MATH 611 HOMEWORK (DUE 9/18)

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Exercise. (Problem 12, Chapter 1.2) The Klein bottle is usually pictured as a subspace of $\mathbb{R}^3$ like the subspace $X \subset \mathbb{R}^3$ shown in the first figure at the right. If one wanted a model that could actually function as a bottle, one would delete the open disk bounded by the circle of self-intersection of X, producing a subspace $Y \subset X$. Show that $\pi_1(X) \approx \mathbb{Z} * \mathbb{Z}$ and that $\pi_1(Y)$ has the presentation $\langle a, b, c \mid aba^{-1}b^{-1}cb^{\epsilon}c^{-1} \rangle$ for $\epsilon = \pm 1$. Show also that $\pi_1(Y)$ is isomorphic to $\pi_1(\mathbb{R}^3 \setminus Z)$ for Z the graph shown in the figure.

Proof. We will construct X from the 1-skeleton in Figure 1. The 1-skeleton has three loops a, b, c, so the fundamental group is $\langle a, b, c \mid \rangle$. The main difference between X and the "proper" Klein bottle is that the loop a actually gets glued on the surface. Thus we will glue the first 2-cell to a, and another 2-cell on the loop $c^{-1}acbab^{-1}$. Therefore, we end up with the fundamental group $\langle a, b, c \mid a, c^{-1}aca^{-1}bab^{-1} \rangle$. Then $\langle a, b, c \mid a, c^{-1}acabab^{-1} \rangle \approx \langle b, c \mid \rangle \approx \mathbb{Z} * \mathbb{Z}$ since the relation $c^{-1}aca^{-1}bab^{-1}$ is trivial by the relation a.

In order to calculate the fundamental group of Y, it suffices to repeat the following step without attaching a 2-cell to a. Thus the fundamental group is $G = \langle a, b, c \mid c^{-1}aca^{-1}bab^{-1} \rangle$.

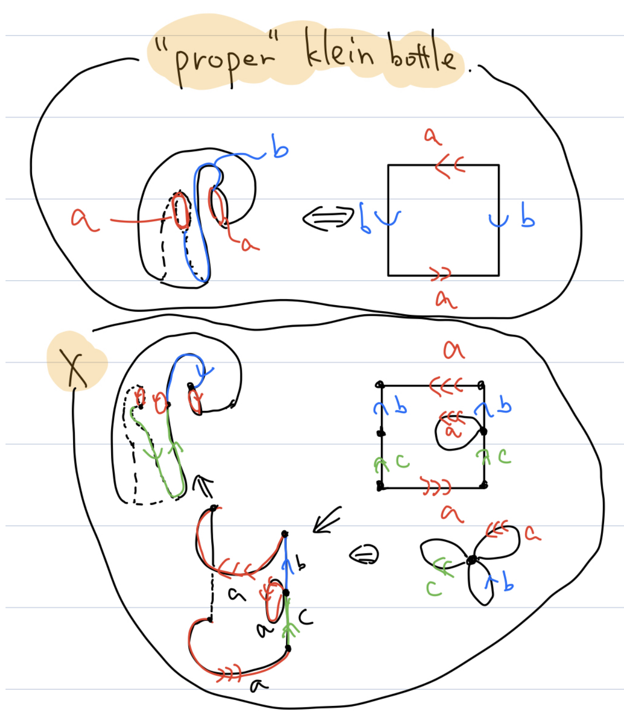

FIGURE 1. Fundamental Group of X

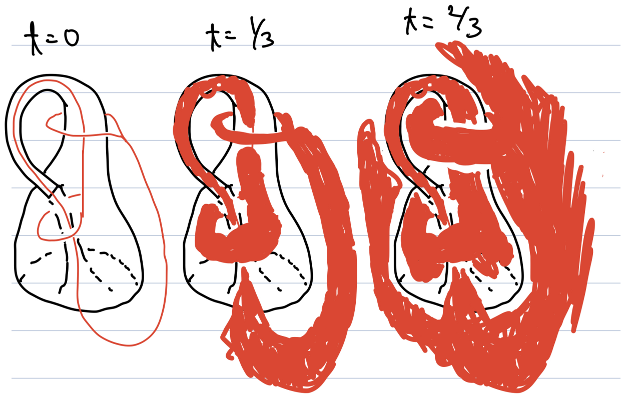

FIGURE 2. Deformation retract

This is isomorphic to the group given in the textbook, $H = \langle a, b, c \mid aba^{-1}b^{-1}cbc^{-1} \rangle$ by $\phi : G \to H$ that maps a to b, b to c, and c to a^{-1}.

We claim that there exists a deformation retract F of $\mathbb{R}^3 - Z$ onto Y. Such an F would map $\mathbb{R}^3 - Z \times I$ into $\mathbb{R}^3 - Z$. Since it is hard to draw how $\mathbb{R}^3 - Z$ deformation retracts, Figure 2 shows the complement of F at each t. In other words, the drawing shows how $\mathbb{R}^3 \setminus F((\mathbb{R}^3 - Z) \times \{t\})$ looks at each t. It is clear from the figure that $\mathbb{R}^3 \setminus F((\mathbb{R}^3 - Z) \times \{t\})$ eventually becomes $\mathbb{R}^3 - Y$. In other words, $F((\mathbb{R}^3 - Z) \times \{1\}) = Y$.

$\square$

Exercise. (Problem 14, Chapter 1.2) Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ of order eight.

Proof. The vertices and edges get identified as in Figure 3. Thus we have two 0-cells and four 1-cells. Since the opposite faces are identified and the cube has 6 faces, we need to glue three 2-cells to the cube. Lastly, we need a 3-cell glued to the three faces. By Proposition 1.26, the fundamental group of a 2-skeleton is the same as the fundamental group of a space obtained by attaching 3-cells, so it suffices to consider the fundamental group we obtain by attaching the three 2-cells to the graph. As in Figure 3, the graph has 4 edges between two vertices. The fundamental group of this is $\langle ab^{-1}, ac, ad \rangle$ because by "shrinking" a we obtain the graph consisting of one vertex and three loops. By attaching a 2-cell to each of the top-bottom pair, left-right pair, and the front-back pair, we obtain

$$\langle ac, ab^{-1}, ad \mid ab^{-1}d^{-1}c, adc^{-1}b^{-1}, acbd \rangle.$$

Thus this is the fundamental group of the given space. We claim that $(ac)^2 = (ab^{-1})^2 = (ad)^2 = (ac)(ab^{-1})(ad)$.

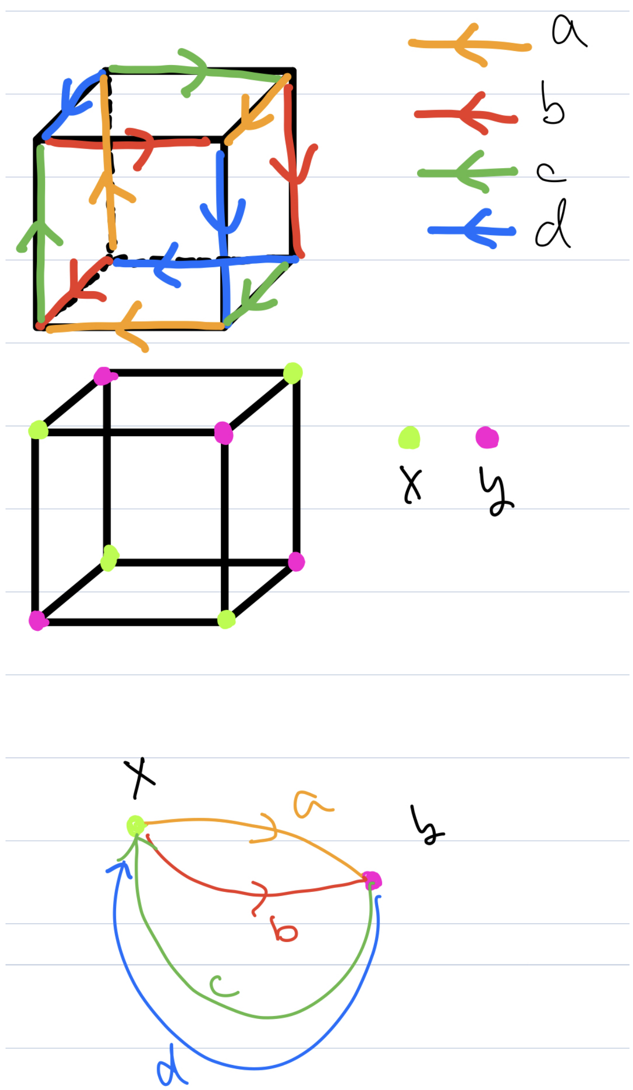

FIGURE 3. Problem 14

- $(ac)^2 = (ab^{-1})^2$?

$$\begin{aligned} ac = d^{-1}b^{-1} &\implies ab^{-1}bc = d^{-1}b^{-1} \\ &\implies ab^{-1}ad = d^{-1}b^{-1} \\ &\implies ab^{-1}a = d^{-1}b^{-1}d^{-1} \\ &\implies ab^{-1}ab^{-1} = d^{-1}b^{-1}d^{-1}b^{-1} \\ &\implies (ab^{-1})^2 = (d^{-1}b^{-1})^2 \\ &\implies (ab^{-1})^2 = (ac)^2. \end{aligned}$$

- $(ac)^2 = (ad)^2$?

$$\begin{aligned} ab^{-1} = c^{-1}d &\implies cab^{-1} = d \\ &\implies ca = db \\ &\implies cac = dbc \\ &\implies cac = dad \\ &\implies acac = adad \\ &\implies (ac)^2 = (ad)^2. \end{aligned}$$

- $(ad)^2 = (ac)(ab^{-1})(ad)$? $(ac)(ab^{-1}) = acc^{-1}d = ad$, so $(ac)(ab^{-1})(ad) = (ad)^2$.

We will simplify this by letting $x = ac, y = ab^{-1}$.

$$\begin{aligned} G &= \langle ac, ab^{-1}, ad \mid ab^{-1}d^{-1}c, adc^{-1}b^{-1}, acbd\rangle \\ &= \langle x, y, xy \mid y((xy)^{-1}x), (xy)(x^{-1}y), x(y^{-1}xy)\rangle \\ &= \langle x, y \mid e, xyx^{-1}y, xy^{-1}xy\rangle \\ &= \langle x, y \mid xyx^{-1}y, xy^{-1}xy\rangle. \end{aligned}$$

$Q_8 = \langle \overline{e}, i, j, k \mid (\overline{e})^2 = e, i^2 = j^2 = k^2 = ijk = \overline{e}\rangle$ is the quaternion group. Let $\phi : G \to Q_8$ be defined such that it preserves multiplication and $\phi(x) = i$ and $\phi(y) = j$.

- Well defined?
  - $\phi(xyx^{-1}y) = iji^{-1}j = ki^{-1}j = ki^3j = ki^2k = -k^2 = e = \phi(e)$.
  - $\phi(xy^{-1}xy) = ij^{-1}ij = ij^{-1}k = i(-j)k = -ijk = e = \phi(e)$.
- Injective?

  Prove this!
- Surjective? It suffices to show that $\overline{e}, i, j, k \in \phi(G)$. We have shown that $\overline{e} = i^2, i, j \in \phi(G)$. Moreover, $k = ij = \phi(xy) \in \phi(G)$. Therefore, ϕ is indeed surjective.

□

Exercise. (Problem 22, Chapter 1.2)

- Show that $\pi_1(\mathbb{R}^3 - K)$ has a presentation with one generator x_i for each strip R_i and one relation of the form $x_i x_j x_i^{-1} = x_k$ for each square S_l, where the indices are as in the figures above.

Proof.

- We will construct the 2-dimensional complex X by first attaching R_i's. We will attach R_i one by one. We begin with a plane $\mathbb{R}^2$ whose fundamental group is 0. A rectangular strip R_i has a fundamental group isomorphic to $\mathbb{Z}$ since it is homotopy equivalent to S^1. Thus it is a free group with one generator. We will calculate the fundamental group of a space we obtain after attaching T to R_i using Van Kampen. The intersection is a rectangle, so the intersection is simply connected. Thus the fundamental group of the new space is simply the free product of T and R_i. Therefore, the fundamental group of the space we obtain by attaching all the R_i's is $\langle x_1, \cdots, x_n \mid\rangle$ where n is the number of R_i's and each x_i corresponds to R_i. Although it is not necessary at this stage, the rotation will be important later. Therefore, we will

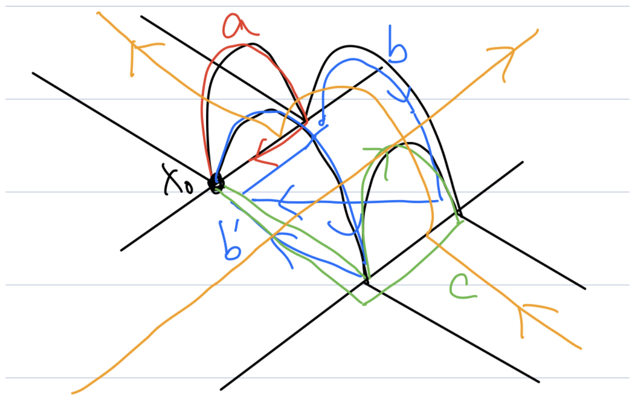

FIGURE 4. Wirtinger presentation

assume that the direction x_i goes around K is consistent with the right-hand rule. It is trivial that this is always possible.

Now, we will attach S_l's and we will do so one by one. The fundamental group of each S_l is 0 since each S_l is simply connected. Thus attaching S_l's does not add any new generators to the fundamental group. Figure 4 shows the intersection between an S_l and the current space X. a, b, b', c denote loops based at x_0, and $[b] = [b']$.

Note that x_0 could have been somewhere else, and it does not matter because X must be path-connected.

Moreover, $[a], [b], [c]$ are exactly the generator of the corresponding rectangular strip because they follow the right-hand rule. We will consider the intersection between S_l and X.

– The loop that goes through the intersection is in the path homotopy class $[a][b][c]^{-1}[b]^{-1}$ in X.
– The loop that goes through the intersection is nulhomotopic in S_l since S_l is simply connected.

By Van Kampen, the new group is $\pi_1(X) * \pi_1(S_l)/(i_X(g)i_{S_l}(g)^{-1})$ where g is any loop in the intersection. Since $\pi_1(S_l) = 0$, $i_{S_l}(g) = e$ for any g. Then $(i_X(g)) = ([abc^{-1}b^{-1}])$ since the intersection is homeomorphic to S^1 and $[a][b][c]^{-1}[b]^{-1}$ is a generator. Since $\pi_1(S_l) = 0$, we have $\pi_1(X)/([a][b][c]^{-1}[b]^{-1})$.

After attaching all the S_l's we will end up with $\langle x_1, \cdots, x_n \mid [a_l][b_l][c_l]^{-1}[b_l]^{-1}\rangle$ where

– For each S_l, we add a relation $[a_l][b_l][c_l]^{-1}[b_l]^{-1}$. Note that this means $[a_l][b_l][c_l]^{-1}[b_l]^{-1} = e$, so $[a_l] = [b_l][c_l][b_l]^{-1}$, and this is exactly the desired relation.
– Each x_i corresponds to a rectangular strip R_i. These are the only generators because S_l's are all simply connected.

- The abelianization of $\pi_1(\mathbb{R}^3 - K)$ turns a relation $x_i x_j x_i^{-1} = x_k$ into $x_j = x_k$. In other words, this implies that, at each square S_l, the generators for the two strips that are "separated" by the middle strip are identified. Let x_i, x_j be two distinct generators. Since K is a knot, there exists a finite sequence $x_i = x_{i_0}, \cdots, x_{i_k} = x_j$ of generators such that the corresponding strips $R_{i_0}, \cdots, R_{i_k}$ are next to each other. (See Figure 5) Since each intersection has a square, $x_{i_t} = x_{i_{t+1}}$ for each t. (For instance, in Figure

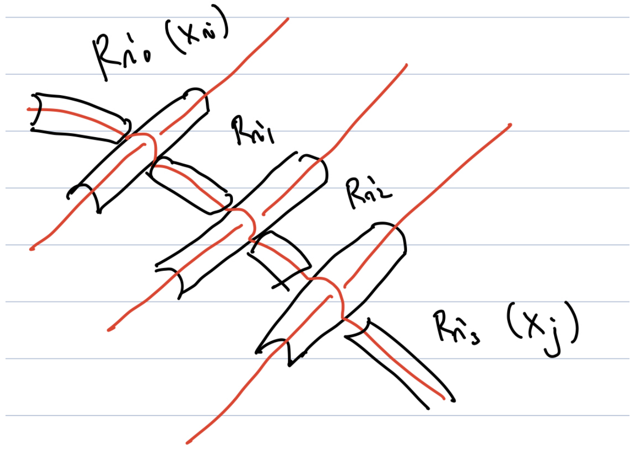

FIGURE 5. Problem 22 (b)

5, $x_{i_0} = x_{i_1}$ because of the intersection between R_{i_0} and R_{i_1}. Similarly, $x_{i_1} = x_{i_2}$ and $x_{i_2} = x_{i_3}$.) Therefore, $x_i = x_{i_0} = x_{i_1} = \cdots = x_{i_k} = x_j$.

This implies that any two generators are identified after the abelianization. Hence, $\pi_1(\mathbb{R}^3 - K)$ is a free group with one generator and no relations, so it is isomorphic to $(\mathbb{Z}, +)$.

$\square$

Exercise. Use the Wirtinger presentation to calculate the fundamental group of the complement of the trefoil knot.

Proof. We will place rectangular strips as in Figure 6.

The first relation we will consider is the upper right intersection. (Magnified in Figure 6.) This relation is $[x_2]^{-1}[x_1][x_2][x_3]^{-1}$, so $[x_1] = [x_2][x_3][x_2]^{-1}$. The other two relations can be obtained in the same manner, and they are $[x_3] = [x_1][x_2][x_1]^{-1}, [x_2] = [x_3][x_1][x_3]^{-1}$. Let a, b, c denote $[x_1], [x_2], [x_3]$, respectively.

$$\begin{aligned}\langle a, b, c \mid a = bcb^{-1}, c = aba^{-1}, b = cac^{-1}\rangle &= \langle b, c \mid c = (bcb^{-1})b(bcb^{-1})^{-1}, b = c(bcb^{-1})c^{-1}\rangle \\ &= \langle b, c \mid c = bc(bc^{-1}b^{-1}), b = c(bcb^{-1})c^{-1}\rangle \\ &= \langle b, c \mid c = bc(bc^{-1}b^{-1}), b = c(bcb^{-1})c^{-1}\rangle\end{aligned}$$

- $c = bc(bc^{-1}b^{-1}) \iff cb = bcbc^{-1} \iff cbc = bcb.$
- $b = cbcb^{-1}c^{-1} \iff bc = cbcb^{-1} \iff bcb = cbc.$

Thus those two relations are identical. Therefore, the fundamental group of the trefoil knot is $\langle b, c \mid bcb = cbc\rangle$.

Let $G = \langle b, c \mid bcb = cbc\rangle, H = \langle x, y \mid x^2 = y^3\rangle$. Let $\phi : H \to G$ be defined such that ϕ maps x to cbc and y to bc and it preserves the multiplicative operation. For instance, $\phi(x^i y^j) = (cbc)^i (bc)^j$. This function is well-defined because $\phi(x^2) = (\phi(x))^2 = (cbc)^2 = (bcb)^2 = (\phi(y))^3 = \phi(y^3)$. Since this function is well-defined and it preserves the multiplicative operation, it is a group homomorphism.

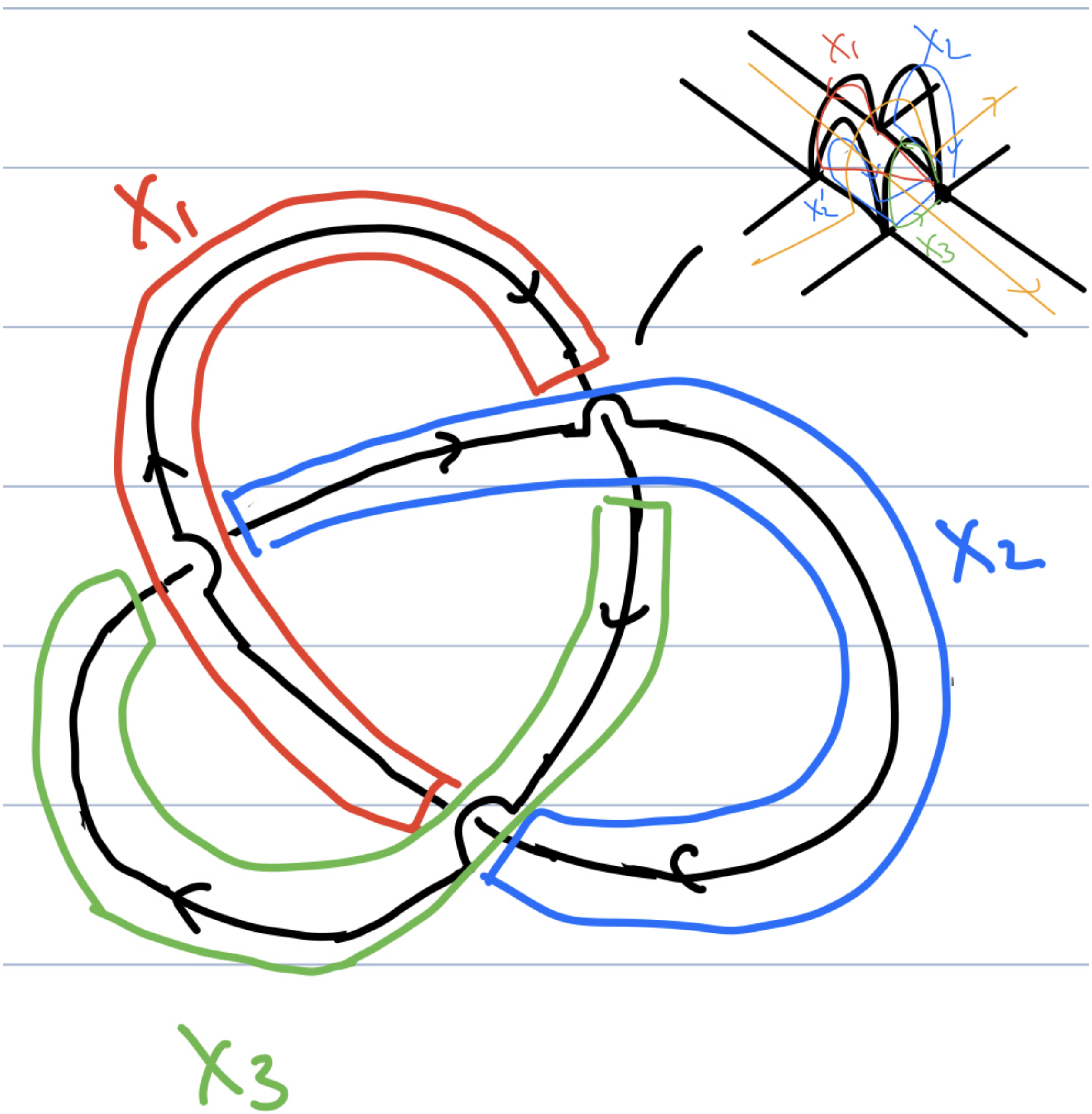

Figure 6. Trefoil

Similarly, define $\psi : G \to H$ such that ϕ maps b to y^2x^{-1} and c to xy^{-1} and it preserves the operation. For instance, $\psi(bcb) = (yxy^{-1})(xy^{-1})(yxy^{-1})$. This function is well-defined because

- $\psi(bcb) = (y^2x^{-1})(xy^{-1})(y^2x^{-1}) = y^3x^{-1} = x^2x^{-1} = x$.
- $\psi(cbc) = (xy^{-1})(y^2x^{-1})(xy^{-1}) = xyy^{-1} = x$.

Moreover,

- $\phi(\psi(b)) = \phi(y^2x^{-1}) = (bc)^2(cbc)^{-1} = bcbcc^{-1}b^{-1}c^{-1} = b$.

- $\phi(\psi(c)) = \phi(xy^{-1}) = (cbc)(bc)^{-1} = c.$
- $\psi(\phi(x)) = \psi(cbc) = xy^{-1}y^2x^{-1}xy^{-1} = x.$
- $\psi(\phi(y)) = \psi(bc) = y^2x^{-1}xy^{-1} = y.$

Therefore, ϕ and ψ are both bijective. In other words, ϕ is an isomorphism between G and H. $\square$