

# MATH 612 (HOMEWORK 1)

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**Exercise.** (Exercise 1(a)) The case of  $G = \mathbb{Z}$  is discussed in Example 2.42.

$$H_k(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0 \text{ and for } k = n \text{ odd} \\ \mathbb{Z}_2 & \text{for } k \text{ odd, } 0 < k < n \\ 0 & \text{otherwise.} \end{cases}$$

Suppose  $n$  is even. For any abelian group  $G$ , we obtain the cellular chain complex

$$0 \rightarrow G \xrightarrow{2} G \xrightarrow{0} \cdots \xrightarrow{2} G \xrightarrow{0} G \rightarrow 0.$$

If  $n$  is odd, we obtain

$$0 \rightarrow G \xrightarrow{0} G \xrightarrow{2} \cdots \xrightarrow{2} G \xrightarrow{0} G \rightarrow 0.$$

- Suppose  $k$  is even and  $2 \leq k \leq n$ . The homology at  $\xrightarrow{0} G \xrightarrow{2}$  is
  - 0 if  $G = \mathbb{Q}, \mathbb{Z}/p^l\mathbb{Z}$  with  $p \neq 2$ .
  - $\mathbb{Z}/2\mathbb{Z}$  if  $G = \mathbb{Z}/2^l$ .
- Suppose  $k$  is odd and  $1 \leq k \leq n - 1$ . The homology at  $\xrightarrow{2} G \xrightarrow{0}$  is
  - $G/2G \cong 0$  if  $G = \mathbb{Q}, \mathbb{Z}/p^l\mathbb{Z}$  with  $p \neq 2$  because multiplication by 2 is an isomorphism.
  - $\mathbb{Z}/2\mathbb{Z}$  if  $G = \mathbb{Z}/2^l$ .
- Suppose  $k = n$  and  $n$  is odd, or  $k = 0$ . The homology at  $\xrightarrow{0} G \xrightarrow{0}$  is  $G$ .

When  $G = \mathbb{Q}$ , the universal coefficient theorem gives an isomorphism  $H_k(X) \otimes \mathbb{Q} \cong H_k(X; \mathbb{Q})$  since  $\mathbb{Q}$  is torsion free.  $\mathbb{Z} \otimes \mathbb{Q} \cong \mathbb{Q}$  and  $\mathbb{Z}/2 \otimes \mathbb{Q} = 0$  because 2 is invertible in  $\mathbb{Q}$ . This agrees with the results above.

**Exercise.** (Exercise 1(b)) As discussed in Example 2.37,  $H_2(N_g; \mathbb{Z}) = 0$ ,  $H_1(N_g; \mathbb{Z}) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$ , and  $H_0(N_g; \mathbb{Z}) = \mathbb{Z}$ . For an abelian group  $G$ , the cellular chain complex is

$$0 \rightarrow G \xrightarrow{d_2} G^g \xrightarrow{d_1} G \rightarrow 0.$$

As discussed in Example 2.37,  $d_2(1) = (2, 2, \dots, 2)$  and  $d_1 = 0$ . If  $1 + 1 = 0$  in  $G$ , then  $H_2(X; G) = H_0(X; G) = G$  and  $H_1(X; G) = G^g$ . Otherwise, then  $H_2(X; G) = 0$ ,  $H_1(X; G) = G^{g-1}$  and  $H_0(X; G) = G$ .

**Exercise.** (Exercise 1(c)) For a  $\mathbb{Z}$ -module  $R$ , we have

$$0 \rightarrow R \xrightarrow{0} R \xrightarrow{a} R \xrightarrow{0} R \rightarrow 0.$$

When  $R = \mathbb{Z}$ , we obtain

$$H_k(X; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0, 2n - 1 \\ \mathbb{Z}_m & \text{for } k \text{ odd, } 0 < k < 2n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

When  $R$  is an abelian group such that  $1 + 1 + \cdots + 1 = 0$  ( $a$  times),  $H_i(X; R) = R$  if  $i = 0, 1, 2, 3$ . Otherwise,  $H_3(X; R) = H_0(X; R) = R$  and all other cohomology groups are 0.