MATH 611 HOMEWORK (DUE 9/18)

HIDENORI SHINOHARA

Exercise. (Problem 12, Chapter 1.2) The Klein bottle is usually pictured as a subspace of $\mathbb{R}^3$ like the subspace $X \subset \mathbb{R}^3$ shown in the first figure at the right. If one wanted a model that could actually function as a bottle, one would delete the open disk bounded by the circle of self-intersection of X, producing a subspace $Y \subset X$. Show that $\pi_1(X) \approx \mathbb{Z} * \mathbb{Z}$ and that $\pi_1(Y)$ has the presentation $\langle a, b, c \mid aba^{-1}b^{-1}cb^{\epsilon}c^{-1} \rangle$ for $\epsilon = \pm 1$. Show also that $\pi_1(Y)$ is isomorphic to $\pi_1(\mathbb{R}^3 \setminus Z)$ for Z the graph shown in the figure.

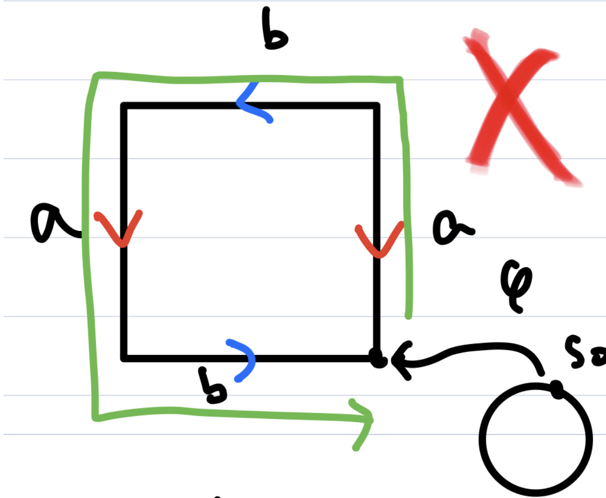
Proof. Tried for 15 minutes. I'm having a hard time figuring out the fundamental group of the Klein bottle using cell complexes. See Figure 1.

□

Exercise. (Problem 14, Chapter 1.2) Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0 cells, four 1 cells, three 2 cells, and one 3 cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ of order eight.

Proof. Tried for 45 minutes. The hardest part is to attach 2-cells to X^1. X^1 ends up being two 0-cells with 4 arrows between them. We need to figure out the relation and show that it's the quaternion group. See Figure 2, 3.

□



$$\langle a, b \mid a^{-1} b a b \rangle$$

$\Downarrow$?!?

$$\pi_1(X) \approx \mathbb{Z} * \mathbb{Z}$$

$$a^{-1} b a b = e$$

$$b a b = a$$

Figure 1. Attempt

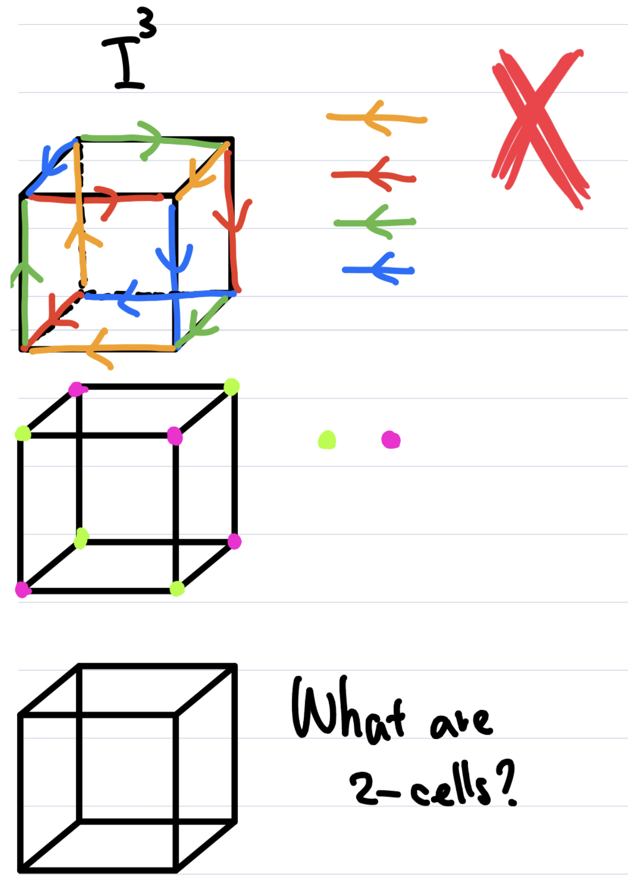


FIGURE 2. Quotient

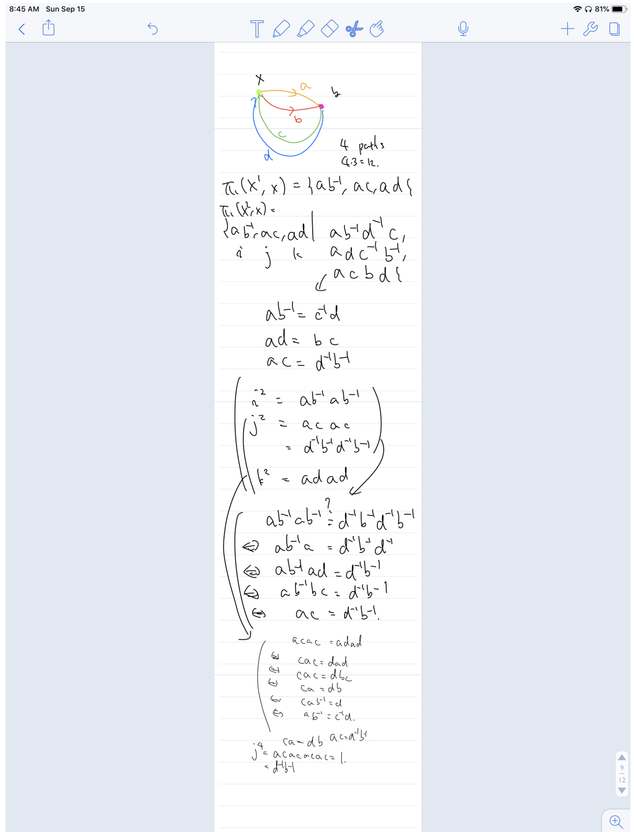


FIGURE 3. Quotient