MATH 611 (DUE 11/6)

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1. Simplicial and Singular Homology

Exercise. (Problem 14) Determine whether there exists a short exact sequence $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$. More generally, determine which abelian groups A fit into a short exact sequence $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ with p prime. What about the case of short exact sequences $0 \to A \to \mathbb{Z}_n \to 0$?

Proof. Let $\phi_1 : \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2, \phi_2 : \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4$ be defined such that $\phi_1(a) = (2a, a)$ and $\phi_2(a, b) = 2b - a$.

Solve this!

□

Exercise. (Problem 15) For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ if and only if the map $A \to B$ is surjective and $D \to E$ is injective. Hence, for a pair of spaces (X, A), the inclusion $A \to X$ induces isomorphisms on all homology groups if and only if $H_n(X, A) = 0$ for all n.

Proof. Suppose $C = 0$. $\mathrm{Im}(\phi_{AB}) = \ker(\phi_{BC}) = B$, so ϕ_{AB} is surjective. $\ker(\phi_{DE}) = \mathrm{Im}(\phi_{CD}) = \{0\}$, so ϕ_{DE} is injective.

On the other hand, suppose ϕ_{AB} is surjective and ϕ_{DE} is injective. $\mathrm{Im}(\phi_{CD}) = \ker(\phi_{DE}) = \{0\}$, so ϕ_{CD} is the zero map. Therefore, $\ker(\phi_{CD}) = C$. $\ker(\phi_{BC}) = \mathrm{Im}(\phi_{AB}) = B$, so ϕ_{BC} is the zero map. Therefore, $\mathrm{Im}(\phi_{BC}) = 0$. Hence, $C = \ker(\phi_{CD}) = \mathrm{Im}(\phi_{BC}) = 0$.

Finish the second part.

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