MATH 611 HOMEWORK (DUE 10/16)

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Exercise. (Problem 16) Given maps $X \to Y \to Z$ such that both $Y \to Z$ and the composition $X \to Z$ are covering spaces, show that $X \to Y$ is a covering space if Z is locally path-connected, and show that this covering space is normal if $X \to Z$ is a normal covering space.

Proof. Let $p : X \to Y, q : Y \to Z$ be given such that q and $q \circ p$ are both covering maps. Let $y_0 \in Y$ be given. It suffices to show that there exists a neighborhood of y_0 that is evenly covered by p. (Hatcher does not require a covering map be surjective.)

Let $z_0 = q(y_0)$. Let U_{z_0} be a locally path-connected neighborhood of z_0 contained in the intersection of the following two neighborhoods:

- A neighborhood of z_0 that is evenly covered by q.
- A neighborhood of z_0 that is evenly covered by $q \circ p$.

Those two neighborhoods of z_0 must exist because q and $q \circ p$ are covering maps. Since Z is locally path-connected, any neighborhood of z_0 contains a path-connected neighborhood of z_0. Therefore, such U_{z_0} must exist. Moreover, any neighborhood contained in an evenly covered neighborhood is evenly covered. Therefore, U_{z_0} is a path-connected neighborhood of z_0 that is evenly covered by both q and $q \circ p$.

Since U_{z_0} is evenly covered by q and $q \circ p$,

- Let $\coprod_\alpha U_{x_\alpha} = (q \circ p)^{-1}(U_{z_0})$ where $q \circ p$ maps each U_{x_α} into U_{z_0} homeomorphically.
- Let $\coprod_\beta U_{y_\beta} = q^{-1}(U_{z_0})$ where q maps each U_{y_β} into U_{z_0} homeomorphically.

Since $z_0 = q(y_0)$ and q is an covering map, there exists U_{y_β} such that $y_0 \in U_{y_\beta}$. For simplicity, we will call it U_{y_0}. In other words, U_{y_0} is a neighborhood of y_0 such that q is a homeomorphism between U_{y_0} and U_{z_0}.

We claim that U_{y_0} is a neighborhood of y_0 that is evenly covered by p by showing that there exists a subset I of the index set such that $p^{-1}(U_{y_0}) = \coprod_{\alpha \in I} U_{x_\alpha}$.

We claim that for all α, $U_{x_\alpha} \subset p^{-1}(U_{y_0})$ or $U_{x_\alpha} \cap p^{-1}(U_{y_0}) = \emptyset$. Let α be given. Suppose $U_{x_\alpha} \cap p^{-1}(U_{y_0}) \neq \emptyset$. Let $x \in U_{x_\alpha} \cap p^{-1}(U_{y_0})$. Let $x' \in U_{x_\alpha}$. We will show that $x' \in p^{-1}(U_{y_0})$.

Since U_{z_0} is path connected and U_{x_α} is homeomorphic to U_{z_0}, U_{x_α} is path connected. Let γ be a path from x to x'. In other words, $\gamma(0) = x$ and $\gamma(1) = x'$. Then $q \circ p \circ \gamma$ is a path in U_{z_0}. Let $z = (q \circ p \circ \gamma)(0), z' = (q \circ p \circ \gamma)(1)$. Then $q \circ p \circ \gamma$ is a path from z to z' in U_{z_0}. Since U_{z_0} and U_{y_0} are homeomorphic by q, there exists a unique point $y \in U_{y_0}$ such that $q(y) = z$. Since q is a covering map, there exists a unique lift $\widetilde{q \circ p \circ \gamma}$ based at y. Let $y' = \widetilde{q \circ p \circ \gamma}(1)$. Then $y' \in U_{y_0}$ because the lift must entirely lie in U_{y_0} because q is a homeomorphism between U_{y_0} and U_{z_0}.

Consider the path $p \circ \gamma$ in Y. Since $(p \circ \gamma)(0) = p(x)$ and $x \in p^{-1}(U_{y_0})$, the initial point of $p \circ \gamma$ is in U_{y_0}. Moreover, $q(p(x)) = z$ and y is the unique point in U_{y_0} such that $q(y) = z$, $y = p(x)$. Since $q \circ (p \circ \gamma) = (q \circ p) \circ \gamma$, $p \circ \gamma$ is also a lift of $q \circ p \circ \gamma$

based at y. By the uniqueness of a lift, $\widetilde{q \circ p \circ \gamma} = p \circ \gamma$. Specifically, this implies that $p(x') = (p \circ \gamma)(1) = \widetilde{q \circ p \circ \gamma}(1) = y' \in U_{y_0}$. Since $p(x') \in U_{y_0}$, $x' \in p^{-1}(U_{y_0})$.

Let $I = \{\alpha \mid U_{x_\alpha} \subset p^{-1}(U_{y_0})\}$. Then we have $\coprod_{\alpha \in I} U_{x_\alpha} \subset p^{-1}(U_{y_0})$.

Since $p^{-1}(U_{y_0}) \subset p^{-1}(q^{-1}(U_{z_0})) = \coprod_\alpha U_{x_\alpha}$, every point in $p^{-1}(U_{y_0})$ is in U_{x_α} for some α. I includes all α such that U_{x_α} intersects with $p^{-1}(U_{y_0})$. Thus $p^{-1}(U_{y_0}) \subset \coprod_{\alpha \in I} U_{x_\alpha}$.

Therefore, $\coprod_{\alpha \in I} U_{x_\alpha} = p^{-1}(U_{y_0})$.

Finally, we will show that p is a homeomorphism between U_{x_α} and U_{y_0}. Let $\alpha \in I$. We claim that $p(U_{x_\alpha}) = U_{y_0}$.

- $p(U_{x_\alpha}) \subset U_{y_0}$ because of how we defined I.
- Let $y \in U_{y_0}$. Since U_{y_0} is path connected, there exists a path γ from y_0 to y in U_{y_0}. Then $q \circ \gamma$ is a path in U_{z_0}. Since $q \circ p$ maps U_{x_α} into U_{z_0} homeomorphically, there exists a unique $x_0 \in U_{x_\alpha}$ such that $(q \circ p)(x_0) = z_0$. By the unique lifting property, there exists a unique lift $\widetilde{q \circ \gamma}$ of $q \circ \gamma$ based at x_0. Again, since $q \circ p$ maps U_{x_α} into U_{z_0} homeomorphically, $\widetilde{q \circ \gamma}$ is in U_{x_α}.

  Then $p \circ (\widetilde{q \circ \gamma})$ is a path in U_{y_0}. Since $q \circ (p \circ (\widetilde{q \circ \gamma})) = (q \circ p) \circ (\widetilde{q \circ \gamma}) = q \circ \gamma$ and q is a homeomorphism between U_{y_0} and U_{z_0}, $\gamma = p \circ (\widetilde{q \circ \gamma})$. Then $p(\widetilde{q \circ \gamma}(1)) = (p \circ (\widetilde{q \circ \gamma}))(1) = \gamma(1) = y$. Thus $y \in p(U_{x_\alpha})$.

We know that $(q \circ p) \mid_{U_{x_\alpha}}$ and $q \mid_{U_{y_0}}$ are homeomorphisms.

$$\begin{aligned}
(q \mid_{U_{y_0}})^{-1} \circ (q \circ p) \mid_{U_{x_\alpha}} &= (q \mid_{U_{y_0}})^{-1} \circ q \mid_{p(U_{x_\alpha})} \circ p \mid_{U_{x_\alpha}} \\
&= (q \mid_{U_{y_0}})^{-1} \circ q \mid_{U_{y_0}} \circ p \mid_{U_{x_\alpha}} \\
&= p \mid_{U_{x_\alpha}}
\end{aligned}$$

Thus $p \mid_{U_{x_\alpha}}$ is a homeomorphism between U_{x_α} and U_{y_0}.

In conclusion, U_{y_0} is a neighborhood of y_0 such that $p^{-1}(U_{y_0})$ is the disjoint union $\coprod_{\alpha \in I} U_{x_\alpha}$ such that p is a homeomorphism between each U_{x_α} and U_{y_0}. Therefore, p is a covering map.

Suppose that $q \circ p$ is normal. By Proposition 1.39(a), it suffices to show that $p_*\pi_1(X)$ is a normal subgroup of $\pi_1(Y)$. Let $p_*([h]) \in \pi_1(X)$ and $[g] \in \pi_1(Y)$. By Proposition 1.39(a), $(q \circ p)_*(\pi_1(X)) = q_*(p_*(\pi_1(X)))$ is a normal subgroup of $\pi_1(Z)$. Therefore, $q_*([g]p_*([h])[g]^{-1}) = q_*([g])q_*(p_*([h]))q_*([g])^{-1} = q_*(p_*([h']))$ for some $[h'] \in \pi_1(X)$. Since q_* is injective by Proposition 1.31, $[g]p_*([h])[g]^{-1} = p_*([h]') \in p_*(\pi_1(X))$. Thus $p_*(\pi_1(X))$ is a normal subgroup of $\pi_1(Y)$. □

Exercise. (Problem 18) For a path-connected, locally path-connected, and semilocally simply-connected space X, call a path-connected covering space $X \to X$ abelian if it is normal and has abelian deck transformation group. Show that X has an abelian covering space that is a covering space of every other abelian covering space of X, and that such a 'universal' abelian covering space is unique up to isomorphism. Describe this covering space explicitly for $X = S^1 \vee S^1$ and $X = S^1 \vee S^1 \vee S^1$.

Proof. We will consider the commutator subgroup $H = [\pi_1(X, x_0), \pi_1(X, x_0)]$ generated by $\{[a, b] \mid a, b \in \pi_1(X, x_0)\}$ of $\pi_1(X, x_0)$. Since H is a subgroup of $\pi_1(X, x_0)$ and X is path-connected, locally path connected, and semilocally simply connected, there exists a path-connected covering space $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ such that $p_*(\pi_1(\tilde{X}, \tilde{x_0})) = H$ by Theorem 1.38.

By Proposition 1.39(b), $G(\tilde{X})$ is isomorphic to the quotient $N(H)/H$.

- Since H is the commutator subgroup, H is a normal subgroup of $\pi_1(X, x_0)$. Thus $N(H) = \pi_1(X, x_0)$. Moreover, Proposition 1.39(a) asserts that $\tilde{X}$ is normal because $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ is normal.
- Since H is the commutator subgroup of $\pi_1(X, x_0) = N(H)$, $N(H)/H$ is abelian.

Therefore, $\tilde{X}$ is an abelian covering space of X.

- Show that $\tilde{X}$ is the 'universal' abelian covering space.
- Show uniqueness.

- What is the hypothesis?
  - X is a path-connected, locally path-connected, semilocally simply-connected space.
- What is the conclusion?
  - There exists a normal covering space of X $p : \tilde{X} \to X$ such that $G(\tilde{X})$ is abelian.
  - X has an abelian covering space that is a covering space of every other abelian covering space of X.
  - A universal abelian covering space is unique up to isomorphism.
  - Find the universal covering space of $S^1 \vee S^1$ and $S^1 \vee S^1 \vee S^1$.
- Introduce suitable notations.
  - $H = p_*(\pi_1(X, x_0))$.
- Separate the various parts of the hypothesis.
- Find the connection between the hypothesis and the conclusion.
  - "X is a path-connected, locally path-connected, semilocally simply-connected space." This condition sounds a lot like Theorem 1.38 on P.67. By using theorem 1.38, we can associate some group to each covering map.
  - "$\tilde{X}$ is a normal covering space of X." By Proposition 1.39 on P.71, $\tilde{X}$ is normal if and only if H is a normal subgroup of $\pi_1(X, x_0)$.
  - $G(\tilde{X})$ is abelian. By Proposition 1.39 on P.71, $G(\tilde{X})$ is isomorphic to the quotient $\pi_1(X, x_0)/H$ because $\tilde{X}$ is normal. Thus $\pi_1(X, x_0)/H$ is abelian.
- Have you seen it before?
  - This might be similar to constructing the universal covering space.
- Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.
  - Showing uniqueness up to isomorphism sounds like the universal covering space theorem.
- Keep only a part of the hypothesis, drop the other part; is the conclusion still valid?
- Could you derive something useful from the hypothesis?
- Could you think of another hypothesis from which you could easily derive the conclusion?
- Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?
- Did you use the whole hypothesis?

□

Lemma 0.1. *Let G be a group and H be a subgroup. If $[a, b] \in H$ for all $a, b \in G$, then H is a normal subgroup of G.*

Proof. Let $g \in G, h \in H$. Then $ghg^{-1} = ghg^{-1}h^{-1}h = [g,h]h$. Since $[g,h] \in H$ and $h \in H$, $ghg^{-1} = [g,h]h \in H$. Thus H is normal. $\square$

Exercise. (Problem 19) Use the preceding problem to show that a closed orientable surface M_g of genus g has a connected normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ (the product of N copies of $\mathbb{Z}$) if and only if $n \leq 2g$. For $n = 3$ and $g \geq 3$, describe such a covering space explicitly as a subspace of $\mathbb{R}^3$ with translations of $\mathbb{R}^3$ as deck transformations.

Proof. Suppose $n \leq 2g$. Then $\pi_1(M_g) = \langle a_1, \cdots, a_{2g} \mid [a_1, a_2] \cdots [a_{2g-1}, a_{2g}] \rangle$. Let H be the subgroup of $\pi_1(M_g)$ generated by $a_1, \cdots, a_{2g-n}$ and the set $\{[a_i, a_j] \mid i \neq j\}$. By Lemma 0.1 above, H is normal. Since H is a subgroup of $\pi_1(M_g)$, there exists a covering space $p : \tilde{M}_g \to M_g$ by Theorem 1.38 such that $p_*(\pi_1(\tilde{M}_g)) = H$.

Therefore, by Proposition 1.39(a), $\tilde{M}_g$ is normal.

By Proposition 1.39(b), $G(\tilde{M}_g)$ is isomorphic to the quotient $N(H)/H$. Since H is normal, $N(H) = \pi_1(M_g)$. Therefore, $G(\tilde{M}_g)$ is isomorphic to $\pi_1(M_g)/H$ where H contains all commutators of $\pi_1(M_g)$. Thus $G(\tilde{M}_g)$ is abelian, so $\tilde{M}_g$ is an abelian covering space.

Moreover,

$$\begin{aligned} G(\tilde{M}_g) &= \pi_1(M_g)/H \\ &= \langle a_1, \cdots, a_{2g} \mid a_1, \cdots, a_{2g-n}, \forall i, j, [a_i, a_j] \rangle \\ &= \langle a_{2g-n+1}, \cdots, a_{2g} \mid \forall i, j, [a_i, a_j] \rangle \\ &\cong \mathbb{Z}^n. \end{aligned}$$

Finish the rest of the problem.

- List examples. $n = 1, 2, g = 1$ and $n = 1, g = 2$ are done. Try others.
- What is the hypothesis? M_g is a closed orientable surface M_g of genus g.
- What is the conclusion? M_g has a connected normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ if and only if $n \leq 2g$.
- Separate the various parts of the hypothesis.

  Closed orientable surface? I don't know what to do with it. Can I just assume that this means $M_g = (S^1 \times S^1) \vee \cdots \vee (S^1 \times S^1)$?
- Find the connection between the hypothesis and the conclusion.
  - The fundamental group of M_g is generated by $2g$ elements with no relations. If we abelianize the fundamental group of M_g, we obtain $\mathbb{Z}^{2g}$.
- Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.
  - The previous problem shows the existence of an abelian covering space, and a normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ is also abelian.
- Keep only a part of the hypothesis, drop the other part; is the conclusion still valid?
- Could you derive something useful from the hypothesis?
- Could you think of another hypothesis from which you could easily derive the conclusion?

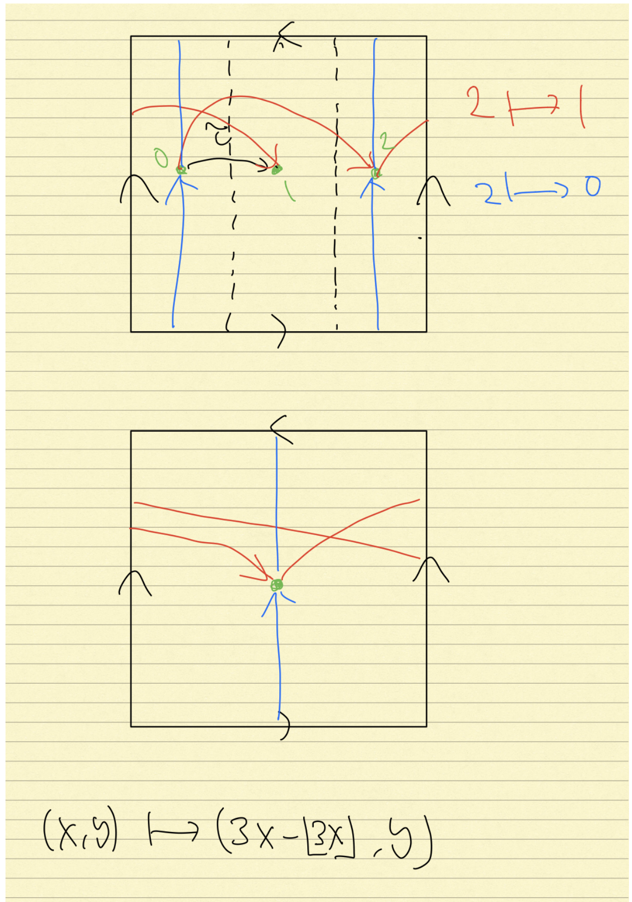


FIGURE 1. Problem 20 (Klein)

- If $g = 1$, then this problem is easy. For $n = 2$, consider the xy plane, and for $n = 1$, consider the infinite chain of squares.

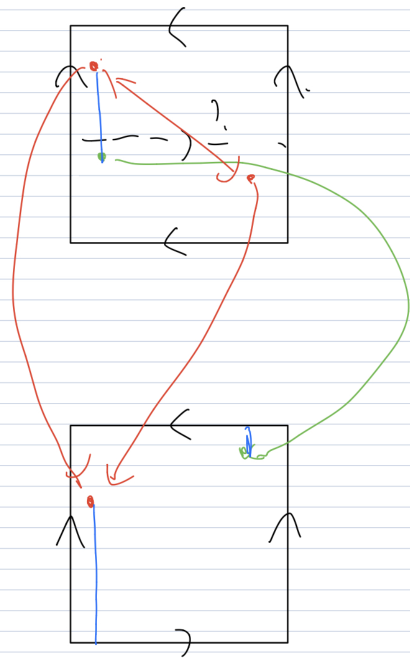
- Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?
- Did you use the whole hypothesis?

□

Exercise. (Problem 20) Construct non-normal covering spaces of the Klein bottle by a Klein bottle and by a torus.

Proof. Figure 1 is the idea that I have for the first part. But I don't know how to show that there exists no deck transformation with that permutation.

□



$$f(x,y) = \begin{cases} (1-x,\ 2y) & (0 \leq y \leq 1/2) \\ (x,\ 2y-1) & (1/2 \leq y \leq 1) \end{cases}$$

FIGURE 2. Problem 20 (Torus)