MATH 612 (HOMEWORK 1)

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Exercise. (Exercise 1(a)) The case of $G = \mathbb{Z}$ is discussed in Example 2.42.

$$H_k(\mathbb{R}\mathrm{P}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0 \text{ and for } k = n \text{ odd} \\ \mathbb{Z}_2 & \text{for } k \text{ odd, } 0 < k < n \\ 0 & \text{otherwise.} \end{cases}$$

Suppose n is even. For any abelian group G, we obtain the cellular chain complex

$$0 \to G \xrightarrow{2} G \xrightarrow{0} \cdots \xrightarrow{2} G \xrightarrow{0} G \to 0.$$

If n is odd, we obtain

$$0 \to G \xrightarrow{0} G \xrightarrow{2} \cdots \xrightarrow{2} G \xrightarrow{0} G \to 0.$$

- Suppose k is even and $2 \leq k \leq n$. The homology at $\xrightarrow{0} G \xrightarrow{2}$ is
  - 0 if $G = \mathbb{Q}, \mathbb{Z}/p^l\mathbb{Z}$ with $p \neq 2$.
  - $\mathbb{Z}/2\mathbb{Z}$ if $G = \mathbb{Z}/2^l$.
- Suppose k is odd and $1 \leq k \leq n - 1$. The homology at $\xrightarrow{2} G \xrightarrow{0}$ is
  - $G/2G \cong 0$ if $G = \mathbb{Q}, \mathbb{Z}/p^l\mathbb{Z}$ with $p \neq 2$ because multiplication by 2 is an isomorphism.
  - $\mathbb{Z}/2\mathbb{Z}$ if $G = \mathbb{Z}/2^l$.
- Suppose $k = n$ and n is odd, or $k = 0$. The homology at $\xrightarrow{0} G \xrightarrow{0}$ is G.

When $G = \mathbb{Q}$, the universal coefficient theorem gives an isomorphism $H_k(X) \otimes Q \cong H_k(X; \mathbb{Q})$ since Q is torsion free. $\mathbb{Z} \otimes \mathbb{Q} \cong \mathbb{Q}$ and $\mathbb{Z}/2 \otimes \mathbb{Q} = 0$ because 2 is invertible in $\mathbb{Q}$. This agrees with the results above.

When $G = \mathbb{Z}/2^l$, we have $0 \to H_k(X) \otimes G \to H_k(X; G) \to \mathrm{Tor}(H_{k-1}(C), G) \to 0$. If $k = n$ and k is odd, $H_k(X) = \mathbb{Z}$, so $\mathbb{Z}/2^l \cong H_k(X; \mathbb{Z}/2^l)$. If $k - 1 = n$ and $k - 1$ is odd, we obtain $0 \to 0 \to H_k(X; \mathbb{Z}/2^l) \to \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/2^l) \to 0$, so $H_k(X; \mathbb{Z}/2^l) = 0$. If k is odd and $0 < k < n$, $0 \to \mathbb{Z}/2 \otimes \mathbb{Z}/2^l \to H_k(X; \mathbb{Z}/2^l) \to \mathrm{Tor}(H_{k-1}(X), \mathbb{Z}/2^l) \to 0$. The Tor is 0 because if $k = 0$, $H_{k-1}(X) = \mathbb{Z}$ and $H_{k-1}(X) = 0$ otherwise. Thus $H_k(X; \mathbb{Z}/2^l) = \mathbb{Z}/2 \otimes \mathbb{Z}/2^l = \mathbb{Z}/2$. In any other cases, the universal coefficient theorem gives the SES $0 \to 0 \to H_n(X; G) \to 0 \to 0$. This agrees with the results above.

Suppose $G = \mathbb{Z}/p^l$. Then the case that $k = n$ and k is odd and the case that $k-1 = n$ and k is odd can be handled in the same way as above. Suppose k is odd and $0 < k < n$. Then $\mathbb{Z}/2 \otimes \mathbb{Z}/p^l = 0$. Moreover, $\mathrm{Tor}(H_{k-1}(X), \mathbb{Z}) = 0$ as discussed above. Thus $H_k(X) = 0$. In any other cases, the universal coefficient theorem gives the SES $0 \to 0 \to H_n(X; G) \to 0 \to 0$. This agrees with the results above.

Exercise. (Exercise 1(b)) As discussed in Example 2.37, $H_2(N_g; \mathbb{Z}) = 0, H_1(N_g; \mathbb{Z}) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$, and $H_0(N_g; \mathbb{Z}) = \mathbb{Z}$. For an abelian group G, the cellular chain complex is

$$0 \to G \xrightarrow{d_2} G^g \xrightarrow{d_1} G \to 0.$$

As discussed in Example 2.37, $d_2(1) = (2, 2, \cdots, 2)$ and $d_1 = 0$. If $G = \mathbb{Z}/p^l$ with $p \neq 2$ or $G = \mathbb{Q}$, then $H_2(X; G) = 0, H_1(X; G) = G^g / \langle(1, \cdots, 1)\rangle = G^{g-1}$ and $H_0(X; G) = G$ because 2^{-1} exists. Suppose $G = \mathbb{Z}/2^l$. Then $H_2(X; G) = \mathbb{Z}/2$ because the kernel is an index-2 subgroup. $H_1(X; G) = G^g / \langle(2a, \cdots, 2a)\rangle = G^{g-1} \otimes \mathbb{Z}/2$, and $H_0(X; G) = G$.

We will verify the results using the universal coefficient theorem.

Suppose $G = \mathbb{Q}$. Then $\mathrm{Tor}(H_{n-1}(C), G) = 0$ for any n. Thus $H_0(X; G) = \mathbb{Z} \otimes \mathbb{Q} = \mathbb{Q}$ and $H_1(X; G) = (\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2) \otimes \mathbb{Q} = (\mathbb{Z} \otimes \mathbb{Q})^{g-1} \oplus (\mathbb{Z}_2 \otimes \mathbb{Q}) = \mathbb{Q}^{g-1}$.

Suppose $G = \mathbb{Z}/p^l$ with $p \neq 2$. When $n = 1$, $H_{n-1}(C) = \mathbb{Z}$, so $\mathrm{Tor}(H_{n-1}(C), G) = 0$. Thus $H_1(C; G) = (\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2) \otimes \mathbb{Z}/p^l = (\mathbb{Z}/p^l)^{g-1}$. When $n = 2$, $H_n(C) = 0$ and $\mathrm{Tor}(H_{n-1}(C), \mathbb{Z}/p^l) = 0$ because multiplication by p^l does not kill any element in $\mathbb{Z}^{g-1} \oplus \mathbb{Z}/2$.

Suppose $G = \mathbb{Z}/2^l$. When $n = 1$, $\mathrm{Tor}(H_{n-1}(C), G) = \mathrm{Tor}(\mathbb{Z}, G) = 0$. Thus $H_n(C; G) = H_n(C) \otimes G = (\mathbb{Z}/2^l)^{g-1} \oplus \mathbb{Z}/2$. When $n = 2$, $H_n(C) = 0$ and $\mathrm{Tor}(H_{n-1}(C), \mathbb{Z}/2^l) = \ker((\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2) \xrightarrow{2^l} (\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2)) = \mathbb{Z}/2$. Thus $H_2(C; G) = \mathbb{Z}/2$.

Exercise. (Exercise 1(c)) For a Z-module R, we have

$$0 \to R \xrightarrow{0} R \xrightarrow{a} R \xrightarrow{0} R \to 0.$$

When $R = \mathbb{Z}$, we obtain

$$H_k(X; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0, 2n-1 \\ \mathbb{Z}_m & \text{for } k \text{ odd, } 0 < k < 2n-1 \\ 0 & \text{otherwise.} \end{cases}$$

When R is an abelian group such that $1 + 1 + \cdots + 1 = 0$ (a times), $H_i(X; R) = R$ if $i = 0, 1, 2, 3$. Otherwise, $H_3(X; R) = H_0(X; R) = R$ and all other cohomology groups are 0.