MATH 601 HOMEWORK (DUE 9/4)

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Exercise. (2.1) Show that the function $g : \mathbb{R} \to S^1$, $g(r) = \exp(2\pi i r)$, where $i^2 = -1$, satisfies the property that $g(r) = g(r')$ if and only if $r \sim r'$. Use this to explicitly construct a bijective map from the orbit space of the action to S^1, $g : \mathbb{R}/\sim = \mathbb{Z}\ \mathbb{R} \to S^1$.

Proof.

- Let $r, r' \in \mathbb{R}$ such that $r \sim r'$. Then $\mathbb{Z} * r = \mathbb{Z} * r'$. Since $0 * r = 0 + r = r \in \mathbb{Z} * r$, there must exist a $k \in \mathbb{Z}$ such that $k * r' = r$. Therefore, $k + r' = r$.

$$\begin{aligned} g(r) &= \exp(2\pi i r) \\ &= \exp(2\pi i(k + r')) \\ &= \exp(2\pi i k + 2\pi i r') \\ &= \exp(2\pi i k)\exp(2\pi i r') \\ &= \exp(2\pi i r') \\ &= g(r'). \end{aligned}$$

- Let $r, r' \in \mathbb{R}$ such that $g(r) = g(r')$.

$$\begin{aligned} \exp(2\pi i r) = \exp(2\pi i r') &\implies \exp(2\pi i(r - r')) = 1 \\ &\implies \cos(2\pi(r - r')) + i\sin(2\pi(r - r')) = 1 \\ &\implies \sin(2\pi(r - r')) = 0 \\ &\implies r - r' \in \mathbb{Z}. \end{aligned}$$

  Let $k = r - r'$. Since $r = 0 + r = 0 * r \in \mathbb{Z} * r$ and $r = k + r' = k * r' \in \mathbb{Z} * r'$, $(\mathbb{Z} * r) \cap (\mathbb{Z} * r') \neq \emptyset$. Since two equivalence classes are either disjoint or identical, this implies that $\mathbb{Z} * r = \mathbb{Z} * r'$. In other words, $r \sim r'$.

TODO □

Exercise. (2.2) Let $* : G \times S \to S$ be a left action of G. Show that $s \star g = g^{-1} * s$ defines a right action of G on S.

Proof. Let $s \in S, g, h \in G$ be given.

$$\begin{aligned}
(s \star g) \star h &= h^{-1} * (s \star g) \\
&= h^{-1} * (g^{-1} * s) \\
&= (h^{-1} g^{-1}) * s \\
&= (gh)^{-1} * s \\
&= s \star (gh).
\end{aligned}$$

Let $e \in G$ denote the identity element and let $s \in S$ be given.

$$\begin{aligned}
s \star e &= e^{-1} * s \\
&= e * s \\
&= s.
\end{aligned}$$

Therefore, $\star$ is indeed a right action of G on S. $\square$