MATH 620 HOMEWORK (DUE 9/10)

HIDENORI SHINOHARA

Exercise. Show that $F_* : T_p\mathbb{R}^n \to T_q\mathbb{R}^m$.

Proof. Let $v_1, v_2 \in T_pU, c \in \mathbb{R}$. Then $v_1 = c_1^j \frac{\partial}{\partial x^j} \mid_p, v_2 = c_2^j \frac{\partial}{\partial x^j} \mid_p$ where $c_i^j \in \mathbb{R}$. Let $\gamma_1(t) = p + t(c_1^1, \cdots, c_1^n), \gamma_2(t) = p + t(c_2^1, \cdots, c_2^n), \gamma = c\gamma_1 + \gamma_2$. Then there exist unique $b_1^1, \cdots, b_1^m, b_2^1, \cdots, b_2^m, b^1, \cdots, b^m \in \mathbb{R}$ such that

- $F_*(v_1) = b_1^s \frac{\partial}{\partial y^s}$.
- $F_*(v_2) = b_2^s \frac{\partial}{\partial y^s}$.
- $F_*(cv_1 + v_2) = b^s \frac{\partial}{\partial y^s}$.

For each s,

$$\begin{aligned}
b_s &= (F_*(cv_1 + v_2))(y^s) \\
&= \frac{d}{dt} y^s \circ F \circ \gamma(t)\Big|_{t=0} \\
&= \frac{d}{dt} F^s \circ \gamma(t)\Big|_{t=0} && \text{(Let } F^s = y^s \circ F.\text{)} \\
&= \frac{\partial F^s}{\partial x^j}\Big|_p (cc_1^j + c_2^j) \\
&= c\frac{\partial F^s}{\partial x^j}\Big|_p c_1^j + \frac{\partial F^s}{\partial x^j}\Big|_p c_2^j \\
&= c\frac{d}{dt} F^s \circ \gamma_1(t)\Big|_p c_1^j + \frac{d}{dt} F^s \circ \gamma_2(t)\Big|_p c_2^j \\
&= c(F_* v_1)(y^s) + (F_* v_2)(y^s) \\
&= cb_1^s + b_2^s.
\end{aligned}$$

Therefore, $F_*(cv_1 + v_2) = cF_*(v_1) + F_*(v_2)$. $\square$