MATH 633 HOMEWORK 6

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Exercise. (1) Define the map $f : H \to \Omega_1$ such that $f(z) = \exp(\log(z)/\alpha)$ where log denotes the principal branch of the complex logarithm function. This is well defined because H does not contain the real line. Moreover, this is holomorphic because it is the composition of holomorphic functions. Finally, $f'(z) = \exp(\log(z)/\alpha)/z \neq 0$ on H. Thus f is conformal.

Exercise. (2) $z \mapsto az + b$ and $z \mapsto cz + d$ are clearly entire. If $c = 0$, then $\phi : z \mapsto (az + b)/(cz + d)$ is entire. If $c \neq 0$, then ϕ is holomorphic everywhere except for $-d/c$ and at $-d/c$, ϕ has a pole because $\phi(-d/c) = \infty$. In other words, it is meromorphic.

Let $\phi : z \mapsto (az + b)/(cz + d)$ and $\psi : z \mapsto (-dz + b)/(cz - a)$. Then $\phi(\psi(z)) = z$ and $\psi(\phi(z)) = z$, and $(-d)(-a) - bc = ad - bc \neq 0$.

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a bijective meromophism. f is actually just holomorphic because f cannot have two poles since it is injective. Let g be a mobius transformation that sends $f(\infty)$ to ∞. Then $g \circ f$ is a bijective map on $\hat{\mathbb{C}}$ into $\hat{\mathbb{C}}$. Since $g \circ f$ sends ∞ to ∞, $g \circ f$ is a bijection on $\mathbb{C}$. If $h = g \circ f$ is a polynomial, it must be linear by the previous homework. Then $f = g^{-1} \circ h$ is a Mobius transformation. Suppose $h = g \circ f$ is not a polynomial. Then $(g \circ f)(\{|z| < 1\})$ is open because $(g \circ f)$ is a continuous bijection. $(g \circ f)(\{|z| > 1\})$ is dense in $\mathbb{C}$ because $z \mapsto (g \circ f)(1/z)$ has an essential singularity around 0. However, this implies there exist $|z_1| > 1, |z_2| < 1$ such that $(g \circ f)(z_1) = (g \circ f)(z_2)$. This is a contradiction because $g \circ f$ is bijective. Therefore, this case is not possible.

Exercise. (3) f is entire, so it has a power series expansion $\sum a_n z^n$. Then f can be extended to a function on $\hat{\mathbb{C}}$ in a canonical way.

If f has a removable singularity at ∞, then f is bounded in a neighborhood N containing ∞. Then N^c is a compact subset of $\mathbb{C}$, so f is bounded on N^c. Therefore, f is bounded on $\mathbb{C}$, so f is constant, which is a contradiction because f must be bijective.

Suppose f has an essential singularity at ∞. Then $f(\hat{\mathbb{C}} \setminus D)$ is dense in $\mathbb{C}$ where D is the unit disk. This implies that $f(\hat{\mathbb{C}} \setminus D) \cap f(D) \neq \emptyset$, which contradicts the bijectivity of f.

Therefore, f has a pole at ∞. By Part (c) of Problem 2, f is a mobius transformation. $c = 0$ because $-d/c$ would be a pole otherwise. Thus $f = (a/d)z + (b/d)$ with $a/d \neq 0$.