

# MATH 612 (HOMEWORK 2)

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## 1. SECTION 3.1

**Exercise.** (Exercise 1) Fix  $G$  and let  $\alpha : H \rightarrow H'$  be given. Let  $0 \rightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \rightarrow 0, 0 \rightarrow G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} H \rightarrow 0$  be free resolutions. By Lemma 3.1(a), we obtain two homomorphisms  $\alpha_1 : F_1 \rightarrow G_1, \alpha_0 : F_0 \rightarrow G_0$  which commutes with  $f_i, g_i, \alpha$ . Then we obtain two chain complexes

$$\begin{aligned} 0 \leftarrow \text{Hom}(F_1, G) \xleftarrow{f_1^*} \text{Hom}(F_0, G) \xleftarrow{f_0^*} \text{Hom}(H, G) \leftarrow 0 \\ 0 \leftarrow \text{Hom}(F_1, G') \xleftarrow{f_1^*} \text{Hom}(F_0, G') \xleftarrow{f_0^*} \text{Hom}(H, G') \leftarrow 0. \end{aligned}$$

with induced maps  $\alpha_1^*, \alpha_0^*, \alpha^*$  forming a chain map from the chain complex on the bottom to the one on the top. Then  $\alpha_1^*$  induces a map from  $\text{Ext}(H', G) \rightarrow \text{Ext}(H, G)$ .

Fix  $H$  and let  $f : G \rightarrow G'$  be given. Let  $0 \rightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \rightarrow 0$  be a free resolution of  $H$ . We obtain two cochain complexes where  $f_*$  is a chain map from the top one to the bottom one.

$$\begin{aligned} 0 \leftarrow \text{Hom}(F_1, G) \xleftarrow{f_1^*} \text{Hom}(F_0, G) \xleftarrow{f_0^*} \text{Hom}(H, G) \leftarrow 0 \\ 0 \leftarrow \text{Hom}(F_1, G') \xleftarrow{f_1^*} \text{Hom}(F_0, G') \xleftarrow{f_0^*} \text{Hom}(H, G') \leftarrow 0. \end{aligned}$$

$f_*$  indeed makes the diagram commute because for any  $\sigma \in \text{Hom}(H, G)$ ,

$$\begin{aligned} f_*(f_0^*(\sigma)) &= f_*(\sigma \circ f_0) \\ &= f \circ (\sigma \circ f_0) \\ &= (f \circ \sigma) \circ f_0 \\ &= f_0^*(f \circ \sigma) \\ &= f_0^*(f_*(\sigma)). \end{aligned}$$

Similarly,  $f_*(f_1^*(\sigma)) = f_1^*(f_*(\sigma))$  for every  $\sigma \in \text{Hom}(F_0, G)$ . Since a chain map induces a homomorphism on cohomology groups,  $f$  induces a map from  $\text{Ext}(H, G) \rightarrow \text{Ext}(H, G')$ .

**Exercise.** (Exercise 1.2)

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H \longrightarrow 0 \\ & & \downarrow \cdot n & & \downarrow \cdot n & & \downarrow \cdot n \\ 0 & \longrightarrow & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H \longrightarrow 0 \end{array}$$

turn into two chain complexes with a chain map

$$\begin{array}{ccccccc}
0 & \longleftarrow & \text{Hom}(F_1, G) & \xleftarrow{f_1^*} & \text{Hom}(F_0, G) & \xleftarrow{f_0^*} & \text{Hom}(H, G) \longleftarrow 0 \\
& & (\cdot n)^* \uparrow & & (\cdot n)^* \uparrow & & (\cdot n)^* \uparrow \\
0 & \longleftarrow & \text{Hom}(F_1, G) & \xleftarrow{f_1^*} & \text{Hom}(F_0, G) & \xleftarrow{f_0^*} & \text{Hom}(H, G) \longleftarrow 0.
\end{array}$$

This diagram commutes because a group homomorphism for abelian groups commute with multiplication by  $n$ . Therefore,  $(\cdot n)^*$  induces a homomorphism on  $\text{Ext}(H, G) = \text{Hom}(F_1, G)/\text{im}(f_1^*)$ . Moreover,  $\forall \phi + \text{im}(f_1^*) \in \text{Ext}(H, G)$ ,

$$(\cdot n)^*(\phi + \text{im}(f_1^*)) = \phi \circ (\cdot n) + \text{im}(f_1^*)$$

where  $(\phi \circ (\cdot n))(x) = \phi(n(x)) = n(\phi(x)) = (n\phi)(x)$  for all  $x \in F_1$ . Therefore, the map induced by  $(\cdot n)^*$  is simply multiplication by  $n$ .

$$\begin{array}{ccccccc}
0 & \longleftarrow & \text{Hom}(F_1, G) & \xleftarrow{f_1^*} & \text{Hom}(F_0, G) & \xleftarrow{f_0^*} & \text{Hom}(H, G) \longleftarrow 0 \\
& & \downarrow (\cdot n)_* & & \downarrow (\cdot n)_* & & \downarrow (\cdot n)_* \\
0 & \longleftarrow & \text{Hom}(F_1, G) & \xleftarrow{f_1^*} & \text{Hom}(F_0, G) & \xleftarrow{f_0^*} & \text{Hom}(H, G) \longleftarrow 0.
\end{array}$$

For every  $\phi \in \text{Hom}(H, G)$  and  $x \in F_0$ ,

$$\begin{aligned}
((\cdot n)_*(f_0^*(\phi)))(x) &= ((\cdot n)_*(\phi \circ f_0))(x) \\
&= n((\phi \circ f_0)(x)) \\
&= n(\phi(f_0(x))) \\
&= ((\cdot n)_*\phi)(f_0(x)) \\
&= f_0^*((\cdot n)_*\phi)(x).
\end{aligned}$$

Similarly,  $(\cdot n)_*$  commutes with  $f_1^*$ , so  $(\cdot n)_*$  is a chain map. For any  $\phi + \text{im}(f_1^*) \in \text{Ext}(H, G)$ ,  $(\cdot n)_*(\phi + \text{im}(f_1^*)) = n\phi + \text{im}(f_1^*)$ , so it is multiplication by  $n$ .

**Exercise.** (Exercise 3.1.3)  $\cdots \xrightarrow{d_2} \mathbb{Z}_4 \xrightarrow{d_1} \mathbb{Z}_4 \xrightarrow{d_0} \mathbb{Z}_2 \rightarrow 0$  is a free resolution where  $d_0 : a \mapsto a$  and  $d_i : a \mapsto 2a$  because  $\ker(d_0) = \text{im}(d_1) = \ker(d_1) = \{0, 2\}$  for each  $i \geq 1$ . Apply  $\text{Hom}(-, \mathbb{Z}_2)$  and replace  $\mathbb{Z}_2^*$  with 0. For any  $\phi \in \text{Hom}(\mathbb{Z}_4, \mathbb{Z}_2)$  and  $x \in \mathbb{Z}_4$ ,  $((\cdot 2)^*(\phi))(x) = (\phi \circ (\cdot 2))(x) = \phi(2x) = \phi(0) = 0$ . Thus  $(\cdot 2)^*(\phi) = 0$ . In other words,  $d_i^* = 0$  for all  $i \geq 1$ , so  $\text{Ext}_{\mathbb{Z}_4}^n(\mathbb{Z}_2, \mathbb{Z}_2) = \text{Hom}(\mathbb{Z}_4, \mathbb{Z}_2)$  which is nontrivial because  $1 \mapsto 1$  is a nontrivial group homomorphism.

**Exercise.** (Exercise 3.1.6(a)) The chain complex we obtain is isomorphic to  $0 \rightarrow \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \rightarrow 0$  where  $\alpha(a, b) = (a + b)(1, 1, -1)$ . If we apply  $\text{Hom}(-, \mathbb{Z})$ , we obtain

- $H^0(T; \mathbb{Z}) = \text{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$ .
- $\alpha^*(\phi) = 0$  if and only if  $\phi(1, 1, -1) = 0$ .  $(a, b, c) \mapsto a - b$  and  $(a, b, c) \mapsto a + c$  form a basis for the subspace consisting of such homomorphisms.  $H^1(T; \mathbb{Z}) = \ker(\alpha^*) = \mathbb{Z} \oplus \mathbb{Z}$ .
- $H^2(T; \mathbb{Z}) = \text{Hom}(\mathbb{Z}^2, \mathbb{Z})/\text{im}(\alpha^*) = \mathbb{Z}$  because  $(a, b) \mapsto a$  and  $(a, b) \mapsto a + b$  form a basis for  $\text{Hom}(\mathbb{Z}^2, \mathbb{Z})$  and  $\text{im}(\alpha^*)$  is spanned by  $(a, b) \mapsto a + b$ .

If we apply  $\text{Hom}(-, \mathbb{Z}_2)$ , we obtain

- $H^0(T; \mathbb{Z}_2) = \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ .
- $\alpha^*(\phi) = 0$  if and only if  $\phi(1, 1, 1) = 0$ .  $(a, b, c) \mapsto a+b$  and  $(a, b, c) \mapsto a+c$  form a basis for the subspace consisting of such homomorphisms.  $H^1(T; \mathbb{Z}_2) = \ker(\alpha^*) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ .
- $H^2(T; \mathbb{Z}_2) = \text{Hom}(\mathbb{Z}_2^2, \mathbb{Z}_2) / \text{im}(\alpha^*) = \mathbb{Z}_2$  because  $(a, b) \mapsto a$  and  $(a, b) \mapsto a+b$  form a basis for  $\text{Hom}(\mathbb{Z}_2^2, \mathbb{Z}_2)$  and  $\text{im}(\alpha^*)$  is spanned by  $(a, b) \mapsto a+b$ .

**Exercise.** (Exercise 3.1.6(b), projective plane) We obtain a chain complex  $0 \rightarrow \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^3 \xrightarrow{\beta} \mathbb{Z}^2 \rightarrow 0$  where  $\alpha(a, b) = (b-a, a-b, a+b)$  and  $\beta(a, b, c) = (a+b, -a-b)$ . By applying  $\text{Hom}(-, \mathbb{Z})$ , we obtain a cochain complex. Each  $\text{Hom}(\mathbb{Z}^k, \mathbb{Z})$  has a basis  $\{\pi_1, \pi_2, \dots, \pi_k\}$  where  $\pi_i$  is a projection on the  $i$ th coordinate. Then  $(\beta^*(\pi_1))(a, b, c) = a+b$ ,  $(\beta^*(\pi_2))(a, b, c) = -a-b$ . Thus  $\ker(\beta^*) = \langle \pi_1 + \pi_2 \rangle$  and  $\text{im}(\beta^*) = \langle \pi_1 + \pi_2 \rangle$ . The kernel and image of  $\alpha$  can be calculated similarly.

- $H^0 = \ker(\beta^*) = \mathbb{Z}$ .
- $H^1 = \ker(\alpha^*) / \text{im}(\beta^*) = \langle \pi_1 + \pi_2 \rangle / \langle \pi_1 + \pi_2 \rangle = 0$ .
- $H_2 = \ker(0) / \text{im}(\alpha^*) = \langle \pi_1, \pi_2 \rangle / \langle \pi_1 - \pi_2, \pi_1 - \pi_2 \rangle = \langle \pi_1 + \pi_2, \pi_1 \mid \pi_1 + \pi_2, 2\pi_1 \rangle = \mathbb{Z}_2$ .

Similarly, we apply  $\text{Hom}(-, \mathbb{Z}_2)$ . Each  $\text{Hom}(\mathbb{Z}^k, \mathbb{Z}_2)$  has a basis  $\{\pi_1, \pi_2, \dots, \pi_k\}$  where  $\pi_i$  is a projection on the  $i$ th coordinate. The calculation of the kernels and images are almost identical as above with the only exception  $\ker(\alpha^*)$ . This is because  $\alpha^*(\pi_i) : (a, b) \mapsto a+b$  for each  $i = 1, 2, 3$ , so the kernel is  $\langle \pi_1 + \pi_2, \pi_1 + \pi_3 \rangle$ .

- $H^0 = \ker(\beta^*) = \langle \pi_1 + \pi_2 \rangle = \mathbb{Z}_2$ .
- $H^1 = \ker(\alpha^*) / \text{im}(\beta^*) = \langle \pi_1 + \pi_2, \pi_1 + \pi_3 \rangle / \langle \pi_1 + \pi_2 \rangle = \mathbb{Z}_2$ .
- $H_2 = \ker(0) / \text{im}(\alpha^*) = \langle \pi_1, \pi_2 \rangle / \langle \pi_1 + \pi_2, \pi_1 + \pi_2 \rangle = \langle \pi_1 \rangle = \mathbb{Z}_2$ .

**Exercise.** (Exercise 3.1.6(b), klein bottle) The chain complex we obtain is  $0 \rightarrow \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \rightarrow 0$  with  $\alpha(a, b) = (a+b, a-b, b-a)$ . Again, we will use the projection map  $\pi_i$  of the  $i$ th coordinate to form bases of the dual spaces.  $\ker 0^* = \mathbb{Z}$ ,  $\text{im } 0^* = 0$ .  $\ker(\alpha^*) = \langle \pi_2 + \pi_3 \rangle$  and  $\text{im}(\alpha^*) = \langle \pi_1 + \pi_2, \pi_1 - \pi_2 \rangle$  because

$$(\alpha^*(\pi_i))(a, b) = \begin{cases} a+b & (i=1) \\ a-b & (i=2) \\ b-a & (i=3). \end{cases}$$

Thus  $H_0 = \mathbb{Z}$ ,  $H_1 = \langle \pi_2 + \pi_3 \rangle / 0 = \mathbb{Z}$  and  $H_2 = \langle \pi_1, \pi_2 \mid \pi_1 + \pi_2, \pi_1 - \pi_2 \rangle = \mathbb{Z}/2$ .

$\ker 0^* = \mathbb{Z}_2$ ,  $\text{im } 0^* = 0$ .  $\ker(\alpha^*) = \langle \pi_1 + \pi_2, \pi_1 + \pi_3 \rangle$  and  $\text{im}(\alpha^*) = \langle \pi_1 + \pi_2 \rangle$ .

Thus  $H_0 = \mathbb{Z}_2$ ,  $H_1 = \langle \pi_1 + \pi_2, \pi_1 + \pi_3 \rangle / 0 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$  and  $H_2 = \langle \pi_1, \pi_2 \mid \pi_1 + \pi_2 \rangle = \mathbb{Z}_2$ .

**Exercise.** (Exercise 3.1.8(a))  $S^0$  consists of two points, so  $\tilde{H}^i(S^0; G) = G^2/G = G$  if  $i = 0$  and 0 otherwise because  $\tilde{H}^0(S^0; G)$  is all functions module constant functions. Suppose we have shown  $\tilde{H}^i(S^k; G) = G$  if  $i = k$  and 0 otherwise. By the long exact sequence of a pair, we obtain  $\tilde{H}^i(D^{k+1}; G) \rightarrow \tilde{H}^i(S^k; G) \rightarrow \tilde{H}^{i+1}(D^{k+1}, S^k; G) \rightarrow \tilde{H}^{i+1}(D^{k+1}; G)$ . Since  $D^{k+1}$  is contractible,  $\tilde{H}^i(D^{k+1}; G) = 0$  for all  $i$ . This induces an isomorphism  $\tilde{H}^i(S^k; G) \cong \tilde{H}^{i+1}(D^{k+1}, S^k; G) = \tilde{H}^{i+1}(S^{k+1}; G) = G$ . Therefore,  $H^k(S^0; G) = G^2$  and 0 if  $k > 0$ , and  $H^k(S^n; G) = G$  if  $k \in \{0, n\}$  and 0 otherwise.

The Mayer-Vietoris sequence gives  $\tilde{H}^k(A; G) \oplus \tilde{H}^k(B; G) \rightarrow \tilde{H}^k(A \cap B; G) \rightarrow \tilde{H}^{k+1}(S^n; G) \rightarrow \tilde{H}^{k+1}(A; G) \oplus \tilde{H}^{k+1}(B; G)$  where  $A, B$  are the northern and southern hemispheres with some extra part so the union of the interiors equals  $S^n$ . Since  $A$  and  $B$  are contractible regardless of the value of  $k$ ,  $\tilde{H}^k(A; G) = \tilde{H}^k(B; G) = \tilde{H}^{k+1}(A; G) = \tilde{H}^{k+1}(B; G) = 0$ . This gives us an isomorphism  $\tilde{H}^k(A \cap B; G) \cong \tilde{H}^{k+1}(S^n; G)$ .  $A \cap B$  is homotopic to  $S^n$ . By induction,  $\tilde{H}^k(A \cap B; G) = G$  if  $k = n$  and 0 otherwise.

**Exercise.** (Exercise 3.1.8(b)) Let  $q$  be the quotient map  $(X, A) \rightarrow (X/A, A/A)$ . Let  $V$  be a neighborhood of  $A$  in  $X$  that deformation retracts onto  $A$ . We have a commutative diagram

$$\begin{array}{ccccc} H^n(X, A) & \longleftarrow & H^n(X, V) & \longrightarrow & H^n(X - A, V - A) \\ q_* \uparrow & & q_* \uparrow & & q_* \uparrow \\ H^n(X/A, A/A) & \longleftarrow & H^n(X/A, V/A) & \longrightarrow & H^n(X/A - A/A, V/A - A/A). \end{array}$$

The upper horizontal map is an isomorphism since in the long exact sequence of the triple  $(X, V \subset A)$  the groups  $H^n(V, A)$  are zero for all  $n$ , because a deformation retraction of  $V$  onto  $A$  gives a homotopy equivalence of pairs  $(V, A) \simeq (A, A)$ , and  $H_n(A, A) = 0$ . The deformation retraction of  $V$  onto  $A$  induces a deformation retraction of  $V/A$  onto  $A/A$ , so the same argument shows that the lower left horizontal map is an isomorphism as well. The other two horizontal maps are isomorphisms directly from excision. The right-hand vertical map  $q_*$  is an isomorphism since  $q$  restricts to a homeomorphism on the complement of  $A$ . From the commutativity of the diagram, it follows that the left-hand  $q_*$  is an isomorphism. Since  $A/A$  is a point,  $H^n(X/A, A/A) \cong \tilde{H}^n(X/A)$ . Therefore,  $H^n(X, A) \cong \tilde{H}^n(X/A)$ .

## 2. SECTION 3.A

**Exercise.** (Exercise 1) If the characteristic of  $F$  is infinity, the Tor functor becomes 0, so the UCT gives us an isomorphism  $H_n(X; \mathbb{Z}) \otimes F \cong H_n(X; F)$ . Therefore, the rank of  $H_n(X; \mathbb{Z})$  equals the dimension of  $H_n(X; F)$ .

Suppose the characteristic of  $F$  is  $p$ . By the UCT,  $H_n(X; F) \cong (H_n(X; \mathbb{Z}) \otimes F) \oplus \text{Tor}(H_{n-1}(X; \mathbb{Z}); F)$ . Suppose  $H_n(X; \mathbb{Z}) = \mathbb{Z}^d \oplus (\oplus_{i=1}^n \mathbb{Z}_{p_i^{k_i}})$  where  $p_1 = \cdots = p_m = p$ .

$$\begin{aligned} \text{Tor}(\mathbb{Z}^d \oplus \mathbb{Z}_{p_1^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_n^{k_n}}; F) &= \oplus_{i=1}^n \text{Tor}(\mathbb{Z}_{p_i^{k_i}}; F) \\ &= \oplus_{i=1}^n \ker(F \xrightarrow{p_i^{k_i}} F) \\ &= F^m. \end{aligned}$$

Also,

$$\begin{aligned} (\mathbb{Z}^d \oplus \mathbb{Z}_{p_1^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_n^{k_n}}) \otimes F &= (\mathbb{Z} \otimes F)^d \oplus (\mathbb{Z}_{p_1^{k_1}} \otimes F) \oplus \cdots \oplus (\mathbb{Z}_{p_n^{k_n}} \otimes F) \\ &= F^d \oplus (\oplus_{i=1}^n (\mathbb{Z}_{p_i^{k_i}} \otimes F)) \\ &= F^d \oplus (\oplus_{i=1}^n (F/p_i^{k_i} F)) \\ &= F^{d+m}. \end{aligned}$$

Therefore,

- Each  $\mathbb{Z}$  summand in  $H_n(X; \mathbb{Z})$  “adds” one to the dimension of  $H_n(X; F)$ .

- Each  $\mathbb{Z}/p^{k_i}$  summand in  $H_n(X; \mathbb{Z})$  “adds” one to the dimension of  $H_n(X; F)$  and adds one to the dimension of  $H_{n+1}(X; F)$ . This gets cancelled out when taking the sum to calculate the Euler characteristic.

**Exercise.** (Exercise 3.A.2) By Proposition 3A.5,  $\text{Tor}(A, \mathbb{Q}/\mathbb{Z}) = \text{Tor}(T(A), \mathbb{Q}, \mathbb{Z})$ . Given the short exact sequence  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$ , we obtain an exact sequence

$$\begin{aligned} 0 \rightarrow \text{Tor}(T(A), \mathbb{Z}) &\rightarrow \text{Tor}(T(A), \mathbb{Z}) \rightarrow \text{Tor}(T(A), \mathbb{Q})/\mathbb{Z} \\ &\rightarrow T(A) \otimes \mathbb{Z} \rightarrow T(A) \otimes \mathbb{Z} \rightarrow T(A) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow 0. \end{aligned}$$

$\text{Tor}(T(A), \mathbb{Q}) = T(A) \otimes \mathbb{Q} = 0$ . Thus  $\text{Tor}(T(A), \mathbb{Q}/\mathbb{Z}) = T(A) \otimes \mathbb{Z} = T(A)$ .