

## MATH 611 HOMEWORK (DUE 9/25)

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**Exercise.** (Problem 4, Chapter 1.3) Construct a simply-connected covering space of the space  $X \subset \mathbb{R}^3$  that is the union of a sphere and a diameter. Do the same when  $X$  is the union of a sphere and a circle intersecting it in two points.

*Proof.* We claim that the space described in Figure 1 is a covering space of  $X$ .

- The shape is an infinitely long chain of spheres and lines. The chain goes infinitely both ways (up and down). This space is clearly simply connected.
- We will map each sphere to the sphere of  $X$ . Each line will be mapped to the diameter up side down. Figure 1 shows how each part gets mapped.
- We claim that such a mapping is a covering map and thus this infinite chain is indeed a covering space. Let  $x \in X$ .

Prove this.

Second part.

□

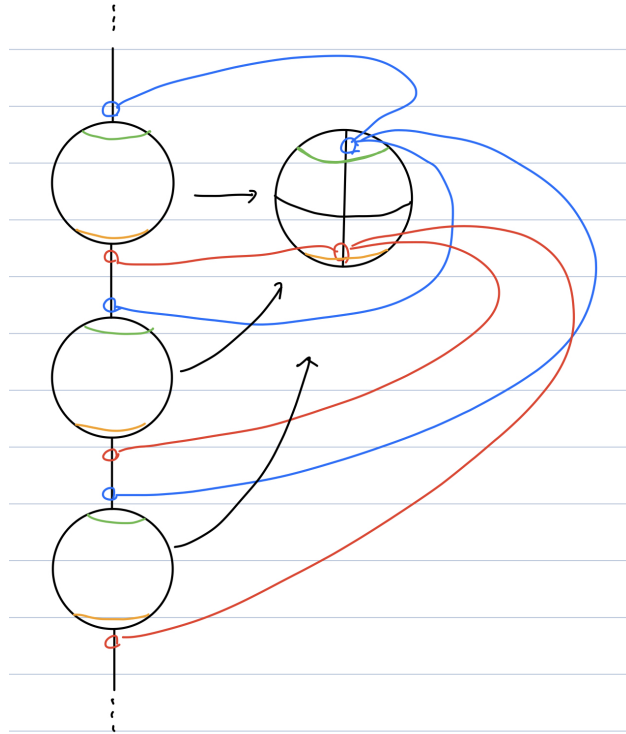


FIGURE 1. Problem 4 (Part 1)

**Exercise.** (Problem 5, Chapter 1.3) Let  $X$  be the subspace of  $\mathbb{R}^2$  consisting of the four sides of the square  $[0, 1] \times [0, 1]$  together with the segments of the vertical lines  $x = 1/2, 1/3, 1/4, \dots$  inside the square. Show that for every covering space  $\tilde{X} \rightarrow X$  there is some neighborhood of the left edge of  $X$  that lifts homeomorphically to  $\tilde{X}$ . Deduce that  $X$  has no simply-connected covering space.

Idea: Open cover of the left edge by evenly covered open sets. Find a finite subcover. By the tube lemma, there exists  $U \times I$  that covers the left edge and a partition  $0 = t_0 < \dots < t_n = 1$  such that  $U \times [t_i, t_{i+1}]$  is contained in an evenly covered neighborhood. Inductively, show  $U \times [0, t_i]$  is in an evenly covered neighborhood. Lift a loop in  $X$  with a vertical line  $x = 1/n$  for some large  $n$ . Then the element maps back to itself by  $p$ . In other words,  $p_*(\pi_1(\tilde{X})) \neq 0$ .

*Proof.*

□

**Exercise.** (Problem 7, Chapter 1.3) Let  $Y$  be the quasi-circle in the figure in the textbook. Collapsing the segment of  $Y$  in the  $y$ -axis to a point gives a quotient map  $f : Y \rightarrow S^1$ . Show that  $f$  does not lift to the covering space  $\mathbb{R} \rightarrow S^1$ , even though  $\pi_1(Y) = 0$ . Thus local path-connectedness of  $Y$  is a necessary hypothesis in the lifting criterion.

The lifting criterion is Proposition 1.33. The only property that  $Y$  is missing is the local path connectedness. But I'm not sure how to make use of it.

*Proof.*

□

**Exercise.** (Problem 8, Chapter 1.3) Let  $\tilde{X}$  and  $\tilde{Y}$  be simply-connected covering spaces of the path-connected, locally path-connected spaces  $X$  and  $Y$ . Show that if  $X \simeq Y$  then  $\tilde{X} \simeq \tilde{Y}$ .

By Proposition 1.33, we can lift the two compositions as in Figure 2. This works because  $\pi_1(\tilde{X}) = \pi_1(\tilde{Y}) = 0$ .

*Proof.*

□

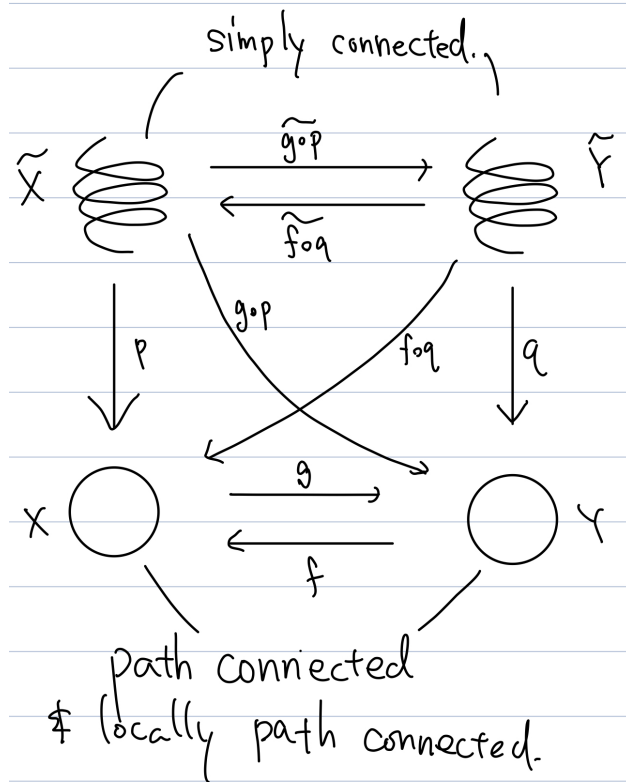


FIGURE 2. delete this!