

# MATH 611 (DUE 11/6)

HIDENORI SHINOHARA

## 1. SIMPLICIAL AND SINGULAR HOMOLOGY

**Exercise.** (Problem 14) Determine whether there exists a short exact sequence  $0 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_8 \oplus \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow 0$ . More generally, determine which abelian groups  $A$  fit into a short exact sequence  $0 \rightarrow \mathbb{Z}_{p^m} \rightarrow A \rightarrow \mathbb{Z}_{p^n} \rightarrow 0$  with  $p$  prime. What about the case of short exact sequences  $0 \rightarrow A \rightarrow \mathbb{Z}_n \rightarrow 0$ ?

*Proof.* Let  $\phi_1 : \mathbb{Z}_4 \rightarrow \mathbb{Z}_8 \oplus \mathbb{Z}_2, \phi_2 : \mathbb{Z}_8 \oplus \mathbb{Z}_2 \rightarrow \mathbb{Z}_4$  be defined such that  $\phi_1(a) = (2a, a)$  and  $\phi_2(a, b) = a + 2b$ . Then  $\ker(\phi_1) = 0, \text{Im}(\phi_1) = \ker(\phi_2) = \{(2k, k) \mid 0 \leq k \leq 3\}$  and  $\text{Im}(\phi_2) = \mathbb{Z}_4$ . Thus this is indeed an exact sequence.

We claim that  $A = \bigoplus_{i=1}^k \mathbb{Z}_{p^{a_i}}$  where  $k \leq 2, a_1 \geq \max\{m, n\}, a_i \geq a_{i+1}, \sum a_i = m + n$  are the only  $\mathbb{Z}$ -modules that satisfy the exact sequence  $0 \rightarrow \mathbb{Z}_{p^m} \xrightarrow{\alpha} A \xrightarrow{\beta} \mathbb{Z}_{p^n} \rightarrow 0$ . It is clear that  $\sum a_i = m + n$  since  $\alpha$  is injective and  $A/\alpha(\mathbb{Z}_{p^m}) = \mathbb{Z}_{p^n}$ .

- First we will show that these  $A$ 's indeed satisfy the exact sequence. When  $k = 1$ ,  $A = \mathbb{Z}_{p^{m+n}}$ . Then with  $\alpha : 1 \mapsto p^n$ , we have  $A/\alpha(\mathbb{Z}_{p^m}) = \mathbb{Z}_{p^n}$ , so we are done. Suppose  $k = 2$ . Then  $A = \mathbb{Z}_{p^{a_1}} \oplus \mathbb{Z}_{p^{a_2}}$ . Define  $\alpha$  such that  $1 \mapsto (p^{a_1-m}, 1)$ . Then  $\alpha$  is injective. Moreover, the order of  $\alpha(1)$  is  $p^m$  in  $A$ ,  $|A/\text{Im}(\alpha)| = p^n$ .  $A/\text{Im}(\alpha) = \langle (1, 0) + \text{Im}(\alpha) \rangle$  because for any  $(a, b) + \text{Im}(\alpha) \in A/\text{Im}(\alpha)$  we have  $(a, b) + \text{Im}(\alpha) = ((a, b) - b\alpha(1)) + \text{Im}(\alpha) = (a - bp^{a_1-m}, 0) + \text{Im}(\alpha)$ . Therefore,  $A/\text{Im}(\alpha)$  is a cyclic group of order  $p^n$ . In other words,  $A/\text{Im}(\alpha) = \mathbb{Z}_{p^n}$ .
- Next, we will show that these  $A$ 's are the only abelian groups to satisfy the exact sequence. Let  $A$  be any abelian group to satisfy the exact sequence. Then  $\alpha$  is injective and  $\beta$  is surjective by the exactness. Let  $u \in A$  such that  $\beta(u) = 1$ . We claim that every element in  $A$  can be uniquely expressed as  $a\alpha(1) + bu$  where  $0 \leq a \leq p^m - 1$  and  $0 \leq b \leq p^n - 1$ .

Let  $0 \leq a \leq p^m - 1, 0 \leq b \leq p^n - 1$  be given such that  $a\alpha(1) + bu = 0$ . Then  $-a\alpha(1) = bu$ , and  $\beta(-a\alpha(1)) = 0$ .  $bu = 0$  implies that  $b\beta(u) = 0$ , so  $b = 0$ . Moreover,  $-a\alpha(1) = 0$  implies that  $\alpha(-a) = 0$ , so  $a = 0$  because  $\alpha$  is injective.

Therefore, whenever  $(a_1, b_1) \neq (a_2, b_2)$ ,  $a_1\alpha(1) + b_1u \neq a_2\alpha(1) + b_2u$ . This implies that there are at least  $p^{m+n}$  elements in  $A$  of this form. By the exactness,  $\mathbb{Z}_{p^n} = A/\mathbb{Z}_{p^m}$ , so  $A$  must contain exactly  $p^{m+n}$  elements. Therefore, every element in  $A$  can be uniquely written in the form.

In other words,  $A$  can be generated by two elements. This implies that  $A = \mathbb{Z}_{p^{a_1}} \oplus \mathbb{Z}_{p^{a_2}}$  for some  $a_1 \geq a_2 \geq 0$ . Moreover, since  $\alpha$  is injective and  $\beta$  is surjective,  $A$  must contain an element of order  $\geq \max\{m, n\}$ . Therefore,  $a_1 \geq \max\{m, n\}$ .

Hence, the  $A$ 's listed above are all the possible abelian groups to satisfy the exact sequence.  $\square$

**Exercise.** (Problem 15) For an exact sequence  $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E$  show that  $C = 0$  if and only if the map  $A \rightarrow B$  is surjective and  $D \rightarrow E$  is injective. Hence, for a pair of spaces  $(X, A)$ , the inclusion  $A \rightarrow X$  induces isomorphisms on all homology groups if and only if  $H_n(X, A) = 0$  for all  $n$ .

*Proof.* Suppose  $C = 0$ .  $\text{Im}(\phi_{AB}) = \ker(\phi_{BC}) = B$ , so  $\phi_{AB}$  is surjective.  $\ker(\phi_{DE}) = \text{Im}(\phi_{CD}) = \{0\}$ , so  $\phi_{DE}$  is injective.

On the other hand, suppose  $\phi_{AB}$  is surjective and  $\phi_{DE}$  is injective.  $\text{Im}(\phi_{CD}) = \ker(\phi_{DE}) = \{0\}$ , so  $\phi_{CD}$  is the zero map. Therefore,  $\ker(\phi_{CD}) = C$ .  $\ker(\phi_{BC}) = \text{Im}(\phi_{AB}) = B$ , so  $\phi_{BC}$  is the zero map. Therefore,  $\text{Im}(\phi_{BC}) = 0$ . Hence,  $C = \ker(\phi_{CD}) = \text{Im}(\phi_{BC}) = 0$ .

By Theorem 2.16 and the discussion at the bottom of P.117(Hatcher), we have a long exact sequence of homology groups

$$(1.1) \quad H_n(A) \xrightarrow{i_*} H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X)$$

for  $n \geq 1$ . Suppose the inclusion induces isomorphisms on all homology groups. Then  $H_n(X, A) = 0$  for all  $n \geq 1$  by the first part. Moreover, we have  $H_1(X, A) \rightarrow H_0(A) \rightarrow H_0(X) \rightarrow H_0(X, A) \rightarrow 0$ . Since  $H_1(X, A) = 0$ , by the first part,  $H_0(X) = 0$ . In order for  $0 \rightarrow H_0(X, A) \rightarrow 0$  to be exact,  $H_0(X, A)$  must be 0. Therefore,  $H_n(X, A) = 0$  for all  $n \geq 0$ .

Suppose that  $H_n(X, A) = 0$  for all  $n \geq 0$ . By exact sequence 1.1 above,  $i_* : H_n(A) \rightarrow H_n(X)$  is surjective for  $n \geq 1$  and injective for  $n \geq 0$ . Thus  $i_*$  is bijective for all  $n \geq 1$ . We have  $H_1(X, A) \rightarrow H_0(A) \rightarrow H_0(X) \rightarrow H_0(X, A)$ . Since  $H_1(X, A) = H_0(X, A) = 0$ ,  $i_*$  must be bijective by the exactness. Therefore, the inclusion induces isomorphisms for all  $n$ .  $\square$

**Exercise.** (Problem 16)

- Show that  $H_0(X, A) = 0$  if and only if  $A$  meets each path-component of  $X$ .
- Show that  $H_1(X, A) = 0$  if and only if  $H_1(A) \rightarrow H_1(X)$  is surjective and each path-component of  $X$  contains at most one path-component of  $A$ .

*Proof.*

- Let  $\gamma_x + C_0(A) \in C_0(X)/C_0(A)$ . Since  $A$  meets each path-component of  $X$ , there exists a path  $\gamma : I \rightarrow X$  that joins a point  $a \in A$  and the image of  $\gamma_x$ . Then  $\gamma$  can be seen as an element of  $C_1(X)$  since  $\gamma$  maps a 1-simplex into  $X$ . Moreover,  $\partial\gamma = \gamma_x - \gamma_a$  where  $\gamma_a \in C_0(A)$  with  $\text{Im}(\gamma_a) = a$ . Therefore,  $\partial(\gamma + C_1(A)) = \gamma_x + C_0(A)$ , so  $\gamma_x + C_0(A) \in \text{Im}(\partial)$ . Hence,  $H_0(X, A) = \ker(\partial_0)/\text{Im}(\partial_1) = (C_0(X)/C_0(A))/(C_0(X)/C_1(A)) = 0$ .

On other hand, suppose that  $A$  does not meet each path component of  $X$ . Let  $x \in X$  be a point in a path component that  $A$  does not intersect. Let  $\gamma_x : \Delta^0 \rightarrow X$  such that  $\text{Im}(\gamma_x) = \{x\}$ . Then  $\gamma_x \in \ker(\partial_0) = C_0(X, A)$ . Let  $\gamma + C_1(A) \in C_1(X, A)$ . Then  $\partial_1(\gamma + C_1(A)) = \partial_1(\gamma) + C_0(A)$ . Let  $\gamma_{x_1}, \gamma_{x_2} \in C_0(X)$  such that  $\partial_1(\gamma) = \gamma_{x_1} - \gamma_{x_2}$ .  $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$  if and only if  $\gamma_{x_1} - \gamma_{x_2} - \gamma_x \in C_0(A)$ .

– If  $\gamma$  lies in the same path component as  $x$ , then so do  $x_1$  and  $x_2$ . Suppose  $x = x_1$ .

Since  $-\gamma_{x_2} \notin C_0(A)$ ,  $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$ . The case when  $x \neq x_1$  and  $x = x_2$  and the case when  $x \neq x_1$  and  $x \neq x_2$  can be proven in a similar way.

– If  $\gamma$  lies in a different path component, then  $\gamma_x \neq \gamma_{x_1}$  and  $\gamma_x \neq \gamma_{x_2}$ . Therefore,

$$\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A).$$

Therefore,  $\gamma_x \notin \text{Im}(\partial_1)$ . Thus  $H_0(X, A) = C_0(X, A)/\text{Im}(\partial_1)$  is not 0.

- Suppose  $H_1(X, A) = 0$ . By the exact sequence  $H_1(A) \xrightarrow{\phi} H_1(X) \rightarrow H_1(X, A) \rightarrow H_0(A) \xrightarrow{\psi} H_1(X)$ , we know that  $\phi$  is surjective and  $\psi$  is injective. Suppose that there is a path component of  $X$  that contains two or more path components of  $A$ . Let  $a, b$  be points in two distinct path components of  $A$  that are contained in a path component of  $X$ . We will regard  $a, b$  as functions  $\Delta^0 \rightarrow A$ . Then  $a, b \in C_0(A)$  and  $[a] \neq [b]$  in  $H_0(A)$  because  $a, b$  are in different path components of  $A$ , so  $a - b \notin \text{Im}(\partial_1)$ . However,  $a, b$  live in the same path component of  $X$ ,  $a - b \in \text{Im}(\partial_1) \subset C_0(X)$ . Therefore,  $\phi([a]) = \phi([b])$  where  $[a] \neq [b]$ . This is a contradiction because  $\phi$  is injective. Therefore, each path component of  $X$  contains at most one path component of  $A$ .

Show the other direction.

□

**Exercise.** (Problem 17)

- Compute the homology groups  $H_n(X, A)$  when  $X$  is  $S^2$  or  $S^1 \times S^1$  and  $A$  is a finite set of points in  $X$ .
- Compute the groups  $H_n(X, A)$  and  $H_n(X, B)$  for  $X$  a closed orientable surface of genus two with  $A$  and  $B$  the circles shown.

*Proof.*

- We will apply Theorem 2.16 to get the exact sequence with  $H_n(A), H_n(X), H_n(X, A)$ .
  - When  $n \geq 3$ ,  $H_n(S^2) \rightarrow H_n(S^2, A) \rightarrow H_{n-1}(A)$  shows that  $H_n(S^2, A)$  is 0 by the exactness since  $H_n(S^2) = H_{n-1}(A) = 0$ .
  - When  $n = 2$ ,  $H_n(A) \rightarrow H_n(S^2) \xrightarrow{\phi} H_n(S^2, A) \rightarrow H_{n-1}(A)$  shows that  $H_n(S^2, A) = H_n(S^2) = \mathbb{Z}$ . This is because  $H_n(A) = H_{n-1}(A) = 0$  so  $\phi$  is an isomorphism by the exactness.
  - By Problem 16,  $H_0(X, A) = 0$ . By the exact sequence  $0 \rightarrow H_1(X, A) \rightarrow H_0(A) \rightarrow H_0(X)$  where  $H_0(A) = \mathbb{Z}^{|A|}$  and  $H_0(X) = \mathbb{Z}$ , we have  $H_1(X, A) = \mathbb{Z}^{|A|-1}$ .

We will first compute the homology groups of a torus using Figure 1.  $C_2 = \{\sigma_1, \sigma_2\}, C_1 = \{a, b, c\}, C_0 = \{v_0\}$ .

- $H_2 = \ker(\partial_2) / \text{Im}(\partial_3) = \langle \sigma_1 - \sigma_2 \rangle / 0 = \mathbb{Z}$ .
- $H_1 = \ker(\partial_1) / \text{Im}(\partial_2) = \langle a, b, c \rangle / \langle b - a + c, c - a + b \rangle = \mathbb{Z}^2$  because  $b - a + c = c - a + b$ .
- $H_0 = \ker(\partial_0) / \text{Im}(\partial_1) = \langle v_0 \rangle / 0 = \mathbb{Z}$ .

Again, we will apply Theorem 2.16 to get the exact sequence with  $H_n(A), H_n(X)$ , and  $H_n(X, A)$ .

- When  $n \geq 3$ ,  $H_n(S^1 \times S^1) \rightarrow H_n(S^1 \times S^1, A) \rightarrow H_{n-1}(A)$  shows that  $H_n(S^1 \times S^1, A)$  is 0 by the exactness since  $H_n(S^1 \times S^1) = H_{n-1}(A) = 0$ .
- When  $n = 2$ ,  $H_n(A) \rightarrow H_n(S^1 \times S^1) \xrightarrow{\phi} H_n(S^1 \times S^1, A) \rightarrow H_{n-1}(A)$  shows that  $H_n(S^1 \times S^1, A) = H_n(S^1 \times S^1) = \mathbb{Z}$ . This is because  $H_n(A) = H_{n-1}(A) = 0$  so  $\phi$  is an isomorphism by the exactness.
- By Problem 16,  $H_0(X, A) = 0$ . We have the exact sequence  $H_1(A) \rightarrow H_1(T^2) \rightarrow H_1(T^2, A) \xrightarrow{\phi} H_0(A) \xrightarrow{\psi} H_0(T^2) \rightarrow H_0(T^2, A)$  where  $H_0(A) = \mathbb{Z}^{|A|}$ ,  $H_0(X) = \mathbb{Z}$ ,  $H_1(T^2) = \mathbb{Z}^2$ , and  $H_1(A) = H_0(T^2, A) = 0$ . Moreover,  $H_1(T^2, A) / \ker(\phi) =$

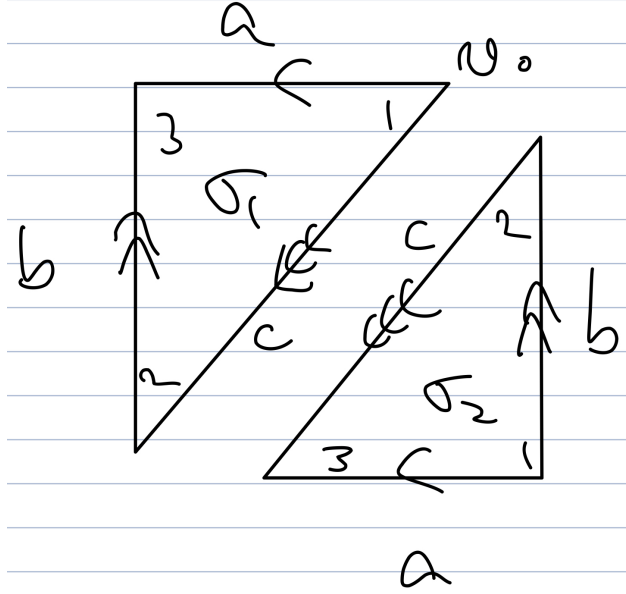


FIGURE 1. Problem 17

$\text{Im}(\phi) = \ker(\psi) = \mathbb{Z}^{|A|-1}$ . Since  $\ker(\phi) = \mathbb{Z}^2$  by the exactness,  $H_1(T^2, A) = \mathbb{Z}^{|A|+1}$ .

- $(X, A)$  is a good pair because  $A$  is a nonempty closed subspace that is a deformation retract of some neighborhood in  $X$ . By Proposition 2.22,  $H_n(X, A) = \tilde{H}_n(X/A)$  for all  $n$ . The quotient space  $X/A$  is  $T^2 \vee T^2$  where  $T^2$  is a torus. By Corollary 2.25,  $\tilde{H}_n(X/A) = \tilde{H}_n(T^2) \oplus \tilde{H}_n(T^2)$ .  $((T^2, p)$  is clearly a good pair for a point  $p \in T^2$ .) We calculate above that

$$H_n(T^2) = \begin{cases} \mathbb{Z} & (n = 0, 2) \\ \mathbb{Z}^2 & (n = 1) \\ 0 & (n \geq 3). \end{cases}$$

Therefore,

$$H_n(X, A) = \tilde{H}_n(X/A) = \tilde{H}_n(T^2) \oplus \tilde{H}_n(T^2) = \begin{cases} \mathbb{Z}^2 & (n = 2) \\ \mathbb{Z}^4 & (n = 1) \\ 0 & (n = 0, n \geq 3). \end{cases}$$

Similarly,  $X/B$  is  $T^2 \vee S^1$  because  $(X, B)$  is a good pair. Moreover,  $(S^1, p)$  is a good pair for a point  $p \in S^1$ , so it suffices to check  $\tilde{H}_n(T^2) \oplus \tilde{H}_n(S^1)$ .

Therefore,

$$H_n(X, B) = \tilde{H}_n(X/B) = \tilde{H}_n(T^2) \oplus \tilde{H}_n(S^1) = \begin{cases} \mathbb{Z} & (n = 2) \\ \mathbb{Z}^3 & (n = 1) \\ 0 & (n = 0, n \geq 3). \end{cases}$$

□

**Exercise.** (Problem 26) Show that  $H_1(X, A)$  is not isomorphic to  $\tilde{H}_1(X/A)$  if  $X = [0, 1]$  and  $A$  is the sequence  $1, 1/2, 1/3, \dots$  together with its limit 0.

*Proof.* We will show that  $H_1(X, A)$  is countable, and  $\tilde{H}_1(X/A) = H_q(X/A)$  is uncountable. We have an exact sequence  $\tilde{H}_1(X) \rightarrow \tilde{H}_1(X, A) \xrightarrow{\phi} \tilde{H}_0(A) \rightarrow \tilde{H}_0(X)$ . Since  $H_1(X, A) = \tilde{H}_1(X, A) = \tilde{H}_0(X) = 0$ ,  $\phi$  is an isomorphism. Thus  $\tilde{H}_1(X, A) = \tilde{H}_0(A) = \ker(\partial_1)/\text{Im}(\partial_2)$ . Since  $A$  is a disjoint union of points,  $\text{Im}(\partial_2) = 0$ .  $\ker(\partial_1) = \{\sum n_i \alpha_i \mid n_i \in \mathbb{Z}, \sum n_i = 0\}$  where  $\alpha_i$  is the point  $1/i$  by the definition of a reduced homology. Then this is generated by  $\{\alpha_1 - \alpha_2, \alpha_1 - \alpha_3, \alpha_1 - \alpha_4, \dots\}$ , so  $\tilde{H}_1(X, A)$  is countable.

We will show the existence of an injective map  $\zeta$  from the direct product  $\prod_{i=1}^{\infty} \mathbb{Z}$  to  $H_1(X/A)$ , which is homeomorphic to the Hawaiian earring. We will refer to the  $n$ th ring  $C_n$  as in Example 1.25. Let  $(k_1, \dots) \in \prod_{i=1}^{\infty} \mathbb{Z}$  be given. Construct the map  $f : I \rightarrow X/A$  that wraps  $k_n$  times around  $C_n$  in the time interval  $[1 - 1/n, 1 - 1/(n+1)]$ . This infinite composition of loops is certainly continuous at each time less than 1, and it is continuous at time 1 since every neighborhood of the basepoint in  $X/A$  contains all but finitely many of the circles  $C_n$ . This shows that  $f \in C_1(X/A)$ . Moreover,  $\partial(f) = v_0 - v_0 = 0$  where  $v_0$  is the origin of the Hawaiian earring. Therefore,  $[f] \in H_1(X/A)$ . We define  $\zeta(k_1, \dots) = [f]$ .

Let  $(k_1, \dots) \neq (l_1, \dots) \in \prod_{i=1}^{\infty} \mathbb{Z}$  be given. Let  $\zeta(k_1, \dots) = f, \zeta(l_1, \dots) = g$  as described above. Let  $i$  be an index such that  $k_i \neq l_i$ . Let  $F : X/A \rightarrow S^1$  be a continuous map that maps  $C_n$  onto  $S_1$  and  $C_i$  to  $-1$  for all  $i$  where  $S_1$  is seen as a subset of  $\mathbb{C}$ . Then  $F$  induces a group homomorphism  $F_* : H_1(X/A) \rightarrow H_1(S^1)$  where  $F_*([f]) = k_n$  and  $F_*([g]) = l_n$ . Since  $F_*([f]) \neq F_*([g])$ ,  $[f] \neq [g]$ . This shows the injectivity of  $\zeta$  and hence  $H_1(X/A)$  must be uncountable.

Therefore,  $H_1(X, A)$  is not isomorphic to  $H_1(X/A)$ . □