MATH 633(HOMEWORK 7)

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Exercise. (1) Suppose f is locally bijective. Let $p \in U$. Then f is bijective in some open set U' satisfying $p \in U' \subset U$. This implies f is injective on U'. By Proposition 1.1, $f' \neq 0$ on U'. In other words, f' is nonzero on U.

Suppose $f'(z) \neq 0$ for all $z \in U$. Let $z_0 \in U$. It suffices to show that f is locally injective. Choose sufficiently small $r > 0$ such that

- $f'(z) \neq 0$ on $D_r(z_0)$.
- $f(z) \neq f(z_0)$ whenever $z \neq z_0$ and $|z - z_0| \leq r$.

This is possible by Theorem 4.8 on P.52. Let $m = \min\{|f(z) - f(z_0)| \mid |z - z_0| = r\}$. First, m must exist because the set is compact, and $m > 0$ because we chose r such that $f(z) \neq f(z_0)$ whenever $0 < |z - z_0| \leq r$. Actually, I don't think this solution will work...

Exercise. (10) Let $\sigma(z) = -i(z+1)/(z-1)$. Then σ sends the unit disk to the upper half-plane with ∞ since $\sigma(a+bi) = (-2b - (a^2+b^2-1)i)/((a-1)^2+b^2)$. On the other hand, $\sigma^{-1} : z \mapsto (z-i)/(z+i)$ sends the upper half plane with ∞ to the unit disk because $|a + (b-1)i| \leq |a + (b+1)i|$ if $b \geq 0$. Therefore, σ is a bijection between the unit disk and $H \cup \{\infty\}$. $F \circ \sigma$ sends the unit disk to the unit disk, and $F(\sigma(0)) = 0$. By Lemma 2.1, $|(F \circ \sigma)(w)| \leq |w|$ for every $w \in D$. Then for every $z \in \mathbb{H}$, $\sigma^{-1}(z) \in D$. Then $|F(z)| = |(F \circ \sigma)(\sigma^{-1}(z))| \leq |\sigma^{-1}(z)| = |(z-i)/(z+i)|$, which is the desired result.

Exercise. (12(a)) Let $a \neq b$ be two fixed points. Let $\sigma(z) = (z-a)/(1-\overline{a}z)$. Then σ sends a to 0 and maps D to D bijectively. Let $g = \sigma \circ f \circ \sigma^{-1}$. g has two fixed points, 0 and $\sigma(b)$. By applying Lemma 2.1, g is a rotation. However, g fixes $\sigma(b) \neq 0$, so g must be the identity map. Then f must be the identity.

Exercise. (12(b)) The map $\sigma : z \mapsto (z-i)/(z+i)$ maps the upper half-plane to the unit disk bijectively. Then $\sigma \circ f \circ \sigma^{-1}$ where $f(z) = z+1$ is a holomorphic bijection on f that has no fixed point because f has no fixed point.

Exercise. (16(a)) The composition of mobius transformations corresponds to the multiplication of the corresponding matrices. Thus it suffices to calculate

$$\begin{aligned}
\begin{bmatrix} -1 & i \\ 1 & i \end{bmatrix}^{-1} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & i \\ 1 & i \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} e^{i\theta}+1 & -i(e^{i\theta}-1) \\ i(e^{i\theta}-1) & e^{i\theta}+1 \end{bmatrix} \\
&\to \frac{1}{2} \begin{bmatrix} 1 & -i(e^{i\theta}-1)/(e^{i\theta}+1) \\ i(e^{i\theta}-1)/(e^{i\theta}+1) & 1 \end{bmatrix} \\
&\to \frac{1}{2} \begin{bmatrix} 1 & -i(i\tan(\theta/2)) \\ i(i\tan(\theta/2)) & 1 \end{bmatrix} \\
&\to \frac{1}{2} \begin{bmatrix} 1 & \tan(\theta/2) \\ -\tan(\theta/2) & 1 \end{bmatrix}.
\end{aligned}$$

Thus the answer is the mobius transformation associated to the last matrix.

Exercise. (16(b)) Let $\alpha = a + bi$.

$$\begin{bmatrix} -1 & i \\ 1 & i \end{bmatrix}^{-1} \begin{bmatrix} -1 & \alpha \\ -\overline{\alpha} & 1 \end{bmatrix} \begin{bmatrix} -1 & i \\ 1 & i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \overline{\alpha} - \alpha & -i(\alpha + \overline{\alpha} - 2) \\ -i(\alpha + \overline{\alpha} + 2) & \alpha - \overline{\alpha} \end{bmatrix}$$
$$\to \begin{bmatrix} b & a-1 \\ a+1 & -b \end{bmatrix}.$$

After multiplying $1/(1 - a^2 - b^2)$ to every term, we obtain a matrix associated to the desired mobius transformation.

Exercise. (16(c)) Let $\alpha = g(0)$. Then ψ_α is an automorphism of the unit disk that sends α to 0. Then $\psi_\alpha \circ g$ is an automorphism of the unit disk that fixes 0. By applying Lemma 2.1 to $\psi_\alpha \circ g$ and its inverse, we obtain that $|\psi_\alpha \circ g| \leq 1$ and $|(\psi_\alpha \circ g)^{-1}| \leq 1$. Thus $|\psi_\alpha \circ g| = 1$. Therefore, $\psi_\alpha \circ g$ is a rotation by Lemma 2.1. By (a), $h = f^{-1} \circ \psi_\alpha \circ g \circ f$ is a Mobius transformation associated to a real matrix with determinant 1. Then $f^{-1} \circ g \circ f = f^{-1} \circ \psi_\alpha^{-1} \circ f \circ h$. By Part (b), $f^{-1} \circ \psi_\alpha^{-1} \circ f$ is a Mobius transformation associated to a real matrix with determinant 1 because $\psi_\alpha^{-1} = \psi_\alpha$. Since the composition of two Mobius transformations corresponds to the product of the two associated matrices, the composition corresponds to a real matrix whose determinant is 1.