MATH 602 HOMEWORK 2

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Lemma 0.1. *Let $R \subset S$ be integral domains and suppose S is integral over R. Then for every $s \in S$, there exists a monic polynomial with coefficients in R and a nonzero constant term that s satisfies.*

Proof. Choose $a_{n-1}, \cdots, a_0 \in R$ such that $s^n + a_{n-1}s^{n-1} + \cdots + a_1 s^1 + a_0 = 0$. If $a_0 = 0$, then $s(s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1) = 0$. Since we are dealing with integral domains, this implies $s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1 = 0$. By repeating this process, we obtain a monic polynomial with coefficients in R and a nonzero constant term that s satisfies. $\square$

Exercise. (Problem 1) We will assume that the problem meant to say "su with $s \in S \setminus \{0\}$" because it would be trivial otherwise. Choose $a_{n-1}, \cdots, a_0 \in R$ such that $u^n + a_{n-1}u^{n-1} + \cdots + a_1 u^1 + a_0 = 0$ with $a_0 \neq 0$. This is possible by Lemma 0.1 that we showed above.

Then $u(a_1 + a_2 u + \cdots + a_{n-1}u^{n-2} + u^{n-1}) = -a_0 \in R$. Since $a_0 \neq 0$, $a_1 + a_2 u + \cdots + a_{n-1}u^{n-2} + u^{n-1}$ is a nonzero element in S. Hence, we showed that some multiple of u lives in R.

Exercise. (Problem 2) Let $R = \mathbb{Z}$ and $S = 2\mathbb{Z}$. $R \setminus S$ is not even an ideal because $0 \notin R \setminus S$. Thus $R \setminus S$ is not a prime ideal.

Exercise. (Problem 3)

Solve this!

Exercise. (Problem 4) Let $p \in \operatorname{Spec}(R)$ such that $I \subset p$. Define $p/I = \{x + I \mid x \in p\} \subset R/I$. By the third isomorphism theorem of rings, p/I is an ideal of R/I. Let $x+I, y+I \in R/I$ and suppose $(x+I)(y+I) \in p/I$. Then $xy + I = z + I$ for some $z \in p$. Thus $xy - z \in I \subset p$ and $z \in p$. Thus $xy \in p$, so $x \in p$ or $y \in p$. This implies $x + I \in p/I$ or $y + I \in p/I$, so $p/I \in \operatorname{Spec}(R/I)$.

On the other hand, let $A/I \subset \operatorname{Spec}(R/I)$ be given. By the third isomorphism theorem of rings, every ideal of R/I must be of the form A/I where A is an ideal of R that contains I. Let $x, y \in R$ and suppose $xy \in A$. Then $xy + I \in A/I$, so $(x+I)(y+I) \in A/I$. Without loss of generality, $x + I \in A/I$. Then $x + I = a + I$ for some $a \in A$. Thus $x - a \in I \subset A$ and $a \in A$, so $x \in A$. Therefore, A is a prime ideal of R containing I.

Exercise. (Problem 5) By the second isomorphism of rings, $R \cap q$ is an ideal in R. Let $x, y \in R$. Suppose $xy \in R \cap q$. Then $xy \in q$, so $x \in q$ or $y \in q$. Without loss of generality, $x \in q$. Then $x \in R \cap q$. Thus $R \cap q$ is a prime ideal of R.

Exercise. (Problem 6) Suppose R is a field. Let $x \in S$ and $x \neq 0$. By Lemma 0.1, we can choose $a_{n-1}, \cdots, a_0 \in R$ with $a_0 \neq 0$ such that $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$. This implies $x((x^{n-1} + a_{n-1}x^{n-1} + \cdots + a_1)/-a_0) = 1$, so x is a unit in S.

Suppose S is a field. Let $x \in R$ and $x \neq 0$. Then $1/x \in S$. Thus $(1/x)^n + a_{n-1}(1/x)^{n-1} + \cdots + a_1(1/x) + a_0 = 0$ for some $a_{n-1}, \cdots, a_0 \in R$ with $a_0 \neq 0$. This implies $1 + x(a_{n-1} +$

$a_{n-2}x + \cdots + a_1x^{n-2} + a_0x^{n-1}) = 0$, so $-(a_{n-1} + a_{n-2}x + \cdots + a_1x^{n-2} + a_0x^{n-1}) = 1/x$. Clearly, $a_{n-1} + a_{n-2}x + \cdots + a_1x^{n-2} + a_0x^{n-1} \in R$, so $1/x \in R$, and thus R is a field.