MATH 633(HOMEWORK 2)

HIDENORI SHINOHARA

Exercise. (Problem 3)

$$\begin{aligned}\int_a^b |z'(t)|dt &= \int_c^d |z'(t(s))|t'(s)ds \\ &= \int_c^d |z'(t(s))t'(s)|ds \\ &= \int_c^d |\tilde{z}'(s)|ds\end{aligned}$$

where $\tilde{z}(s) : [c, d] \to \mathbb{C}$ is a reparametrization of $z(t) : [a, b] \to \mathbb{C}$.

Exercise. (Problem 4a) If $t^\star \in \Omega_1$, then there exists an open neighborhood U of $z(t^\star)$ contained in Ω_1. Then $z^{-1}(U)$ is a neighborhood of $t^\star$ in $[0, 1]$ because z is continuous. Since $z(1) \in \Omega_2$, $t^\star \neq 1$. However, this implies the existence of $\epsilon > 0$ such that $t^\star + \epsilon < 1$ and $z(t^\star + \epsilon) \in \Omega_1$. This is a contradiction.

If $t^\star \in \Omega_2$, then there exists an open neighborhood U of $z(t^\star)$ contained in Ω_2. Since U is open, $z^{-1}(U)$ is a neighborhood of $t^\star$ in $[0, 1]$, so $\exists \epsilon > 0$ such that $z(t^\star - \epsilon) \in \Omega_2$.

In each case, we reached a contradiction, so Ω is not disconnected.

Exercise. (Problem 4b) For every $v \in \Omega_1$, there exists an open set U such that $v \in U \subset \Omega_1$. Then for any $v' \in U$, v and v' can be joined by $\gamma(t) = tv + (1 - t)v'$. Thus $U \subset \Omega_1$, so Ω_1 is open.

Let $v \in \Omega_2$. Suppose that for all $\epsilon > 0$, the open disk at v with the radius ϵ is not contained in Ω_2. Otherwise we are done. Let $v_0 = w$. For every $n \in \mathbb{N}$, choose $v_n \in D(v, 1/n) \setminus \Omega_2$. Then there exists a path between each v_n and w. Moreover, there exists a path between v_n and v_{n-1} for each n and we will call it γ_n. Define $\gamma : [0, 1] \to \Omega$ such that for each $n \in \mathbb{N}$, $\gamma([1 - 1/n, 1 - 1/(n + 1)])$ is the path γ_n and $\gamma(1) = v$. Then γ is a well-defined path from w to v, which is a contradiction because $v \in \Omega_2$.

Clearly, $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1 \cup \Omega_2 = \Omega$. Since $w \in \Omega_1$, $\Omega_1 \neq \emptyset$, so $\Omega_2 = \emptyset$.