

ALGEBRAIC TOPOLOGY II LECTURE NOTES

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1. POINCARÉ DUALITY

Since $H^*(T^n) \cong \wedge_{\mathbb{Z}} M$ where $M = \langle v_1, \dots, v_n \rangle$, we have

k	0	1	2	\dots	$n-1$	n
$\text{rank } H^k(T^n; \mathbb{Z})$	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	\dots	$\binom{n}{n-1}$	$\binom{n}{n}$

This symmetry phenomenon is true in general and very useful. Another example: The cellular complex for \mathbb{CP}^n is $\mathbb{Z} \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0$. Thus

k	0	1	2	\dots	$2n-1$	$2n$
$\text{rank } H^k(\mathbb{CP}^n; \mathbb{Z})$	1	0	1	\dots	0	1

1.1. Orientations.

Definition 1.1. Let M be a triangulable closed n -manifold. Let $\sigma_1, \dots, \sigma_k$ be n -simplices such that $M = \sigma_1 \cup \dots \cup \sigma_k$. Then $\sigma_i \in C_n(M)$ for each i . Suppose that the ordering of the vertices in σ_i and the signs \pm can be chosen such that

$$\sum \pm \partial \sigma_i = 0 \in C_{n-1}(M).$$

Then M is said to be *orientable*.

Example 1.2. A tetrahedron and torus are examples of orientable manifolds.

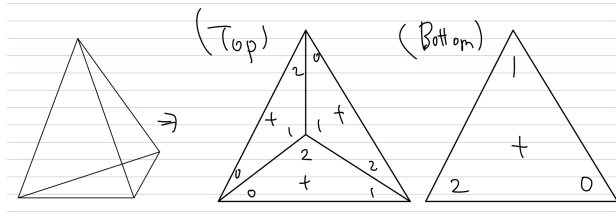


FIGURE 1. Orientation of a tetrahedron

Definition 1.3. Let M be an n -dimensional orientable manifold. Choose $\sigma_i \in C_n(M)$ and signs $\text{sgn}_i \in \{-1, 1\}$ such that $M = \sigma_1 \cup \dots \cup \sigma_k$ and $\sum \text{sgn}_i \partial \sigma_i = 0$. The class represented by $\sum \text{sgn}_i \sigma_i \in \ker(\partial)$ in $H_n(M)$ is called a fundamental class $[M]$.

Theorem 1.4. If M connected, then $[M]$ is a generator of $H_n(M)$.

Proof. By Poincare Duality(which will be discussed later (1.7)), $H_n(M) \cong H^0(M) = \mathbb{Z}$. Let $\sum c_i \sigma_i$ represent a generator of $H_n(M)$ where $c_i \in \mathbb{Z}$. Then $\sum \text{sgn}_i \sigma_i = \lambda \sum c_i \sigma_i = \sum (\lambda c_i) \sigma_i$ for some $\lambda \in \mathbb{Z}$. Since each $\lambda c_i = \text{sgn}_i \in \{-1, 1\}$, λ must be 1 or -1. Therefore, the class represented by $\sum \text{sgn}_i \sigma_i$ is a generator of $H_n(M)$. \square

Corollary 1.5. *There are two fundamental classes for any connected orientable manifold.*

Proof. By (1.4), a fundamental class $[M]$ is a generator of $H_n(M) = \mathbb{Z}$. Since \mathbb{Z} has exactly two generators $1, -1$, M has exactly two fundamental classes. \square

Definition 1.6. Let M be a connected, orientable manifold. Then a choice of a fundamental class is called an orientation of M .

1.2. Poincare Duality(Version 1).

Theorem 1.7. *If M is an orientable closed n -manifold, then*

$$H^k(M; G) \cong H_{n-k}(M; G)$$

for any integer $0 \leq k \leq n$ and an abelian group G .

Remark. *If M is not orientable, Poincare Duality holds when $G = \mathbb{Z}/2$. In other words,*

$$H^k(M; \mathbb{Z}/2) \cong H_{n-k}(M; \mathbb{Z}/2).$$

For instance,

$H_*(\mathbb{RP}^2; \mathbb{Z})$	0	$\mathbb{Z}/2$	\mathbb{Z}
$H^*(\mathbb{RP}^2; \mathbb{Z})$	$\mathbb{Z}/2$	0	\mathbb{Z}

so Poincare Duality does not hold in this case. However, with $\mathbb{Z}/2$,

$H_*(\mathbb{RP}^2; \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H^*(\mathbb{RP}^2; \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

so Poincare Duality holds in this case.