MATH 601 (DUE 11/22)

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1. THE THEOREM ON SYMMETRIC POLYNOMIALS

Exercise. (Problem 1) By substituting $u_4 = 0$, we get $u_1^2u_2u_3 + u_1u_2^2u_3 + u_1u_2u_3^2 = s_3s_1$. s_3s_1 with 4 variables expands to $u_1^2u_2u_3 + u_1^2u_2u_4 + u_1^2u_3u_4 + u_1u_2^2u_3 + u_1u_2^2u_4 + u_1u_2u_3^2 + 4u_1u_2u_3u_4 + u_1u_2u_4^2 + u_1u_3^2u_4 + u_1u_3u_4^2 + u_2^2u_3u_4 + u_2u_3^2u_4 + u_2u_3u_4^2$. Then $s_3s_1 - f$ where f is the original polynomial gives us $4u_1u_2u_3u_4 = 4s_4$. Therefore, $f = s_3s_1 - 4s_4$.

Exercise. (Problem 2) We are given that $|M - xI| = x^3 - ax^2 + bx - c$. This implies that $|M - (-x)I| = -x^3 - ax^2 - bx - c$. Since the determinant function preserves multiplication, $|M - xI||M - (-x)I| = |M^2 - x^2I|$. This implies $|M^2 - x^2I| = -x^6 + (a^2 - 2b)x^4 + (b^2 + 2ac)x^2 + c^2$. Therefore, the characteristic polynomial of M is $-x^3 + (a^2 - 2b)x^2 + (b^2 + 2ac)x + c^2$.

2. Galois Theory VI

Exercise. (Problem 3)

(a) $\{(123), (132), e\}$ is clearly a subgroup of the stabilizer group S_v of v. Since $(12) \notin S_v$, $3 \leq |S_v| \leq 5$. By Lagrange's Theorem, $S_v = \langle (123) \rangle$.
(b) By (i), S_3v contains only $[S_3 : S_v] = 2$ elements. Thus $v' = (12) \cdot v = u_2u_1^2 + u_1u_3^2 + u_3u_2^2$.
(c) By substituting $u_3 = 0$ for $v + v'$, we get $u_1u_2^2 + u_2u_1^2 = s_1s_2$. Then $v + v' - s_1s_2 = -3u_1u_2u_3 = -3s_3$. Therefore, $v + v' = s_1s_2 + 3s_3$.
(d) We will use the fundamental theorem of Galois Theory. $F(v) = K^{\langle (123) \rangle}$, so $|\langle (123) \rangle| = 3 = [K : F(v)]$. Moreover, $|\langle \mathrm{Gal}(K/F) \rangle| = [K : F]$. Therefore, $[F(v) : F] = [K : F]/[K : F(v)] = |\langle \mathrm{Gal}(K/F) \rangle|/3$.
(e) Calculation shows that $vv' = 9s_3^2 + s_3s_1^3 - 6s_3s_1s_2 + s_2^3$. By substituting $s_1 = 0, s_2 = p, s_3 = q$, we get $9q^2 + p^3$.