MATH 602(HOMEWORK 3)

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1. Exercises

Exercise. (Exercise 1) The ideal generated by the three polynomials contains $-yz^4+yz^2+y=(xy^2-xz+y)-y(xy-z^2)+z(x-yz^4)$. However, its leading term $-yz^4$ is not in the ideal generated by the leading terms of the three polynomials.

Exercise. (Exercise 2)

Solve this.

Exercise. (Exercise 3)

Solve this.

Exercise. (Exercise 4) $0 \in \sqrt{0}$, $a, b \in \sqrt{0} \implies (a+b)^{m+n-1} = \sum_{i=0}^{m+n-1} \binom{m+n-1}{i} a^i b^{m+n-1-i} = 0$, and $\forall a \in \sqrt{0}, \forall x \in R, (ax)^n = a^n x^n = 0$, so $\sqrt{0}$ is an ideal.

Exercise. (Exercise 5)

Solve this.

Exercise. (Exercise 6) Tensoring an exact sequence with $M \otimes_A N$ is the same as tensoring it with M first and tensoring the resulting sequence with N later.

Exercise. (Exercise 7) Since $0 \to I \xrightarrow{i} R \xrightarrow{q} R/I \to 0$ is exact, $I \otimes M \to R \otimes M \to (R/I) \otimes M \to 0$ is exact.

$$\begin{aligned}(R/I) \otimes M &= \operatorname{im}(q \otimes \mathrm{Id}) \\ &\cong R \otimes M / \ker(q \otimes \mathrm{Id}) \\ &\cong R \otimes M / \operatorname{im}(i \otimes \mathrm{Id}) \\ &\cong R \otimes M / I \otimes M.\end{aligned}$$

Now consider $\phi : R \otimes M \to M/IM$ that is the composition of $R \otimes M \to M : x \otimes y \mapsto xy$ and $M \to M/IM : x \mapsto x + IM$. In other words, ϕ is $x \otimes y \mapsto xy + IM$. Because the two maps are both surjective, ϕ must be surjective. The kernel of ϕ is $I \otimes M$ because

- For any $x \otimes y \in I \otimes M$, $\phi(x \otimes y) = xy + IM = 0$ since $xy \in IM$.
- If $\phi(x \otimes y) = 0$, then $xy \in IM$. In other words, $xy = x'y'$ for some $x' \in I$ and $y' \in M$. Then $x \otimes y = 1 \otimes xy = 1 \otimes x'y' = x' \otimes y' \in I \otimes M$.

Therefore, $M/IM \cong (R \otimes M)/(I \otimes M) \cong (R/I) \otimes M$.

Exercise. (Exercise 8) Let $pa+qb = 1$ for some $p, q \in \mathbb{Z}$. Then $1 \otimes 1 = (pa+qb) \otimes (pa+qb) = pa \otimes pa + pa \otimes qb + qb \otimes pa + qb \otimes qb = 0+0+0+0 = 0$.

Exercise. (Exercise 9) Let T be a $\mathbb{Z}$-module and $f : \mathbb{Q} \times \mathbb{Q} \to T$ be a bilinear map. Then $f(a/b, c/d) = acf(1/b, 1/d) = acbf(1/b, 1/bd) = acf(1, 1/bd) = f(1, ac/bd)$. Define a bilinear map $h : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ by $(a, b) \mapsto ab$ and a linear map $g : \mathbb{Q} \to T$ by $a/b \mapsto f(1, a/b)$. Then $f = g \circ h$. The universal property of a tensor product is satisfied by $\mathbb{Q}$, so $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

Exercise. (Exercise 10) Let $a_1, \cdots, a_n, b_1, \cdots, b_m$ generate M' and M'', respectively. Let $x_1, \cdots, x_n, y_1, \cdots, y_m \in M$ be chosen such that x_i is the image of a_i and the image of y_j is b_j. We claim that x_i, y_j generate M. Let $x \in M$ be given. Then $q(x) = d_1 b_1 + \cdots + d_m b_m$ for some $d_i \in M$, and thus $q(x - d_1 y_1 - \cdots - d_m y_m) = 0$. Therefore, $x - d_1 y_1 - \cdots - d_m y_m = i(c_1 a_1 + \cdots + c_n a_n) = c_1 x_1 + \cdots + c_n x_n$, so $x = c_1 x_1 + \cdots + c_n x_n + d_1 y_1 + \cdots + d_m y_m$.

Exercise. (Exercise 11) This statement is not true. When $R = \mathbb{Z}$ and $I = (0)$, $I \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.
However, the statement is true if $I \neq 0$. Let $u \in I$ be a nonzero element.
Define $h : I \times K \to K$ by $(a, x/y) \mapsto ax/y$. Let $f \in \mathrm{Hom}(I \times K, T)$ be given.
Define $g : K \to T$ by $x/y \mapsto f(u, x/uy)$. Then

$$\begin{aligned}
(g \circ h)(a, x/y) &= g(h(a, x/y)) \\
&= g(ax/y) \\
&= f(u, \frac{ax}{yu}) \\
&= af(u, \frac{x}{yu}) \\
&= f(au, \frac{x}{yu}) \\
&= uf(a, \frac{x}{yu}) \\
&= f(a, \frac{xu}{yu}) \\
&= f(a, x/y).
\end{aligned}$$

Thus f, g, h commute and thus $K \cong I \otimes K$.