MATH 612(HOMEWORK 4)

HIDENORI SHINOHARA

Exercise. (8) By using cellular cohomology, we obtain

$$H^i(X;\mathbb{Z}) = H^i(Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & (i=0,4), \\ \mathbb{Z}_p & (i=3), \end{cases}$$

$$H^i(X;\mathbb{Z}_p) = H^i(Y;\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & (i=0,2,3,4), \end{cases}$$

Therefore, we cannot distinguish X from Y by looking at the cohomology groups. When using the coefficient $\mathbb{Z}$, cup products are simply 0 because nontrivial cohomology groups are of order 3 and 4. Thus we cannot distinguish X from Y by looking at the cohomology rings of X and Y. Since $H^i(Y;\mathbb{Z}_p) = H^i(S^4;\mathbb{Z}_p) \oplus H^i(M(\mathbb{Z}_p,2);\mathbb{Z}_p)$ and the cup product of elements from different "components" in a wedge sum is 0, cup products in $H^*(Y;\mathbb{Z}_p)$ are all 0. On the other hand, the cup product $\alpha \smile \alpha$ where α is a generator of $H^2(\mathbb{C}P^2;\mathbb{Z}_p)$ is nontrivial because $\alpha \smile \alpha$ is a generator of $H^4(\mathbb{C}P^2;\mathbb{Z}_p)$.

Exercise. (5) Consider the canonical map $\mathbb{Z}_{2k} \to \mathbb{Z}_2$. It induces homomorphisms $\phi : H^i(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k}) \to H^i(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$. By cellular cohomology, $H^0(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k}) = \mathbb{Z}_{2k}$ and $H^i(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k}) = \mathbb{Z}_2$ for $i \geq 1$. Let α denote a generator of $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k})$, which equals the coset represented by k, and let β denote a generator of $H^2(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k})$, which equals the coset represented by 1, and let γ denote a generator of $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$. Then $2\alpha = 2\beta = 0$. Then ϕ on the even dimensions are all isomorphisms because $1 \mapsto 1$.

Suppose k is even. Then $\phi(\alpha) = 0$ because k is even. Moreover, $\phi(\alpha^2) = (\phi(\alpha))^2 = 0$. Since ϕ is an isomorphism on the even dimensions, $\alpha^2 = 0$. Thus $\alpha - k\beta = 0$.

Suppose k is odd. Then the ϕ are isomorphisms on the odd dimensions as well because $\overline{k} \mapsto 1$. Then $\phi(\beta) = \gamma^2 = \phi(\alpha)^2$, so $\alpha^2 = \beta$. Thus $\alpha - k\beta = 0$.

Therefore, we obtained the relations $2\alpha, 2\beta, \alpha^2 - k\beta$.

Exercise. (9) The quotient map $\mathbb{Z} \to \mathbb{Z}_p$ induces a map $f : H^*(X) \to H^*(X;\mathbb{Z}_p)$. Then we have a map $H^*(X) \times \mathbb{Z}_p \to H^*(X;\mathbb{Z}_p)$ defined by $(\alpha, a) \mapsto af(\alpha)$. Since this is bilinear, we obtain a map $\phi : H^*(X) \otimes \mathbb{Z}_p \to H^*(X;\mathbb{Z}_p)$. Suppose $\phi(\alpha \otimes a) = 0$. Without loss of generality, we assume $a = 1$. Then $f(\alpha)(\sigma) = 0$ for any σ. In other words, $\alpha(\sigma) \in p\mathbb{Z}$ for any σ. This implies the existence of $\beta \in H^*(X)$ such that $\alpha = p\beta$. Then $\alpha \otimes 1 = \beta \otimes 0 = 0$. Thus the kernel is 0, so ϕ is injective.

Prove this!

Exercise. (10) Let $X = Y = \mathbb{Z}$ with the discrete topology. Then the only nontrivial cohomology groups are $H^0(X;\mathbb{Z}) = H^0(Y;\mathbb{Z}) = \mathbb{Z}$. Therefore, it suffices to check the cross product map $H^0(X;\mathbb{Z}) \otimes H^0(Y;\mathbb{Z}) \to H^0(X \times Y;\mathbb{Z})$. Every element in $H^0(\mathbb{Z};\mathbb{Z})$ simply represents a map $\mathbb{Z} \to \mathbb{Z}$. Then for each $f \in H^0(X;\mathbb{Z}), g \in H^0(Y;\mathbb{Z})$, $f \times g : (a,b) \mapsto f(a)g(b)$. We claim that this is not surjective.

Let δ be the map such that $\delta(i,j) = \delta_{i,j}$. Then clearly, $\delta \in H^0(X \times Y;\mathbb{Z})$. Suppose that there exists $\sum_{i=1}^n a^i \otimes b^i$ that gets mapped to δ. Let $a_i, b_i \in \mathbb{Z}^n$ (with subscripts instead of superscripts) denote the vectors $a_i = \langle a^1(i), \cdots, a^n(i)\rangle, b_i = \langle b^1(i), \cdots, b^n(i)\rangle$. Then for each $i \in \mathbb{Z}$, the inner product $\langle a_i, b_i\rangle = \delta_{i,j}$. We claim that the set $\{a_i \mid i \in \mathbb{Z}\}$ is linearly independent over $\mathbb{R}$. For simplicity, let $c_1, \cdots, c_m \in \mathbb{R}$ be given such that $\sum_{i=1}^m c_i a_i = 0$. (In general, indices could be taken over any finite subset of $\mathbb{Z}$.) This implies $\sum_{i=1}^m c_i \delta_{i,j} = 0$ by taking the inner product with b_j for each j. Therefore, we obtain a linearly independent set of infinitely many vectors in $\mathbb{R}^n$. This is clearly impossible, so the cross product map cannot be surjective.

Exercise. (11) Let $f : S^{k+l} \to S^k \times S^l$. By the Kunneth formula, $H^*(S^k \times S^l) \cong H^*(S^k) \otimes H^*(S^l)$. Let $\alpha \in H^*(S^k \times S^l)$. By the isomorphism, α corresponds to some $\beta \in H^k(S^k)$ and $\gamma \in H^l(S^l)$ where $\alpha = \beta \times \gamma$. Then $f^*(\alpha) = f^*p_1^*\beta \smile f^*p_2^*\gamma$. Since $H^k(S^{k+l}) = 0$, $f^*p_1^* = 0$. Therefore, $f^*(\alpha) = 0$. In other words, f^* is the zero map.

Since each cohomology group of S^{k+l} is free, the UCT implies $H^{k+l}(S^{k+l}) \cong \mathrm{Hom}(H_{k+l}(S^{k+l}), \mathbb{Z})$. Similarly, $H^{k+l}(S^k \times S^l) \cong \mathrm{Hom}(H_{k+l}(S^k \times S^l), \mathbb{Z})$.

Then f^* can be seen as a homomorphism from $\mathrm{Hom}(H_{k+l}(S^k \times S^l), \mathbb{Z})$ to $\mathrm{Hom}(H_{k+l}(S^{k+l}), \mathbb{Z})$. In other words, f^* and f_* are the dual of each other. Therefore, $f^* = 0$ implies $f_* = 0$.

Exercise. (15) By the Kunneth formula, $\dim H^k(X \times Y; F) = \dim \sum_i H^i(X;F) \otimes H^{k-i}(Y;F) = \sum_i (\dim H^i(X;F))(\dim H^{k-i}(Y;F))$.

Exercise. (18) As discussed in Example 3.7, α_i, β_i generated $H^1(M;\mathbb{Z})$. Therefore, every element in $H^1(M;\mathbb{Z})$ can be expressed as $\sum a_i\alpha_i + \sum b_i\beta_i$. If it is nonzero, at least one of a_i or b_i is nonzero. If $a_i \neq 0$ for some i, multiplying β_i gives us a_i. If $b_i \neq 0$ for some i, multiplying α_i gives us b_i. Thus for every $\alpha \in H^1(M;\mathbb{Z})$, there exists β such that $\alpha\beta \neq 0$.

Suppose $M \simeq X \vee Y$. $H^2(X;\mathbb{Z}) \otimes H^2(Y;\mathbb{Z}) = H^2(X \vee Y;\mathbb{Z}) = H^2(M;\mathbb{Z}) = \mathbb{Z}$. Without loss of generality, $H^2(X;\mathbb{Z}) = \mathbb{Z}$ and $H^2(Y;\mathbb{Z}) = 0$. If $H^1(Y;\mathbb{Z}) = 0$, then $\tilde{H}^0(Y;\mathbb{Z}) \neq 0$. This implies that Y has multiple path components, which contradicts $M \simeq X \vee Y$. If $H^1(Y;\mathbb{Z}) \neq 0$, then let γ denote a nonzero element in $H^1(Y;\mathbb{Z})$. For any $\gamma' \in H^1(Y;\mathbb{Z})$, then $\gamma\gamma' \in H^2(Y;\mathbb{Z}) = 0$. For any $\gamma' \in H^1(X;\mathbb{Z})$, then $\gamma\gamma' = 0$ because γ and γ' are from different "components" of the wedge sum. Therefore, M cannot be homotopy equivalent to $X \vee Y$.