# MATH 620 HOMEWORK DUE 9/5

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**Exercise 0.1.** Prove that $\delta : V \times \cdots \times V \to \mathbb{F}$ is independent of choice of basis $\{e_i\} \subset V$ up to non-zero scalar.

*Proof.* Let $\{e_i\}, \{f_i\}$ be two bases of $V$. Let $v_1, \cdots, v_n \in V$ be given. We must show if $\delta(v_1, \cdots, v_n) = 0$ with both of the bases, or nonzero with both of the bases. Suppose that $\delta(v_1, \cdots, v_n) \neq 0$ with one of the bases, and it is 0 with the other basis. Without loss of generality, we assume that $\{e_i\}$ gives a nonzero value. Let $n \times n$ matrices $(v_j^i), (w_j^i)$ be given such that

$$\begin{aligned}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} &= \begin{bmatrix} v_1^1 & \cdots & v_1^n \\ \vdots & \ddots & \vdots \\ v_1^n & \cdots & v_n^n \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} \\ &= \begin{bmatrix} w_1^1 & \cdots & w_1^n \\ \vdots & \ddots & \vdots \\ w_1^n & \cdots & w_n^n \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}.\end{aligned}$$

Since $\delta(v_1, \cdots, v_n) \neq 0$ with $\{e_i\}$, $\det(v_i^j) \neq 0$. Therefore, the matrix $(v_i^j)$ is invertible.

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} v_1^1 & \cdots & v_1^n \\ \vdots & \ddots & \vdots \\ v_1^n & \cdots & v_n^n \end{bmatrix}^{-1} \begin{bmatrix} w_1^1 & \cdots & w_1^n \\ \vdots & \ddots & \vdots \\ w_1^n & \cdots & w_n^n \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}.$$

Let $A$ denote the product of the two matrices. Then $\det(A) = \det((v_i^j)^{-1}(w_i^j)) = \det(v_i^j)^{-1} \det(w_i^j) = 0$. This implies that the row space of $A$ has a dimension less than $n$. Therefore, $\{e_1, \cdots, e_n\}$ cannot span $V$ whose dimension is $n$.

This is a contradiction, so $\delta$ is independent of choice of basis up to nonzero scaling. □

**Exercise 0.2.** Show that $\{e^{i_1} \otimes \cdots \otimes e^{i_k} \mid 1 \leq i_1, \cdots, i_k \leq n\}$ is a basis of $T^k(V^*)$. Find $\dim T^k(V^*)$.

*Proof.*

- Linearly independent? Suppose $\sum c_{i_1,\cdots,i_k} e^{i_1}\otimes\cdots\otimes e^{i_k} = 0$. Let $1 \le j_1, \cdots, j_k \le n$ be given.

$$
\begin{aligned}
&(\sum_{i_1,\cdots,i_k} c_{i_1,\cdots,i_k} e^{i_1}\otimes\cdots\otimes e^{i_k})(e_{j_1},\cdots,e_{j_k}) = 0\\
&\implies \sum_{i_1,\cdots,i_k} c_{i_1,\cdots,i_k} (e^{i_1}\otimes\cdots\otimes e^{i_k})(e_{j_1},\cdots,e_{j_k}) = 0\\
&\implies \sum_{i_1,\cdots,i_k} c_{i_1,\cdots,i_k} e^{i_1}(e_{j_1})\cdots e^{i_k}(e_{j_k}) = 0\\
&\implies c_{j_1,\cdots,j_k} e^{j_1}(e_{j_1})\cdots e^{j_k}(e_{j_k}) = 0\\
&\implies c_{j_1,\cdots,j_k} = 0.
\end{aligned}
$$

  Therefore, each $c_{i_1,\cdots,i_k} = 0$.
- Span? Let $f \in T^k(V^*)$. We claim that $f = \sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k}) e^{i_1}\otimes\cdots\otimes e^{i_k}$. Let $v_1,\cdots,v_k \in V$ be given. Since $\{e_1,\cdots,e_n\}$ is a

basis of $V$, so each $v_i$ can be represented as $v_i = \sum_j c_i^j e_j$.

$$
\begin{aligned}
&(\sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})e^{i_1}\otimes\cdots\otimes e^{i_k})(v_1,\cdots,v_k)\\
&=(\sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})e^{i_1}\otimes\cdots\otimes e^{i_k})(c_1^j e_j,\cdots,c_k^j e_j)\\
&=\sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})[(e^{i_1}\otimes\cdots\otimes e^{i_k})(c_1^j e_j,\cdots,c_k^j e_j)]\\
&=\sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})[(c_1^j e^{i_1}(e_j))\cdots(c_k^j e^{i_k}(e_j))]\\
&=\sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})[(c_1^{i_1} e^{i_1}(e_{i_1}))\cdots(c_k^{i_k} e^{i_k}(e_{i_k}))]\\
&=\sum_{i_1,\cdots,i_k} f(e_{i_1},\cdots,e_{i_k})c^{i_1}\cdots c^{i_k}\\
&=\sum_{i_1,\cdots,i_k} f(c^{i_1}e_{i_1},\cdots,c^{i_k}e_{i_k})\\
&=\sum_{i_1,\cdots,i_{k-1}} (\sum_{i_k} f(c^{i_1}e_{i_1},\cdots,c^{i_k}e_{i_k}))\\
&=\sum_{i_1,\cdots,i_{k-1}} f(c^{i_1}e_{i_1},\cdots,c^{i_{k-1}}e_{i_{k-1}},\sum_{i_k} c^{i_k}e_{i_k}))\\
&=\sum_{i_1,\cdots,i_{k-1}} f(c^{i_1}e_{i_1},\cdots,c^{i_{k-1}}e_{i_{k-1}},v_k)\\
&\vdots\\
&=f(v_1,\cdots,v_k).
\end{aligned}
$$

The dimension is $n^k$ because each $i_j$ can be any integer between 1 and $n$. □

**Exercise 0.3.** Let $w \in \wedge^2 V^*$.

- Show that there exists a basis $\{e_1,\cdots,e_n\}$ of $V$ with a dual basis $\{e^1,\cdots,e^n\}$ of $V^*$ such that $w = e^1\wedge e^2 + \cdots + e^{2m-1}\wedge e^{2m}$ for some $m \le n/2$.
- $w^l = w\wedge\cdots\wedge w \neq 0$ if and only if $l \le m$.

*Proof.* Let $V_1 = V$. We will pick vectors inductively.

Suppose that we have $V_i$ for some $i \in \mathbb{N}$. If $\forall v, v' \in V_i, w(v,v') = 0$, then we are done. Suppose otherwise. Then there must exist $v, v' \in V_i$ such that $w(v,v') = 1$. Let $e_{2i-1} = v, e_{2i} = v'$. Let $V_{i+1} = \{v \in V \mid w(v,e_{2i-1}) = w(v,e_{2i}) = 0\}$. We will repeat this process with the $V_{i+1}$.

For each $i$, we claim that $\{e_1, \cdots, e_{2i}\}$ is linearly independent. (To-Do)

Since $V$ is an $n$-dimensional vector space, this process will terminate. If not, it would imply the existence of a linearly independent set with more than $n$ vectors. Since the set of all the vectors we found is linearly independent, it can be extended to form a basis of $V$.

Let $\{e_1, \cdots, e_n\}$ be a basis that we obtain by extending the linearly independent set of vectors we found. Let $m$ be chosen such that $2m$ is the number of vectors we found. Let $\{e^1, \cdots e^n\}$ denote the dual basis of $\{e_1, \cdots, e_n\}$. By Proposition 4.1., we know the existence of such a basis and that the dimension of such a basis is $n$. We claim that $w = e^1 \wedge e^2 + \cdots + e^{2m-1} \wedge e^{2m}$.

Because $w$ and $e^1 \wedge e^2 + \cdots + e^{2m-1} \wedge e^{2m}$ are bilinear, it suffices to identify what $(e_i, e_j)$ gets mapped to for each $i, j$. Let $i, j \in \{1, \cdots, n\}$ be given.

- Case 1: The pair $(i, j)$ equals $(2l-1, 2l)$ for some $l \in \{1, \cdots, m\}$. Then $w(e_{2l-1}, e_{2l}) = 1$ because that is how we found $e_{2l-1}, e_2l$. On the other hand,

$$
\begin{aligned}
&(e^1 \wedge e^2 + \cdots + e^{2m-1} \wedge e^{2m})(e_{2l-1}, e_{2l}) \\
&= (e^1 \wedge e^2)(e_{2l-1}, e_{2l}) + \cdots + (e^{2m-1} \wedge e^{2m})(e_{2l-1}, e_{2l}) \\
&= 1.
\end{aligned}
$$

- Case 2: The pair $(i, j)$ equals $(2l, 2l-1)$ for some $l \in \{1, \cdots, m\}$. Since $w$ and $e^1 \wedge e^2 + \cdots + e^{2m-1} \wedge e^{2m}$ are both alternating, $w(e_i, e_j) = -w(e_j, e_i)$ and $(e^1 \wedge e^2 + \cdots + e^{2m-1} \wedge e^{2m})(e_i, e_j) = -(e^1 \wedge e^2 + \cdots + e^{2m-1} \wedge e^{2m})(e_j, e_i)$. Then, by Case 1, they both result in $-1$.
- Case 3: Any other cases.

Therefore, $w = e^1 \wedge e^2 + \cdots + e^{2m-1} \wedge e^{2m}$. □

**Exercise 0.4.** $\omega \wedge \tau = (-1)^{kl} \tau \wedge \omega$

*Proof.*

$$\begin{aligned}
\omega \wedge \tau &= (e^{i_1} \wedge \cdots \wedge e^{i_k}) \wedge (e^{j_1} \wedge \cdots \wedge e^{j_k}) \\
&= e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge e^{j_1} \wedge \cdots \wedge e^{j_k} \\
&= (-1) e^{i_1} \wedge \cdots \wedge e_{i_{k-1}} \wedge e^{j_1} \wedge e^{i_k} \wedge e_{j_1} \wedge \cdots \wedge e^{i_k} \\
&\vdots \\
&= (-1)^k e^{j_1} \wedge e^{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_{j_2} \wedge \cdots \wedge e^{i_k} \\
&= (-1)^{2k} e^{j_1} \wedge e_{j_2} \wedge e^{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_{j_3} \wedge \cdots \wedge e^{i_k} \\
&\vdots \\
&= (-1)^{kl} e^{j_1} \wedge \cdots \wedge e_{j_k} \wedge e^{i_1} \wedge \cdots \wedge e_{i_k} \\
&= (-1)^{kl} \tau \wedge \wedge.
\end{aligned}$$

□

**Exercise 0.5.** Prove that $\{\partial_1, \cdots, \partial_n\}$ is a basis of $T_p\mathbb{R}^n$.

*Proof.*

- Linearly independent? Let $c_1, \cdots, c_n \in \mathbb{R}$ be given. Suppose $c_1\partial_1 + \cdots + c_n\partial_n = 0$. Then $\forall i, 0 = (c_1\partial_1 + \cdots + c_n\partial_n)(x^i) = c_i\partial_i(x^i) = c_i$. Therefore, $c_i = 0$ for each $i$.
- Span? Let $\lambda \in T_p\mathbb{R}^n$ be given. We claim that $\lambda = \sum \lambda(x^i)\partial_i$. Let $f \in \mathscr{C}^\infty$. Then $f(x) = f(p) + \sum_i [\frac{\partial f}{\partial x^i}(p)(x^i - p^i) + g^i(x)(x^i -$

$p^i])$ for some smooth functions $g^i$ by Taylor's formula with remainder. For each $i$, $g_i(p) = 0$.

$$\begin{aligned}
\lambda(f) &= \lambda(f(p)) + \lambda(\sum_i [\frac{\partial f}{\partial x^i}(p)(x^i - p^i) + g^i(x)(x^i - p^i)]) \\
&= \lambda(\sum_i [\frac{\partial f}{\partial x^i}(p)(x^i - p^i) + g^i(x)(x^i - p^i)]) \\
&= \sum_i \frac{\partial f}{\partial x^i}(p)\lambda(x^i - p^i) + \sum_i \lambda(g^i(x)(x^i - p^i)) \\
&= \sum_i \frac{\partial f}{\partial x^i}(p)\lambda(x^i - p^i) + \sum_i [\lambda(g^i(x))(p^i - p^i) + \lambda(x^i - p^i)g^i(p)] \\
&= \sum_i \frac{\partial f}{\partial x^i}(p)\lambda(x^i - p^i) + \sum_i \lambda(x^i - p^i)g^i(p) \\
&= \sum_i \frac{\partial f}{\partial x^i}(p)\lambda(x^i - p^i) \\
&= \sum_i \frac{\partial f}{\partial x^i}(p)(\lambda(x^i) - \lambda(p^i)) \\
&= \sum_i \frac{\partial f}{\partial x^i}(p)\lambda(x^i) \\
&= \sum_i \partial_i(f)\lambda(x^i) \\
&= \sum_i \lambda(x^i)\partial_i(f) \\
&= (\sum_i \lambda(x^i)\partial_i)(f)
\end{aligned}$$

□

**Exercise 0.6.** Show that $\{dx^1, \cdots, dx^n\}$ is a basis of $T_p^*\mathbb{R}^n$ that is dual to $\{\frac{\partial}{\partial x^j}\}_{j=1}^n \subset T_p\mathbb{R}^n$.

*Proof.*

- Dual? Let $i, j \in \{1, \cdots, n\}$. $dx^i(\frac{\partial}{\partial x^j}) = \frac{\partial}{\partial x^j}x^i$. The partial derivative of $x^i$ with respect to $x^j$ is 1 if $i = j$ and 0 otherwise. Thus $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$.

- Linearly independent? Let $c_1, \cdots, c_n \in \mathbb{R}$ be given. Suppose that $c_1 dx^1 + \cdots + c_n dx^n = 0$. For any $i \in \{1, \cdots, n\}$,

$$\begin{aligned}(c_1 dx^1 + \cdots + c_n dx^n)(\partial_i) = 0 &\implies c_1(dx^1(\partial_i)) + \cdots + c_n(dx^n(\partial_i)) = 0 \\ &\implies c_1(\partial_i(x^1)) + \cdots + c_n(\partial_i(x^n)) = 0 \\ &\implies c_i \partial_i(x^i) = 0 \\ &\implies c_i = 0.\end{aligned}$$

Therefore, $c_1 = \cdots = c_n = 0$. Therefore, $\{dx^1, \cdots, dx^n\}$ is indeed linearly independent.

- Span? Let $f \in T_p^*\mathbb{R}^n$ be given. We claim that $f = \sum_{i=1}^n f(\partial_i) dx^i$. Let $\sum_{i=1}^n c_i \partial_i \in T_p\mathbb{R}^n$ be given where $c_i$'s are in $\mathbb{R}$. (It makes sense to assume that every element in $T_p\mathbb{R}^n$ is in this form because we showed earlier that $\{\partial_1, \cdots, \partial_n\}$ is a basis of $T_p\mathbb{R}^n$.)

$$\begin{aligned}(\sum_{i=1}^n f(\partial_i) dx^i)(\sum_{j=1}^n c_j \partial_j) &= \sum_{i=1}^n \left[f(\partial_i) dx^i(\sum_{j=1}^n c_j \partial_j)\right] \\ &= \sum_{i=1}^n f(\partial_i)\big[\sum_{j=1}^n c_j dx^i(\partial_j)\big] \\ &= \sum_{i=1}^n f(\partial_i)\big[\sum_{j=1}^n c_j \partial_j(x^i)\big] \\ &= \sum_{i=1}^n f(\partial_i) c_i \\ &= f(\sum_{i=1}^n c_i \partial_i).\end{aligned}$$

$\square$