MATH 601 HOMEWORK (DUE 9/4)

HIDENORI SHINOHARA

Exercise. (2.1) Show that the function $g : \mathbb{R} \to S^1$, $g(r) = \exp(2\pi i r)$, where $i^2 = -1$, satisfies the property that $g(r) = g(r')$ if and only if $r \sim r'$. Use this to explicitly construct a bijective map from the orbit space of the action to S^1, g : $\mathbb{R}/\sim = \mathbb{Z}\backslash\mathbb{R} \to S^1$.

Proof.

- Let $r, r' \in \mathbb{R}$ such that $r \sim r'$. Let $k \in \mathbb{Z}$ such that $k * r' = r$. Therefore, $k + r' = r$.

$$\begin{aligned} g(r) &= \exp(2\pi i r) \\ &= \exp(2\pi i(k + r')) \\ &= \exp(2\pi i k + 2\pi i r') \\ &= \exp(2\pi i k)\exp(2\pi i r') \\ &= \exp(2\pi i r') \\ &= g(r'). \end{aligned}$$

- Let $r, r' \in \mathbb{R}$ such that $g(r) = g(r')$.

$$\begin{aligned} \exp(2\pi i r) = \exp(2\pi i r') &\implies \exp(2\pi i(r - r')) = 1 \\ &\implies \cos(2\pi(r - r')) + i\sin(2\pi(r - r')) = 1 \\ &\implies \sin(2\pi(r - r')) = 0 \\ &\implies r - r' \in \mathbb{Z} \\ &\implies \exists k \in \mathbb{Z}, r = k * r' \\ &\implies r \sim r'. \end{aligned}$$

Let g : $\mathbb{Z}\backslash\mathbb{R} \to S^1$ be defined such that g$([r]) = g(r)$ for each $[r] \in \mathbb{Z}\backslash\mathbb{R}$.

- Well-defined? Let $[r] = [r'] \in \mathbb{Z}\backslash\mathbb{R}$. Then $r \sim r'$. We showed that $g(r) = g(r')$ if $r \sim r'$ earlier. Therefore, g is indeed well-defined.
- Injective? Let $[r], [r'] \in Z\backslash\mathbb{R}$. Suppose g$([r])$ = g$([r'])$. Then $g(r) = g(r')$. We showed earlier that this implies $r \sim r'$. In other words, $[r] = [r']$. Therefore, g is injective.

- Surjective? Let $z \in S^1$. Express z as $re^{i\theta}$ where $r, \theta \in \mathbb{R}$. Since $|z| = 1$, we can assume that $r = 1$ without loss of generality. (If $r = -1$, then $e^{i\pi} = -1$, so θ can be redefined as $\theta + \pi$.)

  Then $[\theta/2\pi]$ is an element in $\mathbb{Z}\backslash\mathbb{R}$, and g$([\theta/2\pi]) = g(\theta/2\pi) = \exp(2\pi i \cdot \theta/2\pi) = \exp(i\theta) = z$. Therefore, g is indeed surjective.

$\square$

Exercise. (2.2) Let $* : G \times S \to S$ be a left action of G. Show that $s \star g = g^{-1} * s$ defines a right action of G on S.

Proof. Let $s \in S, g, h \in G$ be given.

$$\begin{aligned}(s \star g) \star h &= h^{-1} * (s \star g) \\ &= h^{-1} * (g^{-1} * s) \\ &= (h^{-1} g^{-1}) * s \\ &= (gh)^{-1} * s \\ &= s \star (gh).\end{aligned}$$

Let $e \in G$ denote the identity element and let $s \in S$ be given.

$$\begin{aligned}s \star e &= e^{-1} * s \\ &= e * s \\ &= s.\end{aligned}$$

Therefore, $\star$ is indeed a right action of G on S. $\square$