MATH 620 HOMEWORK (DUE 9/10)

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Exercise. Show that $F_* : T_p\mathbb{R}^n \to T_q\mathbb{R}^m$.

Proof. Let $v_1, v_2 \in T_p\mathbb{R}^n, c \in \mathbb{R}$ be given. We will show that $F_*(cv_1 + v_2) = cF_*(v_1) + F_*(V_2)$. Choose γ_1, γ_2 be paths in U defined on a neighborhood of 0 in $\mathbb{R}$ such that $\gamma_1(0) = \gamma_2(0) = p, \gamma_1'(0) = v_1$ and $\gamma_2'(0) = v_2$. Then $F_*(v_1) = (F(\gamma_1(t)))' \mid_{t=0} = F'(\gamma_1(0))\gamma_1'(0) = F'(p)v_1$, and $F_*(v_2) = (F(\gamma_2(t)))' \mid_{t=0} = F'(\gamma_2(0))\gamma_2'(0) = F'(p)v_2$. Let $\gamma_3 : \mathbb{R} \to U$ be the constant path at p. Let $\gamma = c(\gamma_1 - \gamma_3) + \gamma_2$.

- $\gamma(0) = c\gamma_1(0) - c\gamma(0) + \gamma_2(0) = p$.
- $\gamma'(0) = c\gamma_1'(0) + \gamma_2'(0) = cv_1 + v_2$.

Therefore, $F_*(cv_1 + v_2) = (F \circ (c(\gamma_1 - \gamma_3) + \gamma_2))'(0) = F'(p)(c\gamma_1'(0) + \gamma_2'(0)) = F'(p)(cv_1 + v_2)$.

Hence, $F_*(cv_1 + v_2) = cF_*(v_1) + F_*(v_2)$, so F_* is indeed linear. $\square$