

# MATH 601 HOMEWORK (DUE 10/16)

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## 1. MODULES

**Exercise.** (Problem 2) Consider the  $m \times n$  matrices given below as presentation matrices for  $\mathbb{Z}$ -modules. That is think of the given matrix,  $H$ , as giving a linear transformation,  $\mathbb{Z}^n \rightarrow \mathbb{Z}^m$ ,  $x \mapsto Hx$  and thus giving a presentation of  $\text{Coker}(H) = \mathbb{Z}^m / \text{Im}(H)$ . Give in each case a familiar finitely generated  $\mathbb{Z}$ -module which is isomorphic to the  $\mathbb{Z}$ -module which  $H$  presents.

- $H = 6$ .
- $H = \begin{bmatrix} 2 & 1 \end{bmatrix}$ .
- $H = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ .
- $H = \begin{bmatrix} 4 & 12 \\ 6 & 2 \end{bmatrix}$ .
- $H = \begin{bmatrix} 3 & 6 \\ 8 & 4 \\ 10 & 5 \end{bmatrix}$ .
- $H = \begin{bmatrix} 36 & 12 & 24 \\ 30 & 18 & 24 \\ 15 & -6 & 12 \end{bmatrix}$ .

*Proof.*

- This  $H$  generates the exact sequence

$$\mathbb{Z}^1 \xrightarrow{H} \mathbb{Z}^1 \xrightarrow{p} \mathbb{Z}^1/6\mathbb{Z} \xrightarrow{0} 0$$

where  $p$  is the map  $k \mapsto k + 6\mathbb{Z}$ . Thus  $\mathbb{Z}/6\mathbb{Z}$  is what  $H$  represents.

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**Exercise.** (Problem 3) To what familiar abelian group is the following abelian group isomorphic to? The group generated by  $a, b, c$  for which the module of relations is generated by the following relations,  $6a - 10b + 4c = 0$  and  $8a - 20c = 0$ .

*Proof.* 

- Abelian group =  $\mathbb{Z}$  module.

$$\bullet \quad \mathbb{Z}^2 \xrightarrow{h} \mathbb{Z}^3 \xrightarrow{q} M \xrightarrow{0} 0$$

- $M$  is an abelian group generated by  $a, b, c$  with some relations.
- $\ker(q)$  is the module of relations, and  $\ker(q) = \langle 6a - 10b + 4c, 8a - 20c \rangle$ .
- $h = \begin{bmatrix} 6 & 8 \\ -10 & 0 \\ 2 & -20 \end{bmatrix}$
- $q(1, 0, 0) = a, q(0, 1, 0) = b, q(0, 0, 1) = c$ .
- $M = \langle a, b, c \mid 6a - 10b + 4c, 8a - 20c \rangle$ .
- Is the answer just  $\mathbb{Z}^3 / \langle (6, -10, 4), (8, 0, -20) \rangle$ ? I'm certainly not familiar with that abelian group. If I mod  $\mathbb{Z}^3$  by two independent vectors, does that leave  $\mathbb{Z}$ ? □

**Exercise.** (Problem 4) How many isomorphism classes of abelian groups with  $27783 = 3^4 7^3$  elements are there?

*Proof.* Let  $M$  be an abelian group with 27783 elements. Then  $M$  is a  $\mathbb{Z}$ -module with 27783 elements. By the theorem on PP.8-9 of the Module handout,  $M \simeq \mathbb{Z}/(d_1) \times \cdots \times \mathbb{Z}/(d_n) \times \mathbb{Z}^{m-s}$ . Since  $M$  only contains finitely many elements and  $\mathbb{Z}$  contains infinitely many elements,  $M \simeq \mathbb{Z}/(d_1) \times \cdots \times \mathbb{Z}/(d_n)$ .  $\gcd(a, b) = 1$  if and only if  $\mathbb{Z}/(a)$  is isomorphic to  $\mathbb{Z}/(b)$ .

- $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_9 \times \mathbb{Z}_9, \mathbb{Z}_{27} \times \mathbb{Z}_3, \mathbb{Z}_{81}$ .
- $\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_7, \mathbb{Z}_{49} \times \mathbb{Z}_7, \mathbb{Z}_{343}$ .

Thus the combinations of the above are exactly all the distinct classes of abelian groups with 27783 elements, so there are exactly  $3 \times 5 = 15$  classes. □

## 2. JORDAN CANONICAL FORM

Let  $k$  be a field,  $V$  a finite dimensional  $k$ -vector space, and  $T \in \text{End}_k(V)$  a linear transformation.

**Exercise.** (Problem 1) Show that the set  $\{p(x) \in k[x] \mid p(T) = 0 \in \text{End}_k(V)\}$  is an ideal,  $I \subset k[x]$ . Also, show that  $I \neq 0$ .

*Proof.*

- Claim 1:  $I$  is nonempty. Let  $v_1, \dots, v_n$  be a basis of  $V$ . Such a basis must exist since the dimension of  $V$  is finite. Let  $M$  be the  $n \times n$  matrix associated to  $V$  with respect to the basis  $\{v_1, \dots, v_n\}$ . In other words, for any  $v \in V$ ,  $Mv = T(v)$  where  $Mv$  is the product. Since  $M$  is an  $n \times n$  matrix, the set  $\{M^0, \dots, M^{n^2}\}$  is linearly dependent. Thus there exist  $a_{n^2}, \dots, a_0 \in k$  such that
  - $a_{n^2}M^{n^2} + \cdots + a_0M^0 = 0$ .
  - $a_{n^2}, \dots, a_0$  are not all zero.

Then for any  $v \in V$ ,

$$\begin{aligned} 0 &= (a_{n^2}M^{n^2} + \cdots + a_0M^0)v \\ &= a_{n^2}M^{n^2}v + \cdots + a_0M^0v \\ &= a_{n^2}T^{n^2}(v) + \cdots + a_0T^0(v) \\ &= (a_{n^2}T^{n^2} + \cdots + a_0T^0)(v). \end{aligned}$$

Therefore,  $p(x) = a_{n^2}x^{n^2} + \cdots + a_0x^0 \neq 0$  and  $p(T) = 0$ . Thus  $p(x) \in I$ , so  $I$  is nonempty.

- Claim 2:  $I$  is closed under subtraction. Let  $p(x), q(x) \in I$ . Then  $p(x) - q(x) \in I$  because  $p(T) - q(T) = 0 - 0 = 0$ .
- Claim 3:  $I$  is closed under multiplication by elements in  $k[x]$ . Let  $p(x) \in I, r(x) \in k[x]$ . Then  $p(T)r(T) = 0r(T) = 0$ , so  $r(x)p(x) \in I$ .

By Claim 1 and 2,  $I$  is a subgroup of  $k[x]$  under addition. Then Claim 3 implies that  $I$  is an ideal. By Claim 1,  $I \neq 0$ .  $\square$

**Exercise.** (Problem 2) Let  $p(x) \in k[x]$  be a nonzero polynomial such that  $p(T) = 0 \in \text{End}_k(V)$ . Show that if  $p(x) \in k[x]$  is a product of linear polynomials, then there is a  $k$ -basis for  $V$  with respect to which the matrix for  $T$  is in Jordan normal form.

Since  $k$  is just a field, I can't assume that  $k$  is algebraically closed.

- $p(x) = (x - a_1)^{m_1} \cdots (x - a_n)^{m_n}$ .
- Let  $N = \dim(V)$ .
- Let  $q(\lambda) = \det(T - \lambda \text{Id})$  be the characteristic polynomial of  $T$ .
- Let  $v_1, \dots, v_N$  be a basis of  $V$ .

For each  $i$ ,  $(p(T))(v_i) = 0$ . In other words, there exists a  $j$  such that  $(T - a_j \text{Id})(v) = 0$  for some nonzero  $v$ . This can be found by applying each linear factor to  $v_i$  and figure out the point where it turns into 0. In other words,  $\det(T - a_j \text{Id}) = 0$ . This implies that  $a_j$  is a root of the characteristic polynomial  $q(\lambda)$  of  $T$ . Thus  $\lambda - a_j$  divides  $q(\lambda)$ . But I'm not sure what to do next. We want to find the largest number  $r_j$  such that  $(\lambda - a_j)^{r_j}$  divides  $q(\lambda)$ . What happens next?

*Proof.*

$\square$

**Exercise.** (Problem 3) Suppose that the field  $k$  contains  $m$  distinct  $m$ -th roots of 1. Suppose that  $T^m = \text{Id}_V \in \text{End}_k(V)$ . Show that there is a basis of  $V$  with respect to which, the matrix for  $T$  is diagonal. What can you say about the diagonal entries?

*Proof.*

- Let  $r_1, \dots, r_m$  denote the  $m$  distinct  $m$ th roots of 1.
- Then each  $x - r_i$  divides  $x^m - 1$ . Thus  $x^m - 1 = (x - r_1) \cdots (x - r_m)$ . This means that  $p(x) = x^m - 1$  is a polynomial such that  $p(T) = 0$  and it is a product of linear polynomials. Then I think that we can use an approach similar to the previous problem.
- Let  $M$  denote the diagonal matrix for  $T$ . Then  $M^m$  must be the identity matrix. Moreover, the  $i$ th diagonal entry of  $M^m$  is simply the  $m$ -th power of the  $i$ th diagonal entry of  $M$ . Thus each of the diagonal entries in  $M$  must be an  $m$ -th root of 1. On the other hand, any diagonal matrix where each entry is an  $m$ -th root of 1 becomes the identity when raised to the  $m$ th power.

$\square$

**Exercise.** (Problem 4) Let  $V$  be a 9 dimensional  $k$ -vector space. Let  $T \in \text{End}_k(V)$  have minimal polynomial,  $x^2(x - 1)^3$ . What are the possible Jordan canonical forms for  $T$ ?

*Proof.*

For any  $a, b \in \{0, 1\}$ ,

$$\begin{bmatrix} 1 & 0 & \cdots & & & \\ a & 1 & 0 & \cdots & & \\ 0 & b & 1 & 0 & \cdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & & & & \ddots \end{bmatrix}$$

satisfies  $x^2(x-1)^3$ .

□