

## MATH 611 (DUE 11/6)

HIDENORI SHINOHARA

### 1. SIMPLICIAL AND SINGULAR HOMOLOGY

**Exercise.** (Problem 14) Determine whether there exists a short exact sequence  $0 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_8 \oplus \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow 0$ . More generally, determine which abelian groups  $A$  fit into a short exact sequence  $0 \rightarrow \mathbb{Z}_{p^m} \rightarrow A \rightarrow \mathbb{Z}_{p^n} \rightarrow 0$  with  $p$  prime. What about the case of short exact sequences  $0 \rightarrow A \rightarrow \mathbb{Z}_n \rightarrow 0$ ?

*Proof.* Let  $\phi_1 : \mathbb{Z}_4 \rightarrow \mathbb{Z}_8 \oplus \mathbb{Z}_2, \phi_2 : \mathbb{Z}_8 \oplus \mathbb{Z}_2 \rightarrow \mathbb{Z}_4$  be defined such that  $\phi_1(a) = (2a, a)$  and  $\phi_2(a, b) = a + 2b$ . Then  $\ker(\phi_1) = 0, \text{Im}(\phi_1) = \ker(\phi_2) = \{(2k, k) \mid 0 \leq k \leq 3\}$  and  $\text{Im}(\phi_2) = \mathbb{Z}_4$ . Thus this is indeed an exact sequence.

Finish this!

□

**Exercise.** (Problem 15) For an exact sequence  $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E$  show that  $C = 0$  if and only if the map  $A \rightarrow B$  is surjective and  $D \rightarrow E$  is injective. Hence, for a pair of spaces  $(X, A)$ , the inclusion  $A \rightarrow X$  induces isomorphisms on all homology groups if and only if  $H_n(X, A) = 0$  for all  $n$ .

*Proof.* Suppose  $C = 0$ .  $\text{Im}(\phi_{AB}) = \ker(\phi_{BC}) = B$ , so  $\phi_{AB}$  is surjective.  $\ker(\phi_{DE}) = \text{Im}(\phi_{CD}) = \{0\}$ , so  $\phi_{DE}$  is injective.

On the other hand, suppose  $\phi_{AB}$  is surjective and  $\phi_{DE}$  is injective.  $\text{Im}(\phi_{CD}) = \ker(\phi_{DE}) = \{0\}$ , so  $\phi_{CD}$  is the zero map. Therefore,  $\ker(\phi_{CD}) = C$ .  $\ker(\phi_{BC}) = \text{Im}(\phi_{AB}) = B$ , so  $\phi_{BC}$  is the zero map. Therefore,  $\text{Im}(\phi_{BC}) = 0$ . Hence,  $C = \ker(\phi_{CD}) = \text{Im}(\phi_{BC}) = 0$ .

By Theorem 2.16 and the discussion at the bottom of P.117(Hatcher), we have a long exact sequence of homology groups

$$(1.1) \quad H_n(A) \xrightarrow{i_*} H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X)$$

for  $n \geq 1$ . Suppose the inclusion induces isomorphisms on all homology groups. Then  $H_n(X, A) = 0$  for all  $n \geq 1$  by the first part. Moreover, we have  $H_1(X, A) \rightarrow H_0(A) \rightarrow H_0(X) \rightarrow H_0(X, A) \rightarrow 0$ . Since  $H_1(X, A) = 0$ , by the first part,  $H_0(X) = 0$ . In order for  $0 \rightarrow H_0(X, A) \rightarrow 0$  to be exact,  $H_0(X, A)$  must be 0. Therefore,  $H_n(X, A) = 0$  for all  $n \geq 0$ .

Suppose that  $H_n(X, A) = 0$  for all  $n \geq 0$ . By exact sequence 1.1 above,  $i_* : H_n(A) \rightarrow H_n(X)$  is surjective for  $n \geq 1$  and injective for  $n \geq 0$ . Thus  $i_*$  is bijective for all  $n \geq 1$ . We have  $H_1(X, A) \rightarrow H_0(A) \rightarrow H_0(X) \rightarrow H_0(X, A)$ . Since  $H_1(X, A) = H_0(X, A) = 0$ ,  $i_*$  must be bijective by the exactness. Therefore, the inclusion induces isomorphisms for all  $n$ . □

**Exercise.** (Problem 16)

- Show that  $H_0(X, A) = 0$  if and only if  $A$  meets each path-component of  $X$ .
- Show that  $H_1(X, A) = 0$  if and only if  $H_1(A) \rightarrow H_1(X)$  is surjective and each path-component of  $X$  contains at most one path-component of  $A$ .

*Proof.*

- Let  $\gamma_x + C_0(A) \in C_0(X)/C_0(A)$ . Since  $A$  meets each path-component of  $X$ , there exists a path  $\gamma : I \rightarrow X$  that joins a point  $a \in A$  and the image of  $\gamma_x$ . Then  $\gamma$  can be seen as an element of  $C_1(X)$  since  $\gamma$  maps a 1-simplex into  $X$ . Moreover,  $\partial\gamma = \gamma_x - \gamma_a$  where  $\gamma_a \in C_0(A)$  with  $\text{Im}(\gamma_a) = a$ . Therefore,  $\partial(\gamma + C_1(A)) = \gamma_x + C_0(A)$ , so  $\gamma_x + C_0(A) \in \text{Im}(\partial)$ . Hence,  $H_0(X, A) = \ker(\partial_0)/\text{Im}(\partial_1) = (C_0(X)/C_0(A))/(C_0(X)/C_1(A)) = 0$ .

On other hand, suppose that  $A$  does not meet each path component of  $X$ . Let  $x \in X$  be a point in a path component that  $A$  does not intersect. Let  $\gamma_x : \Delta^0 \rightarrow X$  such that  $\text{Im}(\gamma_x) = \{x\}$ . Then  $\gamma_x \in \ker(\partial_0) = C_0(X, A)$ . Let  $\gamma + C_1(A) \in C_1(X, A)$ . Then  $\partial_1(\gamma + C_1(A)) = \partial_1(\gamma) + C_0(A)$ . Let  $\gamma_{x_1}, \gamma_{x_2} \in C_0(X)$  such that  $\partial_1(\gamma) = \gamma_{x_1} - \gamma_{x_2}$ .  $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$  if and only if  $\gamma_{x_1} - \gamma_{x_2} - \gamma_x \in C_0(A)$ .

- If  $\gamma$  lies in the same path component as  $x$ , then so do  $x_1$  and  $x_2$ . Suppose  $x = x_1$ . Since  $-\gamma_{x_2} \notin C_0(A)$ ,  $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$ . The case when  $x \neq x_1$  and  $x = x_2$  and the case when  $x \neq x_1$  and  $x \neq x_2$  can be proven in a similar way.
- If  $\gamma$  lies in a different path component, then  $\gamma_x \neq \gamma_{x_1}$  and  $\gamma_x \neq \gamma_{x_2}$ . Therefore,  $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$ .

Therefore,  $\gamma_x \notin \text{Im}(\partial_1)$ . Thus  $H_0(X, A) = C_0(X, A)/\text{Im}(\partial_1)$  is not 0.

- Do part (b).

□

**Exercise.** (Problem 17)

- Compute the homology groups  $H_n(X, A)$  when  $X$  is  $S^2$  or  $S^1 \times S^1$  and  $A$  is a finite set of points in  $X$ .
- Compute the groups  $H_n(X, A)$  and  $H_n(X, B)$  for  $X$  a closed orientable surface of genus two with  $A$  and  $B$  the circles shown.

*Proof.*

- We will apply Theorem 2.16 to get the exact sequence with  $H_n(A), H_n(X), H_n(X, A)$ .
  - When  $n \geq 3$ ,  $H_n(S^2) \rightarrow H_n(S^2, A) \rightarrow H_{n-1}(A)$  shows that  $H_n(S^2, A)$  is 0 by the exactness since  $H_n(S^2) = H_{n-1}(A) = 0$ .
  - When  $n = 2$ ,  $H_n(A) \rightarrow H_n(S^2) \xrightarrow{\phi} H_n(S^2, A) \rightarrow H_{n-1}(A)$  shows that  $H_n(S^2, A) = H_n(S^2) = \mathbb{Z}$ . This is because  $H_n(A) = H_{n-1}(A) = 0$  so  $\phi$  is an isomorphism by the exactness.
  - $n = 0$  and  $n = 1$ .

We will first compute the homology groups of a torus using Figure 1.  $C_2 = \{\sigma_1, \sigma_2\}, C_1 = \{a, b, c\}, C_0 = \{v_0\}$ .

- $H_2 = \ker(\partial_2)/\text{Im}(\partial_3) = \langle \sigma_1 - \sigma_2 \rangle / 0 = \mathbb{Z}$ .
- $H_1 = \ker(\partial_1)/\text{Im}(\partial_2) = \langle a, b, c \rangle / \langle b - a + c, c - a + b \rangle = \mathbb{Z}^2$  because  $b - a + c = c - a + b$ .
- $H_0 = \ker(\partial_0)/\text{Im}(\partial_1) = \langle v_0 \rangle / 0 = \mathbb{Z}$ .

Again, we will apply Theorem 2.16 to get the exact sequence with  $H_n(A), H_n(X)$ , and  $H_n(X, A)$ .

- When  $n \geq 3$ ,  $H_n(S^1 \times S^1) \rightarrow H_n(S^1 \times S^1, A) \rightarrow H_{n-1}(A)$  shows that  $H_n(S^1 \times S^1, A)$  is 0 by the exactness since  $H_n(S^1 \times S^1) = H_{n-1}(A) = 0$ .

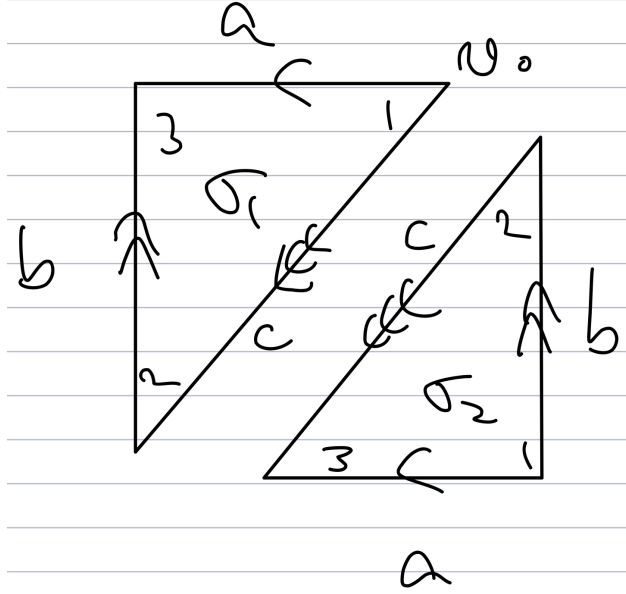


FIGURE 1. Problem 17

- When  $n = 2$ ,  $H_n(A) \rightarrow H_n(S^1 \times S^1) \xrightarrow{\phi} H_n(S^1 \times S^1, A) \rightarrow H_{n-1}(A)$  shows that  $H_n(S^1 \times S^1, A) = H_n(S^1 \times S^1) = \mathbb{Z}$ . This is because  $H_n(A) = H_{n-1}(A) = 0$  so  $\phi$  is an isomorphism by the exactness.

- $n = 0$  and  $n = 1$ .

- Finish this!

□

**Exercise.** (Problem 26) Show that  $H_1(X, A)$  is not isomorphic to  $\tilde{H}_1(X/A)$  if  $X = [0, 1]$  and  $A$  is the sequence  $1, 1/2, 1/3, \dots$  together with its limit 0.

*Proof.* We will show that  $H_1(X, A)$  is countable, and  $\tilde{H}_1(X/A) = H_q(X/A)$  is uncountable.

We have an exact sequence  $\tilde{H}_1(X) \rightarrow \tilde{H}_1(X, A) \xrightarrow{\phi} \tilde{H}_0(A) \rightarrow \tilde{H}_0(X)$ . Since  $H_1(X, A) = \tilde{H}_1(X, A) = \tilde{H}_0(X) = 0$ ,  $\phi$  is an isomorphism. Thus  $\tilde{H}_1(X, A) = \tilde{H}_0(A) = \ker(\partial_1) / \text{Im}(\partial_2)$ . Since  $A$  is a disjoint union of points,  $\text{Im}(\partial_2) = 0$ .  $\ker(\partial_1) = \{\sum n_i \alpha_i \mid n_i \in \mathbb{Z}, \sum n_i = 0\}$  where  $\alpha_i$  is the point  $1/i$  by the definition of a reduced homology. Then this is generated by  $\{\alpha_1 - \alpha_2, \alpha_1 - \alpha_3, \alpha_1 - \alpha_4, \dots\}$ , so  $\tilde{H}_1(X, A)$  is countable.

We will show the existence of an injective map  $\zeta$  from the direct product  $\prod_{i=1}^{\infty} \mathbb{Z}$  to  $H_1(X/A)$ , which is homeomorphic to the Hawaiian earring. We will refer to the  $n$ th ring  $C_n$  as in Example 1.25. Let  $(k_1, \dots) \in \prod_{i=1}^{\infty} \mathbb{Z}$  be given. Construct the map  $f : I \rightarrow X/A$  that wraps  $k_n$  times around  $C_n$  in the time interval  $[1 - 1/n, 1 - 1/(n+1)]$ . This infinite composition of loops is certainly continuous at each time less than 1, and it is continuous at time 1 since every neighborhood of the basepoint in  $X/A$  contains all but finitely many of the circles  $C_n$ . This shows that  $f \in C_1(X/A)$ . Moreover,  $\partial(f) = v_0 - v_0 = 0$  where  $v_0$  is the origin of the Hawaiian earring. Therefore,  $[f] \in H_1(X/A)$ . We define  $\zeta(k_1, \dots) = [f]$ .

Let  $(k_1, \dots) \neq (l_1, \dots) \in \prod_{i=1}^{\infty} \mathbb{Z}$  be given. Let  $\zeta(k_1, \dots) = f, \zeta(l_1, \dots) = g$  as described above. Let  $i$  be an index such that  $k_i \neq l_i$ . Let  $F : X/A \rightarrow S^1$  be a continuous map that

maps  $C_n$  onto  $S_1$  and  $C_i$  to  $-1$  for all  $i$  where  $S_1$  is seen as a subset of  $\mathbb{C}$ . Then  $F$  induces a group homomorphism  $F_* : H_1(X/A) \rightarrow H_1(S^1)$  where  $F_*([f]) = k_n$  and  $F_*([g]) = l_n$ . Since  $F_*([f]) \neq F_*([g])$ ,  $[f] \neq [g]$ . This shows the injectivity of  $\zeta$  and hence  $H_1(X/A)$  must be uncountable.

Therefore,  $H_1(X, A)$  is not isomorphic to  $H_1(X/A)$ . □