ROOT TEST

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1. Absolute Convergence

Example 1.1.

- Does $\sum_{i=1}^{\infty}(\frac{-1}{3})^n$ converge? Yes, geometric.
- Does $\sum_{i=1}^{\infty}\left|(\frac{-1}{3})^n\right|$ converge?

$$\begin{aligned}\sum_{i=1}^{\infty}\left|\left(\frac{-1}{3}\right)^n\right| &= \left|\frac{-1}{3}\right| + \left|\left(\frac{-1}{3}\right)^2\right| + \left|\left(\frac{-1}{3}\right)^3\right| + \cdots \\ &= \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \cdots \\ &= \sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^n.\end{aligned}$$

  Yes, geometric.
- Does $\sum_{i=1}^{\infty}\frac{(-1)^n}{n}$ converge? Yes, geometric.
- Does $\sum_{i=1}^{\infty}\left|\frac{(-1)^n}{n}\right|$ converge?

$$\begin{aligned}\sum_{i=1}^{\infty}\left|\frac{(-1)^n}{3}\right| &= \left|\frac{-1}{1}\right| + \left|\frac{(-1)^2}{2}\right| + \left|\frac{(-1)^3}{3}\right| + \cdots \\ &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \\ &= \sum_{i=1}^{\infty}\frac{1}{n}.\end{aligned}$$

  No, harmonic.

Definition 1.2. $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ is convergent.

Example 1.3.

- $\sum_{i=1}^{\infty}(\frac{-1}{3})^n$ converges and absolutely converges.
- $\sum_{i=1}^{\infty}\frac{(-1)^n}{n}$ converges, but does not absolutely converge.

Remark 1.4. Absolutely convergent $\implies$ Convergence. However, the converse is not always true. (See the example above.)

2. Root Test

Example 2.1.

- Does $\sum_{i=1}^{\infty}(\frac{2}{3})^n$ converge? Yes, geometric.
- Does $\sum_{i=1}^{\infty}(\frac{2}{3n-2})^n$ converge?
  - $(n=1) \implies \frac{2}{3\cdot 1-2} = 2$.
  - $(n=2) \implies (\frac{2}{3\cdot 2-2})^2 = \frac{1}{4}$.
  - $(n=3) \implies (\frac{2}{3\cdot 3-2})^3 = \frac{8}{343}$.

  This doesn't look like a geometric series. How can we tell the convergence?

Remark 2.2. But $\sum_{i=1}^{\infty}(\frac{2}{3n-2})^n$ looks a bit like a geometric series! Recall: $\sum(\text{something})^n$ converges when $|\text{something}| < 1$. If we were to do the same thing, we would want to check $\left|\frac{2}{3n-2}\right|$. This wouldn't make much sense because this would depend on the value of n. It turns out that we need to take the limit $n \to \infty$.

Theorem 2.3. *Let* $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

- $L < 1 \implies$ *absolute convergence.*
- $L > 1 \implies$ *divergent.*
- $L = 1 \implies$ *inconclusive.*

Exercise.

- $\sum_{n=1}^{\infty}(\frac{3n+1}{4-2n})^n$. Diverges since $L = 9/4$.
- $\sum_{n=4}^{\infty}[\frac{(-5)^{1+2n}}{2^{5n-3}}]^n$. Absolutely converges since $L = 25/32$.

Exercise. (Midterm problems) Check if the following sequence absolutely converges, converges, or diverges.

- A: $\sum_{n=1}^{\infty}(\frac{n^{1-3n}}{4^{2n}})^n$.
- B: $\sum_{n=1}^{\infty}(\frac{2^{2n}3^n}{5+10^n})^n$.
- C: $\sum_{n=1}^{\infty}(\frac{5n-3n^3}{7n^3})^n$.
- D: $\sum_{n=1}^{\infty}(\frac{1}{2n})^n$.