MATH 612 (HOMEWORK 1)

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Exercise. (Exercise 1(a)) The case of $G = \mathbb{Z}$ is discussed in Example 2.42.

$$H_k(\mathbb{R}\mathrm{P}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0 \text{ and for } k = n \text{ odd} \\ \mathbb{Z}_2 & \text{for } k \text{ odd, } 0 < k < n \\ 0 & \text{otherwise.} \end{cases}$$

Suppose n is even. For any abelian group G, we obtain the cellular chain complex

$$0 \to G \xrightarrow{2} G \xrightarrow{0} \cdots \xrightarrow{2} G \xrightarrow{0} G \to 0.$$

If n is odd, we obtain

$$0 \to G \xrightarrow{0} G \xrightarrow{2} \cdots \xrightarrow{2} G \xrightarrow{0} G \to 0.$$

- Suppose k is even and $2 \leq k \leq n$. The homology at $\xrightarrow{0} G \xrightarrow{2}$ is
  - 0 if $G = \mathbb{Q}, \mathbb{Z}/p^l\mathbb{Z}$ with $p \neq 2$.
  - $\mathbb{Z}/2\mathbb{Z}$ if $G = \mathbb{Z}/2^l$.
- Suppose k is odd and $1 \leq k \leq n-1$. The homology at $\xrightarrow{2} G \xrightarrow{0}$ is
  - $G/2G \cong 0$ if $G = \mathbb{Q}, \mathbb{Z}/p^l\mathbb{Z}$ with $p \neq 2$ because multiplication by 2 is an isomorphism.
  - $\mathbb{Z}/2\mathbb{Z}$ if $G = \mathbb{Z}/2^l$.
- Suppose $k = n$ and n is odd, or $k = 0$. The homology at $\xrightarrow{0} G \xrightarrow{0}$ is G.

When $G = \mathbb{Q}$, the universal coefficient theorem gives an isomorphism $H_k(X) \otimes Q \cong H_k(X; \mathbb{Q})$ since Q is torsion free. $\mathbb{Z} \otimes \mathbb{Q} \cong \mathbb{Q}$ and $\mathbb{Z}/2 \otimes \mathbb{Q} = 0$ because 2 is invertible in $\mathbb{Q}$. This agrees with the results above.

When $G = \mathbb{Z}/2^l$, we have $0 \to H_k(X) \otimes G \to H_k(X; G) \to \mathrm{Tor}(H_{k-1}(C), G) \to 0$. $\mathrm{Tor}(H_{k-1}(C), G) = \ker(\mathbb{Z}/2^l \xrightarrow{2} \mathbb{Z}/2^l) = \mathbb{Z}/2$. $\mathbb{Z}/2 \otimes \mathbb{Z}/2^l \otimes \mathbb{Z}/2 = \mathbb{Z}/2$ because $\gcd(2, 2^l, 2) = 2$. This agrees with the results above.

Exercise. (Exercise 1(b)) As discussed in Example 2.37, $H_2(N_g; \mathbb{Z}) = 0, H_1(N_g; \mathbb{Z}) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$, and $H_0(N_g; \mathbb{Z}) = \mathbb{Z}$. For an abelian group G, the cellular chain complex is

$$0 \to G \xrightarrow{d_2} G^g \xrightarrow{d_1} G \to 0.$$

As discussed in Example 2.37, $d_2(1) = (2, 2, \cdots, 2)$ and $d_1 = 0$. If $1 + 1 = 0$ in G, then $H_2(X; G) = H_0(X; G) = G$ and $H_1(X; G) = G^g$. Otherwise, then $H_2(X; G) = 0, H_1(X; G) = G^{g-1}$ and $H_0(X; G) = G$.

Exercise. (Exercise 1(c)) For a Z-module R, we have

$$0 \to R \xrightarrow{0} R \xrightarrow{a} R \xrightarrow{0} R \to 0.$$

When $R = \mathbb{Z}$, we obtain

$$H_k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0, 2n-1 \\ \mathbb{Z}_m & \text{for } k \text{ odd, } 0 < k < 2n-1 \\ 0 & \text{otherwise.} \end{cases}$$

When R is an abelian group such that $1 + 1 + \cdots + 1 = 0$ (a times), $H_i(X;R) = R$ if $i = 0, 1, 2, 3$. Otherwise, $H_3(X;R) = H_0(X;R) = R$ and all other cohomology groups are 0.