MATH 611 HOMEWORK (DUE 10/16)

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Exercise. (Problem 16) Given maps $X \to Y \to Z$ such that both $Y \to Z$ and the composition $X \to Z$ are covering spaces, show that $X \to Y$ is a covering space if Z is locally path-connected, and show that this covering space is normal if $X \to Z$ is a normal covering space.

Proof. Let $p : X \to Y, q : Y \to Z$ be given such that q and $q \circ p$ are both covering maps. Let $y_0 \in Y$ be given. It suffices to show that there exists a neighborhood of y_0 that is evenly covered by p. (Hatcher does not require a covering map be surjective.)

Let $z_0 = q(y_0)$. Let U_{z_0} be a locally path-connected neighborhood of z_0 contained in the intersection of the following two neighborhoods:

- A neighborhood of z_0 that is evenly covered by q.
- A neighborhood of z_0 that is evenly covered by $q \circ p$.

Those two neighborhoods of z_0 must exist because q and $q \circ p$ are covering maps. Since Z is locally path-connected, any neighborhood of z_0 contains a path-connected neighborhood of z_0. Therefore, such U_{z_0} must exist. Moreover, any neighborhood contained in an evenly covered neighborhood is evenly covered. Therefore, U_{z_0} is a path-connected neighborhood of z_0 that is evenly covered by both q and $q \circ p$.

Since U_{z_0} is evenly covered by q and $q \circ p$,

- Let $\coprod_\alpha U_{x_\alpha} = (q \circ p)^{-1}(U_{z_0})$ where $q \circ p$ maps each U_{x_α} into U_{z_0} homeomorphically.
- Let $\coprod_\beta U_{y_\beta} = q^{-1}(U_{z_0})$ where q maps each U_{y_β} into U_{z_0} homeomorphically.

Draw a figure.

Since $z_0 = q(y_0)$ and q is an covering map, there exists U_{y_β} such that $y_0 \in U_{y_\beta}$. For simplicity, we will call it U_{y_0}. In other words, U_{y_0} is a neighborhood of y_0 such that q is a homeomorphism between U_{y_0} and U_{z_0}.

We claim that U_{y_0} is a neighborhood of y_0 that is evenly covered by p by showing that there exists a subset I of the index set such that $p^{-1}(U_{y_0}) = \coprod_{\alpha \in I} U_{x_\alpha}$.

Show this!

Show that p is a homeomorphism between U_{x_α} and U_{y_0}. Let $\alpha \in I$. Then $U_{x_\alpha} \subset p^{-1}(U_{y_0})$. Then $p(U_{x_\alpha}) \subset U_{y_0}$.

□

Exercise. (Problem 18) For a path-connected, locally path-connected, and semilocally simply-connected space X, call a path-connected covering space $X \to X$ abelian if it is normal and has abelian deck transformation group. Show that X has an abelian covering space that is a covering space of every other abelian covering space of X, and that such a 'universal' abelian covering space is unique up to isomorphism. Describe this covering space explicitly for $X = S^1 \vee S^1$ and $X = S^1 \vee S^1 \vee S^1$.

Proof. We will consider the commutator subgroup $H = [\pi_1(X, x_0), \pi_1(X, x_0)] = \{[a, b] \mid a, b \in \pi_1(X, x_0)\}$ of $\pi_1(X, x_0)$. Since H is a subgroup of $\pi_1(X, x_0)$ and X is path-connected, locally path connected, and semilocally simply connected, there exists a path-connected covering space $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ such that $p_*(\pi_1(\tilde{X}, \tilde{x_0})) = H$ by Theorem 1.38.

By Proposition 1.39(b), $G(\tilde{X})$ is isomorphic to the quotient $N(H)/H$.

- Since H is the commutator subgroup, H is a normal subgroup of $\pi_1(X, x_0)$. Thus $N(H) = \pi_1(X, x_0)$. Moreover, Proposition 1.39(a) asserts that $\tilde{X}$ is normal because $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ is normal.
- Since H is the commutator subgroup of $\pi_1(X, x_0) = N(H)$, $N(H)/H$ is abelian.

Therefore, $\tilde{X}$ is an abelian covering space of X.

- Show that $\tilde{X}$ is the 'universal' abelian covering space.
- Show uniqueness.

- What is the hypothesis?
  - X is a path-connected, locally path-connected, semilocally simply-connected space.
- What is the conclusion?
  - There exists a normal covering space of X $p : \tilde{X} \to X$ such that $G(\tilde{X})$ is abelian.
  - X has an abelian covering space that is a covering space of every other abelian covering space of X.
  - A universal abelian covering space is unique up to isomorphism.
  - Find the universal covering space of $S^1 \vee S^1$ and $S^1 \vee S^1 \vee S^1$.
- Introduce suitable notations.
  - $H = p_*(\pi_1(X, x_0))$.
- Separate the various parts of the hypothesis.
- Find the connection between the hypothesis and the conclusion.
  - "X is a path-connected, locally path-connected, semilocally simply-connected space." This condition sounds a lot like Theorem 1.38 on P.67. By using theorem 1.38, we can associate some group to each covering map.
  - "$\tilde{X}$ is a normal covering space of X." By Proposition 1.39 on P.71, $\tilde{X}$ is normal if and only if H is a normal subgroup of $\pi_1(X, x_0)$.
  - $G(\tilde{X})$ is abelian. By Proposition 1.39 on P.71, $G(\tilde{X})$ is isomorphic to the quotient $\pi_1(X, x_0)/H$ because $\tilde{X}$ is normal. Thus $\pi_1(X, x_0)/H$ is abelian.
- Have you seen it before?
  - This might be similar to constructing the universal covering space.
- Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.
  - Showing uniqueness up to isomorphism sounds like the universal covering space theorem.
- Keep only a part of the hypothesis, drop the other part; is the conclusion still valid?
- Could you derive something useful from the hypothesis?
- Could you think of another hypothesis from which you could easily derive the conclusion?

- Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?
- Did you use the whole hypothesis?

□

Exercise. (Problem 19) Use the preceding problem to show that a closed orientable surface M_g of genus g has a connected normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ (the product of N copies of $\mathbb{Z}$) if and only if $n \leq 2g$. For $n = 3$ and $g \geq 3$, describe such a covering space explicitly as a subspace of $\mathbb{R}^3$ with translations of $\mathbb{R}^3$ as deck transformations.

Proof. Suppose $n \leq 2g$. Then $\pi_1(M_g) = \langle a_1, \cdots, a_{2g} \mid [a_1, a_2] \cdots [a_{2g-1}, a_{2g}] \rangle$. Let H be the subgroup of $\pi_1(M_g)$ generated by $a_1, \cdots, a_{2g-n}$ and the set $\{[a_i, a_j] \mid i \neq j\}$. Since H is a subgroup of $\pi_1(M_g)$, there exists a covering space $p : \tilde{M}_g \to M_g$ by Theorem 1.38 such that $p_*(\pi_1(\tilde{M}_g)) = H$.

Prove that H is a normal subgroup of $\pi_1(M_g)$.

Therefore, by Proposition 1.39(a), $\tilde{M}_g$ is normal.

By Proposition 1.39(b), $G(\tilde{M}_g)$ is isomorphic to the quotient $N(H)/H$. Since H is normal, $N(H) = \pi_1(M_g)$. Therefore, $G(\tilde{M}_g)$ is isomorphic to $\pi_1(M_g)/H$ where H contains all commutators of $\pi_1(M_g)$. Thus $G(\tilde{M}_g)$ is abelian, so $\tilde{M}_g$ is an abelian covering space.

Moreover,

$$\begin{aligned} G(\tilde{M}_g) &= \pi_1(M_g)/H \\ &= \langle a_1, \cdots, a_{2g} \mid a_1, \cdots, a_{2g-n}, \forall i, j, [a_i, a_j] \rangle \\ &= \langle a_{2g-n+1}, \cdots, a_{2g} \mid \forall i, j, [a_i, a_j] \rangle \\ &\cong \mathbb{Z}^n. \end{aligned}$$

Finish the rest of the problem.

- List examples. $n = 1, 2, g = 1$ and $n = 1, g = 2$ are done. Try others.
- What is the hypothesis? M_g is a closed orientable surface M_g of genus g.
- What is the conclusion? M_g has a connected normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ if and only if $n \leq 2g$.
- Separate the various parts of the hypothesis.

  Closed orientable surface? I don't know what to do with it. Can I just assume that this means $M_g = (S^1 \times S^1) \vee \cdots \vee (S^1 \times S^1)$?
- Find the connection between the hypothesis and the conclusion.
  - The fundamental group of M_g is generated by $2g$ elements with no relations. If we abelianize the fundamental group of M_g, we obtain $\mathbb{Z}^{2g}$.
- Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.
  - The previous problem shows the existence of an abelian covering space, and a normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ is also abelian.
- Keep only a part of the hypothesis, drop the other part; is the conclusion still valid?

- Could you derive something useful from the hypothesis?
- Could you think of another hypothesis from which you could easily derive the conclusion?
  - If $g = 1$, then this problem is easy. For $n = 2$, consider the xy plane, and for $n = 1$, consider the infinite chain of squares.
- Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?
- Did you use the whole hypothesis?

□

Exercise. (Problem 20) Construct non-normal covering spaces of the Klein bottle by a Klein bottle and by a torus.

Proof. Figure 1 is the idea that I have for the first part. But I don't know how to show that there exists no deck transformation with that permutation.

□

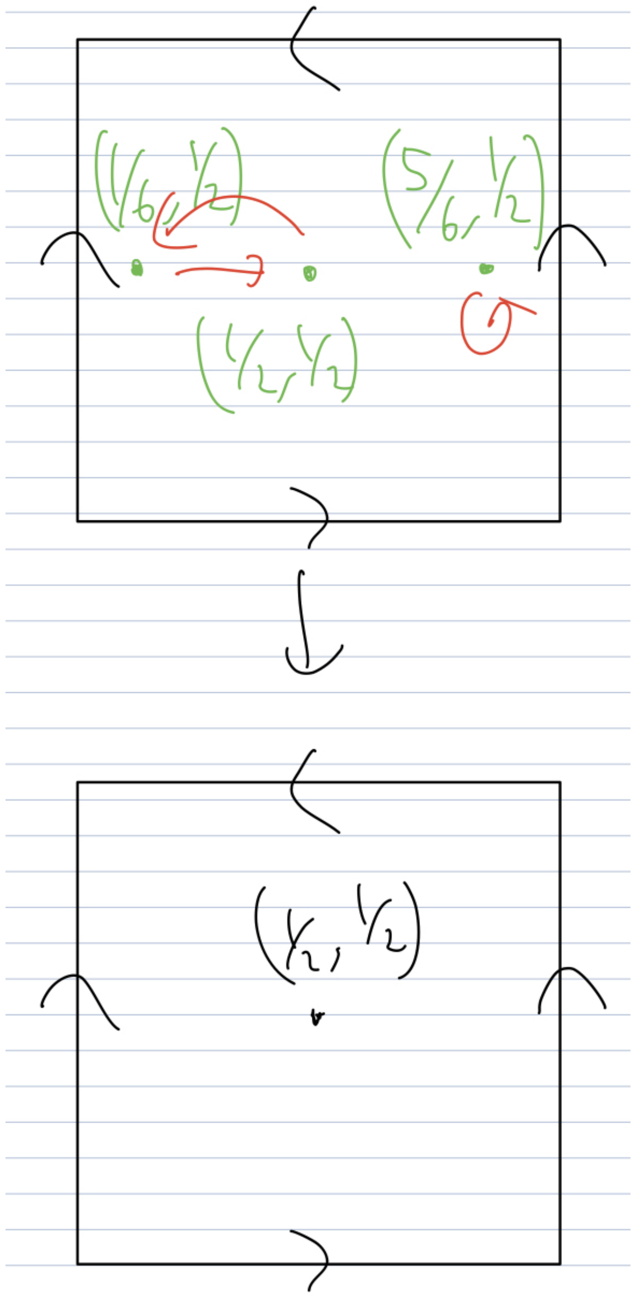
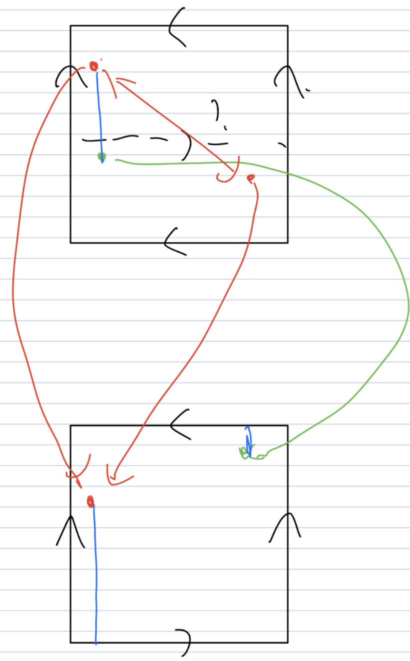


FIGURE 1. Problem 20 (Klein)



$$f(x,y) = \begin{cases} (1-x,\ 2y) & (0 \le y \le 1/2) \\ (x,\ 2y-1) & (1/2 \le y \le 1) \end{cases}$$

Figure 2. Problem 20 (Torus)