MATH 633 (FINAL EXAM)

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Exercise. (1) Since f is holomorphic and $f \neq 0$, $1/f$ is a non-constant, holomorphic function on the region Ω. By the maximum modulus principle, $1/f$ cannot attain a maximum value in Ω. Therefore, f cannot attain a minimum value in Ω.

Exercise. (2) It suffices to show that, for every $R > 0$, f is holomorphic on the open disk centered at 0 with radius R. Let $R > 0$ be given. Let T be a triangle inside the open disk D centered at 0 with radius R. If none of the three edges of T lies on the x or y axis, then $\int_T f(z)dz = 0$. Suppose some of the three edges of T lies on the x and/or y axis. Then $T_n = T + (1+i)/n$ lies in D for any $n \geq N$ for a sufficiently large N. Since none of the three edges of T_n lies on the x or y axis, $\int_{T_n} f = 0$ for any $n \geq N$. Then $\int_T f = \lim_{n\to\infty} \int_{T_n} f = 0$.

Exercise. (3) Since $|f| = |g|$, f and g have poles at the same places. Let z_0 be a point at which f and g are defined and $g(z_0) \neq 0$. Let $\epsilon > 0$ be given such that f and g are defined and $g(z) \neq 0$ for all $|z - z_0| < \epsilon$. If no such ϵ exists, then $f = g = 0$ and we are done. Then the function f/g is holomorphic on the open disk $D(z_0, \epsilon)$. By the maximum modulus principle, f/g must be constant in $D(z_0, \epsilon)$. In other words, there exists a fixed θ at which $f/g = e^{i\theta}$ in $D(z_0, \epsilon)$. If f and g are not always 0, such open sets can be patched to show that $f/g = e^{i\theta}$. Otherwise, $f = g = 0$ so $f = e^{i\theta}g$ for any fixed θ.

Exercise. (6) Let $f = 3z^2$ and $g = z^5 + 1$. Then $|f| > |g|$ on the unit circle. By Rouche's theorem, f and $f + g$ have the same number of zeros inside the unit circle. Clearly, f only has one zero with multiplicity 2. Thus $p = f + g$ has exactly two zeros inside the unit circle.

Let $f = z^5$ and $g = 3z^2 + 1$. Then $|f| > |g|$ on the circle centered at 0 with radius 2 because $|g| \leq 3 \cdot 2 \cdot 2 + 1 = 13 < 32 = |f|$. By Rouche's theorem, f and $f + g$ have the same number of zeros inside C. f clearly has one zero with multiplicity 5, so $p = f + g$ has exactly 5 zeros inside C.

Therefore, in the annulus, p has 5 - 2 = 3 zeros.

Exercise. (7) Let $R > a^2$ be given. Let $T_1 = [-R, R]$ and T_2 be the upper half of the circle centered at 0 with radius R. Let $f(z) = \exp(iz)/(z^2 + a^2)$.

- $\int_{T_1+T_2} f(z)$ can be calculated using residues. The only singularity of f is ia. Since it is a simple pole, the residue is $\lim_{z\to ia}(z - ia)\exp(iz)/(z^2 + a^2) = \exp(-a)/2ia$ by Theorem 1.4 on P.76. By the residue formula, $\int_{T_1+T_2} f(z) = \pi \exp(-a)/2a$.

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$$\begin{aligned}
\left|\int_{T_2} f(z)\right| &= \left|\int_0^1 \frac{\exp(iRe^{\pi it})}{R^2e^{2\pi it}+a^2}R\pi ie^{\pi it}dt\right| \\
&\leq \int_0^1 \left|\frac{\exp(iRe^{\pi it})}{R^2e^{2\pi it}+a^2}R\pi ie^{\pi it}\right|dt \\
&\leq \int_0^1 \frac{|\exp(iRe^{\pi it})|}{|R^2e^{2\pi it}+a^2|}\left|R\pi ie^{\pi it}\right|dt \\
&\leq \int_0^1 \frac{\exp(-\operatorname{Im}(Re^{\pi it}))}{|R^2e^{2\pi it}+a^2|}\left|R\pi ie^{\pi it}\right|dt \\
&\leq \int_0^1 \frac{1}{\exp(R\sin(\pi t))|R^2e^{2\pi it}+a^2|}\left|R\pi ie^{\pi it}\right|dt \\
&\leq \int_0^1 \frac{1}{\exp(R\sin(\pi t))|R^2e^{2\pi it}+a^2|}R\pi dt \\
&\leq \pi\int_0^1 \frac{1}{\exp(R\sin(\pi t))|Re^{2\pi it}+a^2/R|}dt \\
&\to 0.
\end{aligned}$$

Based on these, we obtain that $\int_{T_1} f(z) = \pi e^{-a}/2a$ as $R\to\infty$. The desired integral is the real part of $\int_{T_1} f(z)$, and it is simply $\pi e^{-a}/2a$.

Exercise. (8) By repeatedly applying Theorem 5.3 (P.54), the dth derivative of f is 0 in the open unit disk. By induction, this implies that f is a polynomial of degree at most d.