

## MATH 611 HOMEWORK (DUE 10/16)

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**Exercise.** (Problem 18) For a path-connected, locally path-connected, and semilocally simply-connected space  $X$ , call a path-connected covering space  $\tilde{X} \rightarrow X$  abelian if it is normal and has abelian deck transformation group. Show that  $X$  has an abelian covering space that is a covering space of every other abelian covering space of  $X$ , and that such a 'universal' abelian covering space is unique up to isomorphism. Describe this covering space explicitly for  $X = S^1 \vee S^1$  and  $X = S^1 \vee S^1 \vee S^1$ .

*Proof.*

- What is the hypothesis?
  - $X$  is a path-connected, locally path-connected, semilocally simply-connected space.
- What is the conclusion?
  - There exists a normal covering space of  $X$   $p : \tilde{X} \rightarrow X$  such that  $G(\tilde{X})$  is abelian.
  - $X$  has an abelian covering space that is a covering space of every other abelian covering space of  $X$ .
  - A universal abelian covering space is unique up to isomorphism.
  - Find the universal covering space of  $S^1 \vee S^1$  and  $S^1 \vee S^1 \vee S^1$ .
- Introduce suitable notations.
  - $H = p_*(\pi_1(\tilde{X}, x_0))$ .
- Separate the various parts of the hypothesis.
- Find the connection between the hypothesis and the conclusion.
  - “ $X$  is a path-connected, locally path-connected, semilocally simply-connected space.” This condition sounds a lot like Theorem 1.38 on P.67. By using theorem 1.38, we can associate some group to each covering map.
  - “ $\tilde{X}$  is a normal covering space of  $X$ .” By Proposition 1.39 on P.71,  $\tilde{X}$  is normal if and only if  $H$  is a normal subgroup of  $\pi_1(X, x_0)$ .
  - $G(\tilde{X})$  is abelian. By Proposition 1.39 on P.71,  $G(\tilde{X})$  is isomorphic to the quotient  $\pi_1(X, x_0)/H$  because  $\tilde{X}$  is normal. Thus  $\pi_1(X, x_0)/H$  is abelian.
- Have you seen it before?
  - This might be similar to constructing the universal covering space.
- Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.
  - Showing uniqueness up to isomorphism sounds like the universal covering space theorem.
- Keep only a part of the hypothesis, drop the other part; is the conclusion still valid?
- Could you derive something useful from the hypothesis?
- Could you think of another hypothesis from which you could easily derive the conclusion?

- Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?
- Did you use the whole hypothesis?

