MATH 611 (DUE 11/6)

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1. Simplicial and Singular Homology

Exercise. (Problem 14) Determine whether there exists a short exact sequence $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$. More generally, determine which abelian groups A fit into a short exact sequence $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ with p prime. What about the case of short exact sequences $0 \to A \to \mathbb{Z}_n \to 0$?

Proof. Let $\phi_1 : \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2, \phi_2 : \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4$ be defined such that $\phi_1(a) = (2a, a)$ and $\phi_2(a, b) = a + 2b$. Then $\ker(\phi_1) = 0, \operatorname{Im}(\phi_1) = \ker(\phi_2) = \{(2k, k) \mid 0 \leq k \leq 3\}$ and $\operatorname{Im}(\phi_2) = \mathbb{Z}_4$. Thus this is indeed an exact sequence.

We claim that $A = \oplus_{i=1}^k \mathbb{Z}_{p^{a_i}}$ where $k \leq 2, a_1 \geq \max\{m, n\}, a_i \geq a_{i+1}, \sum a_i = m + n$ are the only $\mathbb{Z}$-modules that satisfy the exact sequence $0 \to \mathbb{Z}_{p^m} \xrightarrow{\alpha} A \xrightarrow{\beta} \mathbb{Z}_{p^n} \to 0$. It is clear that $\sum a_i = m + n$ since α is injective and $A/\alpha(\mathbb{Z}_{p^m}) = \mathbb{Z}_{p^n}$.

- First we will show that these A's indeed satisfy the exact sequence. When $k = 1$, $A = \mathbb{Z}_{p^{m+n}}$. Then with $\alpha : 1 \mapsto p^n$, we have $A/\alpha(\mathbb{Z}_{p^m}) = \mathbb{Z}_{p^n}$, so we are done. Suppose $k = 2$. Then $A = \mathbb{Z}_{p^{a_1}} \oplus \mathbb{Z}_{p^{a_2}}$. Define α such that $1 \mapsto (p^{a_1 - m}, 1)$. Then α is injective. Moreover, the order of $\alpha(1)$ is p^m in A, $|A/\operatorname{Im}(\alpha)| = p^n$. $A/\operatorname{Im}(\alpha) = \langle (1, 0) + \operatorname{Im}(\alpha) \rangle$ because for any $(a, b) + \operatorname{Im}(\alpha) \in A/\operatorname{Im}(\alpha)$ we have $(a, b) + \operatorname{Im}(\alpha) = ((a, b) - b\alpha(1)) + \operatorname{Im}(\alpha) = (a - bp^{a_1 - m}, 0) + \operatorname{Im}(\alpha)$. Therefore, $A/\operatorname{Im}(\alpha)$ is a cyclic group of order p^n. In other words, $A/\operatorname{Im}(\alpha) = \mathbb{Z}_{p^n}$.
- Next, we will show that these A's are the only abelian groups to satisfy the exact sequence. Let A be any abelian group to satisfy the exact sequence. Then α is injective and β is surjective by the exactness. Let $u \in A$ such that $\beta(u) = 1$. We claim that every element in A can be uniquely expressed as $a\alpha(1) + bu$ where $0 \leq a \leq p^m - 1$ and $0 \leq b \leq p^n - 1$.

  Let $0 \leq a \leq p^m - 1, 0 \leq b \leq p^n - 1$ be given such that $a\alpha(1) + bu = 0$. Then $-a\alpha(1) = bu$, and $\beta(-a\alpha(1)) = 0$. $bu = 0$ implies that $b\beta(u) = 0$, so $b = 0$. Moreover, $-a\alpha(1) = 0$ implies that $\alpha(-a) = 0$, so $a = 0$ because α is injective.

  Therefore, whenever $(a_1, b_1) \neq (a_2, b_2)$, $a_1\alpha(1) + b_1 u \neq a_2\alpha(1) + b_2 u$. This implies that are at least p^{m+n} elements in A of this form. By the exactness, $\mathbb{Z}_{p^n} = A/\mathbb{Z}_{p^m}$, so A must contain exactly p^{m+n} elements. Therefore, every element in A can be uniquely written in the form.

  In other words, A can be generated by two elements. This implies that $A = \mathbb{Z}_{p^{a_1}} \oplus \mathbb{Z}_{p^{a_2}}$ for some $a_1 \geq a_2 \geq 0$. Moreover, since α is injective and β is surjective, A must contain an element of order $\geq \max\{m, n\}$. Therefore, $a_1 \geq \max\{m, n\}$.

Hence, the A's listed above are all the possible abelian groups to satisfy the exact sequence.

Finally, we will consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{\alpha} A \xrightarrow{\beta} \mathbb{Z}_n \to 0$. By the exactness, β is surjective. Let $u \in A$ such that $\beta(u) = 1$. Let $x \in A$. Then $\beta(x) = b\beta(u)$ for some b.

Then $\beta(x-bu)=0$, so $x-bu \in \ker(\beta)=\mathrm{Im}(\alpha)$. Therefore, $x-bu = a\alpha(1)$ for some a, and thus every element in A can be expressed as a linear combination of $\alpha(1)$ and u.

Since $\beta(nu)=n\beta(u)=0$ in $\mathbb{Z}_n$, $nu \in \ker(\beta)=\mathrm{Im}(\alpha)$. Choose k such that $nu = k\alpha(1)$.

We will consider the following exact sequence:

$$\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}^2 \xrightarrow{\phi} A \to 0$$

where $\psi(1)=(n,-k)$ and $\phi(x,y)=xu+y\alpha(1)$. Then $\mathrm{Im}(\psi)\subset\ker(\phi)$ since $\phi(n,-k)=nu-k\alpha(1)=0$. Choose (x,y) such that $\phi(x,y)=0$. Then $xu+y\alpha(1)=0$. This implies $xu=-y\alpha(1)\in\mathrm{Im}(\alpha)=\ker(\beta)$, so $n \mid x$. Let $c=x/n$. Then $\phi(cn,-ck)=\phi(x,y)=0$, so $\phi(0,y+ck)=0$. This implies $(y+ck)\alpha(1)=0$, so $\alpha(y+ck)=0$. Since α is injective, $y=ck$. This implies $(x,y)=c(n,-k)$. Therefore, $\mathrm{Im}(\psi)=\ker(\phi)$. Moreover, ϕ is surjective, so this is indeed exact.

This implies that A is a finitely presented $\mathbb{Z}$-module. The Smith normal form of $[n;-k]$ is simply $[\gcd(n,-k);0]$, and this shows that $A\simeq\mathbb{Z}/(\gcd(n,-k))\times\mathbb{Z}/(0)=\mathbb{Z}/(d)\times\mathbb{Z}$ where $d=\gcd(n,-k)$. Therefore, the only $\mathbb{Z}$-modules that might satisfy the given exact sequence is $\mathbb{Z}\times\mathbb{Z}_d$ where d is a divisor of n. Let d be any divisor of n. Then we will show that $A=\mathbb{Z}\times\mathbb{Z}_d$ will satisfy the exact sequence. Let $\alpha(1)=(k,0)$ and $\beta(x,y)=dx+y$ where $k=n/d$.

- α is injective.
- For each $m\in\mathbb{Z}_n$, $\beta(\lfloor m/d\rfloor, m\%d)=m$. Thus β is surjective.
- $\beta(\alpha(m))=\beta(mk,0)=dmk=0$. Therefore, $\mathrm{Im}(\alpha)\subset\ker(\beta)$. Let (x,y) be given such that $\beta(x,y)=0$. Then $n \mid dx+y$. This implies $d\mid dx+y$, so $d\mid y$. Therefore, $y=0$. This implies $n\mid dx$, so $k\mid x$. In other words, $(x,y)\in\mathrm{Im}(\alpha)$.

  Therefore, $\mathrm{Im}(\alpha)=\ker(\beta)$.

Therefore, $\{\mathbb{Z}\times\mathbb{Z}_d \mid d\mid n\}$ is the set of $\mathbb{Z}$-modules that satisfy the exact sequence. □

Exercise. (Problem 15) For an exact sequence $A\to B\to C\to D\to E$ show that $C=0$ if and only if the map $A\to B$ is surjective and $D\to E$ is injective. Hence, for a pair of spaces (X,A), the inclusion $A\to X$ induces isomorphisms on all homology groups if and only if $H_n(X,A)=0$ for all n.

Proof. Suppose $C=0$. $\mathrm{Im}(\phi_{AB})=\ker(\phi_{BC})=B$, so ϕ_{AB} is surjective. $\ker(\phi_{DE})=\mathrm{Im}(\phi_{CD})=\{0\}$, so ϕ_{DE} is injective.

On the other hand, suppose ϕ_{AB} is surjective and ϕ_{DE} is injective. $\mathrm{Im}(\phi_{CD})=\ker(\phi_{DE})=\{0\}$, so ϕ_{CD} is the zero map. Therefore, $\ker(\phi_{CD})=C$. $\ker(\phi_{BC})=\mathrm{Im}(\phi_{AB})=B$, so ϕ_{BC} is the zero map. Therefore, $\mathrm{Im}(\phi_{BC})=0$. Hence, $C=\ker(\phi_{CD})=\mathrm{Im}(\phi_{BC})=0$.

By Theorem 2.16 and the discussion at the bottom of P.117(Hatcher), we have a long exact sequence of homology groups

$$H_n(A)\xrightarrow{i_*}H_n(X)\to H_n(X,A)\to H_{n-1}(A)\xrightarrow{i_*}H_{n-1}(X) \tag{1.1}$$

for $n\geq 1$. Suppose the inclusion induces isomorphisms on all homology groups. Then $H_n(X,A)=0$ for all $n\geq 1$ by the first part. Moreover, we have $H_1(X,A)\to H_0(A)\to H_0(X)\to H_0(X,A)\to 0$. Since $H_1(X,A)=0$, by the first part, $H_0(X)=0$. In order for $0\to H_0(X,A)\to 0$ to be exact, $H_0(X,A)$ must be 0. Therefore, $H_n(X,A)=0$ for all $n\geq 0$.

Suppose that $H_n(X,A)=0$ for all $n\geq 0$. By exact sequence 1.1 above, $i_*: H_n(A)\to H_n(X)$ is surjective for $n\geq 1$ and injective for $n\geq 0$. Thus i_* is bijective for all $n\geq 1$. We have $H_1(X,A) \to H_0(A) \to H_0(X) \to H_0(X,A)$. Since $H_1(X,A) = H_0(X,A) = 0$, i_* must be bijective by the exactness. Therefore, the inclusion induces isomorphisms for all n. ☐

Exercise. (Problem 16)

- Show that $H_0(X,A) = 0$ if and only if A meets each path-component of X.
- Show that $H_1(X,A) = 0$ if and only if $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

Proof.

- Let $\gamma_x + C_0(A) \in C_0(X)/C_0(A)$. Since A meets each path-component of X, there exists a path $\gamma : I \to X$ that joins a point $a \in A$ and the image of γ_x. Then γ can be seen as an element of $C_1(X)$ since γ maps a 1-simplex into X. Moreover, $\partial\gamma = \gamma_x - \gamma_a$ where $\gamma_a \in C_0(A)$ with $\operatorname{Im}(\gamma_a) = a$. Therefore, $\partial(\gamma + C_1(A)) = \gamma_x + C_0(A)$, so $\gamma_x + C_0(A) \in \operatorname{Im}(\partial)$. Hence, $H_0(X,A) = \ker(\partial_0)/\operatorname{Im}(\partial_1) = (C_0(X)/C_0(A))/(C_0(X)/C_1(A)) = 0$.

  On other hand, suppose that A does not meet each path component of X. Let $x \in X$ be a point in a path component that A does not intersect. Let $\gamma_x : \Delta^0 \to X$ such that $\operatorname{Im}(\gamma_x) = \{x\}$. Then $\gamma_x \in \ker(\partial_0) = C_0(X,A)$. Let $\gamma + C_1(A) \in C_1(X,A)$. Then $\partial_1(\gamma + C_1(A)) = \partial_1(\gamma) + C_0(A)$. Let $\gamma_{x_1}, \gamma_{x_2} \in C_0(X)$ such that $\partial_1(\gamma) = \gamma_{x_1} - \gamma_{x_2}$. $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$ if and only if $\gamma_{x_1} - \gamma_{x_2} - \gamma_x \in C_0(A)$.
  - If γ lies in the same path component as x, then so do x_1 and x_2. Suppose $x = x_1$. Since $-\gamma_{x_2} \notin C_0(A)$, $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$. The case when $x \neq x_1$ and $x = x_2$ and the case when $x \neq x_1$ and $x \neq x_2$ can be proven in a similar way.
  - If γ lies in a different path component, then $\gamma_x \neq \gamma_{x_1}$ and $\gamma_x \neq \gamma_{x_2}$. Therefore, $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$.

  Therefore, $\gamma_x \notin \operatorname{Im}(\partial_1)$. Thus $H_0(X,A) = C_0(X,A)/\operatorname{Im}(\partial_1)$ is not 0.
- Suppose $H_1(X,A) = 0$. By the exact sequence $H_1(A) \xrightarrow{\phi} H_1(X) \to H_1(X,A) \to H_0(A) \xrightarrow{\psi} H_1(X)$, we know that ϕ is surjective and ψ is injective. Suppose that there is a path component of X that contains two or more path components of A. Let a, b be points in two distinct path components of A that are contained in a path component of X. We will regard a, b as functions $\Delta^0 \to A$. Then $a, b \in C_0(A)$ and $[a] \neq [b]$ in $H_0(A)$ because a, b are in different path components of A, so $a - b \notin \operatorname{Im}(\partial_1)$. However, a, b live in the same path component of X, $a - b \in \operatorname{Im}(\partial_1) \subset C_0(X)$. Therefore, $\phi([a]) = \phi([b])$ where $[a] \neq [b]$. This is a contradiction because ϕ is injective. Therefore, each path component of X contains at most one path component of A.

  Show the other direction.

☐

Exercise. (Problem 17)

- Compute the homology groups $H_n(X,A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.
- Compute the groups $H_n(X,A)$ and $H_n(X,B)$ for X a closed orientable surface of genus two with A and B the circles shown.

Proof.

- We will apply Theorem 2.16 to get the exact sequence with $H_n(A), H_n(X), H_n(X,A)$.

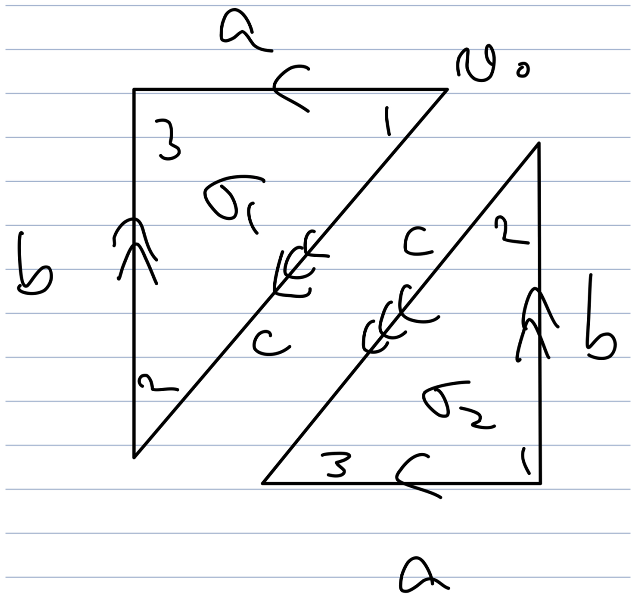

FIGURE 1. Problem 17

– When $n \geq 3$, $H_n(S^2) \to H_n(S^2, A) \to H_{n-1}(A)$ shows that $H_n(S^2, A)$ is 0 by the exactness since $H_n(S^2) = H_{n-1}(A) = 0$.
– When $n = 2$, $H_n(A) \to H_n(S^2) \xrightarrow{\phi} H_n(S^2, A) \to H_{n-1}(A)$ shows that $H_n(S^2, A) = H_n(S^2) = \mathbb{Z}$. This is because $H_n(A) = H_{n-1}(A) = 0$ so ϕ is an isomorphism by the exactness.
– By Problem 16, $H_0(X, A) = 0$. By the exact sequence $0 \to H_1(X, A) \to H_0(A) \to H_0(X)$ where $H_0(A) = \mathbb{Z}^{|A|}$ and $H_0(X) = \mathbb{Z}$, we have $H_1(X, A) = \mathbb{Z}^{|A|-1}$.

We will first compute the homology groups of a torus using Figure 1. $C_2 = \{\sigma_1, \sigma_2\}, C_1 = \{a, b, c\}, C_0 = \{v_0\}$.
– $H_2 = \ker(\partial_2)/\operatorname{Im}(\partial_3) = \langle \sigma_1 - \sigma_2 \rangle /0 = \mathbb{Z}$.
– $H_1 = \ker(\partial_1)/\operatorname{Im}(\partial_2) = \langle a, b, c \rangle / \langle b - a + c, c - a + b \rangle = \mathbb{Z}^2$ because $b - a + c = c - a + b$.
– $H_0 = \ker(\partial_0)/\operatorname{Im}(\partial_1) = \langle v_0 \rangle /0 = \mathbb{Z}$.

Again, we will apply Theorem 2.16 to get the exact sequence with $H_n(A), H_n(X)$, and $H_n(X, A)$.
– When $n \geq 3$, $H_n(S^1 \times S^1) \to H_n(S^1 \times S^1, A) \to H_{n-1}(A)$ shows that $H_n(S^1 \times S^1, A)$ is 0 by the exactness since $H_n(S^1 \times S^1) = H_{n-1}(A) = 0$.
– When $n = 2$, $H_n(A) \to H_n(S^1 \times S^1) \xrightarrow{\phi} H_n(S^1 \times S^1, A) \to H_{n-1}(A)$ shows that $H_n(S^1 \times S^1, A) = H_n(S^1 \times S^1) = \mathbb{Z}$. This is because $H_n(A) = H_{n-1}(A) = 0$ so ϕ is an isomorphism by the exactness.
– By Problem 16, $H_0(X, A) = 0$. We have the exact sequence $H_1(A) \to H_1(T^2) \to H_1(T^2, A) \xrightarrow{\phi} H_0(A) \xrightarrow{\psi} H_0(T^2) \to H_0(T^2, A)$ where $H_0(A) = \mathbb{Z}^{|A|}, H_0(X) = \mathbb{Z}, H_1(T^2) = \mathbb{Z}^2$, and $H_1(A) = H_0(T^2, A) = 0$. Moreover, $H_1(T^2, A)/\ker(\phi) =$

$\mathrm{Im}(\phi) = \ker(\psi) = \mathbb{Z}^{|A|-1}$. Since $\ker(\phi) = \mathbb{Z}^2$ by the exactness, $H_1(T^2, A) = \mathbb{Z}^{|A|+1}$.

- (X, A) is a good pair because A is a nonempty closed subspace that is a deformation retract of some neighborhood in X. By Proposition 2.22, $H_n(X, A) = \tilde{H}_n(X/A)$ for all n. The quotient space X/A is $T^2 \vee T^2$ where T^2 is a torus. By Corollary 2.25, $\tilde{H}_n(X/A) = \tilde{H}_n(T^2) \oplus \tilde{H}_n(T^2)$. (($T^2, p$) is clearly a good pair for a point $p \in T^2$.) We calculate above that

$$H_n(T^2) = \begin{cases} \mathbb{Z} & (n = 0, 2) \\ \mathbb{Z}^2 & (n = 1) \\ 0 & (n \geq 3). \end{cases}$$

Therefore,

$$H_n(X, A) = \tilde{H}_n(X/A) = \tilde{H}_n(T^2) \oplus \tilde{H}_n(T^2) = \begin{cases} \mathbb{Z}^2 & (n = 2) \\ \mathbb{Z}^4 & (n = 1) \\ 0 & (n = 0, n \geq 3). \end{cases}$$

Similarly, X/B is $T^2 \vee S^1$ because (X, B) is a good pair. Moreover, (S^1, p) is a good pair for a point $p \in S^1$, so it suffices to check $\tilde{H}_n(T^2) \oplus \tilde{H}_n(S^1)$.

Therefore,

$$H_n(X, B) = \tilde{H}_n(X/B) = \tilde{H}_n(T^2) \oplus \tilde{H}_n(S^1) = \begin{cases} \mathbb{Z} & (n = 2) \\ \mathbb{Z}^3 & (n = 1) \\ 0 & (n = 0, n \geq 3). \end{cases}$$

□

Exercise. (Problem 26) Show that $H_1(X, A)$ is not isomorphic to $\tilde{H}_1(X/A)$ if $X = [0, 1]$ and A is the sequence $1, 1/2, 1/3, \cdots$ together with its limit 0.

Proof. We will show that $H_1(X, A)$ is countable, and $\tilde{H}_1(X/A) = H_q(X/A)$ is uncountable. We have an exact sequence $\tilde{H}_1(X) \to \tilde{H}_1(X, A) \xrightarrow{\phi} \tilde{H}_0(A) \to \tilde{H}_0(X)$. Since $H_1(X, A) = \tilde{H}_1(X, A) = \tilde{H}_0(X) = 0$, ϕ is an isomorphism. Thus $\tilde{H}_1(X, A) = \tilde{H}_0(A) = \ker(\partial_1)/\mathrm{Im}(\partial_2)$. Since A is a disjoint union of points, $\mathrm{Im}(\partial_2) = 0$. $\ker(\partial_1) = \{\sum n_i \alpha_i \mid n_i \in \mathbb{Z}, \sum n_i = 0\}$ where α_i is the point $1/i$ by the definition of a reduced homology. Then this is generated by $\{\alpha_1 - \alpha_2, \alpha_1 - \alpha_3, \alpha_1 - \alpha_4, \cdots\}$, so $\tilde{H}_1(X, A)$ is countable.

We will show the existence of an injective map ζ from the direct product $\prod_{i=1}^{\infty} \mathbb{Z}$ to $H_1(X/A)$, which is homeomorphic to the Hawaiian earring. We will refer to the nth ring C_n as in Example 1.25. Let $(k_1, \cdots) \in \prod_{i=1}^{\infty} \mathbb{Z}$ be given. Construct the map $f : I \to X/A$ that wraps k_n times around C_n in the time interval $[1 - 1/n, 1 - 1/(n+1)]$. This infinite composition of loops is certainly continuous at each time less than 1, and it is continuous at time 1 since every neighborhood of the basepoint in X/A contains all but finitely many of the circles C_n. This shows that $f \in C_1(X/A)$. Moreover, $\partial(f) = v_0 - v_0 = 0$ where v_0 is the origin of the Hawaiian earring. Therefore, $[f] \in H_1(X/A)$. We define $\zeta(k_1, \cdots) = [f]$.

Let $(k_1, \cdots) \neq (l_1, \cdots) \in \prod_{i=1}^{\infty} \mathbb{Z}$ be given. Let $\zeta(k_1, \cdots) = f, \zeta(l_1, \cdots) = g$ as described above. Let i be an index such that $k_i \neq l_i$. Let $F : X/A \to S^1$ be a continuous map that maps C_n onto S_1 and C_i to -1 for all i where S_1 is seen as a subset of $\mathbb{C}$. Then F induces a group homomorphism $F_* : H_1(X/A) \to H_1(S^1)$ where $F([f]) = k_n$ and $F([g]) = l_n$. Since $F([f]) \neq F([g])$, $[f] \neq [g]$. This shows the injectivity of ζ and hence $H_1(X/A)$ must be uncountable.

Therefore, $H_1(X, A)$ is not isomorphic to $H_1(X/A)$. $\square$