# MATH 611 (DUE 11/6)

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## 1. Simplicial and Singular Homology

**Exercise.** (Problem 14) Determine whether there exists a short exact sequence $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$. More generally, determine which abelian groups $A$ fit into a short exact sequence $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ with $p$ prime. What about the case of short exact sequences $0 \to A \to \mathbb{Z}_n \to 0$?

*Proof.* Let $\phi_1 : \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2, \phi_2 : \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4$ be defined such that $\phi_1(a) = (2a, a)$ and $\phi_2(a, b) = a + 2b$. Then $\ker(\phi_1) = 0, \mathrm{Im}(\phi_1) = \ker(\phi_2) = \{(2k, k) \mid 0 \leq k \leq 3\}$ and $\mathrm{Im}(\phi_2) = \mathbb{Z}_4$. Thus this is indeed an exact sequence.

We claim that $A = \mathbb{Z}_{p^i} \oplus \mathbb{Z}_{p^{m+n-i}}$ where $\max\{m, n\} \leq i$.

- Let $0 \to \mathbb{Z}_{p^m} \xrightarrow{\alpha} A \xrightarrow{\beta} \mathbb{Z}_{p^n} \to 0$ be given. Then $\alpha$ is injective and $\beta$ is surjective by the exactness. Let $u \in A$ such that $\beta(u) = 1$. We claim that every element in $A$ can be uniquely expressed as $a\alpha(1) + bu$ where $0 \leq a \leq p^m - 1$ and $0 \leq b \leq p^n - 1$.

  Let $0 \leq a \leq p^m - 1, 0 \leq b \leq p^n - 1$ be given such that $a\alpha(1) + bu = 0$. Then $-a\alpha(1) = bu$, and $\beta(-a\alpha(1)) = 0$. $bu = 0$ implies that $b\beta(u) = 0$, so $b = 0$. Moreover, $-a\alpha(1) = 0$ implies that $\alpha(-a) = 0$, so $a = 0$ because $\alpha$ is injective.

  Therefore, whenever $(a_1, b_1) \neq (a_2, b_2)$ , $a_1\alpha(1) + b_1u \neq a_2\alpha(1) + b_2u$ . This implies that are at least $p^{m+n}$ elements in $A$ of this form. By the exactness, $\mathbb{Z}_{p^n} = A/\mathbb{Z}_{p^m}$, so $A$ must contain exactly $p^{m+n}$ elements. Therefore, every element in $A$ can be uniquely written in the form.

  How can we use this to prove the rest?

∎

**Exercise.** (Problem 15) For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ if and only if the map $A \to B$ is surjective and $D \to E$ is injective. Hence, for a pair of spaces $(X, A)$, the inclusion $A \to X$ induces isomorphisms on all homology groups if and only if $H_n(X, A) = 0$ for all $n$.

*Proof.* Suppose $C = 0$. $\mathrm{Im}(\phi_{AB}) = \ker(\phi_{BC}) = B$, so $\phi_{AB}$ is surjective. $\ker(\phi_{DE}) = \mathrm{Im}(\phi_{CD}) = \{0\}$, so $\phi_{DE}$ is injective.

On the other hand, suppose $\phi_{AB}$ is surjective and $\phi_{DE}$ is injective. $\mathrm{Im}(\phi_{CD}) = \ker(\phi_{DE}) = \{0\}$, so $\phi_{CD}$ is the zero map. Therefore, $\ker(\phi_{CD}) = C$. $\ker(\phi_{BC}) = \mathrm{Im}(\phi_{AB}) = B$, so $\phi_{BC}$ is the zero map. Therefore, $\mathrm{Im}(\phi_{BC}) = 0$. Hence, $C = \ker(\phi_{CD}) = \mathrm{Im}(\phi_{BC}) = 0$.

By Theorem 2.16 and the discussion at the bottom of P.117(Hatcher), we have a long exact sequence of homology groups

$$H_n(A) \xrightarrow{i_*} H_n(X) \to H_n(X, A) \to H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \tag{1.1}$$

for $n \geq 1$. Suppose the inclusion induces isomorphisms on all homology groups. Then $H_n(X,A) = 0$ for all $n \geq 1$ by the first part. Moreover, we have $H_1(X,A) \to H_0(A) \to H_0(X) \to H_0(X,A) \to 0$. Since $H_1(X,A) = 0$, by the first part, $H_0(X) = 0$. In order for $0 \to H_0(X,A) \to 0$ to be exact, $H_0(X,A)$ must be 0. Therefore, $H_n(X,A) = 0$ for all $n \geq 0$.

Suppose that $H_n(X,A) = 0$ for all $n \geq 0$. By exact sequence 1.1 above, $i_* : H_n(A) \to H_n(X)$ is surjective for $n \geq 1$ and injective for $n \geq 0$. Thus $i_*$ is bijective for all $n \geq 1$. We have $H_1(X,A) \to H_0(A) \to H_0(X) \to H_0(X,A)$. Since $H_1(X,A) = H_0(X,A) = 0$, $i_*$ must be bijective by the exactness. Therefore, the inclusion induces isomorphisms for all $n$. □

**Exercise.** (Problem 16)

- Show that $H_0(X,A) = 0$ if and only if $A$ meets each path-component of $X$.
- Show that $H_1(X,A) = 0$ if and only if $H_1(A) \to H_1(X)$ is surjective and each path-component of $X$ contains at most one path-component of $A$.

*Proof.*

- Let $\gamma_x + C_0(A) \in C_0(X)/C_0(A)$. Since $A$ meets each path-component of $X$, there exists a path $\gamma : I \to X$ that joins a point $a \in A$ and the image of $\gamma_x$. Then $\gamma$ can be seen as an element of $C_1(X)$ since $\gamma$ maps a 1-simplex into $X$. Moreover, $\partial\gamma = \gamma_x - \gamma_a$ where $\gamma_a \in C_0(A)$ with $\mathrm{Im}(\gamma_a) = a$. Therefore, $\partial(\gamma + C_1(A)) = \gamma_x + C_0(A)$, so $\gamma_x + C_0(A) \in \mathrm{Im}(\partial)$. Hence, $H_0(X,A) = \ker(\partial_0)/\mathrm{Im}(\partial_1) = (C_0(X)/C_0(A))/(C_0(X)/C_1(A)) = 0$.

  On other hand, suppose that $A$ does not meet each path component of $X$. Let $x \in X$ be a point in a path component that $A$ does not intersect. Let $\gamma_x : \Delta^0 \to X$ such that $\mathrm{Im}(\gamma_x) = \{x\}$. Then $\gamma_x \in \ker(\partial_0) = C_0(X,A)$. Let $\gamma + C_1(A) \in C_1(X,A)$. Then $\partial_1(\gamma + C_1(A)) = \partial_1(\gamma) + C_0(A)$. Let $\gamma_{x_1}, \gamma_{x_2} \in C_0(X)$ such that $\partial_1(\gamma) = \gamma_{x_1} - \gamma_{x_2}$. $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$ if and only if $\gamma_{x_1} - \gamma_{x_2} - \gamma_x \in C_0(A)$.
  - If $\gamma$ lies in the same path component as $x$, then so do $x_1$ and $x_2$. Suppose $x = x_1$. Since $-\gamma_{x_2} \notin C_0(A)$, $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$. The case when $x \neq x_1$ and $x = x_2$ and the case when $x \neq x_1$ and $x \neq x_2$ can be proven in a similar way.
  - If $\gamma$ lies in a different path component, then $\gamma_x \neq \gamma_{x_1}$ and $\gamma_x \neq \gamma_{x_2}$. Therefore, $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$.

  Therefore, $\gamma_x \notin \mathrm{Im}(\partial_1)$. Thus $H_0(X,A) = C_0(X,A)/\mathrm{Im}(\partial_1)$ is not 0.
- Do part (b).

□

**Exercise.** (Problem 17)

- Compute the homology groups $H_n(X,A)$ when $X$ is $S^2$ or $S^1 \times S^1$ and $A$ is a finite set of points in $X$.
- Compute the groups $H_n(X,A)$ and $H_n(X,B)$ for $X$ a closed orientable surface of genus two with $A$ and $B$ the circles shown.

*Proof.*

- We will apply Theorem 2.16 to get the exact sequence with $H_n(A), H_n(X), H_n(X,A)$.
  - When $n \geq 3$, $H_n(S^2) \to H_n(S^2,A) \to H_{n-1}(A)$ shows that $H_n(S^2,A)$ is 0 by the exactness since $H_n(S^2) = H_{n-1}(A) = 0$.

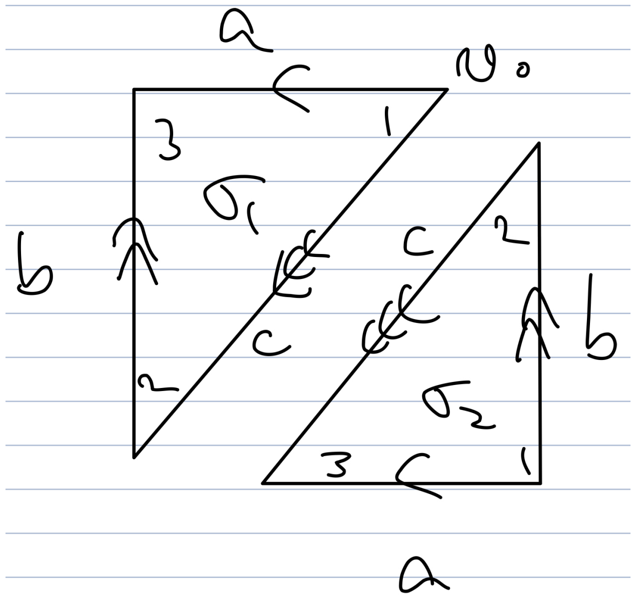


FIGURE 1. Problem 17

- When $n = 2$, $H_n(A) \to H_n(S^2) \xrightarrow{\phi} H_n(S^2, A) \to H_{n-1}(A)$ shows that $H_n(S^2, A) = H_n(S^2) = \mathbb{Z}$. This is because $H_n(A) = H_{n-1}(A) = 0$ so $\phi$ is an isomorphism by the exactness.
- $n = 0$ and $n = 1$.

We will first compute the homology groups of a torus using Figure 1. $C_2 = \{\sigma_1, \sigma_2\}, C_1 = \{a, b, c\}, C_0 = \{v_0\}$.

- $H_2 = \ker(\partial_2)/\operatorname{Im}(\partial_3) = \langle \sigma_1 - \sigma_2 \rangle /0 = \mathbb{Z}$.
- $H_1 = \ker(\partial_1)/\operatorname{Im}(\partial_2) = \langle a, b, c \rangle / \langle b - a + c, c - a + b \rangle = \mathbb{Z}^2$ because $b - a + c = c - a + b$.
- $H_0 = \ker(\partial_0)/\operatorname{Im}(\partial_1) = \langle v_0 \rangle /0 = \mathbb{Z}$.

Again, we will apply Theorem 2.16 to get the exact sequence with $H_n(A), H_n(X)$, and $H_n(X, A)$.

- When $n \geq 3$, $H_n(S^1 \times S^1) \to H_n(S^1 \times S^1, A) \to H_{n-1}(A)$ shows that $H_n(S^1 \times S^1, A)$ is 0 by the exactness since $H_n(S^1 \times S^1) = H_{n-1}(A) = 0$.
- When $n = 2$, $H_n(A) \to H_n(S^1 \times S^1) \xrightarrow{\phi} H_n(S^1 \times S^1, A) \to H_{n-1}(A)$ shows that $H_n(S^1 \times S^1, A) = H_n(S^1 \times S^1) = \mathbb{Z}$. This is because $H_n(A) = H_{n-1}(A) = 0$ so $\phi$ is an isomorphism by the exactness.
- $n = 0$ and $n = 1$.

- Finish this!

□

**Exercise.** (Problem 26) Show that $H_1(X, A)$ is not isomorphic to $\tilde{H}_1(X/A)$ if $X = [0, 1]$ and $A$ is the sequence $1, 1/2, 1/3, \cdots$ together with its limit 0.

*Proof.* We will show that $H_1(X, A)$ is countable, and $\tilde{H}_1(X/A) = H_q(X/A)$ is uncountable. We have an exact sequence $\tilde{H}_1(X) \to \tilde{H}_1(X, A) \xrightarrow{\phi} \tilde{H}_0(A) \to \tilde{H}_0(X)$. Since $H_1(X, A) = \tilde{H}_1(X, A) = \tilde{H}_0(X) = 0$, $\phi$ is an isomorphism. Thus $\tilde{H}_1(X, A) = \tilde{H}_0(A) = \ker(\partial_1)/\operatorname{Im}(\partial_2)$. Since $A$ is a disjoint union of points, $\operatorname{Im}(\partial_2) = 0$. $\ker(\partial_1) = \{\sum n_i \alpha_i \mid n_i \in \mathbb{Z}, \sum n_i = 0\}$ where $\alpha_i$ is the point $1/i$ by the definition of a reduced homology. Then this is generated by $\{\alpha_1 - \alpha_2, \alpha_1 - \alpha_3, \alpha_1 - \alpha_4, \cdots\}$, so $\tilde{H}_1(X, A)$ is countable.

We will show the existence of an injective map $\zeta$ from the direct product $\prod_{i=1}^{\infty} \mathbb{Z}$ to $H_1(X/A)$, which is homeomorphic to the Hawaiian earring. We will refer to the $n$th ring $C_n$ as in Example 1.25. Let $(k_1, \cdots) \in \prod_{i=1}^{\infty} \mathbb{Z}$ be given. Construct the map $f : I \to X/A$ that wraps $k_n$ times around $C_n$ in the time interval $[1 - 1/n, 1 - 1/(n + 1)]$. This infinite composition of loops is certainly continuous at each time less than 1, and it is continuous at time 1 since every neighborhood of the basepoint in $X/A$ contains all but finitely many of the circles $C_n$. This shows that $f \in C_1(X/A)$. Moreover, $\partial(f) = v_0 - v_0 = 0$ where $v_0$ is the origin of the Hawaiian earring. Therefore, $[f] \in H_1(X/A)$. We define $\zeta(k_1, ) = [f]$.

Let $(k_1, \cdots) \neq (l_1, \cdots) \in \prod_{i=1}^{\infty} \mathbb{Z}$ be given. Let $\zeta(k_1, \cdots) = f, \zeta(l_1, \cdots) = g$ as described above. Let $i$ be an index such that $k_i \neq l_i$. Let $F : X/A \to S^1$ be a continuous map that maps $C_n$ onto $S_1$ and $C_i$ to $-1$ for all $i$ where $S_1$ is seen as a subset of $\mathbb{C}$. Then $F$ induces a group homomorphism $F_* : H_1(X/A) \to H_1(S^1)$ where $F([f]) = k_n$ and $F([g]) = l_n$. Since $F([f]) \neq F([g])$, $[f] \neq [g]$. This shows the injectivity of $\zeta$ and hence $H_1(X/A)$ must be uncountable.

Therefore, $H_1(X, A)$ is not isomorphic to $H_1(X/A)$. $\square$