# MATH 602(HOMEWORK 3)

HIDENORI SHINOHARA

## 1. Exercises

**Exercise.** (Exercise 1) The ideal generated by the three polynomials contains $-yz^4+yz^2+y=(xy^2-xz+y)-y(xy-z^2)+z(x-yz^4)$. However, its leading term $-yz^4$ is not in the ideal generated by the leading terms of the three polynomials.

**Exercise.** (Exercise 2) Remainder $= -y^{15}+y^{14}+7y^{13}-7y^{12}-21y^{11}+21y^{10}+35y^9-35y^8-35y^7+35y^6+21y^5-21y^4-7y^3+7y^2+y-1$, $q_1=x^6y^{14}-6x^6y^{12}+15x^6y^{10}-20x^6y^8+15x^6y^6-6x^6y^4+x^6y^2+x^2y^{14}-6x^2y^{12}+15x^2y^{10}-20x^2y^8+15x^2y^6-6x^2y^4+x^2y^2$, $q_2=0$

Remainder $= y^{23}+y^{11}-y+1$, $q_1=x^6y^2+x^5y^5+x^4y^8+x^3y^{11}+x^2y^{14}+x^2y^2+xy^{17}+xy^5+y^{20}+y^8$, $q_2=0$

**Exercise.** (Exercise 3) $z^3-x^2=(y-x^2)-(y-z^3)$ is in the ideal generated by the two polynomials, but the leading term is not in the ideal generated by the two polynomials.

**Exercise.** (Exercise 4) $0\in\sqrt{0}$, $a,b\in\sqrt{0} \implies (a+b)^{m+n-1}=\sum_{i=0}^{m+n-1}\binom{m+n-1}{i}a^ib^{m+n-1-i}=0$, and $\forall a\in\sqrt{0}, \forall x\in R, (ax)^n=a^nx^n=0$, so $\sqrt{0}$ is an ideal.

**Exercise.** (Exercise 5)

Solve this.

**Exercise.** (Exercise 6) Tensoring an exact sequence with $M\otimes_A N$ is the same as tensoring it with $M$ first and tensoring the resulting sequence with $N$ later.

**Exercise.** (Exercise 7) Since $0\to I\xrightarrow{i} R\xrightarrow{q} R/I\to 0$ is exact, $I\otimes M\to R\otimes M\to (R/I)\otimes M\to 0$ is exact.

$$\begin{aligned}(R/I)\otimes M &= \operatorname{im}(q\otimes \mathrm{Id})\\ &\cong R\otimes M/\ker(q\otimes\mathrm{Id})\\ &\cong R\otimes M/\operatorname{im}(i\otimes\mathrm{Id})\\ &\cong R\otimes M/I\otimes M.\end{aligned}$$

Now consider $\phi: R\otimes M\to M/IM$ that is the composition of $R\otimes M\to M: x\otimes y\mapsto xy$ and $M\to M/IM: x\mapsto x+IM$. In other words, $\phi$ is $x\otimes y\mapsto xy+IM$. Because the two maps are both surjective, $\phi$ must be surjective. The kernel of $\phi$ is $I\otimes M$ because

- For any $x\otimes y\in I\otimes M$, $\phi(x\otimes y)=xy+IM=0$ since $xy\in IM$.
- If $\phi(x\otimes y)=0$, then $xy\in IM$. In other words, $xy=x'y'$ for some $x'\in I$ and $y'\in M$. Then $x\otimes y=1\otimes xy=1\otimes x'y'=x'\otimes y'\in I\otimes M$.

Therefore, $M/IM\cong (R\otimes M)/(I\otimes M)\cong (R/I)\otimes M$.

**Exercise.** (Exercise 8) Let $pa+qb=1$ for some $p,q\in\mathbb{Z}$. Then $1\otimes 1=(pa+qb)\otimes(pa+qb)=pa\otimes pa+pa\otimes qb+qb\otimes pa+qb\otimes qb=0+0+0+0=0$.

**Exercise.** (Exercise 9) Let $T$ be a $\mathbb{Z}$-module and $f : \mathbb{Q} \times \mathbb{Q} \to T$ be a bilinear map. Then $f(a/b, c/d) = acf(1/b, 1/d) = acbf(1/b, 1/bd) = acf(1, 1/bd) = f(1, ac/bd)$. Define a bilinear map $h : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ by $(a, b) \mapsto ab$ and a linear map $g : \mathbb{Q} \to T$ by $a/b \mapsto f(1, a/b)$. Then $f = g \circ h$. The universal property of a tensor product is satisfied by $\mathbb{Q}$, so $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

**Exercise.** (Exercise 10) Let $a_1, \cdots, a_n, b_1, \cdots, b_m$ generate $M'$ and $M''$, respectively. Let $x_1, \cdots, x_n, y_1, \cdots, y_m \in M$ be chosen such that $x_i$ is the image of $a_i$ and the image of $y_j$ is $b_j$. We claim that $x_i, y_j$ generate $M$. Let $x \in M$ be given. Then $q(x) = d_1 b_1 + \cdots + d_m b_m$ for some $d_i \in M$, and thus $q(x - d_1 y_1 - \cdots - d_m y_m) = 0$. Therefore, $x - d_1 y_1 - \cdots - d_m y_m = i(c_1 a_1 + \cdots + c_n a_n) = c_1 x_1 + \cdots + c_n x_n$, so $x = c_1 x_1 + \cdots + c_n x_n + d_1 y_1 + \cdots + d_m y_m$.

**Exercise.** (Exercise 11) This statement is not true. When $R = \mathbb{Z}$ and $I = (0)$, $I \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.
However, the statement is true if $I \neq 0$. Let $u \in I$ be a nonzero element.
Define $h : I \times K \to K$ by $(a, x/y) \mapsto ax/y$. Let $f \in \mathrm{Hom}(I \times K, T)$ be given.
Define $g : K \to T$ by $x/y \mapsto f(u, x/uy)$. Then

$$\begin{aligned}
(g \circ h)(a, x/y) &= g(h(a, x/y)) \\
&= g(ax/y) \\
&= f(u, \frac{ax}{yu}) \\
&= af(u, \frac{x}{yu}) \\
&= f(au, \frac{x}{yu}) \\
&= uf(a, \frac{x}{yu}) \\
&= f(a, \frac{xu}{yu}) \\
&= f(a, x/y).
\end{aligned}$$

Thus $f, g, h$ commute and thus $K \cong I \otimes K$.