

# MATH 602(HOMEWORK 1)

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## Exercise. 1

- Let  $p \in V(I \cap J)$ . For any  $\sum_{i=1}^n f_i g_i \in IJ$ , we have  $f_i g_i \in I \cap J$  for each  $i$ . Thus  $(\sum_{i=1}^n f_i g_i)(p) = 0$ , so  $p \in V(IJ)$ . Let  $p \in V(IJ)$ . Let  $f \in I \cap J$ . Then  $f^2 \in IJ$ , so  $(f(p))^2 = 0$ . Thus  $f(p) = 0$ , so  $p \in V(I \cap J)$ . Therefore,  $V(I \cap J) = V(IJ)$ .  
Let  $p \in V(I) \cup V(J)$ . Then either all polynomials in  $I$  vanish at  $p$  or all polynomials in  $J$  vanish at  $p$ . Thus all the polynomials in the intersection must vanish at  $p$ . Thus  $V(I) \cup V(J) \subset V(I \cap J)$ . On the other hand, let  $p \in V(I \cap J) \setminus (V(I) \cup V(J))$ . If no such element exists, we are done. Then every polynomial in the intersection vanishes at  $p$ . Let  $f \in I$  and  $g \in J$  be polynomials that do not vanish at  $p$ . Then  $fg \in I \cap J$ , so  $(fg)(p) = 0$ . However, this is impossible because  $f(p) \neq 0$  and  $g(p) \neq 0$ . Therefore,  $V(I) \cup V(J) = V(I \cap J)$ .
- $p \in V(I + J)$  if and only if  $\forall f \in I + J, f(p) = 0$  if and only if  $\forall f \in I, f(p) = 0$  and  $\forall f \in J, f(p) = 0$  if and only if  $p \in V(I) \cap V(J)$ .
- If every polynomial in  $J$  vanishes at a point, every polynomial in  $I$  must vanish at that point.
- If a polynomial vanishes in  $Y$ , then it must vanish in  $X$ .
- TODO

## Exercise. 2

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$$\begin{aligned}
 y \in (I_1 + I_2)^e &\iff y \in f(I_1 + I_2)B \\
 &\iff \exists x_1, x_2 \in I_1, I_2, b \in B, y = f(x_1 + x_2)b \\
 &\iff \exists x_1, x_2 \in I_1, I_2, b \in B, y = f(x_1)b + f(x_2)b \\
 &\iff y \in I_1^e + I_2^e.
 \end{aligned}$$

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$$\begin{aligned}
 y \in (I_1 \cap I_2)^e &\implies y \in f(I_1 \cap I_2)B \\
 &\implies \exists x \in I_1 \cap I_2, b \in B, y = f(x)b \\
 &\implies (\exists x \in I_1, b \in B, y = f(x)b) \text{ and } (\exists x \in I_2, b \in B, y = f(x)b) \\
 &\implies y \in I_1^e, y \in I_2^e \\
 &\implies y \in I_1^e \cap I_2^e.
 \end{aligned}$$

- $(I_1 I_2)^e = f(I_1 I_2)B = (f(I_1)f(I_2))B = (f(I_1)B)(f(I_2)B)$ .  $f(I_1)f(I_2) = f(I_1 I_2)$  because the product of two ideals consists of a finite sum of elements and  $f$  preserves finite sums.

- Let  $x \in J_1^c + J_2^c$ . Then  $x \in f^{-1}(J_1) + f^{-1}(J_2)$ . Then  $x = a + b$  where  $a \in f^{-1}(J_1)$  and  $b \in f^{-1}(J_2)$ . This implies  $x = a + b$  where  $f(a) \in J_1$  and  $f(b) \in J_2$ . Then,  $f(x) = f(a + b) = f(a) + f(b) \in J_1 + J_2$ , so  $x \in f^{-1}(J_1 + J_2)$ .
- $f^{-1}(J_1 \cap J_2) = f^{-1}(J_1) \cap f^{-1}(J_2)$  from set theory.
- Let  $\sum_{i=1}^n a_i b_i \in J_1^c J_2^c$  where  $a_i \in J_1^c$  and  $b_i \in J_2^c$ . Then  $f(a_i) \in J_1$  and  $f(b_i) \in J_2$ . Thus  $\sum f(a_i) f(b_i) \in J_1 J_2$ . Since  $f$  preserves product and addition,  $f(\sum a_i b_i) \in J_1 J_2$ . Thus  $\sum a_i b_i \in f^{-1}(J_1 J_2) = (J_1 J_2)^c$ .

**Exercise.** 3  $(I : J)$  is nonempty because  $0 \in (I : J)$ .  $(I : J)$  is closed under addition, and for all  $x \in R$ ,  $rJ \subset I \implies x(rJ) = r(xJ) = rJ \subset I$ . Thus  $(I : J)$  is an ideal.

- Lemma: Let  $a, b, c$  be ideals. If  $\forall x \in a, xb \subset c$ , then  $ab \subset c$ .  
Proof: Let  $\sum a_i b_i \in ab$  be given. Then each  $a_i b_i \in c$ . Since  $c$  is closed under addition,  $\sum a_i b_i \in c$ . Therefore,  $ab \subset c$ .
- Let  $x \in a$ . Then  $\forall y \in b, xy \in a$  since  $a$  is an ideal. Then  $xb \subset a$ , so  $x \in (a : b)$ .
- For all  $x \in (a : b)$ ,  $xb \subset a$ . By the Lemma above,  $(a : b)b \subset a$ .
- Let  $x \in ((a : b) : c)$ . Then  $xc \subset (a : b)$ . For all  $xz \in xc, (xz)b \subset a$ . Therefore,  $(xc)b \subset a$  by the Lemma above. Then  $x(cb) \subset a$ , so  $x(bc) \subset a$ . Hence,  $x \in (a : bc)$ .  
On the other hand, suppose  $x \in (a : bc)$ . Then  $x(bc) \subset a$ .  $x(bc) \subset a \implies (xb)c \subset a \implies xb \subset (a : c) \implies x \in ((a : c) : b)$ .  
Therefore,  $((a : b) : c) = (a : bc)$ .  
We showed that  $((a : b) : c) = (a : bc)$ . This implies  $(a : cb) = ((a : c) : b)$ . Since  $(a : bc) = (a : cb)$ , we have  $((a : b) : c) = (a : bc) = (a : cb) = ((a : c) : b)$ .
- For any  $x \in A$ ,

$$\begin{aligned}
x \in (\cap_i a_i : b) &\iff xb \subset \cap_i a_i \\
&\iff \forall i, xb \subset a_i \\
&\iff \forall i, x \subset (a_i : b) \\
&\iff x \subset \cap_i (a_i : b).
\end{aligned}$$

- For any  $x \in A$ ,

$$\begin{aligned}
x \in (a : \sum_i b_i) &\iff x(\sum_i b_i) \subset a \\
&\implies \forall i, xb_i \subset a \\
&\iff \forall i, x \subset (a : b_i) \\
&\iff x \subset \cap_i (a : b_i).
\end{aligned}$$

Therefore, it suffices to show that  $\forall i, xb_i \subset a \implies x(\sum_i b_i) \subset a$ . Let  $y_{i_1} + \dots + y_{i_n} \in \sum_i b_i$  be given where  $y_{i_j} \in b_{i_j}$ . For each  $j$ , since  $xb_{i_j} \subset a$ ,  $xy_{i_j} \in a$ . Since  $a$  is closed under finite addition,  $xy_{i_1} + \dots + xy_{i_n} \in a$ . Therefore,  $\forall i, xb_i \subset a \implies x(\sum_i b_i) \subset a$ , so  $(a : \sum_i b_i) = \cap_i (a : b_i)$ .

- Let  $bf(x) \in (a_1 : a_2)^e$  where  $b \in B$  and  $x \in (a_1 : a_2)$ .

$$\begin{aligned}
xa_2 \subset a_1 &\implies f(xa_2) \subset f(a_1) \\
&\implies f(x)f(a_2) \subset f(a_1) \\
&\implies B(f(x)f(a_2)) \subset Bf(a_1) \\
&\implies f(x)(Bf(a_2)) \subset Bf(a_1) \\
&\implies f(x)a_2^e \subset a_1^e \\
&\implies f(x) \in (a_1^e : a_2^e) \\
&\implies bf(x) \in (a_1^e : a_2^e). \\
x \in (b_1 : b_2)^c &\implies f(x) \in (b_1 : b_2) \\
&\implies f(x)b_2 \in b_1 \\
&\implies f^{-1}(f(x)b_2) \subset f^{-1}(b_1) \\
&\implies xf^{-1}(b_2) \subset f^{-1}(f(x)b_2) \subset f^{-1}(b_1) \\
&\implies xf^{-1}(b_2) \subset f^{-1}(b_1) \\
&\implies x \in (f^{-1}(b_1) : f^{-1}(b_2)) \\
&\implies x \in (b_1^c : b_2^c).
\end{aligned}$$

**Exercise.** (Problem 4) Let  $f = \sum_{i=1}^m a_i x^i, g = \sum_{i=1}^n b_i x^i \notin p[x]$ . Let  $m', n'$  be the smallest integer such that  $a_{m'}, b_{n'} \notin p[x]$ . Such  $m', n'$  must exist because  $f, g \notin p[x]$ . Then the coefficient of  $x^{m'+n'}$  in  $fg$  is  $\sum_{i=0}^{m'+n'} a_i b_{m'+n'-i}$ . Then  $a_i b_{m'+n'-i} \in p$  if and only if  $i \neq m'$ . The coefficient of  $x^{m'+n'}$  in  $fg$  is not in  $p[x]$ . Therefore,  $fg \notin p[x]$ , so  $p[x]$  is a prime ideal.

(0) is a maximal ideal of  $Q$ . However, (0) is not a maximal ideal in  $\mathbb{Q}[x]$  because  $(x)$  is a proper ideal of  $\mathbb{Q}[x]$  that properly contains (0).

**Exercise.** (Problem 5)

- (1) Since  $IJ \subset I$  and  $IJ \subset J$ ,  $IJ \subset I \cap J$ . Let  $x \in I \cap J$ . Let  $a \in I$  and  $b \in J$  such that  $a + b = 1$ . Then  $x, a \in I$  and  $x, b \in J$ . Thus  $ax + xb \in IJ$ . In other words,  $x = (a + b)x \in IJ$ .

The kernel of  $\phi : R \rightarrow R/I \times R/J$  defined by  $x \mapsto (x + I, x + J)$  is  $I \cap J$ . Thus  $R/(I \cap J)$  is isomorphic to  $R/I \times R/J$  and  $I \cap J = IJ$  as shown above.

- (2) Since  $I + J = I + J' = (1)$ ,  $a + b = c + d = 1$  for some  $a, c \in I, b \in J, d \in J'$ . Then  $1 = (a + b)(c + d) = (ac + bc + ad) + bd \in I + JJ'$  because  $ac, bc, ad \in I$  and  $bd \in JJ'$ .
- (3) Suppose  $I + J = (1)$ . Then  $a + b = 1$  for some  $a \in I$  and  $b \in J$ . Let  $m, n \in \mathbb{N}$  be given. Then  $1 = (a + b)^{m+n} \in I^m + J^n$  because  $a^i b^{m+n-i} \in I^m$  if  $i \geq m$  and  $a^i b^{m+n-i} \in J^n$  if  $i \leq m$ . The other direction is trivial.
- (4)  $I_1$  and  $I_1 \cdots I_n$  are not comaximal because  $I_1 \cdots I_n \subset I_1$ . Choose  $a_2, \dots, a_n \in I_1$  and  $b_2, \dots, b_n \in I_2, \dots, I_n$  such that  $a_i + b_i = 1$  for each  $i$ . Then  $1 = (a_2 + b_2) \cdots (a_n + b_n)$ . After expanding  $(a_2 + b_2) \cdots (a_n + b_n)$ , every expression containing some  $a_i$  belongs to  $I_1$ . The only expression that does not contain  $a_i$  is  $b_2 \cdots b_n$ , and it is contained in  $I_2 \cdots I_n$ . Thus  $I_1 + I_2 \cdots I_n = (1)$ .
- (5) We have shown that  $I_1 \cap \cdots \cap I_n = I_1 \cdots I_n$  when  $n = 2$ . Once we show this for some  $n \geq 2$ ,  $I_1$  and  $I_2 \cdots I_{n+1}$  are comaximal, so  $I_1 \cap (I_2 \cap \cdots \cap I_{n+1}) = I_1 \cap (I_2 \cdots I_{n+1}) = I_1 \cdots I_{n+1}$ . By induction, this is true for all  $n$ .

- (6) Define  $\phi : R \rightarrow \prod R/I_i$  by  $\phi(x) = (x + I_1, \dots, x + I_n)$ . Then  $\ker(\phi) = I_1 \cap \dots \cap I_n$ , so  $\ker(\phi) = I_1 \cdots I_n$  by the argument above.

**Exercise.** (Problem 6)  $(1 + x)(1 - x + x^2 + \dots + (-x)^{n-1}) = 1 + (-x)^n = 1 - 0 = 1$ .