

MATH 602 HOMEWORK 4

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Exercise. (1) Let $a/s \in S^{-1}\sqrt{I}$. Then $a^n \in I$ and $s \in S$ for some $n \in \mathbb{N}$. This implies $(a/s)^n \in S^{-1}I$, so $a/s \in \sqrt{S^{-1}I}$.

Let $a/s \in \sqrt{S^{-1}I}$. Then $a^n/s^n \in S^{-1}I$ for some $n \in \mathbb{N}$. Then $a^n \in I$, so $a \in \sqrt{I}$. Since $s \in S$, $a/s \in S^{-1}\sqrt{I}$.

Exercise. (2) Let $\{V_\alpha\}$ be an open cover of $\text{Spec}(R)$. For each α , $\text{Spec}(R) \setminus V_\alpha = V(a_\alpha)$ for some ideal a_α of R . $\text{Spec}(R) = \cup_{\alpha \in I} V_\alpha = U_{\alpha \in I}(\text{Spec}(R) \setminus V(a_\alpha)) = \text{Spec}(R) \setminus V(\cup_{\alpha \in I} a_\alpha) = \text{Spec}(R) \setminus V(\sum a_\alpha)$. In other words, $V(\sum a_\alpha) = \emptyset$. Since every proper ideal is contained in a maximal ideal, $\sum a_\alpha = (1)$. This implies $1 = x_{\alpha_1} + \cdots + x_{\alpha_n}$ for some $\alpha_1, \dots, \alpha_n \in I$ and $x_{\alpha_i} \in a_{\alpha_i}$. Then $\cup V_{\alpha_i} = \text{Spec}(R) \setminus V(\cup a_{\alpha_i}) = \text{Spec}(R) \setminus V(1) = \text{Spec}(R)$. Thus $\text{Spec}(R)$ is indeed compact.

Exercise. (3) Suppose that I is generated by one element x . Then $ax = 0 \implies a = 0$ because A is an integral domain. Therefore, I is a free module with a basis $\{x\}$.

On the other hand, suppose that I is a free module with a basis $\{x_\alpha\}$. Since it is a basis, each $x_\alpha \neq 0$. Moreover, if the basis contains more than 2 elements, $(-x_{\alpha'})x_\alpha + x_\alpha x_{\alpha'} = 0$, so it is not linearly independent. Therefore, the basis must contain exactly one element.

Exercise. (4a) If $m = 0$ in each M_{f_i} , then, for each i , $f_i^{k_i}m = 0$ for some $k_i \geq 0$. Let $k = \max\{k_1, \dots, k_n\}$. Since $1 \in \langle f_1, \dots, f_n \rangle$, 1 can be expressed as a linear combination of N monomials consisting of f_i 's. Then $m = 1m = 1^{Nk}m = 0$ because each monomial in the Nk th power of such a linear combination of N monomials contains at least k appearances of one monomial, which kills m .

Exercise. (6a) $(M : N)$ is nonempty. For any $a, b \in (M : N)$, $(a - b)N = aN + (-b)N = aN + bN \subset M$, so $a - b \in (M : N)$. Finally, for any $a \in (M : N)$, $x \in R$, $(xa)N = a(xN) \subset aN \subset M$, $ax \in (M : N)$.

Exercise. (6b)

$$\begin{aligned}
 a \in \text{Ann}((M + N)/M) &\iff a((M + N)/M) = 0 \\
 &\iff \forall (m + n) + M \in (M + N)/M, a((m + n) + M) = 0 \\
 &\iff \forall (m + n) + M \in (M + N)/M, am + an \in M \\
 &\iff \forall n \in N, an \in M \\
 &\iff aN \subset M \\
 &\iff a \in (M : N).
 \end{aligned}$$

Exercise. (8) Let $b/s \in S^{-1}B$. Then $b \in B$, so $b^n + a_{n-1}b^{n-1} + \cdots + a_1b + a_0 = 0$ where $a_i \in A$. This implies that $(b/s)^n + (a_{n-1}/s)(b/s)^{n-1} + \cdots + (a_1/s^{n-1})(b/s) + a_0/s^n = 0$, thus b/s is integral over $S^{-1}A$.