MATH 601 (DUE 10/9)

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Contents

1. Modules

Exercise. (Problem 1) For each of the $\mathbb{Z}$-modules listed in the handout, answer the questions in the handout.

Proof.

(a) $M = \mathbb{Z}^3 \times \mathbb{Z}/86\mathbb{Z}$.

Solve this problem!

(b) $M = \prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z}$.

Solve this problem!

(c) $M = \mathbb{Z}[1/p] \subset \mathbb{Q}$.

Solve this problem!

(d) $M = \mathbb{Q}/\mathbb{Z}_{(p)}$.

Solve this problem!

□

2. Rings of Fractions

Exercise. (Problem 3) Let $T \subset R$ be the subset consisting of all nonzero divisors.

- Show that T is a multiplicative set.
- Let $s \in T$ and let $S = \{1, s, s^2, s^3, \cdots\} \subset T$. Show that the following rings are isomorphic: $S^{-1}R$, the subring $R[1/s] \subset T^{-1}R$, and the quotient ring $R[x]/(sx - 1)$.

Proof.

- Prove this!
- Prove this!

□

3. The Quadratic Equation

Exercise. (Problem 20)

Exercise. (Problem 21)

Exercise. (Problem 22)

4. Factorization in Integral Domains

Exercise. (Problem 5)

- Let k be a field and let $a \in k$. Construct a k-algebra isomorphism, $k[x,y]/(x-a) \to k[y]$. Justify your answer.
- Let $f(x,y) \in k[x,y]$. What is the image of $f(x,y)$ under the above isomorphism?

Proof.

Ask Professor Schoen after class on Friday.

- Let ϕ be defined such that $\phi(f(x,y)+(x-a)) = f(a,y)$.
  - Well-defined? Let $f(x,y)+(x-a) = g(x,y)+(x-a)$. Then $g(x,y) = f(x,y)+h(x,y)(x-a)$.

$$\begin{aligned}\phi(g(x,y)+(x-a)) &= \phi((f(x,y)+h(x,y)(x-a))+(x-a)) \\ &= f(a,y)+h(a,y)(a-a) \\ &= f(a,y) \\ &= \phi(f(x,y)).\end{aligned}$$

  - k-algebra homomorphism? Let $c \in k$, $f, g \in k[x,y]$ be given.

$$\begin{aligned}\phi(c(f+(x-a)) &= \phi(cf+(x-a)) \\ &= cf(a,y) \\ &= c\phi(f+(x-a)). \\ \phi((f+g)+(x-a)) &= (f+g)(a,y) \\ &= f(a,y)+g(a,y) \\ &= \phi(f+(x-a))+\phi(g+(x-a)). \\ \phi((fg)+(x-a)) &= (fg)(a,y) \\ &= f(a,y)g(a,y) \\ &= \phi(f+(x-a))\phi(g+(x-a)).\end{aligned}$$

- $\phi(f(x,y)+(x-a)) = f(a,y)$.

□

Exercise. (Problem 6)

- Give an example of a field k, an element $a \in k$ and a reducible polynomial $f(x,y) \in k[x,y]$ of degree n in y such that $f(a,y) \in k[y]$ is irreducible and has degree n.

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Proof.

- Let $k = \mathbb{Q}, a = 1, f(x, y) = xy$. Then the degree of $f(x, y)$ in y is 1. $f(x, y) = xy \in k[x, y]$ is reducible since x and y are not units in $k[x, y]$. However, $f(a, y) = 1y = y$ is irreducible in $k[y]$.

$\square$