MATH 611 (DUE 11/13)

HIDENORI SHINOHARA

1. Simplicial and Singular Homology

Exercise. (Problem 27) Let $f : (X, A) \to (Y, B)$ be a map such that both $f : X \to Y, f : A \to B$ are homotopy equivalences.

- Show that $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all n.
- For the case of the inclusion $f : (D^n, S^{n-1}) \to (D^n, D^n \setminus \{0\})$, show that f is not a homotopy equivalence of pairs - there is no $g : (D^n, D^n \setminus \{0\}) \to (D^n, S^{n-1})$ such that fg and gf are homotopic to the identity through maps of pairs.

Proof.

- For each $n \geq 1$, we have an exact sequence $H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X)$ and another one with X, A replaced with Y, B. Moreover, they are connected by homomorphisms $f_* : H_n(A) \to H_n(B), f_* : H_n(X) \to H_n(Y), f_* : H_n(X, A) \to H_n(Y, B)$ such that the diagram commutes. (naturality) Since $f : X \to Y$ and $f : A \to B$ are both homotopy equivalences, $f_* : H_n(X) \to H_n(Y), f_* : H_n(A) \to H_n(B)$ are isomorphisms. By the Five lemma, $f_* : H_n(X, A) \to H_n(X, B)$ is an isomorphism.

  The exact sequence $H_1(A) \to H_1(X) \to H_1(X, A) \to 0$ can be extended to $H_1(A) \to H_1(X) \to H_1(X, A) \to 0 \to 0$ by appending 0 at the end. Using the same argument as above, $f_* : H_1(X, A) \to H_1(Y, B)$ is an isomorphism.
- Suppose $f : (D^n, S^{n-1}) \to (D^n, D^n - \{0\})$ is a homotopy equivalence.

  Prove that $f : S^{n-1} \to D^n$ is a homotopy equivalence.

  Since $f : (D^n, S^{n-1}) \to (D^n, D^n)$ is a homotopy equivalence, we have an isomorphism $f_* : H_n(D^n, S^{n-1}) \to H_n(D^n, D^n)$. $H_n(D^n, S^{n-1}) = H_n(D^n/S^{n-1})$ since (D^n, S^{n-1}) is a good pair. Moreover, $D^n/S^{n-1} = S^n$, so $H_n(D^n, S^{n-1}) = \mathbb{Z}$.

  On the other hand, (D^n, D^n) is a good pair, so $H_n(D^n, D^n) = H_n(D^n/D^n) = H_n(\text{point}) = 0$ since $n \geq 1$.

  Since $\mathbb{Z}$ is not isomorphic to 0, the inclusion is not a homotopy equivalence of pairs.

$\square$