MATH 611 PROBLEM SET 1 (DUE 9/4)

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Exercise 0.1. (Exercise 4, Chapter 0) A deformation retraction in the weak sense of a space X to a subspace A is a homotopy $f_t: X \to X$ such that $f_0 = \operatorname{Id}, f_1(X) \subset A$, and $f_t(A) \subset A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \to X$ is a homotopy equivalence.

Proof. Let $i: A \to X$ denote the inclusion. Let $F: X \times I \to X$ denote the associated map $(x,t) \to f_t(x)$. Then F is a continuous function by the definition of a homotopy.

Let $f: X \to A$ be defined by $f(x) = F(x, 1) = f_1(x)$. This definition makes sense because $f_1(X) \subset A$. We claim that $f_1 \circ i \simeq \operatorname{Id}_A$ and $i \circ f_1 \simeq \operatorname{Id}_X$.

Consider $G: A \times I \to A$ such that G(a,t) = F(a,t) for all $(a,t) \in A \times I$. This definition makes sense because $f_t(A) \subset A$ for all t.

Then G is a homotopy in A between $f \circ i$ and Id_A because:

- G is a restriction of F, so G is continuous.
- $\forall a \in A, G(a,0) = F(a,0) = f_0(a) = \mathrm{Id}_X(a) = \mathrm{Id}_A(a).$
- $\forall a \in A, G(a, 1) = F(a, 1) = f(a) = f(i(a)) = (f \circ i)(a).$

Therefore, $f \circ i \simeq \mathrm{Id}_A$.

F is a homotopy between f_0 and f_1 .

- We are given that $f_0 = \mathrm{Id}_X$.
- For any $x \in X$, $(i \circ f)(x) = i(f(x)) = f(x) = f_1(x)$, so $i \circ f = f_1$.

Therefore, F is a homotopy between Id_X and $i \circ F$, so $i \circ f \simeq \mathrm{Id}_X$. In conclusion, i is indeed a homotopy equivalence.

Exercise 0.2. (Exercise 5, Chapter 0) Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X there exists a neighborhood $V \subset U$ of x such that the inclusion map $V \to U$ is nullhomotopic.

Proof. Let $p \in X$ be a point to which X deformation retracts. Since X deformation retracts to p, there exists a map $F: X \times I \to X$ such that

- $(1) \ \forall x \in X, F(x,0) = x.$
- (2) $\forall x \in X, F(x, 1) = p$.

- $(3) \ \forall t \in I, F(p,t) = p.$
- (4) F is continuous.

Let U be a neighborhood of p. Then $F^{-1}(U)$ is an open subset of the product space $X \times I$. By the 3rd property of F mentioned above, the slice $\{p\} \times I$ is a subset of $F^{-1}(U)$. Since I is compact, there must be a open subset V of X such that $\{p\} \times I \subset V \times I \subset F^{-1}(U)$ by the tube lemma.

We claim that this V is a desired subset.

- V is an open subset of X.
- Since $\{p\} \times I \subset V \times I, p \in V$.
- Since $V \times I \subset F^{-1}(U)$, $F(V \times I) \subset U$. This implies that $\forall v \in V$, $F(v,0) = v \in U$. Therefore, $V \subset U$.
- We claim that the inclusion map $i: V \to U$ is nullhomotopic. Let $e_p: V \to U$ denote the constant map at $p, G: V \times I \to U$ be defined by G(x,t) = F(x,t) for all $x \in V, t \in I$.
 - G indeed maps $V \times I$ into U because $F(V \times I) \subset U$. Therefore, G is well-defined.
 - Since G is the restriction of F to $V \times I$ and F is continuous, G is continuous.

- $\forall x \in V, G(x, 0) = F(x, 0) = x = i(x).$
- $\forall x \in V, G(x, 1) = F(x, 1) = p = e_p(x).$

Thus i is indeed nullhomotopic.

Exercise 0.3. (Exercise 9, Chapter 0) Show that a retract of a contractible space is contractible.

Proof. Let X be a contractible space. Then Id_X is homotopic to a constant map. This implies the existence of a fixed point $p \in X$ and a continuous function $F: X \times I \to X$ such that

- $\forall x \in X, F(x, 0) = x$,
- $\bullet \ \forall x \in X, F(x,1) = p.$

Let $A \subset X$ be a retract of X, and let $r: X \to A$ denote a retraction. In other words, r(X) = A and $r \mid_A = \mathrm{Id}_A$.

Let $G: A \times I \to A$ be the restriction of $r \circ F$ to $A \times I$. This makes sense because F maps $A \times I$ into X, and r maps X into A. We claim that G is a homotopy between Id_A and the constant map $e_{r(p)}$ such that $e_{r(p)}(a) = r(p)$ for all $a \in A$.

- $r \circ F$ is continuous since it is a composition of continuous functions. G is a restriction of a continuous function, so G is continuous.
- $G(a,0) = r(F(a,0)) = r(a) = a = Id_A(a)$.

•
$$G(a,1) = r(F(a,1)) = r(p) = e_{r(p)}(a)$$
.

Therefore, G is indeed a homotopy between Id_A and the constant map at r(p). Since the identity map is homotopic to a constant map, A is contractible.

Exercise 0.4. (Exercise 13, Chapter 0) Show that any two deformation retractions r_t^0 and r_t^1 of a space X onto a subspace A can be joined by a continuous family of deformation retractions r_t^s , $0 \le s \le 1$, of X onto A, where continuity means that the map $X \times I \times I \to X$ sending (x,s,t) to $r_t^s(x)$ is continuous.

Proof. Let $F: X \times I \times I \to X$ be defined such that

$$F(x,t,s) = \begin{cases} r_{t(1-2s)}^{0}(x) & (s \in [0,1/2]) \\ r_{t(2s-1)}^{1}(x) & (s \in [1/2,1]). \end{cases}$$

We claim that F is well-defined and satisfies the desired properties.

- Let s = 1/2. $r_{t(1-2s)}^0(x) = r_0^0(x) = x$ because r_t^0 is a deformation retraction. Similarly, $r_{t(2s-1)}^1(x) = r_0^1(x) = x$ because r_t^0 is a deformation retraction. Therefore, F is well defined when s=1/2. Moreover, by the pasting lemma, F is continuous. This is because the intersection $X \times I \times [0, 1/2] \cap X \times I \times [1/2, 1] =$ $X \times I \times \{1/2\}$ is closed.
- $F(x,t,0) = r_t^0(x)$ for any $x \times t \in X \times I$. $F(x,t,1) = r_t^1(x)$ for any $x \times t \in X \times I$.

Therefore, F maps $X \times I \times I \to X$ continuously sending (x, s, t) to $r_t^s(x)$.

Exercise 0.5. (Exercise 16, Chapter 1.1) Show that there are no retractions $r: X \to A$ in the following cases:

• $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1 .

Proof.

- Suppose that there exists a retract $r: X \to A$. In other words, r is a continuous map such that $r|_A = \operatorname{Id}_A$ and $r(X) \subset A$. Let $\phi:S^1\to A$ be a homeomorphism. Let ω be the loop defined in Theorem 1.7. Then $\pi_1(S^1,(1,0))$ is the infinite cyclic group generated by $[\omega]$. Consider the following two paths:
 - $-\phi\circ\omega:I\to A.$
 - $-e_0: I \to A \text{ such that } e_0(t) = (\phi \circ \omega)(0) = \phi(1,0).$

They are paths in A, and A is a subset of \mathbb{R}^3 , so they are paths in \mathbb{R}^3 . Since \mathbb{R}^3 is convex, we can define a linear homotopy between them. Let $F: I \times I \to \mathbb{R}^3$ such that $F(s,t) = t(\phi \circ$ ω)(s) + (1 - t)e₀(s). Therefore, the two paths are homotopic in \mathbb{R}^3 .

We will consider $\phi^{-1} \circ r \circ F$ that maps $I \times I \to S$. Since it is a composition of continuous functions, it is continuous.

$$(\phi^{-1} \circ r \circ F)(s,0) = \phi^{-1}(r(F(s,0)))$$

$$= \phi^{-1}(r(e_0(s)))$$

$$= \phi^{-1}(e_0(s)) \qquad (since e_0(s) \in A)$$

$$= (1,0).$$

$$(\phi^{-1} \circ r \circ F)(s,1) = \phi^{-1}(r(F(s,1)))$$

$$= \phi^{-1}(r((\phi \circ \omega)(s)))$$

$$= \phi^{-1}(r(\phi(\omega(s))))$$

$$= \phi^{-1}(\phi(\omega(s)))$$

$$= \omega(s).$$

$$(\phi^{-1} \circ r \circ F)(0,t) = \phi^{-1}(r(\phi(1,0)))$$

$$= \phi^{-1}(\phi(1,0))$$

$$= (1,0).$$

$$(\phi^{-1} \circ r \circ F)(1,t) = \phi^{-1}(r(\phi(1,0)))$$

$$= \phi^{-1}(\phi(1,0))$$

$$= (1,0).$$

Therefore, $\phi^{-1} \circ r \circ F$ is a path homotopy between ω and the constant loop at (1,0). This implies that [w] is the identity element in $\pi_1(S^1)$. However, this is a contradiction because Theorem 1.7 states that $\pi_1(S^1)$ is the infinite cyclic group generated by [w].

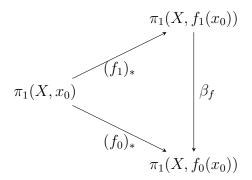
Hence, there exists no such retraction r.

Exercise 0.6. (Exercise 20, Chapter 1.1) Suppose $f_t: X \to X$ is a homotopy such that f_0 and f_1 are each the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$.

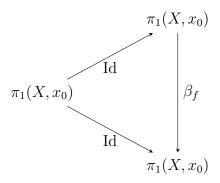
Proof. Let $x_0 \in X$ be given. Let $f: I \to X$ be the loop defined such that $f(t) = f_t(x_0)$.

- $f_t: X \to X$ is a homotopy.
- f is a path formed by the images of the base point x_0 .

By Lemma 1.19, the following diagram commutes.



 $(f_0)_* = (f_1)_* = (\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(X,x_0)}$ by a basic property of induced homomorphisms (P.34 of Hatcher). Since $f_0 = f_1 = \mathrm{Id}_X$, $f_0(x_0) = f_1(x_0) = x_0$. Therefore, the diagram above can be simplified as following:



Let $[g] \in \pi_1(X, x_0)$. Then by the diagram above, we have $\mathrm{Id}([g]) = \mathrm{Id}(\beta_f([g]))$. This implies $[g] = [f \cdot g \cdot \overline{f}]$. Therefore, $[g] \cdot [f] = [f] \cdot [g]$, so [f] commutes with every element in $\pi_1(X, x_0)$. Hence, $[f] \in Z(\pi_1(X, x_0))$.