MATH 601 HOMEWORK (DUE 8/30)

HIDENORI SHINOHARA

Exercise 0.1. Show that a bijective ring homomorphism is an isomorphism in the category of rings.

Proof. Let f be a bijective ring homomorphism from a ring A to a ring B.

Let **C** denote the category of rings. Then A, B are objects of the category **C**. Since $\operatorname{Hom}_{\mathbf{C}}(A, B)$ is defined to be the set of all ring homomorphisms from A to $B, f \in \operatorname{Hom}_{\mathbf{C}}(A, B)$.

We will show that there exists an element $g \in \operatorname{Hom}_{\mathbf{C}}(B, A)$ such that $g \circ f = \operatorname{Id}_A$ and $f \circ g = \operatorname{Id}_B$.

Let a function $g: B \to A$ be defined such that $\forall b \in B, g(b) = a$ where a is an element such that f(a) = b. g is well-defined because:

- f is surjective, so there exists an $a \in A$ such that f(a) = b.
- f is injective, so such an a must be unique.

We claim that this g satisfies the desired properties:

- Claim 1: $g \in \text{Hom}_{\mathbf{C}}(B, A)$. This is equivalent to showing that g is a ring homomorphism. Let $b_1, b_2 \in B$ be given. Let $a_1 = g(b_1), a_2 = g(b_2)$. Then $f(a_1) = b_1$ and $f(a_2) = b_2$.
 - Since f is a ring homomorphism, $f(a_1 + a_2) = f(a_1) + f(a_2) = b_1 + b_2$. Therefore, $g(b_1 + b_2) = a_1 + a_2 = g(b_1) + g(b_2)$.
 - Since f is a ring homomorphism, $f(a_1 \cdot a_2) = f(a_1) \cdot f(a_2) = b_1 \cdot b_2$. Therefore, $g(b_1 \cdot b_2) = a_1 \cdot a_2 = g(b_1) \cdot g(b_2)$.
 - Since f is a ring homomorphism, f(1) = 1. Thus g(1) = 1. Therefore, $g \in \text{Hom}_{\mathbf{C}}(B, A)$.
- Claim 2: $g \circ f = \operatorname{Id}_A$. Let $a \in A$. Let b = f(a). Then g(b) = a, so g(f(a)) = a. This implies that $\forall a \in A, g(f(a)) = a$. Thus $g \circ f = \operatorname{Id}_A$.
- Claim 3: $f \circ g = \operatorname{Id}_B$. Let $b \in B$. Let a = g(b). Then f(a) = b, so f(g(b)) = b. Therefore, $\forall b \in B, f(g(b)) = b$. Thus $f \circ g = \operatorname{Id}_B$.

Therefore, f is indeed an isomorphism in the category of rings. \square

Exercise 0.2. Show that when products exist, they are essentially unique.

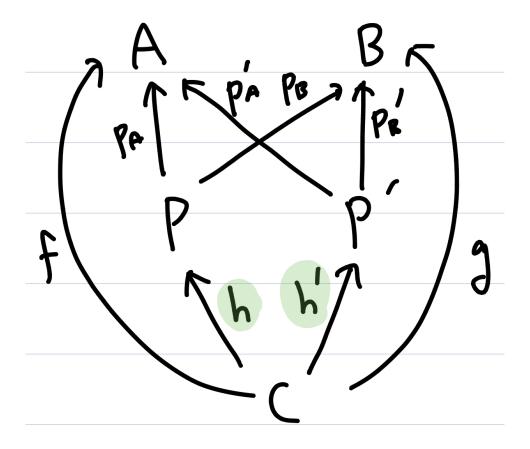


FIGURE 1. Diagram of maps for the second problem

Proof. First, we will consider the case when C = P, $f = p_A$, $g = p_B$.

- There exists a unique map $h_1 \in \operatorname{Hom}_{\mathbf{C}}(C, P) = \operatorname{Hom}_{\mathbf{C}}(P, P)$ such that $f = p_A \circ h_1$. In other words, $p_A = p_A \circ h_1$. By the uniqueness of h_1 and the equality $p_A = p_A \circ \operatorname{Id}_P$, we can conclude that $h_1 = \operatorname{Id}_P$.
- There exists a unique map $h'_1 \in \operatorname{Hom}_{\mathbf{C}}(C, P') = \operatorname{Hom}_{\mathbf{C}}(P, P')$ such that $f = p'_A \circ h'_1$ and $g = p'_B \circ h'_1$. In other words, $p_A = p'_A \circ h'_1$ and $p_B = p'_B \circ h'_1$.

Similarly, we will consider the case when $C = P', f = p'_A, g = p'_B$.

- There exists a unique map $h_2 \in \operatorname{Hom}_{\mathbf{C}}(C, P) = \operatorname{Hom}_{\mathbf{C}}(P', P)$ such that $f = p_A \circ h_2$. In other words, $p'_A = p_A \circ h_2$.
- There exists a unique map $h'_2 \in \operatorname{Hom}_{\mathbf{C}}(C, P') = \operatorname{Hom}_{\mathbf{C}}(P', P')$ such that $f = p'_A \circ h'_2$. In other words, $p'_A = p'_A \circ h'_2$. By the uniqueness of h'_2 and the equality $p'_A = p'_A \circ \operatorname{Id}_{P'}$, we can conclude that $h'_2 = \operatorname{Id}_{P'}$.

$$p_A = p'_A \circ h'_1$$

$$= (p_A \circ h_2) \circ h'_1$$

$$= p_A \circ (h_2 \circ h'_1)$$

We showed earlier that Id_P is the only map that satisfies $p_A = p_A \circ \mathrm{Id}_P$, so $h_2 \circ h_1' = \mathrm{Id}_P$.

$$p'_A = p_A \circ h_2$$

$$= (p'_A \circ h'_1) \circ h_2$$

$$= p'_A \circ (h'_1 \circ h_2).$$

Similarly, $\mathrm{Id}_{P'}$ is the only map such that $p'_A = p'_A \circ \mathrm{Id}_{P'}$, so $h'_1 \circ h_2 = \mathrm{Id}_{P'}$.

We showed that $h'_1 \in \operatorname{Hom}_{\mathbf{C}}(P, P')$ and $h_2 \in \operatorname{Hom}_{\mathbf{C}}(P, P)$ satisfy $h_2 \circ h'_1 = \operatorname{Id}_P$ and $h'_1 \circ h_2 = \operatorname{Id}_{P'}$. In addition, we showed that $p_A = p'_A \circ h'_1$ and $p_B = p'_B \circ h'_1$. Therefore, h'_1 is the desired isomorphism. \square