MATH 601 HOMEWORK (DUE 10/16)

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Contents

1. Jordan Canonical Form

Let k be a field, V a finite dimensional k-vector space, and $T \in \mathrm{End}_k(V)$ a linear transformation.

Exercise. (Problem 1) Show that the set $\{p(x) \in k[x] \mid p(T) = 0 \in \mathrm{End}_k(V)\}$ is an ideal, $I \subset k[x]$. Also, show that $I \neq 0$.

Proof.

- Claim 1: I is nonempty.
  Use Cayley-Hamilton to find a non-trivial element. I'm still trying to understand C-H. One thing I learned today is that the determinant function is independent of the choice of the basis.
- Claim 2: I is closed under subtraction. Let $p(x), q(x) \in I$. Then $p(x) - q(x) \in I$ because $p(T) - q(T) = 0 - 0 = 0$.
- Claim 3: I is closed under multiplication by elements in $k[x]$. Let $p(x) \in I, r(x) \in k[x]$. Then $p(T)r(T) = 0r(T) = 0$, so $r(x)p(x) \in I$.

By Claim 1 and 2, I is a subgroup of $k[x]$ under addition. Then Claim 3 implies that I is an ideal. By Claim 1, $I \neq 0$. □

Exercise. (Problem 2) Let $p(x) \in k[x]$ be a nonzero polynomial such that $p(T) = 0 \in \mathrm{End}_k(V)$. Show that if $p(x) \in k[x]$ is a product of linear polynomials, then there is a k-basis for V with respect to which the matrix for T is in Jordan normal form.

Proof.

I'm not sure what to do here.

- If I use the theorem, this problem will be too easy and the first part of the problem will be unnecessary, so I don't think I can just use the theorem.
- Initially, I thought that I could just do Step 3 and 4 in the proof of the theorem on PP.3-4 of the Jordan Canonical Form handout. However, I realized that Step 3 and 4 require the characteristic polynomial, but $p(x)$ is not necessarily the characteristic polynomial. I don't think there is anything that I can do with $p(x)$ but not with the characteristic polynomial, though. If there is some cool stuff I can do with $p(x)$ then I should be able to do that with the characteristic polynomial because $p(x)$ may be the characteristic polynomial.
- Maybe... I can't assume that k is algebraically closed. But then that means I can't use the C-H theorem for the first part.

□

Exercise. (Problem 3) Suppose that the field k contains m distinct m-th roots of 1. Suppose that $T^m = \mathrm{Id}_V \in \mathrm{End}_k(V)$. Show that there is a basis of V with respect to which, the matrix for T is diagonal. What can you say about the diagonal entries?

Proof.

Some ideas...

- Assume $k = \mathbb{C}$.
- Let $r_l = \exp\left(\frac{2\pi i l}{m}\right)$ for each $l = 1, \cdots, m$.
- $x^m - 1 = (x - r_1)\cdots(x - r_m)$. Thus $T^m - \mathrm{Id}_V = (T - r_1 \mathrm{Id}_V)\cdots(T - r_m \mathrm{Id}_V)$.
- Let M denote the diagonal matrix for T. Then M^m must be the identity matrix. Moreover, each entry of M^m is simply the m-th power of the corresponding entry of M. Thus each of the diagonal entries in M must be an m-th root of 1. On the other hand, any diagonal matrix where each entry is an m-th root of 1 has this property that when raised to the m-th power, it becomes the identity.

□

Exercise. (Problem 4) Let V be a 9 dimensional k-vector space. Let $T \in \mathrm{End}_k(V)$ have minimal polynomial, $x^2(x-1)^3$. What are the possible Jordan canonical forms for T?

Proof.

For any $a, b \in \{0, 1\}$,

$$\begin{bmatrix} 1 & 0 & \cdots & & & \\ a & 1 & 0 & \cdots & & \\ 0 & b & 1 & 0 & \cdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & & & & \ddots \end{bmatrix}$$

satisfies $x^2(x-1)^3$.

□