MATH 611 (DUE 11/13)

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1. Simplicial and Singular Homology

Exercise. (Problem 20) Show that $H(X) = H(SX)$ for all n, where SX is the suspension of X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified.

Proof. It suffices to only prove the second part. Let $(CX)^k$ denote the union of k cones with their bases identified. Specifically, $(CX)^2 = SX$. We claim that $\tilde{H}_n(X) \cong \tilde{H}_{n+1}((CX)^{k+1})$ for all $k \geq 1$.

Let $i \geq 1$. We have $\tilde{H}_i(CX) \to \tilde{H}_i((CX)^{k+1}) \to \tilde{H}_i((CX)^{k+1}, CX) \to \tilde{H}_{i-1}(CX)$. Since CX is contractible by the deformation retract $f_t : (x, s) \mapsto (x, s(1-t))$, $\tilde{H}_i(CX) = \tilde{H}_{i-1}(CX) = 0$. By the exactness, $\tilde{H}_i((CX)^{k+1}) = \tilde{H}_i((CX)^{k+1}, CX)$. Let x be the north pole of CX. We will consider the inclusion $((CX)^{k+1} - x, CX - x) \to ((CX)^{k+1}, CX)$. By the excision theorem, $\tilde{H}_i((CX)^{k+1} - x, CX - x) = \tilde{H}_i((CX)^{k+1}, CX)$.

We will consider the exact sequence $\tilde{H}_i((CX)^{k+1} - x) \to \tilde{H}_i((CX)^{k+1} - x, CX - x) \to \tilde{H}_{i-1}(CX - x) \to \tilde{H}_{i-1}((CX)^{k+1} - x)$. Since $(CX)^{k+1} - x$ is contractible by the same argument as CX, $\tilde{H}_i((CX)^{k+1} - x) = \tilde{H}_{i-1}((CX)^{k+1} - x) = 0$. By the exactness, we have $\tilde{H}((CX)^{k+1} - x, CX - x) = \tilde{H}_{i-1}(CX - x)$. Since $CX - x$ is homeomorphic to X, $\tilde{H}_{i-1}(CX - x) = \tilde{H}_{i-1}(X)$. Hence, $\tilde{H}_i((CX)^{k+1}) = \tilde{H}_{i-1}(X)$. $\square$

Exercise. (Problem 22) Prove by induction on dimension the following facts about the homology of a finite-dimensional CW complex X, using the observation that X^n/X^{n-1} is a wedge sum of n spheres:

- If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.
- $H_n(X)$ is free with basis in bijective correspondence with the n cells if there are no cells of dimension $n-1$ or $n+1$.
- If X has k n-cells, then $H_n(X)$ is generated by at most k elements.

Proof.

- X^0 is a set of points, so it is clear that $H_i(X) = 0$ for $i > 0$. Let $k \geq 0$. Suppose that $H_i(X) = 0$ for $i > k$. Let $n = k+1$. Then we have an exact sequence $H_i(X^{n-1}) \to H_i(X^n) \to H_i(X^n, X^{n-1})$ for any $i > n$. Since (X^n, X^{n-1}) is a good pair, $H_{n+1}(X^n, X^{n-1}) = \tilde{H}_{n+1}(X^n/X^{n-1}) = \tilde{H}_{n+1}(\vee_\alpha S^n) = \oplus_\alpha 0 = 0$. By the inductive hypothesis, $H_i(X^{n-1}) = 0$. Therefore, the exactness of $0 \to H_i(X^n) \to 0$ implies that $H_i(X^n) = 0$ for all $i > n$.

  Moreover, the exact sequence $0 = H_n(X^{n-1}) \to H_n(X^n) \xrightarrow{\phi} H_n(X^n/X^{n-1})$ shows that ϕ is injective. This means that $H_n(X^n/X^{n-1}) = H_n(\wedge_\alpha S^n) = \oplus_\alpha \mathbb{Z}$ contains an isomorphic copy of $H_n(X^n) = H_n(X)$. Therefore, $H_n(X)$ must be free.

- Let $n \geq 1$ be fixed. (It does not make sense to discuss the existence of cells of dimension $0-1=-1$, so it makes sense to assume $n \geq 1$.) Let $P(k)$ be the statement that for any CW complex X of dimension k, if X contains no cells of dimension $n-1$ or $n+1$, then $H_n(X)$ must be free with basis in bijective correspondence with the n-cells.

  When $k < n$, this is true by Part (a) of this problem. Suppose $k = n$. We will consider the long exact sequence $H_n(X^{n-1}) \to H_n(X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$. Since X^{n-1} contains no cells of dimension $n-1$, $X^{n-1} = X^{n-2}$. By Part (a), $H_n(X^{n-1}) = H_{n-1}(X^{n-1}) = 0$. By the exactness, $H_n(X^n) = H_n(X^n, X^{n-1})$. Since (X^n, X^{n-1}) is a good pair, $H_n(X^n, X^{n-1}) = \tilde{H}_n(X^n/X^{n-1}) = \tilde{H}_n(\vee_\alpha S^n)$ where we have one S^n for each n-dimensional cell in X^n. Therefore, $\tilde{H}_n(\vee_\alpha S^n) = \oplus_\alpha \mathbb{Z}$ where we have $\mathbb{Z}$ for each n-dimensional cell in X^n. Hence, $H_n(X^n)$ is free with basis in bijective correspondence with the cells of dimension n in $X^n = X$, so $P(k)$ is true when $k = n$.

  Suppose that $P(k)$ is true for some $k \geq n$. We will prove $P(k+1)$. We will consider the long exact sequence $H_{n+1}(X^{k+1}, X^k) \to H_n(X^k) \to H_n(X^{k+1}) \to H_n(X^{k+1}/X^k)$. Since (X^{k+1}, X^k) is a good pair, $H_{n+1}(X^{k+1}, X^k) = \tilde{H}_{n+1}(X^{k+1}/X^k)$ and $H_n(X^{k+1}, X^k) = \tilde{H}_n(X^{k+1}/X^k)$.
  - By the inductive hypothesis, $H_n(X^k)$ is free with basis in bijective correspondence with the n-cells.
  - Since X^{k+1}/X^k is a wedge sum of S^{k+1} with $k \geq n$, $\tilde{H}_n(X^{k+1}/X^k) = 0$.
  - If $k > n$, then $\tilde{H}_{n+1}(X^{k+1}/X^k) = H_{n+1}(\vee_\alpha S^{k+1}) = 0$. If $k = n$, then X contains no cells of dimension $k+1 = n+1$. Therefore, $\tilde{H}_{n+1}(X^{k+1}/X^k) = 0$. In both cases, $\tilde{H}_{n+1}(X^{k+1}/X^k) = 0$.

  By the exactness, $H_n(X^k)$ is isomorphic to $H_n(X^{k+1})$, so $H_n(X^{k+1}) = H_n(X)$ is free with basis in bijective correspondence with the n-cells.

  By mathematical induction, $P(k)$ is true for any $k \in \mathbb{N}$.

$\square$

Exercise. (Problem 27) Let $f : (X, A) \to (Y, B)$ be a map such that both $f : X \to Y, f : A \to B$ are homotopy equivalences.

- Show that $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all n.
- For the case of the inclusion $f : (D^n, S^{n-1}) \to (D^n, D^n \setminus \{0\})$, show that f is not a homotopy equivalence of pairs - there is no $g : (D^n, D^n \setminus \{0\}) \to (D^n, S^{n-1})$ such that fg and gf are homotopic to the identity through maps of pairs.

Proof.

- For each $n \geq 1$, we have an exact sequence $H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X)$ and another one with X, A replaced with Y, B. Moreover, they are connected by homomorphisms $f_* : H_n(A) \to H_n(B), f_* : H_n(X) \to H_n(Y), f_* : H_n(X, A) \to H_n(Y, B)$ such that the diagram commutes. (naturality) Since $f : X \to Y$ and $f : A \to B$ are both homotopy equivalences, $f_* : H_n(X) \to H_n(Y), f_* : H_n(A) \to H_n(B)$ are isomorphisms. By the Five lemma, $f_* : H_n(X, A) \to H_n(X, B)$ is an isomorphism.

The exact sequence $H_1(A) \to H_1(X) \to H_1(X, A) \to 0$ can be extended to $H_1(A) \to H_1(X) \to H_1(X, A) \to 0 \to 0$ by appending 0 at the end. Using the same argument as above, $f_* : H_1(X, A) \to H_1(Y, B)$ is an isomorphism.

- Suppose $f : (D^n, S^{n-1}) \to (D^n, D^n - \{0\})$ is a homotopy equivalence. Then there exists a $g : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity maps in corresponding domains. Since g is continuous, $g(\overline{D^n - \{0\}}) = \overline{g(D^n - \{0\})} \subset \overline{S^{n-1}} = S^{n-1}$. Therefore, g maps D^n into S^{n-1}. Since f maps S^{n-1} into D^n, $g \circ f$ maps S^{n-1} into S^{n-1}. We know this is homotopic to the identity map from the problem statement. Similarly, $f \circ g$ maps D^n into D^n and we know this is homotopic to the identity map from the problem statement. Therefore, this implies that D^n and S^{n-1} are homotopy equivalent. However, this is false because D^n is contractible but S^{n-1} is not.

  Hence, f cannot be homotopy equivalent.

$\square$