MATH 611 HOMEWORK (DUE 9/25)

HIDENORI SHINOHARA

Exercise. (Problem 4, Chapter 1.3) Construct a simply-connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle intersecting it in two points.

Proof. We claim that the space described in Figure 1 is a covering space of X.

- The shape is an infinitely long chain of spheres and lines. The chain goes infinitely both ways (up and down). This space is clearly simply connected.
- We will map each sphere to the sphere of X. Each line will be mapped to the diameter up side down. Figure 1 shows how each part gets mapped.
- We claim that such a mapping is a covering map and thus this infinite chain is indeed a covering space. Let $x \in X$.
  - If x is on the diameter and disjoint from the sphere, a neighborhood that is disjoint from the sphere is evenly covered.
  - If x is on the sphere and disjoint from the diameter, a neighborhood that is disjoint from the diameter is evenly covered.

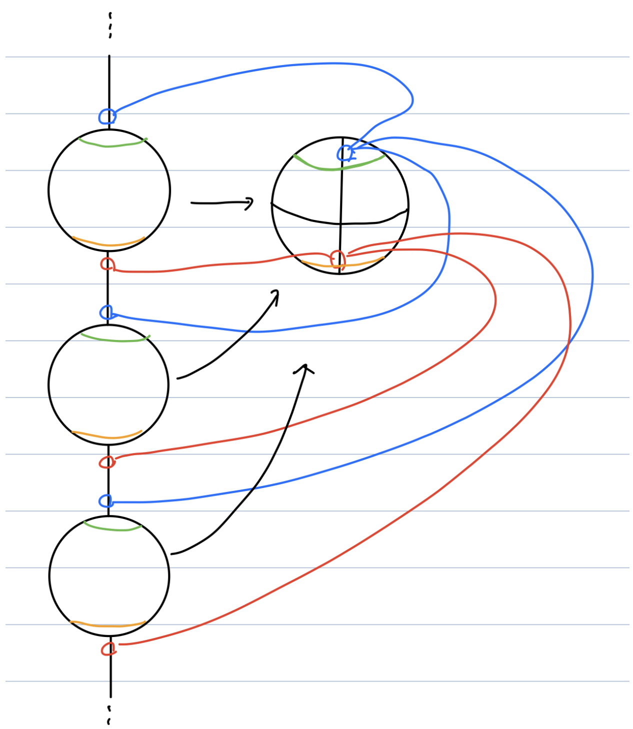

FIGURE 1. Problem 4 (Part 1)

– If x is the north pole, a neighborhood that does not contain the south pole is evenly covered.
– If x is the south pole, a neighborhood that does not contain the north pole is evenly covered.

Therefore, the space described in Figure 1 is a covering space of X.

Second part.

□

Exercise. (Problem 5, Chapter 1.3) Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1]\times[0,1]$ together with the segments of the vertical lines $x = 1/2, 1/3, 1/4, \cdots$ inside the square. Show that for every covering space $X \to X$ there is some neighborhood of the left edge of X that lifts homeomorphically to $\tilde{X}$. Deduce that X has no simply-connected covering space.

Proof. For each $y \in [0,1]$, the point $(0,y)$ has a neighborhood U_y that is evenly covered. Then there exists an open rectangle $R_y \subset \mathbb{R}^2$ such that $y \in R_y \cap Y \subset U_y$. Since any open subset of an evenly covered set is evenly covered, such $R_y \cap Y$ is evenly covered. Let V_y denote $R_y \cap Y$ for each y. $\{V_y \mid y \in [0,1]\}$ is an open cover of the segment $\{0\} \times [0,1]$. Since the segment is compact, there exists a finite subcover, $V_{y_1}, \cdots, V_{y_k}$.

Since we found a finite cover where each of them is rectangular shaped, there exists an $N \in \mathbb{N}$ such that $\forall n \geq N$, $\{1/n\} \times [0,1] \subset \bigcup_{i=1}^{n} V_{y_i}$. Since each $V_{y_1}, \cdots, V_{y_k}$ is a subset of an open rectangle, there must exist a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ such that for all i, $[0, 1/N] \times [t_i, t_{i+1}]$ is contained in an evenly covered open subset. We will inductively show that $[0, 1/N] \times [0, t_i]$ is contained in an evenly covered open subset. □

Exercise. (Problem 7, Chapter 1.3) Let Y be the quasi-circle in the figure in the textbook. Collapsing the segment of Y in the y-axis to a point gives a quotient map $f : Y \to S^1$. Show that f does not lift to the covering space $\mathbb{R} \to S^1$, even though $\pi_1(Y) = 0$. Thus local path-connectedness of Y is a necessary hypothesis in the lifting criterion.

Proof. The lifting criterion is Proposition 1.33. The only property that Y is missing is the local path connectedness. I need to understand the proof because I essentially have to find where the proof goes wrong if local path connectedness is missing. I think what happens is that if $\tilde{f}$ existed, it would have to be unique. Thus we could look into the one function that could possibly be $\tilde{f}$. Since the local connectedness is used to prove continuity of $\tilde{f}$ and Y is not locally connected around the $[-1,1]$ segment, I would guess that that one function is not continuous at a point on the $[-1,1]$ segment. See Figure 2.

□

Exercise. (Problem 8, Chapter 1.3) Let $\tilde{X}$ and $\tilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces X and Y. Show that if $X \simeq Y$ then $\tilde{X} \simeq \tilde{Y}$.

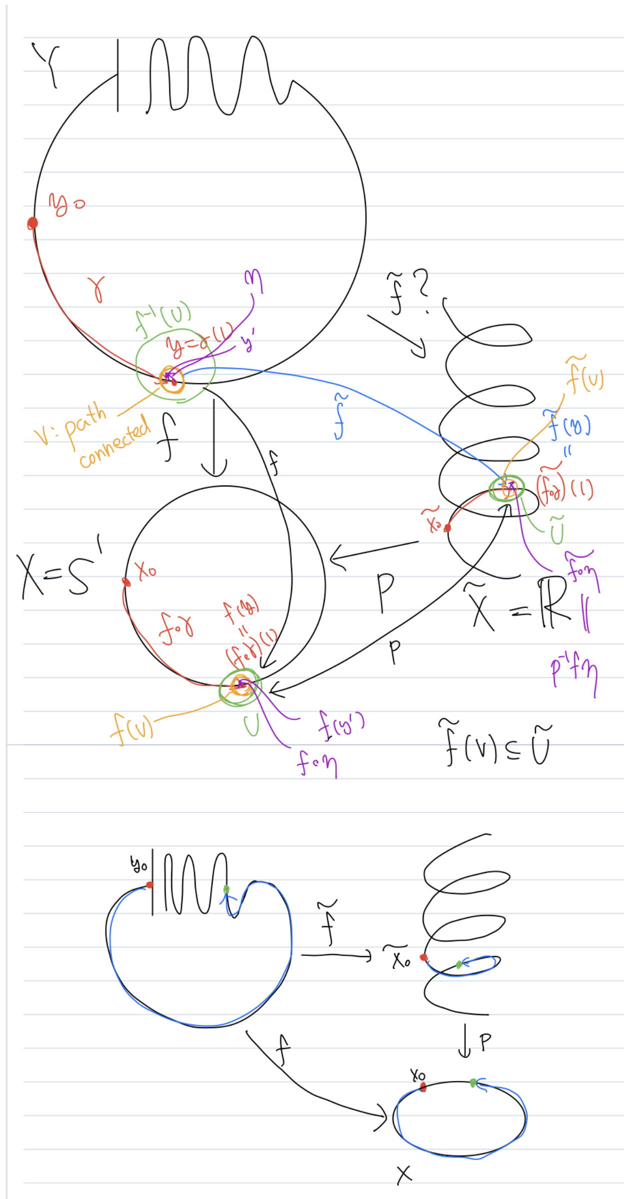

FIGURE 2. Delete this!

Proof. By Proposition 1.33, we can lift the two compositions as in Figure 3. This works because $\pi_1(\tilde{X}) = \pi_1(\tilde{Y}) = 0$. I'm not sure how Exercise 11 (Chapter 0) helps, but I solved the first part of it. Let F be a homotopy between $f \circ g$ and Id, and let H be a homotopy between $h \circ f$ and Id. Let G be defined such that $G_t = h \circ F_{2t} \circ f$ for $t \in [0, 1/2]$, and $G_t = H_{2t-1}$ for $t \in [1/2, 1]$. I should try the second part to see if that helps me.

$\square$

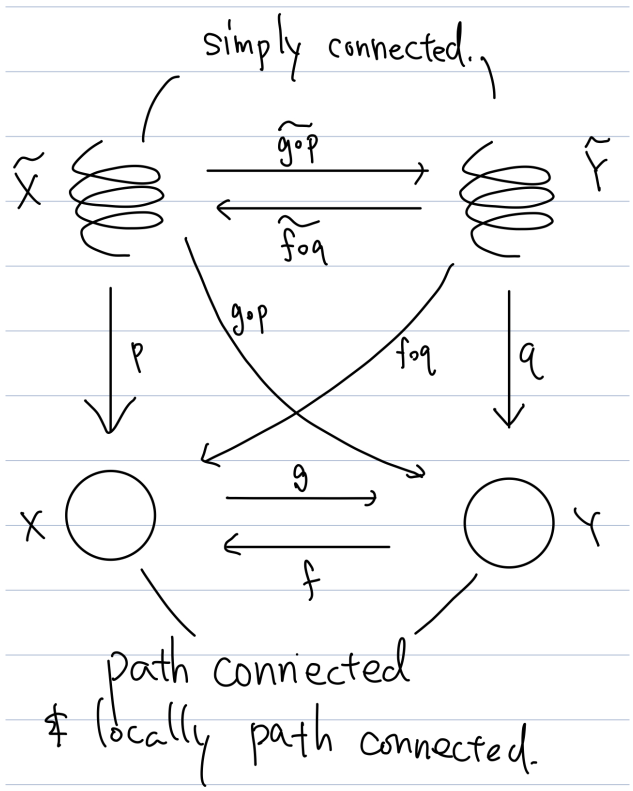

FIGURE 3. delete this!