MATH 602(HOMEWORK 1)

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Exercise. 1

- Let $p \in V(I \cap J)$. For any $\sum_{i=1}^{n} f_i g_i \in IJ$, we have $f_i g_i \in I \cap J$ for each i. Thus $(\sum_{i=1}^{n} f_i g_i)(p) = 0$, so $p \in V(IJ)$. Let $p \in V(IJ)$. Let $f \in I \cap J$. Then $f^2 \in IJ$, so $(f(p))^2 = 0$. Thus $f(p) = 0$, so $p \in V(I \cap J)$. Therefore, $V(I \cap J) = V(IJ)$.

  Let $p \in V(I) \cup V(J)$. Then either all polynomials in I vanish at p or all polynomials in J vanish at p. Thus all the polynomials in the intersection must vanish at p. Thus $V(I) \cup V(J) \subset V(I \cap J)$. On the other hand, let $p \in V(I \cap J) \setminus (V(I) \cup V(J))$. If no such element exists, we are done. Then every polynomial in the intersection vanishes at p. Let $f \in I$ and $g \in J$ be polynomials that do not vanish at p. Then $fg \in I \cap J$, so $(fg)(p) = 0$. However, this is impossible because $f(p) \neq 0$ and $g(p) \neq 0$. Therefore, $V(I) \cup V(J) = V(I \cap J)$.
- $p \in V(I + J)$ if and only if $\forall f \in I + J, f(p) = 0$ if and only if $\forall f \in I, f(p) = 0$ and $\forall f \in J, f(p) = 0$ if and only if $p \in V(I) \cap V(J)$.
- If every polynomial in J vanishes at a point, every polynomial in I must vanish at that point.
- If a polynomial vanishes in Y, then it must vanish in X.
- TODO

Exercise. 2

- $$\begin{aligned} y \in (I_1 + I_2)^e &\iff y \in f(I_1 + I_2)B \\ &\iff \exists x_1, x_2 \in I_1, I_2, b \in B, y = f(x_1 + x_2)b \\ &\iff \exists x_1, x_2 \in I_1, I_2, b \in B, y = f(x_1)b + f(x_2)b \\ &\iff y \in I_1^e + I_2^e. \end{aligned}$$
- $$\begin{aligned} y \in (I_1 \cap I_2)^e &\implies y \in f(I_1 \cap I_2)B \\ &\implies \exists x \in I_1 \cap I_2, b \in B, y = f(x)b \\ &\implies (\exists x \in I_1, b \in B, y = f(x)b) \text{ and } (\exists x \in I_2, b \in B, y = f(x)b) \\ &\implies y \in I_1^e, y \in I_2^e \\ &\implies y \in I_1^e \cap I_2^e. \end{aligned}$$
- $(I_1 I_2)^e = f(I_1 I_2)B = (f(I_1)f(I_2))B = (f(I_1)B)(f(I_2)B)$. $f(I_1)f(I_2) = f(I_1 I_2)$ because the product of two ideals consists of a finite sum of elements and f preserves finite sums.

- Let $x \in J_1^c + J_2^c$. Then $x \in f^{-1}(J_1) + f^{-1}(J_2)$. Then $x = a + b$ where $a \in f^{-1}(J_1)$ and $b \in f^{-1}(J_2)$. This implies $x = a + b$ where $f(a) \in J_1$ and $f(b) \in J_2$. Then, $f(x) = f(a+b) = f(a) + f(b) \in J_1 + J_2$, so $x \in f^{-1}(J_1 + J_2)$.
- $f^{-1}(J_1 \cap J_2) = f^{-1}(J_1) \cap f^{-1}(J_2)$ from set theory.
- Let $\sum_{i=1}^n a_i b_i \in J_1^c J_2^c$ where $a_i \in J_1^c$ and $b_i \in J_2^c$. Then $f(a_i) \in J_1$ and $f(b_i) \in J_2$. Thus $\sum f(a_i) f(b_i) \in J_1 J_2$. Since f preserves product and addition, $f(\sum a_i b_i) \in J_1 J_2$. Thus $\sum a_i b_i \in f^{-1}(J_1 J_2) = (J_1 J_2)^c$.

Exercise. 3 $(I : J)$ is nonempty because $0 \in (I : J)$. $(I : J)$ is closed under addition, and for all $x \in R$, $rJ \subset I \implies x(rJ) = r(xJ) = rJ \subset I$. Thus $(I : J)$ is an ideal.

- Lemma: Let a, b, c be ideas. If $\forall x \in a, xb \subset c$, then $ab \subset c$.

  Proof: Let $\sum a_i b_i \in ab$ be given. Then each $a_i b_i \in c$. Since c is closed under addition, $\sum a_i b_i \in c$. Therefore, $ab \subset c$.
- Let $x \in a$. Then $\forall y \in b, xy \in a$ since a is an ideal. Then $xb \subset a$, so $x \in (a : b)$.
- For all $x \in (a : b)$, $xb \subset a$. By the Lemma above, $(a : b)b \subset a$.
- Let $x \in ((a : b) : c)$. Then $xc \subset (a : b)$. For all $xz \in xc, (xz)b \subset a$. Therefore, $(xc)b \subset a$ by the Lemma above. Then $x(cb) \subset a$, so $x(bc) \subset a$. Hence, $x \in (a : bc)$.

  On the other hand, suppose $x \in (a : bc)$. Then $x(bc) \subset a$. $x(bc) \subset a \implies (xb)c \subset a \implies xb \subset (a : c) \implies x \in ((a : c) : b)$.

  Therefore, $((a : b) : c) = (a : bc)$.

  We showed that $((a : b) : c) = (a : bc)$. This implies $(a : cb) = ((a : c) : b)$. Since $(a : bc) = (a : cb)$, we have $((a : b) : c) = (a : bc) = (a : cb) = ((a : c) : b)$.
- For any $x \in A$,

$$
\begin{aligned}
x \in (\cap_i a_i : b) &\iff xb \subset \cap_i a_i \\
&\iff \forall i, xb \subset a_i \\
&\iff \forall i, x \subset (a_i : b) \\
&\iff x \subset \cap_i (a_i : b).
\end{aligned}
$$

- For any $x \in A$,

$$
\begin{aligned}
x \in (a : \sum_i b_i) &\iff x(\sum_i b_i) \subset a \\
&\implies \forall i, xb_i \subset a \\
&\iff \forall i, x \subset (a : b_i) \\
&\iff x \subset \cap_i (a : b_i).
\end{aligned}
$$

  Therefore, it suffices to show that $\forall i, xb_i \subset a \implies x(\sum_i b_i) \subset a$. Let $y_{i_1} + \cdots + y_{i_n} \in \sum_i b_i$ be given where $y_{i_j} \in b_{i_j}$. For each j, since $xb_{i_j} \subset a$, $xy_{i_j} \in a$. Since a is closed under finite addition, $xy_{i_1} + \cdots + xy_{i_n} \in a$. Therefore, $\forall i, xb_i \subset a \implies x(\sum_i b_i) \subset a$, so $(a : \sum_i b_i) = \cap_i (a : b_i)$.
- Let $bf(x) \in (a_1 : a_2)^e$ where $b \in B$ and $x \in (a_1 : a_2)$.

$$
\begin{aligned}
xa_2 \subset a_1 &\implies f(xa_2) \subset f(a_1) \\
&\implies f(x)f(a_2) \subset f(a_1) \\
&\implies B(f(x)f(a_2)) \subset Bf(a_1) \\
&\implies f(x)(Bf(a_2)) \subset Bf(a_1) \\
&\implies f(x)a_2^e \subset a_1^e \\
&\implies f(x) \in (a_1^e : a_2^e) \\
&\implies bf(x) \in (a_1^e : a_2^e). \\
x \in (b_1 : b_2)^c &\implies f(x) \in (b_1 : b_2) \\
&\implies f(x)b_2 \in b_1 \\
&\implies f^{-1}(f(x)b_2) \subset f^{-1}(b_1) \\
&\implies xf^{-1}(b_2) \subset f^{-1}(f(x)b_2) \subset f^{-1}(b_1) \\
&\implies xf^{-1}(b_2) \subset f^{-1}(b_1) \\
&\implies x \in (f^{-1}(b_1) : f^{-1}(b_2)) \\
&\implies x \in (b_1^c : b_2^c).
\end{aligned}
$$

Exercise. (Problem 4) Let $f = \sum_{i=1}^m a_i x^i, g = \sum_{i=1}^n b_i x^i \notin p[x]$. Let m', n' be the smallest integer such that $a_{m'}, b_{n'} \notin p[x]$. Such m', n' must exist because $f, g \notin p[x]$. Then the coefficient of $x^{m'+n'}$ in fg is $\sum_{i=0}^{m'+n'} a_i b_{m'+n'-i}$. Then $a_i b_{m'+n'-i} \in p$ if and only if $i \neq m'$. The coefficient of $x^{m'+n'}$ in fg is not in $p[x]$. Therefore, $fg \notin p[x]$, so $p[x]$ is a prime ideal.

(0) is a maximal ideal of Q. However, (0) is not a maximal ideal in $\mathbb{Q}[x]$ because (x) is a proper ideal of $\mathbb{Q}[x]$ that properly contains (0).