

# MATH 601 (DUE 10/23)

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## CONTENTS

### 1. Field Extension

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#### 1. FIELD EXTENSION

**Exercise.** (Problem 1) Let  $p$  be a prime number. Let  $K = \mathbb{Z}/p\mathbb{Z}(t)$  be the fraction field of  $\mathbb{Z}/p\mathbb{Z}[t]$ .

- (i) What is the characteristic of  $K$ ?
- (ii) What is the characteristic of any extension field of  $K$ ?
- (iii) Show that the Frobenius endomorphism,  $F : K \rightarrow K$  is not a ring isomorphism.
- (iv) Let  $f(x) = x^p - t \in K[x]$ . Prove that  $f(x)$  is irreducible.
- (v) Prove that  $f(x)$  is not a separable polynomial.
- (vi) Construct an explicit field extension  $K \subset L$  such that  $f(x) \in L[x]$  has a factor of positive degree  $< p$ .
- (vii) With  $f$  and  $L$  above find all the roots of  $f(x)$  in  $L$  and determine their multiplicities.

*Proof.*

- (i) We will prove in general that if  $R \subset S$  are both commutative rings with 1, they have the same characteristic. Let  $i : R \rightarrow S$  be the inclusion map. Let  $\phi : \mathbb{Z} \rightarrow R$  be the unique ring homomorphism.

Then  $i \circ \phi : \mathbb{Z} \rightarrow S$  is a ring homomorphism, and this is the only homomorphism from  $\mathbb{Z}$  to  $S$  by the uniqueness.

$$\begin{aligned} a \in \ker(\phi) &\iff \phi(a) = 0 \\ &\iff i(\phi(a)) = 0 && (i \text{ is injective}) \\ &\iff a \in \ker(i \circ \phi). \end{aligned}$$

Thus  $\ker(\phi) = \ker(i \circ \phi)$ , so  $R$  and  $S$  have the same characteristic.

Therefore,  $\mathbb{Z}/p\mathbb{Z}$  has the same characteristic as  $K$ . The kernel of  $\psi : \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$  is  $(p)$ , so the characteristic of  $K$  is  $p$ .

- (ii) Using the result that we proved in (i), we conclude that the characteristic of any extension field of  $K$  is  $p$ .

(iii) Suppose that it is a ring isomorphism. Let  $a/b \in K$  be chosen such that  $F(a/b) = t$ .

$$\begin{aligned} \left(\frac{a}{b}\right)^p = t &\implies a^p = tb^p \\ &\implies p \deg(a) = \deg(t) + p \deg(b) \\ &\implies p(\deg(a) - \deg(b)) = 1. \end{aligned}$$

However,  $p \geq 2$ , so this is impossible. Therefore,  $F$  is not a ring isomorphism.

- (iv)  $t$  is an irreducible element in  $\mathbb{Z}/p\mathbb{Z}[t]$  because  $t = ab$  implies that the degree of  $a$  or  $b$  must be 0, which implies that one of them is a unit. By Corollary 4 on P.300 (Dummit and Foote),  $\mathbb{Z}/p\mathbb{Z}[t]$  is a principal ideal domain and unique factorization domain. By Proposition 2 on P.284 (Dummit and Foote),  $t$  is a prime element in  $\mathbb{Z}/p\mathbb{Z}[t]$ . By the Eisenstein irreducibility criterion from the Factorization in Integral Domain handout,  $x^p - t$  is irreducible in  $K[x]$  because  $-t \in (t)$  but  $-t \notin (t^2)$ .
- (v)  $f'(x) = px^{p-1} = 0$ . Thus  $f(x) \in \text{GCD}(f(x), f'(x))$  and  $f(x) = x^p - t$  is not a unit. By Lemma 3.2 of the Field Extension handout,  $f(x)$  is not separable. □

**Exercise.** (Problem 2) Let  $F$  be a field of characteristic 0. Let  $f(x) \in F[x]$  be an irreducible polynomial. Then  $f(x)$  is separable.

*Proof.* Let  $f(x) = \sum_{i=0}^n a_i x^i \in F[x]$  be an irreducible polynomial with  $a_n \neq 0$ . Since  $f(x)$  is irreducible,  $f(x)$  is neither a unit nor 0. Since  $F$  is a field, all polynomials of degree 0 are units. Thus  $\deg(f(x)) = n \geq 1$ . It suffices to show that  $\text{GCD}(f(x), f'(x)) = F^*$  by Lemma 3.2. Let  $g(x) \in F[x]$  be given such that  $g(x) \mid f(x), g(x) \mid f'(x)$ . Since  $f(x)$  is irreducible, either  $g(x)$  is a unit or there exists a unit  $u \in F^*$  such that  $g(x) = uf(x)$ . Suppose  $g(x)$  is not a unit. Since  $g(x) \mid f'(x)$ ,  $f'(x) = h(x)g(x) = uh(x)f(x)$  for some  $h(x) \in F[x]$ . Thus  $\deg(f'(x)) = \deg(uh(x)) + \deg(f(x))$ .

- $f'(x) = \sum_{i=1}^n i a_i x^{i-1}, n \geq 1$  and  $a_n \neq 0$ . Since  $F$  is a field of characteristic 0,  $na_n \neq 0$ . Therefore,  $\deg(f'(x)) = n - 1$ .
- $\deg(uh(x)) \geq 0$ .
- $\deg(f(x)) = n$ .

However, this implies that  $n - 1 \geq 0 + n = n$ . This is a contradiction, so  $g(x)$  must be a unit. Therefore,  $\text{GCD}(f(x), f'(x)) = F^*$ . □

**Exercise.** (Problem 3) Let  $F$  be a field. Let  $f(x) \in F[x]$  be an irreducible polynomial which is not separable. Show that  $f'(x) = 0 \in F[x]$ .

*Proof.* Suppose  $f(x)$  is irreducible. Then  $f(x) \neq 0$  and  $f(x)$  is not a unit by definition. Thus  $\deg(f(x)) \geq 1$ .

Since  $f(x)$  is not separable, there exists a non-unit  $g(x) \in F[x]$  such that  $g(x) \mid f(x)$  and  $g(x) \mid f'(x)$  by Lemma 3.2 from the Field Extension handout. Since  $f(x)$  is irreducible and  $g(x)$  is not a unit,  $f(x)$  is the product of  $g(x)$  and a unit. This implies that  $\deg(f(x)) = \deg(g(x))$ .

Since  $g(x) \mid f'(x)$ ,  $f'(x) = h(x)g(x)$ . If  $f'(x) = 0$ , we are done. Suppose otherwise. Then  $\deg(f'(x)) = \deg(h(x)) + \deg(g(x)) = \deg(h(x)) + \deg(f(x)) \geq \deg(f(x))$ . However, by the definition of the  $'$  operator,  $\deg(f'(x)) < \deg(f(x))$ . This is a contradiction, so  $f'(x) = 0$ . □

**Exercise.** (Problem 4) Let  $F$  be a field of prime characteristic  $p$ . Let  $f(x) = \sum_{i=0}^n a_i x^i \in F[x]$  be an irreducible polynomial. Give a necessary and sufficient criterion for  $f(x)$  to be inseparable in terms of the coefficients  $a_i$ .

*Proof.* We claim that  $\forall i, (i \notin p\mathbb{Z} \implies a_i = 0)$  is a necessary and sufficient criterion.

- Suppose  $f(x)$  is inseparable. By Lemma 5.5 from the Field Extension handout,  $f'(x) = 0$ . If  $f'(x) = 0$ , then  $ia_i = 0$  for each  $i$ . Since  $p$  is a prime,  $a_i$  must be 0 if  $i \notin p\mathbb{Z}$ .
- Suppose  $\forall i, (i \notin p\mathbb{Z} \implies a_i = 0)$ . Then  $f'(x) = 0$ , so  $f(x) \mid f(x), f(x) \mid f'(x)$  and  $f(x)$  is not a unit since  $f(x)$  is irreducible. Therefore,  $\text{GCD}(f(x), f'(x)) \neq F^\times$ , so  $f$  is inseparable by Lemma 3.2.

Hence,  $\forall i, (i \notin p\mathbb{Z} \implies a_i = 0)$  is a necessary and sufficient criterion.  $\square$

**Exercise.** (Problem 5) What is the characteristic of the ring  $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$ ?

*Proof.* Then  $\phi$  is injective, so the characteristic is 0.

$1 \mapsto (1, 1, 1)$

$\square$

**Exercise.** (Problem 6) Let  $K$  be a finite field of characteristic  $p$ . Let  $a, b \in K^*$  be two elements which have the same order in this finite group. Show that  $\mathbb{Z}/p[a] = \mathbb{Z}/p[b]$  as subfields of  $K$ .

*Proof.* Can I just say they both have the same number of elements and use the lemma from class?

$\square$