma_1|_{[1,2]}) \\ &= \phi_1(\alpha_2)\phi_2(\alpha_1) \\ &= -1 \cdot 1 = -1. \\ (\phi_1 \smile \phi_2)(\sigma_2) &= \phi_1(\sigma_2|_{[0,1]})\phi_2(\sigma_2|_{[1,2]}) \\ &= \phi_1(\alpha_1)\phi_2(\alpha_2) \\ &= 0. \end{aligned}$$

Therefore,  $[\phi_1 \smile \phi_2] = [\sigma_1^*]$ . Similarly, we obtain  $(\phi_2 \smile \phi_1)(\sigma_1) = 0$  and  $(\phi_2 \smile \phi_1)(\sigma_2) = -1$ . Thus  $[\phi_2 \smile \phi_1] = -[\sigma_1^*]$ .

## 2. POINCARÉ DUALITY

Since  $H^*(T^n) \cong \wedge_{\mathbb{Z}} M$  where  $M = \langle v_1, \dots, v_n \rangle$ , we have

|                                     |                |                |                |         |                  |                |
|-------------------------------------|----------------|----------------|----------------|---------|------------------|----------------|
| $k$                                 | 0              | 1              | 2              | $\dots$ | $n-1$            | $n$            |
| $\text{rank } H^k(T^n; \mathbb{Z})$ | $\binom{n}{0}$ | $\binom{n}{1}$ | $\binom{n}{2}$ | $\dots$ | $\binom{n}{n-1}$ | $\binom{n}{n}$ |

This symmetry phenomenon is true in general and very useful. Another example: The cellular complex for  $\mathbb{CP}^n$  is  $\mathbb{Z} \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0$ . Thus

|                                               |   |   |   |         |        |      |
|-----------------------------------------------|---|---|---|---------|--------|------|
| $k$                                           | 0 | 1 | 2 | $\dots$ | $2n-1$ | $2n$ |
| $\text{rank } H^k(\mathbb{CP}^n; \mathbb{Z})$ | 1 | 0 | 1 | $\dots$ | 0      | 1    |

### 2.1. Orientations.

**Definition 2.1.** Let  $M$  be a triangulable closed  $n$ -manifold. Let  $\sigma_1, \dots, \sigma_k$  be  $n$ -simplices such that  $M = \sigma_1 \cup \dots \cup \sigma_k$ . Then  $\sigma_i \in C_n(M)$  for each  $i$ . Suppose that the ordering of the vertices in  $\sigma_i$  and the signs  $\pm$  can be chosen such that

$$\sum \pm \partial \sigma_i = 0 \in C_{n-1}(M).$$

Then  $M$  is said to be *orientable*.

**Example 2.2.** A tetrahedron and torus are examples of orientable manifolds.

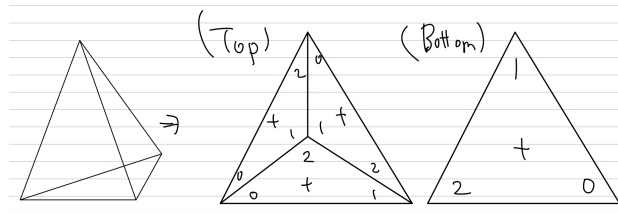


FIGURE 4. Orientation of a tetrahedron

**Definition 2.3.** Let  $M$  be an  $n$ -dimensional orientable manifold. Choose  $\sigma_i \in C_n(M)$  and signs  $\text{sgn}_i \in \{-1, 1\}$  such that  $M = \sigma_1 \cup \dots \cup \sigma_k$  and  $\sum \text{sgn}_i \partial \sigma_i = 0$ . The class represented by  $\sum \text{sgn}_i \sigma_i \in \ker(\partial)$  in  $H_n(M)$  is called a fundamental class  $[M]$ .

**Theorem 2.4.** If  $M$  connected, then  $[M]$  is a generator of  $H_n(M)$ .

*Proof.* By Poincare Duality (which will be discussed later (2.7)),  $H_n(M) \cong H^0(M) = \mathbb{Z}$ . Let  $\sum c_i \sigma_i$  represent a generator of  $H_n(M)$  where  $c_i \in \mathbb{Z}$ . Then  $\sum \text{sgn}_i \sigma_i = \lambda \sum c_i \sigma_i = \sum (\lambda c_i) \sigma_i$  for some  $\lambda \in \mathbb{Z}$ . Since each  $\lambda c_i = \text{sgn}_i \in \{-1, 1\}$ ,  $\lambda$  must be 1 or -1. Therefore, the class represented by  $\sum \text{sgn}_i \sigma_i$  is a generator of  $H_n(M)$ .  $\square$

**Corollary 2.5.** *There are two fundamental classes for any connected orientable manifold.*

*Proof.* By (2.4), a fundamental class  $[M]$  is a generator of  $H_n(M) = \mathbb{Z}$ . Since  $\mathbb{Z}$  has exactly two generators 1, -1,  $M$  has exactly two fundamental classes.  $\square$

**Definition 2.6.** Let  $M$  be a connected, orientable manifold. Then a choice of a fundamental class is called an orientation of  $M$ .

## 2.2. Poincare Duality (Version 1).

**Theorem 2.7.** *If  $M$  is an orientable closed  $n$ -manifold, then*

$$H^k(M; G) \cong H_{n-k}(M; G)$$

*for any integer  $0 \leq k \leq n$  and an abelian group  $G$ .*

**Remark.** *If  $M$  is not orientable, Poincare Duality holds when  $G = \mathbb{Z}/2$ . In other words,*

$$H^k(M; \mathbb{Z}/2) \cong H_{n-k}(M; \mathbb{Z}/2).$$

*For instance,*

|                                  |                |                |              |
|----------------------------------|----------------|----------------|--------------|
| $H_*(\mathbb{RP}^2; \mathbb{Z})$ | 0              | $\mathbb{Z}/2$ | $\mathbb{Z}$ |
| $H^*(\mathbb{RP}^2; \mathbb{Z})$ | $\mathbb{Z}/2$ | 0              | $\mathbb{Z}$ |

*so Poincare Duality does not hold in this case. However, with  $\mathbb{Z}/2$ ,*

|                                    |                |                |                |
|------------------------------------|----------------|----------------|----------------|
| $H_*(\mathbb{RP}^2; \mathbb{Z}/2)$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ |
| $H^*(\mathbb{RP}^2; \mathbb{Z}/2)$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ |

*so Poincare Duality holds in this case.*

**Definition 2.8.** The  $k$ th Betti number of a manifold  $M$  is defined to be  $b_k = \text{rank}(H^k(M; \mathbb{Z}))$ .

**Theorem 2.9.**  $b_k = b_{n-k}$  for all  $k$  if  $M$  is a closed orientable  $n$ -manifold.

*Proof.* By the Universal Coefficient Theorem,  $\text{rank}(H^k(M)) = \text{rank}(H_k(M))$ . By Poincare Duality,  $\text{rank}(H^k(M)) = \text{rank}(H_{n-k}(M))$ . Therefore,  $b_k = \text{rank}(H^k(M)) = \text{rank}(H_{n-k}(M)) = \text{rank}(H^{n-k}(M)) = b_{n-k}$ .  $\square$

**2.3. Cap Products.** There is a nice way to explicitly write down the isomorphism when  $G$  is a commutative ring.

**Definition 2.10.** Let a space  $X$  and a commutative ring  $R$  be given. For any  $k \geq l$ , define the cap product

$$\frown: C_k(X; R) \otimes C^l(X; R) \longrightarrow C_{k-l}(X; R)$$

$$\sigma \otimes \phi \longmapsto \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[v_l, \dots, v_k]}$$

**Theorem 2.11.**  $\partial(\sigma \frown \phi) = (-1)^l((\partial\sigma) \frown \phi - \sigma \frown (\partial\phi))$ .

*Proof.* Let  $k \geq l, \sigma \in C_{k+1}(X; R), \phi \in C^l(X; R)$  be given.

$$\begin{aligned}
(\partial\sigma) \frown \phi &= \left( \sum_j (-1)^j \sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_k]} \right) \frown \phi \\
&= \sum_j (-1)^j (\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_k]} \frown \phi) \\
&= \sum_{j=1}^l (-1)^j \phi(\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} + \sum_{j=l+1}^k (-1)^j \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[v_l, \dots, \hat{v}_j, \dots, v_k]} \\
&= \sum_{j=1}^l (-1)^j \phi(\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} + (-1)^{l+1} \phi(\sigma|_{[v_0, \dots, v_l, \hat{v}_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} \\
&\quad + (-1)^l \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[\hat{v}_l, v_{l+1}, \dots, v_k]} + \sum_{j=l+1}^k (-1)^j \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[v_l, \dots, \hat{v}_j, \dots, v_k]} \\
&= \sum_{j=1}^{l+1} (-1)^j \phi(\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} + \sum_{j=l}^k (-1)^j \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[v_l, \dots, \hat{v}_j, \dots, v_k]}.
\end{aligned}$$

We will compute each summand.

$$\begin{aligned}
\sum_{j=1}^{l+1} (-1)^j \phi(\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} &= \phi \left( \sum_{j=1}^{l+1} (-1)^j \sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{l+1}]} \right) \sigma|_{[v_{l+1}, \dots, v_k]} \\
&= \phi(\partial\sigma|_{[v_0, \dots, v_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} \\
&= \sigma \frown \delta\phi.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{j=l}^k (-1)^j \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[v_l, \dots, \hat{v}_j, \dots, v_k]} &= (-1)^l \sum_{j=0}^{k-l} (-1)^j \phi(\sigma|_{[v_0, \dots, v_{l+j}]} ) \sigma|_{[v_l, \dots, \hat{v}_{l+j}, \dots, v_k]} \\
&= (-1)^l \partial(\phi(\sigma|_{[v_0, \dots, v_{l+j}]})) \sigma|_{[v_l, \dots, v_k]} \\
&= (-1)^l \partial(\sigma \frown \phi).
\end{aligned}$$

Therefore, we obtain the desired result  $\partial(\sigma \frown \phi) = (-1)^l((\partial\sigma) \frown \phi - \sigma \frown (\delta\phi))$ .  $\square$

**Theorem 2.12.**  $\frown: C_k(X; R) \otimes C^l(X; R) \rightarrow C_{k-l}(X; R)$  induces a map  $H_k(X; R) \otimes H^l(X; R) \rightarrow H_{k-l}(X; R)$ .

*Proof.* Let  $[\sigma] \in H_k(X; R)$  and  $[\phi] \in H^l(X; R)$  be given where  $\sigma \in C_k(X; R)$  and  $\phi \in C^l(X; R)$ . Then  $\partial(\sigma) = \delta(\phi) = 0$ . By (2.11),  $\partial(\sigma \frown \phi) = (-1)^l(0 - 0) = 0$ . Therefore,  $\sigma \frown \phi$  represents a class in  $H_{k-l}(X; R)$ .  $\square$

## 2.4. Poincare Duality (Version 2).

**Theorem 2.13.** If  $M$  is an orientable closed  $n$ -manifold, let  $[M] \in H_n(M)$  be a fundamental class for  $M$ , and let  $R$  be a commutative ring. View  $[M] \in H_n(M; R)$ . Then the map

$$H^l(M; R) \longrightarrow H_{n-l}(M; R)$$

$$\phi \longmapsto [M] \frown \phi$$

is an  $R$ -module isomorphism.

If  $M$  is not orientable, then  $[M] \in H_n(M; \mathbb{Z}/2)$  still exists, and we have

$$H^l(M;\mathbb{Z}/2) \longrightarrow H_{n-l}(M;\mathbb{Z}/2)$$

$$\phi \longmapsto [M] \frown \phi$$