MATH 611 (DUE 11/13)

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1. Simplicial and Singular Homology

Exercise. (Problem 22) Prove by induction on dimension the following facts about the homology of a finite-dimensional CW complex X, using the observation that X_n/X_{n1} is a wedge sum of n spheres:

- If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.
- $H_n(X)$ is free with basis in bijective correspondence with the n cells if there are no cells of dimension $n1$ or $n+1$.
- If X has k n-cells, then $H_n(X)$ is generated by at most k elements.

Proof.

- X^0 is a set of points, so it is clear that $H_i(X) = 0$ for $i > 0$. Let $k \geq 0$. Suppose that $H_i(X) = 0$ for $i > k$. Let $n = k+1$. Then we have an exact sequence $H_i(X^{n-1}) \to H_i(X^n) \to H_i(X^n, X^{n-1})$ for any $i > n$. Since (X^n, X^{n-1}) is a good pair, $H_{n+1}(X^n, X^{n-1}) = H_{n+1}(X^n/X^{n-1}) = H_{n+1}(\vee_\alpha S^n) = \oplus_\alpha 0 = 0$. By the inductive hypothesis, $H_i(X^{n-1}) = 0$. Therefore, the exactness of $0 \to H_i(X^n) \to 0$ implies that $H_i(X^n) = 0$ for all $i > n$.

□

Exercise. (Problem 27) Let $f : (X, A) \to (Y, B)$ be a map such that both $f : X \to Y, f : A \to B$ are homotopy equivalences.

- Show that $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all n.
- For the case of the inclusion $f : (D^n, S^{n-1}) \to (D^n, D^n \setminus \{0\})$, show that f is not a homotopy equivalence of pairs - there is no $g : (D^n, D^n \setminus \{0\}) \to (D^n, S^{n-1})$ such that fg and gf are homotopic to the identity through maps of pairs.

Proof.

- For each $n \geq 1$, we have an exact sequence $H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X)$ and another one with X, A replaced with Y, B. Moreover, they are connected by homomorphisms $f_* : H_n(A) \to H_n(B), f_* : H_n(X) \to H_n(Y), f_* : H_n(X, A) \to H_n(Y, B)$ such that the diagram commutes. (naturality) Since $f : X \to Y$ and $f : A \to B$ are both homotopy equivalences, $f_* : H_n(X) \to H_n(Y), f_* : H_n(A) \to H_n(B)$ are isomorphisms. By the Five lemma, $f_* : H_n(X, A) \to H_n(X, B)$ is an isomorphism.

  The exact sequence $H_1(A) \to H_1(X) \to H_1(X, A) \to 0$ can be extended to $H_1(A) \to H_1(X) \to H_1(X, A) \to 0 \to 0$ by appending 0 at the end. Using the same argument as above, $f_* : H_1(X, A) \to H_1(Y, B)$ is an isomorphism.
- Suppose $f : (D^n, S^{n-1}) \to (D^n, D^n - \{0\})$ is a homotopy equivalence. Then there exists a $g : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ such that $f \circ g$ and $g \circ f$ are homotopic to

the identity maps in corresponding domains. Since g is continuous, $g(\overline{D^n - \{0\}}) = \overline{g(D^n - \{0\})} \subset \overline{S^{n-1}} = S^{n-1}$. Therefore, g maps D^n into S^{n-1}. Since f maps S^{n-1} into D^n, $g \circ f$ maps S^{n-1} into S^{n-1}. We know this is homotopic to the identity map from the problem statement. Similarly, $f \circ g$ maps D^n into D^n and we know this is homotopic to the identity map from the problem statement. Therefore, this implies that D^n and S^{n-1} are homotopy equivalent. However, this is false because D^n is contractible but S^{n-1} is not.

Hence, f cannot be homotopy equivalent.

$\square$