MATH 601 (DUE 11/6)

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Contents

1. Galois Theory II (P.2)

Exercise. (Problem 1) Let $f(x) \in F[x]$ be an irreducible polynomial of degree d. Let $F \subset K$ be a field extension such that $f(x)$ factors as a product of linear polynomials in $K[x]$. Show that $f(x)$ is separable if and only if there exist d distinct F-algebra homomorphisms, $F[x]/(f(x)) \to K$.

Proof. Without loss of generality, assume $f(x)$ is monic and $f(x) = \prod_{i=1}^{d}(x - a_i)$ for some $a_i \in K$.

Suppose $f(x)$ is separable. Then $a_i \neq a_j$ for all $i \neq j$. For each i, let $\phi_i : F[x]/(f(x)) \to K$ be an F-algebra homomorphism such that $x \mapsto a_i$ and $a \mapsto a$ for all $a \in F$. Then each ϕ_i is distinct because $\phi_i(x) \neq \phi_j(x)$ whenever $i \neq j$. Thus we showed the existence of d distinct F-algebra homomorphisms.

Show the other direction.

□

Exercise. (Problem 2) Let $F \subset F[v_1, \cdots, v_r] = K$ be an algebraic field extension such that the irreducible manic polynomial, $f_i(x) \in F[x]$, for v_i is separable for each i. Let $F \subset L$ be a splitting field of $f(x) := \prod_{i=1}^{r} f_i(x) \in F[x]$. Let $w \in K$ and let $g(x) \in F[x]$ be the minimal manic polynomial of w. Set $d = \deg(g(x))$. Show that there are exactly d distinct F-algebra homomorphisms, $F[w] \to L$.

Proof. $F[w] \cong F[x]/(g(x))$. If we can show g is separable, we can show the existence of d homomorphisms by Problem 1. The exactness shouldn't be too hard?

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