MATH 633

HIDENORI SHINOHARA

1. Homework 4

Exercise. (Problem 1) $|\exp(f)| = \exp(\mathrm{Re}(f))$. Since $\mathrm{Re}(f)$ is bounded above, $\exp(f)$ is bounded. By Liouville's theorem, $\exp(f)$ is constant. Thus f is constant because f is continuous and $\exp(z) = \exp(w)$ if and only if $z - w = 2k\pi i$ for some $k \in \mathbb{Z}$.

Exercise. (Problem 2) Define

$$v(x, y) = \int_0^y \frac{\partial u}{\partial x}(x, t)dt - \int_0^x \frac{\partial u}{\partial y}(t, 0)dt.$$

This gives us:

$$\begin{aligned}
v_x(x, y) &= \int_0^y \frac{\partial^2 u}{\partial x^2}(x, t)dt - \frac{\partial u}{\partial y}(x, 0) \\
&= -\int_0^y \frac{\partial^2 u}{\partial t^2}(x, t)dt - \frac{\partial u}{\partial y}(x, 0) \\
&= -(\frac{\partial u}{\partial y}(x, y) - \frac{\partial u}{\partial y}(x, 0)) - \frac{\partial u}{\partial y}(x, 0) \\
&= -\frac{\partial u}{\partial y}(x, y) \\
&= -u_y(x, y). \\
v_y(x, y) &= \frac{\partial u}{\partial x}(x, y) - \int_0^x \frac{\partial^2 u}{\partial y^2}(t, 0)dt \\
&= \frac{\partial u}{\partial x}(x, y) + \int_0^x \frac{\partial^2 u}{\partial x^2}(t, 0)dt \\
&= \frac{\partial u}{\partial x}(x, y) + \frac{\partial u}{\partial x}(x, 0) - \frac{\partial u}{\partial x}(x, 0) \\
&= \frac{\partial u}{\partial x}(x, y) \\
&= u_x(x, y).
\end{aligned}$$

By Theorem 2.4, $u + iv$ is holomorphic on D. Given two $v_1, v_2 : D \to \mathbb{R}$ satisfying such properties, $(u + v_1 i) - (u + v_2 i)$ is a holomorphic function whose real value is always 0. By the Cauchy-Riemann equation, the derivative of $i(v_1 - v_2)$ must be 0. In other words, $v_1 - v_2$ must be constant.

Exercise. (Problem 3) Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be defined such that

- $\gamma_1(t) = (-1/R)t - R(1 - t)$.
- $\gamma_2(t) = -e^{-\pi i t}/R$.

- $\gamma_3(t) = Rt - (1-t)/R$.
- $\gamma_4(t) = Re^{\pi i t}$.

Then the 4 curves form a piecewise smooth closed contractible curve γ. Since $\exp(iz)/z$ has no singularity inside γ, $\int_\gamma \exp(iz)/z dz = 0$. We will integrate $\exp(iz)/z$ over γ_i for each $i = 1, \cdots, 4$.

- $$\lim \int_{\gamma_1} \frac{e^{iz}}{z} = \lim \int_{-R}^{-1/R} \frac{\cos x + i \sin x}{x} dx.$$

- $$\lim \int_{\gamma_2} \frac{e^{iz}}{z} = \lim [\int \frac{1}{z} + i \int 1 + \frac{iz}{2!} + \frac{(iz)^2}{3!} + \cdots]$$

  $\int_{\gamma_2} \frac{1}{z} = \int_0^1 \frac{\pi i e^{-\pi i t}}{-e^{-\pi i t}} dt = \pi i$. The function $z \mapsto 1 + \frac{iz}{2!} + \frac{(iz)^2}{3!} + \cdots$ is entire, so it is bounded on the unit disk. In other words, there exists a $B > 0$ such that $\forall |z| < 1$, $\left|1 + \frac{iz}{2!} + \frac{(iz)^2}{3!} + \cdots\right| < B$. Then $\left|\int 1 + \frac{iz}{2!} + \frac{(iz)^2}{3!} + \cdots\right| \leq \int \left|1 + \frac{iz}{2!} + \frac{(iz)^2}{3!} + \cdots\right| < \int_{\gamma_2} B$. As $R \to \infty$, $\int_{\gamma_2} B = 0$ as B does not depend on R. Therefore, $\lim \int_{\gamma_2} \frac{e^{iz}}{z} = \lim \int \frac{1}{z} = \pi i$.

- $$\lim \int_{\gamma_3} \frac{e^{iz}}{z} = \lim \int_{1/R}^{R} \frac{\cos x + i \sin x}{x} dx.$$

- $$\begin{aligned}\left|\int_{\gamma_4} \frac{e^{iz}}{z} dz\right| &= \left|\int_0^1 \frac{e^{iRe^{\pi i t}}}{Re^{\pi i t}} R\pi i e^{\pi i t} dt\right| \\ &\leq \int_0^1 \left|\frac{e^{iRe^{\pi i t}}}{Re^{\pi i t}} R\pi i e^{\pi i t}\right| dt \\ &\leq \pi \int_0^1 \frac{1}{e^{R \sin \pi t}} dt.\end{aligned}$$

  First, $e^{R \sin \pi t}$ is symmetric around $t = 1/2$. Moreover, $\sin \pi t \geq 2t \geq 0$ whenever $t \in [0, 1/2]$ because $2t$ is the straight line approximation of $\sin \pi t$ on $[0, 1/2]$. Then $\pi \int_0^1 \frac{1}{e^{R \sin \pi t}} dt = 2\pi \frac{e^{-2Rt}}{-2Rt}\Big|_0^{1/2} = 0$ as $R \to \infty$.

Therefore, we obtain that $2i \int_0^\infty \frac{\sin x}{x} = \pi i$.

Exercise. (Problem 4) $(z,t) \mapsto (1-t)z + R_0 \frac{z}{|z|} t$ is a homotopy between γ_0 and γ_1. Let $\beta_1, \beta_2, \beta_3$ denote $\alpha_1([R_0, R_1]), \alpha_2([R_1, R_1+2\pi]), \alpha_3([R_1+2\pi, 2R_1 - R_0 + 2\pi])$, respectively.

$$\begin{aligned}
\int_{C_{R_0}(0)} f &= \int_{\gamma_1} f && \text{(Cauchy's Integral Theorem)} \\
&= \int_{\beta_1+\beta_2+\beta_3} f \\
&= \int_{\beta_1} f + \int_{\beta_2} f + \int_{\beta_3} f \\
&= \int_{\beta_1} f + \int_{\beta_2} f - \int_{\beta_1} f \\
&= \int_{\beta_2} f \\
&= \int_{C_{R_1}(z_1)} f.
\end{aligned}$$