MATH 611 (DUE 10/23)

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1. Simplicial and Singular Homology

Exercise. (Problem 2) Show that the Δ-complex obtained from Δ^3 by performing the edge identifications $[v_0, v_1] \sim [v_1, v_3]$ and $[v_0, v_2] \sim [v_2, v_3]$ deformation retracts onto a Klein bottle. Find other pairs of identifications of edges that produce Δ-complexes deformation retracting onto a torus, a 2-sphere, and $\mathbb{R}\mathbf{P}^2$.

Proof. The deformation retraction of Δ^3 onto a Klein bottle is described in 1. We will start by "pushing" Δ^3 from edge (v_1, v_2). This will leave the surface that consists of the triangles $[v_0, v_1, v_3]$ and $[v_0, v_2, v_3]$. (In other words, a diamond shape consisting of the vertices $[v_0, v_1, v_3, v_2]$.) Step 2 in Figure 1 is what Δ^3 should look like after the deformation retract. Step 3 through 6 show why this is a Klein bottle.

Figure 2 shows the identification of edges for a torus, 2-sphere, and $\mathbb{R}\mathbf{P}^2$.

□

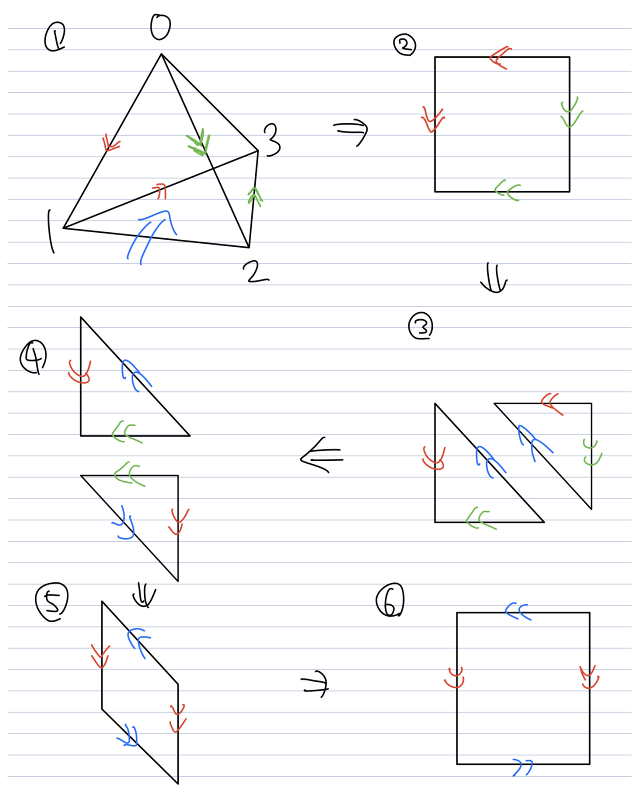

Figure 1. Problem 2(Klein Bottle)

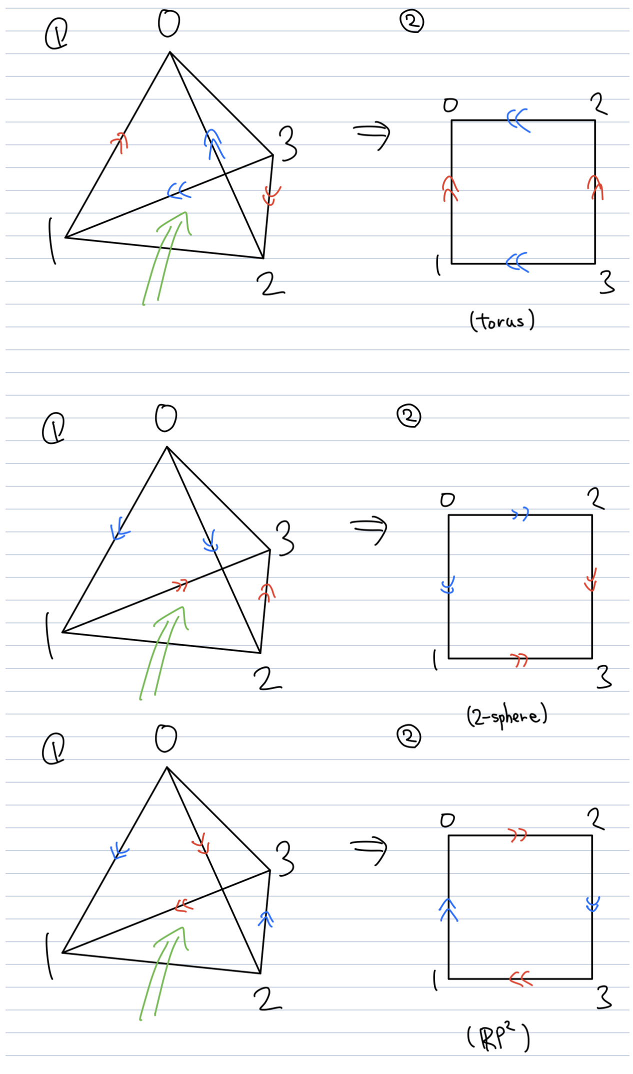

FIGURE 2. Problem 2(Torus, 2-Sphere, $\mathbb{R}\mathbf{P}^2$)

Exercise. (Problem 4) Compute the simplicial homology groups of the triangular parachute obtained from Δ^2 by identifying its three vertices to a single point.

Proof. Let v_0 denote the only vertex, e_1, e_2, e_3 denote the three edges of the parachute, and σ denote the face of the parachute as in Figure 3. $C_k = 0$ for $k \geq 3$ because Δ^2 with the vertices identified does not contain any k-dimensional simplicies for $k \geq 3$. $C_2 = \langle \sigma \rangle, C_1 = \langle e_1, e_2, e_3 \rangle, C_0 = \langle v_0 \rangle$. For each n, ∂_n is defined such that $\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha | [v_0, \cdots, \hat{v}_i, \cdots, v_n]$.

- $\partial_2(\sigma) = e_3 - e_2 + e_1$.
- $\partial_1(e_1) = v - v = 0$. Similarly, $\partial_1(e_2) = \partial_1(e_3) = 0$.
- ∂_0 and ∂_3 are both the zero map.

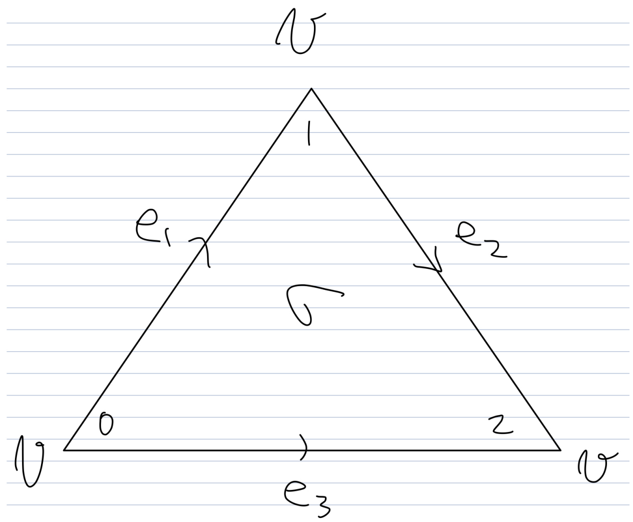

FIGURE 3. Problem 4

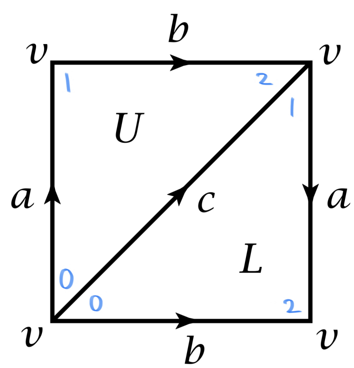

FIGURE 4. Problem 5

Thus

$$H_n = \begin{cases} \{0\} & (n \geq 3) \\ \ker(\partial_2)/\operatorname{Im}(\partial_3) = 0/0 \cong 0 & (n = 2) \\ \ker(\partial_1)/\operatorname{Im}(\partial_2) = \langle e_1, e_2, e_3 \rangle / \langle e_3 - e_2 + e_1 \rangle \cong \langle e_1, e_2, -e_2 + e_1 \rangle \cong \mathbb{Z}^2 & (n = 1) \\ \ker(\partial_0)/\operatorname{Im}(\partial_1) = \langle v \rangle /0 \cong \mathbb{Z} & (n = 0). \end{cases}$$

□

Exercise. (Problem 5) Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure described at the beginning of this section.

Proof. We will use the notations in Figure 4.

$$C_n = \begin{cases} 0 & (n \geq 3) \\ \langle U, L \rangle & (n = 2) \\ \langle a, b, c \rangle & (n = 1) \\ \langle v \rangle & (n = 0). \end{cases}$$

$\partial_n = 0$ for $n \geq 3$ and $n = 0$.

$$\begin{aligned} \partial_2(U) &= \sum_{i=0}^{2} (-1)^i \sigma|[0,1,2] \\ &= \sigma|[1,2] - \sigma|[0,2] + \sigma|[0,1] \\ &= b - c + a. \end{aligned}$$

$$\begin{aligned} \partial_2(L) &= \sum_{i=0}^{2} (-1)^i \sigma|[0,1,2] \\ &= \sigma|[1,2] - \sigma|[0,2] + \sigma|[0,1] \\ &= a - b + c. \end{aligned}$$

$\partial_1(a) = 0$ since $\partial_1(a) = \sigma|[1] - \sigma|[0] = v - v = 0$. Similarly, $\partial_1(b) = \partial_1(c) = 0$. Thus

$$H_n = \begin{cases} \{0\} & (n \geq 3) \\ \langle \sigma \rangle \cong \mathbb{Z} & (n = 2) \\ \langle e_1, e_2, e_3 \rangle \cong \mathbb{Z}^3 & (n = 1) \\ \langle v_0 \rangle \cong \mathbb{Z} & (n = 0). \end{cases}$$

Finish the rest.

□

Exercise. (Problem 7) Find a way of identifying pairs of faces of Δ^3 to produce a Δ-complex structure on S^3 having a single 3-simplex, and compute the simplicial homology groups of this Δ-complex.

Exercise. (Problem 8) Construct a 3 dimensional Δ-complex X from n tetrahedra $T_1, \cdots, T_n$ by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure, so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1}, subscripts being taken mod n. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show the simplicial homology groups of X in dimensions 0, 1, 2, 3 are $\mathbb{Z}, \mathbb{Z}_n, 0, \mathbb{Z}$, respectively.

Proof. □