MATH 612 (HOMEWORK 1)

HIDENORI SHINOHARA

Exercise. (Exercise 1(a)) The case of $G = \mathbb{Z}$ is discussed in Example 2.42.

$$H_k(\mathbb{R}\mathrm{P}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0 \text{ and for } k = n \text{ odd} \\ \mathbb{Z}_2 & \text{for } k \text{ odd, } 0 < k < n \\ 0 & \text{otherwise.} \end{cases}$$

Suppose n is even. For any field F, we obtain the cellular chain complex

$$0 \to F \xrightarrow{2} F \xrightarrow{0} \cdots \xrightarrow{2} F \xrightarrow{0} F \to 0.$$

If the characteristic is 2, then all maps are 0. Therefore, $H_k(\mathbb{R}\mathrm{P}^n; F) = F$ if $k \leq n$ and $H_k(\mathbb{R}\mathrm{P}^n; F) = 0$ otherwise. If the characteristic is not 2, then $H_0(\mathbb{R}\mathrm{P}^n; F) = F$ and all other homology groups are 0. If n is odd, we obtain

$$0 \to F \xrightarrow{0} F \xrightarrow{2} \cdots \xrightarrow{2} F \xrightarrow{0} F \to 0.$$

If the characteristic is 2, $H_k(\mathbb{R}\mathrm{P}^n; F) = F$ if $k \leq n$ and $H_k(\mathbb{R}\mathrm{P}^n; F) = 0$ otherwise. Otherwise, $H_0(\mathbb{R}\mathrm{P}^n; F) = H_n(\mathbb{R}\mathrm{P}^n; F) = F$ and all other homology groups are 0.

Exercise. (Exercise 1(b)) As discussed in Example 2.37, $H_2(N_g; \mathbb{Z}) = 0, H_1(N_g; \mathbb{Z}) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$, and $H_0(N_g; \mathbb{Z}) = \mathbb{Z}$. For a field F, the cellular chain complex is

$$0 \to F \xrightarrow{d_2} F^g \xrightarrow{d_1} F \to 0.$$

As discussed in Example 2.37, $d_2(1) = (2, 2, \cdots, 2)$ and $d_1 = 0$. If the characteristic of F is 2, then $H_2(X; F) = H_0(X; F) = F$ and $H_1(X; F) = F^g$. Otherwise, then $H_2(X; F) = 0, H_1(X; F) = F^{g-1}$ and $H_0(X; F) = F$.

Exercise. (Exercise 1(c)) For a Z-module R, we have

$$0 \to R \xrightarrow{0} R \xrightarrow{a} R \xrightarrow{0} R \xrightarrow{0} .$$

When $R = \mathbb{Z}$, we obtain

$$H_k(X; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0, 2n-1 \\ \mathbb{Z}_m & \text{for } k \text{ odd, } 0 < k < 2n-1 \\ 0 & \text{otherwise.} \end{cases}$$

When R is a field with characteristic that divides a, $H_i(X; R) = R$ if $i = 0, 1, 2, 3$. If R is a field with characteristic that does not divide a, $H_3(X; R) = H_0(X; R) = R$ and all other cohomology groups are 0.