

# MATH 633(HOMEWORK 2)

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**Exercise.** (Problem 2a) Let  $\gamma(t) = Re^{2\pi it}$ .

$$\begin{aligned}\int_{\gamma} z^n dz &= \int_0^1 R^n e^{2\pi i n t} R 2\pi i e^{2\pi i t} dt \\ &= 2\pi i R^{n+1} \int_0^1 e^{2\pi i (n+1)t} dt \\ &= \begin{cases} R^{n+1} \frac{e^{2\pi i (n+1)t}}{n+1} = 0 & (n \neq -1) \\ 2\pi i & (n = -1). \end{cases}\end{aligned}$$

**Exercise.** (Problem 2b) Let  $\gamma(t) = z_0 + Re^{2\pi it}$  where  $|R/z_0| < 1$ .

$$\int_{\gamma} z^n dz = \int_0^1 (z_0 + Re^{2\pi it})^n (z_0 + Re^{2\pi it})' dt$$

When  $n \neq -1$ ,  $(z_0 + Re^{2\pi it})^{n+1}/(n+1)$  is a primitive, so the integral is 0. Suppose  $n = -1$ .

$$\begin{aligned}\int_0^1 \frac{2\pi i R e^{2\pi it}}{z_0 + R e^{2\pi it}} dt &= \int_0^1 \frac{2\pi i R e^{2\pi it}/z_0}{1 + R e^{2\pi it}/z_0} dt \\ &= \int_0^1 \frac{2\pi i R e^{2\pi it}}{z_0} \cdot \sum_{k=0}^{\infty} \left(\frac{-R e^{2\pi it}}{z_0}\right)^k dt \\ &= -2\pi i \sum_{k=0}^{\infty} \int_0^1 \left(\frac{-R e^{2\pi it}}{z_0}\right)^{k+1} dt \\ &= -2\pi i \sum_{k=0}^{\infty} \left(\frac{-R e^{2\pi it}}{z_0}\right)^{k+1} \int_0^1 e^{2\pi i t(k+1)} dt \\ &= 0.\end{aligned}$$

Each  $\int_0^1 e^{2\pi i t(k+1)} dt = 0$  because  $e^{2\pi i t(k+1)}/(2\pi i t(k+1))$  is a primitive.

**Exercise.** (Problem 2c)

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b} \left( \frac{1}{z-a} - \frac{1}{z-b} \right)$$

Thus we will compute two integrals and add them later.

$$\begin{aligned}
\int_{\gamma} \frac{1}{z-a} dz &= \int_{\gamma} \frac{1/z}{1-a/z} dz \\
&= \int_{\gamma} \frac{1}{z} \sum_{k=0}^{\infty} \left(-\frac{a}{z}\right)^k dz \\
&= \sum_{k=0}^{\infty} (-a)^k \int_{\gamma} \frac{1}{z^{k+1}} dz \\
&= 2\pi i
\end{aligned}$$

because  $z^{-(k+1)}$  has a primitive  $z^{-k}/-k$  whenever  $k \neq 0$  and when  $k = 0$  we can use the results above.

$$\begin{aligned}
\int_{\gamma} \frac{1}{b-z} dz &= \int_{\gamma} \frac{1/b}{1-z/b} dz \\
&= \frac{1}{b} \sum_{k=0}^{\infty} \int_{\gamma} \left(\frac{z}{b}\right)^k dz \\
&= \frac{1}{b} \sum_{k=0}^{\infty} \frac{1}{b^k} \int_{\gamma} z^k dz \\
&= 0
\end{aligned}$$

because  $z^{k+1}/(k+1)$  is a primitive.

By putting these together, we conclude that the desired value is  $2\pi i/(a-b)$ .

**Exercise.** (Problem 3)

$$\begin{aligned}
\int_a^b |z'(t)| dt &= \int_c^d |z'(t(s))| t'(s) ds \\
&= \int_c^d |z'(t(s)) t'(s)| ds \\
&= \int_c^d |\tilde{z}'(s)| ds
\end{aligned}$$

where  $\tilde{z}(s) : [c, d] \rightarrow \mathbb{C}$  is a reparametrization of  $z(t) : [a, b] \rightarrow \mathbb{C}$ .

**Exercise.** (Problem 4a) If  $t^* \in \Omega_1$ , then there exists an open neighborhood  $U$  of  $z(t^*)$  contained in  $\Omega_1$ . Then  $z^{-1}(U)$  is a neighborhood of  $t^*$  in  $[0, 1]$  because  $z$  is continuous. Since  $z(1) \in \Omega_2$ ,  $t^* \neq 1$ . However, this implies the existence of  $\epsilon > 0$  such that  $t^* + \epsilon < 1$  and  $z(t^* + \epsilon) \in \Omega_1$ . This is a contradiction.

If  $t^* \in \Omega_2$ , then there exists an open neighborhood  $U$  of  $z(t^*)$  contained in  $\Omega_2$ . Since  $U$  is open,  $z^{-1}(U)$  is a neighborhood of  $t^*$  in  $[0, 1]$ , so  $\exists \epsilon > 0$  such that  $z(t^* - \epsilon) \in \Omega_2$ .

In each case, we reached a contradiction, so  $\Omega$  is not disconnected.

**Exercise.** (Problem 4b) For every  $v \in \Omega_1$ , there exists an open set  $U$  such that  $v \in U \subset \Omega_1$ . Then for any  $v' \in U$ ,  $v$  and  $v'$  can be joined by  $\gamma(t) = tv + (1-t)v'$ . Thus  $U \subset \Omega_1$ , so  $\Omega_1$  is open.

Let  $v \in \Omega_2$ . Suppose that for all  $\epsilon > 0$ , the open disk at  $v$  with the radius  $\epsilon$  is not contained in  $\Omega_2$ . Otherwise we are done. Let  $v_0 = w$ . For every  $n \in \mathbb{N}$ , choose  $v_n \in D(v, 1/n) \setminus \Omega_2$ . Then there exists a path between each  $v_n$  and  $w$ . Moreover, there exists a path between  $v_n$  and  $v_{n-1}$  for each  $n$  and we will call it  $\gamma_n$ . Define  $\gamma : [0, 1] \rightarrow \Omega$  such that for each  $n \in \mathbb{N}$ ,  $\gamma([1 - 1/n, 1 - 1/(n + 1)])$  is the path  $\gamma_n$  and  $\gamma(1) = v$ . Then  $\gamma$  is a well-defined path from  $w$  to  $v$ , which is a contradiction because  $v \in \Omega_2$ .

Clearly,  $\Omega_1 \cap \Omega_2 = \emptyset$  and  $\Omega_1 \cup \Omega_2 = \Omega$ . Since  $w \in \Omega_1$ ,  $\Omega_1 \neq \emptyset$ , so  $\Omega_2 = \emptyset$ .