MATH 612 (HOMEWORK 3)

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Exercise. (3.1.11) Using the cellular homology, we obtain

$$\tilde{H}_i(X) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & (i = n) \\ 0 & (i \neq n). \end{cases}$$

$$\tilde{H}^i(X) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & (i = n+1) \\ 0 & (i \neq n+1). \end{cases}$$

From previous homework,

$$\tilde{H}^i(X/S^n) = \tilde{H}_i(S^{n+1}) = \begin{cases} \mathbb{Z} & (i = n+1) \\ 0 & (i \neq n+1). \end{cases}$$

Thus the map on $\tilde{H}_i(-;\mathbb{Z})$ is the zero map for each i. On the other hand, the long exact sequence of a pair gives us $\tilde{H}^{n+1}(X, S^n;\mathbb{Z}) \xrightarrow{q^*} \tilde{H}^{n+1}(X;\mathbb{Z}) \to \tilde{H}^{n+1}(S^n;\mathbb{Z})$ where $\tilde{H}^{n+1}(S^n;\mathbb{Z}) = 0$, so q^* is surjective. Therefore, it is nontrivial because $\tilde{H}^{n+1}(X;\mathbb{Z}) \neq 0$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_n(X);\mathbb{Z}) & \longrightarrow & H^{n+1}(X;\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(H_{n+1}(X);\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ext}(H_n(X/S^n);\mathbb{Z}) & \longrightarrow & H^{n+1}(X/S^n;\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(H_{n+1}(X/S^n);\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_m & \longrightarrow & \mathbb{Z}_m & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

This splitting is not natural because the middle term in the first sequence is isomorphic to $\mathbb{Z}_m \oplus 0$ and the second one is $0 \oplus \mathbb{Z}$.

The long exact sequence of a pair gives us $\tilde{H}_n(S^n;\mathbb{Z}) \to \tilde{H}_n(X;\mathbb{Z}) \to \tilde{H}_n(X, S^n;\mathbb{Z}) = \tilde{H}_n(S^{n+1};\mathbb{Z}) = 0$ which implies the surjectivity of the induced map. Since $\tilde{H}_n(X;\mathbb{Z}) \neq 0$, the induced map is nonzero.

The map induced on $\tilde{H}^i(-;\mathbb{Z})$ is the zero map for any i because at least one of $\tilde{H}^i(S^n;\mathbb{Z})$ or $\tilde{H}^i(X;\mathbb{Z})$ is 0 for each i.

Exercise. (3.1.13)

Exercise. (3.2.1) $H^0(M_g) = H^2(M_g) = \mathbb{Z}$ and $H^1(M_g) = \mathbb{Z}^{2g}$. Thus the only nontrivial cup products are elements among $H^1(M_g)$. Let $a_1, \cdots, a_g, b_1, \cdots, b_g$ be generators of $H^1(M_g)$. Let q be the quotient map M_g to a wedge sum of g tori. Then $q^* : H^1(M_g) \to H^1(\vee_g M_1) =$

$\oplus_g H^1(M_1)$ where the ith $H^1(M_1)$ is generated by A_i, B_i and $q^*(A_i) = a_i$ and $q^*(B_i) = b_i$. Since cup products are natural, it suffices to check products between $q^*(A_i)$ and $q^*(B_i)$.

How do I calculate these?

Exercise. (3.2.2) Suppose X is the union of contractible open sets $A_1, \cdots, A_n$. Since each A_i is contractible, $H^k(X, A_i; R) = H^k(X; R)$ for all $k \geq 1$.

$$\begin{tikzcd}
H^{k_1}(X, A_1; R) \times \cdots \times H^{k_n}(X, A_n; R) \arrow[r] \arrow[d, "\cong"] & H^{k_1+\cdots+k_n}(X, A_1 \cup \cdots \cup A_n; R) \arrow[d] \\
H^{k_1}(X; R) \times \cdots \times H^{k_n}(X; R) \arrow[r, "f"] & H^{k_1+\cdots+k_n}(X; R).
\end{tikzcd}$$

This diagram commutes by the naturality of a cup product. $H^{k_1+\cdots+k_n}(X, \bigcup_i A_i; R) = H^{k_1+\cdots+k_n}(X, X; R) = 0$ for all $k + l \geq 1$. By the commutativity of this diagram, the function f must be 0.

Exercise. (3.2.3(a)) Suppose otherwise. Let $f : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m$ be such a function. Then f induces a map on $f^* : H^*(\mathbb{R}\mathrm{P}^m) \to H^*(\mathbb{R}\mathrm{P}^n)$. In other words, $f^* : \mathbb{Z}_m[\alpha]/(\alpha^{m+1}) \to \mathbb{Z}_n[\beta]/(\beta^{n+1})$ where α, β are generators of $H^1(\mathbb{R}\mathrm{P}^m)$ and $H^1(\mathbb{R}\mathrm{P}^n)$. $H^1(\mathbb{R}\mathrm{P}^m; \mathbb{Z}_2) = \mathbb{Z}_2 = \{0, \alpha\}$ and $H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) = \mathbb{Z}_2 = \{0, \beta\}$. Since f induces a nontrivial map $H^1(\mathbb{R}\mathrm{P}^m; \mathbb{Z}_2) \to H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$, $f^*(\alpha) = \beta$. However, $f^*(0) = f^*(\alpha^{m+1}) = (f^*(\alpha))^{m+1} = \beta^{m+1} \neq 0$ because $m < n$. This is a contradiction, so such a function does not exist.

$H^1(\mathbb{C}P^n; \mathbb{Z}_2) = 0$ for any n, so there exists no such nontrivial map. The case for $H^2(\mathbb{C}P^n)$ can be argued the same way as above because $H^2(\mathbb{C}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ where α is a generator of $H^2(\mathbb{C}P^n)$.

Exercise. (3.2.3(b)) Suppose $n \geq 2$ because if $n = 1$, then this can be shown using the intermediate value theorem.

$$\begin{tikzcd}
S^n \arrow[r, "g"] \arrow[d, "p"] & S^{n-1} \arrow[d, "p"] \\
\mathbb{R}\mathrm{P}^n \arrow[r, "g"] & \mathbb{R}\mathrm{P}^{n-1}.
\end{tikzcd}$$

Let p denote covering maps. Let γ be a nontrivial loop in $\mathbb{R}\mathrm{P}^n$. Let $a, -a$ denote the end points of the lift $\tilde{\gamma}$. $g(-a) = -g(a)$, so g sends $\tilde{\gamma}$ to a path from $g(a)$ to $g(-a)$. Finally, p pushes it down to a nontrivial loop in $\mathbb{R}\mathrm{P}^{n-1}$. By the commutativity of the diagram, $g(\gamma)$ is a nontrivial path in $\mathbb{R}\mathrm{P}^{n-1}$. Therefore, f induces a nontrivial map from $\pi_1(\mathbb{R}\mathrm{P}^n)(\cong \mathbb{Z}_2)$ to $\pi_1(\mathbb{R}\mathrm{P}^{n-1})(\cong \mathbb{Z}_2)$. Thus f induces an isomorphism. Since the fundamental groups are abelian, the fundamental groups are isomorphic to the first homology groups. By the UCT, $H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) = \mathrm{Hom}(H_1(\mathbb{R}\mathrm{P}^n), \mathbb{Z}_2) = \mathbb{Z}_2$ and $H^1(\mathbb{R}\mathrm{P}^{n-1}; \mathbb{Z}_2) = \mathrm{Hom}(H_1(\mathbb{R}\mathrm{P}^{n-1}), \mathbb{Z}_2) = \mathbb{Z}_2$. Then f induces an isomorphism from $H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ into $H^1(\mathbb{R}\mathrm{P}^{n-1}; \mathbb{Z}_2)$. This is a contradiction as shown in 3(a).

Exercise. (3.2.6)

Exercise. (3.2.7) Let $f : \mathbb{R}\mathrm{P}^3 \to \mathbb{R}\mathrm{P}^2 \vee S^3$ be a homotopy equivalence. Then it induces isomorphisms.

$$\begin{tikzcd}
H^1(\mathbb{R}\mathrm{P}^3;\mathbb{Z}_2) \arrow[d, "f^*"] & \times & H^2(\mathbb{R}\mathrm{P}^3;\mathbb{Z}_2) \arrow[r] \arrow[d, "f^*"] & H^3(\mathbb{R}\mathrm{P}^3;\mathbb{Z}_2) \arrow[d, "f^*"] \\
H^1(\mathbb{R}\mathrm{P}^2 \vee S^3;\mathbb{Z}_2) & \times & H^2(\mathbb{R}\mathrm{P}^2 \vee S^3;\mathbb{Z}_2) \arrow[r] & H^3(\mathbb{R}\mathrm{P}^2 \vee S^3;\mathbb{Z}_2).
\end{tikzcd}$$

The cohomology groups of a wedge sum is the direct sum of cohomology groups of the two spaces. By rewriting the diagram above with generators, we obtain

$$\begin{tikzcd}
\{0,\alpha\} \arrow[d, "f^*"] & \times & \{0,\alpha^2\} \arrow[r] \arrow[d, "f^*"] & \{0,\alpha^3\} \arrow[d, "f^*"] \\
\{0,\beta\} \oplus \{0,\gamma\} & \times & \{0,\beta^2\} \oplus 0 \arrow[r] & 0 \oplus \{0,\gamma^2\}.
\end{tikzcd}$$

This implies f^* sends α^2 to $(\beta^2, 0)$ and α^3 to $(0, \gamma^2)$. However, this implies $(0,0) = (f^*(\alpha^2))^3 = (f^*(\alpha^3))^2 = (0, \gamma^4) = (0, \gamma)$. This is a contradiction because $0 \neq \gamma$.