# MATH 611 HOMEWORK 2 (DUE 9/11)

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**Exercise.** (Problem 1, Section 1.2) Show that the free product $G*H$ of nontrivial groups $G$ and $H$ has trivial center, and that the only elements of $G*H$ of finite order are the conjugates of finite-order elements of $G$ and $H$.

*Proof.* Let $w \in G * H$ be given. Suppose $w$ is not the empty word.

- Suppose the leftmost element of $w$ is in $G$. Let $h \in H$ be given such that $h$ is not the identity element of $H$.
  - Case 1: The rightmost element of $w$ is an element of $G$. Then $wh$ is just a concatenation, so $wh \neq hw$ because the leftmost element of $wh$ is in $G$ and the leftmost element of $hw$ is in $H$.
  - Case 2: The rightmost element of $w$ is an element of $H$, but not $h^{-1}$. Let $h'$ denote the rightmost element of $w$ and $w'$ denote the remaining. Then $w = w'h'$, so $wh = w'(h'h)$. By the definition of a reduced word, the rightmost element of $w'$ is an element of $G$, so the concatenation of $w'$ and $h'h$ is exactly $wh$. The leftmost element of $wh$ is in $G$ and the leftmost element of $hw$ is in $H$, so $wh \neq hw$.
  - Case 3: The rightmost element of $w$ is $h^{-1}$. Then the rightmost element of $w$ disappears in $wh$. In this case, the leftmost element of $w$ stays the same. Therefore, the leftmost element of $wh$ is in $G$ and the leftmost element of $hw$ is in $H$, so $wh \neq hw$.

  In each case, $wh \neq hw$.
- Suppose that the leftmost element of $w$ is in $H$. Let $g \in G$ be given such that $g$ is not the identity element of $G$. Using the exact same logic as above, we can conclude that $wg \neq gw$.

Therefore, $w$ is not in the center of $G * H$, so $Z(G * H) = \{e\}$ where $e$ denotes the empty word.

Let $x$ be a finite-order element in $G$ or $H$. Let $n$ denote the order. Let $w \in G * H$. Then $(wxw^{-1})^n = wx^nw^{-1} = ww^{-1} = e$, so the conjugate of a finite order element in $G$ or $H$ is has finite order.

We will show that every element of finite order in $G*H$ is a conjugate of a finite order element in $G$ or $H$. We will consider the length of a finite-order element.

- Let $w \in G * H$ be a nonempty word of even length. Since adjacent elements must be elements of different groups, the leftmost element of $w$ and rightmost element of $w$ are in different groups. In other words, $w^k$ has the length $k$ times the length of $w$. This implies that the order of $w$ is not finite.
- We will show that every reduced word of length $2k-1$ is a conjugate of a finite order element in $G$ or $H$ for every $k \in \mathbb{N}$. Let $k = 1$. Then it is either just $g$ or $h$ where $g \in G$ or $h \in H$. In each case, it is clear that the order $g$ or $h$ itself is finite. Therefore, it is a conjugate of a finite order element by the empty word.

  Suppose that the claim is true for some $k \in \mathbb{N}$. We will consider a finite-order element of length $2k+1$. Let $w$ denote a reduced word of length $2k+1$. Suppose $w^n = e$ for some $n \in \mathbb{N}$.
  - Case 1: The leftmost element of $w$ is in $G$. Then $w = gw'g'$ where $g, g'$ are in $G$ and $w'$ is a reduced word of length $2k-1$. $g'$ must equal $g^{-1}$. Otherwise, the length of $w^m$ would equal $m \cdot (2k+1) - m$, and it would never equal 0. Consider $g^{-1}wg = w'$. Since $(g^{-1}wg)^n = g^{-1}w^n g = g^{-1}g = e$, the order of $w'$ is finite. By the inductive hypothesis, $w'$ is a conjugate of a finite order element in $G$ or $H$. Since the length of $w'$ is odd and the end elements are in $H$, $w'$ must be a conjugate of a finite order element in $G$. In other words, $w' = axa^{-1}$ for some $a \in G * H$ and $x \in G$. Then $w = g(axa^{-1})g^{-1} = (ga)x(a^{-1}g^{-1}) = (ga)x(ga)^{-1}$. $ga$ is a reduced word because the leftmost element of $a$ is the same as the leftmost element of $w'$, which is in $H$.

    By induction, every reduced word of finite length whose leftmost element is in $G$ is a conjugate of a finite order element in $G$.
  - Case 2: The leftmost element of $w$ is in $H$. By symmetry, every reduced word of finite length whose leftmost element in $H$ is a conjugate of a finite order element in $H$.

  Therefore, the only elements of $G * H$ of finite order are the conjugates of finite-order elements of $G$ and $H$.

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