MATH 612 HOMEWORK 6

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Exercise. (Exercise 3.3.8) Let $y \in B$ and x_i denote the point in B_i such that $f(x_i) = y$ for each i. Let μ_i denote the local orientation at x_i induced by the orientation of M. For each i, we have the following diagrams using excisions, exact sequences of pairs and maps induced by f, inclusions and projections:

$$\begin{tikzcd}
 & H_n(B_i, B_i - x_i) \arrow[r] \arrow[dl] \arrow[d] & H_n(B, B - y) \arrow[d] \\
H_n(M, M - x_i) & H_n(M, M - f^{-1}(y)) \arrow[l] \arrow[r] & H_n(N, N - y) \\
 & H_n(M) \arrow[r] \arrow[u, "j"] \arrow[ul, "\cong"] & H_n(N) \arrow[u]
\end{tikzcd}$$

using the idea for the proof of Proposition 2.30.

$H_n(B_i, B_i - x_i) \to H_n(M, M - x_i)$, $H_n(B, B - y) \to H_n(N, N - y)$ and $H_n(N) \to H_n(N, N - y)$ are all isomorphisms. Since this diagram commutes for each i, $H_n(M, M - f^{-1}(y)) = \oplus_i H_n(B_i, B_i - x_i) = \oplus_i \mathbb{Z}$. Since $1 = [M]$ gets mapped to 1 in $H_n(M, M - x_i)$, $j([M]) = \sum_i k_i(\mu_i)$ where k_i is the map $H_n(B_i, B_i - x_i) \to H_n(M, M - f^{-1}(y))$. Furthermore, $f_*(\mu_i) = \epsilon_i$, so $f_*(k_i(\mu_i)) = \epsilon_i$. Therefore, $f_*([M]) = f_*(\sum k_i(\mu_i)) = \sum f_*(k_i(\mu_i)) = \sum \epsilon_i$.

Exercise. (Exercise 3.3.10) Pick a covering space $\tilde{N}$ as described in the hint and let q denote the covering map. f can be lifted to $\tilde{f} : M \to \tilde{N}$.

If $\tilde{N}$ is infinitely sheeted, it cannot be compact. By Proposition 3.29, $H_i(\tilde{N}) = 0$ for $i = n$. However, this implies that f_*, which sends $[M]$ to $[N]$, factors through 0, which cannot possibly happen. Therefore, $\tilde{N}$ must be finitely sheeted.

$\tilde{f}$ sends $[M]$ to $k[\tilde{N}]$ for some $k \in \mathbb{Z}$. Moreover, $[\tilde{N}]$ gets sent to $p[N]$ by q_* for some p where $|p|$ is the number of sheets. Then f_* sends $[M]$ to $(kp)[N]$. Since the degree of f is 1, $kp = 1$. Therefore, $\tilde{N}$ is a 1-sheeted covering space of N, so $\tilde{N}$ and N are homotopy equivalent. Because $\tilde{N}$ is chosen such that $q_*\pi_1(\tilde{N}) = \operatorname{Im} f_*$, $\operatorname{Im} f_* = \pi_1(N)$.

Exercise. (Exercise 3.3.11) We know that $H^0(M_g) = H^2(M_g) = \mathbb{Z}$ and $H^1(M_g)$ is generated by α_i, β_j with $i, j = 1, \cdots, g$ with some relations as discussed on P.208 of the textbook. Suppose $g < h$. Suppose that a degree 1 map exists. Then for each $i = 1, \cdots, h$, $\gamma = f^*(\gamma) = f^*(\alpha_i \smile \beta_i) = f^*(\alpha_i) \smile f^*(\beta_i)$. Thus there must exist j such that $f^*(\alpha_i) = \alpha_j$ and $f^*(\beta_i) = \beta_j$. For simplicity, we say i gets mapped to j in this case. Since $g < h$, there must exist $i_1 \neq i_2$ such that they both get mapped to the same number j by the pigeon hole principle. However, this implies $0 = f^*(0) = f^*(\alpha_{i_1} \smile \beta_{i_2}) = f^*(\alpha_{i_1}) \smile f^*(\beta_{i_2}) = \alpha_j \smile \beta_j = \gamma$. This is a contradiction, so such a map cannot exist.

If $g \geq h$, then M_g is the connected sum of M_{g-h} and M_h. The map f that contracts the $g - h$ holes to a point has degree 1.