MATH 633 MIDTERM

HIDENORI SHINOHARA

1. Goursat, Cauchy on the disc, and the proofs in Section 5 of Chapter 3.

Proposition 1.1 (Goursat's Theorem)**.** *If Ω is an open set in $\mathbb{C}$, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω, then*

$$\int_T f(z)dz = 0$$

whenever f is holomorphic in Ω.

Proof.

- Let $T^0 = T$. Having created T^i, create 4 triangles from T^i as shown in the textbook with the natural orientation. Then one of the 4 triangles, denoted by T^{i+1}, must satisfy $\left|\int_{T^i} f(z)dz\right| \leq 4\left|\int_{T^{i+1}} f(z)dz\right|$. Since $\{T_i\}$ is a sequence of nonempty compact sets whose diameter diminishes, there must exist a unique point z_0 that belongs to all T^i.
- Since f is holomorphic at z_0, $f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$ where $\psi(z) \to 0$ as $z \to z_0$.
- Since $f(z_0) + f'(z_0)(z - z_0)$ has a primitive, $\int_{T^n} f(z)dz = \int_{T^n} \psi(z)(z - z_0)dz$ for any n. $\left|\int_{T^n} \psi(z)(z - z_0)dz\right| \leq \epsilon_n dp/4^n$ where $\epsilon_n = \sup_{z \in T^n} |\psi(z)|$, d the diameter of T, and p the perimeter of T. $\epsilon_n \to 0$ as $n \to \infty$, so $\left|\int_T f(z)dz\right| \leq \epsilon_n dp = 0$ as $n \to 0$.

$\square$

Proposition 1.2 (Cauchy's Theorem for a Disk)**.** *Suppose f is holomorphic in an open set containing the circle C and its interior. Then*

$$\int_C f(z)dz = 0.$$

Proof. Since f has a primitive, the integral over a closed curve is 0.

Do I need more than this?

$\square$

Proposition 1.3 (Theorem 5.1)**.** *If f is holomorphic in Ω, then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

whenever the two curves γ_0 and γ_1 are homotopic in Ω.

Proof.

- Let $F : (s, t) \mapsto \gamma_s(t)$ be a homotopy between γ_0 and γ_1. Let $\epsilon > 0$ be chosen such that $B(F(s, t), 3\epsilon) \subset \Omega$ for all s, t. Such an ϵ must exist because $F([0, 1]^2)$ is compact.

- Choose $\delta > 0$ such that $\sup_{t\in[0,1]} |\gamma_{s_1}(t) - \gamma_{s_2}(t)| < \epsilon$ whenever $|s_1 - s_2| < \delta$. Such a δ must exist because F is uniformly continuous.
- Pick $|s_1 - s_2| < \delta$. Choose discs $D_0, \cdots, D_n$ of radius 2ϵ and points $\{z_0, \cdots, z_{n+1}\}$, $\{w_0, \cdots, w_{n+1}\}$ on $\gamma_{s_1}, \gamma_{s_2}$, respectively such that $z_i, z_{i+1}, w_i, w_{i+1} \in D_i$. Let F_i denote the primitive of f on D_i. Then $F_{i+1}(z_{i+1}) - F_i(w_{i+1}) = F_{i+1}(z_{i+1}) - F_i(w_{i+1})$.
- $$\begin{aligned}\int_{\gamma_{s_1}} f - \int_{\gamma_{s_2}} f &= \sum_{i=0}^{n}[F_i(z_{i+1}) - F_i(z_i)] - \sum_{i=0}^{n}[F_i(w_{i+1}) - F_i(w_i)] \\ &= \sum_{i=0}^{n}[F_i(z_{i+1}) - F_i(z_i) - F_i(w_{i+1}) + F_i(w_i)] \\ &= F_n(z_{n+1}) - F_n(w_{n+1}) - (F_0(z_0) - F_0(w_0)) \\ &= 0.\end{aligned}$$

□

Proposition 1.4 (Theorem 5.2). *Any holomorphic function in a simply connected domain has a primitive.*

Proof.

- Fix a point z_0 in Ω and define $F(z) = \int_\gamma f(w)dw$ where γ is a path from z_0 to z. Then $F(z+h) - F(z) = \int_\eta f(w)dw$ where η is the path from z to $z+h$.
- Since f is continuous at z, $f(w) = f(z) + \psi(w)$ where $\psi(w) \to 0$ as $w \to z$. Therefore, $F(z+h) - F(z) = f(z)\int_\eta dw + \int_\eta \psi(w)dw = f(z)h + \int_\eta \psi(w)dw$. Since $\left|(\int_\eta \psi(w)dw)/h\right| \leq \sup_{w\in\eta} |\psi(w)| = 0$ as $h \to 0$. Thus $\lim_{h\to 0}(F(z+h) - F(z))/h = f(z)$.

□

2. Liovilles Theorem and the fundamental theorem of algebra

Proposition 2.1 (Corollary 4.5(Liouville's Theorem)). *If f is entire and bounded, then f is constant.*

Proof. It suffices to prove that $f' = 0$ since $\mathcal{C}$ is connected $\forall z_0 \in \mathbb{C}, \forall R > 0, |f'(z_0)| \leq B/R$ by the Cauchy inequalities where B is a bound for f. Let $R \to \infty$. □

Proposition 2.2 (Corollary 4.6(The Fundamental Theorem of Algebra)). *Every non-constant polynomial $P(z) = a_n z^n + \cdots + a_0$ with complex coefficients has a root in $\mathbb{C}$.*

Proof. Proof by contradiction. Consider $P(z)/z^n = a_n + (a_{n-1}/z + \cdots + a_0/z^n)$. As $|z| \to \infty$, the right side approaches $a_n \neq 0$. Thus there exist $c > 0$ and $R > 0$ such that $|P(z)| > c|z|^n$ whenever $|z| > R$. In other words, $|P(z)|$ is bounded below by a positive number when $|z| > R$. On the other hand, $1/P$ is continuous and the disc $|z| \leq R$ is compact, so $1/P$ is bounded below on the disc. By Liouville's theorem, $P(z)$ is constant. Contradiction. □

3. Cauchys integral formula

Proposition 3.1. *Cauchys integral formula Suppose f is holomorphic in an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, then*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

for any point $z \in D$.

Proof. Consider the keyhole $\Gamma_{\delta,\epsilon}$ where δ denotes the width of the corridor and ϵ denotes the radius of the small circle.

- The whole integral: $\int_{\Gamma_{\delta,\epsilon}} F(\zeta) d\zeta = 0$ by Cauchy's theorem.
- Corridors: They cancel out.
- The small circle: $F(\zeta) = \frac{f(\zeta)-f(z)}{\zeta - z} + \frac{f(z)}{\zeta - z}$
  - The first term on the right-hand side is bounded, so the integral over C_ϵ goes to 0 as $\epsilon \to 0$.
  - $\int_{C_\epsilon} f(z)/(\zeta - z) d\zeta = f(z) \int_0^{2\pi} \epsilon i e^{-it}/(\epsilon e^{-it}) dt = -f(z) 2\pi i.$
- The big circle: This is just $\int_C f(\zeta)/(\zeta - z) d\zeta$.

$\square$