

MATH 601 (DUE 10/30)

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1. FACTORING POLYNOMIALS WITH COEFFICIENTS IN FINITE FIELDS

Exercise. (Problem 1) Consider the Frobenius homomorphism, $F_p : \mathbb{F}_q \rightarrow \mathbb{F}_q$. Show that this homomorphism is bijective. If $q = p$, identify it with a familiar homomorphism.

Proof. Since \mathbb{F}_q is finite, it suffices to show that F_p is injective. $F_p(a) = F_p(b) \implies a^p + (-b)^p = 0 \implies a - b = 0$ if $p \geq 3$. The case when $p = 2$ is similar. If $q = p$, $\mathbb{F}_q \cong \mathbb{Z}/p\mathbb{Z}$, which is a cyclic additive group generated by 1. Since $F_p(1) = 1$, F_p must be the identity homomorphism. \square

Exercise. (Problem 2) Let K be a field of characteristic p . Which polynomials $f(x) \in K[x]$ satisfies $f'(x) = 0$?

Proof. $f'(x) = \sum_{i=1}^n i a_i x^{i-1} = 0 \iff (\forall i, i \notin (p) \implies a_i = 0)$ since if $i \in (p)$, $i a_i = 0$ regardless of what a_i is. \square

Exercise. (Problem 3) Suppose that $f(x) \in \mathbb{F}_q[x]$ satisfies $f'(x) = 0$. Show that there exists $g(x) \in \mathbb{F}_q[x]$ with $g^p = f$.

Proof. By Problem 2, $f(x)$ with $f'(x) = 0$ can be always written as $\sum_{i=0}^n a_i x^{pi}$. We will use induction on the degree of polynomials. The base case, $n = 0$, is clear because F_p is bijective. If $f(x) = \sum_{i=0}^{n+1} a_i x^{pi}$, $(F_p^{-1}(a_{n+1})x^{n+1} + g(x))^p = a_{n+1}x^{p(n+1)} + (\sum_{i=0}^n a_i x^{pi})$ where $g(x)$ is the p th root of $\sum_{i=0}^n a_i x^{pi}$, whose existence is given by the inductive hypothesis. \square

Exercise. (Problem 4) Show that there are no inseparable irreducible polynomials, $f(x) \in \mathbb{F}_q[x]$.

Proof. If f is inseparable, $\gcd(f, f') \neq F^\times$. If $f' = 0$, then f has a proper factor by Problem 3. Otherwise, f has a factor of degree between 1 and $\deg(f') = \deg(f) - 1$, so f is not irreducible. \square

Exercise. (Problem 5) Suppose that $f(x) \in \mathbb{F}_q[x]$ and $\gcd(f, f') = f$. How can you reduce the problem of factoring f to a simpler problem?

Proof. If $f' \neq 0$, $f \nmid f'$ because $\deg(f') < \deg(f)$. Thus $\gcd(f, f') = f$ implies $f' = 0$. By Problem 3, $f = g^p$ for some $g \in \mathbb{F}_q[x]$, and thus it suffices to factor g , whose degree is exactly $\deg(f)/p$. \square

2. MODULES

Exercise. (Problem 6) Take four 4×4 matrices with integer entries and check if the abelian group presented by the matrix is cyclic.

Proof.

$$\begin{aligned}
 & \begin{bmatrix} -166 & -74 & 254 & 347 \\ 140 & -93 & 246 & 425 \\ -196 & 57 & -363 & 202 \\ 325 & 257 & 314 & -389 \end{bmatrix} \rightarrow [18444530375 \quad 1 \quad 1 \quad 1] \\
 & \begin{bmatrix} 237 & -81 & 332 & -132 \\ 95 & 268 & 229 & 498 \\ 387 & 213 & 46 & 55 \\ 88 & -126 & -380 & -447 \end{bmatrix} \rightarrow [2610768268 \quad 1 \quad 1 \quad 1] \\
 & \begin{bmatrix} -275 & -22 & -207 & -276 \\ -469 & -342 & 240 & -101 \\ -41 & 455 & 51 & -151 \\ 267 & -450 & 98 & -40 \end{bmatrix} \rightarrow [33644517767 \quad 1 \quad 1 \quad 1] \\
 & \begin{bmatrix} 48 & 29 & 22 & -481 \\ 388 & -468 & -137 & -491 \\ 84 & -352 & 85 & -384 \\ -226 & -486 & 102 & -156 \end{bmatrix} = [13267264454 \quad 1 \quad 1 \quad 1]
 \end{aligned}$$

Each of the groups contains 4 generators, so none of them are cyclic. □

3. GALOIS THEORY

Exercise. (Problem 1) Let $F = \mathbb{Q}$. Let $L = \mathbb{Q}(\sqrt{7}, \sqrt{-11})$. To what familiar group is $\text{Aut}(L/F)$ isomorphic?

Proof. $[K : \mathbb{Q}(\sqrt{7})] = [K : \mathbb{Q}(\sqrt{-11})] = 2$. Since the characteristic of K is not 2, by the argument presented on P.3 of the Galois Theory handout, $\text{Aut}(K/\mathbb{Q}(\sqrt{7}))$ and $\text{Aut}(K/\mathbb{Q}(\sqrt{-11}))$ have 2 elements. For instance, $\alpha = \sqrt{7}$ and the minimal monic polynomial is $x^2 - 7$. This gives $D = 28$ and two automorphisms in $\text{Aut}(K/\mathbb{Q}(\sqrt{7}))$, the identity map, and $\sigma : \sqrt{D} \mapsto -\sqrt{D}$ as discussed in the handout. Similarly, $\text{Aut}(K/\mathbb{Q}(\sqrt{-11}))$ contains the identity map and $\sigma : \sqrt{D} \mapsto -\sqrt{D}$ where $D = -44$.

Finish this proof.

□

Exercise. (Problem 2) Let $F \subset K$ be a field extension.

- (1) Prove in at most two sentences that each $\sigma \in \text{Aut}(K/F)$ is an F -linear transformation of the F -vector space, K .
- (2) Does the same condition hold in general for $\sigma \in \text{Aut}(K)$? Prove or give a counterexample.

Proof.

- (1) For any $a \in F$ and $v, w \in K$, $\sigma(av + w) = \sigma(a)\sigma(v) + \sigma(w) = a\sigma(v) + \sigma(w)$, so σ is indeed an F -linear transformation.
- (2) Let $F = \mathbb{Q}(\sqrt{7})$ and $K = \mathbb{Q}(\sqrt{7}, \sqrt{-11})$. Let $\sigma \in \text{Aut}(K/\mathbb{Q})$ such that $\sigma(\sqrt{7}) = -\sqrt{7}$, $\sigma(\sqrt{-11}) = -\sqrt{-11}$. The existence of such an automorphism is shown in the solution to Problem 1. K is an F -vector space. However, $\sigma(\sqrt{7} \cdot 1) = -\sqrt{7} \neq \sqrt{7} = \sqrt{7}(\sigma(1))$, so σ is not an F -linear transformation.

□

Exercise. (Problem 3) Let $\zeta = \exp(2\pi i/3) \in \mathbb{C}$. Consider the following subfields of \mathbb{C} . Let $F = \mathbb{Q}(\zeta)$. For $i \in \{0, 1, 2\}$, let $K_i = \mathbb{Q}(\zeta^i 7^{1/3})$. Let $L = \mathbb{Q}(7^{1/3}, \zeta 7^{1/3}, \zeta^2 7^{1/3})$.

Proof.

- (1) $[F : \mathbb{Q}] = 3$.
- (2) $\text{Aut}(F/\mathbb{Q})$ consists of two maps, the identity map and another map that swaps ζ and ζ^2 .
- (3) $[K_i : \mathbb{Q}] = 3$ for each i because $\{1, \zeta^i 7^{1/3}, (\zeta^i 7^{1/3})^2\}$ is a \mathbb{Q} -basis.
- (4) Finish the rest!

□