

# MATH 611 PROBLEM SET 1 (DUE 9/4)

HIDENORI SHINOHARA

**Exercise 0.1.** (Exercise 4, Chapter 0) A deformation retraction in the weak sense of a space  $X$  to a subspace  $A$  is a homotopy  $f_t : X \rightarrow X$  such that  $f_0 = \text{Id}$ ,  $f_1(X) \subset A$ , and  $f_t(A) \subset A$  for all  $t$ . Show that if  $X$  deformation retracts to  $A$  in this weak sense, then the inclusion  $A \rightarrow X$  is a homotopy equivalence.

*Proof.* Let  $i : A \rightarrow X$  denote the inclusion. Let  $F : X \times I \rightarrow X$  denote the associated map  $(x, t) \rightarrow f_t(x)$ . Then  $F$  is a continuous function by the definition of a homotopy.

Let  $f : X \rightarrow A$  be defined by  $f(x) = F(x, 1) = f_1(x)$ . This definition makes sense because  $f_1(X) \subset A$ . We claim that  $f_1 \circ i \simeq \text{Id}_A$  and  $i \circ f_1 \simeq \text{Id}_X$ .

Consider  $G : A \times I \rightarrow A$  such that  $G(a, t) = F(a, t)$  for all  $(a, t) \in A \times I$ . This definition makes sense because  $f_t(A) \subset A$  for all  $t$ .

Then  $G$  is a homotopy in  $A$  between  $f \circ i$  and  $\text{Id}_A$  because:

- $G$  is a restriction of  $F$ , so  $G$  is continuous.
- $\forall a \in A, G(a, 0) = F(a, 0) = f_0(a) = \text{Id}_X(a) = \text{Id}_A(a)$ .
- $\forall a \in A, G(a, 1) = F(a, 1) = f(a) = f(i(a)) = (f \circ i)(a)$ .

Therefore,  $f \circ i \simeq \text{Id}_A$ .

$F$  is a homotopy between  $f_0$  and  $f_1$ .

- We are given that  $f_0 = \text{Id}_X$ .
- For any  $x \in X$ ,  $(i \circ f)(x) = i(f(x)) = f(x) = f_1(x)$ , so  $i \circ f = f_1$ .

Therefore,  $F$  is a homotopy between  $\text{Id}_X$  and  $i \circ F$ , so  $i \circ f \simeq \text{Id}_X$ .

In conclusion,  $i$  is indeed a homotopy equivalence.  $\square$

**Exercise 0.2.** (Exercise 5, Chapter 0) Show that if a space  $X$  deformation retracts to a point  $x \in X$ , then for each neighborhood  $U$  of  $x$  in  $X$  there exists a neighborhood  $V \subset U$  of  $x$  such that the inclusion map  $V \rightarrow U$  is nullhomotopic.

*Proof.* Let  $p \in X$  be a point to which  $X$  deformation retracts. Since  $X$  deformation retracts to  $p$ , there exists a map  $F : X \times I \rightarrow X$  such that

- (1)  $\forall x \in X, F(x, 0) = x$ .
- (2)  $\forall x \in X, F(x, 1) = p$ .

- (3)  $\forall t \in I, F(p, t) = p$ .
- (4)  $F$  is continuous.

Let  $U$  be a neighborhood of  $p$ . Then  $F^{-1}(U)$  is an open subset of the product space  $X \times I$ . By the 3rd property of  $F$  mentioned above, the slice  $\{p\} \times I$  is a subset of  $F^{-1}(U)$ . Since  $I$  is compact, there must be an open subset  $V$  of  $X$  such that  $\{p\} \times I \subset V \times I \subset F^{-1}(U)$  by the tube lemma.

We claim that this  $V$  is a desired subset.

- $V$  is an open subset of  $X$ .
- Since  $\{p\} \times I \subset V \times I$ ,  $p \in V$ .
- Since  $V \times I \subset F^{-1}(U)$ ,  $F(V \times I) \subset U$ . This implies that  $\forall v \in V$ ,  $F(v, 0) = v \in U$ . Therefore,  $V \subset U$ .
- We claim that the inclusion map  $i : V \rightarrow U$  is nullhomotopic. Let  $e_p : V \rightarrow U$  denote the constant map at  $p$ ,  $G : V \times I \rightarrow U$  be defined by  $G(x, t) = F(x, t)$  for all  $x \in V, t \in I$ .
  - $G$  indeed maps  $V \times I$  into  $U$  because  $F(V \times I) \subset U$ . Therefore,  $G$  is well-defined.
  - Since  $G$  is the restriction of  $F$  to  $V \times I$  and  $F$  is continuous,  $G$  is continuous.
  - $\forall x \in V, G(x, 0) = F(x, 0) = x = i(x)$ .
  - $\forall x \in V, G(x, 1) = F(x, 1) = p = e_p(x)$ .

Thus  $i$  is indeed nullhomotopic.

□

**Lemma 0.3.** *The neighborhood  $V$  that we find in Problem 5 is connected.*

*Proof.* Suppose otherwise. Let  $A, B$  denote a separation of  $V$ . Without loss of generality, we assume  $p \in A$ . Let  $q \in B$ . ( $B$  must be nonempty since  $A, B$  are a separation.)

Let  $F$  be the homotopy we defined in the solution for Problem 5 from the inclusion map to the constant map at  $p$ . Let  $f : I \rightarrow V$  be defined such that  $f(t) = F(q, t)$ . Then  $f$  is a path from  $f(0) = F(q, 0) = q$  to  $f(1) = F(q, 1) = p$  in  $V$ . Since  $f$  is continuous,  $f^{-1}(A)$  and  $f^{-1}(B)$  are open in  $I$ . Moreover,  $I = f^{-1}(V) = f^{-1}(A) \cup f^{-1}(B)$  and  $\emptyset = f^{-1}(\emptyset) = f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ . Since  $1 \in f^{-1}(p) \subset f^{-1}(A)$  and  $0 \in f^{-1}(q) \subset f^{-1}(B)$ ,  $f^{-1}(A)$  and  $f^{-1}(B)$  are nonempty. Therefore,  $f^{-1}(A)$  and  $f^{-1}(B)$  form a separation of  $I$ . However, this is impossible because  $I$  is connected. □

**Exercise 0.4.** (Exercise 6(a), Chapter 0) Let  $X$  be the subspace of  $\mathbb{R}^2$  consisting of the horizontal segment  $[0, 1] \times \{0\}$  together with all the vertical segments  $\{r\} \times [0, 1 - r]$  for  $r$  a rational number in  $[0, 1]$ . Show

that  $X$  deformation retracts to any point in the segment  $[0, 1] \times \{0\}$ , but not to any other point.

*Proof.* Let  $(a, 0) \in [0, 1] \times \{0\}$  be given. Let  $F : X \times I \rightarrow X$  be defined such that

$$F((x, y), t) = \begin{cases} (x, (1 - 2t)y) & (0 \leq t \leq 1/2) \\ (x + (a - x)(2t - 1), 0) & (1/2 \leq t \leq 1). \end{cases}$$

$F$  is well defined because when  $t = 1/2$ :

- $(x, (1 - 2t)y) = (x, 0)$ .
- $(x + (a - x)(2t - 1), 0) = (x, 0)$ .

Moreover, by the pasting lemma,  $F$  is continuous.

Then  $F$  is a deformation retract of  $X$  onto  $(a, 0)$  because

$$\bullet \quad F((a, 0), t) = \begin{cases} (a, 0(1 - 2t)) = (a, 0) & (t \in [0, 1/2]) \\ (a + (a - a)(2t - 1), 0) = (a, 0) & (t \in [1/2, 1]). \end{cases}$$

Therefore,  $F((a, 0), t) = (a, 0)$  for any  $t \in I$ .

- $F((x, y), 0) = (x, y)$  for any  $(x, y) \in x$ .
- $F((x, y), 1) = (a, 0)$  for any  $(x, y) \in x$ .

Therefore,  $F$  is indeed a deformation retract of  $X$  onto  $(a, 0)$ .

Suppose that there exists a point  $(a, b) \in X$  to which  $X$  deformation retracts onto such that  $b \neq 0$ . Let  $G : X \times I \rightarrow X$  denote such a deformation retract. Consider the open subset  $U = B((a, b), b) \cap X$ . Note that  $U$  is disjoint from the segment  $[0, 1] \times \{0\}$ . Then  $U$  is a neighborhood of  $(a, b)$ , a point to which  $X$  deformation retracts onto. By Problem 5 (Chapter 0), there must exist a neighborhood  $V \subset U$  of  $x$  such that the inclusion map  $V \rightarrow U$  is nullhomotopic. By the Lemma we showed above,  $V$  must be connected. Since  $V$  is an open subset of  $X$ , there must exist an  $r > 0$  such that  $B((a, b), r) \cap X \subset V$ . Let  $c$  be an irrational number in  $(a, a + r)$ . Then  $V \cap ((-\infty, c) \times \mathbb{R})$  and  $V \cap ((c, \infty) \times \mathbb{R})$  form a separation of  $V$ . This is a contradiction, so our initial assumption that  $X$  deformation retracts onto  $(a, b)$  was wrong. Therefore,  $X$  deformation retracts to any point in the segment  $[0, 1] \times \{0\}$ , but not to any other point.  $\square$

**Exercise 0.5.** (Exercise 9, Chapter 0) Show that a retract of a contractible space is contractible.

*Proof.* Let  $X$  be a contractible space. Then  $\text{Id}_X$  is homotopic to a constant map. This implies the existence of a fixed point  $p \in X$  and a continuous function  $F : X \times I \rightarrow X$  such that

- $\forall x \in X, F(x, 0) = x,$
- $\forall x \in X, F(x, 1) = p.$

Let  $A \subset X$  be a retract of  $X$ , and let  $r : X \rightarrow A$  denote a retraction. In other words,  $r(X) = A$  and  $r|_A = \text{Id}_A$ .

Let  $G : A \times I \rightarrow A$  be the restriction of  $r \circ F$  to  $A \times I$ . This makes sense because  $F$  maps  $A \times I$  into  $X$ , and  $r$  maps  $X$  into  $A$ . We claim that  $G$  is a homotopy between  $\text{Id}_A$  and the constant map  $e_{r(p)}$  such that  $e_{r(p)}(a) = r(p)$  for all  $a \in A$ .

- $r \circ F$  is continuous since it is a composition of continuous functions.  $G$  is a restriction of a continuous function, so  $G$  is continuous.
- $G(a, 0) = r(F(a, 0)) = r(a) = a = \text{Id}_A(a).$
- $G(a, 1) = r(F(a, 1)) = r(p) = e_{r(p)}(a).$

Therefore,  $G$  is indeed a homotopy between  $\text{Id}_A$  and the constant map at  $r(p)$ . Since the identity map is homotopic to a constant map,  $A$  is contractible.  $\square$

**Exercise 0.6.** (Exercise 13, Chapter 0) Show that any two deformation retractions  $r_t^0$  and  $r_t^1$  of a space  $X$  onto a subspace  $A$  can be joined by a continuous family of deformation retractions  $r_t^s$ ,  $0 \leq s \leq 1$ , of  $X$  onto  $A$ , where continuity means that the map  $X \times I \times I \rightarrow X$  sending  $(x, s, t)$  to  $r_t^s(x)$  is continuous.

*Proof.* Let  $F : X \times I \times I \rightarrow X$  be defined such that

$$F(x, t, s) = \begin{cases} r_{t(1-2s)}^0(x) & (s \in [0, 1/2]) \\ r_{t(2s-1)}^1(x) & (s \in [1/2, 1]). \end{cases}$$

We claim that  $F$  is well-defined and satisfies the desired properties.

- Let  $s = 1/2$ .  $r_{t(1-2s)}^0(x) = r_0^0(x) = x$  because  $r_t^0$  is a deformation retraction. Similarly,  $r_{t(2s-1)}^1(x) = r_0^1(x) = x$  because  $r_t^0$  is a deformation retraction. Therefore,  $F$  is well defined when  $s = 1/2$ . Moreover, by the pasting lemma,  $F$  is continuous. This is because the intersection  $X \times I \times [0, 1/2] \cap X \times I \times [1/2, 1] = X \times I \times \{1/2\}$  is closed.
- $F(x, t, 0) = r_t^0(x)$  for any  $x \times t \in X \times I$ .
- $F(x, t, 1) = r_t^1(x)$  for any  $x \times t \in X \times I$ .

Therefore,  $F$  maps  $X \times I \times I \rightarrow X$  continuously sending  $(x, s, t)$  to  $r_t^s(x)$ .  $\square$

**Exercise 0.7.** (Exercise 7, Chapter 1.1) Define  $f : S^1 \times I \rightarrow S^1 \times I$  by  $f(\theta, s) = (\theta + 2\pi s, s)$ , so  $f$  restricts to the identity on the two

boundary circles of  $S^1 \times I$ . Show that  $f$  is homotopic to the identity by a homotopy  $f_t$  that is stationary on one of the boundary circles, but not by any homotopy  $f_t$  that is stationary on both boundary circles.

*Proof.* Define  $F : (S^1 \times I) \times I \rightarrow S^1 \times I$  such that  $F((\theta, s), t) = t(\theta, s) + (1-t)f(\theta, s)$ . Then  $F$  is a homotopy between  $f$  and the identity map that is stationary on  $S^1 \times \{0\}$ . This is because  $F((\theta, 0), t) = t(\theta, 0) + (1-t)f(\theta, 0) = (t\theta, 0) + ((1-t)\theta, 0) = (\theta, 0)$  for any  $(\theta, t) \in S^1 \times I$ .

Suppose that there exists a homotopy  $G : (S^1 \times I) \times I \rightarrow S^1 \times I$  between  $f$  and the identity map that is stationary on both boundary circles. Let  $H : I \times I \rightarrow S^1$  be defined such that  $H(s, t) = \pi_1(F((0, t), s))$  where  $\pi_1$  denotes the projection of the first coordinate.

- $H(s, 0) = \pi_1(G((0, 0), s)) = \pi_1(0, 0) = 0$  because  $G$  is stationary on the circle  $S^1 \times \{0\}$ .
- $H(s, 1) = \pi_1(G((0, 1), s)) = \pi_1(0, 1) = 0$  because  $G$  is stationary on the circle  $S^1 \times \{1\}$ .
- $H(0, t) = \pi_1(G((0, t), 0)) = \pi_1(f(0, t)) = \pi_1(2\pi t, t) = 2\pi t$ .
- $H(1, t) = \pi_1(G((0, t), 1)) = \pi_1(0, t) = 0$ .

Then  $t \mapsto H(0, t)$  corresponds to the  $\omega$  in Theorem 1.7, and  $t \mapsto H(1, t)$  corresponds to a constant map. In other words,  $H$  is a homotopy between  $\omega$  and a constant map in  $S^1$ . However, this is a contradiction because Theorem 1.7 states that  $\pi_1(S^1)$  is the infinite cyclic group generated by  $\omega$ . Therefore, such a homotopy  $G$  does not exist.  $\square$

**Exercise 0.8.** (Exercise 16, Chapter 1.1) Show that there are no retractions  $r : X \rightarrow A$  in the following cases:

- $X = \mathbb{R}^3$  with  $A$  any subspace homeomorphic to  $S^1$ .
- $X = S^1 \times D^2$  with  $A$  its boundary torus  $S^1 \times S^1$ .

*Proof.*

- Suppose that there exists a retract  $r : X \rightarrow A$ . In other words,  $r$  is a continuous map such that  $r|_A = \text{Id}_A$  and  $r(X) \subset A$ . Let  $\phi : S^1 \rightarrow A$  be a homeomorphism. Let  $\omega$  be the loop defined in Theorem 1.7. Then  $\pi_1(S^1, (1, 0))$  is the infinite cyclic group generated by  $[\omega]$ . Consider the following two paths:

- $\phi \circ \omega : I \rightarrow A$ .
- $e_0 : I \rightarrow A$  such that  $e_0(t) = (\phi \circ \omega)(0) = \phi(1, 0)$ .

They are paths in  $A$ , and  $A$  is a subset of  $\mathbb{R}^3$ , so they are paths in  $\mathbb{R}^3$ . Since  $\mathbb{R}^3$  is convex, we can define a linear homotopy between them. Let  $F : I \times I \rightarrow \mathbb{R}^3$  such that  $F(s, t) = t(\phi \circ \omega)(s) + (1-t)e_0(s)$ . Therefore, the two paths are homotopic in  $\mathbb{R}^3$ .

We will consider  $\phi^{-1} \circ r \circ F$  that maps  $I \times I \rightarrow S$ . Since it is a composition of continuous functions, it is continuous.

$$\begin{aligned}
(\phi^{-1} \circ r \circ F)(s, 0) &= \phi^{-1}(r(F(s, 0))) \\
&= \phi^{-1}(r(e_0(s))) \\
&= \phi^{-1}(e_0(s)) && (\text{since } e_0(s) \in A) \\
&= (1, 0). \\
(\phi^{-1} \circ r \circ F)(s, 1) &= \phi^{-1}(r(F(s, 1))) \\
&= \phi^{-1}(r((\phi \circ \omega)(s))) \\
&= \phi^{-1}(r(\phi(\omega(s)))) \\
&= \phi^{-1}(\phi(\omega(s))) && (\text{since } \phi(\omega(s)) \in A) \\
&= \omega(s). \\
(\phi^{-1} \circ r \circ F)(0, t) &= \phi^{-1}(r(\phi(1, 0))) \\
&= \phi^{-1}(\phi(1, 0)) \\
&= (1, 0). \\
(\phi^{-1} \circ r \circ F)(1, t) &= \phi^{-1}(r(\phi(1, 0))) \\
&= \phi^{-1}(\phi(1, 0)) \\
&= (1, 0).
\end{aligned}$$

Therefore,  $\phi^{-1} \circ r \circ F$  is a path homotopy between  $\omega$  and the constant loop at  $(1, 0)$ . This implies that  $[w]$  is the identity element in  $\pi_1(S^1)$ . However, this is a contradiction because Theorem 1.7 states that  $\pi_1(S^1)$  is the infinite cyclic group generated by  $[w]$ .

Hence, there exists no such retraction  $r$ .

- Suppose  $X$  retracts onto  $A$ . By Proposition 1.17, the homomorphism  $i_* : \pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$  induced by the inclusion  $i : A \rightarrow X$  is injective. By Theorem 1.7,  $\pi_1(S^1)$  is isomorphic to  $\mathbb{Z}$ .  $\pi_1(D^2) = 0$  because  $D^2$  is a convex subset and thus a linear homotopy connects any paths. By Proposition 1.12,  $\pi_1(X) = \pi_1(S^1) \times \pi_1(D^2) = \mathbb{Z} \times 0 = \mathbb{Z}$  and  $\pi_1(A) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$ . Let  $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$  be any homomorphism. Let  $a = f(1, 0), b = f(0, 1)$ . If  $a = 0$  or  $b = 0$ ,  $f$  is not injective because  $f(0, 0) = 0$ . Suppose otherwise. Then  $f(b, 0) = ab = f(0, a)$ , so  $f$  is not injective.

Therefore, there exists no injection from  $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ . Hence,  $X$  does not retract onto  $A$ .

□

**Exercise 0.9.** (Exercise 20, Chapter 1.1) Suppose  $f_t : X \rightarrow X$  is a homotopy such that  $f_0$  and  $f_1$  are each the identity map. Use Lemma 1.19 to show that for any  $x_0 \in X$ , the loop  $f_t(x_0)$  represents an element of the center of  $\pi_1(X, x_0)$ .

*Proof.* Let  $x_0 \in X$  be given. Let  $f : I \rightarrow X$  be the loop defined such that  $f(t) = f_t(x_0)$ .

- $f_t : X \rightarrow X$  is a homotopy.
- $f$  is a path formed by the images of the base point  $x_0$ .

By Lemma 1.19, the following diagram commutes.

$$\begin{array}{ccc}
 & & \pi_1(X, f_1(x_0)) \\
 & \nearrow (f_1)_* & \downarrow \beta_f \\
 \pi_1(X, x_0) & & \pi_1(X, f_0(x_0)) \\
 & \searrow (f_0)_* &
 \end{array}$$

$(f_0)_* = (f_1)_* = (\text{Id}_X)_* = \text{Id}_{\pi_1(X, x_0)}$  by a basic property of induced homomorphisms (P.34 of Hatcher). Since  $f_0 = f_1 = \text{Id}_X$ ,  $f_0(x_0) = f_1(x_0) = x_0$ . Therefore, the diagram above can be simplified as following:

$$\begin{array}{ccc}
 & & \pi_1(X, x_0) \\
 & \nearrow \text{Id} & \downarrow \beta_f \\
 \pi_1(X, x_0) & & \pi_1(X, x_0) \\
 & \searrow \text{Id} &
 \end{array}$$

Let  $[g] \in \pi_1(X, x_0)$ . Then by the diagram above, we have  $\text{Id}([g]) = \text{Id}(\beta_f([g]))$ . This implies  $[g] = [f \cdot g \cdot \bar{f}]$ . Therefore,  $[g] \cdot [f] = [f] \cdot [g]$ , so  $[f]$  commutes with every element in  $\pi_1(X, x_0)$ . Hence,  $[f] \in Z(\pi_1(X, x_0))$ . □