

# MATH 611 (DUE 11/20)

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## Exercise. (Problem 1)

- As shown in Figure 1, we will let  $A, B$  denote subspaces of  $X$  such that  $X = A \cup B$  and  $A \cap B$  consists of two line segments. (Circled in the figure) Moreover,  $X = \text{int } A \cup \text{int } B$ .

$H_n(A) = 0$  for all  $n \geq 1$ . By Proposition 2.6 (Hatcher), it suffices to consider each path component of  $A \cap B$  separately. Each of them is homeomorphic to  $\Delta^1$ . Thus  $H_n(A \cap B) = H_n(\Delta^1) \oplus H_n(\Delta^1) = 0$  for all  $n \geq 1$ . Using a similar argument,  $H_n(B) = H_n(\Delta^1) \oplus H_n(\Delta^1) = 0$  for all  $n \geq 1$ .

By the exact sequence  $H_n(A) \oplus H_n(B) \rightarrow H_n(A \cup B) \rightarrow H_{n-1}(A \cap B)$ ,  $H_n(A \cup B) = 0$  for all  $n \geq 2$ .

We will consider the exact sequence  $0 \rightarrow H_1(A \cup B) \xrightarrow{\alpha} H_0(A \cap B) \xrightarrow{\beta} H_0(A) \oplus H_0(B)$ . We have 0 because  $H_1(A) \oplus H_1(B) = 0$ .  $H_0(A \cap B) = \mathbb{Z}^2$ ,  $H_0(A) = \mathbb{Z}$ ,  $H_0(B) = \mathbb{Z}^2$  by examining the number of path components. Let  $a, b$  be generators of  $H_0(A \cap B)$ . Then  $\beta(a, b) = (a + b, (a, b))$  because  $a, b$  simply correspond to each path component in  $A \cap B$ . Therefore,  $\beta$  is injective. Since  $\alpha$  is injective by the exactness,  $H_1(A \cup B) = \text{Im}(\alpha) = \ker(\beta) = 0$ . Hence,  $H_1(X) = 0$ .

By examining the number of path components,  $H_0(X) = \mathbb{Z}$ .

- Let  $A, B$  denote the subspaces of  $X$  as in Figure 2. Then  $A \cap B$  is homotopy equivalent to  $S^1$ , and  $B$  is homotopy equivalent to the wedge sum of  $2g$   $S^1$ 's.

For any  $n \geq 3$ ,  $H_n(A) \oplus H_n(B) \rightarrow H_n(A \cup B) \rightarrow H_{n-1}(A \cap B)$  shows that  $H_n(A \cup B) = 0$  because  $H_n(A) = H_n(B) = 0$  and  $H_{n-1}(A \cap B) = H_{n-1}(S^1) = 0$ .

We have  $0 \rightarrow \tilde{H}_2(A \cup B) \xrightarrow{\alpha} \tilde{H}_1(A \cap B) \xrightarrow{\beta} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{\gamma} \tilde{H}_1(A \cup B) \rightarrow 0$ . We have 0's at the end because  $\tilde{H}_2(A) \oplus \tilde{H}_2(B) = \tilde{H}_0(A \cap B) = 0$ . We have  $\tilde{H}_1(A \cap B) = \mathbb{Z}$  and  $\tilde{H}_1(A) \oplus \tilde{H}_1(B) = \bigvee_{i=1}^{2g} \tilde{H}_1(S^1) = \mathbb{Z}^{2g}$ .  $\beta$  maps a generator  $x$  into  $(0, 0)$  because going around  $A \cap B$  once cancels out all the generators of  $\tilde{H}_1(B)$ . For instance, in Figure 2,  $\beta(x) = a + b - a - b + \dots = 0$ . Therefore,  $\beta$  is the zero map.

This implies that  $\gamma$  is injective. By the exactness,  $\gamma$  is surjective. Therefore,  $\gamma$  is isomorphic, and thus  $H_1(A \cup B) = \tilde{H}_1(A \cup B) = \mathbb{Z}^{2g}$ .

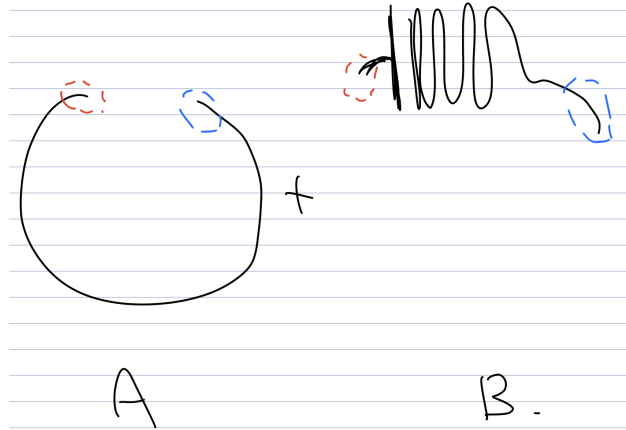


FIGURE 1. Quasi circle



FIGURE 2. Genus  $g$  surface

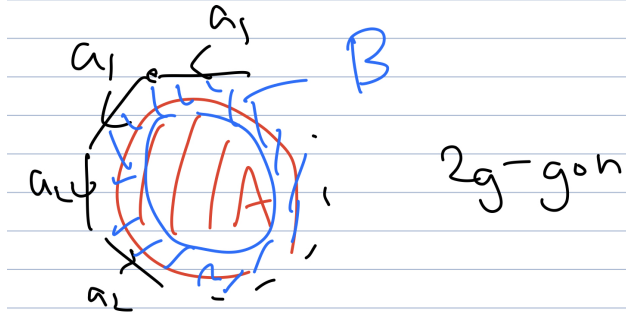


FIGURE 3.  $N_g$

$\alpha$  is injective by the exactness, so  $H_2(A \cup B) = \tilde{H}_2(A \cup B) = \text{Im}(\alpha) = \tilde{H}_1(A \cap B) = \mathbb{Z}$ .  $H_0(A \cap B) = H_0(S^1) = \mathbb{Z}$ .

- Let  $A, B$  denote the subspaces as in Figure 3. Then  $A$  deformation retracts onto a point,  $B$  is homotopy equivalent to  $\vee_g \mathbb{R}P^1$ , which is homotopy equivalent to  $\vee_g S^1$ . Finally,  $A \cap B$  is homotopy equivalent to  $S^1$ . For any  $n \geq 3$ ,  $H_n(A) \oplus H_n(B) \rightarrow H_n(X) \rightarrow H_{n-1}(A \cap B)$  is exact, and  $H_n(A) = H_n(B) = H_{n-1}(A \cap B) = 0$ , so  $H_n(X) = 0$ . Consider  $0 \rightarrow \tilde{H}_2(X) \xrightarrow{\alpha} \tilde{H}_1(A \cap B) \xrightarrow{\beta} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{\gamma} \tilde{H}_1(X) \rightarrow 0$ . We have 0 at the end because  $\tilde{H}_2(A) \oplus \tilde{H}_2(B) = \tilde{H}_0(A \cap B) = 0$ . By exactness,  $\alpha$  is injective and  $\gamma$  is surjective. Let  $a$  be a generator of  $\tilde{H}_1(A \cap B) = \mathbb{Z}$ . Then  $\beta(a) = (0, 2(a_1 + \dots + a_g))$  where  $a_i$ 's are generators of  $\tilde{H}_1(B) = \mathbb{Z}^g$ .

- Since  $\beta$  is injective, so  $0 = \ker(\beta) = \text{Im}(\alpha) = \tilde{H}_2(X) = H_2(X)$ .

- Since  $\gamma$  is surjective,  $\tilde{H}_1(X) = \tilde{H}_1(A) \oplus \tilde{H}_1(B) / \ker(\gamma)$ . Since  $\ker(\gamma) = \text{Im}(\beta)$ , this is  $\langle a_1, \dots, a_g \mid 2(\sum a_i) \rangle$ .

$$\begin{aligned}
 H_1(X) &= \langle a_1, \dots, a_g \mid 2(a_1 + \dots + a_g) \rangle \\
 &= \langle a_1 + \dots + a_g, a_2, \dots, a_g \mid 2(a_1 + \dots + a_g) \rangle \\
 &= \langle b, a_2, \dots, a_g \mid 2b \rangle \\
 &= \mathbb{Z}^{g-1} \oplus (\mathbb{Z}/2\mathbb{Z}).
 \end{aligned}$$

– Since  $X$  consists of one path component,  $H_0(X) = \mathbb{Z}$ .

$\mathbb{R}P^3$ .

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- Let  $A = \mathbb{C}P^n - (0 : \cdots : 0 : 1)$ ,  $B = \{(a_1 : \cdots : a_n : 1) \mid a_i \in \mathbb{C}\}$ . Since  $\{(0 : \cdots : 0 : 1)\}$  is closed,  $A$  is an open set. Moreover,  $(0 : \cdots : 0 : 1)$  is an interior point of  $B$ . Therefore,  $\text{int}(A) \cup \text{int}(B) = \mathbb{C}P^n$ .

$A$  deformation retracts onto  $\mathbb{C}P^{n-1}$  by  $F((a_1 : \cdots : a_n : a_{n+1}), t) = (a_1 : \cdots : a_n : (1-t)a_{n+1})$ .  $B$  is homeomorphic to  $\mathbb{C}^n$ , which is contractible. Finally,  $A \cap B = B \setminus (0 : \cdots : 0 : 1)$ , which is homotopy equivalent to  $\mathbb{C}^n - 0$ . This is homotopy equivalent to  $\mathbb{R}^{2n} - 0$ , so it is  $S^{2n-1}$ .

We claim that  $H_{2k}(\mathbb{C}P^n) = 0$  if  $k > n$  and  $\mathbb{Z}$  if  $k \leq n$ . We will use this to calculate  $H_k(\mathbb{C}P^n)$  by induction on  $n$ . When  $n = 0$ , this is obvious because  $\mathbb{C}P^0$  is a point. Suppose that we have shown this for some  $n - 1$  where  $n \in \mathbb{N}$ . We will prove the case for  $n$ .

- We have the exact sequence  $H_{2n}(A) \oplus H_{2n}(B) \rightarrow H_{2n}(X) \rightarrow H_{2n-1}(A \cap B) \rightarrow H_{2n-1}(A) \oplus H_{2n-1}(B)$ .  $H_{2n}(A) = H_{2n-1}(A) = 0$  by the inductive hypothesis since  $A$  deformation retracts onto  $\mathbb{C}P^{n-1}$ .  $H_{2n}(B) = H_{2n-1}(B) = 0$  because  $B$  is contractible. By the exactness,  $H_{2n}(X) \cong H_{2n-1}(A \cap B) = \mathbb{Z}$  because  $A \cap B$  is homotopy equivalent to  $S^{2n-1}$ .
- We have the exact sequence  $\tilde{H}_{2n-1}(A) \oplus \tilde{H}_{2n-1}(B) \rightarrow \tilde{H}_{2n-1}(X) \rightarrow \tilde{H}_{2n-2}(A \cap B)$ .  $\tilde{H}_{2n-1}(A) = 0$  by the inductive hypothesis.  $\tilde{H}_{2n-1}(B) = 0$ , and  $\tilde{H}_{2n-2}(A \cap B) = 0$ , so  $\tilde{H}_{2n-1}(X) = \tilde{H}_{2n-1}(X) = 0$ .

**Exercise.** (Problem 28 (a)) Let  $A, B$  be the Mobius strip and a torus with a small neighborhood around them so the strip and torus are contained in  $A$  and  $B$ . For any  $n \geq 3$ , the exact sequence  $H_n(A \cap B) \rightarrow H_n(A) \oplus H_n(B) \rightarrow H_n(X) \rightarrow H_{n-1}(A \cap B)$  implies that  $H_n(X) \cong H_n(A) \oplus H_n(B) = 0 \oplus 0 = 0$  because the intersection  $A \cap B$  is homotopic to  $S^1$ , so  $H_n(A \cap B) = H_{n-1}(A \cap B) = 0$ .  $H_0(X) = \mathbb{Z}$  because  $X$  has only one path component.

We will examine the LES

$$\tilde{H}_2(A \cap B) \rightarrow \tilde{H}_2(A) \oplus \tilde{H}_2(B) \xrightarrow{f_1} \tilde{H}_2(X) \xrightarrow{f_2} \tilde{H}_1(A \cap B) \xrightarrow{f_3} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{f_4} \tilde{H}_1(X) \rightarrow \tilde{H}_0(A \cap B).$$

- Since  $\tilde{H}_2(A \cap B) = 0$ , so  $f_1$  is injective.
- $\tilde{H}_1(A \cap B) = \mathbb{Z}$ , and  $f_3(1) = (2, (1, 0))$  because the intersection goes around the mobius strip twice while it only goes around the torus once. Then  $f_3$  is injective, so  $\text{Im}(f_2) = \ker(f_3) = 0$ . This implies that  $\text{Im}(f_1) = \ker(f_2) = H_2$ , so  $f_1$  is surjective.

Therefore,  $f_1$  is bijective, so  $H_2(X) = \tilde{H}_2(X) = \tilde{H}_2(A) \oplus \tilde{H}_2(B) = 0 \oplus \mathbb{Z} = \mathbb{Z}$ .

Finally,  $f_4$ 's surjectivity implies that

$$\begin{aligned} \tilde{H}_1(X) &\cong \tilde{H}_1(A) \oplus \tilde{H}_1(B) / \ker(f_4) \\ &= \mathbb{Z} \oplus \mathbb{Z}^2 / \langle (2, (1, 0)) \rangle \\ &\cong \langle a, b, c \rangle / \langle 2a + b \rangle \\ &\cong \langle a, b, c \mid 2a + b \rangle \\ &\cong \langle a, -2a, c \rangle \\ &\cong \langle a, c \rangle = \mathbb{Z} \oplus \mathbb{Z}. \end{aligned}$$

Thus  $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$ .

**Exercise.** (Problem 28 (b)) Let  $A, B$  be the Mobius strip and  $\mathbb{R}P^2$  with a small neighborhood around them so the strip and  $\mathbb{R}P^2$  are contained in  $A$  and  $B$ . For any  $n \geq 3$ , the exact sequence  $H_n(A \cap B) \rightarrow H_n(A) \oplus H_n(B) \rightarrow H_n(X) \rightarrow H_{n-1}(A \cap B)$  implies that  $H_n(X) \cong H_n(A) \oplus H_n(B) = 0 \oplus 0 = 0$  because the intersection  $A \cap B$  is homotopic to  $S^1$ , so  $H_n(A \cap B) = H_{n-1}(A \cap B) = 0$ . Since  $X = A \cup B$  has one path component,  $H_0(X) = \mathbb{Z}$ . We will consider the LES

$$\tilde{H}_2(A) \oplus \tilde{H}_2(B) \xrightarrow{f_1} \tilde{H}_2(X) \xrightarrow{f_2} \tilde{H}_1(A \cap B) \xrightarrow{f_3} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{f_4} \tilde{H}_1(X) \rightarrow \tilde{H}_0(A \cap B).$$

$\tilde{H}_1(A \cap B) = \mathbb{Z}$ , and  $f_3$  maps 1 to  $(2, 1)$  because the generator wraps around the Mobius strip twice and the  $\mathbb{R}P^2$  once. Then  $f_3$  is injective, so  $f_2$  is the zero map. In other words,  $\ker(f_2) = \tilde{H}_2(X)$ , so  $f_1$  is surjective. Since  $\tilde{H}_2(A) \oplus \tilde{H}_2(B) = 0$ ,  $\tilde{H}_2(X) = 0$ . Thus  $H_2(X) = 0$ .

By the first isomorphism theorem and exactness,

$$\begin{aligned}\tilde{H}_1(X) &= \tilde{H}_1(A) \oplus \tilde{H}_1(B) / \ker(f_4) \\ &= (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) / \langle (2, 1) \rangle \\ &\cong \langle a, b \mid 2b \rangle / \langle 2a + b \rangle \\ &= \langle a, b \mid 2b, 2a + b \rangle \\ &= \langle a, -2a \mid 2(-2a) \rangle \\ &= \langle a \mid 4a \rangle \\ &= \mathbb{Z}_4.\end{aligned}$$

Therefore,  $H_1(X) = \mathbb{Z}_4$ .

**Exercise.** (Problem 29) As shown earlier,

$$H_n(M_g) = \begin{cases} \mathbb{Z}^{2g} & (n = 1) \\ \mathbb{Z} & (n = 0, 2) \\ 0 & (n \geq 3). \end{cases}$$

Let  $R_1, R_2$  be the first and second  $R$  with a small neighborhood around them. Then  $X = R_1 \cup R_2$  and  $R_1 \cap R_2$  is homotopy equivalent to  $M_g$ . Let  $n \geq 3$ . Consider the sequence

$$H_n(R_1) \oplus H_n(R_2) \rightarrow H_n(X) \rightarrow H_{n-1}(R_1 \cap R_2) \rightarrow H_{n-1}(R_1) \oplus H_{n-1}(R_2).$$

A solid  $g$ -torus deformation retracts to the wedge sum of  $g$   $S^1$ 's.  $H_n(R_1) = H_n(R_2) = \oplus_{i=1}^g H_n(S^1) = 0$  for  $n \geq 2$ . By the exactness, we have  $H_n(X) = H_{n-1}(R_1 \cap R_2) = H_{n-1}(M_g)$ . Therefore,  $H_n(X) = 0$  for  $n \geq 4$ , and  $H_3(X) = \mathbb{Z}$ .  $H_0(X) = \mathbb{Z}$  because  $X$  contains only one path component.

Consider the sequence

$$\tilde{H}_2(R_1) \oplus \tilde{H}_2(R_2) \rightarrow \tilde{H}_2(X) \xrightarrow{\alpha} \tilde{H}_1(R_1 \cap R_2) \xrightarrow{\beta} \tilde{H}_1(R_1) \oplus \tilde{H}_1(R_2) \xrightarrow{\gamma} \tilde{H}_1(X) \rightarrow \tilde{H}_0(R_1 \cap R_2).$$

Then this is equivalent to

$$0 \rightarrow \tilde{H}_2(X) \xrightarrow{\alpha} \tilde{H}_1(R_1 \cap R_2) \xrightarrow{\beta} \tilde{H}_1(R_1) \oplus \tilde{H}_1(R_2) \xrightarrow{\gamma} \tilde{H}_1(X) \rightarrow 0.$$

By the exactness,  $\alpha$  is injective and  $\gamma$  is surjective. Let  $a_1, \dots, a_g, b_1, \dots, b_g$  be generators of  $\tilde{H}_1(R_1 \cap R_2)$  where  $a_i$  wraps around the  $i$ th “arm” (or “handle”) and  $b_i$  wraps around the  $i$ th “hole”. Then  $\beta(a_i) = (0, 0)$  because in  $R_1$  and  $R_2$ , each of which is a solid torus, the “arm” gets filled in. On the other hand,  $\beta(b_i) = (b_i, b_i)$  for each  $i$ .

$$\begin{aligned}H_1(X) &= \tilde{H}_1(X) \\ &= \text{Im}(\gamma) \\ &= \tilde{H}_1(R_1) \oplus \tilde{H}_1(R_2) / \ker(\gamma) \\ &= \tilde{H}_1(R_1) \oplus \tilde{H}_1(R_2) / \text{Im}(\beta) \\ &= \langle b_1, \dots, b_g, b'_1, \dots, b'_g \rangle / \langle b_1 + b'_1, \dots, b_g + b'_g \rangle \\ &= \langle b_1, \dots, b_g \rangle \\ &= \mathbb{Z}^g.\end{aligned}$$

Since  $\alpha$  is injective,  $\text{Im}(\alpha)$  is isomorphic to  $\tilde{H}_2(X)$ . Thus  $H_2(X) = \tilde{H}_2(X) = \text{Im}(\alpha) = \ker(\beta) = \langle a_1, \dots, a_g \rangle = \mathbb{Z}^g$ .

- For  $n \geq 4$ , we have  $H_n(R) \rightarrow H_n(R, M_g) \rightarrow H_{n-1}(M_g)$ . As shown earlier,  $H_n(R) = H_{n-1}(M_g) = 0$ , so the exactness implies that  $H_n(R, M_g) = 0$ .
- We will consider  $H_3(R) \rightarrow H_3(R, M_g) \rightarrow H_2(M_g) \rightarrow H_2(R)$ .  $H_3(R) = H_2(R) = 0$ , so  $H_3(R, M_g) = H_2(M_g)$  by the exactness. Thus  $H_3(R, M_g) = \mathbb{Z}$ .
- We will consider  $0 \rightarrow \tilde{H}_2(R, M_g) \xrightarrow{\alpha} \tilde{H}_1(M_g) \xrightarrow{\beta} \tilde{H}_1(R) \xrightarrow{\gamma} \tilde{H}_1(R, M_g) \rightarrow 0$ . (We have 0 on both ends because  $\tilde{H}_2(R) = \tilde{H}_0(M_g) = 0$ . Let  $a_i, b_i$  be generators of  $\tilde{H}_1 M_g$  such that  $a_i$ 's wrap around the handles and  $b_i$ 's wrap around the holes. Using the same discussion as above,  $a_i \mapsto 0$  and  $b_i \mapsto b_i$  by  $\beta$ .
  - By the exactness,  $\alpha$  is injective. Thus  $\tilde{H}_2(R, M_g) = \text{Im}(\alpha) = \ker(\beta) = \langle a_1, \dots, a_g \rangle$ . Therefore,  $\tilde{H}_2(R, M_g) = \mathbb{Z}^g$ .
  - By the exactness,  $\gamma$  is surjective.  $\tilde{H}_1(R, M_g) = \text{Im}(\gamma) = \tilde{H}_1(R) / \ker(\gamma) = \tilde{H}_1(R) / \text{Im}(\beta)$ .  $\tilde{H}_1(R)$  is generated by  $b_1, \dots, b_g$  as it deformation retracts to  $S^1 \vee \dots \vee S^1$ , so  $\beta$  is surjective. Therefore,  $\tilde{H}_1(R, M_g) = 0$ .
- $0 = H_1(R, M_g) \rightarrow H_0(M_g) \xrightarrow{f} H_0(R) \rightarrow H_0(R, M_g)$  is exact. Moreover,  $f$  must be an isomorphism because both  $M_g$  and  $R$  consist of one path component. Therefore, the exactness implies  $H_0(R, M_g) = 0$ .