

# MATH 602(HOMEWORK 3)

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## 1. EXERCISES

**Exercise.** (Exercise 1) The ideal generated by the three polynomials contains  $-yz^4 + yz^2 + y = (xy^2 - xz + y) - y(xy - z^2) + z(x - yz^4)$ . However, its leading term  $-yz^4$  is not in the ideal generated by the leading terms of the three polynomials.

**Exercise.** (Exercise 2) Remainder  $= -y^{15} + y^{14} + 7y^{13} - 7y^{12} - 21y^{11} + 21y^{10} + 35y^9 - 35y^8 - 35y^7 + 35y^6 + 21y^5 - 21y^4 - 7y^3 + 7y^2 + y - 1$ ,  $q_1 = x^6y^{14} - 6x^6y^{12} + 15x^6y^{10} - 20x^6y^8 + 15x^6y^6 - 6x^6y^4 + x^6y^2 + x^2y^{14} - 6x^2y^{12} + 15x^2y^{10} - 20x^2y^8 + 15x^2y^6 - 6x^2y^4 + x^2y^2$ ,  $q_2 = 0$

Remainder  $= y^{23} + y^{11} - y + 1$ ,  $q_1 = x^6y^2 + x^5y^5 + x^4y^8 + x^3y^{11} + x^2y^{14} + x^2y^2 + xy^{17} + xy^5 + y^{20} + y^8$ ,  $q_2 = 0$

**Exercise.** (Exercise 3)  $z^3 - x^2 = (y - x^2) - (y - z^3)$  is in the ideal generated by the two polynomials, but the leading term is not in the ideal generated by the two polynomials.

**Exercise.** (Exercise 4)  $0 \in \sqrt{0}$ ,  $a, b \in \sqrt{0} \implies (a+b)^{m+n-1} = \sum_{i=0}^{m+n-1} \binom{m+n-1}{i} a^i b^{m+n-1-i} = 0$ , and  $\forall a \in \sqrt{0}, \forall x \in R, (ax)^n = a^n x^n = 0$ , so  $\sqrt{0}$  is an ideal.

**Exercise.** (Exercise 5)

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- Suppose otherwise. Pick  $x_j \in I_j \setminus p$  for each  $j$ .  $x = \prod_{j=1}^n x_j \in I_i$  for each  $i$ , so  $x \in p$ . Since  $p$  is a prime,  $p$  must contain at least one  $x_j$ . Contradiction.

**Exercise.** (Exercise 6) Tensoring an exact sequence with  $M \otimes_A N$  is the same as tensoring it with  $M$  first and tensoring the resulting sequence with  $N$  later.

**Exercise.** (Exercise 7) Since  $0 \rightarrow I \xrightarrow{i} R \xrightarrow{q} R/I \rightarrow 0$  is exact,  $I \otimes M \rightarrow R \otimes M \rightarrow (R/I) \otimes M \rightarrow 0$  is exact.

$$\begin{aligned} (R/I) \otimes M &= \text{im}(q \otimes \text{Id}) \\ &\cong R \otimes M / \ker(q \otimes \text{Id}) \\ &\cong R \otimes M / \text{im}(i \otimes \text{Id}) \\ &\cong R \otimes M / I \otimes M. \end{aligned}$$

Now consider  $\phi : R \otimes M \rightarrow M/IM$  that is the composition of  $R \otimes M \rightarrow M : x \otimes y \mapsto xy$  and  $M \rightarrow M/IM : x \mapsto x + IM$ . In other words,  $\phi$  is  $x \otimes y \mapsto xy + IM$ . Because the two maps are both surjective,  $\phi$  must be surjective. The kernel of  $\phi$  is  $I \otimes M$  because

- For any  $x \otimes y \in I \otimes M$ ,  $\phi(x \otimes y) = xy + IM = 0$  since  $xy \in IM$ .
- If  $\phi(x \otimes y) = 0$ , then  $xy \in IM$ . In other words,  $xy = x'y'$  for some  $x' \in I$  and  $y' \in M$ . Then  $x \otimes y = 1 \otimes xy = 1 \otimes x'y' = x' \otimes y' \in I \otimes M$ .

Therefore,  $M/IM \cong (R \otimes M)/(I \otimes M) \cong (R/I) \otimes M$ .

**Exercise.** (Exercise 8) Let  $pa + qb = 1$  for some  $p, q \in \mathbb{Z}$ . Then  $1 \otimes 1 = (pa + qb) \otimes (pa + qb) = pa \otimes pa + pa \otimes qb + qb \otimes pa + qb \otimes qb = 0 + 0 + 0 + 0 = 0$ .

**Exercise.** (Exercise 9) Let  $T$  be a  $\mathbb{Z}$ -module and  $f : \mathbb{Q} \times \mathbb{Q} \rightarrow T$  be a bilinear map. Then  $f(a/b, c/d) = acf(1/b, 1/d) = acbf(1/b, 1/bd) = acf(1, 1/bd) = f(1, ac/bd)$ . Define a bilinear map  $h : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q}$  by  $(a, b) \mapsto ab$  and a linear map  $g : \mathbb{Q} \rightarrow T$  by  $a/b \mapsto f(1, a/b)$ . Then  $f = g \circ h$ . The universal property of a tensor product is satisfied by  $\mathbb{Q}$ , so  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ .

**Exercise.** (Exercise 10) Let  $a_1, \dots, a_n, b_1, \dots, b_m$  generate  $M'$  and  $M''$ , respectively. Let  $x_1, \dots, x_n, y_1, \dots, y_m \in M$  be chosen such that  $x_i$  is the image of  $a_i$  and the image of  $y_j$  is  $b_j$ . We claim that  $x_i, y_j$  generate  $M$ . Let  $x \in M$  be given. Then  $q(x) = d_1b_1 + \dots + d_mb_m$  for some  $d_i \in M$ , and thus  $q(x - d_1y_1 - \dots - d_my_m) = 0$ . Therefore,  $x - d_1y_1 - \dots - d_my_m = i(c_1a_1 + \dots + c_na_n) = c_1x_1 + \dots + c_nx_n$ , so  $x = c_1x_1 + \dots + c_nx_n + d_1y_1 + \dots + d_my_m$ .

**Exercise.** (Exercise 11) This statement is not true. When  $R = \mathbb{Z}$  and  $I = (0)$ ,  $I \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ . However, the statement is true if  $I \neq 0$ . Let  $u \in I$  be a nonzero element. Define  $h : I \times K \rightarrow K$  by  $(a, x/y) \mapsto ax/y$ . Let  $f \in \text{Hom}(I \times K, T)$  be given. Define  $g : K \rightarrow T$  by  $x/y \mapsto f(u, x/uy)$ . Then

$$\begin{aligned} (g \circ h)(a, x/y) &= g(h(a, x/y)) \\ &= g(ax/y) \\ &= f(u, \frac{ax}{yu}) \\ &= af(u, \frac{x}{yu}) \\ &= f(au, \frac{x}{yu}) \\ &= uf(a, \frac{x}{yu}) \\ &= f(a, \frac{xu}{yu}) \\ &= f(a, x/y). \end{aligned}$$

Thus  $f, g, h$  commute and thus  $K \cong I \otimes K$ .

## 2. COMPUTATION

**Exercise.** (Exercise 2)  $xy^2$  and  $4z^3$ .

**Exercise.** (Exercise 3) The Sympy code `groebner([x * y + y**2 - 1, x * y], z, y, x, order = 'grlex')` gives  $\{y^2 - 1, x\}$