MATH 633 HOMEWORK 9

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Exercise. (Problem 1) Let $x \in F_1$. Since Ω is bounded, there exists an $R > 0$ such that $\Omega \subset C(x, R)$. Then $F_1 \setminus C(x, R)$ and $F_2 \setminus C(x, R)$ are disjoint, closed sets whose union is $\mathbb{C} \setminus C(x, R)$, which is connected. Therefore, either $F_1 \setminus C(x, R)$ or $F_2 \setminus C(x, R)$ is empty. In other words, either $F_1 \subset C(x, R)$ or $F_2 \setminus c(x, R)$.

Exercise. (Problem 2(a)) We first assume $\omega = 0$. This is reasonable because the following argument can be extended to general cases by translating every function by ω. If r, θ are continuous, it is clear that α is continuous. Suppose α is continuous. Let $r(t) = |\alpha(t)|$. Then $r(t) : [0, 1] \to (0, \infty)$ is continuous. Moreover, $\alpha(t) = r(t)e^{i\theta(t)}$, so $r(t) = |r(t)| = \left|r(t)e^{i\theta(t)}\right| = |\alpha(t)|$, so this is the only possibility for $r(t)$.

By using the principal branch of logarithm and translation, we can find $\theta(t)$ locally. Since the logarithm function and translation function are both continuous, such local θ's are continuous. Since $[0, 1]$ is compact, we can find a finite cover of $[0, 1]$ such that we have $\theta(t)$ for each open set. Two $\theta(t)$ can be patched for any two overlapping open sets by adding $2k\pi$ for an appropriate value of k. Therefore, we can find θ that is continuous and satisfies $\alpha(t) = r(t)e^{i\theta(t)}$. Any other functions $\gamma(t)$ that satisfy the conditions must satisfy $1 = \alpha(t)/\alpha(t) = (r(t)e^{i\theta(t)})/(r(t)e^{i\gamma(t)}) = e^{i(\theta(t)-\gamma(t))}$, so $\theta(t) - \gamma(t) = 2k\pi$ for some fixed $k \in \mathbb{Z}$.

Hence, we have shown that α is continuous if and only if such continuous r, θ exist and the choice of r, θ are unique up to an additive constant for θ.

Exercise. (Problem 2(b)) Again, we will assume $\omega = 0$. $\alpha(1)/\alpha(0) = (r(1)e^{i\theta(1)})/(r(0)e^{i\theta(0)}) = e^{i(\theta(1)-\theta(0))}$ because $r(1) = |\alpha(1)| = |\alpha(0)| = r(0)$. Since $\alpha(1) = \alpha(0)$, $e^{i(\theta(1)-\theta(0))} = 1$. This implies that $\theta(1) - \theta(0) = 2k\pi$ for a fixed $k \in \mathbb{Z}$. In other words, $(\theta(1) - \theta(0))/2\pi$ is always an integer.

Exercise. (Problem 2(c)) Suppose α_0 and α_1 are homotopic. Let θ_t be the θ function of α_t for each $t \in [0, 1]$. Then $(x, t) \mapsto \theta_t(x)$ is continuous. Therefore, $t \mapsto (\theta_t(1) - \theta_t(0))/2\pi$ is continuous. Since it must be integer valued, it is constant. In other words, α_0 and α_1 have the same winding number.

Suppose α_0 and α_1 have the same winding number.

Exercise. (Problem 3(a)) As $r \to 0$ with $r > 0$, $p(\alpha_r(t))$ is dominated by a_0 for any $t \in [0, 1]$. In other words, $p \circ \alpha_r$ lies in a small disk around a_0 that is disjoint from 0. Thus the winding number is 0 for a sufficiently small r.

As $r \to \infty$, $p(\alpha_r(t))$ is dominated by $a_n\alpha_r(t)^n$ for any $t \in [0, 1]$. Since multiplication by $a_n \neq 0$ is simply a rotation around the origin, $p \circ \alpha_r$ is homotopic to the function that goes around the origin n times in a positive orientation for a sufficiently large r. Thus the winding number is n for large r.

Exercise. (Problem 3(b)) Suppose p has no roots. Let $r_0, r_n > 0$ be chosen such that the winding number is 0 when $r = r_0$ and n when $r = r_n$. Such r_0, r_n exist by Problem 3(a). Then $(s,t) \mapsto (p \circ \alpha_{r_0 t + r_n(1-t)})(s)$ is a homotopy between $p \circ \alpha_{r_0}$ and $p \circ \alpha_{r_n}$ in $\mathbb{C} \setminus 0$. However, this is impossible by Problem 2(c) because the winding numbers are different. Contradiction.