

# ALGEBRAIC TOPOLOGY II LECTURE NOTES

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## 1. POINCARÉ DUALITY

Since  $H^*(T^n) \cong \wedge_{\mathbb{Z}} M$  where  $M = \langle v_1, \dots, v_n \rangle$ , we have

$k$	0	1	2	$\dots$	$n-1$	$n$
$\text{rank } H^k(T^n; \mathbb{Z})$	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\dots$	$\binom{n}{n-1}$	$\binom{n}{n}$

This symmetry phenomenon is true in general and very useful. Another example: The cellular complex for  $\mathbb{CP}^n$  is  $\mathbb{Z} \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0$ . Thus

$k$	0	1	2	$\dots$	$2n-1$	$2n$
$\text{rank } H^k(\mathbb{CP}^n; \mathbb{Z})$	1	0	1	$\dots$	0	1

### 1.1. Orientations.

**Definition 1.1.** Let  $M$  be a triangulable closed  $n$ -manifold. Let  $\sigma_1, \dots, \sigma_k$  be  $n$ -simplices such that  $M = \sigma_1 \cup \dots \cup \sigma_k$ . Then  $\sigma_i \in C_n(M)$  for each  $i$ . Suppose that the ordering of the vertices in  $\sigma_i$  and the signs  $\pm$  can be chosen such that

$$\sum \pm \partial \sigma_i = 0 \in C_{n-1}(M).$$

Then  $M$  is said to be *orientable*.

**Example 1.2.** A tetrahedron and torus are examples of orientable manifolds.

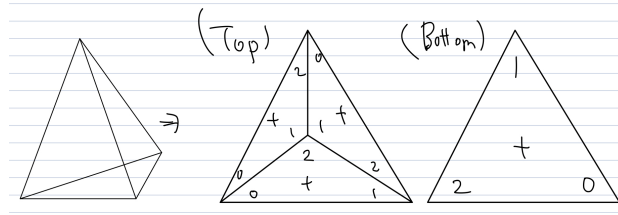


FIGURE 1. Orientation of a tetrahedron

**Definition 1.3.** Let  $M$  be an  $n$ -dimensional orientable manifold. Choose  $\sigma_i \in C_n(M)$  and signs  $\text{sgn}_i \in \{-1, 1\}$  such that  $M = \sigma_1 \cup \dots \cup \sigma_k$  and  $\sum \text{sgn}_i \partial \sigma_i = 0$ . The class represented by  $\sum \text{sgn}_i \sigma_i \in \ker(\partial)$  in  $H_n(M)$  is called a fundamental class  $[M]$ .

**Theorem 1.4.** If  $M$  connected, then  $[M]$  is a generator of  $H_n(M)$ .

*Proof.* By Poincare Duality (which will be discussed later (1.7)),  $H_n(M) \cong H^0(M) = \mathbb{Z}$ . Let  $\sum c_i \sigma_i$  represent a generator of  $H_n(M)$  where  $c_i \in \mathbb{Z}$ . Then  $\sum \text{sgn}_i \sigma_i = \lambda \sum c_i \sigma_i = \sum (\lambda c_i) \sigma_i$  for some  $\lambda \in \mathbb{Z}$ . Since each  $\lambda c_i = \text{sgn}_i \in \{-1, 1\}$ ,  $\lambda$  must be 1 or -1. Therefore, the class represented by  $\sum \text{sgn}_i \sigma_i$  is a generator of  $H_n(M)$ .  $\square$

**Corollary 1.5.** *There are two fundamental classes for any connected orientable manifold.*

*Proof.* By (1.4), a fundamental class  $[M]$  is a generator of  $H_n(M) = \mathbb{Z}$ . Since  $\mathbb{Z}$  has exactly two generators 1, -1,  $M$  has exactly two fundamental classes.  $\square$

**Definition 1.6.** Let  $M$  be a connected, orientable manifold. Then a choice of a fundamental class is called an orientation of  $M$ .

## 1.2. Poincare Duality (Version 1).

**Theorem 1.7.** *If  $M$  is an orientable closed  $n$ -manifold, then*

$$H^k(M; G) \cong H_{n-k}(M; G)$$

*for any integer  $0 \leq k \leq n$  and an abelian group  $G$ .*

**Remark.** *If  $M$  is not orientable, Poincare Duality holds when  $G = \mathbb{Z}/2$ . In other words,*

$$H^k(M; \mathbb{Z}/2) \cong H_{n-k}(M; \mathbb{Z}/2).$$

*For instance,*

$H_*(\mathbb{RP}^2; \mathbb{Z})$	0	$\mathbb{Z}/2$	$\mathbb{Z}$
$H^*(\mathbb{RP}^2; \mathbb{Z})$	$\mathbb{Z}/2$	0	$\mathbb{Z}$

*so Poincare Duality does not hold in this case. However, with  $\mathbb{Z}/2$ ,*

$H_*(\mathbb{RP}^2; \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H^*(\mathbb{RP}^2; \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

*so Poincare Duality holds in this case.*

**Definition 1.8.** The  $k$ th Betti number of a manifold  $M$  is defined to be  $b_k = \text{rank}(H^k(M; \mathbb{Z}))$ .

**Theorem 1.9.**  $b_k = b_{n-k}$  for all  $k$  if  $M$  is a closed orientable  $n$ -manifold.

*Proof.* By the Universal Coefficient Theorem,  $\text{rank}(H^k(M)) = \text{rank}(H_k(M))$ . By Poincare Duality,  $\text{rank}(H^k(M)) = \text{rank}(H_{n-k}(M))$ . Therefore,  $b_k = \text{rank}(H^k(M)) = \text{rank}(H_{n-k}(M)) = \text{rank}(H^{n-k}(M)) = b_{n-k}$ .  $\square$

**1.3. Cap Products.** There is a nice way to explicitly write down the isomorphism when  $G$  is a commutative ring.

**Definition 1.10.** Let a space  $X$  and a commutative ring  $R$  be given. For any  $k \geq l$ , define the cap product

$$\frown: C_k(X; R) \otimes C^l(X; R) \longrightarrow C_{k-l}(X; R)$$

$$\sigma \otimes \phi \longmapsto \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[v_l, \dots, v_k]}$$

**Theorem 1.11.**  $\partial(\sigma \frown \phi) = (-1)^l((\partial\sigma) \frown \phi - \sigma \frown (\partial\phi))$ .

*Proof.* Let  $k \geq l, \sigma \in C_{k+1}(X; R), \phi \in C^l(X; R)$  be given.

$$\begin{aligned}
(\partial\sigma) \frown \phi &= \left( \sum_j (-1)^j \sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_k]} \right) \frown \phi \\
&= \sum_j (-1)^j (\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_k]} \frown \phi) \\
&= \sum_{j=1}^l (-1)^j \phi(\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} + \sum_{j=l+1}^k (-1)^j \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[v_l, \dots, \hat{v}_j, \dots, v_k]} \\
&= \sum_{j=1}^l (-1)^j \phi(\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} + (-1)^{l+1} \phi(\sigma|_{[v_0, \dots, v_l, \hat{v}_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} \\
&\quad + (-1)^l \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[\hat{v}_l, v_{l+1}, \dots, v_k]} + \sum_{j=l+1}^k (-1)^j \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[v_l, \dots, \hat{v}_j, \dots, v_k]} \\
&= \sum_{j=1}^{l+1} (-1)^j \phi(\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} + \sum_{j=l}^k (-1)^j \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[v_l, \dots, \hat{v}_j, \dots, v_k]}.
\end{aligned}$$

We will compute each summand.

$$\begin{aligned}
\sum_{j=1}^{l+1} (-1)^j \phi(\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} &= \phi \left( \sum_{j=1}^{l+1} (-1)^j \sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{l+1}]} \right) \sigma|_{[v_{l+1}, \dots, v_k]} \\
&= \phi(\partial\sigma|_{[v_0, \dots, v_{l+1}]}) \sigma|_{[v_{l+1}, \dots, v_k]} \\
&= \sigma \frown \delta\phi.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{j=l}^k (-1)^j \phi(\sigma|_{[v_0, \dots, v_l]}) \sigma|_{[v_l, \dots, \hat{v}_j, \dots, v_k]} &= (-1)^l \sum_{j=0}^{k-l} (-1)^j \phi(\sigma|_{[v_0, \dots, v_{l+j}]} ) \sigma|_{[v_l, \dots, \hat{v}_{l+j}, \dots, v_k]} \\
&= (-1)^l \partial(\phi(\sigma|_{[v_0, \dots, v_{l+j}]})) \sigma|_{[v_l, \dots, v_k]} \\
&= (-1)^l \partial(\sigma \frown \phi).
\end{aligned}$$

Therefore, we obtain the desired result  $\partial(\sigma \frown \phi) = (-1)^l ((\partial\sigma) \frown \phi - \sigma \frown (\delta\phi))$ . □