MYTITLE

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Contents

1. Modules

Exercise. (Problem 6) Take four 4×4 matrices with integer entries and check if the abelian group presented by the matrix is cyclic.

Proof.

$$\begin{bmatrix} -166 & -74 & 254 & 347 \\ 140 & -93 & 246 & 425 \\ -196 & 57 & -363 & 202 \\ 325 & 257 & 314 & -389 \end{bmatrix} \to \begin{bmatrix} 18444530375 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 237 & -81 & 332 & -132 \\ 95 & 268 & 229 & 498 \\ 387 & 213 & 46 & 55 \\ 88 & -126 & -380 & -447 \end{bmatrix} \to \begin{bmatrix} 2610768268 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -275 & -22 & -207 & -276 \\ -469 & -342 & 240 & -101 \\ -41 & 455 & 51 & -151 \\ 267 & -450 & 98 & -40 \end{bmatrix} \to \begin{bmatrix} 33644517767 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 48 & 29 & 22 & -481 \\ 388 & -468 & -137 & -491 \\ 84 & -352 & 85 & -384 \\ -226 & -486 & 102 & -156 \end{bmatrix} = \begin{bmatrix} 13267264454 & 1 & 1 & 1 \end{bmatrix}$$

Each of the groups contains 4 generators, so none of them are cyclic. $\square$

2. Galois Theory

Exercise. (Problem 1) Let $F = \mathbb{Q}$. Let $L = \mathbb{Q}(\sqrt{7}, \sqrt{-11})$. To what familiar group is $\mathrm{Aut}(L/F)$ is isomorphic?

Proof. $[K : \mathbb{Q}(\sqrt{7})] = [K : \mathbb{Q}(\sqrt{-11})] = 2$. Since the characteristic of K is not 2, by the argument presented on P.3 of the Galous Theory handout, $\mathrm{Aut}(K/\mathbb{Q}(\sqrt{7}))$ and $\mathrm{Aut}(K/\mathbb{Q}(\sqrt{-11}))$ have 2 elements. For instance, $\alpha = \sqrt{7}$ and the minimal monic polynomial is $x^2 - 7$. This gives $D = 28$ and two automorphisms in $\mathrm{Aut}(K/\mathbb{Q}(\sqrt{7}))$, the identity map, and $\sigma : \sqrt{D} \mapsto -\sqrt{D}$ as discussed in the handout. Similarly, $\mathrm{Aut}(K/\mathbb{Q}(\sqrt{-11}))$ contains the identity map and $\sigma : \sqrt{D} \mapsto -\sqrt{D}$ where $D = -44$.

Finish this proof.

□

Exercise. (Problem 2) Let $F \subset K$ be a field extension.

(1) Prove in at most two sentences that each $\sigma \in \mathrm{Aut}(K/F)$ is an F-linear transformation of the F-vector space, K.
(2) Does the same condition hold in general for $\sigma \in \mathrm{Aut}(K)$? Prove or give a counterexample.

Proof.

(1) For any $a \in F$ and $v, w \in K$, $\sigma(av + w) = \sigma(a)\sigma(v) + \sigma(w) = a\sigma(v) + \sigma(w)$, so σ is indeed an F-linear transformation.
(2) Let $F = \mathbb{Q}(\sqrt{7})$ and $K = \mathbb{Q}(\sqrt{7}, \sqrt{-11})$. Let $\sigma \in \mathrm{Aut}(K/\mathbb{Q})$ such that $\sigma(\sqrt{7}) = -\sqrt{7}, \sigma(\sqrt{-11}) = -\sqrt{-11}$. The existence of such an automorphism is shown in the solution to Problem 1. K is an F-vector space. However, $\sigma(\sqrt{7} \cdot 1) = -\sqrt{7} \neq \sqrt{7} = \sqrt{7}(\sigma(1))$, so σ is not an F-linear transformation.

□

Exercise. (Problem 3) Let $\zeta = \exp(2\pi i/3) \in \mathbb{C}$. Consider the following subfields of $\mathbb{C}$. Let $F = \mathbb{Q}(\zeta)$. For $i \in \{0, 1, 2\}$, let $K_i = \mathbb{Q}(\zeta^i 7^{1/3})$. Let $L = \mathbb{Q}(7^{1/3}, \zeta 7^{1/3}, \zeta^2 7^{1/3})$.

Proof.

(1) $[F : \mathbb{Q}] = 3$.
(2) $\mathrm{Aut}(F/\mathbb{Q})$ consists of two maps, the identity map and another map that swaps ζ and ζ^2.
(3) $[K_i : \mathbb{Q}] = 3$ for each i because $\{1, \zeta^i 7^{1/3}, (\zeta^i 7^{i/3})^2\}$ is a $\mathbb{Q}$-basis.

□