MATH 612(HOMEWORK 4)

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Exercise. (8) By using cellular cohomology, we obtain

$$H^i(X;\mathbb{Z}) = H^i(Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & (i = 0, 4), \\ \mathbb{Z}_p & (i = 3), \end{cases}$$

$$H^i(X;\mathbb{Z}_p) = H^i(Y;\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & (i = 0, 2, 3, 4), \end{cases}$$

Therefore, we cannot distinguish X from Y by looking at the cohomology groups. When using the coefficient $\mathbb{Z}$, cup products are simply 0 because nontrivial cohomology groups are of order 3 and 4. Thus we cannot distinguish X from Y by looking at the cohomology rings of X and Y. Since $H^i(Y;\mathbb{Z}_p) = H^i(S^4;\mathbb{Z}_p) \oplus H^i(M(\mathbb{Z}_p, 2);\mathbb{Z}_p)$ and the cup product of elements from different "components" in a wedge sum is 0, cup products in $H^*(Y;\mathbb{Z}_p)$ are all 0. On the other hand, the cup product $\alpha \smile \alpha$ where α is a generator of $H^2(\mathbb{C}P^2;\mathbb{Z}_p)$ is nontrivial because $\alpha \smile \alpha$ is a generator of $H^4(\mathbb{C}P^2;\mathbb{Z}_p)$.

Exercise. (5) Consider the canonical map $\mathbb{Z}_{2k} \to \mathbb{Z}_2$. It induces homomorphisms $\phi : H^i(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \to H^i(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k})$. By cellular cohomology, $H^0(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k}) = \mathbb{Z}_{2k}$ and $H^i(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k}) = \mathbb{Z}_2$ for $i \geq 1$. Let α denote a generator of $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k})$. Then $\phi(\alpha)$ must be a generator of $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$ because ϕ is induced by the map $\overline{1} \mapsto \overline{1}$. Moreover, $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k}) = \mathbb{Z}_2$, so we obtain the relation 2α.

Let β be a generator of $H^2(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k})$. Since $H^2(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k}) = \mathbb{Z}_2$, we obtain the relation 2β. $H^2(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k}) = \{\overline{0}, \overline{k}\}$, and β corresponds to $\overline{k}$.

- If k is even, ϕ sends β which represents the coset $\overline{k}$ to 0 in $\mathbb{Z}_2$, so $\phi(\beta) = 0$.

  Wait, isn't this kinda weird? $\phi(\alpha^2) = \phi(\alpha)^2 = \gamma^2 \neq 0$, so ϕ cannot be the zero map.

- If k is odd, ϕ sends $\overline{k}$ to 1, so ϕ is an isomorphism between $H^2(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k}) = \mathbb{Z}_2$ and $H^2(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) = \mathbb{Z}_2$. Then $\phi(\alpha^2) = \phi(\alpha)^2 = \gamma^2$ and $\phi(\beta) = \gamma^2$, so $\alpha^2 = \beta$. Since $2\beta = 0$, $\alpha^2 = \beta$ implies $\alpha^2 - k\beta = 0$.

Exercise. (10) Let $X = Y = \mathbb{Z}$ with the discrete topology. Then the only nontrivial cohomology groups are $H^0(X;\mathbb{Z}) = H^0(Y;\mathbb{Z}) = \mathbb{Z}$. Therefore, it suffices to check the cross product map $H^0(X;\mathbb{Z}) \otimes H^0(Y;\mathbb{Z}) \to H^0(X \times Y;\mathbb{Z})$. Every element in $H^0(\mathbb{Z};\mathbb{Z})$ simply represents a map $\mathbb{Z} \to \mathbb{Z}$. Then for each $f \in H^0(X;\mathbb{Z}), g \in H^0(Y;\mathbb{Z})$, $f \times g : (a, b) \mapsto f(a)g(b)$. We claim that this is not surjective.

Let δ be the map such that $\delta(i, j) = \delta_{i,j}$. Then clearly, $\delta \in H^0(X \times Y;\mathbb{Z})$. Suppose that there exists $\sum_{i=1}^n a^i \otimes b^i$ that gets mapped to δ. Let $a_i, b_i \in \mathbb{Z}^n$ (with subscripts instead of superscripts) denote the vectors $a_i = \langle a^1(i), \cdots, a^n(i)\rangle, b_i = \langle b^1(i), \cdots, b^n(i)\rangle$. Then for each $i \in \mathbb{Z}$, the inner product $\langle a_i, b_i\rangle = \delta_{i,j}$. We claim that the set $\{a_i \mid i \in \mathbb{Z}\}$ is linearly independent over $\mathbb{R}$. For simplicity, let $c_1, \cdots, c_m \in \mathbb{R}$ be given such that $\sum_{i=1}^{m} c_i a_i = 0$. (In general, indices could be taken over any finite subset of $\mathbb{Z}$.) This implies $\sum_{i=1}^{m} c_i \delta_{i,j} = 0$ by taking the inner product with b_j for each j. Therefore, we obtain a linearly independent set of infinitely many vectors in $\mathbb{R}^n$. This is clearly impossible, so the cross product map cannot be surjective.