MATH 612 (HOMEWORK 3)

HIDENORI SHINOHARA

Exercise. (3.1.11) Using the cellular homology, we obtain

$$\tilde{H}_i(X) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & (i = n) \\ 0 & (i \neq n). \end{cases}$$

$$\tilde{H}^i(X) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & (i = n+1) \\ 0 & (i \neq n+1). \end{cases}$$

From previous homework,

$$\tilde{H}_i(X/S^n) = \tilde{H}_i(S^{n+1}) = \begin{cases} \mathbb{Z} & (i = n+1) \\ 0 & (i \neq n+1). \end{cases}$$

Thus the map on $\tilde{H}_i(-;\mathbb{Z})$ is the zero map for each i. On the other hand, the long exact sequence of a pair gives us $\tilde{H}^{n+1}(X, S^n; \mathbb{Z}) \xrightarrow{q^*} \tilde{H}^{n+1}(X;\mathbb{Z}) \to \tilde{H}^{n+1}(S^n;\mathbb{Z})$ where $\tilde{H}^{n+1}(S^n;\mathbb{Z}) = 0$, so q^* is surjective. Therefore, it is nontrivial because $\tilde{H}^{n+1}(X;\mathbb{Z}) \neq 0$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_n(X);\mathbb{Z}) & \longrightarrow & H^{n+1}(X;\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(H_{n+1}(X);\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ext}(H_n(X/S^n);\mathbb{Z}) & \longrightarrow & H^{n+1}(X/S^n;\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(H_{n+1}(X/S^n);\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_m & \longrightarrow & \mathbb{Z}_m & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

This splitting is not natural because the middle term in the first sequence is isomorphic to $\mathbb{Z}_m \oplus 0$ and the second one is $0 \oplus \mathbb{Z}$.

The long exact sequence of a pair gives us $\tilde{H}_n(S^n;\mathbb{Z}) \to \tilde{H}_n(X;\mathbb{Z}) \to \tilde{H}_n(X, S^n;\mathbb{Z}) = \tilde{H}_n(S^{n+1};\mathbb{Z}) = 0$ which implies the surjectivity of the induced map. Since $\tilde{H}_n(X;\mathbb{Z}) \neq 0$, the induced map is nonzero.

The map induced on $\tilde{H}^i(-;\mathbb{Z})$ is the zero map for any i because at least one of $\tilde{H}^i(S^n;\mathbb{Z})$ or $\tilde{H}^i(X;\mathbb{Z})$ is 0 for each i.

Exercise. (3.1.13)

Exercise. (3.2.1) Suppose X is the union of contractible open sets $A_1, \cdots, A_n$. Since each A_i is contractible, $H^k(X, A_i; R) = H^k(X; R)$ for all $k \geq 1$.

$$\begin{tikzcd}
H^{k_1}(X, A_1; R) \times \cdots \times H^{k_n}(X, A_n; R) \arrow[r] \arrow[d, "\cong"] & H^{k_1+\cdots+k_n}(X, A_1 \cup \cdots \cup A_n; R) \arrow[d] \\
H^{k_1}(X; R) \times \cdots \times H^{k_n}(X; R) \arrow[r, "f"] & H^{k_1+\cdots+k_n}(X; R).
\end{tikzcd}$$

This diagram commutes by the naturality of a cup product. $H^{k_1+\cdots+k_n}(X, \bigcup_i A_i; R) = H^{k_1+\cdots+k_n}(X, X; R) = 0$ for all $k + l \geq 1$. By the commutativity of this diagram, the function f must be 0.

Exercise. (3.2.2)

Exercise. (3.2.3)

Exercise. (3.2.6)

Exercise. (3.2.7)