MATH 612 (HOMEWORK 3)

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Exercise. (3.1.11) Using the cellular homology, we obtain

$$\tilde{H}_i(X) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & (i = n) \\ 0 & (i \neq n). \end{cases}$$

$$\tilde{H}^i(X) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & (i = n+1) \\ 0 & (i \neq n+1). \end{cases}$$

From previous homework,

$$\tilde{H}_i(X/S^n) = \tilde{H}_i(S^{n+1}) = \begin{cases} \mathbb{Z} & (i = n+1) \\ 0 & (i \neq n+1). \end{cases}$$

Thus the map on $\tilde{H}_i(-;\mathbb{Z})$ is the zero map for each i. On the other hand, the long exact sequence of a pair gives us $\tilde{H}^{n+1}(X, S^n;\mathbb{Z}) \xrightarrow{q^*} \tilde{H}^{n+1}(X;\mathbb{Z}) \to \tilde{H}^{n+1}(S^n;\mathbb{Z})$ where $\tilde{H}^{n+1}(S^n;\mathbb{Z}) = 0$, so q^* is surjective. Therefore, it is nontrivial because $\tilde{H}^{n+1}(X;\mathbb{Z}) \neq 0$.

Natural?

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}(H_n(X);\mathbb{Z}) \arrow[r] \arrow[d] & H^{n+1}(X;\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{Hom}(H_{n+1}(X);\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Ext}(H_n(X/S^n);\mathbb{Z}) \arrow[r] & H^{n+1}(X/S^n;\mathbb{Z}) \arrow[r] & \mathrm{Hom}(H_{n+1}(X/S^n);\mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

is

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_m \arrow[r] \arrow[d] & \mathbb{Z}_m \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0. \end{tikzcd}$$

This splitting is not natural because the middle term in the first sequence is isomorphic to $\mathbb{Z}_m \oplus 0$ and the second one is $0 \oplus \mathbb{Z}$.

The long exact sequence of a pair gives us $\tilde{H}_n(S^n;\mathbb{Z}) \to \tilde{H}_n(X;\mathbb{Z}) \to \tilde{H}_n(X, S^n;\mathbb{Z}) = \tilde{H}_n(S^{n+1};\mathbb{Z}) = 0$ which implies the surjectivity of the induced map. Since $\tilde{H}_n(X;\mathbb{Z}) \neq 0$, the induced map is nonzero.

The map induced on $\tilde{H}^i(-;\mathbb{Z})$ is the zero map for any i because at least one of $\tilde{H}^i(S^n;\mathbb{Z})$ or $\tilde{H}^i(X;\mathbb{Z})$ is 0 for each i.

Exercise. (3.1.13)

Exercise. (3.2.1)

Exercise. (3.2.2)

Exercise. (3.2.3)

Exercise. (3.2.6)

Exercise. (3.2.7)