MATH 611 (DUE 10/2)

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Exercise. (Problem 10, Chapter 1.3) Find all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$, up to isomorphisms of covering spaces without base points.

Proof. Let $X = S^1 \vee S^1$. By the discussion on P.70 of the textbook, we know that n-sheeted covering spaces of X are classified by equivalence classes of homomorphisms $\pi_1(X, x_0) \to S_n$. Let a, b denote paths in X as in Figure 1. We can identify each homomorphism ϕ by checking what ϕ maps a and b to. (Strictly speaking, $\pi_1(X, x_0)$ is generated by $[a], [b]$, but we will abuse notations by writing a and b instead of $[a], [b]$.)

The following are all the cases. Figure 2 shows the corresponding graphs.

- Case 1: $\phi_1(a) = \phi_1(b) = (1)$. The space that corresponds to this homomorphism is disconnected.
- Case 2: $\phi_2(a) = (12), \phi_2(b) = (1)$. This generates a connected covering space.
- Case 3: $\phi_3(a) = (1), \phi_3(b) = (12)$. This generates a connected covering space.
- Case 4: $\phi_4(a) = (12), \phi_4(b) = (12)$. This generates a connected covering space.

$\phi_1 \neq \phi_2$ and $(12)\phi_1(12) \neq \phi_2$, so ϕ_1 and ϕ_2 are not conjugates of each other. Similarly, ϕ_2 and ϕ_3 are not conjugates of each other, and neither are ϕ_1 and ϕ_3.

Thus the three graphs corresponding to Case 2, 3 and 4 in Figure 2 are all the 2-sheeted covering spaces of X.

We will take the exact same approach for the case of 3. If a certain vertex is fixed in both $\phi(a)$ and $\phi(b)$, then such a vertex is disjoint from the rest of the graph. We will use that property to reduce the possibilities.

- Case 1: $\phi_1 : a \mapsto (1), b \mapsto (1)$ The following maps are conjugates of ϕ_1
  - $a \mapsto (1), b \mapsto (1)$

  This graph is not connected because every vertex is fixed.
- Case 2: $\phi_2 : a \mapsto (12), b \mapsto (1)$ The following maps are conjugates of ϕ_2
  - $a \mapsto (23), b \mapsto (1)$
  - $a \mapsto (13), b \mapsto (1)$
  - $a \mapsto (12), b \mapsto (1)$

  This graph is not connected because vertex 3 is fixed.
- Case 3: $\phi_3 : a \mapsto (1), b \mapsto (12)$ The following maps are conjugates of ϕ_3

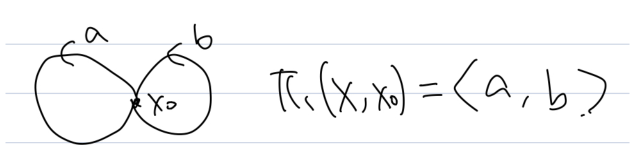

FIGURE 1. Problem 10 ($X = S^1 \vee S^1$)

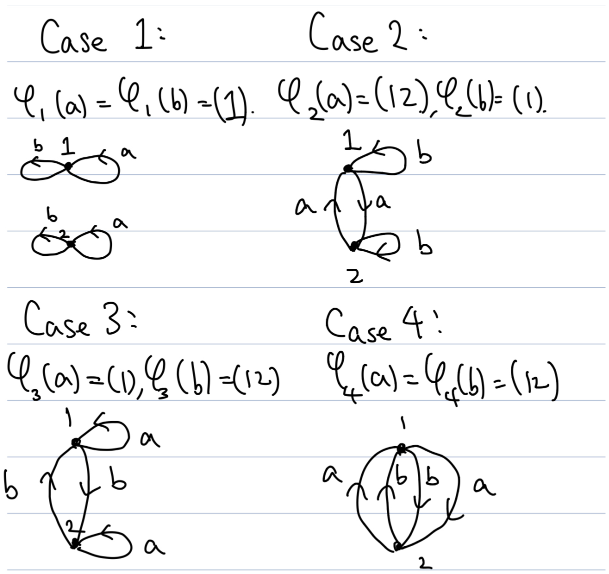

FIGURE 2. Problem 10 (2-sheeted covers)

  - $a \mapsto (1), b \mapsto (12)$
  - $a \mapsto (1), b \mapsto (23)$
  - $a \mapsto (1), b \mapsto (13)$

  This is the same as Case 2.
- Case 4: $\phi_4 : a \mapsto (12), b \mapsto (13)$ The following maps are conjugates of ϕ_4
  - $a \mapsto (13), b \mapsto (12)$
  - $a \mapsto (12), b \mapsto (23)$
  - $a \mapsto (12), b \mapsto (13)$
  - $a \mapsto (13), b \mapsto (23)$
  - $a \mapsto (23), b \mapsto (12)$
  - $a \mapsto (23), b \mapsto (13)$

  See Figure 3.
- Case 5: $\phi_5 : a \mapsto (12), b \mapsto (123)$ The following maps are conjugates of ϕ_5
  - $a \mapsto (23), b \mapsto (123)$
  - $a \mapsto (12), b \mapsto (123)$
  - $a \mapsto (12), b \mapsto (132)$
  - $a \mapsto (13), b \mapsto (132)$
  - $a \mapsto (13), b \mapsto (123)$
  - $a \mapsto (23), b \mapsto (132)$

  See Figure 3.
- Case 6: $\phi_6 : a \mapsto (123), b \mapsto (12)$ The following maps are conjugates of ϕ_6
  - $a \mapsto (123), b \mapsto (13)$
  - $a \mapsto (132), b \mapsto (12)$
  - $a \mapsto (132), b \mapsto (23)$
  - $a \mapsto (132), b \mapsto (13)$
  - $a \mapsto (123), b \mapsto (12)$
  - $a \mapsto (123), b \mapsto (23)$

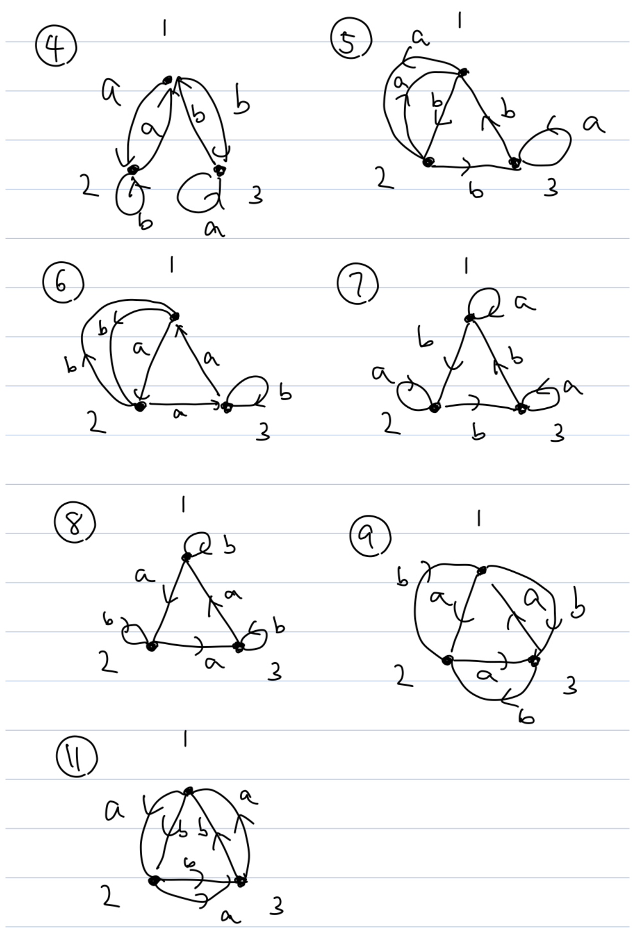

FIGURE 3. Problem 10 (3-sheeted)

See Figure 3.

- Case 7: $\phi_7 : a \mapsto (1), b \mapsto (123)$ The following maps are conjugates of ϕ_7
  - $a \mapsto (1), b \mapsto (132)$
  - $a \mapsto (1), b \mapsto (123)$

  See Figure 3.
- Case 8: $\phi_8 : a \mapsto (123), b \mapsto (1)$ The following maps are conjugates of ϕ_8
  - $a \mapsto (132), b \mapsto (1)$
  - $a \mapsto (123), b \mapsto (1)$

  See Figure 3.
- Case 9: $\phi_9 : a \mapsto (123), b \mapsto (132)$ The following maps are conjugates of ϕ_9
  - $a \mapsto (123), b \mapsto (132)$
  - $a \mapsto (132), b \mapsto (123)$

  See Figure 3.
- Case 10: $\phi_{10} : a \mapsto (23), b \mapsto (23)$ The following maps are conjugates of ϕ_{10}
  - $a \mapsto (12), b \mapsto (12)$
  - $a \mapsto (23), b \mapsto (23)$
  - $a \mapsto (13), b \mapsto (13)$

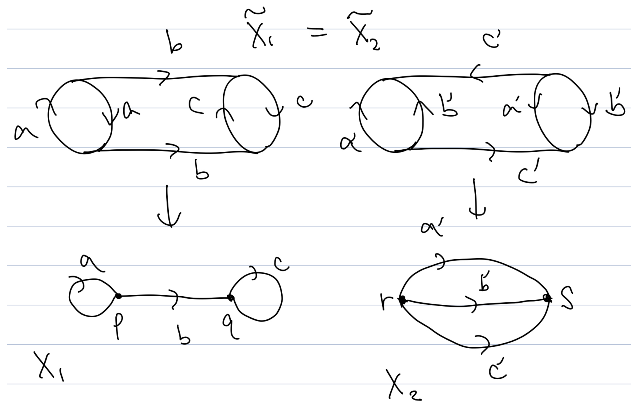

FIGURE 4. Problem 11

Vertex 1 is disconnected from the rest of the graph since it is fixed.

- Case 11: $\phi_{11} : a \mapsto (123), b \mapsto (123)$ The following maps are conjugates of ϕ_{11}
  - $a \mapsto (132), b \mapsto (132)$
  - $a \mapsto (123), b \mapsto (123)$

  See Figure 3.

Since there are 6 elements in S_3, there are 36 possible homomorphisms. The list above contains all of them. Therefore, Figure 3 lists all the possible 3-sheeted covers. $\square$

Exercise. (Problem 11, Chapter 1.3) Construct finite graphs X_1 and X_2 having a common finite-sheeted covering space $\tilde{X}_1 = \tilde{X}_2$, but such that there is no space having both X_1 and X_2 as covering spaces.

Proof. Figure 4 shows X_1, X_2 and $\tilde{X}_1 = \tilde{X}_2$.

We claim that there exists no space having both X_1 and X_2 as covering spaces. On the contrary, suppose there exists such a space X with covering maps $p_1 : X_1 \to X, p_2 : X_2 \to X$. Then every point in X must have a neighborhood that homeomorphic to an open subset of X_1. Since X_1 is a graph, that means X is locally a line and a vertex with edges. In other words, X must be a graph.

There must exist a neighborhood of $p_1(p)$ and a neighborhood of p such that they are homeomorphic. Since p is a vertex of degree 3, $p_1(p)$ must be a vertex of degree 3 as well. Similarly, $p_1(q)$ must be a vertex of degree 3 as well.

Since p, q are the only vertices of X_1, X contains at most two vertices and their degrees must be 3. Since the sum of degrees of all vertices must be even from elementary graph theory, X must contain two vertices of degree 3.

If X only consists of loops, then the degree of each vertex will be even. Thus the two vertices must be joined by at least one edge. Then if one vertex has a loop, the other must have a loop as well in order to have degree 3. If there exists another edge joining the two vertices, there must be a third one in order for the two vertices to have degree 3. Therefore, X_1, X_2 are the only graphs with two vertices of degree 3.

Suppose that X_1 is a covering space of X_2 with a covering map $f : X_1 \to X_2$. Without loss of generality, $f(p) = r, f(q) = s$. Consider the path a' in X_2. Lifting a' to X_1 will result in a path from p to q. This implies that f maps points on the path b into points on a path a'.

Now consider the path b' in X_2. Lifting b' to X_1 will again result in a path from p to q. This implies that f maps points on the path b into points on a path b'.

This implies that every point on the path b must be mapped to r or s. This is a contradiction because f is continuous and $\{b(t) \mid t \in [0,1]\}$ is connected, but $\{r, s\}$ is disconnected.

Thus X_1 is not a covering space of X_2.

Similarly, suppose that X_2 is a covering space of X_1 with a covering map $g : X_2 \to X_1$. Without loss of generality, $g(r) = p, g(s) = q$. This implies $g^{-1}(p) = \{r\}$, so the number of sheets is 1. In other words, g is injective. Consider the path a in X_1. Lifting a to X_2 results into a loop based at r. Since $a : I \to X_1$ is injective, $\tilde{a} : I \to X_2$ is injective since $g \circ \tilde{a} = a$. Then $\tilde{a}(t) = s$ for some $t \in [0,1]$, so $a(t) = g(\tilde{a}(t)) = g(s) = q$. However, q is not a point on a. This is a contradiction, so X_2 is not a covering space of X_1.

Hence, there exists no space that has both X_1 and X_2 as covering spaces. □

Exercise. (Problem 14, Chapter 1.3) Find all the connected covering spaces of $\mathbb{R}\mathbf{P}^2 \vee \mathbb{R}\mathbf{P}^2$.

Proof. Let $X = \mathbb{R}\mathbf{P}^2 \vee \mathbb{R}\mathbf{P}^2$. By Theorem 1.38 of the textbook, it suffices to check all the conjugacy classes of subgroups of $\pi_1(X, x_0)$ and corresponding covering spaces.

Since $\pi_1(\mathbb{R}\mathbf{P}^2) = \langle a \mid a^2 \rangle$, $\pi_1(X, x_0) = \langle a, b \mid a^2 = b^2 = e \rangle$ by Van Kampen. Since $a^2 = b^2 = e$, we can express each element in $\pi_1(X, x_0)$ uniquely as a word which alternates a, b.

We will use three properties of elements in $\pi_1(X, x_0)$:

- Every word can be expressed as a finite alternating sequence of a and b.

  Maybe explain more.

- The parity of the length of a word does not depend on its representation because $a^2 = b^2 = 1$.

  Maybe explain more.

- The parity of the length of a word is invariant under conjugation because conjugating a word appends an even number of letters.
- The parity of the number of a in a word is invariant under conjugation because conjugating a word appends an even number of a.

  Maybe explain more.

- Similarly, the parity of the number of b in a word is invariant under conjugation because conjugating a word appends an even number of b.

We will call a subgroup even if the length of each element in it is even, and we will call a subgroup odd otherwise. Since the parity of the length of a word is invariant under conjugation, even subgroups and odd subgroups are not conjugates of each other. Therefore, we will classify even subgroups and odd subgroups separately.

- Let $H \subset G$ be an even subgroup. Every even length word can be expressed as $(ab)^k$ for some $k \in \mathbb{Z}$. Since $(ab)^k = e$ if and only if $k = 0$, this number k is unique to each word of odd length. Let $\phi : H \to \mathbb{Z}$ be defined such that $\phi((ab)^k) = k$. Then $\phi(H)$ is a subgroup of $(\mathbb{Z}, +)$. Since $\mathbb{Z}$ is an infinite cyclic group, $\phi(H) = (n)$ for some non-negative integer n. Thus $H = \langle (ab)^n \rangle$. For each n, Figure

shows the covering space corresponding to $\langle (ab)^n \rangle$. Since the number of sheets is the same as the index of the subset by Proposition 1.32 of the textbook, $[\pi_1(X, x_0) : \langle (ab)^n \rangle]$ is infinity when $n = 0$ and the index is $2n$ when $n \geq 1$. Since the index of a subgroup is invariant under conjugation, $\langle (ab)^n \rangle \not\sim \langle (ab)^m \rangle$ whenever $n \neq m$.

- Let $H \subset G$ be an odd subgroup. Every odd element is an alternating sequence of a and b which starts and ends with the same letter. Thus, every odd element is conjugate to a or b. Let $w \in H$ be a word of odd length. Then w is xax^{-1} or xbx^{-1} for some x. Then $x^{-1}Hx$ contains a or b. This implies that every odd subgroup is conjugate to an odd subgroup containing either a or b. (possibly both) Since we are classifying subgroups up to conjugation, we can assume that H contains a or b without loss of generality.

  We will first assume that $a \in H$ since the argument for the case $b \in H$ will be the same. Let H' be a subset of H containing only words of even length. Since $e \in H'$ and xy^{-1} is another word of even length for each $x, y \in H'$, H' is a subgroup of H. Then H' is an even subgroup of G, so by the argument above, $H' = \langle (ab)^n \rangle$ for some $n \geq 0$.

  Let $w \in H$.
  - If the length of w is even, $w \in H' = \langle (ab)^n \rangle = \langle a, (ab)^n \rangle$.
  - If the length of w is odd, $aw \in H' = \langle (ab)^n \rangle = \langle a, (ab)^n \rangle$. Since $aw \in \langle a, (ab)^n \rangle$, $w \in \langle a, (ab)^n \rangle$.

  Therefore, $H = \langle a, (ab)^n \rangle$.
  - When $n = 0$, $H = \langle a \rangle$.
  - When $n = 1$, $H = \langle a, ab \rangle = \langle a, b \rangle = G$.
  - When $n \geq 2$, $H = \langle a, (ab)^n \rangle$.

  Using the same argument, if $b \in H$, we can conclude that H is one of the following:
  - When $n = 0$, $H = \langle b \rangle$.
  - When $n = 1$, $H = \langle b, ab \rangle = \langle a, b \rangle = G$.
  - When $n \geq 2$, $H = \langle b, (ab)^n \rangle$.

  By putting these together, if H is an odd subgroup, w

$\square$