MATH 633(HOMEWORK 7)

HIDENORI SHINOHARA

Exercise. (1) Suppose f is locally bijective. For every $p \in U$, f is locally injective. Therefore, f' is nonzero in the neighborhood around p. In other words, f' is nonzero on U.

The other direction

Exercise. (10) Let $\sigma(z) = -i(z+1)/(z-1)$. Then σ sends the unit disk to the upper half-plane with ∞ since $\sigma(a+bi) = (-2b-(a^2+b^2-1)i)/((a-1)^2+b^2)$. On the other hand, $\sigma^{-1} : z \mapsto (z-i)/(z+i)$ sends the upper half plane with ∞ to the unit disk because $|a+(b-1)i| \leq |a+(b+1)i|$ if $b \geq 0$. Therefore, σ is a bijection between the unit disk and $H \cup \{\infty\}$. $F \circ \sigma$ sends the unit disk to the unit disk, and $F(\sigma(0)) = 0$. By Lemma 2.1, $|(F \circ \sigma)(w)| \leq |w|$ for every $w \in D$. Then for every $z \in \mathbb{H}$, $\sigma^{-1}(z) \in D$. Then $F(z) = |(F \circ \sigma)(\sigma^{-1}(z))| \leq |\sigma^{-1}(z)| = |(z-i)/(z+i)|$, which is the desired result.

Exercise. (12(a)) Let $a \neq b$ be two fixed points. Let $\sigma(z) = (z-a)/(1-\overline{a}z)$. Then σ sends a to 0 and maps D to D bijectively. Let $g = \sigma \circ f \circ \sigma^{-1}$. g has two fixed points, 0 and $\sigma(b)$. By applying Lemma 2.1, g is a rotation. However, g fixes $\sigma(b) \neq 0$, so g must be the identity map. Then f must be the identity.

Exercise. (12(b)) The map $\sigma : z \mapsto (z-i)/(z+i)$ maps the upper half-plane to the unit disk bijectively. Then $\sigma \circ f \circ \sigma^{-1}$ where $f(z) = z+1$ is a holomorphic bijection on f that has no fixed point because f has no fixed point.