MATH 611 PROBLEM SET 1 (DUE 9/4)

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Exercise 0.1. (Exercise 4, Chapter 0) A deformation retraction in the weak sense of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \mathrm{Id}, f_1(X) \subset A$, and $f_t(A) \subset A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \to X$ is a homotopy equivalence.

Proof. Let $i : A \to X$ denote the inclusion. Let $F : X \times I \to X$ denote the associated map $(x, t) \to f_t(x)$. Then F is a continuous function by the definition of a homotopy.

Let $f : X \to A$ be defined by $f(x) = F(x, 1) = f_1(x)$. This definition makes sense because $f_1(X) \subset A$. We claim that $f_1 \circ i \simeq \mathrm{Id}_A$ and $i \circ f_1 \simeq \mathrm{Id}_X$.

Consider $G : A \times I \to A$ such that $G(a, t) = F(a, t)$ for all $(a, t) \in A \times I$. This definition makes sense because $f_t(A) \subset A$ for all t.

Then G is a homotopy in A between $f \circ i$ and Id_A because:

- G is a restriction of F, so G is continuous.
- $\forall a \in A, G(a, 0) = F(a, 0) = f_0(a) = \mathrm{Id}_X(a) = \mathrm{Id}_A(a)$.
- $\forall a \in A, G(a, 1) = F(a, 1) = f(a) = f(i(a)) = (f \circ i)(a)$.

Therefore, $f \circ i \simeq \mathrm{Id}_A$.

F is a homotopy between f_0 and f_1.

- We are given that $f_0 = \mathrm{Id}_X$.
- For any $x \in X$, $(i \circ f)(x) = i(f(x)) = f(x) = f_1(x)$, so $i \circ f = f_1$.

Therefore, F is a homotopy between Id_X and $i \circ F$, so $i \circ f \simeq \mathrm{Id}_X$.

In conclusion, i is indeed a homotopy equivalence. $\square$

Exercise 0.2. (Exercise 5, Chapter 0) Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X there exists a neighborhood $V \subset U$ of x such that the inclusion map $V \to U$ is nullhomotopic.

Proof. Let $p \in X$ be a point to which X deformation retracts. Since X deformation retracts to p, there exists a map $F : X \times I \to X$ such that

(1) $\forall x \in X, F(x, 0) = x$.
(2) $\forall x \in X, F(x, 1) = p$.

(3) $\forall t \in I, F(p,t) = p$.
(4) F is continuous.

Let U be a neighborhood of p. Then $F^{-1}(U)$ is an open subset of the product space $X \times I$. By the 3rd property of F mentioned above, the slice $\{p\} \times I$ is a subset of $F^{-1}(U)$. Since I is compact, there must be a open subset V of X such that $\{p\} \times I \subset V \times I \subset F^{-1}(U)$ by the tube lemma.

We claim that this V is a desired subset.

- V is an open subset of X.
- Since $\{p\} \times I \subset V \times I$, $p \in V$.
- Since $V \times I \subset F^{-1}(U)$, $F(V \times I) \subset U$. This implies that $\forall v \in V$, $F(v,0) = v \in U$. Therefore, $V \subset U$.
- We claim that the inclusion map $i : V \to U$ is nullhomotopic. Let $e_p : V \to U$ denote the constant map at p, $G : V \times I \to U$ be defined by $G(x,t) = F(x,t)$ for all $x \in V, t \in I$.
  - G indeed maps $V \times I$ into U because $F(V \times I) \subset U$. Therefore, G is well-defined.
  - Since G is the restriction of F to $V \times I$ and F is continuous, G is continuous.
  - $\forall x \in V, G(x,0) = F(x,0) = x = i(x)$.
  - $\forall x \in V, G(x,1) = F(x,1) = p = e_p(x)$.

  Thus i is indeed nullhomotopic.

□

Lemma 0.3. *The neighborhood V that we find in Problem 5 is connected.*

Proof. Suppose otherwise. Let A, B denote a separation of V. Without loss of generality, we assume $p \in A$. Let $q \in B$. (B must be nonempty since A, B are a separation.)

Let F be the homotopy we defined in the solution for Problem 5 from the inclusion map to the constant map at p. Let $f : I \to V$ be defined such that $f(t) = F(q,t)$. Then f is a path from $f(0) = F(q,0) = q$ to $f(1) = F(q,1) = p$ in V. Since f is continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are open in I. Moreover, $I = f^{-1}(V) = f^{-1}(A) \cup f^{-1}(B)$ and $\emptyset = f^{-1}(\emptyset) = f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. Since $1 \in f^{-1}(p) \subset f^{-1}(A)$ and $0 \in f^{-1}(q) \subset f^{-1}(B)$, $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty. Therefore, $f^{-1}(A)$ and $f^{-1}(B)$ form a separation of I. However, this is impossible because I is connected. □

Exercise 0.4. (Exercise 6(a), Chapter 0) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1]\times\{0\}$, but not to any other point.

Proof. Let $(a,0)\in[0,1]\times\{0\}$ be given. Let $F : X\times I\to X$ be defined such that

$$F((x,y),t)=\begin{cases}(x,(1-2t)y) & (0\le t\le 1/2)\\ (x+(a-x)(2t-1),0) & (1/2\le t\le 1).\end{cases}$$

F is well defined because when $t=1/2$:

- $(x,(1-2t)y)=(x,0)$.
- $(x+(a-x)(2t-1),0)=(x,0)$.

Moreover, by the pasting lemma, F is continuous.

Then F is a deformation retract of X onto $(a,0)$ because

- $$F((a,0),t)=\begin{cases}(a,0(1-2t))=(a,0) & (t\in[0,1/2])\\ (a+(a-a)(2t-1),0)=(a,0) & (t\in[1/2,1]).\end{cases}$$

  Therefore, $F((a,0),t)=(a,0)$ for any $t\in I$.
- $F((x,y),0)=(x,y)$ for any $(x,y)\in x$.
- $F((x,y),1)=(a,0)$ for any $(x,y)\in x$.

Therefore, F is indeed a deformation retract of X onto $(a,0)$.

Suppose that there exists a point $(a,b)\in X$ to which X deformation retracts onto such that $b\neq 0$. Let $G : X\times I\to X$ denote such a deformation retract. Consider the open subset $U=B((a,b),b)\cap X$. Note that U is disjoint from the segment $[0,1]\times\{0\}$. Then U is a neighborhood of (a,b), a point to which X deformation retracts onto. By Problem 5 (Chapter 0), there must exist a neighborhood $V\subset U$ of x such that the inclusion map $V\to U$ is nullhomotopic. By the Lemma we showed above, V must be connected. Since V is an open subset of X, there must exist an $r>0$ such that $B((a,b),r)\cap X\subset V$. Let c be an irrational number in $(a,a+r)$. Then $V\cap((-\infty,c)\times\mathbb{R})$ and $V\cap((c,\infty)\times\mathbb{R})$ form a separation of V. This is a contradiction, so our initial assumption that X deformation retracts onto (a,b) was wrong. Therefore, X deformation retracts to any point in the segment $[0,1]\times\{0\}$, but not to any other point. □

Exercise 0.5. (Exercise 9, Chapter 0) Show that a retract of a contractible space is contractible.

Proof. Let X be a contractible space. Then Id_X is homotopic to a constant map. This implies the existence of a fixed point $p\in X$ and a continuous function $F : X\times I\to X$ such that

- $\forall x \in X, F(x, 0) = x$,
- $\forall x \in X, F(x, 1) = p$.

Let $A \subset X$ be a retract of X, and let $r : X \to A$ denote a retraction. In other words, $r(X) = A$ and $r\mid_A = \mathrm{Id}_A$.

Let $G : A \times I \to A$ be the restriction of $r \circ F$ to $A \times I$. This makes sense because F maps $A \times I$ into X, and r maps X into A. We claim that G is a homotopy between Id_A and the constant map $e_{r(p)}$ such that $e_{r(p)}(a) = r(p)$ for all $a \in A$.

- $r \circ F$ is continuous since it is a composition of continuous functions. G is a restriction of a continuous function, so G is continuous.
- $G(a, 0) = r(F(a, 0)) = r(a) = a = \mathrm{Id}_A(a)$.
- $G(a, 1) = r(F(a, 1)) = r(p) = e_{r(p)}(a)$.

Therefore, G is indeed a homotopy between Id_A and the constant map at $r(p)$. Since the identity map is homotopic to a constant map, A is contractible. □

Exercise 0.6. (Exercise 13, Chapter 0) Show that any two deformation retractions r_t^0 and r_t^1 of a space X onto a subspace A can be joined by a continuous family of deformation retractions r_t^s, $0 \le s \le 1$, of X onto A, where continuity means that the map $X \times I \times I \to X$ sending (x, s, t) to $r_t^s(x)$ is continuous.

Proof. Let $F : X \times I \times I \to X$ be defined such that

$$F(x, t, s) = \begin{cases} r^0_{t(1-2s)}(x) & (s \in [0, 1/2]) \\ r^1_{t(2s-1)}(x) & (s \in [1/2, 1]). \end{cases}$$

We claim that F is well-defined and satisfies the desired properties.

- Let $s = 1/2$. $r^0_{t(1-2s)}(x) = r^0_0(x) = x$ because r^0_t is a deformation retraction. Similarly, $r^1_{t(2s-1)}(x) = r^1_0(x) = x$ because r^0_t is a deformation retraction. Therefore, F is well defined when $s = 1/2$. Moreover, by the pasting lemma, F is continuous. This is because the intersection $X \times I \times [0, 1/2] \cap X \times I \times [1/2, 1] = X \times I \times \{1/2\}$ is closed.
- $F(x, t, 0) = r^0_t(x)$ for any $x \times t \in X \times I$.
- $F(x, t, 1) = r^1_t(x)$ for any $x \times t \in X \times I$.

Therefore, F maps $X \times I \times I \to X$ continuously sending (x, s, t) to $r^s_t(x)$. □

Exercise 0.7. (Exercise 7, Chapter 1.1) Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles.

Proof. Define $F : (S^1 \times I) \times I \to S^1 \times I$ such that $F((\theta, s), t) = t(\theta, s) + (1-t)f(\theta, s)$. Then F is a homotopy between f and the identity map that is stationary on $S^1 \times \{0\}$. This is because $F((\theta, 0), t) = t(\theta, 0) + (1-t)f(\theta, 0) = (t\theta, 0) + ((1-t)\theta, 0) = (\theta, 0)$ for any $(\theta, t) \in S^1 \times I$.

Suppose that there exists a homotopy $G : (S^1 \times I) \times I \to S^1 \times I$ between f and the identity map that is stationary on both boundary circles. Let $H : I \times I \to S^1$ be defined such that $H(s, t) = \pi_1(F((0, t), s))$ where π_1 denotes the projection of the first coordinate.

- $H(s, 0) = \pi_1(G((0, 0), s)) = \pi_1(0, 0) = 0$ because G is stationary on the circle $S^1 \times \{0\}$.
- $H(s, 1) = \pi_1(G((0, 1), s)) = \pi_1(0, 1) = 0$ because G is stationary on the circle $S^1 \times \{1\}$.
- $H(0, t) = \pi_1(G((0, t), 0)) = \pi_1(f(0, t)) = \pi_1(2\pi t, t) = 2\pi t$.
- $H(1, t) = \pi_1(G((0, t), 1)) = \pi_1(0, t) = 0$.

Then $t \mapsto H(0, t)$ corresponds to the ω in Theorem 1.7, and $t \mapsto H(1, t)$ corresponds to a constant map. In other words, H is a homotopy between ω and a constant map in S^1. However, this is a contradiction because Theorem 1.7 states that $\pi_1(S^1)$ is the infinite cyclic group generated by ω. Therefore, such a homotopy G does not exist. □

Exercise 0.8. (Exercise 16, Chapter 1.1) Show that there are no retractions $r : X \to A$ in the following cases:

- $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.
- $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.

Proof.

- Suppose that X retracts onto A. By Proposition 1.17, the homomorphism $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $i : A \to X$ is injective. Since A and S^1 are homeomorhpic, $\pi_1(S^1)$ and $\pi_1(A)$ are isomorphic to each other. By Theorem 1.7, $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$. On the other hand, $\pi_1(\mathbb{R}^3) = 0$ because $\mathbb{R}^3$ is convex. This implies the existence of an injective homomorphism from $\mathbb{Z}$ into 0, which is impossible. Therefore, X does not retract onto A.
- Suppose X retracts onto A. By Proposition 1.17, the homomorphism $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $i : A \to X$ is injective. By Theorem 1.7, $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$. $\pi_1(D^2) = 0$ because D^2 is a convex subset and

thus a linear homotopy connects any paths. By Proposition 1.12, $\pi_1(X) = \pi_1(S^1) \times \pi_1(D^2) = \mathbb{Z} \times 0 = \mathbb{Z}$ and $\pi_1(A) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be any homomorphism. Let $a = f(1, 0), b = f(0, 1)$. If $a = 0$ or $b = 0$, f is not injective because $f(0, 0) = 0$. Suppose otherwise. Then $f(b, 0) = ab = f(0, a)$, so f is not injective.

Therefore, there exists no injection from $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. Hence, X does not retract onto A.

□

Exercise 0.9. (Exercise 20, Chapter 1.1) Suppose $f_t : X \to X$ is a homotopy such that f_0 and f_1 are each the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$.

Proof. Let $x_0 \in X$ be given. Let $f : I \to X$ be the loop defined such that $f(t) = f_t(x_0)$.

- $f_t : X \to X$ is a homotopy.
- f is a path formed by the images of the base point x_0.

By Lemma 1.19, the following diagram commutes.

$$\begin{tikzcd}
& \pi_1(X, f_1(x_0)) \arrow[dd, "\beta_f"] \\
\pi_1(X, x_0) \arrow[ur, "(f_1)_*"'] \arrow[dr, "(f_0)_*"'] & \\
& \pi_1(X, f_0(x_0))
\end{tikzcd}$$

$(f_0)_* = (f_1)_* = (\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(X,x_0)}$ by a basic property of induced homomorphisms (P.34 of Hatcher). Since $f_0 = f_1 = \mathrm{Id}_X$, $f_0(x_0) = f_1(x_0) = x_0$. Therefore, the diagram above can be simplified as following:

$$\begin{tikzcd}
 & \pi_1(X, x_0) \arrow[dd, "\beta_f"] \\
\pi_1(X, x_0) \arrow[ur, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\
 & \pi_1(X, x_0)
\end{tikzcd}$$

Let $[g] \in \pi_1(X, x_0)$. Then by the diagram above, we have $\mathrm{Id}([g]) = \mathrm{Id}(\beta_f([g]))$. This implies $[g] = [f \cdot g \cdot \overline{f}]$. Therefore, $[g] \cdot [f] = [f] \cdot [g]$, so $[f]$ commutes with every element in $\pi_1(X, x_0)$. Hence, $[f] \in Z(\pi_1(X, x_0))$. $\square$