

MATH 611 HOMEWORK (DUE 10/16)

HIDENORI SHINOHARA

Exercise. (Problem 18) For a path-connected, locally path-connected, and semilocally simply-connected space X , call a path-connected covering space $\tilde{X} \rightarrow X$ abelian if it is normal and has abelian deck transformation group. Show that X has an abelian covering space that is a covering space of every other abelian covering space of X , and that such a ‘universal’ abelian covering space is unique up to isomorphism. Describe this covering space explicitly for $X = S^1 \vee S^1$ and $X = S^1 \vee S^1 \vee S^1$.

Proof.

- What is the hypothesis?
 - X is a path-connected, locally path-connected, semilocally simply-connected space.
- What is the conclusion?
 - There exists a normal covering space of X $p : \tilde{X} \rightarrow X$ such that $G(\tilde{X})$ is abelian.
 - X has an abelian covering space that is a covering space of every other abelian covering space of X .
 - A universal abelian covering space is unique up to isomorphism.
 - Find the universal covering space of $S^1 \vee S^1$ and $S^1 \vee S^1 \vee S^1$.
- Introduce suitable notations.
 - $H = p_*(\pi_1(X, x_0))$.
- Separate the various parts of the hypothesis.
- Find the connection between the hypothesis and the conclusion.
 - “ X is a path-connected, locally path-connected, semilocally simply-connected space.” This condition sounds a lot like Theorem 1.38 on P.67. By using theorem 1.38, we can associate some group to each covering map.
 - “ \tilde{X} is a normal covering space of X .” By Proposition 1.39 on P.71, \tilde{X} is normal if and only if H is a normal subgroup of $\pi_1(X, x_0)$.
 - $G(\tilde{X})$ is abelian. By Proposition 1.39 on P.71, $G(\tilde{X})$ is isomorphic to the quotient $\pi_1(X, x_0)/H$ because \tilde{X} is normal. Thus $\pi_1(X, x_0)/H$ is abelian.
- Have you seen it before?
 - This might be similar to constructing the universal covering space.
- Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.
 - Showing uniqueness up to isomorphism sounds like the universal covering space theorem.
- Keep only a part of the hypothesis, drop the other part; is the conclusion still valid?
- Could you derive something useful from the hypothesis?
- Could you think of another hypothesis from which you could easily derive the conclusion?

- Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?
- Did you use the whole hypothesis?

□

Exercise. (Problem 19) Use the preceding problem to show that a closed orientable surface M_g of genus g has a connected normal covering space with deck transformation group isomorphic to \mathbb{Z}^n (the product of n copies of \mathbb{Z}) if and only if $n \leq 2g$. For $n = 3$ and $g \geq 3$, describe such a covering space explicitly as a subspace of \mathbb{R}^3 with translations of \mathbb{R}^3 as deck transformations.

Proof.

- What is the hypothesis? M_g is a closed orientable surface M_g of genus g .
- What is the conclusion? M_g has a connected normal covering space with deck transformation group isomorphic to \mathbb{Z}^n if and only if $n \leq 2g$.
- Separate the various parts of the hypothesis.

Closed orientable surface? I don't know what to do with it. Can I just assume that this means $M_g = S^1 \wedge \cdots \wedge S^1$?

- Find the connection between the hypothesis and the conclusion.
 - The fundamental group of M_g is generated by $2g$ elements with no relations. If we abelianize the fundamental group of M_g , we obtain \mathbb{Z}^{2g} .
- Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.
 - The previous problem shows the existence of an abelian covering space, and a normal covering space with deck transformation group isomorphic to \mathbb{Z}^n is also abelian.
- Keep only a part of the hypothesis, drop the other part; is the conclusion still valid?
- Could you derive something useful from the hypothesis?
- Could you think of another hypothesis from which you could easily derive the conclusion?
 - If $g = 1$, then this problem is easy. For $n = 2$, consider the xy plane, and for $n = 1$, consider the infinite chain of squares.
- Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?
- Did you use the whole hypothesis?

□

Exercise. (Problem 20) Construct non-normal covering spaces of the Klein bottle by a Klein bottle and by a torus.

Proof. Figure 1 is the idea that I have for the first part. But I don't know how to show that there exists no deck transformation with that permutation.

□

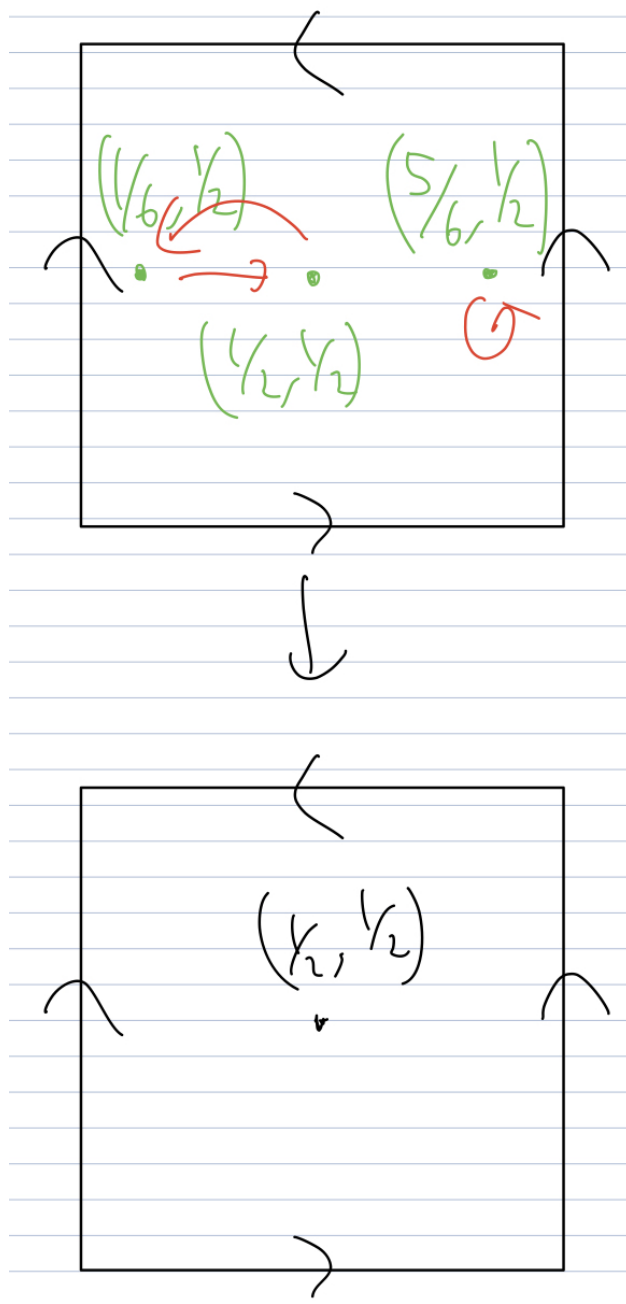


FIGURE 1. Problem 20 (Klein)

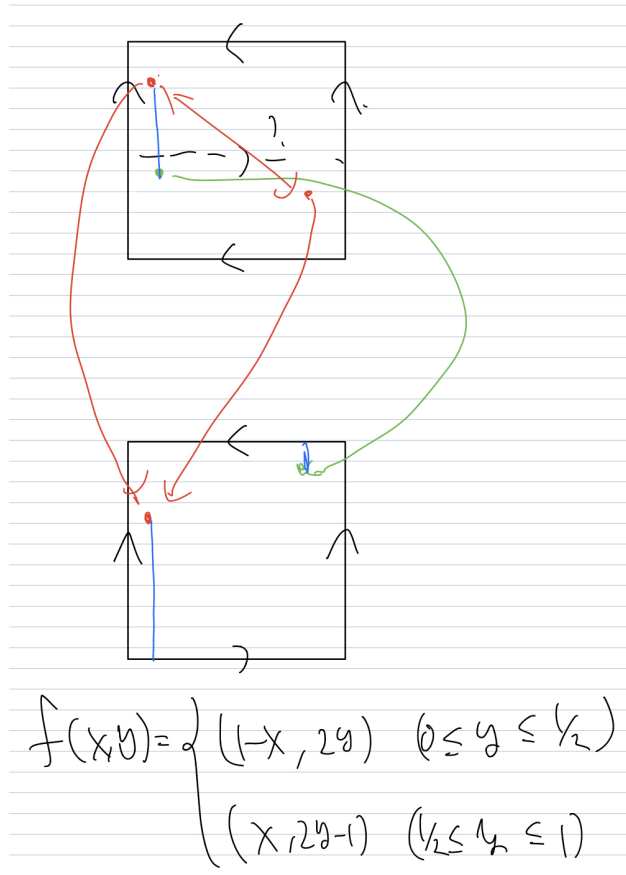


FIGURE 2. Problem 20 (Torus)