MATH 633 HOMEWORK 3

HIDENORI SHINOHARA

Exercise. (Problem 1) A simply connected space is clearly piecewise smooth simply connected. Let Ω be piecewise smooth simply connected and $\gamma_1, \gamma_2 : [0,1] \to \Omega$ be two continuous curves with the same end points. Since Ω is open, $\gamma_1(t)$ has an open ball around it that is contained in Ω for each $t \in [0,1]$. Since $[0,1]$ is compact and γ_1 is continuous, $\gamma_1([0,1])$ is compact. Hence, there is a finite partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that $\gamma_1([t_i, t_{i+1}])$ is contained in an open ball $\subset \Omega$ for each i. Then γ_1 is homotopic to the curve $\gamma_{1'}$ that consists of n straight lines, ith of which is the line between $\gamma_1(t_i)$ and $\gamma_1(t_{i-1})$ where $i = 1, \cdots, n$. This can be shown by the "straight-line" homotopy because $\gamma_1([t_{i-1}, t_i])$ and the ith straight line are in an open ball contained in Ω.

A similar argument can be applied to show that γ_2 is homotopic to a curve $\gamma_{2'}$ that consists of finitely many straight lines. A curve consisting of finitely many straight lines is clearly piecewise smooth.

Therefore, $\gamma_1 \sim \gamma_{1'} \sim \gamma_{2'} \sim \gamma_2$. Thus Ω is simply connected.

Exercise. (Problem 2) Define $T(x,y) = x + iy$.

$$\begin{aligned}
\int_S f dz &= \int_0^1 f(t)dt + \int_0^1 f(it)(it)'dt + \int_0^1 f(1+it)(1+it)'dt + \int_0^1 f(t+i)(t+i)'dt \\
&= \int_0^1 f(t) + f(t+i)dt + i\int_0^1 f(it) + f(1+it)dt \\
&= \int_0^1 f(T(x,0)) + f(T(x,1))dx + i\int_0^1 f(T(0,y)) + f(T(1,y))dy \\
&= \int_0^1 u(T(x,0)) + u(T(x,1))dx + i\int_0^1 u(T(0,y)) + u(T(1,y))dy \\
&+ i\int_0^1 v(T(x,0)) + v(T(x,1))dx - \int_0^1 v(T(0,y)) + v(T(1,y))dy \\
&= \int_0^1 u(T(x,0)) + u(T(x,1))dx - \int_0^1 v(T(0,y)) + v(T(1,y))dy \\
&+ i(\int_0^1 u(T(0,y)) + u(T(1,y))dy + \int_0^1 v(T(x,0)) + v(T(x,1))dx) \\
&= \int_S u \circ T dx + \int_S -v \circ T dy + i(\int_S v \circ T dx + \int_S u \circ T) \\
&= \int_{\operatorname{int} S} -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + i(\int_{\operatorname{int} S} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) \\
&= 0.
\end{aligned}$$

Exercise. (Problem 4)

- Ω_1 is simply connected because any two continuous curves with the same end points are joined by the straight-line homotopy.
- Ω_2 is not simply connected because Ω_2 is homeomorphic to S^1 which has a nontrivial fundamental group. In other words, $\phi : \theta \mapsto (a+b)e^{2\pi i\theta}/2$ is a continuous curve in Ω that is not homotopic to the constant curve at $(a+b)/2$.
- Ω_3 is not simply connected because it is not connected.