MATH 602 HOMEWORK 4

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Exercise. (1) Let $a/s \in S^{-1}\sqrt{I}$. Then $a^n \in I$ and $s \in S$ for some $n \in \mathbb{N}$. This implies $(a/s)^n \in S^{-1}I$, so $a/s \in \sqrt{S^{-1}I}$.

Let $a/s \in \sqrt{S^{-1}I}$. Then $a^n/s^n \in S^{-1}I$ for some $n \in \mathbb{N}$. Then $a^n \in I$, so $a \in \sqrt{I}$. Since $s \in S$, $a/s \in S^{-1}\sqrt{I}$.

Exercise. (3) Suppose that I is generated by one element x. Then $ax = 0 \implies a = 0$ because A is an integral domain. Therefore, I is a free module with a basis $\{x\}$.

On the other hand, suppose that I is a free module with a basis $\{x_\alpha\}$. Since it is a basis, each $x_\alpha \neq 0$. Moreover, if the basis contains more than 2 elements, $(-x_{\alpha'})x_\alpha + x_\alpha x_{\alpha'} = 0$, so it is not linearly independent. Therefore, the basis must contain exactly one element.

Exercise. (6a) $(M : N)$ is nonempty. For any $a, b \in (M : N)$, $(a - b)N = aN + (-b)N = aN + bN \subset M$, so $a - b \in (M : N)$. Finally, for any $a \in (M : N), x \in R$, $(xa)N = a(xN) \subset aN \subset M$, $ax \in (M : N)$.

Exercise. (6b)

$$\begin{aligned}
a \in \operatorname{Ann}((M+N)/M) &\iff a((M+N)/M) = 0 \\
&\iff \forall (m+n) + M \in (M+N)/M, a((m+n)+M) = 0 \\
&\iff \forall (m+n) + M \in (M+N)/M, am + an \in M \\
&\iff \forall n \in N, an \in M \\
&\iff aN \subset M \\
&\iff a \in (M : N).
\end{aligned}$$

Exercise. (8) Let $b/s \in S^{-1}B$. Then $b \in B$, so $b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0$ where $a_i \in A$. This implies that $(b/s)^n + (a_{n-1}/s)(b/s)^{n-1} + \cdots + (a_1/s^{n-1})(b/s) + a_0/s^n = 0$, thus b/s is integral over $S^{-1}A$.