# MATH 611 (DUE 10/2)

HIDENORI SHINOHARA

**Exercise.** (Problem 10, Chapter 1.3) Find all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$, up to isomorphisms of covering spaces without base points.

*Proof.* Let $X = S^1 \vee S^1$. By the discussion on P.70 of the textbook, we know that $n$-sheeted covering spaces of $X$ are classified by equivalence classes of homomorphisms $\pi_1(X, x_0) \to S_n$. Let $a, b$ denote paths in $X$ as in Figure 1. We can identify each homomorphism $\phi$ by checking what $\phi$ maps $a$ and $b$ to. (Strictly speaking, $\pi_1(X, x_0)$ is generated by $[a], [b]$, but we will abuse notations by writing $a$ and $b$ instead of $[a], [b]$.)

The following are all the cases. Figure 2 shows the corresponding graphs.

- Case 1: $\phi_1(a) = \phi_1(b) = (1)$. The space that corresponds to this homomorphism is disconnected.
- Case 2: $\phi_2(a) = (12), \phi_2(b) = (1)$. This generates a connected covering space.
- Case 3: $\phi_3(a) = (1), \phi_3(b) = (12)$. This generates a connected covering space.
- Case 4: $\phi_4(a) = (12), \phi_4(b) = (12)$. This generates a connected covering space.

$\phi_1 \neq \phi_2$ and $(12)\phi_1(12) \neq \phi_2$, so $\phi_1$ and $\phi_2$ are not conjugates of each other. Similarly, $\phi_2$ and $\phi_3$ are not conjugates of each other, and neither are $\phi_1$ and $\phi_3$.

Thus the three graphs corresponding to Case 2, 3 and 4 in Figure 2 are all the 2-sheeted covering spaces of $X$.

We will take the exact same approach for the case of 3. If a certain vertex is fixed in both $\phi(a)$ and $\phi(b)$, then such a vertex is disjoint from the rest of the graph. We will use that property to reduce the possibilities.

- Case 1: $\phi_1 : a \mapsto (1), b \mapsto (1)$ The following maps are conjugates of $\phi_1$
  - $a \mapsto (1), b \mapsto (1)$

  This graph is not connected because every vertex is fixed.
- Case 2: $\phi_2 : a \mapsto (12), b \mapsto (1)$ The following maps are conjugates of $\phi_2$
  - $a \mapsto (23), b \mapsto (1)$
  - $a \mapsto (13), b \mapsto (1)$
  - $a \mapsto (12), b \mapsto (1)$

  This graph is not connected because vertex 3 is fixed.
- Case 3: $\phi_3 : a \mapsto (1), b \mapsto (12)$ The following maps are conjugates of $\phi_3$

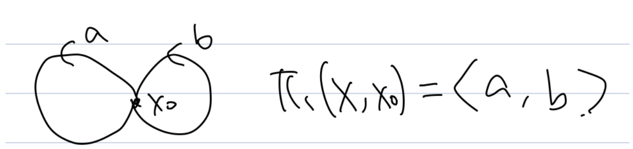


FIGURE 1. Problem 10 ($X = S^1 \vee S^1$)

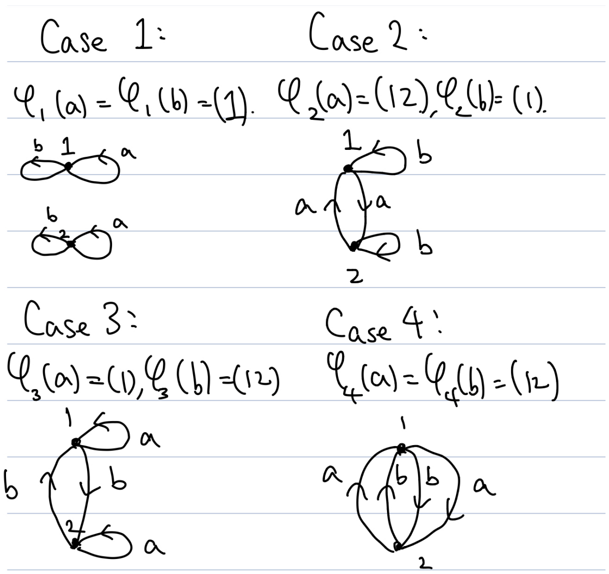


Figure 2. Problem 10 (2-sheeted covers)

  - $a \mapsto (1), b \mapsto (12)$
  - $a \mapsto (1), b \mapsto (23)$
  - $a \mapsto (1), b \mapsto (13)$

  This is the same as Case 2.
- Case 4: $\phi_4 : a \mapsto (12), b \mapsto (13)$ The following maps are conjugates of $\phi_4$
  - $a \mapsto (13), b \mapsto (12)$
  - $a \mapsto (12), b \mapsto (23)$
  - $a \mapsto (12), b \mapsto (13)$
  - $a \mapsto (13), b \mapsto (23)$
  - $a \mapsto (23), b \mapsto (12)$
  - $a \mapsto (23), b \mapsto (13)$

  See Figure 3.
- Case 5: $\phi_5 : a \mapsto (12), b \mapsto (123)$ The following maps are conjugates of $\phi_5$
  - $a \mapsto (23), b \mapsto (123)$
  - $a \mapsto (12), b \mapsto (123)$
  - $a \mapsto (12), b \mapsto (132)$
  - $a \mapsto (13), b \mapsto (132)$
  - $a \mapsto (13), b \mapsto (123)$
  - $a \mapsto (23), b \mapsto (132)$

  See Figure 3.
- Case 6: $\phi_6 : a \mapsto (123), b \mapsto (12)$ The following maps are conjugates of $\phi_6$
  - $a \mapsto (123), b \mapsto (13)$
  - $a \mapsto (132), b \mapsto (12)$
  - $a \mapsto (132), b \mapsto (23)$
  - $a \mapsto (132), b \mapsto (13)$
  - $a \mapsto (123), b \mapsto (12)$
  - $a \mapsto (123), b \mapsto (23)$

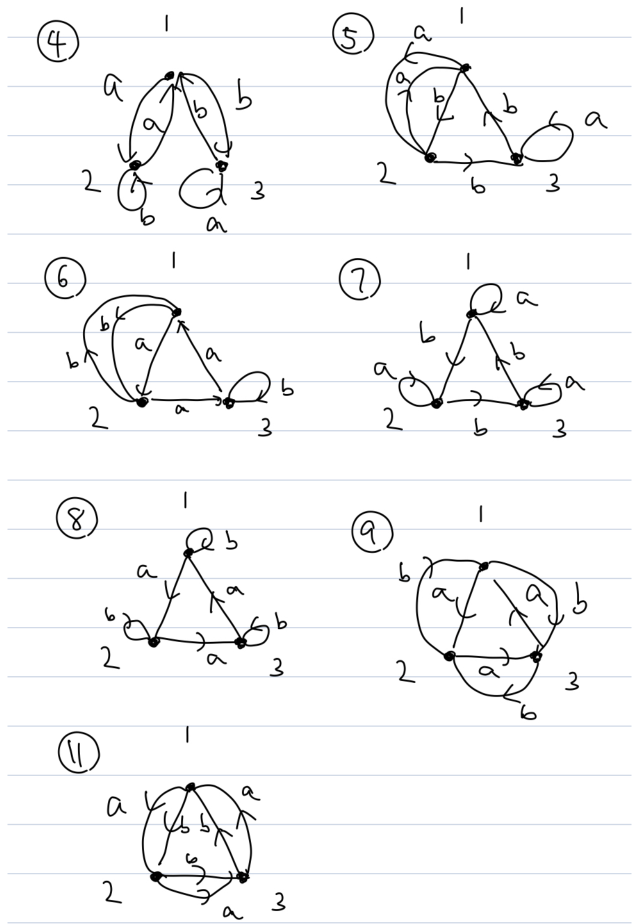


FIGURE 3. Problem 10 (3-sheeted)

See Figure 3.

- Case 7: $\phi_7 : a \mapsto (1), b \mapsto (123)$ The following maps are conjugates of $\phi_7$
  - $a \mapsto (1), b \mapsto (132)$
  - $a \mapsto (1), b \mapsto (123)$

  See Figure 3.
- Case 8: $\phi_8 : a \mapsto (123), b \mapsto (1)$ The following maps are conjugates of $\phi_8$
  - $a \mapsto (132), b \mapsto (1)$
  - $a \mapsto (123), b \mapsto (1)$

  See Figure 3.
- Case 9: $\phi_9 : a \mapsto (123), b \mapsto (132)$ The following maps are conjugates of $\phi_9$
  - $a \mapsto (123), b \mapsto (132)$
  - $a \mapsto (132), b \mapsto (123)$

  See Figure 3.
- Case 10: $\phi_{10} : a \mapsto (23), b \mapsto (23)$ The following maps are conjugates of $\phi_{10}$
  - $a \mapsto (12), b \mapsto (12)$
  - $a \mapsto (23), b \mapsto (23)$
  - $a \mapsto (13), b \mapsto (13)$

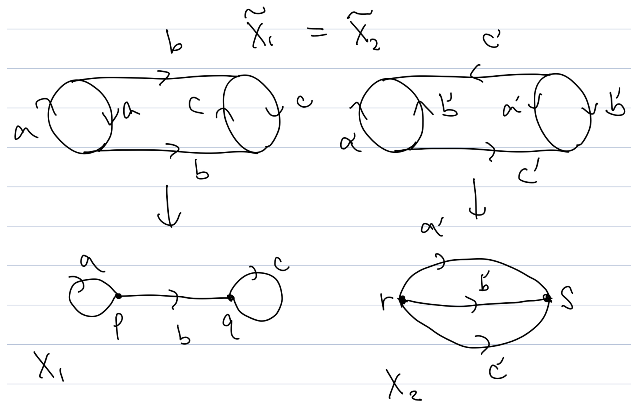


FIGURE 4. Problem 11

Vertex 1 is disconnected from the rest of the graph since it is fixed.

- Case 11: $\phi_{11} : a \mapsto (123), b \mapsto (123)$ The following maps are conjugates of $\phi_{11}$
  - $a \mapsto (132), b \mapsto (132)$
  - $a \mapsto (123), b \mapsto (123)$

  See Figure 3.

Since there are 6 elements in $S_3$, there are 36 possible homomorphisms. The list above contains all of them. Therefore, Figure 3 lists all the possible 3-sheeted covers. □

**Exercise.** (Problem 11, Chapter 1.3) Construct finite graphs $X_1$ and $X_2$ having a common finite-sheeted covering space $\tilde{X}_1 = \tilde{X}_2$, but such that there is no space having both $X_1$ and $X_2$ as covering spaces.

*Proof.* Figure 4 shows $X_1, X_2$ and $\tilde{X}_1 = \tilde{X}_2$.

We claim that there exists no space having both $X_1$ and $X_2$ as covering spaces. On the contrary, suppose there exists such a space $X$ with covering maps $p_1 : X_1 \to X, p_2 : X_2 \to X$. Then every point in $X$ must have a neighborhood that homeomorphic to an open subset of $X_1$. Since $X_1$ is a graph, that means $X$ is locally a line and a vertex with edges. In other words, $X$ must be a graph.

There must exist a neighborhood of $p_1(p)$ and a neighborhood of $p$ such that they are homeomorphic. Since $p$ is a vertex of degree 3, $p_1(p)$ must be a vertex of degree 3 as well. Similarly, $p_1(q)$ must be a vertex of degree 3 as well.

Since $p, q$ are the only vertices of $X_1$, $X$ contains at most two vertices and their degrees must be 3. Since the sum of degrees of all vertices must be even from elementary graph theory, $X$ must contain two vertices of degree 3.

If $X$ only consists of loops, then the degree of each vertex will be even. Thus the two vertices must be joined by at least one edge. Then if one vertex has a loop, the other must have a loop as well in order to have degree 3. If there exists another edge joining the two vertices, there must be a third one in order for the two vertices to have degree 3. Therefore, $X_1, X_2$ are the only graphs with two vertices of degree 3.

Suppose that $X_1$ is a covering space of $X_2$ with a covering map $f : X_1 \to X_2$. Without loss of generality, $f(p) = r, f(q) = s$. Consider the path $a'$ in $X_2$. Lifting $a'$ to $X_1$ will result in a path from $p$ to $q$. This implies that $f$ maps points on the path $b$ into points on a path $a'$.

Now consider the path $b'$ in $X_2$. Lifting $b'$ to $X_1$ will again result in a path from $p$ to $q$. This implies that $f$ maps points on the path $b$ into points on a path $b'$.

This implies that every point on the path $b$ must be mapped to $r$ or $s$. This is a contradiction because $f$ is continuous and $\{b(t) \mid t \in [0,1]\}$ is connected, but $\{r, s\}$ is disconnected.

Thus $X_1$ is not a covering space of $X_2$.

Similarly, suppose that $X_2$ is a covering space of $X_1$ with a covering map $g : X_2 \to X_1$. Without loss of generality, $g(r) = p, g(s) = q$. This implies $g^{-1}(p) = \{r\}$, so the number of sheets is 1. In other words, $g$ is injective. Consider the path $a$ in $X_1$. Lifting $a$ to $X_2$ results into a loop based at $r$. Since $a : I \to X_1$ is injective, $\tilde{a} : I \to X_2$ is injective since $g \circ \tilde{a} = a$. Then $\tilde{a}(t) = s$ for some $t \in [0,1]$, so $a(t) = g(\tilde{a}(t)) = g(s) = q$. However, $q$ is not a point on $a$. This is a contradiction, so $X_2$ is not a covering space of $X_1$.

Hence, there exists no space that has both $X_1$ and $X_2$ as covering spaces. □

**Exercise.** (Problem 14, Chapter 1.3) Find all the connected covering spaces of $\mathbb{R}\mathbf{P}^2 \vee \mathbb{R}\mathbf{P}^2$.

*Proof.* Let $X = \mathbb{R}\mathbf{P}^2 \vee \mathbb{R}\mathbf{P}^2$. By Theorem 1.38 of the textbook, it suffices to check all the conjugacy classes of subgroups of $\pi_1(X, x_0)$.

Since $\pi_1(\mathbb{R}\mathbf{P}^2) = \langle a \mid a^2 \rangle$, $\pi_1(X, x_0) = \langle a, b \mid a^2 = b^2 = e \rangle$ by Van Kampen. Since $a^2 = b^2 = e$, we can express each element in $\pi_1(X, x_0)$ uniquely as a word which alternates $a, b$.

Here are all the conjugacy classes of subgroups:

(1) Conjugacy class represented by $\langle e \rangle$.
(2) Conjugacy class represented by $\langle a \rangle$. This conjugacy class contains $\langle bab \rangle, \langle ababa \rangle, \langle bababab \rangle, \cdots$.
(3) Conjugacy class represented by $\langle b \rangle$. This conjugacy class contains $\langle aba \rangle, \langle babab \rangle, \langle abababa \rangle, \cdots$.
(4) For each $k \in \mathbb{N}$, there exists a conjugacy class that only consists of $\left\langle (ab)^k \right\rangle$.
(5) For each word $w$ that starts and ends with $b$, there exists a conjugacy class represented by $\langle a, w \rangle$ For each $w$, $\langle bab, bwb \rangle, \langle ababa, abwba \rangle, \cdots$ are the elements in the conjugacy class of $\langle a, w \rangle$. Each conjugacy class of this type contains finitely many elements. For instance, when $w = bababab$, $\langle a, bababab \rangle, \langle bab, ababa \rangle, \langle ababa, bab \rangle, \langle bababab, a \rangle$ are the only elements in this class.
(6) Conjugacy class that only consists of $\pi_1(X, x_0)$. The covering space that corresponds to this is $X$ itself. Since this is a trivial case, Figure 5 does not contain this case.

Figure 5 shows covering spaces corresponding to each conjugacy class.

We will prove that we have listed all the conjugacy classes, and that the classes we listed are indeed disjoint from each other.

- All classes?
- First, it is easy to see that conjugacy class 1 is indeed disjoint from the other classes.

  We will check that the other classes are disjoint. The parity of the number of $a$ in a word is independent of a representation of a word. For instance, $aaa = a$, and each of them contains an odd number of $a$.

  Prove this?

  Moreover, the parity of the number of $a$ in a word is invariant under conjugation because conjugation appends an even number of $a$.

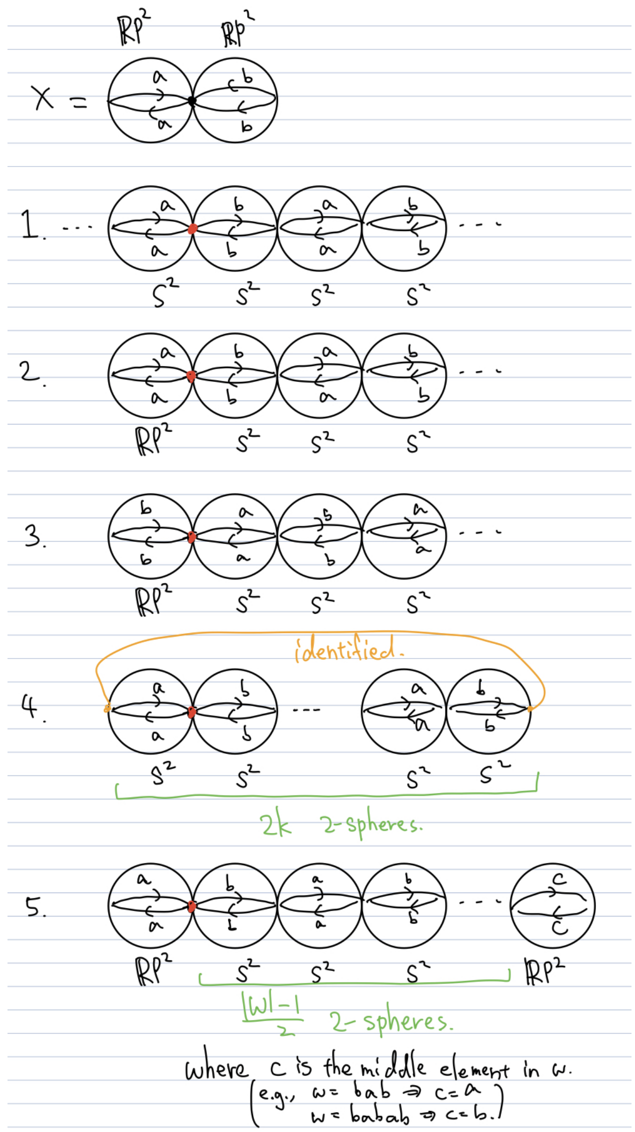


FIGURE 5. Problem 14

Similarly, the parity of the number of $b$ in a word is invariant under conjugation. Using this property, we can see that conjugacy class 2 is indeed disjoint from conjugacy class 3. Similarly, this implies that $2 \neq 4, 3 \neq 4, 4 \neq 5$. Thus it remains to show that conjugacy class 2 is disjoint from any conjugacy class 5, and any conjugacy classes 4 are disjoint for different $k$ values.

- $2 \neq 5$? For any word $w$, $w \langle a \rangle w^{-1} = \langle waw^{-1} \rangle$.
  * If $waw^{-1}$ starts and ends with $a$, then $\langle waw^{-1} \rangle$ does not contain any element that starts and ends with $b$.
  * If $waw^{-1}$ starts with $a$ and ends with $b$, then $\langle waw^{-1} \rangle$ does not contain $a$.
  * If $waw^{-1}$ starts with $b$ and ends with $a$, then $\langle waw^{-1} \rangle$ does not contain $a$.
  * If $waw^{-1}$ starts with $b$ and ends with $b$, then $\langle waw^{-1} \rangle$ does not contain $a$.
- 4?