MATH 611 HOMEWORK 2 (DUE 9/11)

HIDENORI SHINOHARA

Exercise. (Problem 1, Section 1.2) Show that the free product $G*H$ of nontrivial groups G and H has trivial center, and that the only elements of $G*H$ of finite order are the conjugates of finite-order elements of G and H.

Proof. Let $w \in G * H$ be given. Suppose w is not the empty word.

- Suppose the leftmost element of w is in G. Let $h \in H$ be given such that h is not the identity element of H.
  - Case 1: The rightmost element of w is an element of G. Then wh is just a concatenation, so $wh \neq hw$ because the leftmost element of wh is in G and the leftmost element of hw is in H.
  - Case 2: The rightmost element of w is an element of H, but not h^{-1}. Let h' denote the rightmost element of w and w' denote the remaining. Then $w = w'h'$, so $wh = w'(h'h)$. By the definition of a reduced word, the rightmost element of w' is an element of G, so the concatenation of w' and $h'h$ is exactly wh. The leftmost element of wh is in G and the leftmost element of hw is in H, so $wh \neq hw$.
  - Case 3: The rightmost element of w is h^{-1}. Then the rightmost element of w disappears in wh. In this case, the leftmost element of w stays the same. Therefore, the leftmost element of wh is in G and the leftmost element of hw is in H, so $wh \neq hw$.

  In each case, $wh \neq hw$.
- Suppose that the leftmost element of w is in H. Let $g \in G$ be given such that g is not the identity element of G. Using the exact same logic as above, we can conclude that $wg \neq gw$.

Therefore, w is not in the center of $G * H$, so $Z(G * H) = \{e\}$ where e denotes the empty word.

Let x be a finite-order element in G or H. Let n denote the order. Let $w \in G * H$. Then $(wxw^{-1})^n = wx^nw^{-1} = ww^{-1} = e$, so the conjugate of a finite order element in G or H is has finite order.

We will show that every element of finite order in $G*H$ is a conjugate of a finite order element in G or H. We will consider the length of a finite-order element.

- Let $w \in G * H$ be a nonempty word of even length. Since adjacent elements must be elements of different groups, the leftmost element of w and rightmost element of w are in different groups. In other words, w^k has the length k times the length of w. This implies that the order of w is not finite.
- We will show that every reduced word of length $2k-1$ is a conjugate of a finite order element in G or H for every $k \in \mathbb{N}$. Let $k = 1$. Then it is either just g or h where $g \in G$ or $h \in H$. In each case, it is clear that the order g or h itself is finite. Therefore, it is a conjugate of a finite order element by the empty word.

  Suppose that the claim is true for some $k \in \mathbb{N}$. We will consider a finite-order element of length $2k+1$. Let w denote a reduced word of length $2k+1$. Suppose $w^n = e$ for some $n \in \mathbb{N}$.
  - Case 1: The leftmost element of w is in G. Then $w = gw'g'$ where g, g' are in G and w' is a reduced word of length $2k-1$. g' must equal g^{-1}. Otherwise, the length of w^m would equal $m \cdot (2k+1) - m$, and it would never equal 0. Consider $g^{-1}wg = w'$. Since $(g^{-1}wg)^n = g^{-1}w^n g = g^{-1}g = e$, the order of w' is finite. By the inductive hypothesis, w' is a conjugate of a finite order element in G or H. Since the length of w' is odd and the end elements are in H, w' must be a conjugate of a finite order element in G. In other words, $w' = axa^{-1}$ for some $a \in G * H$ and $x \in G$. Then $w = g(axa^{-1})g^{-1} = (ga)x(a^{-1}g^{-1}) = (ga)x(ga)^{-1}$. ga is a reduced word because the leftmost element of a is the same as the leftmost element of w', which is in H.

    By induction, every reduced word of finite length whose leftmost element is in G is a conjugate of a finite order element in G.
  - Case 2: The leftmost element of w is in H. By symmetry, every reduced word of finite length whose leftmost element in H is a conjugate of a finite order element in H.

  Therefore, the only elements of $G * H$ of finite order are the conjugates of finite-order elements of G and H.

□

Exercise. (Problem 4, Chapter 1.2) Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 - X)$.

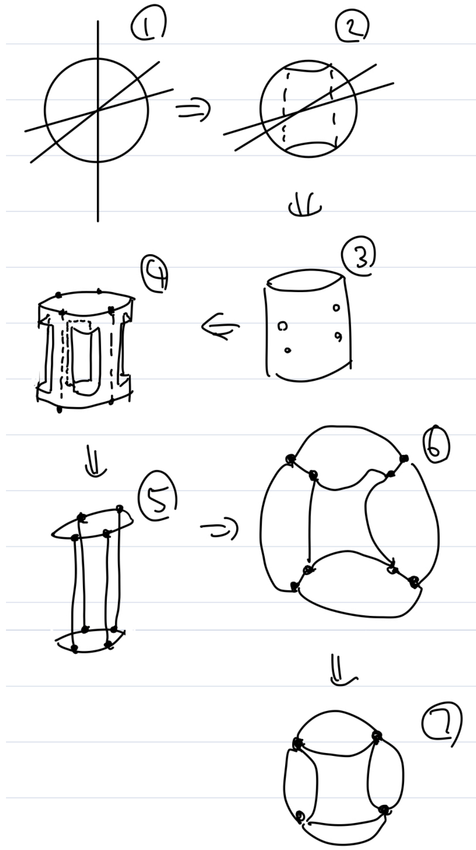

FIGURE 1. Transformation of S^2 with holes

Proof. When $n = 1$, the space can be deformation retracted to S^1. Thus $\pi_1(\mathbb{R}^3 - X)$ is $\mathbb{Z}$ when $n = 1$. Suppose $n \geq 2$.

Figure 1 shows how we will transform the space.

- 1 First, $\mathbb{R}^3$ can be deformation retracted to S^2.
- 1 $\to$ 2. Deformation retraction. Pick one of the lines and expand the hole.
- 2 $\to$ 3. Deformation retraction. Make the side thinner to create a tunnel with $2(n-1)$ holes on the side. The number of holes on the side is always $2(n-1)$ because there are $n-1$ lines after picking the first line and each line creates two holes.
- 3 $\to$ 4. Deformation retraction. Place the $2(n-1)$ holes evenly and expand each hole to create a "slit".
- 4 $\to$ 5. Deformation retraction. Expand each hole so we are remained with a graph.
- 5 $\to$ 6. Homeomorphism.

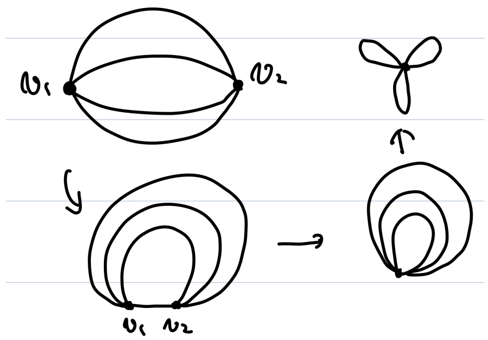

FIGURE 2. Case 1: $n = 2$

- $6 \to 7$. Homotopy equivalence. Shrink each of the short edges so they turn into points. We end up with a graph with $2(n-1)$ vertices $v_1, \cdots, v_{2(n-1)}$ such that for each v_i, we draw two edges to the next vertex. In total, the graph has $4(n-1)$ edges. In case that $2(n-1) = 2$, we draw 2 (undirected) edges from v_1 to v_2 and 2 (undirected) edges from v_2 to v_1, so there are be 4 edges between v_1 and v_2.

We will calculate the fundamental group of such a space. Let G_n denote such a graph for each n. We claim that the fundamental group of G_n is $\mathbb{Z} * \cdots * \mathbb{Z}$ ($2n-1$ times).

- Let $n = 2$. As in Figure 2, G_2 has the same fundamental group as a graph with one vertex and three loops. By Van Kampen's theorem, the fundamental group of two circles joined at a point is $\mathbb{Z} * \mathbb{Z}$ since the intersection is just a point. Similarly, the fundamental group of three circles joined at a point is $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, which is $3 = 2n - 1$ times.
- Suppose the statement is true for some $n \geq 2$. Consider G_{n+1}. As in Figure 3, G_{n+1} can be transformed into G_n with two loops attached to one vertex without changing the fundamental group. By Van Kampen's theorem, the fundamental group of such a graph is $\pi_1(G_n) * \mathbb{Z} * \mathbb{Z}$. In other words, $\mathbb{Z} * \cdots * \mathbb{Z}$ ($2(n+1) - 1$ times).

$\square$

Exercise. (Problem 8, Chapter 1.2) Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

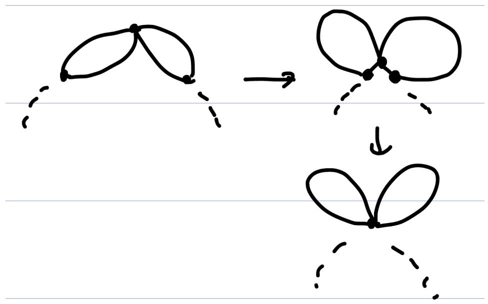

FIGURE 3. Inductive step

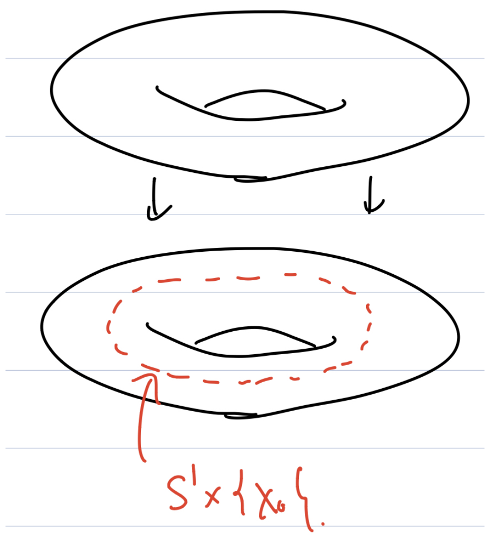

FIGURE 4. Two tori

Proof. We consider a torus that is on another of the identical size such that they contact each other on the circle $S^1 \times \{x_0\}$ for some x_0. Let T_1 denote one of the tori and T_2 the other. Let $X = T_1 \cup T_2$. (Figure 4) Each torus is path connected, and the intersection $S^1 \times \{x_0\}$ is also path connected. Let $p = (0, x_0)$.

- Let w_1 be a loop that goes around $S^1 \times \{x_0\}$ based at p in T_1.
- Let w_2 be a loop that goes around $\{0\} \times S^1$ based at p in T_1.
- Let w_3 be a loop that goes around $S^1 \times \{x_0\}$ based at p in T_2.
- Let w_4 be a loop that goes around $\{0\} \times S^1$ based at p in T_2.

Since $\pi_1(S^1 \times S^1) = \pi(S^1) \times \pi(S^1)$, $\pi_1(T_1, p) = \{[w_1], [w_2] \mid [w_1][w_2] = [w_2][w_1]\}$ and $\pi_1(T_2, p) = \{[w_3], [w_4] \mid [w_3][w_4] = [w_4][w_3]\}$. The fundamental group of the intersection $S^1 \times \{0\}$ is the group generated by $[w_1]$, (or alternatively $[w_3]$). Let i_1, i_2 denote the injections from $\pi_1(T_1, p)$ and $\pi_1(T_2, p)$ into $\pi_1(T_1, p) * \pi_1(T_2, p)$, respectively. By Van Kampen's theorem, $\pi_1(T_1 \cup T_2, p) = \pi_1(T_1, p) * \pi_1(T_2, p)/\langle i_1([w_1])i_2([w_3])^{-1}\rangle$. To simplify the notations, let $a = [w_1], b = [w_2], c = [w_3], d = [w_4]$ and $N = \langle i_1(a)i_2(c)\rangle$. Then $\pi_1(T_1 \cup T_2, p) = \langle a, b \mid ab = ba\rangle * \langle c, d \mid cd = dc\rangle/N$.

Let $(x_1, x_2, \cdots, x_n)N \in \pi_1(T_1 \cup T_2, p)$. For each i,

- Case 1: x_i is an element of $\langle a, b \mid ab = ba\rangle$. Then $(x_i)N \in \langle a, b \mid ab = ba\rangle * \langle d\rangle/N$.
- Case 2: x_i is an element of $\langle c, d \mid cd = dc\rangle$. By the commutativity, $x_i = c^i d^j$ for some $i, j \in \mathbb{Z}$. Then $x_iN = (c^i d^j)N = (c^iN)(d^jN) = (a^iN)(d^jN) = (a^i, d^j)N$, and $(a^i, d^j)N \in \langle a, b \mid ab = ba\rangle * \langle d\rangle/N$.

Therefore, each element in $\pi_1(T_1 \cup T_2, p)$ can be represented as cosets by words generated by a, b, d. Hence, $\pi_1(T_1 \cup T_2, p) = (\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$. $\square$