MATH 633(HOMEWORK 2)

HIDENORI SHINOHARA

Exercise. (Problem 2a) Let $\gamma(t) = Re^{2\pi it}$.

$$\int_\gamma z^n dz = \int_0^1 R^n e^{2\pi int} R 2\pi i e^{2\pi it} dt$$

$$= 2\pi i R^{n+1} \int_0^1 e^{2\pi i(n+1)t} dt$$

$$= \begin{cases} R^{n+1} \frac{e^{2\pi i(n+1)t}}{n+1} = 0 & (n \neq -1) \\ 2\pi i & (n = -1). \end{cases}$$

Exercise. (Problem 2b) Let $\gamma(t) = z_0 + Re^{2\pi it}$ where $|R/z_0| < 1$.

$$\int_\gamma z^n dz = \int_0^1 (z_0 + Re^{2\pi it})^n (z_0 + Re^{e\pi it})' dt$$

When $n \neq -1$, $(z_0 + Re^{2\pi it})^{n+1}/(n+1)$ is a primitive, so the integral is 0. Suppose $n = -1$.

$$\int_0^1 \frac{2\pi i Re^{2\pi it}}{z_0 + Re^{2\pi it}} dt = \int_0^1 \frac{2\pi i Re^{2\pi it}/z_0}{1 + Re^{2\pi it}/z_0} dt$$

$$= \int_0^1 \frac{2\pi i Re^{2\pi it}}{z_0} \cdot \sum_{k=0}^\infty (\frac{-Re^{2\pi it}}{z_0})^k dt$$

$$= -2\pi i \sum_{k=0}^\infty \int_0^1 (\frac{-Re^{2\pi it}}{z_0})^{k+1} dt$$

$$= -2\pi i \sum_{k=0}^\infty (\frac{-Re^{2\pi it}}{z_0})^{k+1} \int_0^1 e^{2\pi it(k+1)} dt$$

$$= 0.$$

Each $\int_0^1 e^{2\pi it(k+1)} dt = 0$ because $e^{2\pi it(k+1)}/(2\pi it(k+1))$ is a primitive.

Exercise. (Problem 3)

$$\int_a^b |z'(t)| dt = \int_c^d |z'(t(s))| t'(s) ds$$

$$= \int_c^d |z'(t(s)) t'(s)| ds$$

$$= \int_c^d |\tilde{z}'(s)| ds$$

1

where $\tilde{z}(s) : [c, d] \to \mathbb{C}$ is a reparametrization of $z(t) : [a, b] \to \mathbb{C}$.

Exercise. (Problem 4a) If $t^\star \in \Omega_1$, then there exists an open neighborhood U of $z(t^\star)$ contained in Ω_1. Then $z^{-1}(U)$ is a neighborhood of $t^\star$ in $[0, 1]$ because z is continuous. Since $z(1) \in \Omega_2$, $t^\star \neq 1$. However, this implies the existence of $\epsilon > 0$ such that $t^\star + \epsilon < 1$ and $z(t^\star + \epsilon) \in \Omega_1$. This is a contradiction.

If $t^\star \in \Omega_2$, then there exists an open neighborhood U of $z(t^\star)$ contained in Ω_2. Since U is open, $z^{-1}(U)$ is a neighborhood of $t^\star$ in $[0, 1]$, so $\exists \epsilon > 0$ such that $z(t^\star - \epsilon) \in \Omega_2$.

In each case, we reached a contradiction, so Ω is not disconnected.

Exercise. (Problem 4b) For every $v \in \Omega_1$, there exists an open set U such that $v \in U \subset \Omega_1$. Then for any $v' \in U$, v and v' can be joined by $\gamma(t) = tv + (1 - t)v'$. Thus $U \subset \Omega_1$, so Ω_1 is open.

Let $v \in \Omega_2$. Suppose that for all $\epsilon > 0$, the open disk at v with the radius ϵ is not contained in Ω_2. Otherwise we are done. Let $v_0 = w$. For every $n \in \mathbb{N}$, choose $v_n \in D(v, 1/n) \setminus \Omega_2$. Then there exists a path between each v_n and w. Moreover, there exists a path between v_n and v_{n-1} for each n and we will call it γ_n. Define $\gamma : [0, 1] \to \Omega$ such that for each $n \in \mathbb{N}$, $\gamma([1 - 1/n, 1 - 1/(n + 1)])$ is the path γ_n and $\gamma(1) = v$. Then γ is a well-defined path from w to v, which is a contradiction because $v \in \Omega_2$.

Clearly, $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1 \cup \Omega_2 = \Omega$. Since $w \in \Omega_1$, $\Omega_1 \neq \emptyset$, so $\Omega_2 = \emptyset$.