

# MATH 611 FINAL

HIDENORI SHINOHARA

**Lemma 0.1.**  $H_1(\vee_g S^1) = \mathbb{Z}^g$  and  $H_n(\vee_g S^1) = 0$  for  $n \geq 2$ .

*Proof.* This can be shown using induction. When  $g = 1$ , this is obvious. Suppose we have shown this for some  $g \in \mathbb{N}$ . Split  $\vee_{g+1} S^1$  into  $\vee_g S^1$  and  $S^1$  with a small neighborhood. Figure

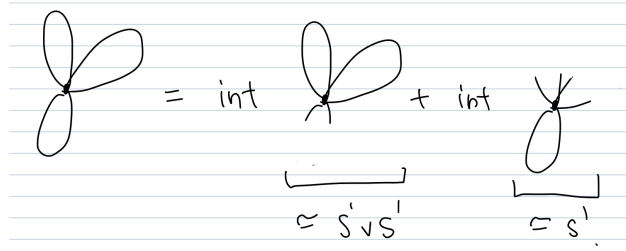


FIGURE 1.  $\vee_g S^1$

1 shows the case when  $g = 2$ . The intersection deformation retracts to a point. For simplicity, we will denote each component  $\vee_g S^1$  and  $S^1$  although they come with some extra neighborhood. The Mayer-Vietoris sequence  $H_n(\vee_g S^1) \oplus H_n(S^1) \rightarrow H_n(\vee_{g+1} S^1) \rightarrow H_{n-1}((\vee_g S^1) \cap S^1)$  gives  $H_n(\vee_{g+1} S^1)$  for all  $n \geq 2$  because  $H_n(\vee_g S^1) = H_n(S^1) = H_n((\vee_g S^1) \cap S^1) = 0$ . We will consider the exact sequence  $\tilde{H}_1((\vee_g S^1) \cap S^1) \rightarrow \tilde{H}_1(\vee_g S^1) \oplus \tilde{H}_1(S^1) \rightarrow \tilde{H}_1(\vee_{g+1} S^1) \rightarrow \tilde{H}_0((\vee_g S^1) \cap S^1)$ . Since  $\tilde{H}_1((\vee_g S^1) \cap S^1) = \tilde{H}_0((\vee_g S^1) \cap S^1) = 0$ ,  $\tilde{H}_1(\vee_{g+1} S^1) = \tilde{H}_1(\vee_g S^1) \oplus \tilde{H}_1(S^1) = \mathbb{Z}^g \oplus \mathbb{Z} = \mathbb{Z}^{g+1}$ .  $\square$

**Exercise.** (Problem 1(a)) We will use the 1-skeletons in Figure 2 to calculate the fundamental group of  $S$ . The fundamental group of the left side with a 2-cell attached is  $\langle a_1, b_1, \dots, a_g, b_g, c \mid [a_1, b_1] \cdots [a_g, b_g] c \rangle$ , and the right side is  $\langle gf, f^{-1}h \mid gfh^{-1}f \rangle$ . By Van Kampen, the fundamental group of  $S$  is

$$\langle a_1, b_1, \dots, a_g, b_g, c, gf, f^{-1}h \mid [a_1, b_1] \cdots [a_g, b_g] c, gfh^{-1}f, c(gh)^{-1} \rangle$$

where  $c(gh)^{-1}$  corresponds to  $i_{\alpha\beta}(c)i_{\beta\alpha}(c)^{-1}$  because we identify  $c$  with  $gh$ .

**Exercise.** (Problem 1(b)) Let  $A = \Sigma_g \setminus D^2$  and  $B$  be a Mobius strip  $M$  with some neighborhood from  $\Sigma_g$  such that  $\text{Int}(A) \cup \text{Int}(B) = S$  as in Figure 3. Then  $A$  is homotopy equivalent to the wedge sum of  $2g$   $S^1$ 's. Moreover,  $B$  is homotopy equivalent to  $S^1$  and so is  $A \cap B$ . We will consider the Mayer-Vietoris sequence formed by  $A, B \subset X$ .

We will start with the sequence  $H_n(A) \oplus H_n(B) \rightarrow H_n(A \cup B) \rightarrow H_{n-1}(A \cap B)$  where  $n-1 \geq 2$ . Then  $H_n(A) = H_n(B) = H_{n-1}(A \cap B) = 0$  for  $n \geq 3$ . By exactness,  $H_n(A \cup B) = 0$  when  $n \geq 3$ .

We will consider the following exact sequence:

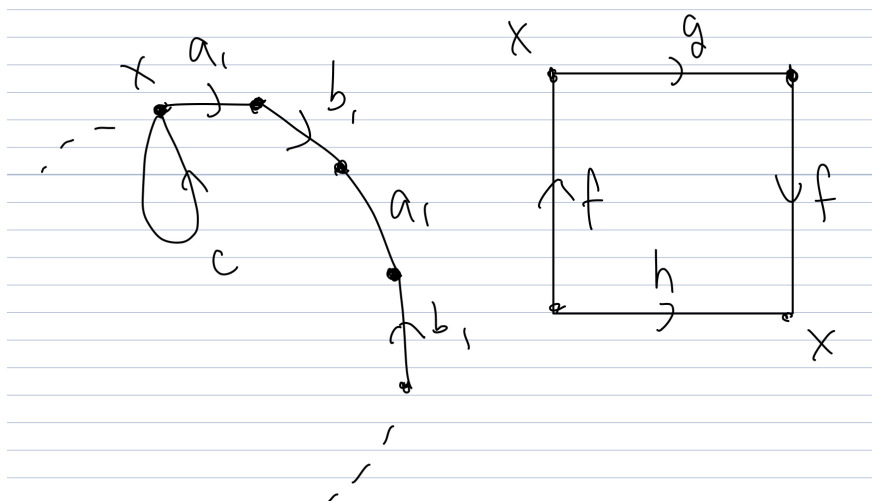


FIGURE 2. Problem 1(a)

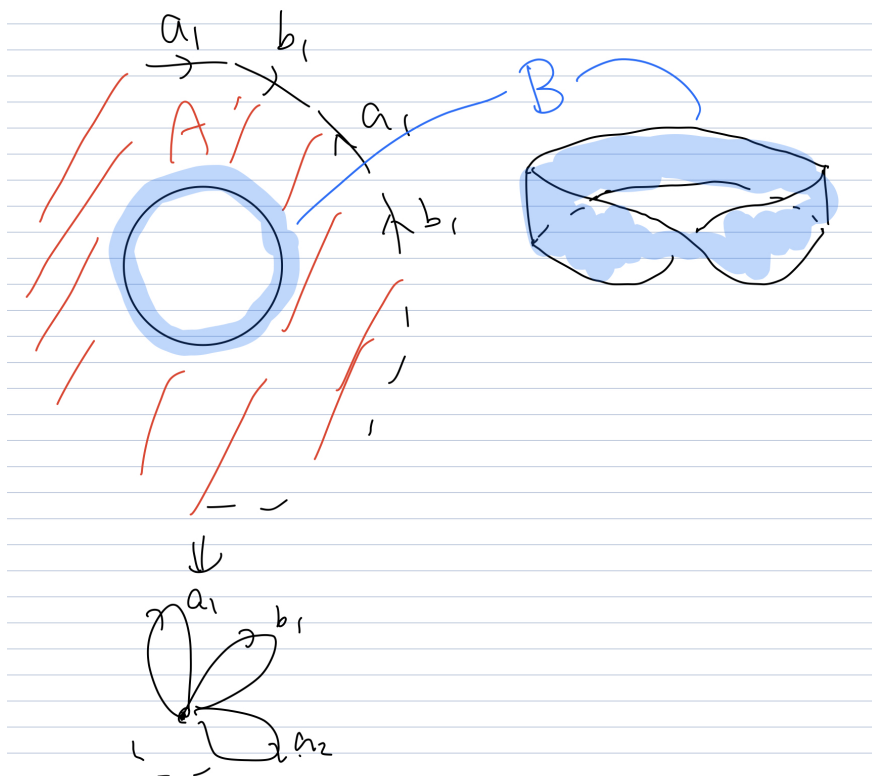


FIGURE 3.  $M_g$  with the Mobius band

$$\begin{aligned} \tilde{H}_2(A \cap B) &\rightarrow \tilde{H}_2(A) \oplus \tilde{H}_2(B) \rightarrow \tilde{H}_2(X) \xrightarrow{\alpha} \\ \tilde{H}_1(A \cap B) &\xrightarrow{\beta} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{\gamma} \tilde{H}_1(X) \rightarrow \\ \tilde{H}_0(A \cap B). & \quad 2 \end{aligned}$$

Then  $\tilde{H}_2(A) = \tilde{H}_2(B) = \tilde{H}_0(A \cap B) = 0$ . Thus the above sequence can be simplified to

$$0 \rightarrow \tilde{H}_2(X) \xrightarrow{\alpha} \tilde{H}_1(A \cap B) \xrightarrow{\beta} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{\gamma} \tilde{H}_1(X) \rightarrow 0.$$

Since the sequence is exact,  $\alpha$  must be injective and  $\gamma$  must be surjective. We will examine  $\beta$  to calculate the homology groups. Since  $A \cap B$  is homotopy equivalent to  $S^1$ ,  $\tilde{H}_1(A \cap B) = \mathbb{Z}$ . By Lemma 0.1,  $\tilde{H}_1(A) = \mathbb{Z}^{2g}$ . Finally,  $\tilde{H}_1(B) = \mathbb{Z}$ . Let  $a_1, b_1, \dots, a_g, b_g$  denote generators of  $\mathbb{Z}^{2g}$  and let  $a$  denote a generator of  $\tilde{H}_1(B)$ . A generator of  $\tilde{H}_1(A \cap B)$  goes around the intersection once, which is homotopy equivalent to  $a_1 + b_1 - a_1 - b_1 + \dots = 0$  inside  $A$ . A generator of  $\tilde{H}_1(A \cap B)$  goes around the Mobius strip twice inside  $B$ . Therefore,  $\beta$  sends a generator of  $\tilde{H}_1(A \cap B)$  to  $(0, 2a)$ .

Since  $\text{Im}(\alpha) = \ker(\beta) = 0$  and  $\alpha$  is injective,  $\tilde{H}_2(A \cup B) = 0$ . Since  $\gamma$  is surjective and  $\text{Im}(\beta) = \ker(\gamma)$ ,  $\tilde{H}_1(A \cup B) = \mathbb{Z}^{2g} \oplus \mathbb{Z} / \langle (0, 2) \rangle = \mathbb{Z}^{2g} \oplus (\mathbb{Z}/2\mathbb{Z})$ . Since  $H_n = \tilde{H}_n$  when  $n \geq 2$  and  $X$  is path connected, we have

$$H_n(X) = \begin{cases} 0 & (n \geq 2) \\ \mathbb{Z}^{2g} \oplus (\mathbb{Z}/2\mathbb{Z}) & (n = 1) \\ \mathbb{Z} & (n = 0). \end{cases}$$

**Exercise.** (Problem 1(c)) We will use Theorem 2.44 and the remark on P.147 [Hatcher].  $\chi(S) = 1 - 2g$  based on the calculation from Part (b). Therefore,  $\chi(S)$  is odd.  $\chi(S^2) = 1 - 0 + 1 = 2$  because  $H_0(S^2) = H_2(S^2) = \mathbb{Z}$ . This is even, so  $S$  cannot be homeomorphic to  $S^2$ . As mentioned on P.147 [Hatcher], the Euler characteristic of a closed orientable surface is even. Therefore,  $S$  must be homeomorphic to  $N_k$  for some  $k$ .  $\chi(N_k) = 2 - k$ , so  $2 - k = 1 - 2g \implies k = 1 + 2g$ . Therefore,  $S$  is homeomorphic to  $N_{1+2g}$ .

**Exercise.** (Problem 2(a)) Figure 4 shows how  $K_{3,3}$  is homotopy equivalent to  $S^1 \vee S^1 \vee S^1 \vee S^1$ . Thus the Van Kampen theorem implies that the fundamental group is the free group generated by 4 elements  $\langle a, b, c, d \rangle$  where each generator corresponds to each  $S^1$ .

**Exercise.** (Problem 2(b)) From Figure 4, it is clear that attaching four 2-cells, each killing one  $S^1$ , will give a simply connected space. We claim that 4 is the smallest number.

When we attach 2-cells to the graph, the fundamental group of the resulting space is  $\langle a, b, c, d \rangle / \langle r_1, r_2, \dots \rangle$  where each  $r_i$  is the relation given by a product of  $a, b, c, d$  in the order the boundary of the  $i$ th 2-cell was attached. Therefore, it suffices to show that  $\langle a, b, c, d \rangle / \langle r_1, r_2, r_3 \rangle \neq 0$ . On the contrary, suppose that it is.

If  $\langle a, b, c, d \rangle / \langle r_1, r_2, r_3 \rangle = 0$ , then  $\langle a, b, c, d \rangle = \langle r_1, r_2, r_3 \rangle$ . We will consider the surjective group homomorphism  $\phi : \langle a, b, c, d \rangle \rightarrow \mathbb{Z}^4$  defined by  $a \mapsto (1, 0, 0, 0)$ ,  $b \mapsto (0, 1, 0, 0)$ ,  $\dots$ ,  $d \mapsto (0, 0, 0, 1)$ . Each  $r_1, r_2, r_3$  is a product of  $a, b, c, d$ , so  $\phi(r_i) = (d_{i,1}, d_{i,2}, d_{i,3}, d_{i,4})$  for some  $d_{i,j} \in \mathbb{Z}$ . Since  $\langle a, b, c, d \rangle = \langle r_1, r_2, r_3 \rangle$ ,  $\phi(\langle a, b, c, d \rangle) = \phi(\langle r_1, r_2, r_3 \rangle)$ . Since  $\phi$  is surjective,  $\phi(r_1), \phi(r_2), \phi(r_3)$  generate  $\mathbb{Z}^4$ . However, this implies  $\{\phi(r_1), \phi(r_2), \phi(r_3)\}$  is a basis of  $\mathbb{R}^4$  because  $\{(1, 0, 0, 0), \dots, (0, 0, 0, 1)\}$  is. This is clearly a contradiction, so we need at least four 2-cells.

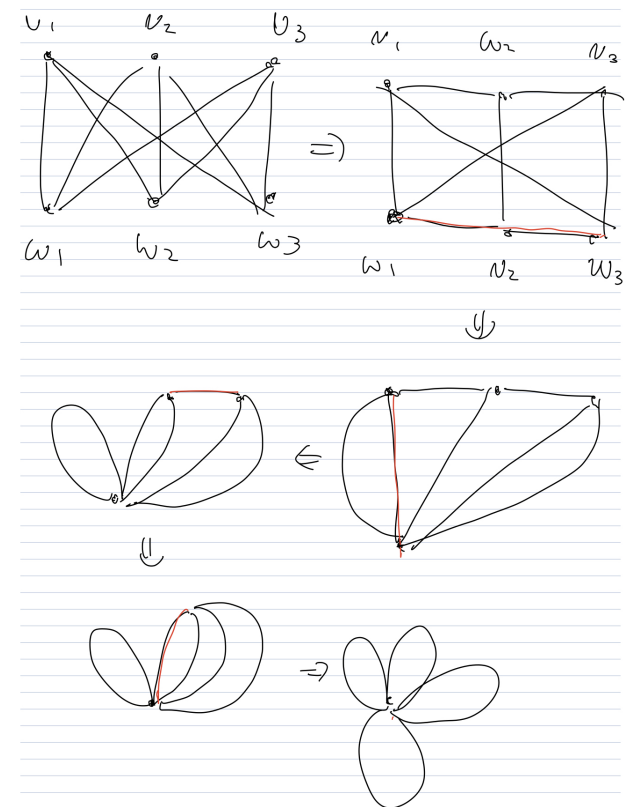


FIGURE 4.  $K_{3,3}$

**Exercise.** (Problem 5(a)) Let  $X = S^1 \times S^2$  and  $Y = S^1 \vee S^2 \vee S^3$ .

$$\begin{aligned} \pi_1(S^1 \times S^2) &= \pi_1(S^1) \times \pi_1(S^2) && \text{(Proposition 1.12)} \\ &= \mathbb{Z} \times 0 \\ &= \mathbb{Z}. \end{aligned}$$

$$\begin{aligned} \pi_1(S^1 \vee S^2 \vee S^3) &= \pi_1(S^1) * \pi_1(S^2) * \pi_1(S^3) && \text{(Van Kampen)} \\ &= \mathbb{Z} * 0 * 0 \\ &= \mathbb{Z}. \end{aligned}$$

$X$  and  $Y$  are both path connected, so  $H_0(X) = H_0(Y) = \mathbb{Z}$ .

We will consider two subspaces of  $X$  the union of whose interiors equals  $X$ . Identify each point of  $X = S^1 \times S^2$  by a pair of coordinates  $(\theta, (x, y, z))$  where  $\theta$  is the angle in  $S^1$  and  $(x, y, z)$  satisfies  $x^2 + y^2 + z^2 = 1$ . Let  $A = \{(\theta, (x, y, z)) \mid -\epsilon \leq \theta \leq \pi + \epsilon\}$ ,  $B = \{(\theta, (x, y, z)) \mid \pi - \epsilon \leq \theta \leq 2\pi + \epsilon\}$  where  $\epsilon > 0$  is a small number. Then each  $A$  and  $B$  deformation retracts to a space homeomorphic to  $S^2$ .  $A \cap B$  consists of two path components, each of which deformation retracts to a space homeomorphic to  $S^2$ . The homology groups of  $A \cap B$  are relatively easy to calculate because  $H_n(A \cap B) = H_n(S^2 \amalg S^2) = H_n(S^2) \oplus H_n(S^2)$  by Proposition 2.6 for any  $n$ . Moreover, it is clear that  $\text{Int}(A) \cup \text{Int}(B) = X$ . We will consider the Mayer-Vietoris sequence formed by  $A, B \subset X$ .

First, we will consider the sequence  $H_n(A) \oplus H_n(B) \rightarrow H_n(X) \rightarrow H_{n-1}(A \cap B)$  for each  $n \geq 4$ .  $H_n(A) = H_n(B) = H_{n-1}(A \cap B) = 0$  for  $n \geq 4$ . By the exactness,  $H_n(X) = 0$  for all  $n \geq 4$ . Next, we will consider the following sequence:

$$\begin{aligned} \tilde{H}_3(A \cap B) &\rightarrow \tilde{H}_3(A) \oplus \tilde{H}_3(B) \rightarrow \tilde{H}_3(X) \xrightarrow{\alpha} \\ \tilde{H}_2(A \cap B) &\xrightarrow{\beta} \tilde{H}_2(A) \oplus \tilde{H}_2(B) \xrightarrow{\gamma} \tilde{H}_2(X) \rightarrow \\ \tilde{H}_1(A \cap B) &\rightarrow \tilde{H}_1(A) \oplus \tilde{H}_1(B) \rightarrow \tilde{H}_1(X) \rightarrow \\ \tilde{H}_0(A \cap B) &\rightarrow \tilde{H}_0(A) \oplus \tilde{H}_0(B). \end{aligned}$$

$\tilde{H}_3(A \cap B) = \tilde{H}_3(A) = \tilde{H}_3(B) = \tilde{H}_1(A \cap B) = \tilde{H}_1(A) = \tilde{H}_1 = \tilde{H}_0(A) = \tilde{H}_0(B) = 0$ , and  $\tilde{H}_0(A \cap B)$ . By replacing the exact sequence with those values and splitting the sequence into two for readability, we obtain the following sequences:

$$\begin{aligned} 0 \rightarrow \tilde{H}_3(X) &\xrightarrow{\alpha} \tilde{H}_2(A \cap B) \xrightarrow{\beta} \tilde{H}_2(A) \oplus \tilde{H}_2(B) \xrightarrow{\gamma} \tilde{H}_2(X) \rightarrow 0, \\ 0 \rightarrow \tilde{H}_1(X) &\rightarrow \mathbb{Z} \rightarrow 0. \end{aligned}$$

By the exactness, we can conclude that  $\tilde{H}_1(X) \cong \mathbb{Z}$ . We will examine the homomorphism  $\beta$  to understand the sequence.  $\tilde{H}_2(A \cap B) = \langle [a], [b] \mid [[a], [b]] \rangle$  where each  $a, b$  lives in  $A \cap B$  and  $a$  lives in one of the path components of  $A \cap B$  and  $b$  lives in the other. Moreover,  $[a] = [b]$  in  $\tilde{H}_2(A)$  and  $\tilde{H}_2(B)$ . (Based on orientation,  $[a] = -[b]$ , but we can simply change the orientation of  $[b]$  in that case.) Then  $\beta(c_1[a] + c_2[b]) = ((c_1 + c_2)[a], (c_1 + c_2)[a])$ . This gives us that  $\text{Im}(\alpha) = \ker(\beta) = \{c[a] - c[b] \mid c \in \mathbb{Z}\} = \mathbb{Z}$ . By the exactness,  $\alpha$  is injective, so  $\tilde{H}_3(X) = \mathbb{Z}$ . Moreover,  $\ker(\gamma) = \text{Im}(\beta) = \{(c[a], c[a]) \mid c \in \mathbb{Z}\}$ . By the exactness,  $\gamma$  is surjective, so  $\tilde{H}_2(X) = (\tilde{H}_2(A) \oplus \tilde{H}_2(B)) / \text{Im}(\beta) = \langle [a] \rangle \oplus \langle [a] \rangle / \langle ([a], [a]) \rangle = \mathbb{Z}$ . Since reduced homology groups and homology groups are identical when  $n \geq 2$ , we have

$$H_n(X) = \begin{cases} \mathbb{Z} & (n = 0, 1, 2, 3) \\ 0 & (n \geq 4). \end{cases}$$

By Corollary 2.25,  $\tilde{H}_n(S^1 \vee S^2 \vee S^3) = \tilde{H}_n(S^1) \otimes \tilde{H}_n(S^2) \otimes \tilde{H}_n(S^3)$ . Therefore,

$$\tilde{H}_n(Y) = \begin{cases} \mathbb{Z} & (n = 1, 2, 3) \\ 0 & (n = 0, n \geq 4). \end{cases}$$

For  $n \geq 1$ ,  $\tilde{H}_n(Y) = H_n(Y)$ , so  $H_0(Y) = H_1(Y) = H_2(Y) = H_3(Y) = \mathbb{Z}$  and  $H_n(Y) = 0$  for all  $n \geq 4$ .

**Exercise.** (Problem 5(b)) We claim that the universal cover is  $\mathbb{R} \times S^2$ .  $p(\theta, (x, y, z)) = ((\cos \theta, \sin \theta), (x, y, z))$  is a covering map. Moreover,  $\pi_1(\mathbb{R} \times S^2) = \pi_1(\mathbb{R}) \times \pi_1(S^2) = 0 \times 0 = 0$ , so  $\mathbb{R} \times S^2$  is simply connected. Therefore,  $\mathbb{R} \times S^2$  is indeed a universal cover of  $X$ .

$\mathbb{R} \times S^2$  is homeomorphic to  $(0, 1) \times S^2$ . This space deformation retracts to  $S^2$  because  $(0, 1) \times S^2$  is homeomorphic to an open ball with its center removed. Thus their homology groups are  $H_2(\tilde{X}) = H_0(\tilde{X}) = \mathbb{Z}$  and  $H_n(\tilde{X}) = 0$  for all other  $n$ .

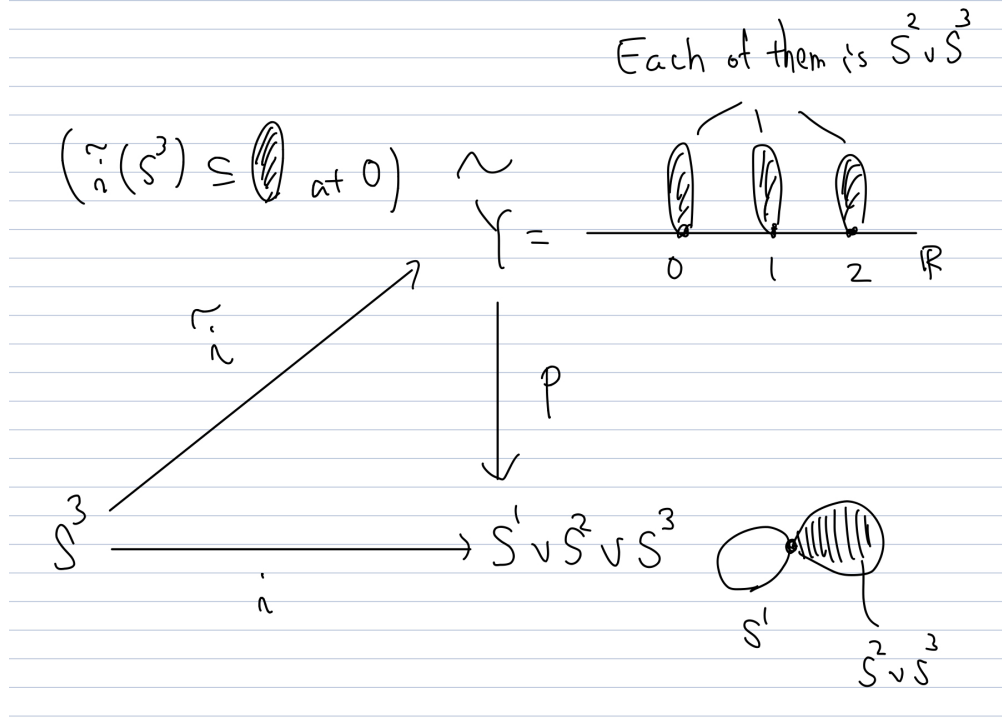


FIGURE 5. Problem 5(c)

**Exercise.** (Problem 5(c)) We claim that the universal covering space is the real line with  $S^2 \vee S^3$  attached to each of its integral points (Figure 5). Since  $S^2$  and  $S^3$  are both contractible, the wedge sum must be contractible. Attaching contractible spaces to each integral point of  $\mathbb{R}$ , which itself is contractible, gives a contractible space. The covering map  $p$  can be defined in an obvious way. Every point on  $\mathbb{R}$  can be mapped to  $S^1$  by  $\theta \rightarrow (\cos(\theta), \sin(\theta))$ , and each copy of  $S^2 \vee S^3$  can be mapped identically to  $S^2 \vee S^3$ . The  $i$  in Figure 5 is the obvious inclusion map, and  $\tilde{i}$  sends  $S^3$  into the copy of  $S^2 \vee S^3$  that is attached to 0 on  $\mathbb{R}$ . (It does not matter which copy, but it is necessary to specify which.) Then the diagram clearly commutes.

By the Mayer-Vietoris sequence, we have an exact sequence  $H_3((S^1 \vee S^2) \cap S^3) \rightarrow H_3(S^1 \vee S^2) \oplus H_3(S^3) \xrightarrow{\psi} H_3(S^1 \vee S^2 \vee S^3) \rightarrow H_2((S^1 \vee S^2) \cap S^3)$ . (To be precise, we need  $S^1 \vee S^2$  with a small neighborhood and  $S^3$  with a small neighborhood, such that the union of the interiors is  $S^1 \vee S^2 \vee S^3$  and the intersection deformation retracts onto a point.) Then  $H_n((S^1 \vee S^2) \cap S^3) = 0$  for  $n = 2, 3$ . Therefore,  $\psi$  is an isomorphism.  $H_3(S^1 \vee S^2) = 0$  by the Mayer-Vietoris sequence  $0 = H_3(S^1) \oplus H_3(S^2) \rightarrow H_3(S^1 \vee S^2) \rightarrow H_3(S^1 \cap S^2) = 0$  where  $S^1, S^2 \subset S^1 \vee S^2$  are technically  $S^1$  and  $S^2$  with a small neighborhood. Therefore, instead of  $\psi$ , we can consider the map  $\psi' : H_3(S^3) \rightarrow H_3(S^1 \vee S^2 \vee S^3)$  defined by  $\psi'(x) = \psi(0, x)$ . By construction of the Mayer-Vietoris sequence,  $\psi'$  is induced by the inclusion map  $i$ . Since homology is a covariant functor,  $p^*$  and  $\tilde{i}^*$ , which are induced by  $p$  and  $\tilde{i}$ , must commute with  $\psi' = i^*$ . In other words,  $i^* = \psi' = p^* \circ \tilde{i}^*$ . Since  $i^*$  is an isomorphism,  $\tilde{i}^*$  must be injective. This implies  $H_3(\tilde{Y})$  contains an isomorphic copy of  $H_3(S^3) = \mathbb{Z}$ .

We calculate in Part (b) that  $H_3(\tilde{X}) = 0$ . Therefore,  $H_3(\tilde{X}) \neq H_3(\tilde{Y})$ .