# MATH 633

HIDENORI SHINOHARA

## 1. Homework 4

**Exercise.** (Problem 1) $|\exp(f)| = \exp(\mathrm{Re}(f))$. Since $\mathrm{Re}(f)$ is bounded above, $\exp(f)$ is bounded. By Liouville's theorem, $\exp(f)$ is constant. Thus $f$ is constant because $f$ is continuous and $\exp(z) = \exp(w)$ if and only if $z - w = 2k\pi i$ for some $k \in \mathbb{Z}$.

**Exercise.** (Problem 2) Define

$$v(x, y) = \int_0^y \frac{\partial u}{\partial x}(x, t)dt - \int_0^x \frac{\partial u}{\partial y}(t, 0)dt.$$

This gives us:

$$\begin{aligned}
v_x(x, y) &= \int_0^y \frac{\partial^2 u}{\partial x^2}(x, t)dt - \frac{\partial u}{\partial y}(x, 0) \\
&= -\int_0^y \frac{\partial^2 u}{\partial t^2}(x, t)dt - \frac{\partial u}{\partial y}(x, 0) \\
&= -(\frac{\partial u}{\partial y}(x, y) - \frac{\partial u}{\partial y}(x, 0)) - \frac{\partial u}{\partial y}(x, 0) \\
&= -\frac{\partial u}{\partial y}(x, y) \\
&= -u_y(x, y). \\
v_y(x, y) &= \frac{\partial u}{\partial x}(x, y) - \int_0^x \frac{\partial^2 u}{\partial y^2}(t, 0)dt \\
&= \frac{\partial u}{\partial x}(x, y) + \int_0^x \frac{\partial^2 u}{\partial x^2}(t, 0)dt \\
&= \frac{\partial u}{\partial x}(x, y) + \frac{\partial u}{\partial x}(x, 0) - \frac{\partial u}{\partial x}(x, 0) \\
&= \frac{\partial u}{\partial x}(x, y) \\
&= u_x(x, y).
\end{aligned}$$

By Theorem 2.4, $u + iv$ is holomorphic on $D$. Given two $v_1, v_2 : D \to \mathbb{R}$ satisfying such properties, $(u + v_1 i) - (u + v_2 i)$ is a holomorphic function whose real value is always 0. By the Cauchy-Riemann equation, the derivative of $i(v_1 - v_2)$ must be 0. In other words, $v_1 - v_2$ must be constant.