

# MATH 611 HOMEWORK (DUE 10/16)

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**Exercise.** (Problem 16) Given maps  $X \rightarrow Y \rightarrow Z$  such that both  $Y \rightarrow Z$  and the composition  $X \rightarrow Z$  are covering spaces, show that  $X \rightarrow Y$  is a covering space if  $Z$  is locally path-connected, and show that this covering space is normal if  $X \rightarrow Z$  is a normal covering space.

*Proof.* Let  $p : X \rightarrow Y, q : Y \rightarrow Z$  be given such that  $q$  and  $q \circ p$  are both covering maps. Let  $y_0 \in Y$  be given. It suffices to show that there exists a neighborhood of  $y_0$  that is evenly covered by  $p$ . (Hatcher does not require a covering map be surjective.)

Let  $z_0 = q(y_0)$ . Let  $U_{z_0}$  be a locally path-connected neighborhood of  $z_0$  contained in the intersection of the following two neighborhoods:

- A neighborhood of  $z_0$  that is evenly covered by  $q$ .
- A neighborhood of  $z_0$  that is evenly covered by  $q \circ p$ .

Those two neighborhoods of  $z_0$  must exist because  $q$  and  $q \circ p$  are covering maps. Since  $Z$  is locally path-connected, any neighborhood of  $z_0$  contains a path-connected neighborhood of  $z_0$ . Therefore, such  $U_{z_0}$  must exist. Moreover, any neighborhood contained in an evenly covered neighborhood is evenly covered. Therefore,  $U_{z_0}$  is a path-connected neighborhood of  $z_0$  that is evenly covered by both  $q$  and  $q \circ p$ .

Since  $U_{z_0}$  is evenly covered by  $q$  and  $q \circ p$ ,

- Let  $\coprod_{\alpha} U_{x_{\alpha}} = (q \circ p)^{-1}(U_{z_0})$  where  $q \circ p$  maps each  $U_{x_{\alpha}}$  into  $U_{z_0}$  homeomorphically.
- Let  $\coprod_{\beta} U_{y_{\beta}} = q^{-1}(U_{z_0})$  where  $q$  maps each  $U_{y_{\beta}}$  into  $U_{z_0}$  homeomorphically.

Since  $z_0 = q(y_0)$  and  $q$  is an covering map, there exists  $U_{y_{\beta}}$  such that  $y_0 \in U_{y_{\beta}}$ . For simplicity, we will call it  $U_{y_0}$ . In other words,  $U_{y_0}$  is a neighborhood of  $y_0$  such that  $q$  is a homeomorphism between  $U_{y_0}$  and  $U_{z_0}$ .

We claim that  $U_{y_0}$  is a neighborhood of  $y_0$  that is evenly covered by  $p$  by showing that there exists a subset  $I$  of the index set such that  $p^{-1}(U_{y_0}) = \coprod_{\alpha \in I} U_{x_{\alpha}}$ .

We claim that for all  $\alpha$ ,  $U_{x_{\alpha}} \subset p^{-1}(U_{y_0})$  or  $U_{x_{\alpha}} \cap p^{-1}(U_{y_0}) = \emptyset$ . Let  $\alpha$  be given. Suppose  $U_{x_{\alpha}} \cap p^{-1}(U_{y_0}) \neq \emptyset$ . Let  $x \in U_{x_{\alpha}} \cap p^{-1}(U_{y_0})$ . Let  $x' \in U_{x_{\alpha}}$ . We will show that  $x' \in p^{-1}(U_{y_0})$ .

Since  $U_{z_0}$  is path connected and  $U_{x_{\alpha}}$  is homeomorphic to  $U_{z_0}$ ,  $U_{x_{\alpha}}$  is path connected. Let  $\gamma$  be a path from  $x$  to  $x'$ . In other words,  $\gamma(0) = x$  and  $\gamma(1) = x'$ . Then  $q \circ p \circ \gamma$  is a path in  $U_{z_0}$ . Let  $z = (q \circ p \circ \gamma)(0), z' = (q \circ p \circ \gamma)(1)$ . Then  $q \circ p \circ \gamma$  is a path from  $z$  to  $z'$  in  $U_{z_0}$ . Since  $U_{z_0}$  and  $U_{y_0}$  are homeomorphic by  $q$ , there exists a unique point  $y \in U_{y_0}$  such that  $q(y) = z$ . Since  $q$  is a covering map, there exists a unique lift  $\widetilde{q \circ p \circ \gamma}$  based at  $y$ . Let  $y' = \widetilde{q \circ p \circ \gamma}(1)$ . Then  $y' \in U_{y_0}$  because the lift must entirely lie in  $U_{y_0}$  because  $q$  is a homeomorphism between  $U_{y_0}$  and  $U_{z_0}$ .

Consider the path  $p \circ \gamma$  in  $Y$ . Since  $(p \circ \gamma)(0) = p(x)$  and  $x \in p^{-1}(U_{y_0})$ , the initial point of  $p \circ \gamma$  is in  $U_{y_0}$ . Moreover,  $q(p(x)) = z$  and  $y$  is the unique point in  $U_{y_0}$  such that  $q(y) = z$ ,  $y = p(x)$ . Since  $q \circ (p \circ \gamma) = (q \circ p) \circ \gamma$ ,  $p \circ \gamma$  is also a lift of  $q \circ p \circ \gamma$

based at  $y$ . By the uniqueness of a lift,  $q \circ \widetilde{p \circ \gamma} = p \circ \gamma$ . Specifically, this implies that  $p(x') = (p \circ \gamma)(1) = q \circ \widetilde{p \circ \gamma}(1) = y' \in U_{y_0}$ . Since  $p(x') \in U_{y_0}$ ,  $x' \in p^{-1}(U_{y_0})$ .

Let  $I = \{\alpha \mid U_{x_\alpha} \subset p^{-1}(U_{y_0})\}$ . Then we have  $\coprod_{\alpha \in I} U_{x_\alpha} \subset p^{-1}(U_{y_0})$ .

Since  $p^{-1}(U_{y_0}) \subset p^{-1}(q^{-1}(U_{z_0})) = \coprod_{\alpha} U_{x_\alpha}$ , every point in  $p^{-1}(U_{y_0})$  is in  $U_{x_\alpha}$  for some  $\alpha$ .  $I$  includes all  $\alpha$  such that  $U_{x_\alpha}$  intersects with  $p^{-1}(U_{y_0})$ . Thus  $p^{-1}(U_{y_0}) \subset \coprod_{\alpha \in I} U_{x_\alpha}$ .

Therefore,  $\coprod_{\alpha \in I} U_{x_\alpha} = p^{-1}(U_{y_0})$ .

Finally, we will show that  $p$  is a homeomorphism between  $U_{x_\alpha}$  and  $U_{y_0}$ . Let  $\alpha \in I$ . We claim that  $p(U_{x_\alpha}) = U_{y_0}$ .

- $p(U_{x_\alpha}) \subset U_{y_0}$  because of how we defined  $I$ .
- Let  $y \in U_{y_0}$ . Since  $U_{y_0}$  is path connected, there exists a path  $\gamma$  from  $y_0$  to  $y$  in  $U_{y_0}$ . Then  $q \circ \gamma$  is a path in  $U_{z_0}$ . Since  $q \circ p$  maps  $U_{x_\alpha}$  into  $U_{z_0}$  homeomorphically, there exists a unique  $x_0 \in U_{x_\alpha}$  such that  $(q \circ p)(x_0) = z_0$ . By the unique lifting property, there exists a unique lift  $\widetilde{q \circ \gamma}$  of  $q \circ \gamma$  based at  $x_0$ . Again, since  $q \circ p$  maps  $U_{x_\alpha}$  into  $U_{z_0}$  homeomorphically,  $\widetilde{q \circ \gamma}$  is in  $U_{x_\alpha}$ . Then  $p \circ (\widetilde{q \circ \gamma})$  is a path in  $U_{y_0}$ . Since  $q \circ (p \circ (\widetilde{q \circ \gamma})) = (q \circ p) \circ (\widetilde{q \circ \gamma}) = q \circ \gamma$  and  $q$  is a homeomorphism between  $U_{y_0}$  and  $U_{z_0}$ ,  $\gamma = p \circ (\widetilde{q \circ \gamma})$ . Then  $p(\widetilde{q \circ \gamma}(1)) = (p \circ (\widetilde{q \circ \gamma}))(1) = \gamma(1) = y$ . Thus  $y \in p(U_{x_\alpha})$ .

We know that  $(q \circ p)|_{U_{x_\alpha}}$  and  $q|_{U_{y_0}}$  are homeomorphisms.

$$\begin{aligned} (q|_{U_{y_0}})^{-1} \circ (q \circ p)|_{U_{x_\alpha}} &= (q|_{U_{y_0}})^{-1} \circ q|_{p(U_{x_\alpha})} \circ p|_{U_{x_\alpha}} \\ &= (q|_{U_{y_0}})^{-1} \circ q|_{U_{y_0}} \circ p|_{U_{x_\alpha}} \\ &= p|_{U_{x_\alpha}} \end{aligned}$$

Thus  $p|_{U_{x_\alpha}}$  is a homeomorphism between  $U_{x_\alpha}$  and  $U_{y_0}$ .

In conclusion,  $U_{y_0}$  is a neighborhood of  $y_0$  such that  $p^{-1}(U_{y_0})$  is the disjoint union  $\coprod_{\alpha \in I} U_{x_\alpha}$  such that  $p$  is a homeomorphism between each  $U_{x_\alpha}$  and  $U_{y_0}$ . Therefore,  $p$  is a covering map.

Suppose that  $q \circ p$  is normal. By Proposition 1.39(a), it suffices to show that  $p_*\pi_1(X)$  is a normal subgroup of  $\pi_1(Y)$ . Let  $p_*([h]) \in \pi_1(X)$  and  $[g] \in \pi_1(Y)$ . By Proposition 1.39(a),  $(q \circ p)_*(\pi_1(X)) = q_*(p_*(\pi_1(X)))$  is a normal subgroup of  $\pi_1(Z)$ . Therefore,  $q_*([g]p_*([h])[g]^{-1}) = q_*([g])q_*(p_*([h]))q_*([g]^{-1}) = q_*(p_*([h']))$  for some  $[h'] \in \pi_1(X)$ . Since  $q_*$  is injective by Proposition 1.31,  $[g]p_*([h])[g]^{-1} = p_*([h']) \in p_*(\pi_1(X))$ . Thus  $p_*(\pi_1(X))$  is a normal subgroup of  $\pi_1(Y)$ .  $\square$

**Exercise.** (Problem 18) For a path-connected, locally path-connected, and semilocally simply-connected space  $X$ , call a path-connected covering space  $X \rightarrow X$  abelian if it is normal and has abelian deck transformation group. Show that  $X$  has an abelian covering space that is a covering space of every other abelian covering space of  $X$ , and that such a ‘universal’ abelian covering space is unique up to isomorphism. Describe this covering space explicitly for  $X = S^1 \vee S^1$  and  $X = S^1 \vee S^1 \vee S^1$ .

*Proof.* We will consider the commutator subgroup  $H = [\pi_1(X, x_0), \pi_1(X, x_0)] = \{[a, b] \mid a, b \in \pi_1(X, x_0)\}$  of  $\pi_1(X, x_0)$ . Since  $H$  is a subgroup of  $\pi_1(X, x_0)$  and  $X$  is path-connected, locally path connected, and semilocally simply connected, there exists a path-connected covering space  $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  such that  $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$  by Theorem 1.38.

By Proposition 1.39(b),  $G(\tilde{X})$  is isomorphic to the quotient  $N(H)/H$ .

- Since  $H$  is the commutator subgroup,  $H$  is a normal subgroup of  $\pi_1(X, x_0)$ . Thus  $N(H) = \pi_1(X, x_0)$ . Moreover, Proposition 1.39(a) asserts that  $\tilde{X}$  is normal because  $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$  is normal.
- Since  $H$  is the commutator subgroup of  $\pi_1(X, x_0) = N(H)$ ,  $N(H)/H$  is abelian.

Therefore,  $\tilde{X}$  is an abelian covering space of  $X$ .

- Show that  $\tilde{X}$  is the ‘universal’ abelian covering space.
- Show uniqueness.

- What is the hypothesis?
  - $X$  is a path-connected, locally path-connected, semilocally simply-connected space.
- What is the conclusion?
  - There exists a normal covering space of  $X$   $p : \tilde{X} \rightarrow X$  such that  $G(\tilde{X})$  is abelian.
  - $X$  has an abelian covering space that is a covering space of every other abelian covering space of  $X$ .
  - A universal abelian covering space is unique up to isomorphism.
  - Find the universal covering space of  $S^1 \vee S^1$  and  $S^1 \vee S^1 \vee S^1$ .
- Introduce suitable notations.
  - $H = p_*(\pi_1(X, x_0))$ .
- Separate the various parts of the hypothesis.
- Find the connection between the hypothesis and the conclusion.
  - “ $X$  is a path-connected, locally path-connected, semilocally simply-connected space.” This condition sounds a lot like Theorem 1.38 on P.67. By using theorem 1.38, we can associate some group to each covering map.
  - “ $\tilde{X}$  is a normal covering space of  $X$ .” By Proposition 1.39 on P.71,  $\tilde{X}$  is normal if and only if  $H$  is a normal subgroup of  $\pi_1(X, x_0)$ .
  - $G(\tilde{X})$  is abelian. By Proposition 1.39 on P.71,  $G(\tilde{X})$  is isomorphic to the quotient  $\pi_1(X, x_0)/H$  because  $\tilde{X}$  is normal. Thus  $\pi_1(X, x_0)/H$  is abelian.
- Have you seen it before?
  - This might be similar to constructing the universal covering space.
- Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.
  - Showing uniqueness up to isomorphism sounds like the universal covering space theorem.
- Keep only a part of the hypothesis, drop the other part; is the conclusion still valid?
- Could you derive something useful from the hypothesis?
- Could you think of another hypothesis from which you could easily derive the conclusion?
- Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?
- Did you use the whole hypothesis?

□

**Lemma 0.1.** *Let  $G$  be a group and  $H$  be a subgroup. If  $[a, b] \in H$  for all  $a, b \in G$ , then  $H$  is a normal subgroup of  $G$ .*

*Proof.* Let  $g \in G, h \in H$ . Then  $ghg^{-1} = ghg^{-1}h^{-1}h = [g, h]h$ . Since  $[g, h] \in H$  and  $h \in H$ ,  $ghg^{-1} = [g, h]h \in H$ . Thus  $H$  is normal.  $\square$

**Exercise.** (Problem 19) Use the preceding problem to show that a closed orientable surface  $M_g$  of genus  $g$  has a connected normal covering space with deck transformation group isomorphic to  $\mathbb{Z}^n$  (the product of  $N$  copies of  $\mathbb{Z}$ ) if and only if  $n \leq 2g$ . For  $n = 3$  and  $g \geq 3$ , describe such a covering space explicitly as a subspace of  $\mathbb{R}^3$  with translations of  $\mathbb{R}^3$  as deck transformations.

*Proof.* Suppose  $n \leq 2g$ . Then  $\pi_1(M_g) = \langle a_1, \dots, a_{2g} \mid [a_1, a_2] \cdots [a_{2g-1}, a_{2g}] \rangle$ . Let  $H$  be the subgroup of  $\pi_1(M_g)$  generated by  $a_1, \dots, a_{2g-n}$  and the set  $\{[a_i, a_j] \mid i \neq j\}$ . By Lemma 0.1 above,  $H$  is normal. Since  $H$  is a subgroup of  $\pi_1(M_g)$ , there exists a covering space  $p: \tilde{M}_g \rightarrow M_g$  by Theorem 1.38 such that  $p_*(\pi_1(\tilde{M}_g)) = H$ .

Therefore, by Proposition 1.39(a),  $\tilde{M}_g$  is normal.

By Proposition 1.39(b),  $G(\tilde{M}_g)$  is isomorphic to the quotient  $N(H)/H$ . Since  $H$  is normal,  $N(H) = \pi_1(M_g)$ . Therefore,  $G(\tilde{M}_g)$  is isomorphic to  $\pi_1(M_g)/H$  where  $H$  contains all commutators of  $\pi_1(M_g)$ . Thus  $G(\tilde{M}_g)$  is abelian, so  $\tilde{M}_g$  is an abelian covering space.

Moreover,

$$\begin{aligned} G(\tilde{M}_g) &= \pi_1(M_g)/H \\ &= \langle a_1, \dots, a_{2g} \mid a_1, \dots, a_{2g-n}, \forall i, j, [a_i, a_j] \rangle \\ &= \langle a_{2g-n+1}, \dots, a_{2g} \mid \forall i, j, [a_i, a_j] \rangle \\ &\cong \mathbb{Z}^n. \end{aligned}$$

Finish the rest of the problem.

- List examples.  $n = 1, 2, g = 1$  and  $n = 1, g = 2$  are done. Try others.
- What is the hypothesis?  $M_g$  is a closed orientable surface  $M_g$  of genus  $g$ .
- What is the conclusion?  $M_g$  has a connected normal covering space with deck transformation group isomorphic to  $\mathbb{Z}^n$  if and only if  $n \leq 2g$ .
- Separate the various parts of the hypothesis.

Closed orientable surface? I don't know what to do with it. Can I just assume that this means  $M_g = (S^1 \times S^1) \vee \cdots \vee (S^1 \times S^1)$ ?

- Find the connection between the hypothesis and the conclusion.
  - The fundamental group of  $M_g$  is generated by  $2g$  elements with no relations. If we abelianize the fundamental group of  $M_g$ , we obtain  $\mathbb{Z}^{2g}$ .
- Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.
  - The previous problem shows the existence of an abelian covering space, and a normal covering space with deck transformation group isomorphic to  $\mathbb{Z}^n$  is also abelian.
- Keep only a part of the hypothesis, drop the other part; is the conclusion still valid?
- Could you derive something useful from the hypothesis?
- Could you think of another hypothesis from which you could easily derive the conclusion?

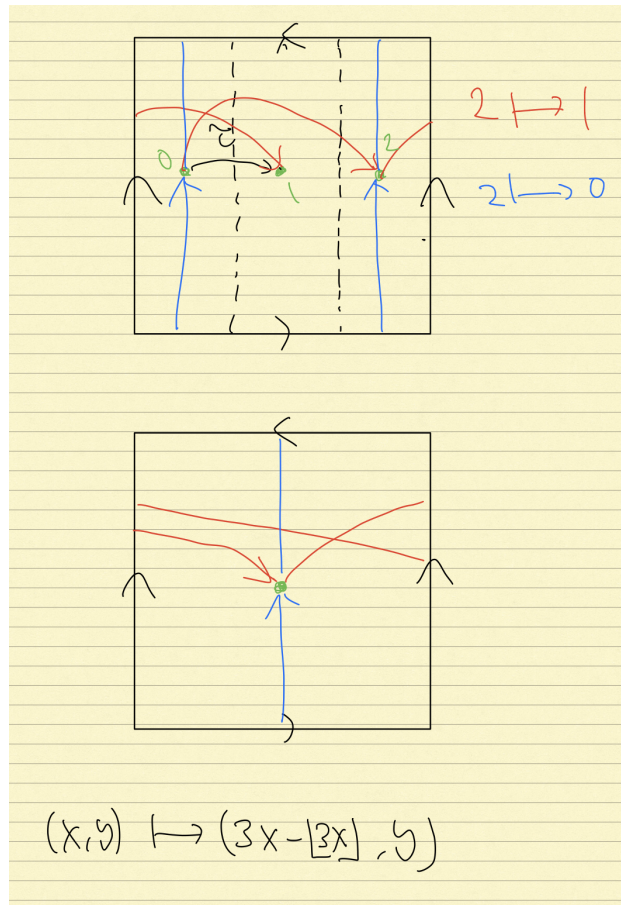


FIGURE 1. Problem 20 (Klein)

- If  $g = 1$ , then this problem is easy. For  $n = 2$ , consider the  $xy$  plane, and for  $n = 1$ , consider the infinite chain of squares.
- Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?
- Did you use the whole hypothesis?

□

**Exercise.** (Problem 20) Construct non-normal covering spaces of the Klein bottle by a Klein bottle and by a torus.

*Proof.* Figure 1 is the idea that I have for the first part. But I don't know how to show that there exists no deck transformation with that permutation.

□

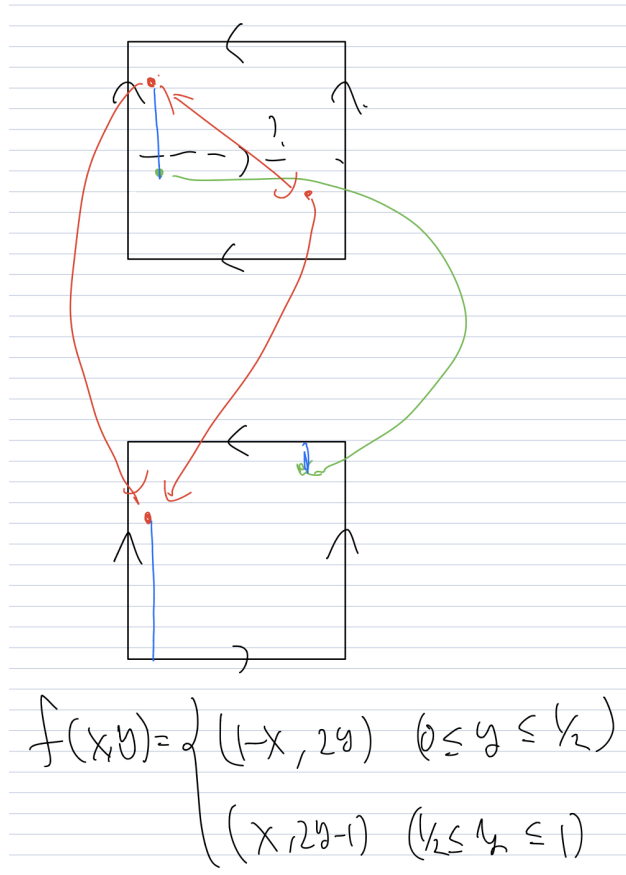


FIGURE 2. Problem 20 (Torus)