

# MATH 611 (DUE 10/23)

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## 1. SIMPLICIAL AND SINGULAR HOMOLOGY

**Exercise.** (Problem 2) Show that the  $\Delta$ -complex obtained from  $\Delta^3$  by performing the edge identifications  $[v_0, v_1] \sim [v_1, v_3]$  and  $[v_0, v_2] \sim [v_2, v_3]$  deformation retracts onto a Klein bottle. Find other pairs of identifications of edges that produce  $\Delta$ -complexes deformation retracting onto a torus, a 2-sphere, and  $\mathbb{RP}^2$ .

*Proof.*

Maybe something like this? Either way, I noticed that it looks like it contains  $2 \mathbb{RP}^2$ .

□

**Exercise.** (Problem 4) Compute the simplicial homology groups of the triangular parachute obtained from  $\Delta^2$  by identifying its three vertices to a single point.

*Proof.* Let  $v_0$  denote the only vertex,  $e_1, e_2, e_3$  denote the three edges of the parachute, and  $\sigma$  denote the face of the parachute.  $C_k = 0$  for  $k \geq 3$  because  $\Delta^2$  with the vertices identified does not contain any  $k$ -dimensional simplices.  $C_2 = \langle \sigma \rangle, C_1 = \langle e_1, e_2, e_3 \rangle, C_0 = \langle v_0 \rangle$ . Let  $n \in \mathbb{N}$ .  $\partial_n$  is defined such that  $\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha| [v_0, \dots, \hat{v}_i, \dots, v_n]$ . Since there is only one vertex,  $\partial_n$  is the zero map.

This argument doesn't work. Check the torus example from class. It only has one vertex, but  $\partial_n$  is not the zero map for some  $n$ .

Therefore,  $H_n = \ker(\partial_n) / \text{Im}(\partial_{n+1}) = C_n / \langle 0 \rangle = C_n$ . Thus

$$H_n = \begin{cases} \{0\} & (n \geq 3) \\ \langle \sigma \rangle \cong \mathbb{Z} & (n = 2) \\ \langle e_1, e_2, e_3 \rangle \cong \mathbb{Z}^3 & (n = 1) \\ \langle v_0 \rangle \cong \mathbb{Z} & (n = 0). \end{cases}$$

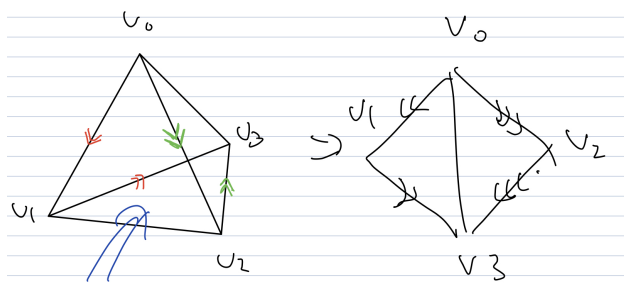


FIGURE 1. mycaption

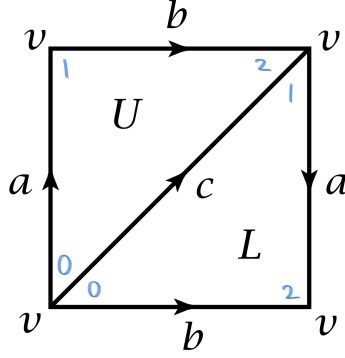


FIGURE 2. Problem 5

I'm not sure if this is correct.

□

**Exercise.** (Problem 5) Compute the simplicial homology groups of the Klein bottle using the  $\Delta$ -complex structure described at the beginning of this section.

*Proof.* We will use the notations in Figure 2.

$$C_n = \begin{cases} 0 & (n \geq 3) \\ \langle U, L \rangle & (n = 2) \\ \langle a, b, c \rangle & (n = 1) \\ \langle v \rangle & (n = 0). \end{cases}$$

$\partial_n = 0$  for  $n \geq 3$  and  $n = 0$ .

$$\begin{aligned} \partial_2(U) &= \sum_{i=0}^2 (-1)^i \sigma|[0, 1, 2] \\ &= \sigma|[1, 2] - \sigma|[0, 2] + \sigma|[0, 1] \\ &= b - c + a. \end{aligned}$$

$$\begin{aligned} \partial_2(L) &= \sum_{i=0}^2 (-1)^i \sigma|[0, 1, 2] \\ &= \sigma|[1, 2] - \sigma|[0, 2] + \sigma|[0, 1] \\ &= a - b + c. \end{aligned}$$

$\partial_1(a) = 0$  since  $\partial_1(a) = \sigma|[1] - \sigma|[0] = v - v = 0$ . Similarly,  $\partial_1(b) = \partial_1(c) = 0$ . Thus

$$H_n = \begin{cases} \{0\} & (n \geq 3) \\ \langle \sigma \rangle \cong \mathbb{Z} & (n = 2) \\ \langle e_1, e_2, e_3 \rangle \cong \mathbb{Z}^3 & (n = 1) \\ \langle v_0 \rangle \cong \mathbb{Z} & (n = 0). \end{cases}$$

Finish the rest.

□

**Exercise.** (Problem 7) Find a way of identifying pairs of faces of  $\Delta^3$  to produce a  $\Delta$ -complex structure on  $S^3$  having a single 3-simplex, and compute the simplicial homology groups of this  $\Delta$ -complex.