

# MATH 620 HOMEWORK DUE 9/5

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**Exercise 0.1.** Prove that  $\delta : V \times \cdots \times V \rightarrow \mathbb{F}$  is independent of choice of basis  $\{e_i\} \subset V$  up to non-zero scalar.

*Proof.* Let  $\{e_i\}, \{f_i\}$  be two bases of  $V$ . Let  $v_1, \dots, v_n \in V$  be given. We must show if  $\delta(v_1, \dots, v_n) = 0$  with both of the bases, or nonzero with both of the bases. Suppose that  $\delta(v_1, \dots, v_n) \neq 0$  with one of the bases, and it is 0 with the other basis. Without loss of generality, we assume that  $\{e_i\}$  gives a nonzero value. Let  $n \times n$  matrices  $(v_j^i), (w_j^i)$  be given such that

$$\begin{aligned} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} &= \begin{bmatrix} v_1^1 & \cdots & v_1^n \\ \vdots & \ddots & \vdots \\ v_1^n & \cdots & v_n^n \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} \\ &= \begin{bmatrix} w_1^1 & \cdots & w_1^n \\ \vdots & \ddots & \vdots \\ w_1^n & \cdots & w_n^n \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}. \end{aligned}$$

Since  $\delta(v_1, \dots, v_n) \neq 0$  with  $\{e_i\}$ ,  $\det(v_i^j) \neq 0$ . Therefore, the matrix  $(v_i^j)$  is invertible.

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} v_1^1 & \cdots & v_1^n \\ \vdots & \ddots & \vdots \\ v_1^n & \cdots & v_n^n \end{bmatrix}^{-1} \begin{bmatrix} w_1^1 & \cdots & w_1^n \\ \vdots & \ddots & \vdots \\ w_1^n & \cdots & w_n^n \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}.$$

Let  $A$  denote the product of the two matrices. Then  $\det(A) = \det((v_i^j)^{-1}(w_i^j)) = \det(v_i^j)^{-1} \det(w_i^j) = 0$ . This implies that the row space of  $A$  has a dimension less than  $n$ . Therefore,  $\{e_1, \dots, e_n\}$  cannot span  $V$  whose dimension is  $n$ .

This is a contradiction, so  $\delta$  is independent of choice of basis up to nonzero scaling.  $\square$

**Exercise 0.2.** Show that  $\{e^{i_1} \otimes \cdots \otimes e^{i_k} \mid 1 \leq i_1, \dots, i_k \leq n\}$  is a basis of  $T^k(V^*)$ . Find  $\dim T^k(V^*)$ .

*Proof.*

- Linearly independent? Suppose  $\sum c_{i_1, \dots, i_k} e^{i_1} \otimes \dots \otimes e^{i_k} = 0$ . Let  $1 \leq j_1, \dots, j_k \leq n$  be given.

$$\begin{aligned}
 & \left( \sum_{i_1, \dots, i_k} c_{i_1, \dots, i_k} e^{i_1} \otimes \dots \otimes e^{i_k} \right) (e_{j_1}, \dots, e_{j_k}) = 0 \\
 & \implies \sum_{i_1, \dots, i_k} c_{i_1, \dots, i_k} (e^{i_1} \otimes \dots \otimes e^{i_k}) (e_{j_1}, \dots, e_{j_k}) = 0 \\
 & \implies \sum_{i_1, \dots, i_k} c_{i_1, \dots, i_k} e^{i_1}(e_{j_1}) \dots e^{i_k}(e_{j_k}) = 0 \\
 & \implies c_{j_1, \dots, j_k} e^{j_1}(e_{j_1}) \dots e^{j_k}(e_{j_k}) = 0 \\
 & \implies c_{j_1, \dots, j_k} = 0.
 \end{aligned}$$

Therefore, each  $c_{i_1, \dots, i_k} = 0$ .

- Span? Let  $f \in T^k(V^*)$ . We claim that  $f = \sum_{i_1, \dots, i_k} f(e_{i_1}, \dots, e_{i_k}) e^{i_1} \otimes \dots \otimes e^{i_k}$ . Let  $v_1, \dots, v_k \in V$  be given. Since  $\{e_1, \dots, e_n\}$  is a

basis of  $V$ , so each  $v_i$  can be represented as  $v_i = \sum_j c_i^j e_j$ .

$$\begin{aligned}
& \left( \sum_{i_1, \dots, i_k} f(e_{i_1}, \dots, e_{i_k}) e^{i_1} \otimes \dots \otimes e^{i_k} \right) (v_1, \dots, v_k) \\
&= \left( \sum_{i_1, \dots, i_k} f(e_{i_1}, \dots, e_{i_k}) e^{i_1} \otimes \dots \otimes e^{i_k} \right) (c_1^j e_j, \dots, c_k^j e_j) \\
&= \sum_{i_1, \dots, i_k} f(e_{i_1}, \dots, e_{i_k}) [(e^{i_1} \otimes \dots \otimes e^{i_k})(c_1^j e_j, \dots, c_k^j e_j)] \\
&= \sum_{i_1, \dots, i_k} f(e_{i_1}, \dots, e_{i_k}) [(c_1^j e^{i_1}(e_j)) \dots (c_k^j e^{i_k}(e_j))] \\
&= \sum_{i_1, \dots, i_k} f(e_{i_1}, \dots, e_{i_k}) [(c_1^{i_1} e^{i_1}(e_{i_1})) \dots (c_k^{i_k} e^{i_k}(e_{i_k}))] \\
&= \sum_{i_1, \dots, i_k} f(e_{i_1}, \dots, e_{i_k}) c^{i_1} \dots c^{i_k} \\
&= \sum_{i_1, \dots, i_k} f(c^{i_1} e_{i_1}, \dots, c^{i_k} e_{i_k}) \\
&= \sum_{i_1, \dots, i_{k-1}} \left( \sum_{i_k} f(c^{i_1} e_{i_1}, \dots, c^{i_k} e_{i_k}) \right) \\
&= \sum_{i_1, \dots, i_{k-1}} f(c^{i_1} e_{i_1}, \dots, c^{i_{k-1}} e_{i_{k-1}}, \sum_{i_k} c^{i_k} e_{i_k}) \\
&= \sum_{i_1, \dots, i_{k-1}} f(c^{i_1} e_{i_1}, \dots, c^{i_{k-1}} e_{i_{k-1}}, v_k) \\
&\vdots \\
&= f(v_1, \dots, v_k).
\end{aligned}$$

The dimension is  $n^k$  because each  $i_j$  can be any integer between 1 and  $n$ .  $\square$

**Exercise 0.3.** Let  $w \in \wedge^2 V^*$ .

- Show that there exists a basis  $\{e_1, \dots, e_n\}$  of  $V$  with a dual basis  $\{e^1, \dots, e^n\}$  of  $V^*$  such that  $w = e^1 \wedge e^2 + \dots + e^{2m-1} \wedge e^{2m}$  for some  $m \leq n/2$ .
- $w^l = w \wedge \dots \wedge w \neq 0$  if and only if  $l \leq m$ .

*Proof.* Let  $V_1 = V$ . We will pick vectors inductively.

Suppose that we have  $V_i$  for some  $i \in \mathbb{N}$ . If  $\forall v, v' \in V_i, w(v, v') = 0$ , then we are done. Suppose otherwise. Then there must exist  $v, v' \in V_i$  such that  $w(v, v') = 1$ . Let  $e_{2i-1} = v, e_{2i} = v'$ . Let  $V_{i+1} = \{v \in V \mid w(v, e_{2i-1}) = w(v, e_{2i}) = 0\}$ . We will repeat this process with the  $V_{i+1}$ .

For each  $i$ , we claim that  $\{e_1, \dots, e_{2i}\}$  is linearly independent. TODO!  
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Since  $V$  is an  $n$ -dimensional vector space, this process will terminate. If not, it would imply the existence of a linearly independent set with more than  $n$  vectors. Since the set of all the vectors we found is linearly independent, it can be extended to form a basis of  $V$ .

Let  $\{e_1, \dots, e_n\}$  be a basis that we obtain by extending the linearly independent set of vectors we found. Let  $m$  be chosen such that  $2m$  is the number of vectors we found. Let  $\{e^1, \dots, e^n\}$  denote the dual basis of  $\{e_1, \dots, e_n\}$ . By Proposition 4.1., we know the existence of such a basis and that the dimension of such a basis is  $n$ . We claim that  $w = e^1 \wedge e^2 + \dots + e^{2m-1} \wedge e^{2m}$ .

Because  $w$  and  $e^1 \wedge e^2 + \dots + e^{2m-1} \wedge e^{2m}$  are bilinear, it suffices to identify what  $(e_i, e_j)$  gets mapped to for each  $i, j$ . Let  $i, j \in \{1, \dots, n\}$  be given.

- Case 1: The pair  $(i, j)$  equals  $(2l-1, 2l)$  for some  $l \in \{1, \dots, m\}$ . Then  $w(e_{2l-1}, e_{2l}) = 1$  because that is how we found  $e_{2l-1}, e_{2l}$ . On the other hand,

$$\begin{aligned} & (e^1 \wedge e^2 + \dots + e^{2m-1} \wedge e^{2m})(e_{2l-1}, e_{2l}) \\ &= (e^1 \wedge e^2)(e_{2l-1}, e_{2l}) + \dots + (e^{2m-1} \wedge e^{2m})(e_{2l-1}, e_{2l}) \\ &= 1. \end{aligned}$$

- Case 2: The pair  $(i, j)$  equals  $(2l, 2l-1)$  for some  $l \in \{1, \dots, m\}$ . Since  $w$  and  $e^1 \wedge e^2 + \dots + e^{2m-1} \wedge e^{2m}$  are both alternating,  $w(e_i, e_j) = -w(e_j, e_i)$  and  $(e^1 \wedge e^2 + \dots + e^{2m-1} \wedge e^{2m})(e_i, e_j) = -(e^1 \wedge e^2 + \dots + e^{2m-1} \wedge e^{2m})(e_j, e_i)$ . Then, by Case 1, they both result in  $-1$ .
- Case 3: Any other cases. Then  $(e^1 \wedge e^2 + \dots + e^{2m-1} \wedge e^{2m})(e_i, e_j) = 0$ . We claim that  $w(e_i, e_j) = 0$ . If  $e_i = e_j$ , then  $w(e_i, e_j) = -w(e_j, e_i) = -w(e_i, e_j)$ , so  $w(e_i, e_j) = 0$ . Suppose otherwise. Without loss of generality, assume  $i < j$ . If  $2m < i < j$ , then  $w(e_i, e_j) = 0$  because of the terminating condition. Suppose  $i \leq 2m$ . Then  $e_j$  was in

Suppose  $w(e_i, e_j) \neq 0$ . Then at least one of  $e_i, e_j$  is in  $\{e_1, \dots, e_{2m}\}$ .

TODO\*\*\*\*\*

Therefore,  $w = e^1 \wedge e^2 + \dots + e^{2m-1} \wedge e^{2m}$ . □

**Exercise 0.4.** Prove that  $\{\partial_1, \dots, \partial_n\}$  is a basis of  $T_p \mathbb{R}^n$ .

*Proof.* TODO □

**Exercise 0.5.** Show that  $\{dx^1, \dots, dx^n\}$  is a basis of  $T_p^*\mathbb{R}^n$  that is dual to  $\{\frac{\partial}{\partial x^j}\}_{j=1}^n \subset T_p\mathbb{R}^n$ .

*Proof.*

- Dual? Let  $i, j \in \{1, \dots, n\}$ .  $dx^i(\frac{\partial}{\partial x^j}) = \frac{\partial}{\partial x^j}x^i$ . The partial derivative of  $x^i$  with respect to  $x^j$  is 1 if  $i = j$  and 0 otherwise. Thus  $dx^i(\frac{\partial}{\partial x^j}) = \delta_j^i$ .
- Linearly independent? Let  $c_1, \dots, c_n \in \mathbb{R}$  be given. Suppose that  $c_1dx^1 + \dots + c_ndx^n = 0$ . For any  $i \in \{1, \dots, n\}$ ,

$$\begin{aligned} (c_1dx^1 + \dots + c_ndx^n)(\partial_i) &= 0 \implies c_1(dx^1(\partial_i)) + \dots + c_n(dx^n(\partial_i)) = 0 \\ &\implies c_1(\partial_i(x^1)) + \dots + c_n(\partial_i(x^n)) = 0 \\ &\implies c_i\partial_i(x^i) = 0 \\ &\implies c_i = 0. \end{aligned}$$

Therefore,  $c_1 = \dots = c_n = 0$ . Therefore,  $\{dx^1, \dots, dx^n\}$  is indeed linearly independent.

- Span? Let  $f \in T_p^*\mathbb{R}^n$  be given. We claim that  $f = \sum_{i=1}^n f(\partial_i)dx^i$ . Let  $\sum_{i=1}^n c_i\partial_i \in T_p\mathbb{R}^n$  be given where  $c_i$ 's are in  $\mathbb{R}$ . (It makes sense to assume that every element in  $T_p\mathbb{R}^n$  is in this form because we showed earlier that  $\{\partial_1, \dots, \partial_n\}$  is a basis of  $T_p\mathbb{R}^n$ .)

$$\begin{aligned} \left(\sum_{i=1}^n f(\partial_i)dx^i\right)\left(\sum_{j=1}^n c_j\partial_j\right) &= \sum_{i=1}^n \left[f(\partial_i)dx^i\left(\sum_{j=1}^n c_j\partial_j\right)\right] \\ &= \sum_{i=1}^n f(\partial_i)\left[\sum_{j=1}^n c_jdx^i(\partial_j)\right] \\ &= \sum_{i=1}^n f(\partial_i)\left[\sum_{j=1}^n c_j\partial_j(x^i)\right] \\ &= \sum_{i=1}^n f(\partial_i)c_i \\ &= f\left(\sum_{i=1}^n c_i\partial_i\right). \end{aligned}$$

□