MATH 620 (9/17)

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Exercise. Prove $\omega = g^{-1}dg$.

Proof.

$$\begin{aligned}
g^{-1}dg &= g^{-1}d\begin{bmatrix} 1 & 0 \\ x & A \end{bmatrix} \\
&= g^{-1}\begin{bmatrix} 0 & 0 \\ dx & dA \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ x & A \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 \\ dx & dA \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ x & A \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 \\ \omega^i A_i & \omega_i^j A_j \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ -A^{-1}x & A^{-1} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \omega^i A_i & \omega_i^j A_j \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ \omega^i & \omega_j^i \end{bmatrix} \\
&= \omega.
\end{aligned}$$

□

Exercise. Prove that $V \times \mathrm{Aut}(V)$ is a group.

Proof.

- Associativity. Let $(u, \phi), (v, \psi), (w, \rho)$ be given.

$$\begin{aligned}
((u, \phi) \cdot (v, \psi)) \cdot (w, \rho) &= (u + \phi v, \phi \circ \psi) \cdot (w, \rho) \\
&= (u + \phi v + (\phi \circ \psi)w, (\phi \circ \psi) \circ \rho) \\
&= (u + \phi v + (\phi \circ \psi)w, \phi \circ (\psi \circ \rho)) \\
&= (u + \phi v + \phi(\psi w), \phi \circ (\psi \circ \rho)) \\
&= (u + \phi(v + \psi w), \phi \circ (\psi \circ \rho)) \\
&= (u, \phi) \cdot (v + \psi w, \psi \circ \rho) \\
&= (u, \phi) \cdot ((v, \psi) \cdot (w, \rho)).
\end{aligned}$$

- Identity. Let $u = 0, \phi = \mathrm{Id}$. Then for any $(v, \psi) \in V \times \mathrm{Aut}(V)$,
  - $(u, \phi) \cdot (v, \psi) = (u + \phi v, \phi \circ \psi) = (0 + \mathrm{Id} \circ v, \mathrm{Id} \circ \psi) = (v, \psi)$,
  - $(v, psi) \cdot (u, \phi) = (v + \psi u, \psi \circ \phi) = (v + \mathrm{Id} \circ u, \psi \circ \mathrm{Id}) = (v, \psi)$.

  Thus (u, ϕ) is the identity.

- Inverse. Let $(u, \phi) \in V \times \mathrm{Aut}(V)$ be given. Then $\phi^{-1} \in \mathrm{Aut}(V)$, and thus $-\phi^{-1}(u) \in V$.
  - $(-\phi^{-1}(u), \phi^{-1}) \cdot (u, \phi) = (-\phi^{-1}(u) + \phi^{-1}(u), \phi^{-1} \circ \phi) = (0, \mathrm{Id})$.
  - $(u, \phi) \cdot (-\phi^{-1}(u), \phi^{-1}) = (u + \phi(-\phi^{-1}(u)), \phi \circ \phi^{-1}) = (0, \mathrm{Id})$.

Therefore, $V \times \mathrm{Aut}(V)$ forms a group. $\square$