

# MATH 611 (DUE 10/23)

HIDENORI SHINOHARA

## 1. SIMPLICIAL AND SINGULAR HOMOLOGY

**Exercise.** (Problem 2) Show that the  $\Delta$ -complex obtained from  $\Delta^3$  by performing the edge identifications  $[v_0, v_1] \sim [v_1, v_3]$  and  $[v_0, v_2] \sim [v_2, v_3]$  deformation retracts onto a Klein bottle. Find other pairs of identifications of edges that produce  $\Delta$ -complexes deformation retracting onto a torus, a 2-sphere, and  $\mathbb{RP}^2$ .

*Proof.* The deformation retraction of  $\Delta^3$  onto a Klein bottle is described in 1. We will start by “pushing”  $\Delta^3$  from edge  $(v_1, v_2)$ . This will leave the surface that consists of the triangles  $[v_0, v_1, v_3]$  and  $[v_0, v_2, v_3]$ . (In other words, a diamond shape consisting of the vertices  $[v_0, v_1, v_3, v_2]$ .) Step 2 in Figure 1 is what  $\Delta^3$  should look like after the deformation retract. Step 3 through 6 show why this is a Klein bottle.

Figure 2 shows the identification of edges for a torus, 2-sphere, and  $\mathbb{RP}^2$ .

□

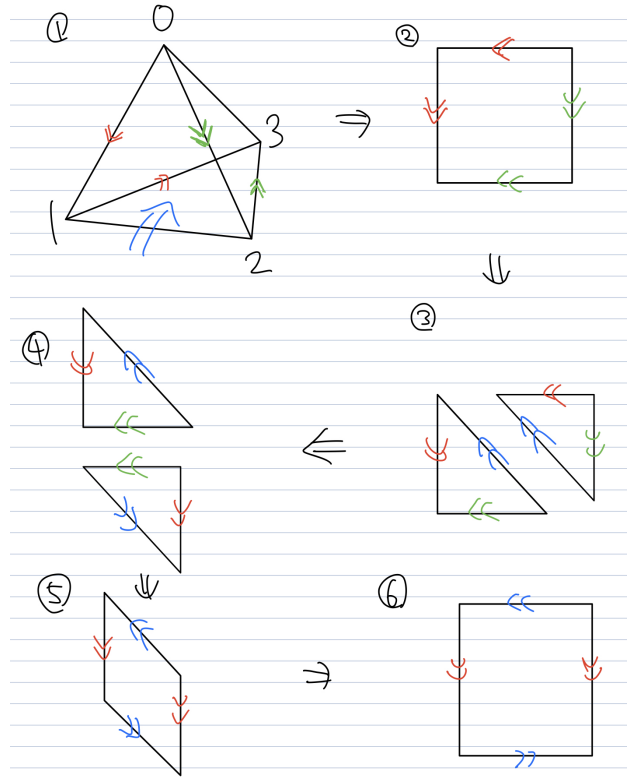


FIGURE 1. Problem 2(Klein Bottle)

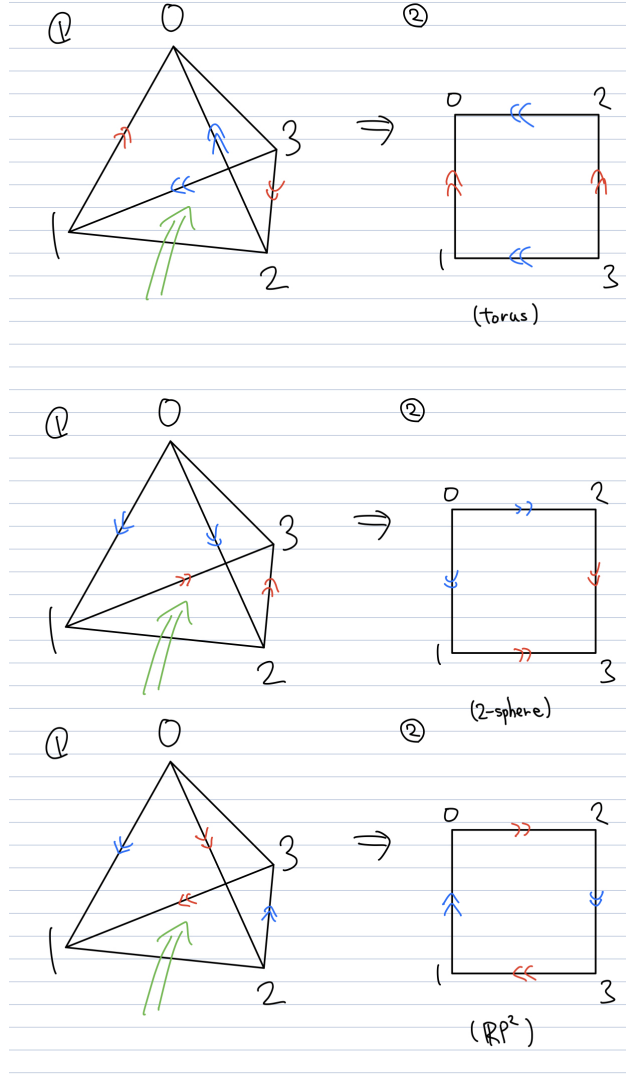


FIGURE 2. Problem 2(Torus, 2-Sphere,  $\mathbb{RP}^2$ )

**Exercise.** (Problem 4) Compute the simplicial homology groups of the triangular parachute obtained from  $\Delta^2$  by identifying its three vertices to a single point.

*Proof.* Let  $v_0$  denote the only vertex,  $e_1, e_2, e_3$  denote the three edges of the parachute, and  $\sigma$  denote the face of the parachute as in Figure 3.  $C_k = 0$  for  $k \geq 3$  because  $\Delta^2$  with the vertices identified does not contain any  $k$ -dimensional simplices for  $k \geq 3$ .  $C_2 = \langle \sigma \rangle, C_1 = \langle e_1, e_2, e_3 \rangle, C_0 = \langle v_0 \rangle$ . For each  $n$ ,  $\partial_n$  is defined such that  $\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha | [v_0, \dots, \hat{v}_i, \dots, v_n]$ .

- $\partial_2(\sigma) = e_3 - e_2 + e_1$ .
- $\partial_1(e_1) = v - v = 0$ . Similarly,  $\partial_1(e_2) = \partial_1(e_3) = 0$ .
- $\partial_0$  and  $\partial_3$  are both the zero map.

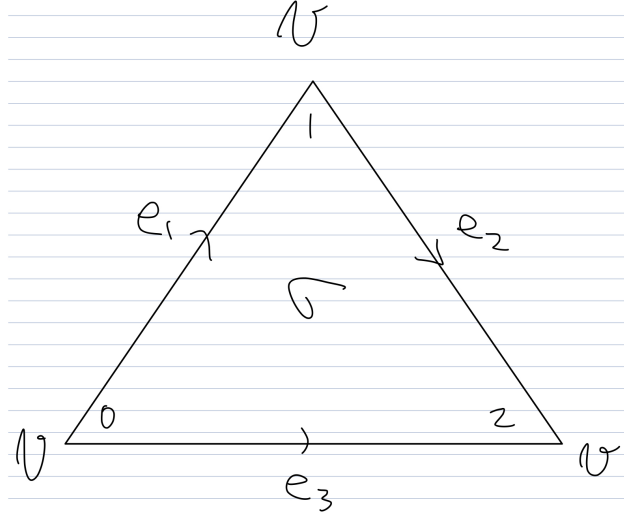


FIGURE 3. Problem 4

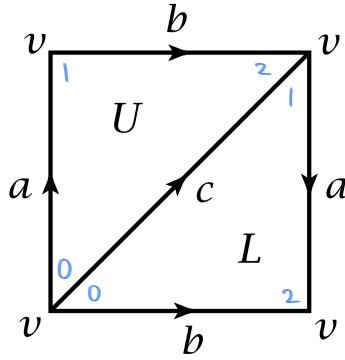


FIGURE 4. Problem 5

Thus

$$H_n = \begin{cases} \{0\} & (n \geq 3) \\ \ker(\partial_2) / \text{Im}(\partial_3) = 0/0 \cong 0 & (n = 2) \\ \ker(\partial_1) / \text{Im}(\partial_2) = \langle e_1, e_2, e_3 \rangle / \langle e_3 - e_2 + e_1 \rangle \cong \langle e_1, e_2, -e_2 + e_1 \rangle \cong \mathbb{Z}^2 & (n = 1) \\ \ker(\partial_0) / \text{Im}(\partial_1) = \langle v \rangle / 0 \cong \mathbb{Z} & (n = 0). \end{cases}$$

□

**Exercise.** (Problem 5) Compute the simplicial homology groups of the Klein bottle using the  $\Delta$ -complex structure described at the beginning of this section.

*Proof.* We will use the notations in Figure 4.

$$C_n = \begin{cases} 0 & (n \geq 3) \\ \langle U, L \rangle & (n = 2) \\ \langle a, b, c \rangle & (n = 1) \\ \langle v \rangle & (n = 0). \end{cases}$$

$\partial_n = 0$  for  $n \geq 3$  and  $n = 0$ .

$$\begin{aligned} \partial_2(U) &= \sum_{i=0}^2 (-1)^i \sigma|[0, 1, 2] \\ &= \sigma|[1, 2] - \sigma|[0, 2] + \sigma|[0, 1] \\ &= b - c + a. \end{aligned}$$

$$\begin{aligned} \partial_2(L) &= \sum_{i=0}^2 (-1)^i \sigma|[0, 1, 2] \\ &= \sigma|[1, 2] - \sigma|[0, 2] + \sigma|[0, 1] \\ &= a - b + c. \end{aligned}$$

$\partial_1(a) = 0$  since  $\partial_1(a) = \sigma|[1] - \sigma|[0] = v - v = 0$ . Similarly,  $\partial_1(b) = \partial_1(c) = 0$ . Thus  $H_n = \{0\}$  if  $(n \geq 3)$ .  $H_2 = \ker(\partial_2)/\text{Im}(\partial_3) = 0/0 \cong 0$ .

$$\begin{aligned} H_1 &= \ker(\partial_1)/\text{Im}(\partial_2) \\ &= \langle a, b, c \rangle / \langle b - c + a, a - b + c \rangle \\ &\cong \langle a, b, a + b \mid a - b + (a + b) \rangle \\ &\cong \langle a, b \mid 2a \rangle \\ &\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}. \end{aligned}$$

$$H_0 = \ker(\partial_0)/\text{Im}(\partial_1) = \langle v \rangle / 0 \cong \mathbb{Z}. \quad \square$$

**Exercise.** (Problem 7) Find a way of identifying pairs of faces of  $\Delta^3$  to produce a  $\Delta$ -complex structure on  $S^3$  having a single 3-simplex, and compute the simplicial homology groups of this  $\Delta$ -complex.

**Exercise.** (Problem 8) Construct a 3 dimensional  $\Delta$ -complex  $X$  from  $n$  tetrahedra  $T_1, \dots, T_n$  by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure, so that each  $T_i$  shares a common vertical face with its two neighbors  $T_{i-1}$  and  $T_{i+1}$ , subscripts being taken mod  $n$ . Then identify the bottom face of  $T_i$  with the top face of  $T_{i+1}$  for each  $i$ . Show the simplicial homology groups of  $X$  in dimensions 0, 1, 2, 3 are  $\mathbb{Z}, \mathbb{Z}_n, 0, \mathbb{Z}$ , respectively.

*Proof.* □