MATH 611 (DUE 11/13)

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1. Simplicial and Singular Homology

Exercise. (Problem 27) Let $f : (X, A) \to (Y, B)$ be a map such that both $f : X \to Y, f : A \to B$ are homotopy equivalences.

- Show that $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all n.
- For the case of the inclusion $f : (D^n, S^{n-1}) \to (D^n, D^n \setminus \{0\})$, show that f is not a homotopy equivalence of pairs - there is no $g : (D^n, D^n \setminus \{0\}) \to (D^n, S^{n-1})$ such that fg and gf are homotopic to the identity through maps of pairs.

Proof.

- For each $n \geq 1$, we have an exact sequence $H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X)$ and another one with X, A replaced with Y, B. Moreover, they are connected by homomorphisms $f_* : H_n(A) \to H_n(B), f_* : H_n(X) \to H_n(Y), f_* : H_n(X, A) \to H_n(Y, B)$ such that the diagram commutes. (naturality) Since $f : X \to Y$ and $f : A \to B$ are both homotopy equivalences, $f_* : H_n(X) \to H_n(Y), f_* : H_n(A) \to H_n(B)$ are isomorphisms. By the Five lemma, $f_* : H_n(X, A) \to H_n(X, B)$ is an isomorphism.

  The exact sequence $H_1(A) \to H_1(X) \to H_1(X, A) \to 0$ can be extended to $H_1(A) \to H_1(X) \to H_1(X, A) \to 0 \to 0$ by appending 0 at the end. Using the same argument as above, $f_* : H_1(X, A) \to H_1(Y, B)$ is an isomorphism.
- Suppose $f : (D^n, S^{n-1}) \to (D^n, D^n - \{0\})$ is a homotopy equivalence. Then there exists a $g : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity maps in corresponding domains. Since g is continuous, $g(\overline{D^n - \{0\}}) = \overline{g(D^n - \{0\})} \subset \overline{S^{n-1}} = S^{n-1}$. Therefore, g maps D^n into S^{n-1}. Since f maps S^{n-1} into D^n, $g \circ f$ maps S^{n-1} into S^{n-1}. We know this is homotopic to the identity map from the problem statement. Similarly, $f \circ g$ maps D^n into D^n and we know this is homotopic to the identity map from the problem statement. Therefore, this implies that D^n and S^{n-1} are homotopy equivalent. However, this is false because D^n is contractible but S^{n-1} is not.

  Hence, f cannot be homotopy equivalent.

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