MATH 620 HOMEWORK DUE 9/5

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Exercise 0.1. Prove that $\{\partial_1, \cdots, \partial_n\}$ is a basis of $T_p\mathbb{R}^n$.

Proof. TODO $\square$

Exercise 0.2. Show that $\{dx^1, \cdots, dx^n\}$ is a basis of $T_p^*\mathbb{R}^n$ that is dual to $\{\frac{\partial}{\partial x^j}\}_{j=1}^n \subset T_p\mathbb{R}^n$.

Proof.

- Dual? Let $i, j \in \{1, \cdots, n\}$. $dx^i(\frac{\partial}{\partial x^j}) = \frac{\partial}{\partial x^j}x^i$. The partial derivative of x^i with respect to x^j is 1 if $i = j$ and 0 otherwise. Thus $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$.
- Linearly independent? Let $c_1, \cdots, c_n \in \mathbb{R}$ be given. Suppose that $c_1 dx^1 + \cdots + c_n dx^n = 0$. For any $i \in \{1, \cdots, n\}$,

$$\begin{aligned}
(c_1 dx^1 + \cdots + c_n dx^n)(\partial_i) = 0 &\implies c_1(dx^1(\partial_i)) + \cdots + c_n(dx^n(\partial_i)) = 0 \\
&\implies c_1(\partial_i(x^1)) + \cdots + c_n(\partial_i(x^n)) = 0 \\
&\implies c_i\partial_i(x^i) = 0 \\
&\implies c_i = 0.
\end{aligned}$$

  Therefore, $c_1 = \cdots = c_n = 0$. Therefore, $\{dx^1, \cdots, dx^n\}$ is indeed linearly independent.
- Span? Let $f \in T_p^*\mathbb{R}^n$ be given. We claim that $f = \sum_{i=1}^n f(\partial_i)dx^i$. Let $\sum_{i=1}^n c_i\partial_i \in T_p\mathbb{R}^n$ be given where c_i's are in $\mathbb{R}$. (It makes

sense to assume that every element in $T_p\mathbb{R}^n$ is in this form because we showed earlier that $\{\partial_1, \cdots, \partial_n\}$ is a basis of $T_p\mathbb{R}^n$.)

$$\begin{aligned}
(\sum_{i=1}^n f(\partial_i)dx^i)(\sum_{j=1}^n c_j\partial_j) &= \sum_{i=1}^n \big[f(\partial_i)dx^i(\sum_{j=1}^n c_j\partial_j)\big] \\
&= \sum_{i=1}^n f(\partial_i)\big[\sum_{j=1}^n c_j dx^i(\partial_j)\big] \\
&= \sum_{i=1}^n f(\partial_i)\big[\sum_{j=1}^n c_j \partial_j(x^i)\big] \\
&= \sum_{i=1}^n f(\partial_i)c_i \\
&= f(\sum_{i=1}^n c_i\partial_i).
\end{aligned}$$

□