MATH 633 (HOMEWORK 5)

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Exercise. (Problem 1)

Exercise. (Problem 2) The desired equation can be obtained by integrating $\frac{e^{iz}}{z^2+a^2}$ over the closed curve γ consisting of the interval $\gamma_1 = [-R, R]$ and the arc $\gamma_2 = Re^{\pi it}$ with $t \in [0, 1]$ as $R \to \infty$ and comparing the real part. In the following calculation, we assume R is sufficiently large.

$$\int_\gamma \frac{e^{iz}}{z^2+a^2} = \int_\gamma \frac{(e^{iz})/(z+ia)}{z-ia} dz$$

$$= 2\pi i \frac{e^{i(ia)}}{2ia}$$

$$= \pi \frac{e^{-a}}{a}.$$

$$\int_{\gamma_1} \frac{e^{iz}}{z^2+a^2} = \int_{-R}^{R} \frac{e^{ix}}{x^2+a^2} dx$$

$$= \int_{-R}^{R} \frac{\cos x}{x^2+a^2} dx + i \int_{-R}^{R} \frac{\sin x}{x^2+a^2} dy.$$

$$\left| \int_{\gamma_2} \frac{e^{iz}}{z^2+a^2} dz \right| \leq \int_0^1 \frac{\left| e^{i\gamma_2(t)} \right|}{|Re^{2\pi it}+a^2|} |\gamma_2'(t)| dz$$

$$= R \int_0^1 \frac{e^{-R\sin \pi t}}{|Re^{2\pi it}+a^2|} dz$$

$$\leq R \int_0^1 \frac{e^{-R\sin \pi t}}{R/2} dz$$

$$= 2 \int_0^1 e^{-R\sin \pi t} dz$$

$$= 2 \frac{e^{-R\sin \pi t}}{-R\sin \pi} \Big|_0^1$$

$$= 0 \qquad \qquad (\text{as } R \to \infty).$$

Exercise. (Problem 3) $p(z) = az + b$ with $a \neq 0$ are the only bijective polynomials.

By the fundamental theorem of algebra, every polynomial $p(z)$ with coefficients in $\mathbb{C}$ is of the form $a \prod_{i=1}^{n}(z - a_i)$ for $a \neq 0, a_1, \cdots, a_n \in \mathbb{C}$. If $a_i \neq a_j$ for some i, j, then p cannot be injective. Thus any bijective polynomials must be of the form $a(z - b)^n$ for some $a \neq 0$ and $b \in \mathbb{C}$. If $n \geq 2$, then $p(\omega + b) = a\omega^n = a$ where $\omega = e^{2\pi ij/n}$ where $j = 0, \cdots, n - 1$. Thus $n = 1$ if the polynomial is injective. In other words, any bijective polynomial must be linear.

On the other hand, it is clear that any non-constant linear function is bijective.