MATH 602(HOMEWORK 3)

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1. Exercises

Exercise. (Exercise 1) The ideal generated by the three polynomials contains $-yz^4+yz^2+y = (xy^2-xz+y)-y(xy-z^2)+z(x-yz^4)$. However, its leading term $-yz^4$ is not in the ideal generated by the leading terms of the three polynomials.

Exercise. (Exercise 2)

Solve this.

Exercise. (Exercise 3)

Solve this.

Exercise. (Exercise 4) $0 \in \sqrt{0}, a, b \in \sqrt{0} \implies (a+b)^{m+n-1} = \sum_{i=0}^{m+n-1}\binom{m+n-1}{i}a^i b^{m+n-1-i} = 0$, and $\forall a \in \sqrt{0}, \forall x \in R, (ax)^n = a^n x^n = 0$, so $\sqrt{0}$ is an ideal.

Exercise. (Exercise 5)

Solve this.

Exercise. (Exercise 6) Tensoring an exact sequence with $M \otimes_A N$ is the same as tensoring it with M first and tensoring the resulting sequence with N later.

Exercise. (Exercise 7) Since $0 \to I \xrightarrow{i} R \xrightarrow{q} R/I \to 0$ is exact, $I \otimes M \to R \otimes M \to (R/I) \otimes M \to 0$ is exact.

$$\begin{aligned}
(R/I) \otimes M &= \operatorname{im}(q \otimes \mathrm{Id}) \\
&\cong R \otimes M / \ker(q \otimes \mathrm{Id}) \\
&\cong R \otimes M / \operatorname{im}(i \otimes \mathrm{Id}) \\
&\cong R \otimes M / I \otimes M.
\end{aligned}$$

Now consider $\phi : R \otimes M \to M/IM$ that is the composition of $R \otimes M \to M : x \otimes y \mapsto xy$ and $M \to M/IM : x \mapsto x + IM$. In other words, ϕ is $x \otimes y \mapsto xy + IM$. Because the two maps are both surjective, ϕ must be surjective. The kernel of ϕ is $I \otimes M$ because

- For any $x \otimes y \in I \otimes M$, $\phi(x \otimes y) = xy + IM = 0$ since $xy \in IM$.
- If $\phi(x \otimes y) = 0$, then $xy \in IM$. In other words, $xy = x'y'$ for some $x' \in I$ and $y' \in M$. Then $x \otimes y = 1 \otimes xy = 1 \otimes x'y' = x' \otimes y' \in I \otimes M$.

Therefore, $M/IM \cong (R \otimes M)/(I \otimes M) \cong (R/I) \otimes M$.

Exercise. (Exercise 8) Let $pa+qb = 1$ for some $p, q \in \mathbb{Z}$. Then $1 \otimes 1 = (pa+qb) \otimes (pa+qb) = pa \otimes pa + pa \otimes qb + qb \otimes pa + qb \otimes qb = 0 + 0 + 0 + 0 = 0$.

Exercise. (Exercise 9)

Finish this!

Exercise. (Exercise 10) Let $a_1, \cdots, a_n, b_1, \cdots, b_m$ generate M' and M'', respectively. Let $x_1, \cdots, x_n, y_1, \cdots, y_m \in M$ be chosen such that x_i is the image of a_i and the image of y_j is b_j. We claim that x_i, y_j generate M. Let $x \in M$ be given. Then $q(x) = d_1 b_1 + \cdots + d_m b_m$ for some $d_i \in M$, and thus $q(x - d_1 y_1 - \cdots - d_m y_m) = 0$. Therefore, $x - d_1 y_1 - \cdots - d_m y_m = i(c_1 a_1 + \cdots + c_n a_n) = c_1 x_1 + \cdots + c_n x_n$, so $x = c_1 x_1 + \cdots + c_n x_n + d_1 y_1 + \cdots + d_m y_m$.