MATH 601 (DUE 9/25)

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Exercise. (Problem 1) Define $\gamma : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[\sqrt{2}]$ by $\gamma(a + b\sqrt{2}) = a - b\sqrt{2}$. Show that γ is a ring isomorphism and compute its inverse.

Proof. Let $a + b\sqrt{2}, c + d\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ be given.

$$\begin{aligned}
\gamma((a + b\sqrt{2}) + (c + d\sqrt{2})) &= \gamma((a + c) + (b + d)\sqrt{2}) \\
&= (a + c) - (b + d)\sqrt{2} \\
&= (a - b\sqrt{2}) + (c - d\sqrt{2}) \\
&= \gamma(a + b\sqrt{2}) + \gamma(c + d\sqrt{2}). \\
\gamma((a + b\sqrt{2})(c + d\sqrt{2})) &= \gamma((ac + 2bd) + (ad + bc)\sqrt{2}) \\
&= (ac + 2bd) - (ad + bc)\sqrt{2} \\
&= (ac + 2(-b)(-d)) + (a(-d) + (-b)c)\sqrt{2} \\
&= (a - b\sqrt{2})(c - d\sqrt{2}) \\
&= \gamma(a + b\sqrt{2})\gamma(c + d\sqrt{2}).
\end{aligned}$$

Moreover, $\gamma(1) = 1 - 0\sqrt{2} = 1$. Therefore, γ is a ring homomorphism. For any $a + b\sqrt{2}$, $\gamma(\gamma(a + b\sqrt{2})) = \gamma(a - b\sqrt{2}) = a + b\sqrt{2}$. Therefore, γ has an inverse, and the inverse of γ is γ. This implies that γ is bijective.

In conclusion, γ is an isomorphism and its inverse is itself. $\square$