

# MATH 612 (HOMEWORK 3)

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**Exercise.** (3.1.11) Using the cellular homology, we obtain

$$\begin{aligned}\tilde{H}_i(X) &= \begin{cases} \mathbb{Z}/m\mathbb{Z} & (i = n) \\ 0 & (i \neq n). \end{cases} \\ \tilde{H}^i(X) &= \begin{cases} \mathbb{Z}/m\mathbb{Z} & (i = n+1) \\ 0 & (i \neq n+1). \end{cases}\end{aligned}$$

From previous homework,

$$\tilde{H}^i(X/S^n) = \tilde{H}_i(S^{n+1}) = \begin{cases} \mathbb{Z} & (i = n+1) \\ 0 & (i \neq n+1). \end{cases}$$

Thus the map on  $\tilde{H}_i(-; \mathbb{Z})$  is the zero map for each  $i$ . On the other hand, the long exact sequence of a pair gives us  $\tilde{H}^{n+1}(X, S^n; \mathbb{Z}) \xrightarrow{q^*} \tilde{H}^{n+1}(X; \mathbb{Z}) \rightarrow \tilde{H}^{n+1}(S^n; \mathbb{Z})$  where  $\tilde{H}^{n+1}(S^n; \mathbb{Z}) = 0$ , so  $q^*$  is surjective. Therefore, it is nontrivial because  $\tilde{H}^{n+1}(X; \mathbb{Z}) \neq 0$ .

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ext}(H_n(X); \mathbb{Z}) & \longrightarrow & H^{n+1}(X; \mathbb{Z}) & \longrightarrow & \text{Hom}(H_{n+1}(X); \mathbb{Z}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ext}(H_n(X/S^n); \mathbb{Z}) & \longrightarrow & H^{n+1}(X/S^n; \mathbb{Z}) & \longrightarrow & \text{Hom}(H_{n+1}(X/S^n); \mathbb{Z}) \longrightarrow 0 \end{array}$$

is

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}_m & \longrightarrow & \mathbb{Z}_m & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \longrightarrow 0. \end{array}$$

This splitting is not natural because the middle term in the first sequence is isomorphic to  $\mathbb{Z}_m \oplus 0$  and the second one is  $0 \oplus \mathbb{Z}$ .

The long exact sequence of a pair gives us  $\tilde{H}_n(S^n; \mathbb{Z}) \rightarrow \tilde{H}_n(X; \mathbb{Z}) \rightarrow \tilde{H}_n(X, S^n; \mathbb{Z}) = \tilde{H}_n(S^{n+1}; \mathbb{Z}) = 0$  which implies the surjectivity of the induced map. Since  $\tilde{H}_n(X; \mathbb{Z}) \neq 0$ , the induced map is nonzero.

The map induced on  $\tilde{H}^i(-; \mathbb{Z})$  is the zero map for any  $i$  because at least one of  $\tilde{H}^i(S^n; \mathbb{Z})$  or  $\tilde{H}^i(X; \mathbb{Z})$  is 0 for each  $i$ .

**Exercise.** (3.1.13)

**Exercise.** (3.2.1)

**Exercise.** (3.2.2) Suppose  $X$  is the union of contractible open sets  $A_1, \dots, A_n$ . Since each  $A_i$  is contractible,  $H^k(X, A_i; R) = H^k(X; R)$  for all  $k \geq 1$ .

$$\begin{array}{ccc}
H^{k_1}(X, A_1; R) \times \cdots \times H^{k_n}(X, A_n; R) & \longrightarrow & H^{k_1+\cdots+k_n}(X, A_1 \cup \cdots \cup A_n; R) \\
\downarrow \cong & & \downarrow \\
H^{k_1}(X; R) \times \cdots \times H^{k_n}(X; R) & \xrightarrow{f} & H^{k_1+\cdots+k_n}(X; R).
\end{array}$$

This diagram commutes by the naturality of a cup product.  $H^{k_1+\cdots+k_n}(X, \bigcup_i A_i; R) = H^{k_1+\cdots+k_n}(X, X; R) = 0$  for all  $k + l \geq 1$ . By the commutativity of this diagram, the function  $f$  must be 0.

**Exercise.** (3.2.3(a)) Suppose otherwise. Let  $f : \mathbb{RP}^n \rightarrow \mathbb{RP}^m$  be such a function. Then  $f$  induces a map on  $f^* : H^*(\mathbb{RP}^m) \rightarrow H^*(\mathbb{RP}^n)$ . In other words,  $f^* : \mathbb{Z}_m[\alpha]/(\alpha^{m+1}) \rightarrow \mathbb{Z}_n[\beta]/(\beta^{n+1})$  where  $\alpha, \beta$  are generators of  $H^1(\mathbb{RP}^m)$  and  $H^1(\mathbb{RP}^n)$ .  $H^1(\mathbb{RP}^m; \mathbb{Z}_2) = \mathbb{Z}_2 = \{0, \alpha\}$  and  $H^1(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2 = \{0, \beta\}$ . Since  $f$  induces a nontrivial map  $H^1(\mathbb{RP}^m; \mathbb{Z}_2) \rightarrow H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ ,  $f^*(\alpha) = \beta$ . However,  $f^*(0) = f^*(\alpha^{m+1}) = (f^*(\alpha))^{m+1} = \beta^{m+1} \neq 0$  because  $m < n$ . This is a contradiction, so such a function does not exist.

$H^1(\mathbb{CP}^n; \mathbb{Z}_2) = 0$  for any  $n$ , so there exists no such nontrivial map. The case for  $H^2(\mathbb{CP}^n)$  can be argued the same way as above because  $H^2(\mathbb{CP}^n; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$  where  $\alpha$  is a generator of  $H^2(\mathbb{CP}^n)$ .

**Exercise.** (3.2.3(b))

**Exercise.** (3.2.6)

**Exercise.** (3.2.7)