# MATH 601 (DUE 10/9)

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## 1. Rings of Fractions

**Exercise.** (Problem 3) Let $T \subset R$ be the subset consisting of all non zero divisors.

- Show that $T$ is a multiplicative set.
-

*Proof.*

- – Let $a, b \in T$. Let $c \in R$ be given. If $(ab)c = 0$, then $a(bc) = 0$. Since $a$ is a non zero divisor, $bc = 0$. Since $b$ is a non zero divisor, $c = 0$. Since $R$ is a commutative ring throughout this handout, there is no need to check the case that $c(ab) = 0$. Thus $ab$ is a non zero divisor, so $T$ is closed under multiplication.
  – $1 \in T$ since $\forall c \in R, c \cdot 1 = 0 \implies c = 0$.

  Therefore, $T$ is indeed a multiplicative set.

□

## 2. Modules

**Exercise.** (Problem 1) For each of the $\mathbb{Z}$-modules listed in the handout, answer the questions in the handout.

*Proof.*

(a) $M = \mathbb{Z}^3 \times \mathbb{Z}/86\mathbb{Z}$.

Solve this problem!

(b) $M = \prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z}$.

Solve this problem!

(c) $M = \mathbb{Z}[1/p] \subset \mathbb{Q}$.

Solve this problem!

(d) $M = \mathbb{Q}/\mathbb{Z}_{(p)}$.

Solve this problem!

$\square$

## 3. Rings of Fractions

**Exercise.** (Problem 3) Let $T \subset R$ be the subset consisting of all nonzero divisors.

- Show that $T$ is a multiplicative set.
- Let $s \in T$ and let $S = \{1, s, s^2, s^3, \cdots\} \subset T$. Show that the following rings are isomorphic: $S^{-1}R$, the subring $R[1/s] \subset T^{-1}R$, and the quotient ring $R[x]/(sx-1)$.

*Proof.*

- Prove this!
- Prove this!

$\square$

## 4. The Quadratic Equation

**Exercise.** (Problem 20)

**Exercise.** (Problem 21)

**Exercise.** (Problem 22)

## 5. Factorization in Integral Domains

**Exercise.** (Problem 5)

- Let $k$ be a field and let $a \in k$. Construct a $k$-algebra isomorphism, $k[x,y]/(x-a) \to k[y]$. Justify your answer.
- Let $f(x,y) \in k[x,y]$. What is the image of $f(x,y)$ under the above isomorphism?

*Proof.*

- Let $\phi$ be defined such that $\phi(f(x,y) + (x-a)) = f(a,y)$.
  - Well-defined? Let $f(x,y) + (x-a) = g(x,y) + (x-a)$. Then $g(x,y) = f(x,y) + h(x,y)(x-a)$.

$$\begin{aligned}\phi(g(x,y) + (x-a)) &= \phi((f(x,y) + h(x,y)(x-a)) + (x-a)) \\ &= f(a,y) + h(a,y)(a-a) \\ &= f(a,y) \\ &= \phi(f(x,y)).\end{aligned}$$

– $k$-algebra homomorphism? Let $c \in k$, $f, g \in k[x, y]$ be given.

$$\begin{aligned}
\phi(c(f + (x - a)) &= \phi(cf + (x - a)) \\
&= cf(a, y) \\
&= c\phi(f + (x - a)). \\
\phi((f + g) + (x - a)) &= (f + g)(a, y) \\
&= f(a, y) + g(a, y) \\
&= \phi(f + (x - a)) + \phi(g + (x - a)). \\
\phi((fg) + (x - a)) &= (fg)(a, y) \\
&= f(a, y)g(a, y) \\
&= \phi(f + (x - a))\phi(g + (x - a)).
\end{aligned}$$

- $\phi(f(x, y) + (x - a)) = f(a, y)$.

□

**Exercise.** (Problem 6)

- Give an example of a field $k$, an element $a \in k$ and a reducible polynomial $f(x, y) \in k[x, y]$ of degree $n$ in $y$ such that $f(a, y) \in k[y]$ is irreducible and has degree $n$.
- Suppose given a polynomial $f \in k[x, y]$ which when viewed as an element of $k(x)[y]$ has degree $n$ (in $y$) and content 1. Suppose there is some $a \in k$ such that $f(a, y) \in k[y]$ is irreducible and has degree $n$. Show that $f(x, y) \in k[x, y]$ is irreducible.
- Give an example of a field $k$, an element, $a \in k$, and a reducible polynomial $f(x, y) \in k[x, y]$, which when viewed as an element of $k(x)[y]$ has degree $n$ and content 1 such that $f(a, y) \in k[y]$ is irreducible.

*Proof.*

- Let $k = \mathbb{Q}, a = 1, f(x, y) = xy$. Then the degree of $f(x, y)$ in $y$ is 1. $f(x, y) = xy \in k[x, y]$ is reducible since $x$ and $y$ are not units in $k[x, y]$. However, $f(a, y) = 1y = y$ is irreducible in $k[y]$.
- Choose $f_1, \cdots, f_n \in k[x]$ such that $f(x, y) = f_n(x)y^n + \cdots + f_1(x)y^1 + f_0(x)$. Then $f(a, y) = f_n(a)y^n + \cdots + f_1(a)y^1 + f_0(a)$. Let $h_1(x, y), h_2(x, y) \in k[x]$ be given such that $f(x, y) = h_1(x, y)h_2(x, y)$. Then $f(a, y) = h_1(a, y)h_2(a, y)$. Then $h_1(a, y)$ or $h_2(a, y)$ is a unit in $k[y]$ since $f(a, y)$ is irreducible in $k[y]$. Without loss of generality, we will assume $h_1(a, y)$ is a unit in $k[y]$.

  It is given that $\deg_y(f(a, y))$, the degree of $f(a, y)$ in $y$, is $n$. Thus $\deg_y(h_1(a, y)) + \deg_y(h_2(a, y)) = n$. Since $\deg_y(h_1(a, y)) = 0$, $\deg_y(h_2(a, y)) = n$. Therefore, $\deg_y(h_2(x, y)) \geq n$.

  On the other hand, $\deg_y(f(x, y)) = \deg_y(h_1(x, y)) + \deg_y(h_2(x, y))$, so $\deg_y(h_2(x, y)) \leq n$. Thus $\deg_y(h_2(x, y)) = n$. Let $g_1(x), \cdots, g_n(x) \in k[x]$ such that $h_2(x, y) = g_n(x)y^n + \cdots + g_1(x)y^1 + g_0(x)$. Then $f(x, y) = h_1(x, y)h_2(x, y) = (h_1(x, y)g_n(x))y^n + \cdots + (h_1(x, y)g_1(x))y^1 + h_1(x, y)g_0(x)$.

  Since $\deg_y(h_2(x, y)) = n$, $\deg_y(h_1(x, y)) = 0$. Thus, $h_1(x, y) \in k[x]$, so $h_1(x, y)g_i(x) \in k[x]$ for each $i$. Therefore, $h_1(x, y)g_i(x) = f_i(x)$ for each $i$.

  Let $p \in k[x]$ be an irreducible. If $p \mid h_1(x, y)$, then $p \mid f_i(x) = h_1(x, y)g_i(x)$ for each $i$, so $\text{ord}_p(f_i) \geq 1$ for each $i$. Therefore, $o_p(f(x, y)) \geq 1$, and thus $p \mid \text{cont}(f(x, y))$.

However, since $\text{cont}(f(x,y)) = 1$, $p \nmid h_1(x,y)$. Thus $h_1(x,y)$ is a unit in $k[x]$ since it cannot be divided by any irreducibles. Since $h_1(x,y)$ is a unit in $k[x]$ and $k[y]$, it must consist only of a constant term, which is a unit in $k$. Hence, $h_1(x,y)$ is a unit in $k[x,y]$.

We have shown that for any $h_1(x,y), h_2(x,y) \in k[x,y]$, $h_1 h_2 = f$ implies one of $h_1$ or $h_2$ is a unit. Therefore, $f(x,y)$ is an irreducible in $k[x,y]$.

- Let $k = \mathbb{Q}, a = 1, f(x,y) = (x-1)y^2 + y$. Then $f(x,y)$, which when viewed as an element of $k(x)[y]$ has degree 1.
  - The coefficient of $y$ is 1, and $\text{ord}_p(1) = 0$ for any $p$ because $1 \in k[x]^*$.
  - The coefficient of $y^2$, when $f(x,y)$ is viewed as an element of $k(x)[y]$ is $x-1$. Thus for any irreducible element $p \in k[x]$, $\text{ord}_p(x-1) \geq 0$.

Therefore, $o_p(f(x)) = 0$ for any irreducible element $p \in k[x]$. Thus $\text{cont}(f(x,y)) = 1$.

$f(a,y) = y \in k[y]$. This is irreducible because if $f_1 f_2 = y$ for some $f_1, f_2 \in k[y]$, then $\deg(f_1) + \deg(f_2) = 1$ implies that one of $f_1$ or $f_2$ is a unit in $k$.

□