MATH 611 FINAL

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Lemma 0.1. *$H_1(\vee_g S^1) = \mathbb{Z}^g$ and $H_n(\vee_g S^1) = 0$ for $n \geq 2$.*

Proof. This can be shown using induction. When $g = 1$, this is obvious. Suppose we have shown this for some $g \in \mathbb{N}$.

Finish this!

□

Exercise. (Problem 1(b)) Let $A = \Sigma_g \setminus D^2$ and B be a Mobius strip M with some "extra points" from Σ_g such that $\int(A) \cup \int(B) = S$ as in Figure 1. Then A is homotopy equivalent to the wedge sum of $2g$ S^1's. Moreover, B is homotopy equivalent to S^1 and so is $A \cap B$. We will consider the Mayer-Vietoris sequence formed by $A, B \subset X$.

We will start with the sequence $H_n(A) \oplus H_n(B) \to H_n(A \cup B) \to H_{n-1}(A \cap B)$ where $n-1 \geq 2$. Then $H_n(A) = H_n(B) = H_{n-1}(A \cap B) = 0$ for $n \geq 3$. By exactness, $H_n(A \cup B) = 0$ when $n \geq 3$.

We will consider the following exact sequence:

$$\tilde{H}_2(A \cap B) \to \tilde{H}_2(A) \oplus \tilde{H}_2(B) \to \tilde{H}_2(X) \xrightarrow{\alpha}$$
$$\tilde{H}_1(A \cap B) \xrightarrow{\beta} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{\gamma} \tilde{H}_1(X) \to$$
$$\tilde{H}_0(A \cap B).$$

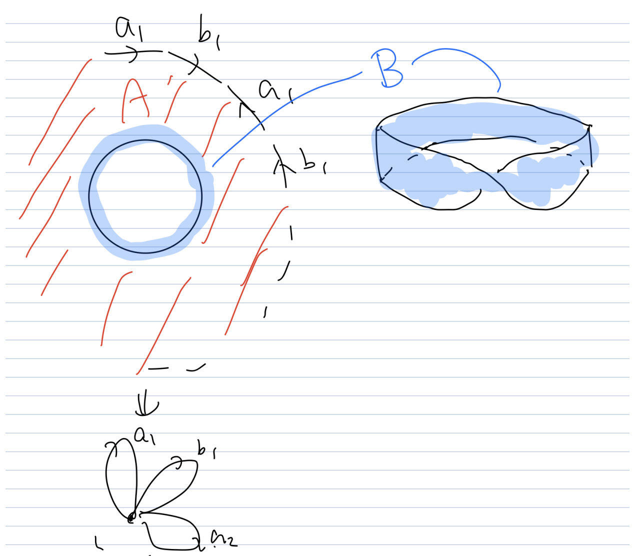

FIGURE 1. M_g with the Mobius band

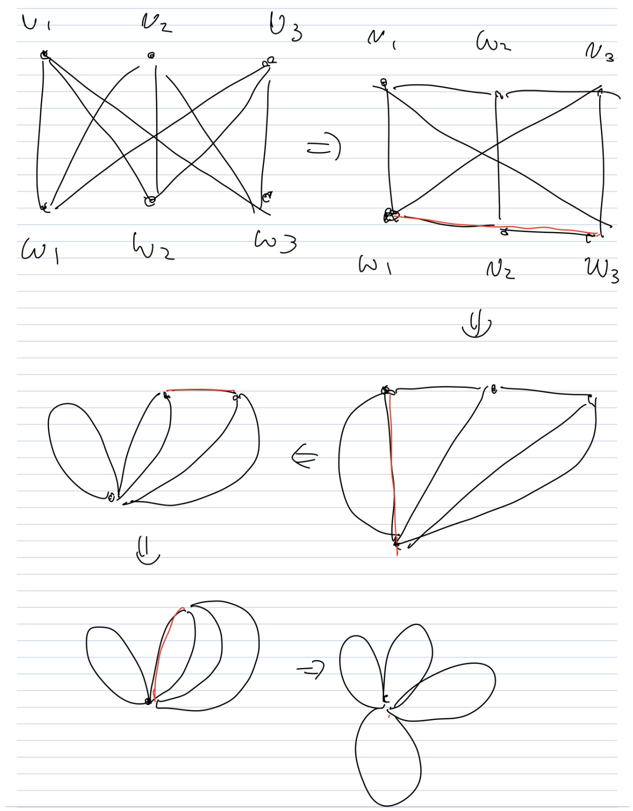

FIGURE 2. $K_{3,3}$

Then $\tilde{H}_2(A) = \tilde{H}_2(B) = \tilde{H}_0(A \cap B) = 0$. Thus the above sequence can be simplified to

$$0 \to \tilde{H}_2(X) \alpha \tilde{H}_1(A \cap B) \xrightarrow{\beta} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{\gamma} \tilde{H}_1(X) \to 0.$$

Since the sequence is exact, α must be injective and γ must be surjective. We will examine β to calculate the homology groups. Since $A \cap B$ is homotopy equivalent to S^1, $\tilde{H}_1(A \cap B) = \mathbb{Z}$.

Finish the lemma if I want to cite it.

By Lemma 0.1, $\tilde{H}_1(A) = \mathbb{Z}^{2g}$. Finally, $\tilde{H}_1(B) = \mathbb{Z}$. Let $a_1, b_1, \cdots, a_g, b_g$ denote generators of $\mathbb{Z}^{2g}$ and let a denote a generator of $\tilde{H}_1(B)$. A generator of $\tilde{H}_1(A \cap B)$ goes around the intersection once, which is homotopy equivalent to $a_1 + b_1 - a_1 - b_1 + \cdots = 0$ inside A. A generator of $\tilde{H}_1(A \cap B)$ goes around the Mobius strip twice inside B. Therefore, β sends a generator of $\tilde{H}_1(A \cap B)$ to $(0, 2a)$.

This is different from what I thought the answer should be...? Find out how this argument differs from the one in my notes from yesterday.

Exercise. (Problem 2) Figure 2 shows how $K_{3,3}$ is homotopy equivalent to $S_1 \vee S_1 \vee S_1 \vee S_1$. Thus the Van Kampen theorem implies that the fundamental group is the free group generated by 4 elements $\langle a, b, c, d \rangle$ where each generator corresponds to each S_1.

Exercise. (Problem 5(a)) Let $X = S^1 \times S^2$ and $Y = S^1 \vee S^2 \vee S^3$.

$$\begin{aligned}
\pi_1(S^1 \times S^2) &= \pi_1(S^1) \times \pi_1(S^2) && \text{(Proposition 1.12)} \\
&= \mathbb{Z} \times 0 \\
&= \mathbb{Z}. \\
\pi_1(S^1 \vee S^2 \vee S^3) &= \pi_1(S^1) * \pi_1(S^2) * \pi_1(S^3) && \text{(Van Kampen)} \\
&= \mathbb{Z} * 0 * 0 \\
&= \mathbb{Z}.
\end{aligned}$$

X and Y are both path connected, so $H_0(X) = H_0(Y) = \mathbb{Z}$.

We will consider two subspaces of X the union of whose interiors equals X. Identify each point of $X = S^1 \times S^2$ by a pair of coordinates $(\theta, (x, y, z))$ where θ is the angle in S^1 and (x, y, z) satisfies $x^2 + y^2 + z^2 = 1$. Let $A = \{(\theta, (x, y, z)) \mid -\epsilon \leq \theta \leq \pi + \epsilon\}, B = \{(\theta, (x, y, z)) \mid \pi - \epsilon \leq \theta \leq 2\pi + \epsilon\}$ where $\epsilon > 0$ is a small number. Then each A and B deformation retracts to a space homeomorphic to S^2. $A \cap B$ consists of two path components, each of which deformation retracts to a space homeomorphic to S^2. The homology groups of $A \cap B$ are relatively easy to calculate because $H_n(A \cap B) = H_n(S^2 \coprod S^2) = H_n(S^2) \oplus H_n(S^2)$ by Proposition 2.6 for any n. Moreover, it is clear that $\int(A) \cup \int(B) = X$. We will consider the Mayer-Vietoris sequence formed by $A, B \subset X$.

First, we will consider the sequence $H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B)$ for each $n \geq 4$. $H_n(A) = H_n(B) = H_{n-1}(A \cap B) = 0$ for $n \geq 4$ By the exactness, $H_n(X) = 0$ for all $n \geq 4$. Next, we will consider the following sequence:

$$\begin{aligned}
&\tilde{H}_3(A \cap B) \to \tilde{H}_3(A) \oplus \tilde{H}_3(B) \to \tilde{H}_3(X) \xrightarrow{\alpha} \\
&\tilde{H}_2(A \cap B) \xrightarrow{\beta} \tilde{H}_2(A) \oplus \tilde{H}_2(B) \xrightarrow{\gamma} \tilde{H}_2(X) \to \\
&\tilde{H}_1(A \cap B) \to \tilde{H}_1(A) \oplus \tilde{H}_1(B) \to \tilde{H}_1(X) \to \\
&\tilde{H}_0(A \cap B) \to \tilde{H}_0(A) \oplus \tilde{H}_0(B).
\end{aligned}$$

$\tilde{H}_3(A \cap B) = \tilde{H}_3(A) = \tilde{H}_3(B) = \tilde{H}_1(A \cap B) = \tilde{H}_1(A) = \tilde{H}_1 = \tilde{H}_0(A) = \tilde{H}_0(B) = 0$, and $\tilde{H}_0(A \cap B)$. By replacing the exact sequence with those values and splitting the sequence into two for readability, we obtain the following sequences:

$$0 \to \tilde{H}_3(X) \xrightarrow{\alpha} \tilde{H}_2(A \cap B) \xrightarrow{\beta} \tilde{H}_2(A) \oplus \tilde{H}_2(B) \xrightarrow{\gamma} \tilde{H}_2(X) \to 0,$$
$$0 \to \tilde{H}_1(X) \to \mathbb{Z} \to 0.$$

By the exactness, we can conclude that $\tilde{H}_1(X) \cong \mathbb{Z}$. We will examine the homomorphism β to understand the sequence. $\tilde{H}_2(A \cap B) = \langle [a], [b] \mid [[a], [b]] \rangle$ where each a, b lives in $A \cap B$ and a lives in one of the path components of $A \cap B$ and b lives in the other. Moreover, $[a] = [b]$ in $\tilde{H}_2(A)$ and $\tilde{H}_2(B)$. (Based on orientation, $[a] = -[b]$, but we can simply change the orientation of $[b]$ in that case.) Then $\beta(c_1[a] + c_2[b]) = ((c_1 + c_2)[a], (c_1 + c_2)[a])$. This gives us that $\text{Im}(\alpha) = \ker(\beta) = \{c[a] - c[b] \mid c \in \mathbb{Z}\} = \mathbb{Z}$. By the exactness, α is injective, so $\tilde{H}_3(X) = \mathbb{Z}$. Moreover, $\ker(\gamma) = \text{Im}(\beta) = \{(c[a], c[a]) \mid c \in \mathbb{Z}\}$. By the exactness, γ is surjective, so $\tilde{H}_2(X) = (\tilde{H}_2(A) \oplus \tilde{H}_2(B)) / \text{Im}(\beta) = \langle [a] \rangle \oplus \langle [a] \rangle / \langle ([a], [a]) \rangle = \mathbb{Z}$. Since

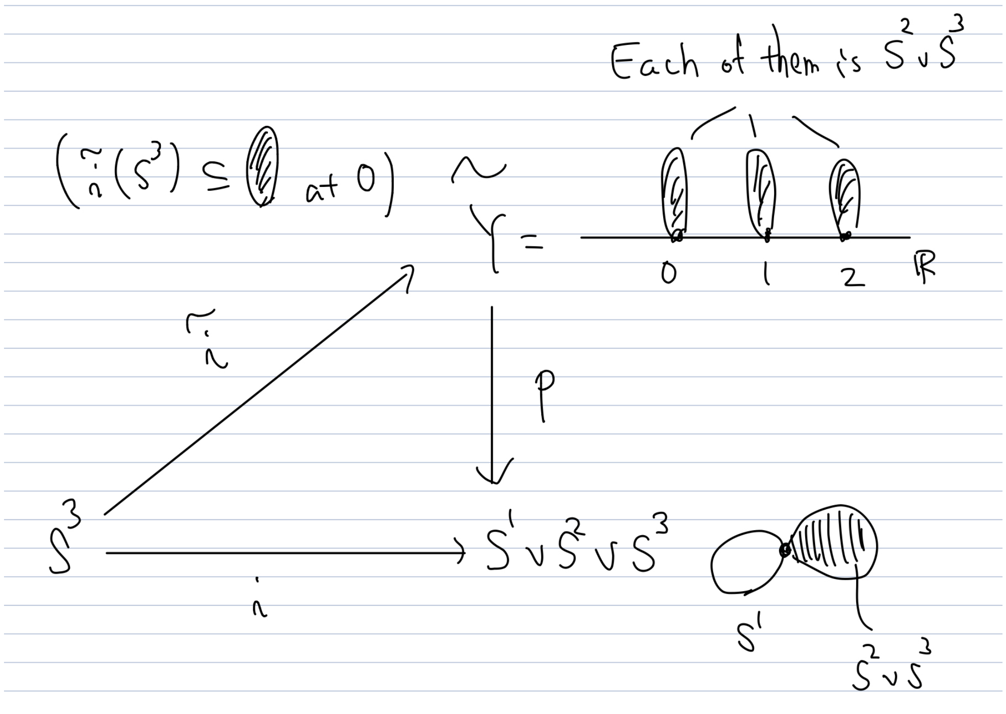

FIGURE 3. Problem 5(c)

reduced homology groups and homology groups are identical when $n \geq 2$, we have

$$H_n(X) = \begin{cases} \mathbb{Z} & (n = 0, 1, 2, 3) \\ 0 & (n \geq 4). \end{cases}$$

By Corollary 2.25, $\tilde{H}_n(S^1 \vee S^2 \vee S^3) = \tilde{H}_n(S^1) \otimes \tilde{H}_n(S^2) \otimes \tilde{H}_n(S^3)$.
Therefore,

$$\tilde{H}_n(Y) = \begin{cases} \mathbb{Z} & (n = 1, 2, 3) \\ 0 & (n = 0, n \geq 4). \end{cases}$$

For $n \geq 1$, $\tilde{H}_n(Y) = H_n(Y)$, so $H_0(Y) = H_1(Y) = H_2(Y) = H_3(Y) = \mathbb{Z}$ and $H_n(Y) = 0$ for all $n \geq 4$.

Exercise. (Problem 5(b)) We claim that the universal cover is $\mathbb{R} \times S^2$. $p(\theta, (x, y, z)) = ((\cos\theta, \sin\theta), (x, y, z))$ is a covering map. Moreover, $\pi_1(\mathbb{R} \times S^2) = \pi_1(\mathbb{R}) \times \pi_1(S^2) = 0 \times 0 = 0$, so $\mathbb{R} \times S^2$ is simply connected. Therefore, $\mathbb{R} \times S^2$ is indeed a universal cover of X.

$\mathbb{R} \times S^2$ is homeomorphic to $(0, 1) \times S^2$. This space deformation retracts to S^2 because $(0, 1) \times S^2$ is homeomorphic to an open ball with its center removed. Thus their homology groups are $H_2(\tilde{X}) = H_0(\tilde{X}) = \mathbb{Z}$ and $H_n(\tilde{X}) = 0$ for all other n.

Exercise. (Problem 5(c)) We claim that the universal covering space is the real line with $S^2 \vee S^3$ attached to each of its integral points (Figure 3). Since S^2 and S^3 are both contractible, the wedge sum must be contractible. Attaching contractible spaces to each integral point of $\mathbb{R}$, which itself is contractible, gives a contractible space. The covering map p can be defined in an obvious way. Every point on $\mathbb{R}$ can be mapped to S^1 by $\theta \to (\cos(\theta), \sin(\theta))$, and each copy of $S^2 \vee S^3$ can be mapped identically to $S^2 \vee S^3$. The i in Figure 3 is the obvious inclusion map, and $\tilde{i}$ sends S^3 into the copy of $S^2 \vee S^3$ that is attached to 0 on $\mathbb{R}$. (It does not matter which copy, but it is necessary to specify which.) Then the diagram clearly commutes.

By the Mayer-Vietoris sequence, we have an exact sequence $H_3((S^1 \vee S^2) \cap S^3) \to H_3(S^1 \vee S^2) \oplus H_3(S^3) \xrightarrow{\psi} H_3(S^1 \vee S^2 \vee S^3) \to H_2((S^1 \vee S^2) \cap S^3)$. (To be precise, we need $S^1 \vee S^2$ with a small neighborhood and S^3 with a small neighborhood, such that the union of the interiors is $S^1 \vee S^2 \vee S^3$ and the intersection deformation retracts onto a point.) Then $H_n((S^1 \vee S^2) \cap S^3) = 0$ for $n = 2, 3$. Therefore, ψ is an isomorphism. $H_3(S^1 \vee S^2) = 0$ by the Mayer-Vietoris sequence $0 = H_3(S^1) \oplus H_3(S^2) \to H_3(S^1 \vee S^2) \to H_3(S^1 \cap S^2) = 0$ where $S^1, S^2 \subset S^1 \vee S^2$ are technically S^1 and S^2 with a small neighborhood. Therefore, instead of ψ, we can consider the map $\psi' : H_3(S^3) \to H_3(S^1 \vee S^2 \vee S^3)$ defined by $\psi'(x) = \psi(0, x)$. By construction of the Mayer-Vietoris sequence, ψ' is induced by the inclusion map i. Since homology is a covariant functor, p^* and $\tilde{i}^*$, which are induced by p and $\tilde{i}$, must commute with $\psi' = i^*$. In other words, $i^* = \psi' = p^* \circ \tilde{i}^*$. Since i^* is an isomorphism, $\tilde{i}^*$ must be injective. This implies $H_3(\tilde{Y})$ contains an isomorphic copy of $H_3(S^3) = \mathbb{Z}$.

We calculate in Part (b) that $H_3(\tilde{X}) = 0$. Therefore, $H_3(\tilde{X}) \neq H_3(\tilde{Y})$.