MATH 612 (HOMEWORK 2)

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1. Section 3.1

Exercise. (Exercise 1) Fix G and let $\alpha : H \to H'$ be given. Let $0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0, 0 \to G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} H \to 0$ be free resolutions. By Lemma 3.1(a), we obtain two homomorphisms $\alpha_1 : F_1 \to G_1, \alpha_0 : F_0 \to G_0$ which commutes with f_i, g_i, α. Then we obtain two chain complexes

$$0 \leftarrow \mathrm{Hom}(F_1, G) \xleftarrow{f_1^*} \mathrm{Hom}(F_0, G) \xleftarrow{f_0^*} \mathrm{Hom}(H, G) \leftarrow 0$$
$$0 \leftarrow \mathrm{Hom}(F_1, G') \xleftarrow{f_1^*} \mathrm{Hom}(F_0, G') \xleftarrow{f_0^*} \mathrm{Hom}(H, G') \leftarrow 0.$$

with induced maps $\alpha_1^*, \alpha_0^*, \alpha^*$ forming a chain map from the chain complex on the bottom to the one on the top. Then α_1^* induces a map from $\mathrm{Ext}(H', G) \to \mathrm{Ext}(H, G)$.

Fix H and let $f : G \to G'$ be given. Let $0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$ be a free resolution of H. We obtain two cochain complexes where f_* is a chain map from the top one to the bottom one.

$$0 \leftarrow \mathrm{Hom}(F_1, G) \xleftarrow{f_1^*} \mathrm{Hom}(F_0, G) \xleftarrow{f_0^*} \mathrm{Hom}(H, G) \leftarrow 0$$
$$0 \leftarrow \mathrm{Hom}(F_1, G') \xleftarrow{f_1^*} \mathrm{Hom}(F_0, G') \xleftarrow{f_0^*} \mathrm{Hom}(H, G') \leftarrow 0.$$

f_* indeed makes the diagram commute because for any $\sigma \in \mathrm{Hom}(H, G)$,

$$\begin{aligned} f_*(f_0^*(\sigma)) &= f_*(\sigma \circ f_0) \\ &= f \circ (\sigma \circ f_0) \\ &= (f \circ \sigma) \circ f_0 \\ &= f_0^*(f \circ \sigma) \\ &= f_0^*(f_*(\sigma)). \end{aligned}$$

Similarly, $f_*(f_1^*(\sigma)) = f_1^*(f_*(\sigma))$ for every $\sigma \in \mathrm{Hom}(F_0, G)$. Since a chain map induces a homomorphism on cohomology groups, f induces a map from $\mathrm{Ext}(H, G) \to \mathrm{Ext}(H, G')$.

Exercise. (Exercise 1.2)

$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[r, "f_1"] \arrow[d, "\cdot n"] & F_0 \arrow[r, "f_0"] \arrow[d, "\cdot n"] & H \arrow[r] \arrow[d, "\cdot n"] & 0 \\ 0 \arrow[r] & F_1 \arrow[r, "f_1"] & F_0 \arrow[r, "f_0"] & H \arrow[r] & 0 \end{tikzcd}$$

turn into two chain complexes with a chain map

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathrm{Hom}(F_1, G) & \underset{f_1^*}{\longleftarrow} & \mathrm{Hom}(F_0, G) & \underset{f_0^*}{\longleftarrow} & \mathrm{Hom}(H, G) & \longleftarrow & 0 \\
 & & {\scriptstyle (\cdot n)^*}\uparrow & & {\scriptstyle (\cdot n)^*}\uparrow & & {\scriptstyle (\cdot n)^*}\uparrow & & \\
0 & \longleftarrow & \mathrm{Hom}(F_1, G) & \underset{f_1^*}{\longleftarrow} & \mathrm{Hom}(F_0, G) & \underset{f_0^*}{\longleftarrow} & \mathrm{Hom}(H, G) & \longleftarrow & 0.
\end{array}$$

This diagram commutes because a group homomorphism for abelian groups commute with multiplication by n. Therefore, $(\cdot n)^*$ induces a homomorphism on $\mathrm{Ext}(H, G) = \mathrm{Hom}(F_1, G)/\operatorname{im}(f_1^*)$. Moreover, $\forall \phi + \operatorname{im}(f_1^*) \in \mathrm{Ext}(H, G)$,

$$(\cdot n)^*(\phi + \operatorname{im}(f_1^*)) = \phi \circ (\cdot n) + \operatorname{im}(f_1^*)$$

where $(\phi \circ (\cdot n))(x) = \phi(n(x)) = n(\phi(x)) = (n\phi)(x)$ for all $x \in F_1$. Therefore, the map induced by $(\cdot n)^*$ is simply multiplication by n.

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathrm{Hom}(F_1, G) & \underset{f_1^*}{\longleftarrow} & \mathrm{Hom}(F_0, G) & \underset{f_0^*}{\longleftarrow} & \mathrm{Hom}(H, G) & \longleftarrow & 0 \\
 & & \downarrow{\scriptstyle (\cdot n)_*} & & \downarrow{\scriptstyle (\cdot n)_*} & & \downarrow{\scriptstyle (\cdot n)_*} & & \\
0 & \longleftarrow & \mathrm{Hom}(F_1, G) & \underset{f_1^*}{\longleftarrow} & \mathrm{Hom}(F_0, G) & \underset{f_0^*}{\longleftarrow} & \mathrm{Hom}(H, G) & \longleftarrow & 0.
\end{array}$$

For every $\phi \in \mathrm{Hom}(H, G)$ and $x \in F_0$,

$$\begin{aligned}
((\cdot n)_*(f_0^*(\phi)))(x) &= ((\cdot n)_*(\phi \circ f_0))(x) \\
&= n((\phi \circ f_0)(x)) \\
&= n(\phi(f_0(x))) \\
&= ((\cdot n)_*\phi)(f_0(x)) \\
&= f_0^*((\cdot n)_*\phi)(x).
\end{aligned}$$

Similarly, $(\cdot n)_*$ commutes with f_1^*, so $(\cdot n)_*$ is a chain map. For any $\phi + \operatorname{im}(f_1^*) \in \mathrm{Ext}(H, G)$, $(\cdot n)_*(\phi + \operatorname{im}(f_1^*)) = n\phi + \operatorname{im}(f_1^*)$, so it is multiplication by n.

Exercise. (Exercise 3.1.3) $\cdots \xrightarrow{d_2} \mathbb{Z}_4 \xrightarrow{d_1} \mathbb{Z}_4 \xrightarrow{d_0} \mathbb{Z}_2 \to 0$ is a free resolution where $d_0 : a \mapsto a$ and $d_i : a \mapsto 2a$ because $\ker(d_0) = \operatorname{im}(d_i) = \ker(d_i) = \{0, 2\}$ for each $i \geq 1$. Apply $\mathrm{Hom}(-, \mathbb{Z}_2)$ and replace $\mathbb{Z}_2^*$ with 0. For any $\phi \in \mathrm{Hom}(\mathbb{Z}_4, \mathbb{Z}_2)$ and $x \in \mathbb{Z}_4$, $((\cdot 2)^*(\phi))(x) = (\phi \circ (\cdot 2))(x) = \phi(2x) = \phi(0) = 0$. Thus $(\cdot 2)^*(\phi) = 0$. In other words, $d_i^* = 0$ for all $i \geq 1$, so $\mathrm{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathrm{Hom}(\mathbb{Z}_4, \mathbb{Z}_2)$ which is nontrivial because $1 \mapsto 1$ is a nontrivial group homomorphism.

Exercise. (Exercise 3.1.6(a)) The chain complex we obtain is isomorphic to $0 \to \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0$ where $\alpha(a, b) = (a + b)(1, 1, -1)$. If we apply $\mathrm{Hom}(-, \mathbb{Z})$, we obtain

- $H^0(T; \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$.
- $\alpha^*(\phi) = 0$ if and only if $\phi(1, 1, -1) = 0$. $(a, b, c) \mapsto a - b$ and $(a, b, c) \mapsto a + c$ form a basis for the subspace consisting of such homomorphisms. $H^1(T; \mathbb{Z}) = \ker(\alpha^*) = \mathbb{Z} \oplus \mathbb{Z}$.
- $H^2(T; \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z})/\operatorname{im}(\alpha^*) = \mathbb{Z}$ because $(a, b) \mapsto a$ and $(a, b) \mapsto a + b$ form a basis for $\mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z})$ and $\operatorname{im}(\alpha^*)$ is spanned by $(a, b) \mapsto a + b$.

If we apply $\mathrm{Hom}(-, \mathbb{Z}_2)$, we obtain

- $H^0(T; \mathbb{Z}_2) = \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$.
- $\alpha^*(\phi) = 0$ if and only if $\phi(1,1,1) = 0$. $(a,b,c) \mapsto a+b$ and $(a,b,c) \mapsto a+c$ form a basis for the subspace consisting of such homomorphisms. $H^1(T; \mathbb{Z}_2) = \ker(\alpha^*) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
- $H^2(T; \mathbb{Z}_2) = \mathrm{Hom}(\mathbb{Z}_2^2, \mathbb{Z}_2)/\operatorname{im}(\alpha^*) = \mathbb{Z}_2$ because $(a,b) \mapsto a$ and $(a,b) \mapsto a+b$ form a basis for $\mathrm{Hom}(\mathbb{Z}_2^2, \mathbb{Z}_2)$ and $\operatorname{im}(\alpha^*)$ is spanned by $(a,b) \mapsto a+b$.

Exercise. (Exercise 3.1.6(b), projective plane) We obtain a chain complex $0 \to \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^3 \xrightarrow{\beta} \mathbb{Z}^2 \to 0$ where $\alpha(a,b) = (b-a, a-b, a+b)$ and $\beta(a,b,c) = (a+b, -a-b)$. By applying $\mathrm{Hom}(-, \mathbb{Z})$, we obtain a cochain complex. Each $\mathrm{Hom}(\mathbb{Z}^k, \mathbb{Z})$ has a basis $\{\pi_1, \pi_2, \cdots, \pi_k\}$ where π_i is a projection on the ith coordinate. Then $(\beta^*(\pi_1))(a,b,c) = a+b, (\beta^*(\pi_2))(a,b,c) = -a-b$. Thus $\ker(\beta^*) = \langle \pi_1 + \pi_2 \rangle$ and $\operatorname{im}(\beta^*) = \langle \pi_1 + \pi_2 \rangle$. The kernel and image of α can be calculated similarly.

- $H^0 = \ker(\beta^*) = \mathbb{Z}$.
- $H^1 = \ker(\alpha^*)/\operatorname{im}(\beta^*) = \langle \pi_1 + \pi_2 \rangle / \langle \pi_1 + \pi_2 \rangle = 0$.
- $H_2 = \ker(0)/\operatorname{im}(\alpha^*) = \langle \pi_1, \pi_2 \rangle / \langle \pi_1 - \pi_2, \pi_1 - \pi_2 \rangle = \langle \pi_1 + \pi_2, \pi_1 \mid \pi_1 + \pi_2, 2\pi_1 \rangle = \mathbb{Z}_2$.

Similarly, we apply $\mathrm{Hom}(-\mathbb{Z}_2)$. Each $\mathrm{Hom}(\mathbb{Z}^k, \mathbb{Z}_2)$ has a basis $\{\pi_1, \pi_2, \cdots, \pi_k\}$ where π_i is a projection on the ith coordinate. The calculation of the kernels and images are almost identical as above with the only exception $\ker(\alpha^*)$. This is because $\alpha^*(\pi_i) : (a,b) \mapsto a+b$ for each $i = 1, 2, 3$, so the kernel is $\langle \pi_1 + \pi_2, \pi_1 + \pi_3 \rangle$.

- $H^0 = \ker(\beta^*) = \langle \pi_1 + \pi_2 \rangle = \mathbb{Z}_2$.
- $H^1 = \ker(\alpha^*)/\operatorname{im}(\beta^*) = \langle \pi_1 + \pi_2, \pi_1 + \pi_3 \rangle / \langle \pi_1 + \pi_2 \rangle = \mathbb{Z}_2$.
- $H_2 = \ker(0)/\operatorname{im}(\alpha^*) = \langle \pi_1, \pi_2 \rangle / \langle \pi_1 + \pi_2, \pi_1 + \pi_2 \rangle = \langle \pi_1 \rangle = \mathbb{Z}_2$.

Exercise. (Exercise 3.1.6(b), klein bottle) The chain complex we obtain is $0 \to \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0$ with $\alpha(a,b) = (a+b, a-b, b-a)$. Again, we will use the projection map π_i of the ith coordinate to form bases of the dual spaces. $\ker 0^* = \mathbb{Z}, \operatorname{im} 0^* = 0$. $\ker(\alpha^*) = \langle \pi_2 + \pi_3 \rangle$ and $\operatorname{im}(\alpha^*) = \langle \pi_1 + \pi_2, \pi_1 - \pi_2 \rangle$ because

$$(\alpha^*(\pi_i))(a,b) = \begin{cases} a+b & (i=1) \\ a-b & (i=2) \\ b-a & (i=3). \end{cases}$$

Thus $H_0 = \mathbb{Z}$, $H_1 = \langle \pi_2 + \pi_3 \rangle / 0 = \mathbb{Z}$ and $H_2 = \langle \pi_1, \pi_2 \mid \pi_1 + \pi_2, \pi_1 - \pi_2 \rangle = \mathbb{Z}/2$.
$\ker 0^* = \mathbb{Z}_2, \operatorname{im} 0^* = 0$. $\ker(\alpha^*) = \langle \pi_1 + \pi_2, \pi_1 + \pi_3 \rangle$ and $\operatorname{im}(\alpha^*) = \langle \pi_1 + \pi_2 \rangle$.
Thus $H_0 = \mathbb{Z}_2$, $H_1 = \langle \pi_1 + \pi_2, \pi_1 + \pi_3 \rangle / 0 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $H_2 = \langle \pi_1, \pi_2 \mid \pi_1 + \pi_2 \rangle = \mathbb{Z}_2$.

Exercise. (Exercise 3.1.8(a)) S^0 consists of two points, so $\tilde{H}^i(S^0; G) = G^2/G = G$ if $i = 0$ and 0 otherwise because $\tilde{H}^0(S^0; G)$ is all functions module constant functions. Suppose we have shown $\tilde{H}^i(S^k; G) = G$ if $i = k$ and 0 otherwise. By the long exact sequence of a pair, we obtain $\tilde{H}^i(D^{k+1}; G) \to \tilde{H}^i(S^k; G) \to \tilde{H}^{i+1}(D^{k+1}, S^k; G) \to \tilde{H}^{i+1}(D^{k+1}; G)$. Since D^{k+1} is contractible, $\tilde{H}^i(D^{k+1}; G) = 0$ for all i. This induces an isomorphism $\tilde{H}^i(S^k; G) \cong \tilde{H}^{i+1}(D^{k+1}, S^k; G) = \tilde{H}^{i+1}(S^{k+1}; G) = G$. Therefore, $H^k(S^0; G) = G^2$ and 0 if $k > 0$, and $H^k(S^n; G) = G$ if $k \in \{0, n\}$ and 0 otherwise.

The Mayer-Vietoris sequence gives $\tilde{H}^k(A;G)\oplus\tilde{H}^k(B;G) \to \tilde{H}^k(A\cap B;G) \to \tilde{H}^{k+1}(S^n;G) \to \tilde{H}^{k+1}(A;G)\oplus\tilde{H}^{k+1}(B;G)$ where A, B are the northern and southern hemispheres with some extra part so the union of the interiors equals S^n. Since A and B are contractible regardless of the value of k, $\tilde{H}^k(A;G) = \tilde{H}^k(B;G) = \tilde{H}^{k+1}(A;G) = \tilde{H}^k(B;G) = 0$. This gives us an isomorphism $\tilde{H}^k(A\cap B;G) \cong \tilde{H}^{k+1}(S^n;G)$. $A\cap B$ is homotopic to S^n. By induction, $\tilde{H}^k(A\cap B;G) = G$ if $k = n$ and 0 otherwise.

Exercise. (Exercise 3.1.8(b)) Let q be the quotient map $(X, A) \to (X/A, A/A)$. Let V be a neighborhood of A in X that deformation retracts onto A. We have a commutative diagram

$$\begin{tikzcd}
H^n(X,A) & H^n(X,V) \arrow[l] \arrow[r] & H^n(X-A, V-A) \\
H^n(X/A,A/A) \arrow[u, "q^*"] & H^n(X/A,V/A) \arrow[l] \arrow[r] \arrow[u, "q^*"] & H^n(X/A-A/A, V/A-A/A). \arrow[u, "q^*"]
\end{tikzcd}$$

The upper horizontal map is an isomorphism since in the long exact sequence of the triple $(X, V < A)$ the groups $H^n(V, A)$ are zero for all n, because a deformation retraction of V onto A gives a homotopy equivalence of pairs $(V, A) \simeq (A, A)$, and $H_n(A, A) = 0$. The deformation retraction of V onto A induces a deformation retraction of V/A onto A/A, so the same argument shows that the lower left horizontal map is an isomorphism as well. The other two horizontal maps are isomorphisms directly from excision. The right-hand vertical map q_* is an isomorphism since q restricts to a homeomorphism on the complement of A. From the commutativity of the diagram, it follows that the left-hand q_* is an isomorphism. Since A/A is a point, $H^n(X/A, A/A) \cong \tilde{H}^n(X/A)$. Therefore, $H^n(X, A) \cong \tilde{H}^n(X/A)$.

Exercise. (Exercise 3.1.8(c)) Let $r : X \to A, i : A \to X$ be the retract and inclusion. Then $i^* : H^n(X;G) \to H^n(A;G)$ is injective because $\mathrm{Id} = (ri)^* = i^*r^*$. Thus the boundary map of the long exact sequence must be 0 by the exactness, so we obtain a short exact sequence $0 \to H^n(X, A;G) \to H^n(X;G) \to H^n(A;G) \to 0$. The relation $\mathrm{Id} = i^*r^*$ implies that the short exact sequence splits by the split lemma. Therefore, $H^n(X;G) = H^n(X, A;G) \oplus H^n(A;G)$.

2. Section 3.A

Exercise. (Exercise 1) If the characteristic of F is infinity, the Tor functor becomes 0, so the UCT gives us an isomorphism $H_n(X;\mathbb{Z})\otimes F \cong H_n(X;F)$. Therefore, the rank of $H_n(X;\mathbb{Z})$ equals the dimension of $H_n(X;F)$.

Suppose the characteristic of F is p. By the UCT, $H_n(X;F) \cong (H_n(X;\mathbb{Z})\otimes F) \oplus \mathrm{Tor}(H_{n-1}(X;\mathbb{Z});F)$. Suppose $H_n(X;\mathbb{Z}) = \mathbb{Z}^d \oplus (\oplus_{i=1}^n \mathbb{Z}_{p_i^{k_i}})$ where $p_1 = \cdots = p_m = p$.

$$\begin{aligned}
\mathrm{Tor}(\mathbb{Z}^d \oplus \mathbb{Z}_{p_1^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_n^{k_n}}; F) &= \oplus_{i=1}^n \mathrm{Tor}(\mathbb{Z}_{p_i^{k_i}}; F) \\
&= \oplus_{i=1}^n \ker(F \xrightarrow{p_i^{k_i}} F) \\
&= F^m.
\end{aligned}$$

Also,

$$\begin{aligned}
(\mathbb{Z}^d \oplus \mathbb{Z}_{p_1^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_n^{k_n}}) \otimes F &= (\mathbb{Z} \otimes F)^d \oplus (\mathbb{Z}_{p_1^{k_1}} \otimes F) \oplus \cdots \oplus (\mathbb{Z}_{p_n^{k_n}} \otimes F) \\
&= F^d \oplus (\oplus_{i=1}^n (\mathbb{Z}_{p_i^{k_i}} \otimes F)) \\
&= F^d \oplus (\oplus_{i=1}^n (F/p_i^{k_i} F)) \\
&= F^{d+m}.
\end{aligned}$$

Therefore,

- Each $\mathbb{Z}$ summand in $H_n(X;\mathbb{Z})$ “adds” one to the dimension of $H_n(X;F)$.
- Each $\mathbb{Z}/p^{k_i}$ summand in $H_n(X;\mathbb{Z})$ “adds” one to the dimension of $H_n(X;F)$ and adds one to the dimension of $H_{n+1}(X;F)$. This gets cancelled out when taking the sum to calculate the Euler characteristic.

Exercise. (Exercise 3.A.2) By Proposition 3A.5, $\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z}) = \mathrm{Tor}(T(A), \mathbb{Q}/\mathbb{Z})$. Using the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, we obtain an exact sequence

$$\begin{aligned}
0 \to \mathrm{Tor}(T(A), \mathbb{Z}) \to \mathrm{Tor}(T(A), \mathbb{Q}) \to \mathrm{Tor}(T(A), \mathbb{Q}/\mathbb{Z}) \\
\to T(A) \otimes \mathbb{Z} \to T(A) \otimes \mathbb{Q} \to T(A) \otimes \mathbb{Q}/\mathbb{Z} \to 0.
\end{aligned}$$

$\mathrm{Tor}(T(A), \mathbb{Q}) = T(A) \otimes \mathbb{Q} = 0$. Thus $\mathrm{Tor}(T(A), \mathbb{Q}/\mathbb{Z}) = T(A) \otimes \mathbb{Z} = T(A)$.