MATH 602 HOMEWORK 4

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Exercise. (1) Let $a/s \in S^{-1}\sqrt{I}$. Then $a^n \in I$ and $s \in S$ for some $n \in \mathbb{N}$. This implies $(a/s)^n \in S^{-1}I$, so $a/s \in \sqrt{S^{-1}I}$.

Let $a/s \in \sqrt{S^{-1}I}$. Then $a^n/s^n \in S^{-1}I$ for some $n \in \mathbb{N}$. Then $a^n \in I$, so $a \in \sqrt{I}$. Since $s \in S$, $a/s \in S^{-1}\sqrt{I}$.

Exercise. (2) Let $\{V_\alpha\}$ be an open cover of $\mathrm{Spec}(R)$. For each α, $\mathrm{Spec}(R) \setminus V_\alpha = V(a_\alpha)$ for some ideal a_α of R. $\mathrm{Spec}(R) = \cup_{\alpha \in I} V_\alpha = U_{\alpha \in I}(\mathrm{Spec}(R) \setminus V(a_\alpha)) = \mathrm{Spec}(R) \setminus V(\cup a_\alpha) = \mathrm{Spec}(R) \setminus V(\sum a_\alpha)$. In other words, $V(\sum a_\alpha) = \emptyset$. Since every proper ideal is contained in a maximal ideal, $\sum a_\alpha = (1)$. This implies $1 = x_{\alpha_1} + \cdots + x_{\alpha_n}$ for some $\alpha_1, \cdots, \alpha_n \in I$ and $x_{\alpha_i} \in a_{\alpha_i}$. Then $\cup V_{\alpha_i} = \mathrm{Spec}(R) \setminus V(\cup a_{\alpha_i}) = \mathrm{Spec}(R) \setminus V(1) = \mathrm{Spec}(R)$. Thus $\mathrm{Spec}(R)$ is indeed compact.

Exercise. (3) Suppose that I is generated by one element x. Then $ax = 0 \implies a = 0$ because A is an integral domain. Therefore, I is a free module with a basis $\{x\}$.

On the other hand, suppose that I is a free module with a basis $\{x_\alpha\}$. Since it is a basis, each $x_\alpha \neq 0$. Moreover, if the basis contains more than 2 elements, $(-x_{\alpha'})x_\alpha + x_\alpha x_{\alpha'} = 0$, so it is not linearly independent. Therefore, the basis must contain exactly one element.

Exercise. (4a) If $m = 0$ in each M_{f_i}, then, for each i, $f_i^{k_i} m = 0$ for some $k_i \geq 0$. Let $k = \max\{k_1, \cdots, k_n\}$. Since $1 \in \langle f_1, \cdots, f_n \rangle$, 1 can be expressed as a linear combination of N monomials consisting of f_i's. Then $m = 1m = 1^{Nk} m = 0$ because each monomial in the Nkth power of such a linear combination of N monomials contains at least k appearances of one monomial, which kills m.

Exercise. (5) We will consider the R/I-module M/IM. Let $\hat{m}$ be a maximal ideal in R/I. Then $\hat{m}$ is of the form $\{x + I \mid x \in m\}$ for some maximal ideal m of R containing I. $(M/IM)_m = M_m/(IM)_m$ by Corollary 3.4(iii). Since $M_m = 0$ for any maximal ideal m containing I, $(M/IM)_m = 0$. By Proposition 3.8[Atiyah], $M/IM = 0$, so $M = IM$.

Exercise. (6a) $(M : N)$ is nonempty. For any $a, b \in (M : N)$, $(a - b)N = aN + (-b)N = aN + bN \subset M$, so $a - b \in (M : N)$. Finally, for any $a \in (M : N), x \in R$, $(xa)N = a(xN) \subset aN \subset M$, $ax \in (M : N)$.

Exercise. (6b)

$$\begin{aligned}
a \in \mathrm{Ann}((M+N)/M) &\iff a((M+N)/M) = 0 \\
&\iff \forall (m+n) + M \in (M+N)/M, a((m+n)+M) = 0 \\
&\iff \forall (m+n) + M \in (M+N)/M, am + an \in M \\
&\iff \forall n \in N, an \in M \\
&\iff aN \subset M \\
&\iff a \in (M : N).
\end{aligned}$$

Exercise. (6c) First, we assume that J is generated by a single element x. Then $Rx = R/\operatorname{Ann}(x)$. Then $S^{-1}(Rx) = S^{-1}R/S^{-1}\operatorname{Ann}(x)$. On the other hand, $S^{-1}(Rx)$ is an ideal of $S^{-1}R$ generated by x, so $(S^{-1}R)x \cong S^{-1}R/\operatorname{Ann}(S^{-1}Rx)$. Therefore, $S^{-1}\operatorname{Ann}(x) = \operatorname{Ann}(S^{-1}Rx)$. In other words, $S^{-1}\operatorname{Ann}(J) = \operatorname{Ann}(S^{-1}J)$.

Moreover, if J_1, J_2 are generated by single elements,

$$\begin{aligned} S^{-1}\operatorname{Ann}(J_1 + J_2) &= S^{-1}(\operatorname{Ann}(J_1) \cap \operatorname{Ann}(J_2)) \\ &= S^{-1}\operatorname{Ann}(J_1) \cap S^{-1}\operatorname{Ann}(J_2) \\ &= \operatorname{Ann}(S^{-1}J_1) \cap \operatorname{Ann}(S^{-1}J_2) \\ &= \operatorname{Ann}(S^{-1}J_1 + S^{-1}J_2) \\ &= \operatorname{Ann}(S^{-1}(J_1 + J_2)). \end{aligned}$$

By induction, $S^{-1}\operatorname{Ann}(J) = \operatorname{Ann}(S^{-1}J)$ for any finitely generated ideal. Then

$$\begin{aligned} S^{-1}(I : J) &= S^{-1}\operatorname{Ann}((I + J)/I) \\ &= \operatorname{Ann}(S^{-1}(I + J)/S^{-1}I) \\ &= \operatorname{Ann}((S^{-1}I + S^{-1}J)/S^{-1}I) \\ &= (S^{-1}I : S^{-1}J). \end{aligned}$$

Exercise. (7) Let $q \in V(p)$. Suppose $M_q = 0$. Let $m/s \in (A - q)^{-1}M$. Then $tm = 0$ for some $t \in A - q$. In other words, for each $m \in M$, there exists $t \in A - q$ such that $tm = 0$.

Since $p \subset q$, for each m, the t must live in $A - p$. Therefore, $M_p = 0$. However, this is a contradiction because $p \in \operatorname{Supp}(M)$. Thus $q \in \operatorname{Supp}(M)$.

Exercise. (8) Let $b/s \in S^{-1}B$. Then $b \in B$, so $b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0$ where $a_i \in A$. This implies that $(b/s)^n + (a_{n-1}/s)(b/s)^{n-1} + \cdots + (a_1/s^{n-1})(b/s) + a_0/s^n = 0$, thus b/s is integral over $S^{-1}A$.