MATH 633(HOMEWORK 2)

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Exercise. (Problem 1)

$$\begin{aligned}
\frac{\partial}{\partial x}(T^{-1}\circ f\circ T)(x,y) &= \lim_{h\to 0}\frac{T^{-1}(f(T(x+h,y))) - T^{-1}(f(T(x,y)))}{h}\\
&= \lim_{h\to 0}\frac{T^{-1}(f(x+h+iy)) - T^{-1}(f(x+iy))}{h}\\
&= \lim_{h\to 0}\frac{T^{-1}(f(x+h+iy) - f(x+iy))}{h}\\
&= T^{-1}(\lim_{h\to 0}\frac{f(x+h+iy) - f(x+iy)}{h})\\
&= T^{-1}(f'(x+iy))\\
&= T^{-1}(f'(T(x,y))).
\end{aligned}$$

Using the same argument again, we obtain $\frac{\partial^2}{\partial x^2}(T^{-1}\circ f\circ T)(x,y) = T^{-1}(f''(T(x,y)))$.

$$\begin{aligned}
\frac{\partial}{\partial y}(T^{-1}\circ f\circ T)(x,y) &= \lim_{h\to 0}\frac{T^{-1}(f(T(x,y+h))) - T^{-1}(f(T(x,y)))}{h}\\
&= \lim_{h\to 0}\frac{T^{-1}(f(x+i(y+h))) - T^{-1}(f(x+iy))}{h}\\
&= \lim_{h\to 0}\frac{T^{-1}(i(f(x+i(y+h)) - f(x+iy)))}{ih}\\
&= T^{-1}(i\lim_{h\to 0}\frac{f(x+i(y+h)) - f(x+iy)}{ih})\\
&= T^{-1}(if'(x+iy))\\
&= T^{-1}(if'(T(x,y))).
\end{aligned}$$

Using the same argument again, we obtain $\frac{\partial^2}{\partial y^2}(T^{-1}\circ f\circ T)(x,y) = T^{-1}(-f''(T(x,y))) = -T^{-1}(f''(T(x,y)))$ because $i^2 = -1$.

Thus $\frac{\partial^2}{\partial x^2}(T^{-1}\circ f\circ T)(x,y) + \frac{\partial^2}{\partial y^2}(T^{-1}\circ f\circ T)(x,y) = 0$.

Exercise. (Problem 2a) Let $\gamma(t) = Re^{2\pi it}$.

$$\begin{aligned}\int_\gamma z^n dz &= \int_0^1 R^n e^{2\pi int} R2\pi i e^{2\pi it} dt \\ &= 2\pi i R^{n+1} \int_0^1 e^{2\pi i(n+1)t} dt \\ &= \begin{cases} R^{n+1} \frac{e^{2\pi i(n+1)t}}{n+1} = 0 & (n \neq -1) \\ 2\pi i & (n = -1). \end{cases}\end{aligned}$$

Exercise. (Problem 2b) Let $\gamma(t) = z_0 + Re^{2\pi it}$ where $|R/z_0| < 1$.

$$\int_\gamma z^n dz = \int_0^1 (z_0 + Re^{2\pi it})^n (z_0 + Re^{e\pi it})' dt$$

When $n \neq -1$, $(z_0 + Re^{2\pi it})^{n+1}/(n+1)$ is a primitive, so the integral is 0. Suppose $n = -1$.

$$\begin{aligned}\int_0^1 \frac{2\pi i Re^{2\pi it}}{z_0 + Re^{2\pi it}} dt &= \int_0^1 \frac{2\pi i Re^{2\pi it}/z_0}{1 + Re^{2\pi it}/z_0} dt \\ &= \int_0^1 \frac{2\pi i Re^{2\pi it}}{z_0} \cdot \sum_{k=0}^{\infty} (\frac{-Re^{2\pi it}}{z_0})^k dt \\ &= -2\pi i \sum_{k=0}^{\infty} \int_0^1 (\frac{-Re^{2\pi it}}{z_0})^{k+1} dt \\ &= -2\pi i \sum_{k=0}^{\infty} (\frac{-Re^{2\pi it}}{z_0})^{k+1} \int_0^1 e^{2\pi it(k+1)} dt \\ &= 0.\end{aligned}$$

Each $\int_0^1 e^{2\pi it(k+1)} dt = 0$ because $e^{2\pi it(k+1)}/(2\pi it(k+1))$ is a primitive.

Exercise. (Problem 2c)

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}(\frac{1}{z-a} - \frac{1}{z-b})$$

Thus we will compute two integrals and add them later.

$$\begin{aligned}\int_\gamma \frac{1}{z-a} dz &= \int_\gamma \frac{1/z}{1 - a/z} dz \\ &= \int_\gamma \frac{1}{z} \sum_{k=0}^{\infty} (-\frac{a}{z})^k dz \\ &= \sum_{k=0}^{\infty} (-a)^k \int_\gamma \frac{1}{z^{k+1}} dz \\ &= 2\pi i\end{aligned}$$

because $z^{-(k+1)}$ has a primitive $z^{-k}/-k$ whenever $k \neq 0$ and when $k = 0$ we can use the results above.

$$\begin{aligned}\int_\gamma \frac{1}{b-z}dz &= \int_\gamma \frac{1/b}{1-z/b}dz \\ &= \frac{1}{b}\sum_{k=0}^{\infty}\int_\gamma (\frac{z}{b})^k dz \\ &= \frac{1}{b}\sum_{k=0}^{\infty}\frac{1}{b^k}\int_\gamma z^k dz \\ &= 0\end{aligned}$$

because $z^{k+1}/(k+1)$ is a primitive.

By putting these together, we conclude that the desired value is $2\pi i/(a-b)$.

Exercise. (Problem 3)

$$\begin{aligned}\int_a^b |z'(t)|dt &= \int_c^d |z'(t(s))|t'(s)ds \\ &= \int_c^d |z'(t(s))t'(s)|ds \\ &= \int_c^d |\tilde{z}'(s)|ds\end{aligned}$$

where $\tilde{z}(s) : [c,d] \to \mathbb{C}$ is a reparametrization of $z(t) : [a,b] \to \mathbb{C}$.

Exercise. (Problem 4a) If $t^\star \in \Omega_1$, then there exists an open neighborhood U of $z(t^\star)$ contained in Ω_1. Then $z^{-1}(U)$ is a neighborhood of $t^\star$ in $[0,1]$ because z is continuous. Since $z(1) \in \Omega_2$, $t^\star \neq 1$. However, this implies the existence of $\epsilon > 0$ such that $t^\star + \epsilon < 1$ and $z(t^\star + \epsilon) \in \Omega_1$. This is a contradiction.

If $t^\star \in \Omega_2$, then there exists an open neighborhood U of $z(t^\star)$ contained in Ω_2. Since U is open, $z^{-1}(U)$ is a neighborhood of $t^\star$ in $[0,1]$, so $\exists \epsilon > 0$ such that $z(t^\star - \epsilon) \in \Omega_2$.

In each case, we reached a contradiction, so Ω is not disconnected.

Exercise. (Problem 4b) For every $v \in \Omega_1$, there exists an open set U such that $v \in U \subset \Omega_1$. Then for any $v' \in U$, v and v' can be joined by $\gamma(t) = tv + (1-t)v'$. Thus $U \subset \Omega_1$, so Ω_1 is open.

Let $v \in \Omega_2$. Suppose that for all $\epsilon > 0$, the open disk at v with the radius ϵ is not contained in Ω_2. Otherwise we are done. Let $v_0 = w$. For every $n \in \mathbb{N}$, choose $v_n \in D(v, 1/n) \setminus \Omega_2$. Then there exists a path between each v_n and w. Moreover, there exists a path between v_n and v_{n-1} for each n and we will call it γ_n. Define $\gamma : [0,1] \to \Omega$ such that for each $n \in \mathbb{N}$, $\gamma([1-1/n, 1-1/(n+1)])$ is the path γ_n and $\gamma(1) = v$. Then γ is a well-defined path from w to v, which is a contradiction because $v \in \Omega_2$.

Clearly, $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1 \cup \Omega_2 = \Omega$. Since $w \in \Omega_1$, $\Omega_1 \neq \emptyset$, so $\Omega_2 = \emptyset$.