MATH 601 (DUE 10/30)

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Contents

1. Factoring Polynomials with coefficients in Finite Fields

Exercise. (Problem 1) Consider the Frobenius homomorphism, $F_p : \mathbb{F}_q \to \mathbb{F}_q$. Show that this homomorphism is bijective. If $q = p$, identify it with a familiar homomorphism.

Proof. Since $\mathbb{F}_q$ is finite, it suffices to show that F_p is injective. $F_p(a) = F_p(b) \implies a^p + (-b)^p = 0 \implies a - b = 0$ if $p \geq 3$. The case when $p = 2$ is similar. If $q = p$, $\mathbb{F}_q \cong \mathbb{Z}/p\mathbb{Z}$, which is a cyclic additive group generated by 1. Since $F_p(1) = 1$, F_p must be the identity homomorphism. $\square$

Exercise. (Problem 2) Let K be a field of characteristic p. Which polynomials $f(x) \in K[x]$ satisfies $f'(x) = 0$?

Proof. $f'(x) = \sum_{i=1}^{n} i a_i x^i = 0 \iff (\forall i, i \notin (p) \implies a_i = 0)$ since if $i \in (p)$, $ia_i = 0$ regardless of what a_i is. $\square$

Exercise. (Problem 3) Suppose that $f(x) \in \mathbb{F}_q[x]$ satisfies $f'(x) = 0$. Show that there exists $g(x) \in \mathbb{F}_q[x]$ with $g^p = f$.

Proof. By Problem 2, $f(x)$ with $f'(x) = 0$ can be always written as $\sum_{i=0}^{n} a_i x^{pi}$. We will use induction on the degree of polynomials. The base case, $n = 0$, is clear because F_p is bijective. If $f(x) = \sum_{i=0}^{n+1} a_i x^{pi}$, $(F_p^{-1}(a_{n+1})x^{n+1} + g(x))^p = a_{n+1}x^{p(n+1)} + (\sum_{i=0}^{n} a_i x^{pi})$ where $g(x)$ is the pth root of $\sum_{i=0}^{n} a_i x^{pi}$, whose existence is given by the inductive hypothesis. $\square$

Exercise. (Problem 4) Show that there are no inseparable irreducible polynomials, $f(x) \in \mathbb{F}_q[x]$.

Proof. If f is inseparable, $\gcd(f, f') \neq F^\times$. If $f' = 0$, then f has a proper factor by Problem 3. Otherwise, f has a factor of degree between 1 and $\deg(f') = \deg(f) - 1$, so f is not irreducible. $\square$

Exercise. (Problem 5) Suppose that $f(x) \in \mathbb{F}_q[x]$ and $\gcd(f, f') = f$. How can you reduce the problem of factoring f to a simpler problem?

Proof. If $f' \neq 0$, $f \nmid f'$ because $\deg(f') < \deg(f)$. Thus $\gcd(f, f') = f$ implies $f' = 0$. By Problem 3, $f = g^p$ for some $g \in \mathbb{F}_q[x]$, and thus it suffices to factor g, whose degree is exactly $\deg(f)/p$. $\square$

Exercise. (Problem 6) Let L be a field and $f(x) = \prod_{i=1}^{n}(x - a_i)^{m_i} \in L[x]$, where the a_i's are pairwise distinct. Compute $d(x) = \gcd(f(x), f'(x))$.

Proof. Let p be the characteristic of the finite field L. Since $L[x]$ is a UFD, every divisor of $f(x)$ is associate to a product of $(x - a_i)$'s, and so is $d(x)$. Let j be given. Then $f' = m_j(x - a_j)^{m_j - 1}g(x) + (x - a_j)^{m_j}g(x)'$ where $g(x) = \prod_{i \neq j}(x - a_i^{m_i})$. If $p \mid m_j$, then $(x - a_j)^{m_j}$ divides both f and f'. If $p \nmid m_j$, then $b_j = m_j - 1$ is the largest integer such that $(x - a_j)^{b_j}$ divides both f and f'. Therefore, $d(x) = \prod_{i=1}^{n}(x - a_i)^{m_i - c_i}$ where $c_i = 0$ if $p \mid m_i$ and $c_i = 1$ otherwise. □

Exercise. (Problem 7) A polynomial, $f(x)$, is said to be square free if it can be written as a product of irreducible factors, no two of which are associate. For $f(x) \in \mathbb{F}_q[x]$, find a criterion in terms of $gcd(f(x), f'(x))$ for $f(x)$ to be square free.

Proof. Let $f = \prod f_i$ be square free. Let j be given. $f' = f_j'g + f_jg'$ where $g = \prod_i f_i$. Since f_j is irreducible, f_j is separable by Problem 4. Thus $\gcd(f_j, f_j') = F^\times$, so $f_j \nmid f_j'$. Thus $f_j \nmid f'$.

Since all divisors of f are associate to some product of f_i's, $\gcd(f, f') = 1$.

On the other hand, suppose f is not square free. Then $f = g^2h$ for some irreducible g and some h. $f' = g(2g'h + gh')$, so $g \mid \gcd(f, f')$.

Therefore, f is square free if and only if $\gcd(f, f') = 1$. □

2. Modules

Exercise. (Problem 6) Take four 4×4 matrices with integer entries and check if the abelian group presented by the matrix is cyclic.

Proof.

$$\begin{bmatrix} -166 & -74 & 254 & 347 \\ 140 & -93 & 246 & 425 \\ -196 & 57 & -363 & 202 \\ 325 & 257 & 314 & -389 \end{bmatrix} \to \begin{bmatrix} 18444530375 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 237 & -81 & 332 & -132 \\ 95 & 268 & 229 & 498 \\ 387 & 213 & 46 & 55 \\ 88 & -126 & -380 & -447 \end{bmatrix} \to \begin{bmatrix} 2610768268 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -275 & -22 & -207 & -276 \\ -469 & -342 & 240 & -101 \\ -41 & 455 & 51 & -151 \\ 267 & -450 & 98 & -40 \end{bmatrix} \to \begin{bmatrix} 33644517767 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 48 & 29 & 22 & -481 \\ 388 & -468 & -137 & -491 \\ 84 & -352 & 85 & -384 \\ -226 & -486 & 102 & -156 \end{bmatrix} = \begin{bmatrix} 13267264454 & 1 & 1 & 1 \end{bmatrix}$$

Each of the groups contains 4 generators, so none of them are cyclic. □

3. Galois Theory

Exercise. (Problem 1) Let $F = \mathbb{Q}$. Let $L = \mathbb{Q}(\sqrt{7}, \sqrt{-11})$. To what familiar group is $\mathrm{Aut}(L/F)$ is isomorphic?

Proof. $[K : \mathbb{Q}(\sqrt{7})] = [K : \mathbb{Q}(\sqrt{-11})] = 2$. Since the characteristic of K is not 2, by the argument presented on P.3 of the Galous Theory handout, $\mathrm{Aut}(K/\mathbb{Q}(\sqrt{7}))$ and $\mathrm{Aut}(K/\mathbb{Q}(\sqrt{-11}))$ have 2 elements. For instance, $\alpha = \sqrt{7}$ and the minimal monic polynomial is x^2-7. This gives $D = 28$ and two automorphisms in $\mathrm{Aut}(K/\mathbb{Q}(\sqrt{7}))$, the identity map, and $\sigma : \sqrt{D} \mapsto -\sqrt{D}$ as discussed in the handout. Similarly, $\mathrm{Aut}(K/\mathbb{Q}(\sqrt{-11}))$ contains the identity map and $\sigma : \sqrt{D} \mapsto -\sqrt{D}$ where $D = -44$.

Finish this proof.

□

Exercise. (Problem 2) Let $F \subset K$ be a field extension.

(1) Prove in at most two sentences that each $\sigma \in \mathrm{Aut}(K/F)$ is an F-linear transformation of the F-vector space, K.

(2) Does the same condition hold in general for $\sigma \in \mathrm{Aut}(K)$? Prove or give a counterexample.

Proof.

(1) For any $a \in F$ and $v, w \in K$, $\sigma(av + w) = \sigma(a)\sigma(v) + \sigma(w) = a\sigma(v) + \sigma(w)$, so σ is indeed an F-linear transformation.

(2) Let $F = \mathbb{Q}(\sqrt{7})$ and $K = \mathbb{Q}(\sqrt{7}, \sqrt{-11})$. Let $\sigma \in \mathrm{Aut}(K/\mathbb{Q})$ such that $\sigma(\sqrt{7}) = -\sqrt{7}, \sigma(\sqrt{-11}) = -\sqrt{-11}$. The existence of such an automorphism is shown in the solution to Problem 1. K is an F-vector space. However, $\sigma(\sqrt{7} \cdot 1) = -\sqrt{7} \neq \sqrt{7} = \sqrt{7}(\sigma(1))$, so σ is not an F-linear transformation.

□

Exercise. (Problem 3) Let $\zeta = \exp(2\pi i/3) \in \mathbb{C}$. Consider the following subfields of $\mathbb{C}$. Let $F = \mathbb{Q}(\zeta)$. For $i \in \{0, 1, 2\}$, let $K_i = \mathbb{Q}(\zeta^i 7^{1/3})$. Let $L = \mathbb{Q}(7^{1/3}, \zeta 7^{1/3}, \zeta^2 7^{1/3})$.

Proof.

(1) $[F : \mathbb{Q}] = 3$.

(2) $\mathrm{Aut}(F/\mathbb{Q})$ consists of two maps, the identity map and another map that swaps ζ and ζ^2.

(3) $[K_i : \mathbb{Q}] = 3$ for each i because $\{1, \zeta^i 7^{1/3}, (\zeta^i 7^{i/3})^2\}$ is a $\mathbb{Q}$-basis.

(4) Finish the rest!

□