

MATH 633 (HOMEWORK 1)

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Exercise. (Problem 1)

- From basic topology, the closure of A is equal to the intersection of all closed sets containing A . The intersection of closed sets is closed, so \overline{A} is closed.
- Let $z \in \overline{A} \setminus A$. z is a limit point of A and $A \subset B$, so z is a limit point of B . Since B is closed, $z \in B$.

Exercise. (Problem 2)

- Not open, not closed, not compact. The boundary is $\{x + iy \mid |x| = |y| = 1\}$.
- Not open. Closed. Compact. The boundary is A .
- Not open. Closed. Not compact. The boundary is the real line.
- Open. Not closed. Not compact. The boundary is $\{0\}$.

Exercise. (Problem 3)

- $f'(z) = -1/z^2$.
- $|z|^2 \cdot (1/z) = \overline{z}$, which is not differentiable anywhere on \mathbb{C} . Since $1/z$ is differentiable everywhere on $z \neq 0$, $|z|^2$ is not differentiable anywhere on $z \neq 0$. Thus $|z|$ is not differentiable anywhere on $z \neq 0$.

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

does not exist. This is because the limit is 1 with $h_n = 1/n$, but the limit is -1 with $h_n = -1/n$. Therefore, $|z|$ is nowhere differentiable.

Exercise. (Problem 4)

- $a \implies b$. Define

$$\psi(h) = \begin{cases} \frac{f(z_0+h) - f(z_0)}{h} - w & (h \neq 0) \\ 0 & (h = 0). \end{cases}$$

Then ψ is defined on a neighborhood of 0 because z_0 is an interior point of U . Moreover, $\lim_{h \rightarrow 0} \psi(h) = 0$.

- $b \implies c$. Locally, $\frac{|f(z_0+h) - f(z_0) - wh|}{|h|} = \psi(h)$, so the limit is 0 as $h \rightarrow 0$.
- $c \implies a$.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(z_0+h) - f(z_0)}{h} = w &\iff \lim_{h \rightarrow 0} \frac{f(z_0+h) - f(z_0) - wh}{h} = 0 \\ &\iff \lim_{h \rightarrow 0} \frac{|f(z_0+h) - f(z_0) - wh|}{|h|} = 0. \end{aligned}$$

Thus $c \implies a$.

Exercise. (Problem 5(i)(ii))

$$\begin{aligned}
& \lim_{h \rightarrow 0} \frac{(f+g)(z_0+h) - (f+g)(z_0)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(z_0+h) - f(z_0)}{h} + \lim_{h \rightarrow 0} \frac{g(z_0+h) - g(z_0)}{h} \\
&= f'(z_0) + g'(z_0). \\
& \lim_{h \rightarrow 0} \frac{(fg)(z_0+h) - (fg)(z_0)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(z_0+h)g(z_0+h) - f(z_0+h)g(z_0) + f(z_0+h)g(z_0) - f(z_0)g(z_0)}{h} \\
&= \lim_{h \rightarrow 0} f(z_0+h) \frac{g(z_0+h) - g(z_0)}{h} + g(z_0) \lim_{h \rightarrow 0} \frac{f(z_0+h) - f(z_0)}{h} \\
&= f(z_0)g'(z_0) + g(z_0)f'(z_0).
\end{aligned}$$

Exercise. (Problem 5(iii))

$$\begin{aligned}
& \lim_{h \rightarrow 0} \frac{1/g(z_0+h) - 1/g(z_0)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{g(z_0)g(z_0+h)} \frac{g(z_0) - g(z_0+h)}{h} \\
&= \frac{g'(z_0)}{g^2(z_0)}.
\end{aligned}$$

By applying Problem 5(ii), we obtain the quotient rule.