MATH 612(HOMEWORK 4)

HIDENORI SHINOHARA

Exercise. (8) By using cellular cohomology, we obtain

$$H^i(X;\mathbb{Z}) = H^i(Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & (i = 0, 4), \\ \mathbb{Z}_p & (i = 3), \end{cases}$$

$$H^i(X;\mathbb{Z}_p) = H^i(Y;\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & (i = 0, 2, 3, 4), \end{cases}$$

Therefore, we cannot distinguish X from Y by looking at the cohomology groups. When using the coefficient $\mathbb{Z}$, cup products are simply 0 because nontrivial cohomology groups are of order 3 and 4. Thus we cannot distinguish X from Y by looking at the cohomology rings of X and Y. Since $H^i(Y;\mathbb{Z}_p) = H^i(S^4;\mathbb{Z}_p) \oplus H^i(M(\mathbb{Z}_p, 2);\mathbb{Z}_p)$ and the cup product of elements from different "components" in a wedge sum is 0, cup products in $H^*(Y;\mathbb{Z}_p)$ are all 0. On the other hand, the cup product $\alpha \smile \alpha$ where α is a generator of $H^2(\mathbb{C}P^2;\mathbb{Z}_p)$ is nontrivial because $\alpha \smile \alpha$ is a generator of $H^4(\mathbb{C}P^2;\mathbb{Z}_p)$.

Exercise. (5) Consider the canonical map $\mathbb{Z}_{2k} \to \mathbb{Z}_2$. It induces a chain map between the cellular chain complexes of $\mathbb{R}\mathrm{P}^\infty$ over $\mathbb{Z}_{2k}$ and $\mathbb{Z}_2$. Moreover, they induce homomorphisms $\phi : H^i(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \to H^i(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k})$. By cellular cohomology, $H^0(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k}) = \mathbb{Z}_{2k}$ and $H^i(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k}) = \mathbb{Z}_2$ for $i \geq 1$. Let γ denote a generator of $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$. Then $\phi(\gamma)$ must be a generator of $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k})$ because ϕ is induced by the map $1 \mapsto 1$. Let $\alpha = \phi(\gamma)$. $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k}) = \mathbb{Z}_2$, so we obtain the relation 2α.

Let β be a generator of $H^2(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k})$. Since $H^2(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k}) = \mathbb{Z}_2$, we obtain the relation 2β.

How do I obtain the relation $\alpha^2 - k\beta$? More specifically, is $\phi : H^2 \to H^2$ an isomorphism or the zero map?