MATH 612 (HOMEWORK 3)

HIDENORI SHINOHARA

Exercise. (3.1.11) Using the cellular homology, we obtain

$$\tilde{H}_i(X) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & (i = n) \\ 0 & (i \neq n). \end{cases}$$

$$\tilde{H}^i(X) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & (i = n+1) \\ 0 & (i \neq n+1). \end{cases}$$

From previous homework,

$$\tilde{H}^i(X/S^n) = \tilde{H}_i(S^{n+1}) = \begin{cases} \mathbb{Z} & (i = n+1) \\ 0 & (i \neq n+1). \end{cases}$$

Thus the map on $\tilde{H}_i(-;\mathbb{Z})$ is the zero map for each i. On the other hand, the long exact sequence of a pair gives us $\tilde{H}^{n+1}(X, S^n;\mathbb{Z}) \xrightarrow{q^*} \tilde{H}^{n+1}(X;\mathbb{Z}) \to \tilde{H}^{n+1}(S^n;\mathbb{Z})$ where $\tilde{H}^{n+1}(S^n;\mathbb{Z}) = 0$, so q^* is surjective. Therefore, it is nontrivial because $\tilde{H}^{n+1}(X;\mathbb{Z}) \neq 0$.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_n(X);\mathbb{Z}) \arrow[r] \arrow[d] & H^{n+1}(X;\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{Hom}(H_{n+1}(X);\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_n(X/S^n);\mathbb{Z}) \arrow[r] & H^{n+1}(X/S^n;\mathbb{Z}) \arrow[r] & \mathrm{Hom}(H_{n+1}(X/S^n);\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

is

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_m \arrow[r] \arrow[d] & \mathbb{Z}_m \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

This splitting is not natural because the middle term in the first sequence is isomorphic to $\mathbb{Z}_m \oplus 0$ and the second one is $0 \oplus \mathbb{Z}$.

The long exact sequence of a pair gives us $\tilde{H}_n(S^n;\mathbb{Z}) \to \tilde{H}_n(X;\mathbb{Z}) \to \tilde{H}_n(X, S^n;\mathbb{Z}) = \tilde{H}_n(S^{n+1};\mathbb{Z}) = 0$ which implies the surjectivity of the induced map. Since $\tilde{H}_n(X;\mathbb{Z}) \neq 0$, the induced map is nonzero.

The map induced on $\tilde{H}^i(-;\mathbb{Z})$ is the zero map for any i because at least one of $\tilde{H}^i(S^n;\mathbb{Z})$ or $\tilde{H}^i(X;\mathbb{Z})$ is 0 for each i.

Exercise. (3.1.13) Let $\Phi : \langle X, Y\rangle \to \mathrm{Hom}(H_1(X), H_1(Y))$ denote the map in the problem statement.

- Φ is well-defined because homotopy equivalent maps induce the same homomorphisms on homology classes.

- Let $f, g \in \langle X, Y\rangle$ be given such that $f_* = g_*$. Let $q : \pi_1(X) \to H_1(X)$ be the canonical quotient map as $H_1(X)$ is the abelianization of $\pi_1(X)$. Since $\pi_1(Y) = G$ is abelian, $\pi_1(Y) = H_1(Y)$. This implies that $f_* \circ q, g_* \circ q$ are both homomorphisms from $\pi_1(X)$ to $\pi_1(Y)$. By Proposition 1B.9, such homomorphisms must be induced by a map $(X, x_0) \to (Y, y_0)$ that is unique up to homotopy fixing the base point. In other words, $f = g$ in $\langle X, Y\rangle$.
- For any $\phi \in \mathrm{Hom}(H_1(X), H_1(Y))$, we obtain $\phi\circ q \in \mathrm{Hom}(\pi_1(X), \pi_1(Y))$. By Proposition 1B.9, there exists a map $f \in \langle X, Y\rangle$ that induces $\phi\circ q$. Then $f_* : H_1(X) \to H_1(Y)$ equals ϕ since each equivalence class in H_1 and π_1 denotes a path in the corresponding space and the induced map by f simply maps a path into another path in the other space while respecting the equivalence class the path is in.

Exercise. (3.2.1) $H^0(M_g) = H^2(M_g) = \mathbb{Z}$ and $H^1(M_g) = \mathbb{Z}^{2g}$. Thus the only nontrivial cup products are elements among $H^1(M_g)$. Let $a_1, \cdots, a_g, b_1, \cdots, b_g$ be generators of $H^1(M_g)$. Let q be the quotient map $M_g \to \vee_g M_1$. Then $q^* : H^1(\vee_g M_1) \to H^1(M_g)$. Since $H^1(\vee_g M_1) = \oplus_g H^1(M_1)$, let A_i, B_i denote generators of the ith $H^1(M_1)$ such that $q^*(A_i) = a_i$ and $q^*(B_i) = b_i$. $H^2(\vee_g M_1) = \oplus_g H^2(M_1)$, and let c_i denote a generator of the ith $H^2(M_1)$ such that $\{C_1, \cdots, C_g\}$ generate $H^2(M_g)$ and $q^*(C_i) = c_i$. Since cup products are natural, they commute with q^*.

- $a_i \smile a_i = q^*(A_i) \smile q^*(A_i) = q^*(A_i \smile A_i) = q^*(0) = 0$.
- $b_i \smile b_i = q^*(B_i) \smile q^*(B_i) = q^*(B_i \smile B_i) = q^*(0) = 0$.
- $a_i \smile b_i = q^*(A_i) \smile q^*(B_i) = q^*(A_i \smile B_i) = q^*(C_i) = c_i$.
- All other cases are 0 because the cup product of elements from different "components" when dealing with a wedge sum of spaces is 0 as discussed in class.

Exercise. (3.2.2) Suppose X is the union of contractible open sets $A_1, \cdots, A_n$. Since each A_i is contractible, $H^k(X, A_i; R) = H^k(X; R)$ for all $k \geq 1$.

$$\begin{array}{ccc} H^{k_1}(X, A_1; R) \times \cdots \times H^{k_n}(X, A_n; R) & \longrightarrow & H^{k_1+\cdots+k_n}(X, A_1 \cup \cdots \cup A_n; R) \\ \downarrow\cong & & \downarrow \\ H^{k_1}(X; R) \times \cdots \times H^{k_n}(X; R) & \xrightarrow{\quad f \quad} & H^{k_1+\cdots+k_n}(X; R). \end{array}$$

This diagram commutes by the naturality of a cup product. $H^{k_1+\cdots+k_n}(X, \bigcup_i A_i; R) = H^{k_1+\cdots+k_n}(X, X; R) = 0$ for all $k + l \geq 1$. By the commutativity of this diagram, the function f must be 0.

Exercise. (3.2.3(a)) Suppose otherwise. Let $f : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m$ be such a function. Then f induces a map on $f^* : H^*(\mathbb{R}\mathrm{P}^m) \to H^*(\mathbb{R}\mathrm{P}^n)$. In other words, $f^* : \mathbb{Z}_m[\alpha]/(\alpha^{m+1}) \to \mathbb{Z}_n[\beta]/(\beta^{n+1})$ where α, β are generators of $H^1(\mathbb{R}\mathrm{P}^m)$ and $H^1(\mathbb{R}\mathrm{P}^n)$. $H^1(\mathbb{R}\mathrm{P}^m; \mathbb{Z}_2) = \mathbb{Z}_2 = \{0, \alpha\}$ and $H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) = \mathbb{Z}_2 = \{0, \beta\}$. Since f induces a nontrivial map $H^1(\mathbb{R}\mathrm{P}^m; \mathbb{Z}_2) \to H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$, $f^*(\alpha) = \beta$. However, $f^*(0) = f^*(\alpha^{m+1}) = (f^*(\alpha))^{m+1} = \beta^{m+1} \neq 0$ because $m < n$. This is a contradiction, so such a function does not exist.

$H^1(\mathbb{C}P^n; \mathbb{Z}_2) = 0$ for any n, so there exists no such nontrivial map. The case for $H^2(\mathbb{C}P^n)$ can be argued the same way as above because $H^2(\mathbb{C}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ where α is a generator of $H^2(\mathbb{C}P^n)$.

Exercise. (3.2.3(b)) Suppose $n \geq 2$ because if $n = 1$, then this can be shown using the intermediate value theorem.

$$\begin{array}{ccc} S^n & \xrightarrow{g} & S^{n-1} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \mathbb{R}\mathrm{P}^n & \xrightarrow{g} & \mathbb{R}\mathrm{P}^{n-1}. \end{array}$$

Let p denote covering maps. Let γ be a nontrivial loop in $\mathbb{R}\mathrm{P}^n$. Let $a, -a$ denote the end points of the lift $\tilde{\gamma}$. $g(-a) = -g(a)$, so g sends $\tilde{\gamma}$ to a path from $g(a)$ to $g(-a)$. Finally, p pushes it down to a nontrivial loop in $\mathbb{R}\mathrm{P}^{n-1}$. By the commutativity of the diagram, $g(\gamma)$ is a nontrivial path in $\mathbb{R}\mathrm{P}^{n-1}$. Therefore, f induces a nontrivial map from $\pi_1(\mathbb{R}\mathrm{P}^n)(\cong \mathbb{Z}_2)$ to $\pi_1(\mathbb{R}\mathrm{P}^{n-1})(\cong \mathbb{Z}_2)$. Thus f induces an isomorphism. Since the fundamental groups are abelian, the fundamental groups are isomorphic to the first homology groups. By the UCT, $H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) = \mathrm{Hom}(H_1(\mathbb{R}\mathrm{P}^n),\mathbb{Z}_2) = \mathbb{Z}_2$ and $H^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{Z}_2) = \mathrm{Hom}(H_1(\mathbb{R}\mathrm{P}^{n-1}),\mathbb{Z}_2) = \mathbb{Z}_2$. Then f induces an isomorphism from $H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2)$ into $H^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{Z}_2)$. This is a contradiction as shown in 3(a).

Exercise. (3.2.6) For simplicity, we will abuse a notation and let g be the quotient of the map $(z_0,\cdots,z_n) \mapsto (z_0^d,\cdots,z_n^d)$ for any n. We will first consider the case when $n=1$. Then $\mathbb{C}P^1$ is homeomorphic to S^2, so $g^* : H^2(\mathbb{C}P^1;\mathbb{Z}) \to H^2(\mathbb{C}P^1;\mathbb{Z})$ is simply multiplication by d since $H^2(\mathbb{C}P^1;\mathbb{Z}) = \mathbb{Z}$. Consider the inclusion $i : \mathbb{C}P^1 \to \mathbb{C}P^n$. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} H^2(\mathbb{C}P^1;\mathbb{Z}) & \xleftarrow[i^*]{} & H^2(\mathbb{C}P^n;\mathbb{Z}) \\ {\scriptstyle g^*=(\cdot d)}\uparrow & & {\scriptstyle g^*}\uparrow \\ H^2(\mathbb{C}P^1;\mathbb{Z}) & \xleftarrow[i^*]{} & H^2(\mathbb{C}P^n;\mathbb{Z}). \end{array}$$

Let α, β denote generators of $H^2(\mathbb{C}P^1;\mathbb{Z}), H^2(\mathbb{C}P^n;\mathbb{Z})$. Then $i^*(\beta) = \alpha$. Since the diagram commutes, this shows that $g^*(\beta) = d\beta$. Therefore, $g^*(\beta^k) = (g^*(\beta))^k = (d\beta)^k = d^k\beta^k$ for any $\beta^k \in H^*(\mathbb{C}P^n;\mathbb{Z})$.

Exercise. (3.2.7) Let $f : \mathbb{R}\mathrm{P}^3 \to \mathbb{R}\mathrm{P}^2 \vee S^3$ be a homotopy equivalence. Then it induces isomorphisms.

$$\begin{array}{ccccc} H^1(\mathbb{R}\mathrm{P}^3;\mathbb{Z}_2) & \times & H^2(\mathbb{R}\mathrm{P}^3;\mathbb{Z}_2) & \longrightarrow & H^3(\mathbb{R}\mathrm{P}^3;\mathbb{Z}_2) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ H^1(\mathbb{R}\mathrm{P}^2 \vee S^3;\mathbb{Z}_2) & \times & H^2(\mathbb{R}\mathrm{P}^2 \vee S^3;\mathbb{Z}_2) & \longrightarrow & H^3(\mathbb{R}\mathrm{P}^2 \vee S^3;\mathbb{Z}_2). \end{array}$$

The cohomology groups of a wedge sum is the direct sum of cohomology groups of the two spaces. By rewriting the diagram above with generators, we obtain

$$\begin{array}{ccccc} \{0,\alpha\} & \times & \{0,\alpha^2\} & \longrightarrow & \{0,\alpha^3\} \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \{0,\beta\} \oplus \{0,\gamma\} & \times & \{0,\beta^2\} \oplus 0 & \longrightarrow & 0 \oplus \{0,\gamma^2\}. \end{array}$$

This implies f^* sends α^2 to $(\beta^2, 0)$ and α^3 to $(0,\gamma^2)$. However, this implies $(0,0) = (f^*(\alpha^2))^3 = (f^*(\alpha^3))^2 = (0,\gamma^4) = (0,\gamma)$. This is a contradiction because $0 \neq \gamma$.