MATH 611 HOMEWORK (DUE 9/18)

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Exercise. (Problem 12, Chapter 1.2) The Klein bottle is usually pictured as a subspace of $\mathbb{R}^3$ like the subspace $X \subset \mathbb{R}^3$ shown in the first figure at the right. If one wanted a model that could actually function as a bottle, one would delete the open disk bounded by the circle of self-intersection of X, producing a subspace $Y \subset X$. Show that $\pi_1(X) \approx \mathbb{Z} * \mathbb{Z}$ and that $\pi_1(Y)$ has the presentation $\langle a, b, c \mid aba^{-1}b^{-1}cb^{\epsilon}c^{-1} \rangle$ for $\epsilon = \pm 1$. Show also that $\pi_1(Y)$ is isomorphic to $\pi_1(\mathbb{R}^3 \setminus Z)$ for Z the graph shown in the figure.

Proof. We will construct X from the 1-skeleton in Figure 1. The 1-skeleton has three loops a, b, c, so the fundamental group is $\langle a, b, c \mid \rangle$. The main difference between X and the "proper" Klein bottle is that the loop a actually gets glued on the surface. Thus we will glue the first 2-cell to a, and another 2-cell on the loop $c^{-1}acbab^{-1}$. Therefore, we end up with the fundamental group $\langle a, b, c \mid a, c^{-1}aca^{-1}bab^{-1} \rangle$. Then $\langle a, b, c \mid a, c^{-1}acabab^{-1} \rangle \approx \langle b, c \mid \rangle \approx \mathbb{Z} * \mathbb{Z}$ since the relation $c^{-1}aca^{-1}bab^{-1}$ is trivial by the relation a.

In order to calculate the fundamental group of Y, it suffices to repeat the following step without attaching a 2-cell to a. Thus the fundamental group is $G = \langle a, b, c \mid c^{-1}aca^{-1}bab^{-1} \rangle$.

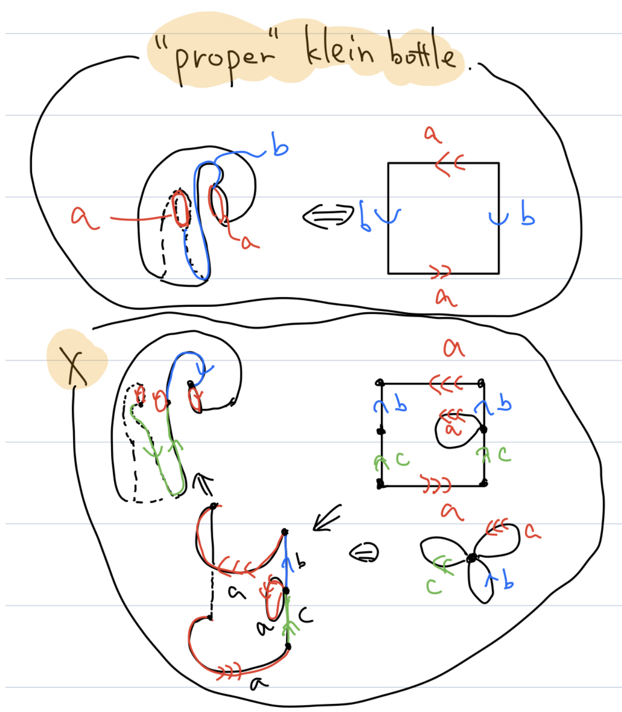

FIGURE 1. Fundamental Group of X

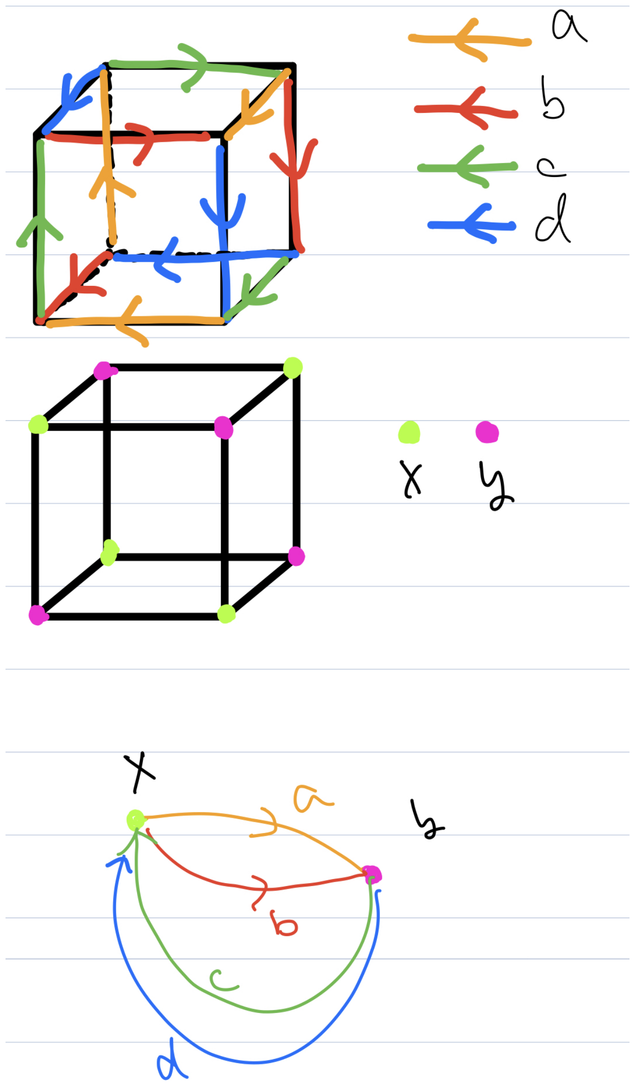

FIGURE 2. Problem 14

This is isomorphic to the group given in the textbook, $H = \langle a, b, c \mid aba^{-1}b^{-1}cbc^{-1} \rangle$ by $\phi : G \to H$ that maps a to b, b to c, and c to a^{-1}. □

Exercise. (Problem 14, Chapter 1.2) Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ of order eight.

Proof. The vertices and edges get identified as in Figure 2. Thus we have two 0-cells and four 1-cells. Since the opposite faces are identified and the cube has 6 faces, we need to glue three 2-cells to the cube. Lastly, we need a 3-cell glued to the three faces. By Proposition 1.26, the fundamental group of a 2-skeleton is the same as the fundamental group of a space obtained by attaching 3-cells, so it suffices to consider the fundamental group we obtain by attaching the three 2-cells to the graph. As in Figure 2, the graph has 4 edges between two vertices. The fundamental group of this is $\langle ab^{-1}, ac, ad \rangle$ because by "shrinking" a we obtain the graph consisting of one vertex and three loops. By attaching a 2-cell to each of the top-bottom pair, left-right pair, and the front-back pair, we obtain

$$\langle ac, ab^{-1}, ad \mid ab^{-1}d^{-1}c, adc^{-1}b^{-1}, acbd \rangle.$$

Thus this is the fundamental group of the given space. We claim that $(ac)^2 = (ab^{-1})^2 = (ad)^2 = (ac)(ab^{-1})(ad)$.

- $(ac)^2 = (ab^{-1})^2$?

$$\begin{aligned} ac = d^{-1}b^{-1} &\implies ab^{-1}bc = d^{-1}b^{-1} \\ &\implies ab^{-1}ad = d^{-1}b^{-1} \\ &\implies ab^{-1}a = d^{-1}b^{-1}d^{-1} \\ &\implies ab^{-1}ab^{-1} = d^{-1}b^{-1}d^{-1}b^{-1} \\ &\implies (ab^{-1})^2 = (d^{-1}b^{-1})^2 \\ &\implies (ab^{-1})^2 = (ac)^2. \end{aligned}$$

- $(ac)^2 = (ad)^2$?

$$\begin{aligned} ab^{-1} = c^{-1}d &\implies cab^{-1} = d \\ &\implies ca = db \\ &\implies cac = dbc \\ &\implies cac = dad \\ &\implies acac = adad \\ &\implies (ac)^2 = (ad)^2. \end{aligned}$$

- $(ad)^2 = (ac)(ab^{-1})(ad)$? $(ac)(ab^{-1}) = acc^{-1}d = ad$, so $(ac)(ab^{-1})(ad) = (ad)^2$.

Moreover, we claim that $(ac)^2 \neq e$ and $(ac)^4 = e$.

- $(ac)^2 \neq e$.

  Prove this!

- $(ac)^4 = e$.

  Prove this!

$\square$