

MATH 611 (DUE 10/23)

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1. SIMPLICIAL AND SINGULAR HOMOLOGY

Exercise. (Problem 2) Show that the Δ -complex obtained from Δ^3 by performing the edge identifications $[v_0, v_1] \sim [v_1, v_3]$ and $[v_0, v_2] \sim [v_2, v_3]$ deformation retracts onto a Klein bottle. Find other pairs of identifications of edges that produce Δ -complexes deformation retracting onto a torus, a 2-sphere, and \mathbb{RP}^2 .

Proof. The deformation retraction of Δ^3 onto a Klein bottle is described in 1. We will start by “pushing” Δ^3 from edge (v_1, v_2) . This will leave the surface that consists of the triangles $[v_0, v_1, v_3]$ and $[v_0, v_2, v_3]$. (In other words, a diamond shape consisting of the vertices $[v_0, v_1, v_3, v_2]$.) Step 2 in Figure 1 is what Δ^3 should look like after the deformation retract. Step 3 through 6 show why this is a Klein bottle.

Figure 2 shows the identification of edges for a torus, 2-sphere, and \mathbb{RP}^2 .

□

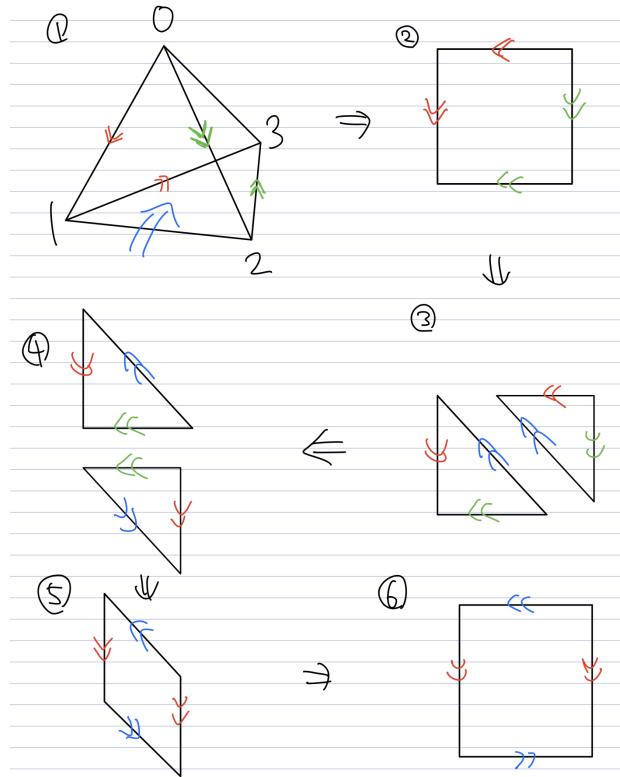


FIGURE 1. Problem 2(Klein Bottle)

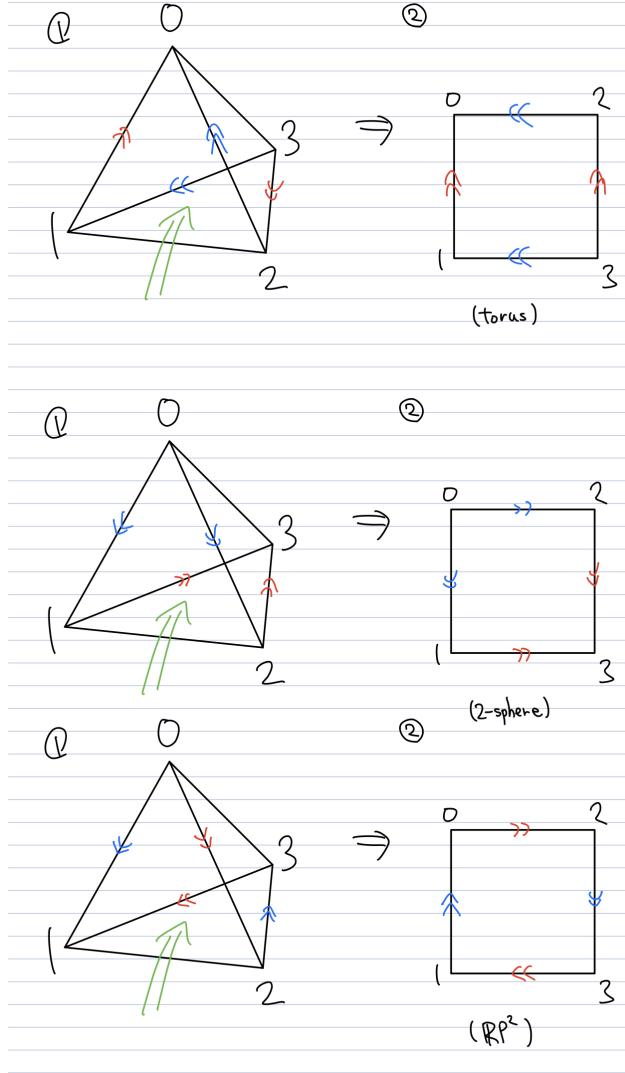


FIGURE 2. Problem 2(Torus, 2-Sphere, \mathbb{RP}^2)

Exercise. (Problem 4) Compute the simplicial homology groups of the triangular parachute obtained from Δ^2 by identifying its three vertices to a single point.

Proof. Let v_0 denote the only vertex, e_1, e_2, e_3 denote the three edges of the parachute, and σ denote the face of the parachute as in Figure 3. $C_k = 0$ for $k \geq 3$ because Δ^2 with the vertices identified does not contain any k -dimensional simplices for $k \geq 3$. $C_2 = \langle \sigma \rangle, C_1 = \langle e_1, e_2, e_3 \rangle, C_0 = \langle v_0 \rangle$. For each n , ∂_n is defined such that $\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha | [v_0, \dots, \hat{v}_i, \dots, v_n]$.

- $\partial_2(\sigma) = e_3 - e_2 + e_1$.
- $\partial_1(e_1) = v - v = 0$. Similarly, $\partial_1(e_2) = \partial_1(e_3) = 0$.
- ∂_0 and ∂_3 are both the zero map.

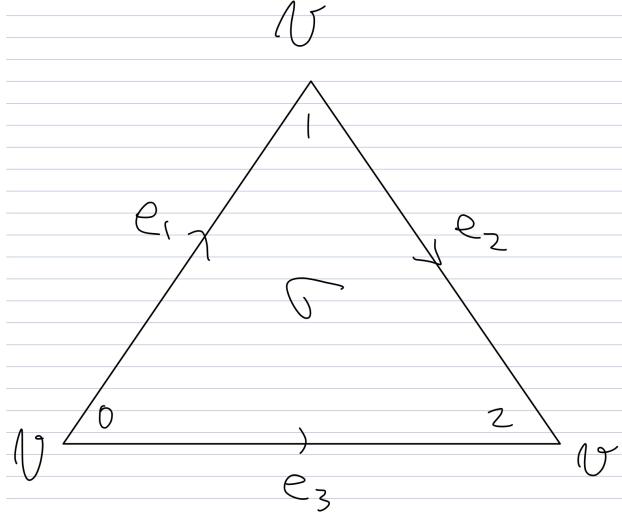


FIGURE 3. Problem 4

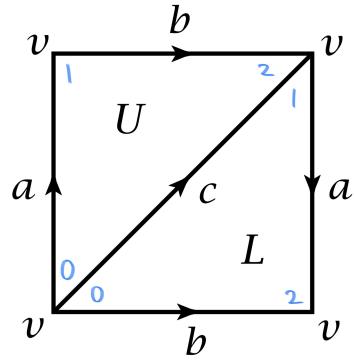


FIGURE 4. Problem 5

Thus

$$H_n = \begin{cases} \{0\} & (n \geq 3) \\ \ker(\partial_2) / \text{Im}(\partial_3) = 0/0 \cong 0 & (n = 2) \\ \ker(\partial_1) / \text{Im}(\partial_2) = \langle e_1, e_2, e_3 \rangle / \langle e_3 - e_2 + e_1 \rangle \cong \langle e_1, e_2, -e_2 + e_1 \rangle \cong \mathbb{Z}^2 & (n = 1) \\ \ker(\partial_0) / \text{Im}(\partial_1) = \langle v \rangle / 0 \cong \mathbb{Z} & (n = 0). \end{cases}$$

□

Exercise. (Problem 5) Compute the simplicial homology groups of the Klein bottle using the Δ -complex structure described at the beginning of this section.

Proof. We will use the notations in Figure 4.

$$C_n = \begin{cases} 0 & (n \geq 3) \\ \langle U, L \rangle & (n = 2) \\ \langle a, b, c \rangle & (n = 1) \\ \langle v \rangle & (n = 0). \end{cases}$$

$\partial_n = 0$ for $n \geq 3$ and $n = 0$.

$$\begin{aligned} \partial_2(U) &= \sum_{i=0}^2 (-1)^i \sigma |[0, 1, 2]| \\ &= \sigma |[1, 2] - \sigma |[0, 2] + \sigma |[0, 1]| \\ &= b - c + a. \\ \partial_2(L) &= \sum_{i=0}^2 (-1)^i \sigma |[0, 1, 2]| \\ &= \sigma |[1, 2] - \sigma |[0, 2] + \sigma |[0, 1]| \\ &= a - b + c. \end{aligned}$$

$\partial_1(a) = 0$ since $\partial_1(a) = \sigma |[1] - \sigma |[0]| = v - v = 0$. Similarly, $\partial_1(b) = \partial_1(c) = 0$. Thus $H_n = \{0\}$ if $(n \geq 3)$. $H_2 = \ker(\partial_2)/\text{Im}(\partial_3) = 0/0 \cong 0$.

$$\begin{aligned} H_1 &= \ker(\partial_1)/\text{Im}(\partial_2) \\ &= \langle a, b, c \rangle / \langle b - c + a, a - b + c \rangle \\ &\cong \langle a, b, a + b \mid a - b + (a + b) \rangle \\ &\cong \langle a, b \mid 2a \rangle \\ &\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}. \end{aligned}$$

$$H_0 = \ker(\partial_0)/\text{Im}(\partial_1) = \langle v \rangle / 0 \cong \mathbb{Z}.$$

□

Exercise. (Problem 7) Find a way of identifying pairs of faces of Δ^3 to produce a Δ -complex structure on S^3 having a single 3-simplex, and compute the simplicial homology groups of this Δ -complex.

Proof. We will identify $[0, 2, 3] \sim [1, 2, 3]$ and $[0, 1, 2] \sim [0, 1, 3]$ of the tetrahedra T in Figure 5. Then we have

$$\begin{aligned} C_3 &= \{T\} \\ C_2 &= \{f_1, f_2\} \\ C_1 &= \{e_1, e_2, e_3\} \\ C_0 &= \{v_1, v_2\}. \end{aligned}$$

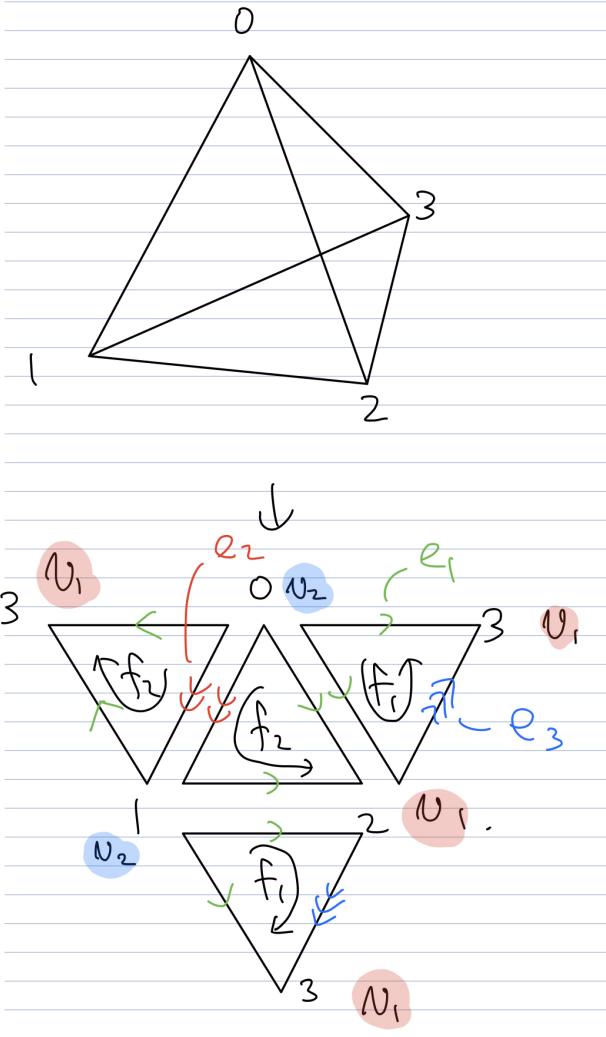


FIGURE 5. Problem 7

We will examine ∂ .

$$\partial_3(T) = [1, 2, 3] - [0, 2, 3] + [0, 1, 3] - [0, 1, 2] = f_1 - f_1 + f_2 - f_2 = 0.$$

$$\partial_2(f_1) = [2, 3] - [0, 3] + [0, 2] = e_3 - e_1 + e_1 = e_3.$$

$$\partial_2(f_2) = [1, 2] - [0, 2] + [0, 1] = e_1 - e_1 + e_2 = e_2.$$

$$\partial_1(e_1) = [3] - [0] = v_1 - v_2.$$

$$\partial_1(e_2) = [1] - [0] = v_2 - v_2 = 0.$$

$$\partial_1(e_3) = [3] - [2] = v_1 - v_1 = 0.$$

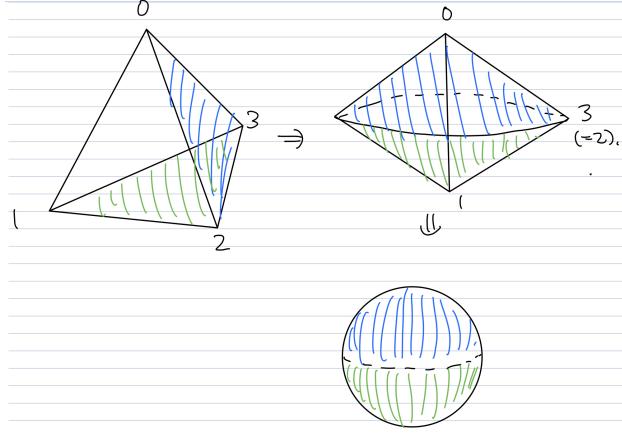


FIGURE 6. Problem 7(S^3)

Therefore,

$$H_3 = \langle T \rangle / 0 = \mathbb{Z}.$$

$$H_2 = 0/0 = 0.$$

$$H_1 = \langle e_1, e_3 \rangle / \langle e_2, e_3 \rangle = 0.$$

$$H_1 = \langle v_1, v_2 \rangle / \langle v_1 - v_2 \rangle = \mathbb{Z}.$$

As shown in Figure 6, it is isomorphic to a 3-ball where the boundary of the northern hemisphere is identified with the boundary of the southern hemisphere by the reflection along the equator. Therefore, this figure is indeed an S^3 .

If I have time, describe this more carefully.

□

Exercise. (Problem 8) Construct a 3 dimensional Δ -complex X from n tetrahedra T_1, \dots, T_n by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure, so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1} , subscripts being taken mod n . Then identify the bottom face of T_i with the top face of T_{i+1} for each i . Show the simplicial homology groups of X in dimensions 0, 1, 2, 3 are $\mathbb{Z}, \mathbb{Z}_n, 0, \mathbb{Z}$, respectively.

Proof. Let T_0, \dots, T_{n-1} denote the n tetrahedra. Let $v_0, v_1, e_0, \dots, e_{n+1}, f_0, \dots, f_{2n-1}$ denote the vertices and edges as in Figure 7. (It has 4 tetrahedra, but they all represent the same T_i . I wrote four of them only because the figure would be too complicated if I denoted all the vertices, edges, faces in one picture.)

Then we have

- $C_3 = \{T_0, \dots, T_{n-1}\}$.
- $C_2 = \{f_0, \dots, f_{2n-1}\}$.
- $C_1 = \{e_0, \dots, e_{n+1}\}$.
- $C_0 = \{v_0, v_1\}$.

Now we will examine ∂ .

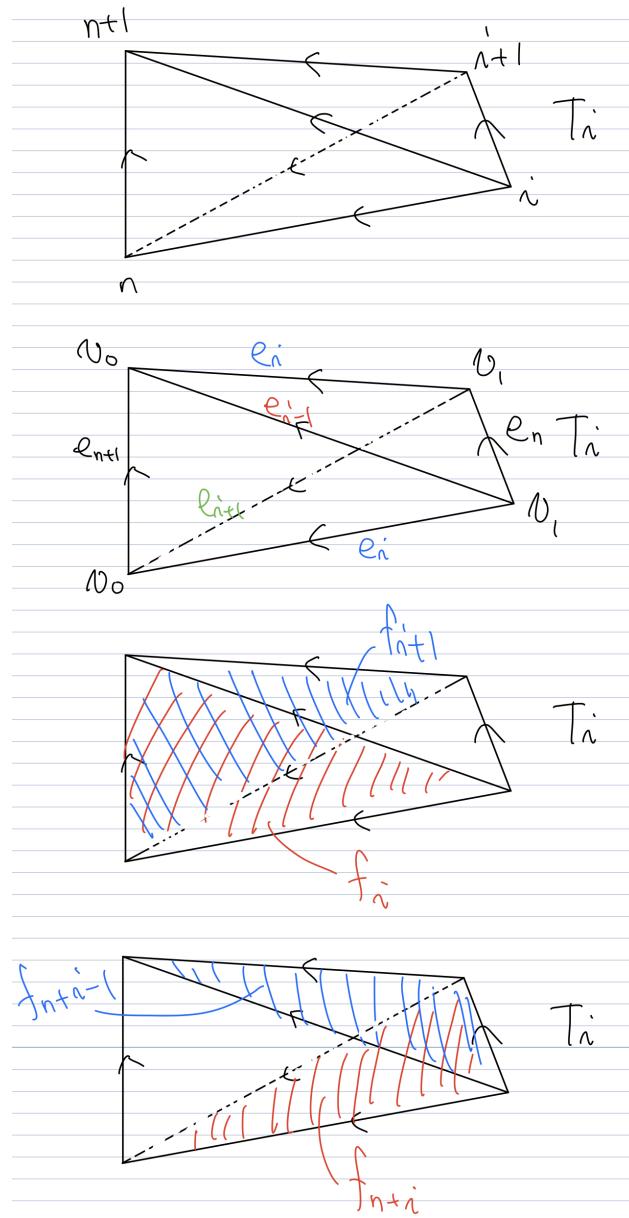


FIGURE 7. Problem 8

$$\begin{aligned}\partial_3(T_i) &= [i+1, n, n+1] - [i, n, n+1] + [i, i+1, n+1] - [i, i+1, n] \\ &= f_{i+1} - f_i + f_{n+i-1} - f_{n+i}.\end{aligned}$$

$$\partial_2(f_i) = [n, n+1] - [i, n+1] + [i, n] = e_{n+1} - e_{i-1} + e_i.$$

$$\partial_2(f_{n+i}) = [i+1, n] - [i, n] + [i, i+1] = e_{i+1} - e_i + e_n.$$

$$\partial_1(e_i) = \begin{cases} v_0 - v_1 & (0 \leq i \leq n-1) \\ 0 & (i = n, n+1). \end{cases}$$

Therefore,

$$\begin{aligned}\partial_3(\sum a_i T_i) &= \sum(a_{i-1} - a_i)f_i + \sum(a_{i+1} - a_i)f_{n+i}. \\ \partial_2(\sum a_i f_i + \sum b_i f_{n+i}) &= e_{n+1} \sum a_i + e_n \sum b_i + \sum(a_i - a_{i+1} + b_{i-1} - b_i)e_i. \\ \partial_1(\sum a_i e_i) &= (\sum_{i=0}^{n-1} a_i)v_0 - (\sum_{i=0}^{n-1} a_i)v_1.\end{aligned}$$

Hence,

- $H_3 = \langle T_0 + \cdots + T_{n-1} \rangle / 0 = \mathbb{Z}$.
- Let $\sum a_i f_i + \sum b_i f_{n+i} \in \ker(\partial_2)$. Then $\sum a_i = \sum b_i = 0$, and $b_{i-1} + a_i = b_i + a_{i+1}$ for each i . By induction, $b_{i-1} + a_i = b_{i+j} + a_{i+j}$ for each i, j . $\sum a_i = \sum b_i$ implies that $\sum(a_i + b_{i-1}) = 0$. Thus for each i , $n(a_i + b_{i-1}) = 0$. This implies that $a_i + b_{i-1} = 0$ for each i . In other words, $b_i = -a_{i+1}$ for each i . Moreover, if $\sum a_i = \sum b_i = 0$ and $b_i = -a_{i+1}$ for each i , then $\partial(\sum a_i f_i + \sum b_i f_{n+i}) = 0$. Therefore, $\ker(\partial_2) = \{\sum a_i f_i + \sum -a_i + 1 f_{n+i} \mid \sum a_i = 0\}$.

On the other hand, $\text{Im}(\partial_3) = \{\sum(a_{i-1} - a_i)f_i + \sum(a_{i+1} - a_i)f_{n+i}\}$. Let $A_i = a_{i-1} - a_i$ for each i . Then $\text{Im}(\partial_3) = \{\sum A_i f_i + \sum(-A_{i+1})f_{n+i} \mid \sum A_i = 0\}$.

It is clear that $\ker(\partial_2) = \text{Im}(\partial_3)$. Thus $H_2 = 0$.

H₁

- $H_0 = \langle v_0, v_1 \rangle / \langle v_0 - v_1 \rangle = \mathbb{Z}$.

□