MATH 601 HOMEWORK (DUE 10/16)

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Contents

1. Modules

Exercise. (Problem 2) Consider the $m \times n$ matrices given below as presentation matrices for $\mathbb{Z}$-modules. That is think of the given matrix, H, as giving a linear transformation, $\mathbb{Z}^n \to \mathbb{Z}^m$, $x \mapsto Hx$ and thus giving a presentation of $\operatorname{Coker}(H) = \mathbb{Z}^m / \operatorname{Im}(H)$. Give in each case a familiar finitely generated $\mathbb{Z}$-module which is isomorphic to the $\mathbb{Z}$-module which H presents.

- $H = 6$.
- $H = \begin{bmatrix} 2 & 1 \end{bmatrix}$.
- $H = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$.
- $H = \begin{bmatrix} 4 & 12 \\ 6 & 2 \end{bmatrix}$.
- $H = \begin{bmatrix} 3 & 6 \\ 8 & 4 \\ 10 & 5 \end{bmatrix}$.
- $H = \begin{bmatrix} 36 & 12 & 24 \\ 30 & 18 & 24 \\ 15 & -6 & 12 \end{bmatrix}$.

Proof.

- This H generates the exact sequence
$$\mathbb{Z}^1 \xrightarrow{H} \mathbb{Z}^1 \xrightarrow{p} \mathbb{Z}^1/6\mathbb{Z} \xrightarrow{0} 0$$
where p is the map $k \mapsto k + 6\mathbb{Z}$. Thus $\mathbb{Z}/6\mathbb{Z}$ is what H represents.
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□

2. Jordan Canonical Form

Let k be a field, V a finite dimensional k-vector space, and $T \in \operatorname{End}_k(V)$ a linear transformation.

Exercise. (Problem 1) Show that the set $\{p(x) \in k[x] \mid p(T) = 0 \in \mathrm{End}_k(V)\}$ is an ideal, $I \subset k[x]$. Also, show that $I \neq 0$.

Proof.

- Claim 1: I is nonempty. Let $v_1, \cdots, v_n$ be a basis of V. Such a basis must exist since the dimension of V is finite. Let M be the $n \times n$ matrix associated to V with respect to the basis $\{v_1, \cdots, v_n\}$. In other words, for any $v \in V$, $Mv = T(v)$ where Mv is the product. Since M is an $n \times n$ matrix, the set $\{M^0, \cdots, M^{n^2}\}$ is linearly dependent. Thus there exist $a_{n^2}, \cdots, a_0 \in k$ such that
  - $a_{n^2}M^{n^2} + \cdots + a_0M^0 = 0$.
  - $a_{n^2}, \cdots, a_0$ are not all zero.

  Then for any $v \in V$,

$$\begin{aligned} 0 &= (a_{n^2}M^{n^2} + \cdots + a_0M^0)v \\ &= a_{n^2}M^{n^2}v + \cdots + a_0M^0v \\ &= a_{n^2}T^{n^2}(v) + \cdots + a_0T^0(v) \\ &= (a_{n^2}T^{n^2} + \cdots + a_0T^0)(v). \end{aligned}$$

  Therefore, $p(x) = a_{n^2}x^{n^2} + \cdots + a_0x^0 \neq 0$ and $p(T) = 0$. Thus $p(x) \in I$, so I is nonempty.
- Claim 2: I is closed under subtraction. Let $p(x), q(x) \in I$. Then $p(x) - q(x) \in I$ because $p(T) - q(T) = 0 - 0 = 0$.
- Claim 3: I is closed under multiplication by elements in $k[x]$. Let $p(x) \in I, r(x) \in k[x]$. Then $p(T)r(T) = 0r(T) = 0$, so $r(x)p(x) \in I$.

By Claim 1 and 2, I is a subgroup of $k[x]$ under addition. Then Claim 3 implies that I is an ideal. By Claim 1, $I \neq 0$. □

Exercise. (Problem 2) Let $p(x) \in k[x]$ be a nonzero polynomial such that $p(T) = 0 \in \mathrm{End}_k(V)$. Show that if $p(x) \in k[x]$ is a product of linear polynomials, then there is a k-basis for V with respect to which the matrix for T is in Jordan normal form.

Proof.

Since k is just a field, I can't assume that k is algebraically closed.

- $p(x) = (x - a_1)^{m_1} \cdots (x - a_n)^{m_n}$.
- Let $N = \dim(V)$.
- Let $q(\lambda) = \det(T - \lambda\,\mathrm{Id})$ be the characteristic polynomial of T.
- Let $v_1, \cdots, v_N$ be a basis of V.

For each i, $(p(T))(v_i) = 0$. In other words, there exists a j such that $(T - a_j\,\mathrm{Id})(v) = 0$ for some nonzero v. This can be found by applying each linear factor to v_i and figure out the point where it turns into 0. In other words, $\det(T - a_j\,\mathrm{Id}) = 0$. This implies that a_j is a root of the characteristic polynomial $q(\lambda)$ of T. Thus $\lambda - a_j$ divides $q(\lambda)$.
But I'm not sure what to do next. We want to find the largest number r_j such that $(\lambda - a_j)^{r_j}$ divides $q(\lambda)$. What happens next?

□

Exercise. (Problem 3) Suppose that the field k contains m distinct m-th roots of 1. Suppose that $T^m = \mathrm{Id}_V \in \mathrm{End}_k(V)$. Show that there is a basis of V with respect to which, the matrix for T is diagonal. What can you say about the diagonal entries?

Proof.

- Let $r_i, \cdots, r_m$ denote the m distinct mth roots of 1.
- Then each $x - r_i$ divides $x^m - 1$. Thus $x^m - 1 = (x - r_1) \cdots (x - r_m)$. This means that $p(x) = x^m - 1$ is a polynomial such that $p(T) = 0$ and it is a product of linear polynomials. Then I think that we can use an approach similar to the previous problem.
- Let M denote the diagonal matrix for T. Then M^m must be the identity matrix. Moreover, the ith diagonal entry of M^m is simply the m-th power of the ith diagonal entry of M. Thus each of the diagonal entries in M must be an m-th root of 1. On the other hand, any diagonal matrix where each entry is an m-th root of 1 becomes the identity when raised to the mth power.

$\square$

Exercise. (Problem 4) Let V be a 9 dimensional k-vector space. Let $T \in \mathrm{End}_k(V)$ have minimal polynomial, $x^2(x-1)^3$. What are the possible Jordan canonical forms for T?

Proof.

For any $a, b \in \{0, 1\}$,

$$\begin{bmatrix} 1 & 0 & \cdots & & & \\ a & 1 & 0 & \cdots & & \\ 0 & b & 1 & 0 & \cdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & & & & \ddots \end{bmatrix}$$

satisfies $x^2(x-1)^3$.

$\square$