MATH 620 HOMEWORK DUE 9/5

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Exercise 0.1. Show that $\{e^{i_1} \otimes \cdots \otimes e^{i_k} \mid 1 \leq i_1, \cdots, i_k \leq n\}$ is a basis of $T^k(V^*)$. Find $\dim T^k(V^*)$.

Proof.

- Linearly independent? Suppose $\sum c_{i_1,\cdots,i_k} e^{i_1} \otimes \cdots \otimes e^{i_k} = 0$. Let $1 \leq j_1, \cdots, j_k \leq n$ be given.

$$
\begin{aligned}
&(\sum_{i_1,\cdots,i_k} c_{i_1,\cdots,i_k} e^{i_1} \otimes \cdots \otimes e^{i_k})(e_{j_1}, \cdots, e_{j_k}) = 0 \\
&\implies \sum_{i_1,\cdots,i_k} c_{i_1,\cdots,i_k} (e^{i_1} \otimes \cdots \otimes e^{i_k})(e_{j_1}, \cdots, e_{j_k}) = 0 \\
&\implies \sum_{i_1,\cdots,i_k} c_{i_1,\cdots,i_k} e^{i_1}(e_{j_1}) \cdots e^{i_k}(e_{j_k}) = 0 \\
&\implies c_{j_1,\cdots,j_k} e^{j_1}(e_{j_1}) \cdots e^{j_k}(e_{j_k}) = 0 \\
&\implies c_{j_1,\cdots,j_k} = 0.
\end{aligned}
$$

  Therefore, each $c_{i_1,\cdots,i_k} = 0$.
- Span? Let $f \in T^k(V^*)$. We claim that $f = \sum_{i_1,\cdots,i_k} f(e_{i_1}, \cdots, e_{i_k}) e^{i_1} \otimes \cdots \otimes e^{i_k}$. Let $v_1, \cdots, v_k \in V$ be given. Since $\{e_1, \cdots, e_n\}$ is a

basis of V, so each v_i can be represented as $v_i = \sum_j c_i^j e_j$.

$$
\begin{aligned}
&(\sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})e^{i_1}\otimes\cdots\otimes e^{i_k})(v_1,\cdots,v_k)\\
&= (\sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})e^{i_1}\otimes\cdots\otimes e^{i_k})(c_1^j e_j,\cdots,c_k^j e_j)\\
&= \sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})[(e^{i_1}\otimes\cdots\otimes e^{i_k})(c_1^j e_j,\cdots,c_k^j e_j)]\\
&= \sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})[(c_1^j e^{i_1}(e_j))\cdots(c_k^j e^{i_k}(e_j))]\\
&= \sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})[(c_1^{i_1} e^{i_1}(e_{i_1}))\cdots(c_k^{i_k} e^{i_k}(e_{i_k}))]\\
&= \sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})c^{i_1}\cdots c^{i_k}\\
&= \sum_{i_1,\cdots,i_k} f(c^{i_1}e_{i_1},\cdots,c^{i_k}e_{i_k})\\
&= \sum_{i_1,\cdots,i_{k-1}} (\sum_{i_k} f(c^{i_1}e_{i_1},\cdots,c^{i_k}e_{i_k}))\\
&= \sum_{i_1,\cdots,i_{k-1}} f(c^{i_1}e_{i_1},\cdots,c^{i_{k-1}}e_{i_{k-1}},\sum_{i_k} c^{i_k}e_{i_k}))\\
&= \sum_{i_1,\cdots,i_{k-1}} f(c^{i_1}e_{i_1},\cdots,c^{i_{k-1}}e_{i_{k-1}},v_k)\\
&\vdots\\
&= f(v_1,\cdots,v_k).
\end{aligned}
$$

The dimension is n^k because each i_j can be any integer between 1 and n. □

Exercise 0.2. Prove that $\{\partial_1,\cdots,\partial_n\}$ is a basis of $T_p\mathbb{R}^n$.

Proof. TODO □

Exercise 0.3. Show that $\{dx^1,\cdots,dx^n\}$ is a basis of $T_p^*\mathbb{R}^n$ that is dual to $\{\frac{\partial}{\partial x^j}\}_{j=1}^n \subset T_p\mathbb{R}^n$.

Proof.

- Dual? Let $i,j\in\{1,\cdots,n\}$. $dx^i(\frac{\partial}{\partial x^j}) = \frac{\partial}{\partial x^j}x^i$. The partial derivative of x^i with respect to x^j is 1 if $i=j$ and 0 otherwise. Thus $dx^i(\frac{\partial}{\partial x^j}) = \delta_j^i$.

- Linearly independent? Let $c_1, \cdots, c_n \in \mathbb{R}$ be given. Suppose that $c_1 dx^1 + \cdots + c_n dx^n = 0$. For any $i \in \{1, \cdots, n\}$,

$$\begin{aligned}(c_1 dx^1 + \cdots + c_n dx^n)(\partial_i) = 0 &\implies c_1(dx^1(\partial_i)) + \cdots + c_n(dx^n(\partial_i)) = 0 \\ &\implies c_1(\partial_i(x^1)) + \cdots + c_n(\partial_i(x^n)) = 0 \\ &\implies c_i \partial_i(x^i) = 0 \\ &\implies c_i = 0.\end{aligned}$$

  Therefore, $c_1 = \cdots = c_n = 0$. Therefore, $\{dx^1, \cdots, dx^n\}$ is indeed linearly independent.
- Span? Let $f \in T_p^*\mathbb{R}^n$ be given. We claim that $f = \sum_{i=1}^n f(\partial_i) dx^i$. Let $\sum_{i=1}^n c_i \partial_i \in T_p\mathbb{R}^n$ be given where c_i's are in $\mathbb{R}$. (It makes sense to assume that every element in $T_p\mathbb{R}^n$ is in this form because we showed earlier that $\{\partial_1, \cdots, \partial_n\}$ is a basis of $T_p\mathbb{R}^n$.)

$$\begin{aligned}(\sum_{i=1}^n f(\partial_i) dx^i)(\sum_{j=1}^n c_j \partial_j) &= \sum_{i=1}^n \big[f(\partial_i) dx^i(\sum_{j=1}^n c_j \partial_j)\big] \\ &= \sum_{i=1}^n f(\partial_i) \big[\sum_{j=1}^n c_j dx^i(\partial_j)\big] \\ &= \sum_{i=1}^n f(\partial_i) \big[\sum_{j=1}^n c_j \partial_j(x^i)\big] \\ &= \sum_{i=1}^n f(\partial_i) c_i \\ &= f(\sum_{i=1}^n c_i \partial_i).\end{aligned}$$

$\square$