

# MATH 602(HOMEWORK 1)

HIDENORI SHINOHARA

## Exercise. 1

- Let  $p \in V(I \cap J)$ . For any  $\sum_{i=1}^n f_i g_i \in IJ$ , we have  $f_i g_i \in I \cap J$  for each  $i$ . Thus  $(\sum_{i=1}^n f_i g_i)(p) = 0$ , so  $p \in V(IJ)$ . Let  $p \in V(IJ)$ . Let  $f \in I \cap J$ . Then  $f^2 \in IJ$ , so  $(f(p))^2 = 0$ . Thus  $f(p) = 0$ , so  $p \in V(I \cap J)$ . Therefore,  $V(I \cap J) = V(IJ)$ .

Let  $p \in V(I) \cup V(J)$ . Then either all polynomials in  $I$  vanish at  $p$  or all polynomials in  $J$  vanish at  $p$ . Thus all the polynomials in the intersection must vanish at  $p$ . Thus  $V(I) \cup V(J) \subset V(I \cap J)$ . On the other hand, let  $p \in V(I \cap J) \setminus (V(I) \cup V(J))$ . If no such element exists, we are done. Then every polynomial in the intersection vanishes at  $p$ . Let  $f \in I$  and  $g \in J$  be polynomials that do not vanish at  $p$ . Then  $fg \in I \cap J$ , so  $(fg)(p) = 0$ . However, this is impossible because  $f(p) \neq 0$  and  $g(p) \neq 0$ . Therefore,  $V(I) \cup V(J) = V(I \cap J)$ .

- $p \in V(I + J)$  if and only if  $\forall f \in I + J, f(p) = 0$  if and only if  $\forall f \in I, f(p) = 0$  and  $\forall f \in J, f(p) = 0$  if and only if  $p \in V(I) \cap V(J)$ .
- If every polynomial in  $J$  vanishes at a point, every polynomial in  $I$  must vanish at that point.
- If a polynomial vanishes in  $Y$ , then it must vanish in  $X$ .
- TODO

## Exercise. 2

•

$$\begin{aligned} y \in (I_1 + I_2)^e &\iff y \in f(I_1 + I_2)B \\ &\iff \exists x_1, x_2 \in I_1, I_2, b \in B, y = f(x_1 + x_2)b \\ &\iff \exists x_1, x_2 \in I_1, I_2, b \in B, y = f(x_1)b + f(x_2)b \\ &\iff y \in I_1^e + I_2^e. \end{aligned}$$

•

$$\begin{aligned} y \in (I_1 \cap I_2)^e &\implies y \in f(I_1 \cap I_2)B \\ &\implies \exists x \in I_1 \cap I_2, b \in B, y = f(x)b \\ &\implies (\exists x \in I_1, b \in B, y = f(x)b) \text{ and } (\exists x \in I_2, b \in B, y = f(x)b) \\ &\implies y \in I_1^e, y \in I_2^e \\ &\implies y \in I_1^e \cap I_2^e. \end{aligned}$$

- $(I_1 I_2)^e = f(I_1 I_2)B = (f(I_1)f(I_2))B = (f(I_1)B)(f(I_2)B)$ .  $f(I_1)f(I_2) = f(I_1 I_2)$  because the product of two ideals consists of a finite sum of elements and  $f$  preserves finite sums.

- Let  $x \in J_1^c + J_2^c$ . Then  $x \in f^{-1}(J_1) + f^{-1}(J_2)$ . Then  $x = a + b$  where  $a \in f^{-1}(J_1)$  and  $b \in f^{-1}(J_2)$ . This implies  $x = a + b$  where  $f(a) \in J_1$  and  $f(b) \in J_2$ . Then,  $f(x) = f(a + b) = f(a) + f(b) \in J_1 + J_2$ , so  $x \in f^{-1}(J_1 + J_2)$ .