MATH 611 HOMEWORK (DUE 9/25)

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Exercise. (Problem 4, Chapter 1.3) Construct a simply-connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle intersecting it in two points.

Proof. We claim that the space described in Figure 1 is a covering space of X.

- The shape is an infinitely long chain of spheres and lines. The chain goes infinitely both ways (up and down). This space is clearly simply connected.
- We will map each sphere to the sphere of X. Each line will be mapped to the diameter up side down. Figure 1 shows how each part gets mapped.
- We claim that such a mapping is a covering map and thus this infinite chain is indeed a covering space. Let $x \in X$.

  Prove this.

Second part.

□

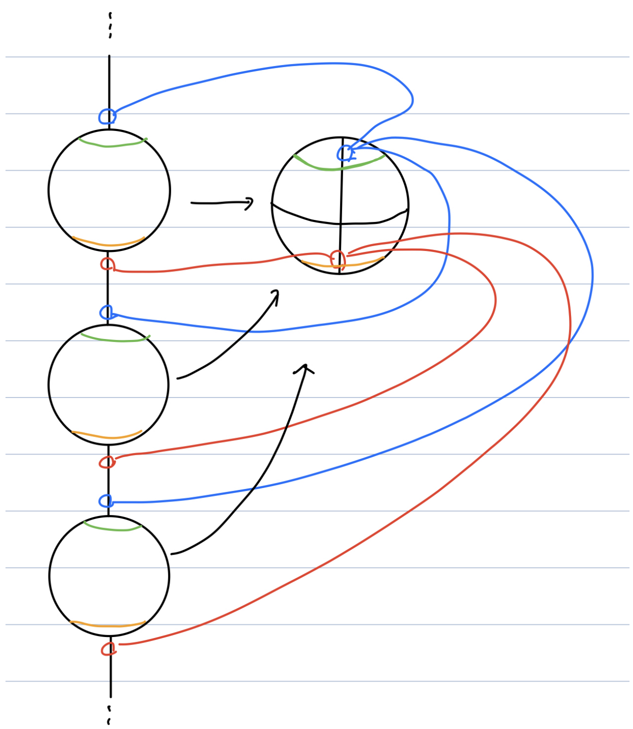

Figure 1. Problem 4 (Part 1)

Exercise. (Problem 5, Chapter 1.3) Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1]\times[0,1]$ together with the segments of the vertical lines $x = 1/2, 1/3, 1/4, \cdots$ inside the square. Show that for every covering space $X \to X$ there is some neighborhood of the left edge of X that lifts homeomorphically to $\tilde{X}$. Deduce that X has no simply-connected covering space.

Proof. Idea: Open cover of the left edge by evenly covered open sets. Find a finite subcover. By the tube lemma, there exists $U \times I$ that covers the left edge and a partition $0 = t_0 < \cdots < t_n = 1$ such that $U \times [t_i, t_{i+1}]$ is contained in an evenly covered neighborhood. Inductively, show $U \times [0, t_i]$ is in an evenly covered neighborhood. Lift a loop in X with a vertical line $x = 1/n$ for some large n. Then the element maps back to itself by p. In other words, $p_*(\pi_1(\tilde{X})) \neq 0$.

□

Exercise. (Problem 7, Chapter 1.3) Let Y be the quasi-circle in the figure in the textbook. Collapsing the segment of Y in the y-axis to a point gives a quotient map $f : Y \to S^1$. Show that f does not lift to the covering space $\mathbb{R} \to S^1$, even though $\pi_1(Y) = 0$. Thus local path-connectedness of Y is a necessary hypothesis in the lifting criterion.

Proof. The lifting criterion is Proposition 1.33. The only property that Y is missing is the local path connectedness. But I'm not sure how to make use of it.

□

Exercise. (Problem 8, Chapter 1.3) Let $\tilde{X}$ and $\tilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces X and Y. Show that if $X \simeq Y$ then $\tilde{X} \simeq \tilde{Y}$.

Proof. I suspected that Proposition 1.33 might be useful. But I think it's actually pretty useful for this problem because the covering spaces are simply connected, so $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ is only possible if Y is simply connected. But then this means X is simply connected because they have the same homotopy type. However, this is not true in general. For instance, $X = Y = S^1$ and $\tilde{X} = \tilde{Y} = \mathbb{R}$.

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