MATH 602 (HOMEWORK 5)

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Exercise. (1) This can be proved using induction. The base case $m = 1$ is trivial. Suppose that the proposition has been shown for some $m \in \mathbb{N}$. We will show the $(m+1)$ case. By the definition of a determinant,

$$\Delta = \sum_{k=1}^{m+1} (-1)^{k+1} \det(M_{k,1})$$

where $M_{k,1}$ is the matrix obtained by deleting the kth row and 1st column. We can apply the inductive hypothesis to each $M_{k,1}$ because, for instance, when $k = 1$,

$$\begin{aligned}
\det(M_{1,1}) &= \det \begin{bmatrix} \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^m \\ & \ddots & & \\ \alpha_{m+1} & \alpha_{m+1}^2 & \cdots & \alpha_{m+1}^m \end{bmatrix} \\
&= \alpha_2 \cdots \alpha_{m+1} \det \begin{bmatrix} 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{m-1} \\ & \ddots & & & \\ 1 & \alpha_{m+1} & \alpha_{m+1}^2 & \cdots & \alpha_{m+1}^{m-1} \end{bmatrix} \\
&= \alpha_2 \cdots \alpha_{m+1} \prod_{2 \le i < j \le m} (\alpha_j - \alpha_i).
\end{aligned}$$

A similar argument can be applied to other cases and we obtain

$$\Delta = \sum_{k=1}^{m+1} (-1)^{k+1} (\alpha_1 \cdots \hat{\alpha_k} \cdots \alpha_m) \prod_{i<j, i \neq k, j \neq k} (\alpha_j - \alpha_i).$$

It can be observed that, for each $k = 1, \cdots, m+1$, the kth term $(\alpha_1 \cdots \hat{\alpha_k} \cdots \alpha_m) \prod_{i<j,i\neq k,j\neq k}(\alpha_j - \alpha_i)$ does not contain any α_k. On the other hand, for any $l \neq k$, every term that we obtain when expanding the lth term contains α_k. Therefore, it suffices to show that, for each k, the sum of all the terms in $\prod_{1 \le i<j \le m+1}(\alpha_j - \alpha_i)$ that do not contain α_k is equal to the kth term in the above expression.

$$\begin{aligned}
\prod_{1 \le i < j \le m+1} (\alpha_j - \alpha_i) &= \prod_{k+1 \le j} (\alpha_j - \alpha_k) \prod_{j \le k-1} (\alpha_k - \alpha_j) \prod_{1 \le i<j \le m+1, i \neq k, j \neq k} (\alpha_j - \alpha_i) \\
&= (-1)^{k-1} \prod_{j \neq k} (\alpha_j - \alpha_k) \prod_{1 \le i<j \le m+1, i \neq k, j \neq k} (\alpha_j - \alpha_i) \\
&= (-1)^{k-1} (\alpha_1 \cdots \hat{\alpha_k} \cdots \alpha_{m+1}) \prod_{1 \le i<j \le m+1, i \neq k, j \neq k} (\alpha_j - \alpha_i) + \alpha_k F(\alpha_1, \cdots, \alpha_{m+1}) \\
&= (-1)^{k+1} (\alpha_1 \cdots \hat{\alpha_k} \cdots \alpha_{m+1}) \prod_{1 \le i<j \le m+1, i \neq k, j \neq k} (\alpha_j - \alpha_i) + \alpha_k F(\alpha_1, \cdots, \alpha_{m+1})
\end{aligned}$$

for some polynomial F.

$\Delta^2 \neq \prod_{i\neq j}(\alpha_j - \alpha_i)$ in general. Let $\alpha_1 = 0, \alpha_2 = 1$. Then $\det(A)^2 = \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^2 = 1$. On the other hand, $\prod_{i\neq j}(\alpha_j - \alpha_i) = (0-1)(1-0) = -1$.

Exercise. (5) Since R is Noetherian, $\sqrt{I}$ is generated by finitely many elements. Let $g_1, \cdots, g_n$ denote a set of generators of $\sqrt{I}$.

For each i, there exists $m_i \geq 1$ such that $g_i^{m_i} \in I$. Let $N = \sum m_i$. Then $(\sqrt{I})^N = \sqrt{I} \cdots \sqrt{I}$ consists of elements of the form $(\sum_{i=1}^n x_{1,i} g_i) \cdots (\sum_{i=1}^n x_{N,i} g_i)$. Each term that we obtain by expanding it is of the from $x g_1^{k_1} \cdots g_n^{k_n}$ for some $k_1, \cdots, k_n$ with $k_1 + \cdots + k_n = N$. This implies that for at least one i, $m_i \geq k_i$, so each term in the expansion belongs to I. Therefore, every element in $(\sqrt{I})^N$ is in I.

Exercise. (6) Let $ab \in \sqrt{q}$. Then $a^n b^n \in q$ for some $n \in \mathbb{N}$. Then $a^n \in q$ or $(b^n)^m \in q$ for some $m \in \mathbb{N}$. If $a^n \in q$, then $a \in \sqrt{q}$. If $b^{nm} \in q$, then $b \in \sqrt{q}$. Therefore, $\sqrt{q}$ is prime.

Let $f : A \to B$ be given and q be a primary ideal of B. Let $ab \in f^{-1}(q)$. Then $f(a)f(b) \in q$, so $f(a) \in q$ or $(f(b))^m \in q$ for some $m \geq 1$. If $f(a) \in q$, then $a \in f^{-1}(q)$. If $f(b^m) \in q$, then $b^m \in f^{-1}(q)$. Therefore, $f^{-1}(q)$ is primary.

Exercise. (7) Since $\sqrt{I}$ is maximal, $I \neq R$.

Let $x + I, y + I \in A/I$ be two nonzero elements such that $(x + I)(y + I) = 0$. In other words, $xy \in I$. Since $I \subset \sqrt{I}$, $(x + \sqrt{I})(y + \sqrt{I}) = 0$. Since $\sqrt{I}$ is maximal, $A/\sqrt{I}$ is a field. Therefore, $x + \sqrt{I} = 0$ or $y + \sqrt{I} = 0$. In other words, $x \in \sqrt{I}$ or $y \in \sqrt{I}$. If $x \in \sqrt{I}$, then $x + I$ is nilpotent in $A + I$. Suppose $x \notin \sqrt{I}$. Since $\sqrt{I}$ is maximal, $(x) + \sqrt{I} = (1)$. Therefore, $ax + b = 1$ for some $a \in R$ and $b \in \sqrt{I}$. Since $b \in \sqrt{I}$, $b^n \in I$ for some $n \geq 1$. Therefore $1 = ((ax + b) + I)^n = (ax + b)^n + I = xc + I$ for some element c since $b^n + I = 0$. However, this implies $0 = (x + I)(y + I)(c + I) = y + I$, which is a contradiction. Therefore, $x + I$ must be nilpotent in $A + I$. By symmetry, $y + I$ must be nilpotent in $A + I$.

We have shown that every zero divisor in A/I is nilpotent, which is precisely the definition of a primary ideal.

Exercise. (9) Let $x \in (q : b)$. Then $xb \in q$. Since $b \notin q$, $x^n \in q$ for some $n \geq 1$. However, this implies $x \in p$. Since $(q : b) \subset p$, $\sqrt{(q : b)} \subset \sqrt{p} = p$. Clearly, $q \subset (q : b)$, so $p = \sqrt{q} \subset \sqrt{(q : b)}$. Therefore, $p = \sqrt{(q : b)}$.

We will now show that $\sqrt{(q : b)}$ is primary. Let x, y be chosen such that $xy \in (q : b)$. If $y^n \in (q : b)$ for some $n \geq 1$, we are done. In other words, if $y \in \sqrt{(q : b)} = p$, then we are done. Suppose otherwise. Then $xyb \in q$, so $(xb)y \in q$. This implies $xb \in q$ because $y \notin \sqrt{q}$. This implies $x \in (q : b)$, and we are done.