MATH 601 (DUE 10/23)

HIDENORI SHINOHARA

Contents

1. Field Extension

Exercise. (Problem 1) Let p be a prime number. Let $K = \mathbb{Z}/p\mathbb{Z}(t)$ be the fraction field of $\mathbb{Z}/p\mathbb{Z}[t]$.

(i) What is the characteristic of K?
(ii) What is the characteristic of any extension field of K?
(iii) Show that the Frobenius endormophism, $F : K \to K$ is not a ring isomorphism.
(iv) Let $f(x) = x^p - t \in K[x]$. Prove that $f(x)$ is irreducible.
(v) Prove that $f(x)$ is not a separable polynomial.
(vi) Construct an explicit field extension $K \subset L$ such that $f(x) \in L[x]$ has a factor of positive degree $< p$.
(vii) With f and L above find all the roots of $f(x)$ in L and determine their multiplicities.

Proof.

(i) We will write $k \cdot 1$ to denote $1 + 1 + \cdots + 1$ (k times). Since $p \cdot 1 = 0$ in K, the characteristic of K is at most p. Let k denote the characteristic of K. Let $i : \mathbb{Z}/p\mathbb{Z} \to (\mathbb{Z}/p\mathbb{Z})[t], i' : \mathbb{Z}/p\mathbb{Z}[t] \to K$ be inclusions. Then $i' \circ i : \mathbb{Z}/p\mathbb{Z} \to K$ is an injective ring homomorphism. $k \cdot 1 \neq 0$ in $\mathbb{Z}/p\mathbb{Z}$. Thus $(i' \circ i)(k \cdot 1) = k \cdot (i' \circ i)(1) = k' \cdot 1 = 0$. Since $i' \circ i$ is injective, this implies $k \cdot 1 = 0$. Therefore, $k \geq p$, so k must be equal to p.

∎