MATH 611 (DUE 10/23)

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1. Simplicial and Singular Homology

Exercise. (Problem 2) Show that the Δ-complex obtained from Δ^3 by performing the edge identifications $[v_0, v_1] \sim [v_1, v_3]$ and $[v_0, v_2] \sim [v_2, v_3]$ deformation retracts onto a Klein bottle. Find other pairs of identifications of edges that produce Δ-complexes deformation retracting onto a torus, a 2-sphere, and $\mathbb{R}\mathbf{P}^2$.

Proof. Maybe something like this? Either way, I noticed that it looks like it contains 2 $\mathbb{R}\mathbf{P}^2$.

□

Exercise. (Problem 4) Compute the simplicial homology groups of the triangular parachute obtained from Δ^2 by identifying its three vertices to a single point.

Proof. Let v_0 denote the only vertex, e_1, e_2, e_3 denote the three edges of the parachute, and σ denote the face of the parachute. $C_k = 0$ for $k \geq 3$ because Δ^2 with the vertices identified does not contain any k-dimensional simplicies. $C_2 = \langle \sigma \rangle, C_1 = \langle e_1, e_2, e_3 \rangle, C_0 = \langle v_0 \rangle$. Let $n \in \mathbb{N}$. ∂_n is defined such that $\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha \mid [v_0, \cdots, \hat{v}_i, \cdots, v_n]$. Since there is only one vertex, ∂_n is the zero map.

This argument doesn't work. Check the torus example from class. It only has one vertex, but ∂_n is not the zero map for some n.

Therefore, $H_n = \ker(\partial_n) / \operatorname{Im}(\partial_{n+1}) = C_n / \langle 0 \rangle = C_n$. Thus

$$H_n = \begin{cases} \{0\} & (n \geq 3) \\ \langle \sigma \rangle \cong \mathbb{Z} & (n = 2) \\ \langle e_1, e_2, e_3 \rangle \cong \mathbb{Z}^3 & (n = 1) \\ \langle v_0 \rangle \cong \mathbb{Z} & (n = 0). \end{cases}$$

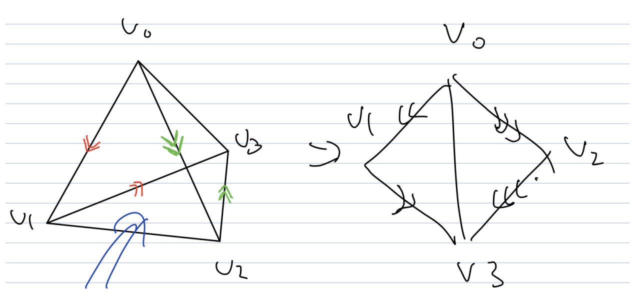

Figure 1. mycaption

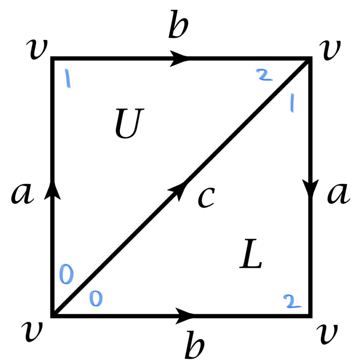

FIGURE 2. Problem 5

I'm not sure if this is correct.

□

Exercise. (Problem 5) Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure described at the beginning of this section.

Proof. We will use the notations in Figure 2.

$$C_n = \begin{cases} 0 & (n \geq 3) \\ \langle U, L \rangle & (n = 2) \\ \langle a, b, c \rangle & (n = 1) \\ \langle v \rangle & (n = 0). \end{cases}$$

$\partial_n = 0$ for $n \geq 3$ and $n = 0$.

$$\begin{aligned} \partial_2(U) &= \sum_{i=0}^{2} (-1)^i \sigma \mid [0,1,2] \\ &= \sigma \mid [1,2] - \sigma \mid [0,2] + \sigma \mid [0,1] \\ &= b - c + a. \end{aligned}$$

$$\begin{aligned} \partial_2(L) &= \sum_{i=0}^{2} (-1)^i \sigma \mid [0,1,2] \\ &= \sigma \mid [1,2] - \sigma \mid [0,2] + \sigma \mid [0,1] \\ &= a - b + c. \end{aligned}$$

$\partial_1(a) = 0$ since $\partial_1(a) = \sigma|[1] - \sigma|[0] = v - v = 0$. Similarly, $\partial_1(b) = \partial_1(c) = 0$. □