MATH 602 (HOMEWORK 5)

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Exercise. (1) This can be proved using induction. The base case $m = 1$ is trivial. Suppose that the proposition has been shown for some $m \in \mathbb{N}$. We will show the $(m+1)$ case. By the definition of a determinant,

$$\Delta = \sum_{k=1}^{m+1} (-1)^{k+1} \det(M_{k,1})$$

where $M_{k,1}$ is the matrix obtained by deleting the kth row and 1st column. We can apply the inductive hypothesis to each $M_{k,1}$ because, for instance, when $k = 1$,

$$\begin{aligned}
\det(M_{1,1}) &= \det \begin{bmatrix} \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^m \\ & \ddots & & \\ \alpha_{m+1} & \alpha_{m+1}^2 & \cdots & \alpha_{m+1}^m \end{bmatrix} \\
&= \alpha_2 \cdots \alpha_{m+1} \det \begin{bmatrix} 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{m-1} \\ & \ddots & & & \\ 1 & \alpha_{m+1} & \alpha_{m+1}^2 & \cdots & \alpha_{m+1}^{m-1} \end{bmatrix} \\
&= \alpha_2 \cdots \alpha_{m+1} \prod_{2 \le i < j \le m} (\alpha_j - \alpha_i).
\end{aligned}$$

A similar argument can be applied to other cases and we obtain

$$\Delta = \sum_{k=1}^{m+1} (-1)^{k+1} (\alpha_1 \cdots \hat{\alpha_k} \cdots \alpha_m) \prod_{i<j, i \ne k, j \ne k} (\alpha_j - \alpha_i).$$

It can be observed that, for each $k = 1, \cdots, m+1$, the kth term $(\alpha_1 \cdots \hat{\alpha_k} \cdots \alpha_m) \prod_{i<j,i\ne k,j\ne k}(\alpha_j - \alpha_i)$ does not contain any α_k. On the other hand, for any $l \ne k$, every term that we obtain when expanding the lth term contains α_k. Therefore, it suffices to show that, for each k, the sum of all the terms in $\prod_{1\le i<j\le m+1}(\alpha_j - \alpha_i)$ that do not contain α_k is equal to the kth term in the above expression.

$$\begin{aligned}
\prod_{1 \le i < j \le m+1} (\alpha_j - \alpha_i) &= \prod_{k+1 \le j} (\alpha_j - \alpha_k) \prod_{j \le k-1} (\alpha_k - \alpha_j) \prod_{1 \le i < j \le m+1, i \ne k, j \ne k} (\alpha_j - \alpha_i) \\
&= (-1)^{k-1} \prod_{j \ne k} (\alpha_j - \alpha_k) \prod_{1 \le i < j \le m+1, i \ne k, j \ne k} (\alpha_j - \alpha_i) \\
&= (-1)^{k-1} (\alpha_1 \cdots \hat{\alpha_k} \cdots \alpha_{m+1}) \prod_{1 \le i < j \le m+1, i \ne k, j \ne k} (\alpha_j - \alpha_i) + \alpha_k F(\alpha_1, \cdots, \alpha_{m+1}) \\
&= (-1)^{k+1} (\alpha_1 \cdots \hat{\alpha_k} \cdots \alpha_{m+1}) \prod_{1 \le i < j \le m+1, i \ne k, j \ne k} (\alpha_j - \alpha_i) + \alpha_k F(\alpha_1, \cdots, \alpha_{m+1})
\end{aligned}$$

for some polynomial F.

$\Delta^2 \neq \prod_{i \neq j}(\alpha_j - \alpha_i)$ in general. Let $\alpha_1 = 0, \alpha_2 = 1$. Then $\det(A)^2 = \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^2 = 1$. On the other hand, $\prod_{i \neq j}(\alpha_j - \alpha_i) = (0-1)(1-0) = -1$.

Exercise. (6) Let $ab \in \sqrt{q}$. Then $a^n b^n \in q$ for some $n \in \mathbb{N}$. Then $a^n \in q$ or $(b^n)^m \in q$ for some $m \in \mathbb{N}$. If $a^n \in q$, then $a \in \sqrt{q}$. If $b^{nm} \in q$, then $b \in \sqrt{q}$. Therefore, $\sqrt{q}$ is prime.

Let $f : A \to B$ be given and q be a primary ideal of B. Let $ab \in f^{-1}(q)$. Then $f(a)f(b) \in q$, so $f(a) \in q$ or $(f(b))^m \in q$ for some $m \geq 1$. If $f(a) \in q$, then $a \in f^{-1}(q)$. If $f(b^m) \in q$, then $b^m \in f^{-1}(q)$. Therefore, $f^{-1}(q)$ is primary.