MATH 611 (DUE 11/6)

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1. Simplicial and Singular Homology

Exercise. (Problem 14) Determine whether there exists a short exact sequence $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$. More generally, determine which abelian groups A fit into a short exact sequence $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ with p prime. What about the case of short exact sequences $0 \to A \to \mathbb{Z}_n \to 0$?

Proof. Let $\phi_1 : \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2, \phi_2 : \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4$ be defined such that $\phi_1(a) = (2a, a)$ and $\phi_2(a, b) = 2b - a$. Then $\ker(\phi_1) = 0, \operatorname{Im}(\phi_1) = \ker(\phi_2) = \{(2k, k) \mid 0 \leq k \leq 3\}$ and $\operatorname{Im}(\phi_2) = \mathbb{Z}_4$. Thus this is indeed an exact sequence.

Finish this!

□

Exercise. (Problem 15) For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ if and only if the map $A \to B$ is surjective and $D \to E$ is injective. Hence, for a pair of spaces (X, A), the inclusion $A \to X$ induces isomorphisms on all homology groups if and only if $H_n(X, A) = 0$ for all n.

Proof. Suppose $C = 0$. $\operatorname{Im}(\phi_{AB}) = \ker(\phi_{BC}) = B$, so ϕ_{AB} is surjective. $\ker(\phi_{DE}) = \operatorname{Im}(\phi_{CD}) = \{0\}$, so ϕ_{DE} is injective.

On the other hand, suppose ϕ_{AB} is surjective and ϕ_{DE} is injective. $\operatorname{Im}(\phi_{CD}) = \ker(\phi_{DE}) = \{0\}$, so ϕ_{CD} is the zero map. Therefore, $\ker(\phi_{CD}) = C$. $\ker(\phi_{BC}) = \operatorname{Im}(\phi_{AB}) = B$, so ϕ_{BC} is the zero map. Therefore, $\operatorname{Im}(\phi_{BC}) = 0$. Hence, $C = \ker(\phi_{CD}) = \operatorname{Im}(\phi_{BC}) = 0$.

For each n, we have an exact sequence $0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$ where i is induced by the inclusion map $A \to X$ and j is the canonical quotient map. Moreover, the diagram formed by the exact sequence for each n joined by ∂ is commutative by the definition of ∂.

Apply Theorem 2.16!

□

Exercise. (Problem 16)

- Show that $H_0(X, A) = 0$ if and only if A meets each path-component of X.
- Do Part (b).

Proof.

- Let $\gamma_x + C_0(A) \in C_0(X)/C_0(A)$. Since A meets each path-component of X, there exists a path $\gamma : I \to X$ that joins a point $a \in A$ and the image of γ_x. Then γ can be seen as an element of $C_1(X)$ since γ maps a 1-simplex into X. Moreover, $\partial\gamma = \gamma_x - \gamma_a$ where $\gamma_a \in C_0(A)$ with $\operatorname{Im}(\gamma_a) = a$. Therefore, $\partial(\gamma + C_1(A)) = \gamma_x + C_0(A)$, so $\gamma_x +$

$C_0(A) \in \operatorname{Im}(\partial)$. Hence, $H_0(X, A) = \ker(\partial_0)/\operatorname{Im}(\partial_1) = (C_0(X)/C_0(A))/(C_0(X)/C_1(A)) = 0$.

Do the opposite direction.

- Do part (b).

$\square$