MATH 601 (DUE 10/2)

HIDENORI SHINOHARA

Contents

1. The Quadratic Equation $x^2 - 2y^2 = n$

Exercise. (Problem 15) Find a solution to $x^2 - 2y^2 = 7$.

Proof. $3^2 - 2 \cdot 1^2 = 9 - 2 = 7$. Thus $(x, y) = (3, 1)$ is a solution to $x^2 - 2y^2 = 7$. $\square$

Exercise. (Problem 16) Is 7 irreducible in $\mathbb{Z}[\sqrt{2}]$? If not, find a factorization into irreducible elements.

Proof. By Problem 3 from the previous assignment, we know that $\alpha \in \mathbb{Z}[\sqrt{2}]$ is a unit if and only if $N(\alpha) = \pm 1$. We will use this result in this solution.

By Problem 15, we know that $7 = (3 + \sqrt{2})(3 - \sqrt{2})$. Since $N(3 + \sqrt{2}) = N(3 - \sqrt{2}) = 7 \neq \pm 1$, 7 can be expressed as a product of two non-unit elements, so 7 is not irreducible.

Suppose $3 + \sqrt{2} = (a + b\sqrt{2})(c + d\sqrt{2})$ for some $a, b, c, d \in \mathbb{Z}$. By Problem 2 from the previous assignment, we know that $N(3 + \sqrt{2}) = N(a + b\sqrt{2})N(c + d\sqrt{2})$. Since N maps $\mathbb{Z}[\sqrt{2}]$ into integers, exactly one of $N(a + b\sqrt{2})$ and $N(c + d\sqrt{2}$ must be 1 or -1, and the other one is 7 or -7. Therefore, one of $a + b\sqrt{2}$ or $c + d\sqrt{2}$ is a unit, so $3 + \sqrt{2}$ is irreducible.

Similarly, if $3 - \sqrt{2} = (a' + b'\sqrt{2})(c' + d'\sqrt{2})$, then $7 = N(3 - \sqrt{2}) = N(a' + b'\sqrt{2})N(c' + d'\sqrt{2})$. Therefore, one of $a' + b'\sqrt{2}$ or $c' + d'\sqrt{2}$ is a unit, so $3 - \sqrt{2}$ is irreducible. $\square$