MATH 601 (DUE 10/9)

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1. Modules

Exercise. (Problem 1) For each of the $\mathbb{Z}$-modules listed in the handout, answer the questions in the handout.

Proof.

(a) $M = \mathbb{Z}^3 \times \mathbb{Z}/86\mathbb{Z}$.

Solve this problem!

(b) $M = \prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z}$.

Solve this problem!

(c) $M = \mathbb{Z}[1/p] \subset \mathbb{Q}$.

Solve this problem!

(d) $M = \mathbb{Q}/\mathbb{Z}_{(p)}$.

Solve this problem!

□

2. Rings of Fractions

Exercise. (Problem 3) Let $T \subset R$ be the subset consisting of all nonzero divisors.

- Show that T is a multiplicative set.
- Let $s \in T$ and let $S = \{1, s, s^2, s^3, \cdots\} \subset T$. Show that the following rings are isomorphic: $S^{-1}R$, the subring $R[1/s] \subset T^{-1}R$, and the quotient ring $R[x]/(sx-1)$.

Proof.

- Prove this!
- Prove this!

□

3. The Quadratic Equation

Exercise. (Problem 20)

Exercise. (Problem 21)

Exercise. (Problem 22)

4. Factorization in Integral Domains

Exercise. (Problem 5)

- Let k be a field and let $a \in k$. Construct a k-algebra isomorphism, $k[x,y]/(x-a) \to k[y]$. Justify your answer.
- Let $f(x,y) \in k[x,y]$. What is the image of $f(x,y)$ under the above isomorphism?

Proof.

- Let ϕ be defined such that $\phi(f(x,y)+(x-a)) = f(a,y)$.
  - Well-defined? Let $f(x,y)+(x-a) = g(x,y)+(x-a)$. Then $g(x,y) = f(x,y) + h(x,y)(x-a)$.

$$\begin{aligned}\phi(g(x,y)+(x-a)) &= \phi((f(x,y)+h(x,y)(x-a))+(x-a)) \\ &= f(a,y)+h(a,y)(a-a) \\ &= f(a,y) \\ &= \phi(f(x,y)).\end{aligned}$$

  - k-algebra homomorphism? Let $c \in k$, $f, g \in k[x,y]$ be given.

$$\begin{aligned}\phi(c(f+(x-a)) &= \phi(cf+(x-a)) \\ &= cf(a,y) \\ &= c\phi(f+(x-a)). \\ \phi((f+g)+(x-a)) &= (f+g)(a,y) \\ &= f(a,y)+g(a,y) \\ &= \phi(f+(x-a))+\phi(g+(x-a)). \\ \phi((fg)+(x-a)) &= (fg)(a,y) \\ &= f(a,y)g(a,y) \\ &= \phi(f+(x-a))\phi(g+(x-a)).\end{aligned}$$

- $\phi(f(x,y)+(x-a)) = f(a,y)$.

□

Exercise. (Problem 6)

- Give an example of a field k, an element $a \in k$ and a reducible polynomial $f(x,y) \in k[x,y]$ of degree n in y such that $f(a,y) \in k[y]$ is irreducible and has degree n.

- Suppose given a polynomial $f \in k[x,y]$ which when viewed as an element of $k(x)[y]$ has degree n (in y) and content 1. Suppose there is some $a \in k$ such that $f(a,y) \in k[y]$ is irreducible and has degree n. Show that $f(x,y) \in k[x,y]$ is irreducible.
- Give an example of a field k, an element, $a \in k$, and a reducible polynomial $f(x,y) \in k[x,y]$, which when viewed as an element of $k(x)[y]$ has degree n and content 1 such that $f(a,y) \in k[y]$ is irreducible.

Proof.

- Let $k = \mathbb{Q}, a = 1, f(x,y) = xy$. Then the degree of $f(x,y)$ in y is 1. $f(x,y) = xy \in k[x,y]$ is reducible since x and y are not units in $k[x,y]$. However, $f(a,y) = 1y = y$ is irreducible in $k[y]$.
- Choose $f_1, \cdots, f_n \in k[x]$ such that $f(x,y) = f_n(x)y^n + \cdots + f_1(x)y^1 + f_0(x)$. Then $f(a,y) = f_n(a)y^n + \cdots + f_1(a)y^1 + f_0(a)$. We are given that $f(a,y)$ has degree n in y, so $f_n(a) \neq 0$. Let $h_1(x,y), h_2(x,y) \in k[x]$ be given such that $f(x,y) = h_1(x,y)h_2(x,y)$. Then $f(a,y) = h_1(a,y)h_2(a,y)$. Then $h_1(a,y)$ or $h_2(a,y)$ is a unit in $k[y]$ since $f(a,y)$ is irreducible in $k[y]$. Without loss of generality, we will assume $h_1(a,y)$ is a unit in $k[y]$. Then $h_1(a,y)$ is a unit in k.

  $\deg_y(f(a,y))$, the degree of $f(a,y)$ in y, is n. Thus $\deg_y(h_1(a,y)) + \deg_y(h_2(a,y)) = n$. Therefore, $\deg_y(h_2(x,y)) \geq n$.

  On the other hand, $\deg_y(f(x,y)) = \deg_y(h_1(x,y)) + \deg_y(h_2(x,y))$, so $\deg_y(h_2(x,y)) \leq n$. Thus $\deg_y(h_2(x,y)) = n$ and this implies that $\deg_y(h_1(x,y)) = 0$. Let $g_1(x0, \cdots, g_n(x) \in k[x]$ such that $h_2(x,y) = g_n(x)y^n + \cdots + g_1(x)y^1 + g_0(x)$. Then $f(x,y) = h_1(x,y)h_2(x,y) = (h_1(x,y)g_n(x))y^n + \cdots + (h_1(x,y)g_1(x))y^1 + h_1(x,y)g_0(x)$.

  Since $\deg_y(h_1(x,y)) = 0$, $h_1(x,y) \in k[x]$, so $h_1(x,y)g_i(x) \in k[x]$ for each i. Therefore, $h_1(x,y)g_i(x) = f_i(x)$ for each i.

  I know that the idea is that $h_1(x,y)$ divides each coefficient $f_i(x)$ and the content of f is 1, $h_1(x,y) \mid 1$, so $h_1(x,y)$ is a unit in k. However, I noticed that this problem simply says "field k" but we defined on $S^{-1}R$ where R is a UFD and $S = R \setminus \{0\}$. What's R in this case?

- Let $k = \mathbb{Q}, a = 1, f(x,y) = (x-1)y^2 + y$. Then $f(x,y)$, which when viewed as an element of $k(x)[y]$ has degree 1. The content is 1 since
  - The coefficient of y is 1, and $\mathrm{ord}_p(1) = 0$ for any p.
  - $x - 1$ is a prime in $k(x)[y]$ since

□