

# MATH 601 (DUE 12/6)

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### 1. GALOIS THEORY VI

**Exercise.** (Problem 1) Let  $u_1, u_2, u_3, u_4$  be the variables of the elementary symmetric polynomials  $s_1, s_2, s_3, s_4$ . Then  $f(x) = (x - u_1)(x - u_2)(x - u_3)(x - u_4)$ . For any permutation  $\sigma \in S_4$ ,  $\phi \in \text{Aut}(F(u_1, \dots, u_n))$  determined by  $\phi(u_i) = u_{\sigma_i}$  is an automorphism that fixes  $F$  because every elementary symmetric polynomial  $s_i$  is symmetric. Therefore, the Galois group is isomorphic to  $S_4$ .

The roots of  $f(x)$  are expressible by radicals relative to  $F$  because, as shown in Problem 3 below,  $S_4$  is solvable.

**Exercise.** (Problem 2)  $f(x) = x^6 - 2$  is irreducible over  $\mathbb{Q}$  by Eisenstein ( $p = 2$ ). The roots are  $\{\zeta^i \sqrt[6]{2} \mid i = 0, \dots, 5\}$  where  $\zeta = e^{2\pi i/6} = (1 + \sqrt{-3})/2$ . Then the splitting field  $L$  is  $\mathbb{Q}(\zeta^0 \sqrt[6]{2}, \dots, \zeta^5 \sqrt[6]{2}) = \mathbb{Q}(\zeta, \sqrt[6]{2})$ . Let  $\sigma \in \text{Aut}(L/\mathbb{Q})$ . The minimal polynomial of  $\sqrt[6]{2}$  is  $x^6 - 2$ , so  $\sigma(\sqrt[6]{2}) = \zeta^i \sqrt[6]{2}$  for some  $i$ . The minimal polynomial of  $\zeta$  is  $x^2 - x + 1$ , so  $\sigma(\zeta) = \zeta, \bar{\zeta}$ . Thus there are  $6 \cdot 2 = 12$  automorphisms. This is isomorphic to  $D_6$  because  $\sqrt[6]{2} \mapsto \zeta \sqrt[6]{2}$  corresponds to rotation and  $\zeta \mapsto \bar{\zeta}$  corresponds to reflection.

**Exercise.** (Problem 3) As discussed in the Galois Theory IV handout, the only transitive subgroups of  $S_4$  are  $S_4, A_4, V_4, C_4$ , and groups with 8 elements. Clearly,  $V_4, C_4$  are solvable. We showed below (Problem 2 from the Cauchy handout) that every  $p$ -group is solvable. Thus any group with 8 elements is solvable. The handout mentions  $V_4 S_4$ , so clearly  $V_4 \trianglelefteq A_4$ .

Moreover,  $A_4/V_4$  has only 3 elements, so it is abelian. Thus  $\{e\} \subset V_4 \subset A_4 \subset S_4$  is a filtration because  $A_4$  is an index-2 subgroup of  $S_4$ . Therefore, all the transitive subgroups of  $S_4$  are solvable, so all the roots of any quartic polynomial are expressible by radicals.

### 2. CAUCHY'S THEOREM, FINITE $p$ -GROUPS, THE SYLOW THEOREMS

**Exercise.** (Problem 2) Let a prime number  $p$  be given. We will show that any group  $G$  of order  $p^n$  for some  $n$  is solvable by induction on  $n$ . When  $n = 1$ ,  $G \cong \mathbb{Z}_p$ , which is abelian, so it is solvable. Suppose we have shown the proposition for some  $n \in \mathbb{N}$ , and let  $G$  be a group of order  $p^{n+1}$ . By Corollary 1 right above this problem statement in the handout, the center  $H$  of  $G$  is a nontrivial subgroup. Moreover,  $H$  is clearly a normal subgroup of  $G$ . Thus it makes sense to consider  $G/H$ . The order of  $G/H$  must be  $p^m$  for some  $1 \leq m \leq n$ . By the inductive hypothesis,  $G/H$  is solvable. Since every subgroup of  $G/H$

can be realized as the quotient of a subgroup of  $G$  by  $H$  [Theorem 20(1), P.99, Dummit and Foote], there must exist a sequence of subgroups  $H = G_0 \leq G_1 \leq \cdots \leq G_l = G$  such that  $G_0/H \trianglelefteq G_1/H \trianglelefteq \cdots \trianglelefteq G_l/H$  and  $(G_{i+1}/H)/(G_i/H)$  is abelian for each  $i$ . By Theorem 19 [P.98, Dummit and Foote],  $(G_{i+1}/H)/(G_i/H) \cong G_{i+1}/G_i$ , so  $G_{i+1}/G_i$  is abelian for each  $i$ .  $G_i/H \trianglelefteq G_{i+1}/H$  implies  $G_i \trianglelefteq G_{i+1}$  for each  $i$  by Theorem 20(5) [P.99, Dummit and Foote].

We showed the existence of a sequence  $H = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_l = G$  such that  $G_{i+1}/G_i$  is abelian for each  $i$ . By the inductive hypothesis, there exists a similar sequence of subgroups from  $\{e\}$  to  $H$ . Therefore,  $G$  is solvable.

**Exercise.** (Problem 3) Let  $m = 3, p = 7$ . Then  $|G| = 21 = pm$  with  $p \nmid m$ . Let  $t$  be the number of Sylow  $p$ -subgroups. By the third Sylow theorem,  $t \mid m$  and  $t \equiv 1 \pmod{p}$ . The only number that satisfies this is 1, so every group of order 21 has a unique Sylow 7-subgroup.

**Exercise.** (Problem 4) Using the same idea as Problem 2 above, we will construct a filtration. Let  $G$  be an extension of  $H$  by  $Q$ . Suppose  $H$  and  $Q$  are both solvable. Since  $Q$  is solvable, there exists a filtration  $\{e\} = Q_0 \trianglelefteq \cdots \trianglelefteq Q_n = Q$ . Let  $\phi$  be an isomorphism from  $Q$  to  $G/H$ . Then the  $\phi(Q_i)$ 's form a filtration of  $G/H$  and  $\phi(Q_i) = G_i/H$  for some subgroup  $G_i$  by the same theorems that we used in Problem 2. Moreover,  $G_i$ 's form a filtration from  $H$  to  $G$ . Since  $H$  is solvable, there exists a filtration from  $\{e\}$  to  $H$ . By concatenating them, we obtain a filtration from  $\{e\}$  to  $G$ , so  $G$  is solvable.

**Exercise.** (Problem 5) By Problem 3,  $G$  has a unique subgroup  $H$  of order 7. Since conjugation preserves the order of a group, the group must be normal. Then  $H \trianglelefteq G$  and  $G/H \cong \mathbb{Z}_3$ . Any group of prime order is abelian and thus solvable. Therefore,  $G$  is an extension of a solvable group  $\mathbb{Z}_7$  by a solvable group  $\mathbb{Z}_3$ , so it must be solvable.

**Exercise.** (Problem 7) Since we are given that  $\mathbb{Q}(\alpha)$  is the splitting field, every root of  $f(x)$  can be expressed by multiplying, adding, dividing and subtracting rational numbers and  $\alpha$ . This implies that  $\sigma \in G = \text{Aut}(\mathbb{Q}(\alpha)/\mathbb{Q})$  is uniquely determined by  $\sigma(\alpha)$ . Therefore,  $|G| \leq 80$ .

The Galois group should have exactly 80 elements by the fundamental theorem of Galois theory.  $80 = 5 \cdot 2^4$ . Use Problem 4.

**Exercise.** (Problem 8)  $A_5$  is a simple non-abelian group, so it is not solvable. [P.3, Galois Theory VI]

$|A_5| = 5!/2 = 60$ . Let  $G = A_5 \times \mathbb{Z}/5\mathbb{Z}$ . Then  $G$  has 300 elements and  $H = \{(x, 0) \in G\}$  is a subgroup of  $G$  that is isomorphic to  $A_5$ . By lemma 1 [P.4, Galois Theory V], a solvable group cannot contain an unsolvable subgroup. Therefore,  $G$  is an unsolvable group of order 300.

**Exercise.** (Problem 9)

- (1) By the third Sylow theorem, the number  $t$  of Sylow  $p$ -subgroups of  $G$  satisfies  $t \mid q$  and  $t \equiv 1 \pmod{p}$ . Thus  $t = 1$ . Thus the subgroup  $H$  of  $G$  with  $p$  elements is normal because conjugation preserves the order of a group.  $G/H$  is a cyclic group of order  $q$ , so let  $x + H$  be a generator. Then every element  $g \in G$  satisfies  $g + H = x^i + H$  for a unique  $i \in \{0, \dots, q-1\}$ . Then the map  $G \rightarrow \mathbb{Z}_q$  such that  $g \mapsto i$  is a surjective group homomorphism. A surjective homomorphism  $G \rightarrow \mathbb{Z}_q$  can be constructed in a similar fashion.

- (2) The problem statement simply says the existence of a homomorphism, which can be achieved by the “zero” map  $g \mapsto e$ . We will instead show the existence of a surjective homomorphism. In (1), we showed the existence of surjective homomorphisms  $\phi_p : G \rightarrow C_p$  and  $\phi_q : G \rightarrow C_q$ . We have trivial homomorphisms  $\psi_p : C_p \times C_q \rightarrow C_p$  and  $\psi_q : C_p \times C_q \rightarrow C_q$  defined by  $\psi_p(a, b) \rightarrow a$  and  $\psi_q(a, b) \rightarrow b$ . By the universal mapping property of the product, there must exist a unique group homomorphism  $\Phi : G \rightarrow C_p \times C_q$  such that  $\phi_p, \phi_q, \psi_p, \psi_q, \Phi$  all commute. Since  $\phi_p = \psi_p \circ \Phi$  and  $\phi_q = \psi_q \circ \Phi$  are both surjective,  $\Phi$  must be surjective.
- (3) Since  $|G| = pq$ ,  $\Phi$  must be bijective, so it is an isomorphism.
- (4) Clearly,  $C_p$  and  $C_q$  are isomorphic to  $\mathbb{Z}/p$  and  $\mathbb{Z}/q$ . Then the map  $(a, b) \mapsto qa + b$  is an isomorphism from  $\mathbb{Z}/p \times \mathbb{Z}/q$  into  $\mathbb{Z}/pq$ .  $\mathbb{Z}/pq$  is isomorphic to  $C_{pq}$ . Therefore,  $G$  is isomorphic to  $C_{pq}$ .

**Exercise.** (Problem 10) By the Corollary 1 indicated in the hint, we obtain a nontrivial center  $C$  of  $G$ . By Lagrange,  $|C| = p, p^2$ . If  $|C| = p^2$ , then  $G$  is abelian, so  $G$  must be isomorphic to  $\mathbb{Z}/(p^2)$  or  $(\mathbb{Z}/p)^2$ . Suppose  $|C| = p$ . Since  $C$  is normal, we will consider  $G/C$ , which is isomorphic to  $\mathbb{Z}/p$ . Let  $x + C$  be a generator of  $G/C$  and  $y$  be a generator of  $C$ . Then every element in  $G$  can be expressed as  $x^i y^j$  for some  $i, j \in \mathbb{Z}/p$ . However, this implies that  $C = G$  because for any  $i, j, k, l$ ,  $(x^i y^j)(x^k y^l) = x^i x^k y^j y^l = x^k x^i y^l y^j = (x^k y^l)(x^i y^j)$  because a power of  $y$  commutes with any element. This is a contradiction, so  $|C| \neq p$ .

**Exercise.** (Problem 11) It suffices to show that every group of order 132 is solvable because it implies that every subgroup of a group of order 132 is solvable. Let  $p = 11, m = 12$  and apply the third Sylow theorem. Then  $t \mid 12$  and  $t \equiv 1 \pmod{p}$  is satisfied only by 1 or 12.

- Suppose  $t = 1$ . Let  $H$  be the subgroup of order 11. Then  $H$  is normal and  $G/H$  is a group of order 12. By Sylow, the number  $t_4$  of subgroups of order 4 of  $G/H$  is either 1 or 3 and the number  $t_3$  of subgroups of order 3 is either 1 or 4. If  $t_4 = 1$  or  $t_3 = 1$ , then we obtain a normal subgroup  $H'$  and  $(G/H)/H'$  has 3 or 4 elements, which mean  $(G/H)$  is solvable. By Problem 4,  $G$  is solvable. Suppose  $t_4 = 3$  and  $t_3 = 4$ . Then we have 3 distinct subgroups  $A_1, A_2, A_3$  of order 4 and 4 distinct subgroups  $B_1, \dots, B_4$  of order 3. By Lagrange, for any  $i \neq j$ ,  $B_i \cap B_j = \{e\}$ , and, for any  $i, j$ ,  $B_i \cap A_j = \{e\}$ . However, this implies that  $G$  contains more than 12 elements. (i.e.,  $B_1 = \{v_1, v_2, v_3\}, B_2 = \{v_1, v_4, v_5\}, \dots, B_4 = \{v_1, v_8, v_9\}$  and  $A_1 = \{v_1, v_{10}, v_{11}, v_{12}\}$ .) Therefore, this case is impossible.
- Suppose  $t = 12$ .

I think this is only possible if  $G$  is abelian. But I don't know how to proceed.