MATH 611 FINAL

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Lemma 0.1. *$H_0(\vee_g S^1) = \mathbb{Z}, H_1(\vee_g S^1) = \mathbb{Z}^g$ and $H_n(\vee_g S^1) = 0$ for $n \geq 2$.*

Proof. For any g, it is clear that $H_0(\vee_g S^1) = \mathbb{Z}$ by the number of path components.

We will consider the nth homology groups for $n \geq 1$.

When $g = 1$, the proposition is obvious. Suppose we have shown this for some $g \in \mathbb{N}$. Split $\vee_{g+1} S^1$ into $\vee_g S^1$ and S^1 with a small neighborhood. Figure 1 shows the case when

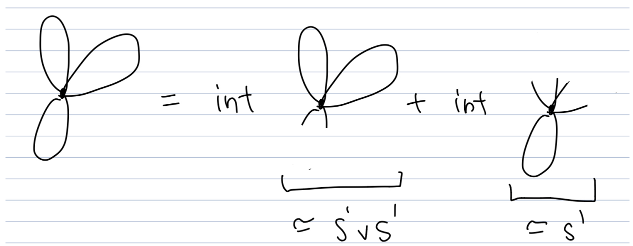

FIGURE 1. $\vee_g S^1$

$g = 2$. The intersection deformation retracts to a point. For simplicity, we will denote each component $\vee_g S^1$ and S^1 although they come with some extra neighborhood. The Mayer-Vietoris sequence $H_n(\vee_g S^1) \oplus H_n(S^1) \to H_n(\vee_{g+1} S^1) \to H_{n-1}((\vee_g S^1) \cap S^1)$ gives $H_n(\vee_{g+1} S^1)$ for all $n \geq 2$ because $H_n(\vee_g S^1) = H_n(S^1) = H_n((\vee_g S^1) \cap S^1) = 0$. We will consider the exact sequence $\tilde{H}_1((\vee_g S^1) \cap S^1) \to \tilde{H}_1(\vee_g S^1) \oplus \tilde{H}_1(S^1) \to \tilde{H}_1(\vee_{g+1} S^1) \to \tilde{H}_0((\vee_g S^1) \cap S^1)$. Since $\tilde{H}_1((\vee_g S^1) \cap S^1) = \tilde{H}_0((\vee_g S^1) \cap S^1) = 0$, $\tilde{H}_1(\vee_{g+1} S^1) = \tilde{H}_1(\vee_g S^1) \oplus \tilde{H}_1(S^1) = \mathbb{Z}^g \oplus \mathbb{Z} = \mathbb{Z}^{g+1}$. □

Exercise. (Problem 1(a)) We will use the 1-skeletons in Figure 2 to calculate the fundamental group of S. The fundamental group of the left side with a 2-cell attached is $\langle a_1, b_1, \cdots, a_g, b_g, c \mid [a_1, b_1] \cdots [a_g, b_g] c \rangle$, and the right side is $\langle gf, f^{-1}h \mid gfh^{-1}f \rangle$. By Van Kampen, the fundamental group of S is

$$\left\langle a_1, b_1, \cdots, a_g, b_g, c, gf, f^{-1}h \mid [a_1, b_1] \cdots [a_g, b_g] c, gfh^{-1}f, c(gh)^{-1} \right\rangle$$

where $c(gh)^{-1}$ corresponds to $i_{\alpha\beta}(c) i_{\beta\alpha}(c)^{-1}$ because we identify c with gh.

Exercise. (Problem 1(b)) Let $A = \Sigma_g \setminus D^2$ and B be a Mobius strip M with some neighborhood from Σ_g such that $\mathrm{Int}(A) \cup \mathrm{Int}(B) = S$ as in Figure 3. Then A is homotopy equivalent to the wedge sum of $2g$ S^1's. Moreover, B is homotopy equivalent to S^1 and so is $A \cap B$. We will consider the Mayer-Vietoris sequence formed by $A, B \subset X$.

We will start with the sequence $H_n(A) \oplus H_n(B) \to H_n(A \cup B) \to H_{n-1}(A \cap B)$ where $n-1 \geq 2$. Then $H_n(A) = H_n(B) = H_{n-1}(A \cap B) = 0$ for $n \geq 3$. By exactness, $H_n(A \cup B) = 0$ when $n \geq 3$.

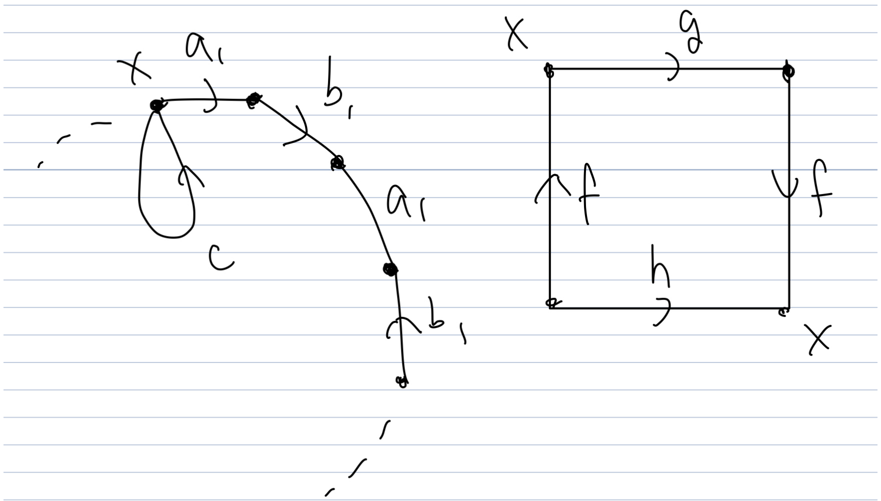

FIGURE 2. Problem 1(a)

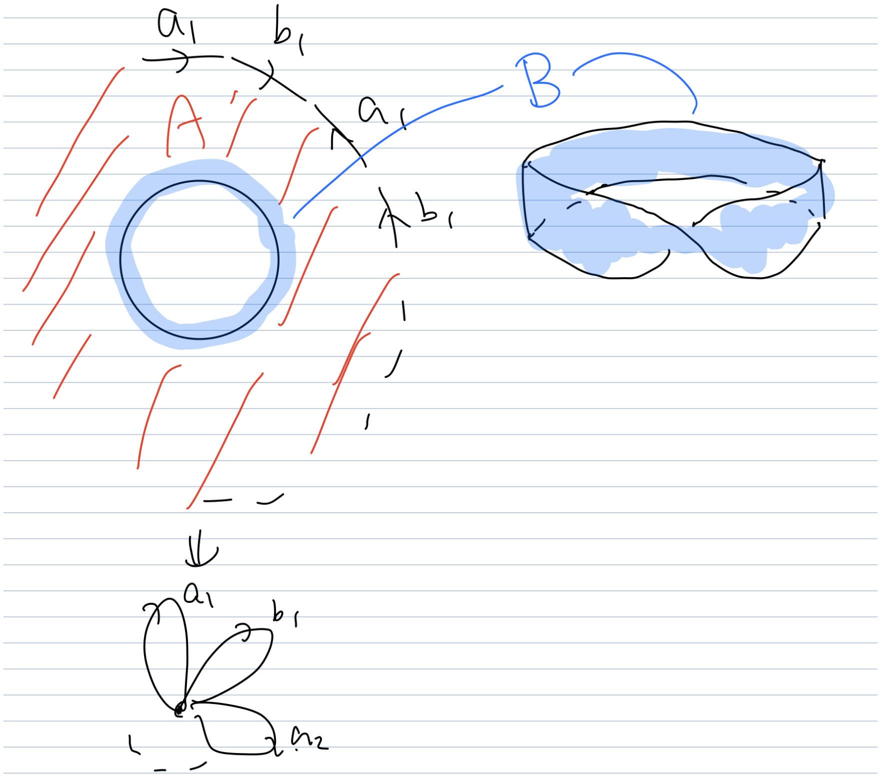

FIGURE 3. M_g with the Mobius band

We will consider the following exact sequence:

$$\tilde{H}_2(A\cap B) \to \tilde{H}_2(A) \oplus \tilde{H}_2(B) \to \tilde{H}_2(X) \xrightarrow{\alpha}$$
$$\tilde{H}_1(A\cap B) \xrightarrow{\beta} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{\gamma} \tilde{H}_1(X) \to$$
$$\tilde{H}_0(A\cap B).$$

Then $\tilde{H}_2(A) = \tilde{H}_2(B) = \tilde{H}_0(A\cap B) = 0$. Thus the above sequence can be simplified to

$$0 \to \tilde{H}_2(X) \xrightarrow{\alpha} \tilde{H}_1(A\cap B) \xrightarrow{\beta} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{\gamma} \tilde{H}_1(X) \to 0.$$

Since the sequence is exact, α must be injective and γ must be surjective. We will examine β to calculate the homology groups. Since $A\cap B$ is homotopy equivalent to S^1, $\tilde{H}_1(A\cap B) = \mathbb{Z}$. By Lemma 0.1, $\tilde{H}_1(A) = \mathbb{Z}^{2g}$. Finally, $\tilde{H}_1(B) = \mathbb{Z}$. Let $a_1, b_1, \cdots, a_g, b_g$ denote generators of $\mathbb{Z}^{2g}$ and let a denote a generator of $\tilde{H}_1(B)$. A generator of $\tilde{H}_1(A\cap B)$ goes around the intersection once, which is homotopy equivalent to $a_1 + b_1 - a_1 - b_1 + \cdots = 0$ inside A. A generator of $\tilde{H}_1(A\cap B)$ goes around the Mobius strip twice inside B. Therefore, β sends a generator of $\tilde{H}_1(A\cap B)$ to $(0, 2a)$.

Since $\mathrm{Im}(\alpha) = \ker(\beta) = 0$ and α is injective, $\tilde{H}_2(A\cup B) = 0$. Since γ is surjective and $\mathrm{Im}(\beta) = \ker(\gamma)$, $\tilde{H}_1(A\cup B) = \mathbb{Z}^{2g} \oplus \mathbb{Z}/\langle(0,2)\rangle = \mathbb{Z}^{2g} \oplus (\mathbb{Z}/2\mathbb{Z})$. Since $H_n = \tilde{H}_n$ when $n \geq 2$ and X is path connected, we have

$$H_n(X) = \begin{cases} 0 & (n \geq 2) \\ \mathbb{Z}^{2g} \oplus (\mathbb{Z}/2\mathbb{Z}) & (n = 1) \\ \mathbb{Z} & (n = 0). \end{cases}$$

Exercise. (Problem 1(c)) We will use Theorem 2.44 and the remark on P.147 [Hatcher]. $\mathcal{X}(S) = 1 - 2g$ based on the calculation from Part (b). Therefore, $\mathcal{X}(S)$ is odd. $\mathcal{X}(S^2) = 1 - 0 + 1 = 2$ because $H_0(S^2) = H_2(S^2) = \mathbb{Z}$. This is even, so S cannot be homeomorphic to S^2. As mentioned on P.147 [Hatcher], the Euler characteristic of a closed orientable surface is even. Therefore, S must be homeomorphic to N_k for some k. $\mathcal{X}(N_k) = 2 - k$, so $2 - k = 1 - 2g \implies k = 1 + 2g$. Therefore, S is homeomorphic to N_{1+2g}.

Exercise. (Problem 2(a)) Figure 4 shows how $K_{3,3}$ is homotopy equivalent to $S^1 \vee S^1 \vee S^1 \vee S^1$. Thus the Van Kampen theorem implies that the fundamental group is the free group generated by 4 elements $\langle a, b, c, d\rangle$ where each generator corresponds to each S_1.

Exercise. (Problem 2(b)) From Figure 4, it is clear that attaching four 2-cells, each killing one S^1, will give a simply connected space. We claim that 4 is the smallest number.

When we attach 2-cells to the graph, the fundamental graph of the resulting space is $\langle a, b, c, d\rangle / \langle r_1, r_2, \cdots\rangle$ where each r_i is the relation given by a product of a, b, c, d in the order the boundary of the ith 2-cell was attached. Therefore, it suffices to show that $\langle a, b, c, d\rangle / \langle r_1, r_2, r_3\rangle \neq 0$. On the contrary, suppose that it is.

If $\langle a, b, c, d\rangle / \langle r_1, r_2, r_3\rangle = 0$, then $\langle a, b, c, d\rangle = \langle r_1, r_2, r_3\rangle$. We will consider the surjective group homomorphism $\phi : \langle a, b, c, d\rangle \to \mathbb{Z}^4$ defined by $a \mapsto (1,0,0,0), b \mapsto (0,1,0,0), \cdots, d \mapsto (0,0,0,1)$. Each r_1, r_2, r_3 is a product of a, b, c, d, so $\phi(r_i) = (d_{i,1}, d_{i,2}, d_{i,3}, d_{i,4})$ for some $d_{i,j} \in \mathbb{Z}$. Since $\langle a, b, c, d\rangle = \langle r_1, r_2, r_3\rangle$, $\phi(\langle a, b, c, d\rangle) = \phi(\langle r_1, r_2, r_3\rangle)$. Since ϕ is surjective, $\phi(r_1), \phi(r_2), \phi(r_3)$ generate $\mathbb{Z}^4$. However, this implies $\{\phi(r_1), \phi(r_2), \phi(r_3)\}$ is a basis of $\mathbb{R}^4$

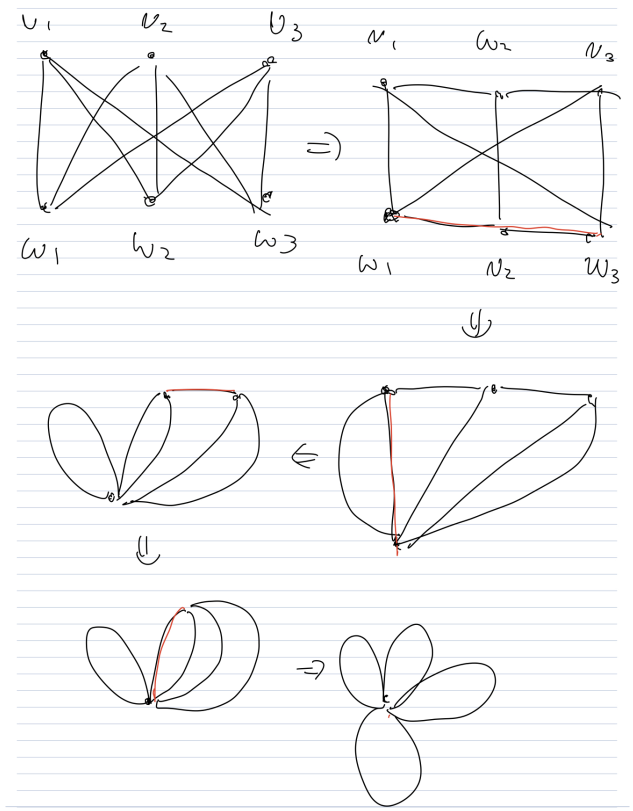

FIGURE 4. $K_{3,3}$

because $\{(1,0,0,0), \cdots, (0,0,0,1)\}$ is. This is clearly a contradiction, so we need at least four 2-cells.

Exercise. (Problem 3) Figure 5 shows what X looks like. (It does not include all the faces in order to avoid cluttering the figure.) X clearly deformation retracts to a point. Let $x \in X$. For any $n \geq 1$, the exact sequence $\tilde{H}_n(X) \to \tilde{H}_n(X, X \setminus \{x\}) \to \tilde{H}_{n-1}(X \setminus \{x\}) \to \tilde{H}_{n-1}(X)$ shows that $\tilde{H}_n(X, X\{x\}) \cong \tilde{H}_{n-1}(X \setminus \{x\})$ because $\tilde{H}_n(X) = \tilde{H}_{n-1}(X) = 0$.

What happens when $n = 0$?

We will try to calculate $\tilde{H}_n(X \setminus \{x\})$ for each $n \geq 1$. There are five cases:

(1) Suppose $x = v_i$ for some i. Then $X \setminus \{x\}$ deformation retracts to a point, so $\tilde{H}_n(X, X \setminus \{x\}) = \tilde{H}_{n-1}(X \setminus \{x\}) = \tilde{H}_{n-1}(\cdot) = 0$ for all $n \geq 1$.

(2) Suppose $x \in \mathrm{Int}([v_i, v_j])$ for some $i \neq j$. In other words, x lies in the edge $v_i v_j$, and $x \neq v_i$ and $x \neq v_j$. This case is exactly the same as above because $X \setminus \{x\}$ deformation retracts to a point,

(3) Suppose x is on one of the faces. In other words, $v \in \mathrm{Int}([v, v_i, v_j])$ for some $i \neq j$. The space is homotopy equivalent to S^1, so $\tilde{H}_n(X, X \setminus \{x\}) = \tilde{H}_{n-1}(S^1)$. Therefore, $\tilde{H}_n(X, X \setminus \{x\}) = \mathbb{Z}$ when $n = 2$ and 0 otherwise.

(4) Suppose $x = v$. Then the space is homotopy equivalent to the 1-skeleton of the 3-simplex. In other words, $X \setminus \{x\}$ deformation retracts to a space consisting of 4 edges $[v, v_1], [v, v_2], [v, v_3], [v, v_4]$. Using a similar argument as Problem 2(a), we can see that

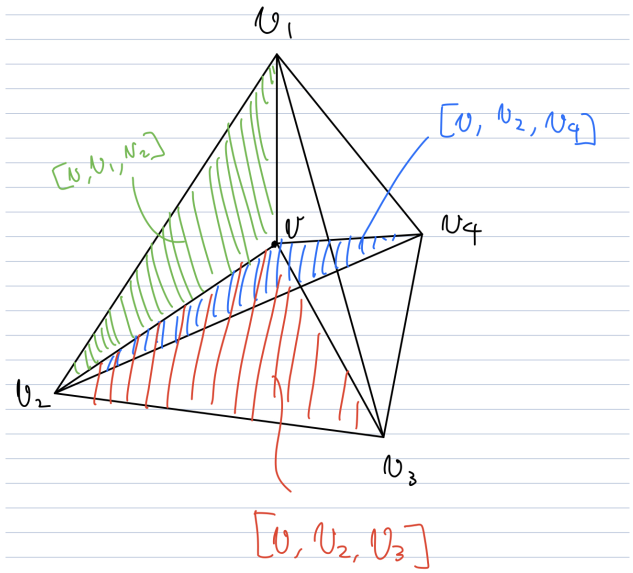

FIGURE 5. Problem 3

it is homotopy equivalent to $S^1 \vee S^1 \vee S^1$. By Lemma 0.1, $\tilde{H}_n(X, X \setminus \{x\}) = \mathbb{Z}^3$ when $n = 2$ and 0 otherwise.

(5) Suppose x is on one of the edges from v. In other words, $x \in \mathrm{Int}([v, v_i])$ for some i. Without loss of generality, $i = 2$. Then the 3 faces shown in Figure 5 deformation retract to the edges $[v, v_i], [v_2, v_i]$ for each $i = 1, 3, 4$. Using a similar argument as Problem 2(a), we can see that it is homotopy equivalent to $S^1 \vee S^1$. By Lemma 0.1, $\tilde{H}_n(X, X \setminus \{x\}) = \mathbb{Z}^2$ when $n = 2$ and 0 otherwise.

Exercise. (Problem 5(a)) Let $X = S^1 \times S^2$ and $Y = S^1 \vee S^2 \vee S^3$.

$$\begin{aligned}
\pi_1(S^1 \times S^2) &= \pi_1(S^1) \times \pi_1(S^2) && \text{(Proposition 1.12)} \\
&= \mathbb{Z} \times 0 \\
&= \mathbb{Z}. \\
\pi_1(S^1 \vee S^2 \vee S^3) &= \pi_1(S^1) * \pi_1(S^2) * \pi_1(S^3) && \text{(Van Kampen)} \\
&= \mathbb{Z} * 0 * 0 \\
&= \mathbb{Z}.
\end{aligned}$$

X and Y are both path connected, so $H_0(X) = H_0(Y) = \mathbb{Z}$.

We will consider two subspaces of X the union of whose interiors equals X. Identify each point of $X = S^1 \times S^2$ by a pair of coordinates $(\theta, (x, y, z))$ where θ is the angle in S^1 and (x, y, z) satisfies $x^2 + y^2 + z^2 = 1$. Let $A = \{(\theta, (x, y, z)) \mid -\epsilon \le \theta \le \pi + \epsilon\}, B = \{(\theta, (x, y, z)) \mid \pi - \epsilon \le \theta \le 2\pi + \epsilon\}$ where $\epsilon > 0$ is a small number. Then each A and B deformation retracts to a space homeomorphic to S^2. $A \cap B$ consists of two path components, each of which deformation retracts to a space homeomorphic to S^2. The homology groups of $A \cap B$ are relatively easy to calculate because $H_n(A \cap B) = H_n(S^2 \coprod S^2) = H_n(S^2) \oplus H_n(S^2)$ by Proposition 2.6 for any n. Moreover, it is clear that $\mathrm{Int}(A) \cup \mathrm{Int}(B) = X$. We will consider the Mayer-Vietoris sequence formed by $A, B \subset X$.

First, we will consider the sequence $H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B)$ for each $n \geq 4$. $H_n(A) = H_n(B) = H_{n-1}(A \cap B) = 0$ for $n \geq 4$ By the exactness, $H_n(X) = 0$ for all $n \geq 4$. Next, we will consider the following sequence:

$$\begin{aligned}
&\tilde{H}_3(A \cap B) \to \tilde{H}_3(A) \oplus \tilde{H}_3(B) \to \tilde{H}_3(X) \xrightarrow{\alpha} \\
&\tilde{H}_2(A \cap B) \xrightarrow{\beta} \tilde{H}_2(A) \oplus \tilde{H}_2(B) \xrightarrow{\gamma} \tilde{H}_2(X) \to \\
&\tilde{H}_1(A \cap B) \to \tilde{H}_1(A) \oplus \tilde{H}_1(B) \to \tilde{H}_1(X) \to \\
&\tilde{H}_0(A \cap B) \to \tilde{H}_0(A) \oplus \tilde{H}_0(B).
\end{aligned}$$

$\tilde{H}_3(A \cap B) = \tilde{H}_3(A) = \tilde{H}_3(B) = \tilde{H}_1(A \cap B) = \tilde{H}_1(A) = \tilde{H}_1 = \tilde{H}_0(A) = \tilde{H}_0(B) = 0$, and $\tilde{H}_0(A \cap B)$. By replacing the exact sequence with those values and splitting the sequence into two for readability, we obtain the following sequences:

$$0 \to \tilde{H}_3(X) \xrightarrow{\alpha} \tilde{H}_2(A \cap B) \xrightarrow{\beta} \tilde{H}_2(A) \oplus \tilde{H}_2(B) \xrightarrow{\gamma} \tilde{H}_2(X) \to 0,$$
$$0 \to \tilde{H}_1(X) \to \mathbb{Z} \to 0.$$

By the exactness, we can conclude that $\tilde{H}_1(X) \cong \mathbb{Z}$. We will examine the homomorphism β to understand the sequence. $\tilde{H}_2(A \cap B) = \langle [a], [b] \mid [[a], [b]] \rangle$ where each a, b lives in $A \cap B$ and a lives in one of the path components of $A \cap B$ and b lives in the other. Moreover, $[a] = [b]$ in $\tilde{H}_2(A)$ and $\tilde{H}_2(B)$. (Based on orientation, $[a] = -[b]$, but we can simply change the orientation of $[b]$ in that case.) Then $\beta(c_1[a] + c_2[b]) = ((c_1 + c_2)[a], (c_1 + c_2)[a])$. This gives us that $\operatorname{Im}(\alpha) = \ker(\beta) = \{c[a] - c[b] \mid c \in \mathbb{Z}\} = \mathbb{Z}$. By the exactness, α is injective, so $\tilde{H}_3(X) = \mathbb{Z}$. Moreover, $\ker(\gamma) = \operatorname{Im}(\beta) = \{(c[a], c[a]) \mid c \in \mathbb{Z}\}$. By the exactness, γ is surjective, so $\tilde{H}_2(X) = (\tilde{H}_2(A) \oplus \tilde{H}_2(B)) / \operatorname{Im}(\beta) = \langle [a] \rangle \oplus \langle [a] \rangle \,/\, \langle ([a], [a]) \rangle = \mathbb{Z}$. Since reduced homology groups and homology groups are identical when $n \geq 2$, we have

$$H_n(X) = \begin{cases} \mathbb{Z} & (n = 0, 1, 2, 3) \\ 0 & (n \geq 4). \end{cases}$$

By Corollary 2.25, $\tilde{H}_n(S^1 \vee S^2 \vee S^3) = \tilde{H}_n(S^1) \otimes \tilde{H}_n(S^2) \otimes \tilde{H}_n(S^3)$.
Therefore,

$$\tilde{H}_n(Y) = \begin{cases} \mathbb{Z} & (n = 1, 2, 3) \\ 0 & (n = 0, n \geq 4). \end{cases}$$

For $n \geq 1$, $\tilde{H}_n(Y) = H_n(Y)$, so $H_0(Y) = H_1(Y) = H_2(Y) = H_3(Y) = \mathbb{Z}$ and $H_n(Y) = 0$ for all $n \geq 4$.

Exercise. (Problem 5(b)) We claim that the universal cover is $\mathbb{R} \times S^2$. $p(\theta, (x, y, z)) = ((\cos\theta, \sin\theta), (x, y, z))$ is a covering map. Moreover, $\pi_1(\mathbb{R} \times S^2) = \pi_1(\mathbb{R}) \times \pi_1(S^2) = 0 \times 0 = 0$, so $\mathbb{R} \times S^2$ is simply connected. Therefore, $\mathbb{R} \times S^2$ is indeed a universal cover of X.

$\mathbb{R} \times S^2$ is homeomorphic to $(0, 1) \times S^2$. This space deformation retracts to S^2 because $(0, 1) \times S^2$ is homeomorphic to an open ball with its center removed. Thus their homology groups are $H_2(\tilde{X}) = H_0(\tilde{X}) = \mathbb{Z}$ and $H_n(\tilde{X}) = 0$ for all other n.

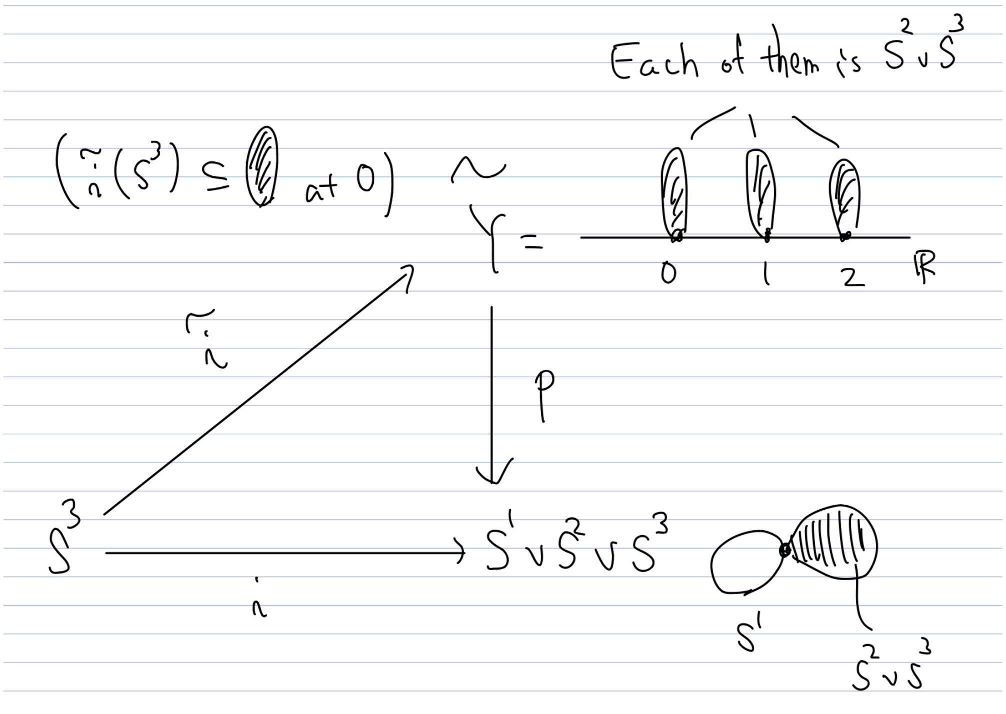

FIGURE 6. Problem 5(c)

Exercise. (Problem 5(c)) We claim that the universal covering space is the real line with $S^2 \vee S^3$ attached to each of its integral points (Figure 6). Since S^2 and S^3 are both contractible, the wedge sum must be contractible. Attaching contractible spaces to each integral point of $\mathbb{R}$, which itself is contractible, gives a contractible space. The covering map p can be defined in an obvious way. Every point on $\mathbb{R}$ can be mapped to S^1 by $\theta \to (\cos(\theta), \sin(\theta))$, and each copy of $S^2 \vee S^3$ can be mapped identically to $S^2 \vee S^3$. The i in Figure 6 is the obvious inclusion map, and $\tilde{i}$ sends S^3 into the copy of $S^2 \vee S^3$ that is attached to 0 on $\mathbb{R}$. (It does not matter which copy, but it is necessary to specify which.) Then the diagram clearly commutes.

By the Mayer-Vietoris sequence, we have an exact sequence $H_3((S^1 \vee S^2) \cap S^3) \to H_3(S^1 \vee S^2) \oplus H_3(S^3) \xrightarrow{\psi} H_3(S^1 \vee S^2 \vee S^3) \to H_2((S^1 \vee S^2) \cap S^3)$. (To be precise, we need $S^1 \vee S^2$ with a small neighborhood and S^3 with a small neighborhood, such that the union of the interiors is $S^1 \vee S^2 \vee S^3$ and the intersection deformation retracts onto a point.) Then $H_n((S^1 \vee S^2) \cap S^3) = 0$ for $n = 2, 3$. Therefore, ψ is an isomorphism. $H_3(S^1 \vee S^2) = 0$ by the Mayer-Vietoris sequence $0 = H_3(S^1) \oplus H_3(S^2) \to H_3(S^1 \vee S^2) \to H_3(S^1 \cap S^2) = 0$ where $S^1, S^2 \subset S^1 \vee S^2$ are technically S^1 and S^2 with a small neighborhood. Therefore, instead of ψ, we can consider the map $\psi' : H_3(S^3) \to H_3(S^1 \vee S^2 \vee S^3)$ defined by $\psi'(x) = \psi(0, x)$. By construction of the Mayer-Vietoris sequence, ψ' is induced by the inclusion map i. Since homology is a covariant functor, p^* and $\tilde{i}^*$, which are induced by p and $\tilde{i}$, must commute with $\psi' = i^*$. In other words, $i^* = \psi' = p^* \circ \tilde{i}^*$. Since i^* is an isomorphism, $\tilde{i}^*$ must be injective. This implies $H_3(\tilde{Y})$ contains an isomorphic copy of $H_3(S^3) = \mathbb{Z}$.

We calculate in Part (b) that $H_3(\tilde{X}) = 0$. Therefore, $H_3(\tilde{X}) \neq H_3(\tilde{Y})$.