MATH 633 MIDTERM

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1. Goursat, Cauchy on the disc, and the proofs in Section 5 of Chapter 3.

Proposition 1.1 (Goursat's Theorem). *If Ω is an open set in $\mathbb{C}$, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω, then*

$$\int_T f(z)dz = 0$$

whenever f is holomorphic in Ω.

Proof.

- Let $T^0 = T$. Having created T^i, create 4 triangles from T^i as shown in the textbook with the natural orientation. Then one of the 4 triangles, denoted by T^{i+1}, must satisfy $\left|\int_{T^i} f(z)dz\right| \leq 4\left|\int_{T^{i+1}} f(z)dz\right|$. Since $\{T_i\}$ is a sequence of nonempty compact sets whose diameter diminishes, there must exist a unique point z_0 that belongs to all T^i.
- Since f is holomorphic at z_0, $f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$ where $\psi(z) \to 0$ as $z \to z_0$.
- Since $f(z_0) + f'(z_0)(z - z_0)$ has a primitive, $\int_{T^n} f(z)dz = \int_{T^n} \psi(z)(z - z_0)dz$ for any n. $\left|\int_{T^n} \psi(z)(z - z_0)dz\right| \leq \epsilon_n dp/4^n$ where $\epsilon_n = \sup_{z \in T^n} |\psi(z)|$, d the diameter of T, and p the perimeter of T. $\epsilon_n \to 0$ as $n \to \infty$, so $\left|\int_T f(z)dz\right| \leq \epsilon_n dp = 0$ as $n \to 0$.

□

Proposition 1.2 (Cauchy's Theorem for a Disk). *Suppose f is holomorphic in an open set containing the circle C and its interior. Then*

$$\int_C f(z)dz = 0.$$

Proof. Since f has a primitive, the integral over a closed curve is 0.

Do I need more than this?

□