MATH 612 (HOMEWORK 1)

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Exercise. (Exercise 1(a)) The case of $G = \mathbb{Z}$ is discussed in Example 2.42.

$$H_k(\mathbb{R}\mathrm{P}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0 \text{ and for } k = n \text{ odd} \\ \mathbb{Z}_2 & \text{for } k \text{ odd, } 0 < k < n \\ 0 & \text{otherwise.} \end{cases}$$

Suppose n is even. For any abelian group G, we obtain the cellular chain complex

$$0 \to G \xrightarrow{2} G \xrightarrow{0} \cdots \xrightarrow{2} G \xrightarrow{0} G \to 0.$$

If n is odd, we obtain

$$0 \to G \xrightarrow{0} G \xrightarrow{2} \cdots \xrightarrow{2} G \xrightarrow{0} G \to 0.$$

- Suppose k is even and $2 \leq k \leq n$. The homology at $\xrightarrow{0} G \xrightarrow{2}$ is
  - 0 if $G = \mathbb{Q}, \mathbb{Z}/p^l\mathbb{Z}$ with $p \neq 2$.
  - $\mathbb{Z}/2\mathbb{Z}$ if $G = \mathbb{Z}/2^l$.
- Suppose k is odd and $1 \leq k \leq n-1$. The homology at $\xrightarrow{2} G \xrightarrow{0}$ is
  - $G/2G \cong 0$ if $G = \mathbb{Q}, \mathbb{Z}/p^l\mathbb{Z}$ with $p \neq 2$ because multiplication by 2 is an isomorphism.
  - $\mathbb{Z}/2\mathbb{Z}$ if $G = \mathbb{Z}/2^l$.
- Suppose $k = n$ and n is odd, or $k = 0$. The homology at $\xrightarrow{0} G \xrightarrow{0}$ is G.

When $G = \mathbb{Q}$, the universal coefficient theorem gives an isomorphism $H_k(X) \otimes Q \cong H_k(X; \mathbb{Q})$ since Q is torsion free. $\mathbb{Z} \otimes \mathbb{Q} \cong \mathbb{Q}$ and $\mathbb{Z}/2 \otimes \mathbb{Q} = 0$ because 2 is invertible in $\mathbb{Q}$. This agrees with the results above.

When $G = \mathbb{Z}/2^l$, we have $0 \to H_k(X) \otimes G \to H_k(X; G) \to \mathrm{Tor}(H_{k-1}(C), G) \to 0$. If $k = n$ and k is odd, $H_k(X) = \mathbb{Z}$, so $\mathbb{Z}/2^l \cong H_k(X; \mathbb{Z}/2^l)$. If $k - 1 = n$ and $k - 1$ is odd, we obtain $0 \to 0 \to H_k(X; \mathbb{Z}/2^l) \to \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/2^l) \to 0$, so $H_k(X; \mathbb{Z}/2^l) = 0$. If k is odd and $0 < k < n$, $0 \to \mathbb{Z}/2 \otimes \mathbb{Z}/2^l \to H_k(X; \mathbb{Z}/2^l) \to \mathrm{Tor}(H_{k-1}(X), \mathbb{Z}/2^l) \to 0$. The Tor is 0 because if $k = 0$, $H_{k-1}(X) = \mathbb{Z}$ and $H_{k-1}(X) = 0$ otherwise. Thus $H_k(X; \mathbb{Z}/2^l) = \mathbb{Z}/2 \otimes \mathbb{Z}/2^l = \mathbb{Z}/2$. In any other cases, the universal coefficient theorem gives the SES $0 \to 0 \to H_n(X; G) \to 0 \to 0$. This agrees with the results above.

Suppose $G = \mathbb{Z}/p^l$. Then the case that $k = n$ and k is odd and the case that $k - 1 = n$ and k is odd can be handled in the same way as above. Suppose k is odd and $0 < k < n$. Then $\mathbb{Z}/2 \otimes \mathbb{Z}/p^l = 0$. Moreover, $\mathrm{Tor}(H_{k-1}(X), \mathbb{Z}) = 0$ as discussed above. Thus $H_k(X) = 0$. In any other cases, the universal coefficient theorem gives the SES $0 \to 0 \to H_n(X; G) \to 0 \to 0$. This agrees with the results above.

Exercise. (Exercise 1(b)) As discussed in Example 2.37, $H_2(N_g; \mathbb{Z}) = 0, H_1(N_g; \mathbb{Z}) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$, and $H_0(N_g; \mathbb{Z}) = \mathbb{Z}$. For an abelian group G, the cellular chain complex is

$$0 \to G \xrightarrow{d_2} G^g \xrightarrow{d_1} G \to 0.$$

As discussed in Example 2.37, $d_2(1) = (2, 2, \cdots, 2)$ and $d_1 = 0$. If $G = \mathbb{Z}/p^l$ with $p \neq 2$ or $G = \mathbb{Q}$, then $H_2(X; G) = 0, H_1(X; G) = G^g / \langle(1, \cdots, 1)\rangle = G^{g-1}$ and $H_0(X; G) = G$ because 2^{-1} exists. Suppose $G = \mathbb{Z}/2^l$. Then $H_2(X; G) = \mathbb{Z}/2$ because the kernel is an index-2 subgroup. $H_1(X; G) = G^g / \langle(2a, \cdots, 2a)\rangle = G^{g-1} \otimes \mathbb{Z}/2$, and $H_0(X; G) = G$.

We will verify the results using the universal coefficient theorem.

Suppose $G = \mathbb{Q}$. Then $\mathrm{Tor}(H_{n-1}(C), G) = 0$ for any n. Thus $H_0(X; G) = \mathbb{Z} \otimes \mathbb{Q} = \mathbb{Q}$ and $H_1(X; G) = (\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2) \otimes \mathbb{Q} = (\mathbb{Z} \otimes \mathbb{Q})^{g-1} \oplus (\mathbb{Z}_2 \otimes \mathbb{Q}) = \mathbb{Q}^{g-1}$.

Suppose $G = \mathbb{Z}/p^l$ with $p \neq 2$. When $n = 1$, $H_{n-1}(C) = \mathbb{Z}$, so $\mathrm{Tor}(H_{n-1}(C), G) = 0$. Thus $H_1(C; G) = (\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2) \otimes \mathbb{Z}/p^l = (\mathbb{Z}/p^l)^{g-1}$. When $n = 2$, $H_n(C) = 0$ and $\mathrm{Tor}(H_{n-1}(C), \mathbb{Z}/p^l) = 0$ because multiplication by p^l does not kill any element in $\mathbb{Z}^{g-1} \oplus \mathbb{Z}/2$.

Suppose $G = \mathbb{Z}/2^l$. When $n = 1$, $\mathrm{Tor}(H_{n-1}(C), G) = \mathrm{Tor}(\mathbb{Z}, G) = 0$. Thus $H_n(C; G) = H_n(C) \otimes G = (\mathbb{Z}/2^l)^{g-1} \oplus \mathbb{Z}/2$. When $n = 2$, $H_n(C) = 0$ and $\mathrm{Tor}(H_{n-1}(C), \mathbb{Z}/2^l) = \ker((\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2) \xrightarrow{2^l} (\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2)) = \mathbb{Z}/2$. Thus $H_2(C; G) = \mathbb{Z}/2$.

Exercise. (Exercise 1(c)) For a Z-module R, we have

$$0 \to R \xrightarrow{0} R \xrightarrow{a} R \xrightarrow{0} R \to 0.$$

Clearly, $H_k(X; R) = 0$ for $k \geq 4$ for any R.

- When $k = 0, 3$, $H_k(X; R) = R/0 = R$.
- $H_2(X; R) = \ker(R \xrightarrow{a} R)$. When $R = \mathbb{Z}, \mathbb{Q}$, the kernel is 0. When $R = \mathbb{Z}/p^k$, the kernel is isomorphic to $\mathbb{Z}/\gcd(p^k, a)$.
- $H_1(X; R) = R/aR$. Thus $H_1(X; \mathbb{Q}) = 0$. $H_1(X; \mathbb{Z}) = \mathbb{Z}/a\mathbb{Z}$. When $R = \mathbb{Z}/p^k$, we obtain $(\mathbb{Z}/p^k)/a(\mathbb{Z}/p^k) = \mathbb{Z}/\gcd(p^k, a)$.

Use the UCT to verify the results.

Exercise. (Exercise 3(a)) $e_i \mapsto t^i x$ is an isomorphism between $C_1^{CW}(X)$ and a free module generated over $\mathbb{Z}[t, t^{-1}]$ where x is the only element in a basis. Similarly, $f_i \mapsto t^i y$ gives an isomorphism. With this identification, the boundary map $f_i \mapsto -e_i + 2e_{i+1}$ becomes $(\sum a_i t^{b_i})x \mapsto (\sum a_i(-t^{b_i} + 2t^{b_i+1}))x$ which is clearly $\mathbb{Z}[t, t^{-1}]$-linear. Moreover, the property that $d^2 = 0$ is clearly preserved after the identification, so the homology groups, which are just the kernels modulo the images, must be $\mathbb{Z}[t, t^{-1}]$-modules.

Exercise. (Exercise 3(b)) Since $d_2 : 2f_0 \mapsto -e_0 + 2e_1$, $x \mapsto -x + 2tx = (2t-1)x$ after the identification described above. Then for all $\alpha \in \mathbb{Z}[t, t^{-1}]$, $d_2(\alpha x) = 0 \implies (2t-1)\alpha = 0 \implies \alpha = 0$. Thus $H_2(X) = 0$.

$d_1 = 0$ because there is only one 0-cell. Thus $H_1(X) = \mathbb{Z}[t, t^{-1}]/(2t-1)$. This is isomorphic to $\mathbb{Z}[1/2]$ because the kernel of the homomorphism $\phi : \mathbb{Z}[t, t^{-1}] \mapsto \mathbb{Z}[1/2]$ defined by $t \mapsto 1/2$ is $(2t-1)$.

Exercise. (Exercise 3(c)) We will use the universal coefficient theorem with the values we have calculated: $H_2(X) = 0, H_1(X) = \mathbb{Z}[1/2], H_0(X) = \mathbb{Z}$.

- $\mathbb{Q}$. The UCT states $H_k(X, \mathbb{Q}) = (H_k(X) \otimes \mathbb{Q}) \oplus \mathrm{Tor}(H_{k-1}(X), \mathbb{Q})$. $\mathrm{Tor}(H_{k-1}(X), \mathbb{Q}) = 0$ because $\mathbb{Q}$ is torsion-free.

$$H_k(X, \mathbb{Q}) = \begin{cases} 0 & (k = 2) \\ \mathbb{Q} & (k = 0, 1). \end{cases}$$

- $\mathbb{Z}/p^k$ with $p \neq 2$. The UCT states $H_k(X, \mathbb{Z}/p^k) = (H_k(X) \otimes \mathbb{Z}/p^k) \oplus \mathrm{Tor}(H_{k-1}(X), \mathbb{Z}/p^k)$. Since $\mathrm{Tor}(H_{k-1}(X), \mathbb{Z}/p^k) = \ker(\mathbb{Z}[1/2] \xrightarrow{p^k} \mathbb{Z}[1/2]) = 0$, it suffices to consider the tensor product.

$$H_k(X, \mathbb{Z}/p^k) = \begin{cases} 0 & (k = 2) \\ \mathbb{Z}/p^k & (k = 0, 1). \end{cases}$$

- $\mathbb{Z}/2^k$. The UCT states $H_k(X, \mathbb{Z}/2^k) = (H_k(X) \otimes \mathbb{Z}/2^k) \oplus \mathrm{Tor}(H_{k-1}(X), \mathbb{Z}/2^k)$. Again, $\mathrm{Tor}(H_{k-1}(X), \mathbb{Z}/2^k) = \ker(\mathbb{Z}[1/2] \xrightarrow{2^k} \mathbb{Z}[1/2]) = 0$.

$$H_k(X, \mathbb{Z}/2^k) = \begin{cases} 0 & (k = 2, k = 1) \\ \mathbb{Z}/2^k & (k = 0). \end{cases}$$

  When $k = 1$, $\mathbb{Z}[1/2] \otimes \mathbb{Z}/2^k = 0$ because $a \otimes b = a/2^k \otimes 2^k b = 0$.