MATH 633 (HOMEWORK 1)

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Exercise. (Problem 1)

- Let x be a limit point of $\overline{A}$. If $x \in A$, $x \in \overline{A}$ and we are done. Suppose $x \notin A$. Then $x_n \to x$ for some $\{x_n\} \subset \overline{A}$. If infinitely many terms in $\{x_n\}$ are in A, then x is a limit point of A. Suppose otherwise. Then for all sufficiently large $n \in \mathbb{N}$, there exists $\{x_{n,i}\} \subset A$ that converges to x_n, so we can pick sufficiently large j_n such that $|x_n - x_{n,j_n}| < 1/n$. Then $\{x_{n,j_n}\}$ is a sequence of points in A that converges to x, so x is a limit point of A. Thus $x \in \overline{A}$.
- Let $z \in \overline{A} \setminus A$. z is a limit point of A and $A \subset B$, so z is a limit point of B. Since B is closed, $z \in B$.

Exercise. (Problem 2)

- Not open, not closed, not compact. The boundary is $\{x + iy | |x| = |y| = 1\}$.
- Not open. Closed. Compact. The boundary is A.
- Not open. Closed. Not compact. The boundary is the real line.
- Open. Not closed. Not compact. The boundary is $\{0\}$.

Exercise. (Problem 3)

- $f'(z) = -1/z^2$.
- $|z|^2 \cdot (1/z) = \overline{z}$, which is not differentiable anywhere on $\mathbb{C}$. Since $1/z$ is differentiable everywhere on $z \neq 0$, $|z|^2$ is not differentiable anywhere on $z \neq 0$. Thus $|z|$ is not differentiable anywhere on $z \neq 0$.

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist. This is because the limit is 1 with $h_n = 1/n$, but the limit is -1 with $h_n = -1/n$. Therefore, $|z|$ is nowhere differentiable.

Exercise. (Problem 4)

- $a \implies b$. Define

$$\psi(h) = \begin{cases} \frac{f(z_0+h)-f(z_0)}{h} - w & (h \neq 0) \\ 0 & (h = 0). \end{cases}$$

Then ψ is defined on a neighborhood of 0 because z_0 is an interior point of U. Moreover, $\lim_{h \to 0} \psi(h) = 0$.

- $b \implies c$. Locally, $\frac{|f(z_0+h)-f(z_0)-wh|}{|h|} = \psi(h)$, so the limit is 0 as $h \to 0$.

- $c \implies a$.

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = w \iff \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)-wh}{h} = 0$$
$$\iff \lim_{h\to 0} \frac{|f(z_0+h)-f(z_0)-wh|}{|h|} = 0.$$

Thus $c \implies a$.

Exercise. (Problem 5(i)(ii))

$$\lim_{h\to 0} \frac{(f+g)(z_0+h)-(f+g)(z_0)}{h}$$
$$= \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} + \lim_{h\to 0} \frac{g(z_0+h)-g(z_0)}{h}$$
$$= f'(z_0) + g'(z_0).$$

$$\lim_{h\to 0} \frac{(fg)(z_0+h)-(fg)(z_0)}{h}$$
$$= \lim_{h\to 0} \frac{f(z_0+h)g(z_0+h)-f(z_0+h)g(z_0)+f(z_0+h)g(z_0)-f(z_0)g(z_0)}{h}$$
$$= \lim_{h\to 0} f(z_0+h)\frac{g(z_0+h)-g(z_0)}{h} + g(z_0)\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$
$$= f(z_0)g'(z_0) + g(z_0)f'(z_0).$$

Exercise. (Problem 5(iii))

$$\lim_{h\to 0} \frac{1/g(z_0+h)-1/g(z_0)}{h}$$
$$= \lim_{h\to 0} \frac{1}{g(z_0)g(z_0+h)} \frac{g(z_0)-g(z_0+h)}{h}$$
$$= \frac{g'(z_0)}{g^2(z_0)}.$$

By applying Problem 5(ii), we obtain the quotient rule.