

MATH 601 HOMEWORK (DUE 9/11)

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Exercise. (1) Show that 2×2 matrices give a functor, M_2 , from the category of rings to itself, $R \mapsto M_2(R)$.

Proof. Let R, R' be rings and $\phi \in \text{Hom}(R, R')$. Let $M_2(\phi) : M_2(R) \rightarrow M_2(R')$ be defined such that

$$(M_2(\phi)) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{bmatrix}.$$

We claim that M_2 is indeed a functor.

- Claim 1: For any $\phi \in \text{Hom}(R, R')$, $M_2(\phi) \in \text{Hom}(M_2(R), M_2(R'))$.
In other words, we want to show that $M_2(\phi)$ is a ring homomorphism for any ϕ .

$$\begin{aligned} (M_2(\phi)) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) &= (M_2(\phi)) \left(\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \right) \\ &= \begin{bmatrix} \phi(a+e) & \phi(b+f) \\ \phi(c+g) & \phi(d+h) \end{bmatrix} \\ &= \begin{bmatrix} \phi(a) + \phi(e) & \phi(b) + \phi(f) \\ \phi(c) + \phi(g) & \phi(d) + \phi(h) \end{bmatrix} \\ &= \begin{bmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{bmatrix} + \begin{bmatrix} \phi(e) & \phi(f) \\ \phi(g) & \phi(h) \end{bmatrix} \\ &= (M_2(\phi)) \begin{bmatrix} a & b \\ c & d \end{bmatrix} + (M_2(\phi)) \begin{bmatrix} e & f \\ g & h \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
& (M_2(\phi)) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) \\
&= (M_2(\phi)) \left(\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} \right) \\
&= \begin{bmatrix} \phi(ae + bg) & \phi(af + bh) \\ \phi(ce + dg) & \phi(cf + dh) \end{bmatrix} \\
&= \begin{bmatrix} \phi(a)\phi(e) + \phi(b)\phi(g) & \phi(a)\phi(f) + \phi(b)\phi(h) \\ \phi(c)\phi(e) + \phi(d)\phi(g) & \phi(c)\phi(f) + \phi(d)\phi(h) \end{bmatrix} \\
&= \begin{bmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{bmatrix} \begin{bmatrix} \phi(e) & \phi(f) \\ \phi(g) & \phi(h) \end{bmatrix} \\
&= (M_2(\phi)) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) (M_2(\phi)) \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} \right)
\end{aligned}$$

Therefore, $M_2(\phi)$ is indeed a ring homomorphism.

- For any ring R and the identity function Id_R , $M_2(\text{Id}_R)$ is the identity map on $M_2(R)$ because it maps each element in a given matrix to itself.
- Let $f \in \text{Hom}(A, B)$, $g \in \text{Hom}(B, C)$.

$$\begin{aligned}
(M_2(f \circ g)) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) &= \begin{bmatrix} (f \circ g)(a) & (f \circ g)(b) \\ (f \circ g)(c) & (f \circ g)(d) \end{bmatrix} \\
&= \begin{bmatrix} f(g(a)) & f(g(b)) \\ f(g(c)) & f(g(d)) \end{bmatrix} \\
&= M_2(f) \left(\begin{bmatrix} g(a) & g(b) \\ g(c) & g(d) \end{bmatrix} \right) \\
&= M_2(f) \left(M_2(g) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \\
&= (M_2(f) \circ M_2(g)) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right).
\end{aligned}$$

Therefore, M_2 is indeed a functor. □