MATH 620 HOMEWORK DUE 9/5

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Exercise 0.1. Prove that $\delta : V \times \cdots \times V \to \mathbb{F}$ is independent of choice of basis $\{e_i\} \subset V$ up to non-zero scalar.

Proof. Let $\{e_i\}, \{f_i\}$ be two bases of V. Let $v_1, \cdots, v_n \in V$ be given. We must show if $\delta(v_1, \cdots, v_n) = 0$ with both of the bases, or nonzero with both of the bases. Suppose that $\delta(v_1, \cdots, v_n) \neq 0$ with one of the bases, and it is 0 with the other basis. Without loss of generality, we assume that $\{e_i\}$ gives a nonzero value. Let $n \times n$ matrices $(v_j^i), (w_j^i)$ be given such that

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1^1 & \cdots & v_1^n \\ \vdots & \ddots & \vdots \\ v_1^n & \cdots & v_n^n \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

$$= \begin{bmatrix} w_1^1 & \cdots & w_1^n \\ \vdots & \ddots & \vdots \\ w_1^n & \cdots & w_n^n \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}.$$

Since $\delta(v_1, \cdots, v_n) \neq 0$ with $\{e_i\}$, $\det(v_i^j) \neq 0$. Therefore, the matrix (v_i^j) is invertible.

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} v_1^1 & \cdots & v_1^n \\ \vdots & \ddots & \vdots \\ v_1^n & \cdots & v_n^n \end{bmatrix}^{-1} \begin{bmatrix} w_1^1 & \cdots & w_1^n \\ \vdots & \ddots & \vdots \\ w_1^n & \cdots & w_n^n \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}.$$

Let A denote the product of the two matrices. Then $\det(A) = \det((v_i^j)^{-1}(w_i^j)) = \det(v_i^j)^{-1}\det(w_i^j) = 0$. This implies that the row space of A has a dimension less than n. Therefore, $\{e_1, \cdots, e_n\}$ cannot span V whose dimension is n.

This is a contradiction, so δ is independent of choice of basis up to nonzero scaling. $\square$

Exercise 0.2. Show that $\{e^{i_1} \otimes \cdots \otimes e^{i_k} \mid 1 \leq i_1, \cdots, i_k \leq n\}$ is a basis of $T^k(V^*)$. Find $\dim T^k(V^*)$.

Proof.

- Linearly independent? Suppose $\sum c_{i_1,\cdots,i_k} e^{i_1} \otimes \cdots \otimes e^{i_k} = 0$. Let $1 \leq j_1, \cdots, j_k \leq n$ be given.

$$
\begin{aligned}
&(\sum_{i_1,\cdots,i_k} c_{i_1,\cdots,i_k} e^{i_1} \otimes \cdots \otimes e^{i_k})(e_{j_1}, \cdots, e_{j_k}) = 0 \\
&\implies \sum_{i_1,\cdots,i_k} c_{i_1,\cdots,i_k} (e^{i_1} \otimes \cdots \otimes e^{i_k})(e_{j_1}, \cdots, e_{j_k}) = 0 \\
&\implies \sum_{i_1,\cdots,i_k} c_{i_1,\cdots,i_k} e^{i_1}(e_{j_1}) \cdots e^{i_k}(e_{j_k}) = 0 \\
&\implies c_{j_1,\cdots,j_k} e^{j_1}(e_{j_1}) \cdots e^{j_k}(e_{j_k}) = 0 \\
&\implies c_{j_1,\cdots,j_k} = 0.
\end{aligned}
$$

  Therefore, each $c_{i_1,\cdots,i_k} = 0$.
- Span? Let $f \in T^k(V^*)$. We claim that $f = \sum_{i_1,\cdots,i_k} f(e_{i_1}, \cdots, e_{i_k}) e^{i_1} \otimes \cdots \otimes e^{i_k}$. Let $v_1, \cdots, v_k \in V$ be given. Since $\{e_1, \cdots, e_n\}$ is a

basis of V, so each v_i can be represented as $v_i = \sum_j c_i^j e_j$.

$$
\begin{aligned}
&(\sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})e^{i_1}\otimes\cdots\otimes e^{i_k})(v_1,\cdots,v_k) \\
&= (\sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})e^{i_1}\otimes\cdots\otimes e^{i_k})(c_1^j e_j,\cdots,c_k^j e_j) \\
&= \sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})[(e^{i_1}\otimes\cdots\otimes e^{i_k})(c_1^j e_j,\cdots,c_k^j e_j)] \\
&= \sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})[(c_1^j e^{i_1}(e_j))\cdots(c_k^j e^{i_k}(e_j))] \\
&= \sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})[(c_1^{i_1} e^{i_1}(e_{i_1}))\cdots(c_k^{i_k} e^{i_k}(e_{i_k}))] \\
&= \sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})c^{i_1}\cdots c^{i_k} \\
&= \sum_{i_1,\cdots,i_k} f(c^{i_1}e_{i_1},\cdots,c^{i_k}e_{i_k}) \\
&= \sum_{i_1,\cdots,i_{k-1}} (\sum_{i_k} f(c^{i_1}e_{i_1},\cdots,c^{i_k}e_{i_k})) \\
&= \sum_{i_1,\cdots,i_{k-1}} f(c^{i_1}e_{i_1},\cdots,c^{i_{k-1}}e_{i_{k-1}},\sum_{i_k} c^{i_k}e_{i_k})) \\
&= \sum_{i_1,\cdots,i_{k-1}} f(c^{i_1}e_{i_1},\cdots,c^{i_{k-1}}e_{i_{k-1}},v_k) \\
&\vdots \\
&= f(v_1,\cdots,v_k).
\end{aligned}
$$

The dimension is n^k because each i_j can be any integer between 1 and n. ☐

Exercise 0.3. Prove that $\{\partial_1,\cdots,\partial_n\}$ is a basis of $T_p\mathbb{R}^n$.

Proof. TODO ☐

Exercise 0.4. Show that $\{dx^1,\cdots,dx^n\}$ is a basis of $T_p^*\mathbb{R}^n$ that is dual to $\{\frac{\partial}{\partial x^j}\}_{j=1}^n \subset T_p\mathbb{R}^n$.

Proof.

- Dual? Let $i,j \in \{1,\cdots,n\}$. $dx^i(\frac{\partial}{\partial x^j}) = \frac{\partial}{\partial x^j}x^i$. The partial derivative of x^i with respect to x^j is 1 if $i=j$ and 0 otherwise. Thus $dx^i(\frac{\partial}{\partial x^j}) = \delta_j^i$.

- Linearly independent? Let $c_1, \cdots, c_n \in \mathbb{R}$ be given. Suppose that $c_1 dx^1 + \cdots + c_n dx^n = 0$. For any $i \in \{1, \cdots, n\}$,

$$\begin{aligned}(c_1 dx^1 + \cdots + c_n dx^n)(\partial_i) = 0 &\implies c_1(dx^1(\partial_i)) + \cdots + c_n(dx^n(\partial_i)) = 0\\ &\implies c_1(\partial_i(x^1)) + \cdots + c_n(\partial_i(x^n)) = 0\\ &\implies c_i \partial_i(x^i) = 0\\ &\implies c_i = 0.\end{aligned}$$

  Therefore, $c_1 = \cdots = c_n = 0$. Therefore, $\{dx^1, \cdots, dx^n\}$ is indeed linearly independent.
- Span? Let $f \in T_p^*\mathbb{R}^n$ be given. We claim that $f = \sum_{i=1}^n f(\partial_i) dx^i$. Let $\sum_{i=1}^n c_i \partial_i \in T_p\mathbb{R}^n$ be given where c_i's are in $\mathbb{R}$. (It makes sense to assume that every element in $T_p\mathbb{R}^n$ is in this form because we showed earlier that $\{\partial_1, \cdots, \partial_n\}$ is a basis of $T_p\mathbb{R}^n$.)

$$\begin{aligned}(\sum_{i=1}^n f(\partial_i) dx^i)(\sum_{j=1}^n c_j \partial_j) &= \sum_{i=1}^n \left[f(\partial_i) dx^i(\sum_{j=1}^n c_j \partial_j)\right]\\ &= \sum_{i=1}^n f(\partial_i)\big[\sum_{j=1}^n c_j dx^i(\partial_j)\big]\\ &= \sum_{i=1}^n f(\partial_i)\big[\sum_{j=1}^n c_j \partial_j(x^i)\big]\\ &= \sum_{i=1}^n f(\partial_i) c_i\\ &= f(\sum_{i=1}^n c_i \partial_i).\end{aligned}$$

$\square$