MATH 612(HOMEWORK 5)

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Exercise. (2.2.7) Let $f(x_1, \cdots, x_n) = (-x_1, x_2, x_3, \cdots, x_n)$. Then

$$\begin{array}{ccc} \mathbb{R}^n \setminus \{0\} & \xrightarrow{f} & \mathbb{R}^n \setminus \{0\} \\ \downarrow{r} & & \downarrow{r} \\ S^{n-1} & \xrightarrow{\text{reflection}} & S^{n-1} \end{array}$$

where r is the obvious deformation retraction. By (e) on P.134, the reflection map induces -1 on $H^{n-1}(S^{n-1})$. By naturality, f_* is -1.

Similarly, let $f(x_1, \cdots, x_n) = (cx_1, x_2, x_3, \cdots, x_n)$ with $c > 0$. Then

$$\begin{array}{ccc} \mathbb{R}^n \setminus \{0\} & \xrightarrow{f} & \mathbb{R}^n \setminus \{0\} \\ \downarrow{r} & & \downarrow{r} \\ S^{n-1} & \xrightarrow{g} & S^{n-1} \end{array}$$

where r is the obvious deformation retraction. Then g is a function that is homotopy equivalent to the identity map on S^{n-1}. By (e) on P.134, g induces the identity map on $H^{n-1}(S^{n-1})$. By naturality, f_* is 1.

Using the exact same argument, $(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) \mapsto (x_1, \cdots, x_j, \cdots, x_i, \cdots, x_n)$ induces -1 because a reflection is -1 and $(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) \mapsto (x_1, \cdots, x_i, \cdots, x_j + x_i, \cdots, x_n)$ induces 1 because homotopy equivalent maps induce the same map. Therefore, we have shown that elementary matrices induce 1 or -1 based on the sign of their determinants. Any invertible linear operation can be written as a product of elementary matrices and since $(fg)_* = f_* g_*$ the given invertible linear operation induces 1 or -1 based on the sign of their determinants.

3.3 (p. 257): 1, 2, 3. (We will talk a lot about the concept of orientability in class the Monday after break, but feel to start reading up.

And also the following: Show that there exists a homeomorphism $f : CP^n \to CP^n$ whose induced map on $H^{2n}(CP^n; Z)$ is multiplication by -1 iff n is odd.