MATH 611 HOMEWORK (DUE 9/18)

HIDENORI SHINOHARA

Exercise. (Problem 12, Chapter 1.2) The Klein bottle is usually pictured as a subspace of $\mathbb{R}^3$ like the subspace $X \subset \mathbb{R}^3$ shown in the first figure at the right. If one wanted a model that could actually function as a bottle, one would delete the open disk bounded by the circle of self-intersection of X, producing a subspace $Y \subset X$. Show that $\pi_1(X) \approx \mathbb{Z} * \mathbb{Z}$ and that $\pi_1(Y)$ has the presentation $\langle a, b, c \mid aba^{-1}b^{-1}cb^{\epsilon}c^{-1} \rangle$ for $\epsilon = \pm 1$. Show also that $\pi_1(Y)$ is isomorphic to $\pi_1(\mathbb{R}^3 \setminus Z)$ for Z the graph shown in the figure.

Proof. We will construct X from the 1-skeleton in Figure 1. The 1-skeleton has three loops a, b, c, so the fundamental group is $\langle a, b, c \mid \rangle$. The main difference between X and the "proper" Klein bottle is that the loop a actually gets glued on the surface. Thus we will glue the first 2-cell to around a, and another 2-cell on the loop $c^{-1}acbab^{-1}$. Therefore, we end up with the fundamental group $\langle a, b, c \mid a, c^{-1}acabab^{-1} \rangle$. Then $\langle a, b, c \mid a, c^{-1}acabab^{-1} \rangle \approx \langle b, c \mid \rangle \approx \mathbb{Z} * \mathbb{Z}$ since the relation $c^{-1}acabab^{-1}$ is trivial.

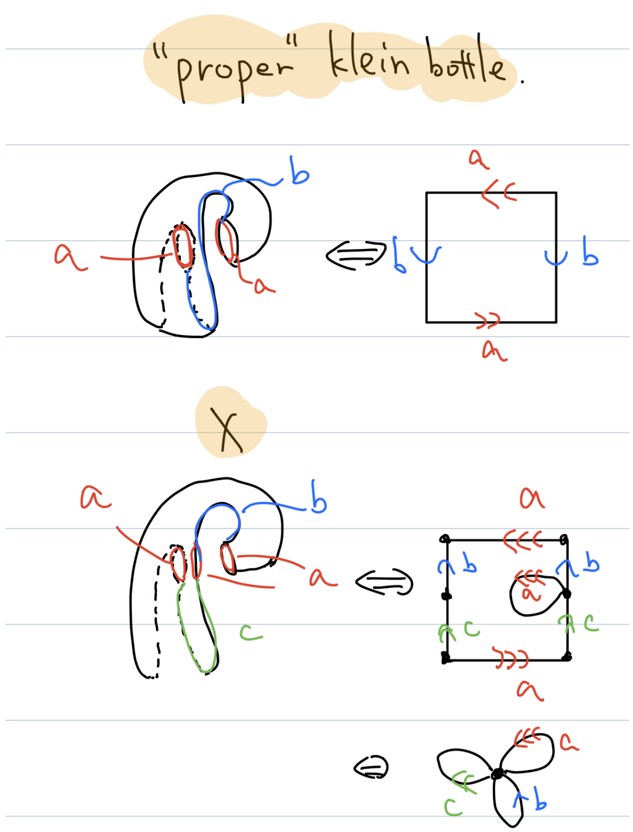

FIGURE 1. Fundamental Group of X

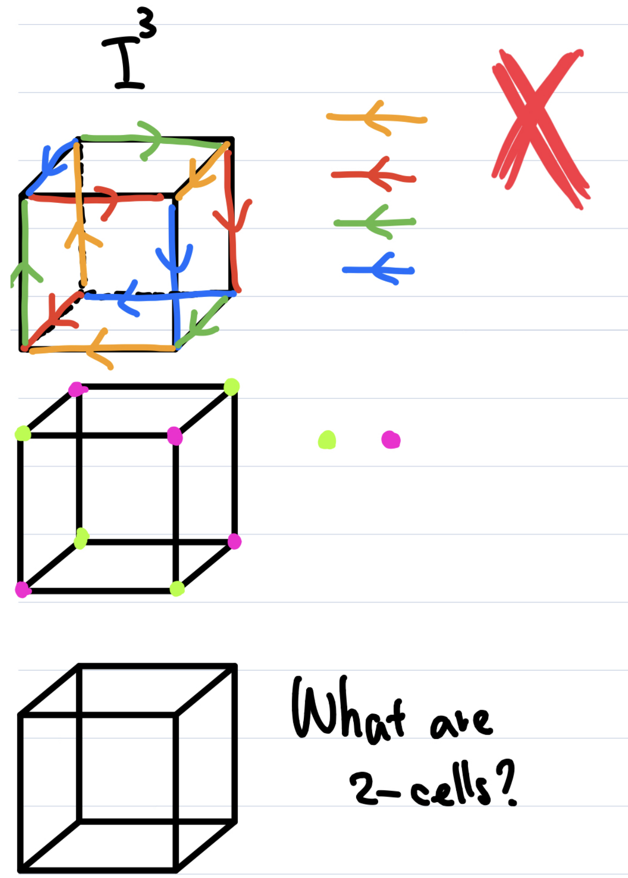

FIGURE 2. Quotient

□

Exercise. (Problem 14, Chapter 1.2) Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0 cells, four 1 cells, three 2 cells, and one 3 cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ of order eight.

Proof. Tried for 45 minutes. The hardest part is to attach 2-cells to X^1. X^1 ends up being two 0-cells with 4 arrows between them. We need to figure out the relation and show that it's the quaternion group. See Figure 2, 3. Tried for 30 more minutes after Yupeng's advice. The first thing that I figured out is that if I put $i = ac$ and $j = ab^{-1}$ and $k = ad$, then $i^2 = j^2 = k^2 = ijk$. I think that I also need to show that $i^2 \neq e$ and $i^4 = e$.

□

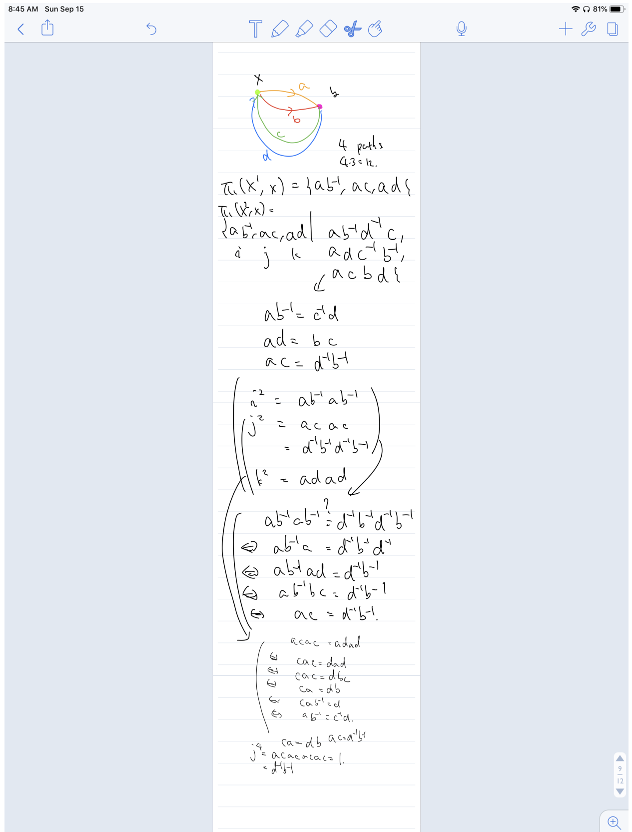

FIGURE 3. Quotient