

# MATH 611 HOMEWORK (DUE 9/18)

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**Exercise.** (Problem 12, Chapter 1.2) The Klein bottle is usually pictured as a subspace of  $\mathbb{R}^3$  like the subspace  $X \subset \mathbb{R}^3$  shown in the first figure at the right. If one wanted a model that could actually function as a bottle, one would delete the open disk bounded by the circle of self-intersection of  $X$ , producing a subspace  $Y \subset X$ . Show that  $\pi_1(X) \approx \mathbb{Z} * \mathbb{Z}$  and that  $\pi_1(Y)$  has the presentation  $\langle a, b, c \mid aba^{-1}b^{-1}cb^\epsilon c^{-1} \rangle$  for  $\epsilon = \pm 1$ . Show also that  $\pi_1(Y)$  is isomorphic to  $\pi_1(\mathbb{R}^3 \setminus Z)$  for  $Z$  the graph shown in the figure.

*Proof.* We will construct  $X$  from the 1-skeleton in Figure 1. The 1-skeleton has three loops  $a, b, c$ , so the fundamental group is  $\langle a, b, c \mid \rangle$ . The main difference between  $X$  and the “proper” Klein bottle is that the loop  $a$  actually gets glued on the surface. Thus we will glue the first 2-cell to  $a$ , and another 2-cell on the loop  $c^{-1}acbab^{-1}$ . Therefore, we end up with the fundamental group  $\langle a, b, c \mid a, c^{-1}aca^{-1}bab^{-1} \rangle$ . Then  $\langle a, b, c \mid a, c^{-1}acabab^{-1} \rangle \approx \langle b, c \mid \rangle \approx \mathbb{Z} * \mathbb{Z}$  since the relation  $c^{-1}aca^{-1}bab^{-1}$  is trivial by the relation  $a$ .

In order to calculate the fundamental group of  $Y$ , it suffices to repeat the following step without attaching a 2-cell to  $a$ . Thus the fundamental group is  $G = \langle a, b, c \mid c^{-1}aca^{-1}bab^{-1} \rangle$ .

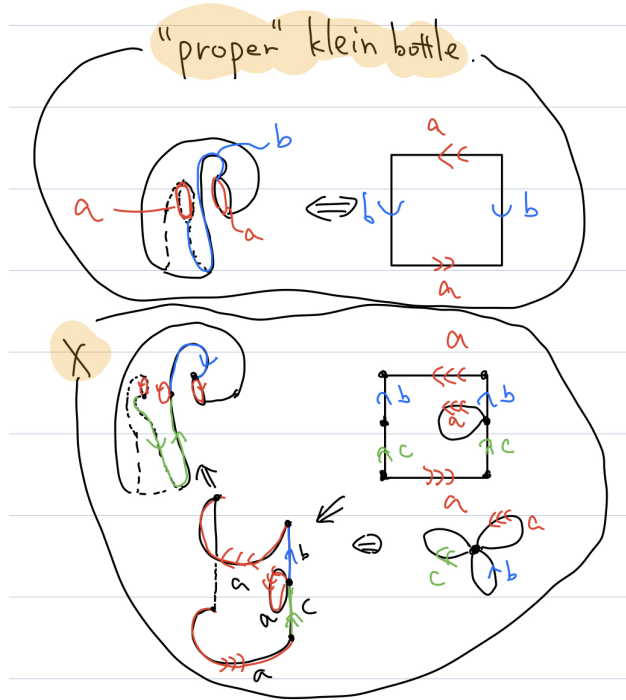


FIGURE 1. Fundamental Group of  $X$

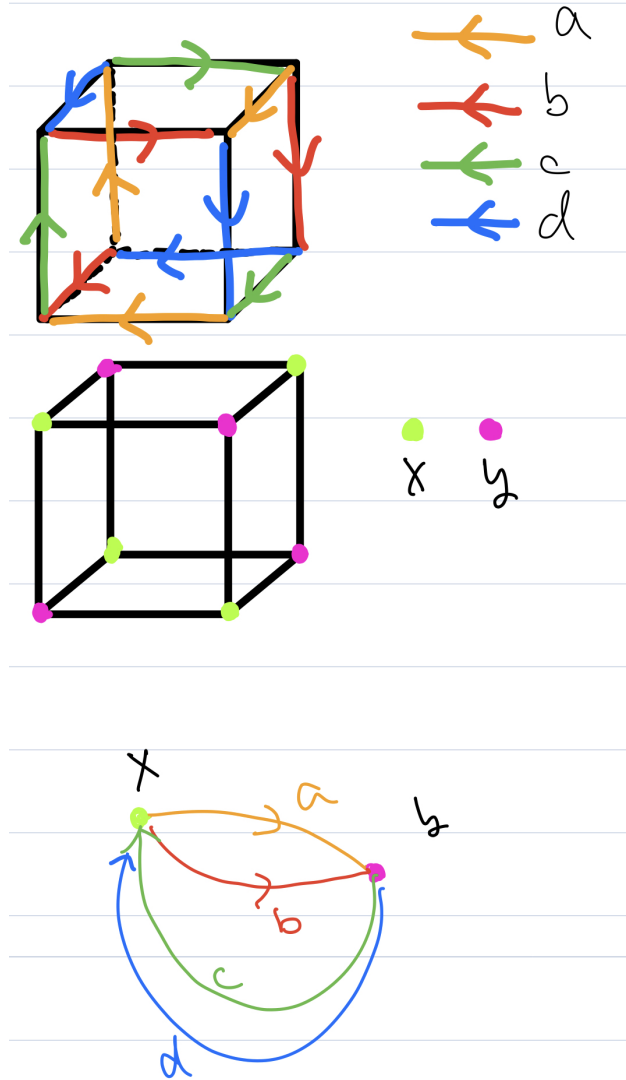


FIGURE 2. Problem 14

This is isomorphic to the group given in the textbook,  $H = \langle a, b, c \mid aba^{-1}b^{-1}cbc^{-1} \rangle$  by  $\phi: G \rightarrow H$  that maps  $a$  to  $b$ ,  $b$  to  $c$ , and  $c$  to  $a^{-1}$ .  $\square$

**Exercise.** (Problem 14, Chapter 1.2) Consider the quotient space of a cube  $I^3$  obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space  $X$  is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that  $\pi_1(X)$  is the quaternion group  $\{\pm 1, \pm i, \pm j, \pm k\}$  of order eight.

*Proof.* The vertices and edges get identified as in Figure 2. Thus we have two 0-cells and four 1-cells. Since the opposite faces are identified and the cube has 6 faces, we need to glue three 2-cells to the cube. Lastly, we need a 3-cell glued to the three faces. By Proposition 1.26, the fundamental group of a 2-skeleton is the same as the fundamental group of a space obtained by attaching 3-cells, so it suffices to consider the fundamental group we obtain by

attaching the three 2-cells to the graph. As in Figure 2, the graph has 4 edges between two vertices. The fundamental group of this is  $\langle ab^{-1}, ac, ad \rangle$  because by “shrinking”  $a$  we obtain the graph consisting of one vertex and three loops. By attaching a 2-cell to each of the top-bottom pair, left-right pair, and the front-back pair, we obtain

$$\langle ac, ab^{-1}, ad \mid ab^{-1}d^{-1}c, adc^{-1}b^{-1}, acbd \rangle.$$

Thus this is the fundamental group of the given space. We claim that  $(ac)^2 = (ab^{-1})^2 = (ad)^2 = (ac)(ab^{-1})(ad)$ .

- $(ac)^2 = (ab^{-1})^2$ ?

$$\begin{aligned} ac = d^{-1}b^{-1} &\implies ab^{-1}bc = d^{-1}b^{-1} \\ &\implies ab^{-1}ad = d^{-1}b^{-1} \\ &\implies ab^{-1}a = d^{-1}b^{-1}d^{-1} \\ &\implies ab^{-1}ab^{-1} = d^{-1}b^{-1}d^{-1}b^{-1} \\ &\implies (ab^{-1})^2 = (d^{-1}b^{-1})^2 \\ &\implies (ab^{-1})^2 = (ac)^2. \end{aligned}$$

- $(ac)^2 = (ad)^2$ ?

$$\begin{aligned} ab^{-1} = c^{-1}d &\implies cab^{-1} = d \\ &\implies ca = db \\ &\implies cac = dbc \\ &\implies cac = dad \\ &\implies acac = adad \\ &\implies (ac)^2 = (ad)^2. \end{aligned}$$

- $(ad)^2 = (ac)(ab^{-1})(ad)$ ?  $(ac)(ab^{-1}) = acc^{-1}d = ad$ , so  $(ac)(ab^{-1})(ad) = (ad)^2$ .

Moreover, we claim that  $(ac)^2 \neq e$  and  $(ac)^4 = e$ .

- $(ac)^2 \neq e$ .

Prove this!

- $(ac)^4 = e$ .

Prove this!

□

**Exercise.** (Problem 22, Chapter 1.2)

- Show that  $\pi_1(\mathbb{R}^3 - K)$  has a presentation with one generator  $x_i$  for each strip  $R_i$  and one relation of the form  $x_i x_j x_i^{-1} = x_k$  for each square  $S_l$ , where the indices are as in the figures above.

*Proof.*

- We will construct the 2-dimensional complex  $X$  by first attaching  $R_i$ 's. We will attach  $R_i$  one by one. We begin with a plane  $\mathbb{R}^2$  whose fundamental group is 0. A rectangular strip  $R_i$  has a fundamental group isomorphic to  $\mathbb{Z}$  since it is homotopy

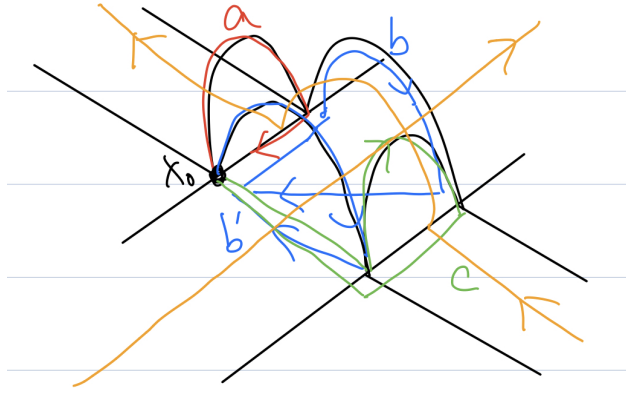


FIGURE 3. Wirtinger presentation

equivalent to  $S^1$ . Thus it is a free group with one generator. We will calculate the fundamental group of a space we obtain after attaching  $T$  to  $R_i$  using Van Kampen. The intersection is a rectangle, so the intersection is simply connected. Thus the fundamental group of the new space is simply the free product of  $T$  and  $R_i$ . Therefore, the fundamental group of the space we obtain by attaching all the  $R_i$ 's is  $\langle x_1, \dots, x_n \rangle$  where  $n$  is the number of  $R_i$ 's and each  $x_i$  corresponds to  $R_i$ .

Now, we will attach  $S_l$ 's and we will do so one by one. The fundamental group of each  $S_l$  is 0 since each  $S_l$  is simply connected. Thus attaching  $S_l$ 's does not add any new generators to the fundamental group. Figure 3 shows the intersection between an  $S_l$  and the current space  $X$ .  $a, b, b', c$  denote loops based at  $x_0$ , and  $[b] = [b']$ . Moreover,  $[a], [b], [c]$  are exactly the generator of the corresponding rectangular strip. We will consider the intersection between  $S_l$  and  $X$ .

- The loop that goes through the intersection is path homotopic to  $abc^{-1}b^{-1}$  in  $X$ .
- The loop that goes through the intersection is nullhomotopic in  $S_l$  since  $S_l$  is simply connected.

(By using the right-hand rule to determine the direction of the generator, the sign of the power of each generator will be consistent.) By Van Kampen, the new group is  $\pi_1(X) * \pi_1(S_l) / (i_X(g)i_{S_l}(g)^{-1})$  where  $g$  is any loop in the intersection. Since  $\pi_1(S_l) = 0$ ,  $i_{S_l}(g) = e$  for any  $g$ . Then  $(i_X(g)) = [abc^{-1}b^{-1}]$  since the intersection is homeomorphic to  $S^1$  and  $[abc^{-1}b^{-1}]$  is a generator. Since  $\pi_1(S_l) = 0$ , we have  $\pi_1(X) / ([a][b][c^{-1}][b^{-1}])$ .

After attaching all the  $S_l$ 's we will end up with  $\langle x_1, \dots, x_n \mid [a_l][b_l][c_l^{-1}][b_l^{-1}] \rangle$  where

- For each  $S_l$ , we add a relation  $[a_l][b_l][c_l^{-1}][b_l^{-1}]$ . Note that this means  $[a_l][b_l][c_l^{-1}][b_l^{-1}] = e$ , so  $[a_l] = [b_l][c_l][b_l^{-1}]$ , and this is exactly the desired relation.
- Each  $x_i$  corresponds to a rectangular strip  $R_i$ . These are the only generators because  $S_l$ 's are all simply connected.
- The abelianization of  $\pi_1(\mathbb{R}^3 - K)$  turns a relation  $x_i x_j x_i^{-1} = x_k$  into  $x_j = x_k$ . In other words, this implies that, at each square  $S_l$ , the generators for the two strips that are “separated” by the middle strip are identified. Let  $x_i, x_j$  be two distinct generators. Since  $K$  is a knot, there exists a finite sequence  $x_i = x_{i_0}, \dots, x_{i_k} = x_j$  of generators such that the corresponding strips  $R_{i_0}, \dots, R_{i_k}$  are next to each other. (See Figure

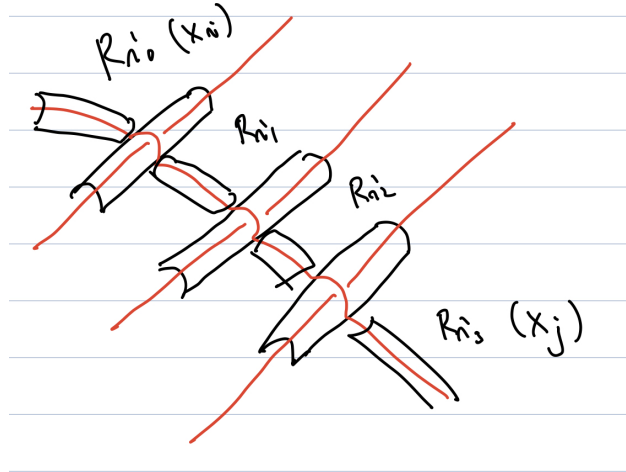


FIGURE 4. Problem 22 (b)

4) Since each intersection has a square,  $x_{i_t} = x_{i_{t+1}}$  for each  $t$ . (For instance, in Figure 4,  $x_{i_0} = x_{i_1}$  because of the intersection between  $R_{i_0}$  and  $R_{i_1}$ . Similarly,  $x_{i_1} = x_{i_2}$  and  $x_{i_2} = x_{i_3}$ .) Therefore,  $x_i = x_{i_0} = x_{i_1} = \cdots = x_{i_k} = x_j$ .

This implies that any two generators are identified after the abelianization. Hence,  $\pi_1(\mathbb{R}^3 - K)$  is a free group with one generator and no relations, so it is isomorphic to  $(\mathbb{Z}, +)$ .

□