MATH 611 (DUE 11/6)

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1. Simplicial and Singular Homology

Exercise. (Problem 14) Determine whether there exists a short exact sequence $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$. More generally, determine which abelian groups A fit into a short exact sequence $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ with p prime. What about the case of short exact sequences $0 \to A \to \mathbb{Z}_n \to 0$?

Proof. Let $\phi_1 : \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2, \phi_2 : \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4$ be defined such that $\phi_1(a) = (2a, a)$ and $\phi_2(a, b) = a + 2b$. Then $\ker(\phi_1) = 0, \mathrm{Im}(\phi_1) = \ker(\phi_2) = \{(2k, k) \mid 0 \leq k \leq 3\}$ and $\mathrm{Im}(\phi_2) = \mathbb{Z}_4$. Thus this is indeed an exact sequence.

Finish this!

□

Exercise. (Problem 15) For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ if and only if the map $A \to B$ is surjective and $D \to E$ is injective. Hence, for a pair of spaces (X, A), the inclusion $A \to X$ induces isomorphisms on all homology groups if and only if $H_n(X, A) = 0$ for all n.

Proof. Suppose $C = 0$. $\mathrm{Im}(\phi_{AB}) = \ker(\phi_{BC}) = B$, so ϕ_{AB} is surjective. $\ker(\phi_{DE}) = \mathrm{Im}(\phi_{CD}) = \{0\}$, so ϕ_{DE} is injective.

On the other hand, suppose ϕ_{AB} is surjective and ϕ_{DE} is injective. $\mathrm{Im}(\phi_{CD}) = \ker(\phi_{DE}) = \{0\}$, so ϕ_{CD} is the zero map. Therefore, $\ker(\phi_{CD}) = C$. $\ker(\phi_{BC}) = \mathrm{Im}(\phi_{AB}) = B$, so ϕ_{BC} is the zero map. Therefore, $\mathrm{Im}(\phi_{BC}) = 0$. Hence, $C = \ker(\phi_{CD}) = \mathrm{Im}(\phi_{BC}) = 0$.

By Theorem 2.16 and the discussion at the bottom of P.117(Hatcher), we have a long exact sequence of homology groups

$$H_n(A) \xrightarrow{i_*} H_n(X) \to H_n(X, A) \to H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \tag{1.1}$$

for $n \geq 1$. Suppose the inclusion induces isomorphisms on all homology groups. Then $H_n(X, A) = 0$ for all $n \geq 1$ by the first part. Moreover, we have $H_1(X, A) \to H_0(A) \to H_0(X) \to H_0(X, A) \to 0$. Since $H_1(X, A) = 0$, by the first part, $H_0(X) = 0$. In order for $0 \to H_0(X, A) \to 0$ to be exact, $H_0(X, A)$ must be 0. Therefore, $H_n(X, A) = 0$ for all $n \geq 0$.

Suppose that $H_n(X, A) = 0$ for all $n \geq 0$. By exact sequence 1.1 above, $i_* : H_n(A) \to H_n(X)$ is surjective for $n \geq 1$ and injective for $n \geq 0$. Thus i_* is bijective for all $n \geq 1$. We have $H_1(X, A) \to H_0(A) \to H_0(X) \to H_0(X, A)$. Since $H_1(X, A) = H_0(X, A) = 0$, i_* must be bijective by the exactness. Therefore, the inclusion induces isomorphisms for all n. □

Exercise. (Problem 16)

- Show that $H_0(X, A) = 0$ if and only if A meets each path-component of X.
- Show that $H_1(X, A) = 0$ if and only if $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

Proof.

- Let $\gamma_x + C_0(A) \in C_0(X)/C_0(A)$. Since A meets each path-component of X, there exists a path $\gamma : I \to X$ that joins a point $a \in A$ and the image of γ_x. Then γ can be seen as an element of $C_1(X)$ since γ maps a 1-simplex into X. Moreover, $\partial\gamma = \gamma_x - \gamma_a$ where $\gamma_a \in C_0(A)$ with $\operatorname{Im}(\gamma_a) = a$. Therefore, $\partial(\gamma + C_1(A)) = \gamma_x + C_0(A)$, so $\gamma_x + C_0(A) \in \operatorname{Im}(\partial)$. Hence, $H_0(X, A) = \ker(\partial_0)/\operatorname{Im}(\partial_1) = (C_0(X)/C_0(A))/(C_0(X)/C_1(A)) = 0$.

  On other hand, suppose that A does not meet each path component of X. Let $x \in X$ be a point in a path component that A does not intersect. Let $\gamma_x : \Delta^0 \to X$ such that $\operatorname{Im}(\gamma_x) = \{x\}$. Then $\gamma_x \in \ker(\partial_0) = C_0(X, A)$. Let $\gamma + C_1(A) \in C_1(X, A)$. Then $\partial_1(\gamma + C_1(A)) = \partial_1(\gamma) + C_0(A)$. Let $\gamma_{x_1}, \gamma_{x_2} \in C_0(X)$ such that $\partial_1(\gamma) = \gamma_{x_1} - \gamma_{x_2}$. $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$ if and only if $\gamma_{x_1} - \gamma_{x_2} - \gamma_x \in C_0(A)$.
  - If γ lies in the same path component as x, then so do x_1 and x_2. Suppose $x = x_1$. Since $-\gamma_{x_2} \notin C_0(A)$, $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$. The case when $x \neq x_1$ and $x = x_2$ and the case when $x \neq x_1$ and $x \neq x_2$ can be proven in a similar way.
  - If γ lies in a different path component, then $\gamma_x \neq \gamma_{x_1}$ and $\gamma_x \neq \gamma_{x_2}$. Therefore, $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$.

  Therefore, $\gamma_x \notin \operatorname{Im}(\partial_1)$. Thus $H_0(X, A) = C_0(X, A)/\operatorname{Im}(\partial_1)$ is not 0.
- Do part (b).

$\square$

Exercise. (Problem 17)

- Compute the homology groups $H_n(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.
- Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus two with A and B the circles shown.

Proof.

- We will apply Theorem 2.16 to get the exact sequence with $H_n(A), H_n(X), H_n(X, A)$.
  - When $n \geq 3$, $H_n(S^2) \to H_n(S^2, A) \to H_{n-1}(A)$ shows that $H_n(S^2, A)$ is 0 by the exactness since $H_n(S^2) = H_{n-1}(A) = 0$.
  - When $n = 2$, $H_n(A) \to H_n(S^2) \xrightarrow{\phi} H_n(S^2, A) \to H_{n-1}(A)$ shows that $H_n(S^2, A) = H_n(S^2) = \mathbb{Z}$. This is because $H_n(A) = H_{n-1}(A) = 0$ so ϕ is an isomorphism by the exactness.
  - $n = 0$ and $n = 1$.

  We will first compute the homology groups of a torus using Figure 1. $C_2 = \{\sigma_1, \sigma_2\}, C_1 = \{a, b, c\}, C_0 = \{v_0\}$.
  - $H_2 = \ker(\partial_2)/\operatorname{Im}(\partial_3) = \langle \sigma_1 - \sigma_2 \rangle / 0 = \mathbb{Z}$.
  - $H_1 = \ker(\partial_1)/\operatorname{Im}(\partial_2) = \langle a, b, c \rangle / \langle b - a + c, c - a + b \rangle = \mathbb{Z}^2$ because $b - a + c = c - a + b$.
  - $H_0 = \ker(\partial_0)/\operatorname{Im}(\partial_1) = \langle v_0 \rangle / 0 = \mathbb{Z}$.
- Finish this!

$\square$

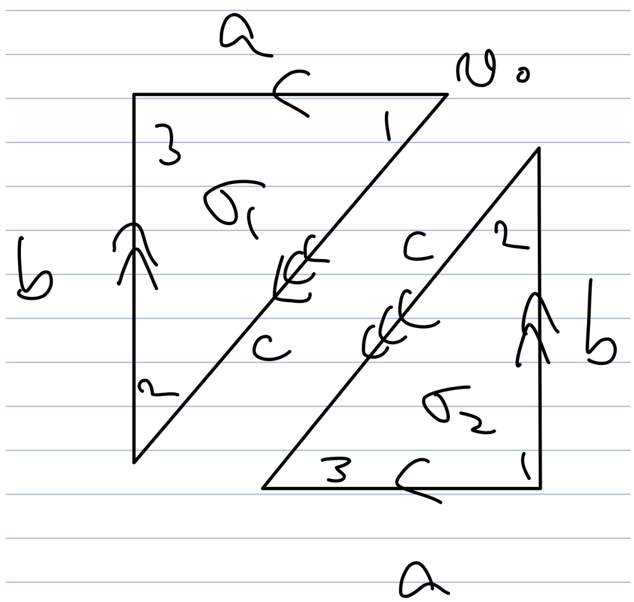

FIGURE 1. Problem 17