MATH 601 HOMEWORK (DUE 8/30)

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Exercise 0.1. Show that a bijective ring homomorphism is an isomorphism in the category of rings.

Proof. Let f be a bijective ring homomorphism from a ring A to a ring B.

Let $\mathbf{C}$ denote the category of rings. Then A, B are objects of the category $\mathbf{C}$. Since $\mathrm{Hom}_{\mathbf{C}}(A, B)$ is defined to be the set of all ring homomorphisms from A to B, $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$.

We will show that there exists an element $g \in \mathrm{Hom}_{\mathbf{C}}(B, A)$ such that $g \circ f = \mathrm{Id}_A$ and $f \circ g = \mathrm{Id}_B$.

Let a function $g : B \to A$ be defined such that $\forall b \in B, g(b) = a$ where a is an element such that $f(a) = b$. g is well-defined because:

- f is surjective, so there exists an $a \in A$ such that $f(a) = b$.
- f is injective, so such an a must be unique.

We claim that this g satisfies the desired properties:

- Claim 1: $g \in \mathrm{Hom}_{\mathbf{C}}(B, A)$. This is equivalent to showing that g is a ring homomorphism. Let $b_1, b_2 \in B$ be given. Let $a_1 = g(b_1), a_2 = g(b_2)$. Then $f(a_1) = b_1$ and $f(a_2) = b_2$.
  - Since f is a ring homomorphism, $f(a_1 + a_2) = f(a_1) + f(a_2) = b_1 + b_2$. Therefore, $g(b_1 + b_2) = a_1 + a_2 = g(b_1) + g(b_2)$.
  - Since f is a ring homomorphism, $f(a_1 \cdot a_2) = f(a_1) \cdot f(a_2) = b_1 \cdot b_2$. Therefore, $g(b_1 \cdot b_2) = a_1 \cdot a_2 = g(b_1) \cdot g(b_2)$.
  - Since f is a ring homomorphism, $f(1) = 1$. Thus $g(1) = 1$.

  Therefore, $g \in \mathrm{Hom}_{\mathbf{C}}(B, A)$.
- Claim 2: $g \circ f = \mathrm{Id}_A$. Let $a \in A$. Let $b = f(a)$. Then $g(b) = a$, so $g(f(a)) = a$. This implies that $\forall a \in A, g(f(a)) = a$. Thus $g \circ f = \mathrm{Id}_A$.
- Claim 3: $f \circ g = \mathrm{Id}_B$. Let $b \in B$. Let $a = g(b)$. Then $f(a) = b$, so $f(g(b)) = b$. Therefore, $\forall b \in B, f(g(b)) = b$. Thus $f \circ g = \mathrm{Id}_B$.

Therefore, f is indeed an isomorphism in the category of rings. $\square$

Exercise 0.2. Show that when products exist, they are essentially unique.

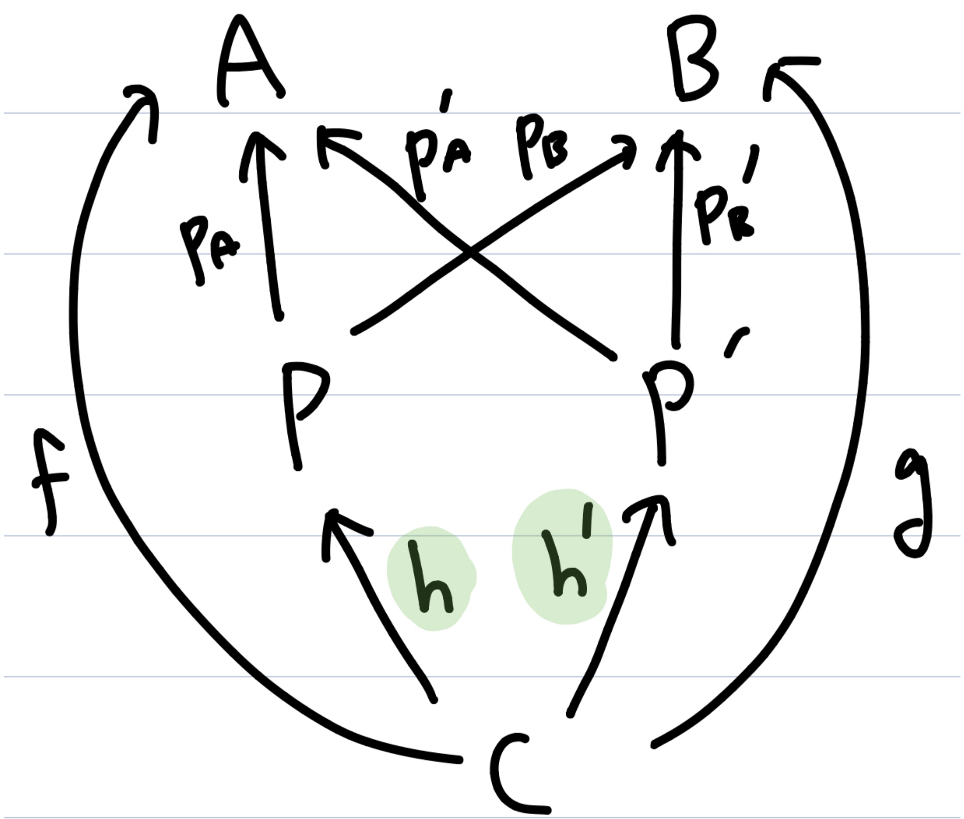

FIGURE 1. Diagram of maps for the second problem

Proof. First, we will consider the case when $C = P, f = p_A, g = p_B$.

- There exists a unique map $h_1 \in \mathrm{Hom}_{\mathbf{C}}(C, P) = \mathrm{Hom}_{\mathbf{C}}(P, P)$ such that $f = p_A \circ h_1$. In other words, $p_A = p_A \circ h_1$. By the uniqueness of h_1 and the equality $p_A = p_A \circ \mathrm{Id}_P$, we can conclude that $h_1 = \mathrm{Id}_P$.
- There exists a unique map $h_1' \in \mathrm{Hom}_{\mathbf{C}}(C, P') = \mathrm{Hom}_{\mathbf{C}}(P, P')$ such that $f = p_A' \circ h_1'$ and $g = p_B' \circ h_1'$ In other words, $p_A = p_A' \circ h_1'$ and $p_B = p_B' \circ h_1'$.

Similarly, we will consider the case when $C = P', f = p_A', g = p_B'$.

- There exists a unique map $h_2 \in \mathrm{Hom}_{\mathbf{C}}(C, P) = \mathrm{Hom}_{\mathbf{C}}(P', P)$ such that $f = p_A \circ h_2$. In other words, $p_A' = p_A \circ h_2$.
- There exists a unique map $h_2' \in \mathrm{Hom}_{\mathbf{C}}(C, P') = \mathrm{Hom}_{\mathbf{C}}(P', P')$ such that $f = p_A' \circ h_2'$. In other words, $p_A' = p_A' \circ h_2'$. By the uniqueness of h_2' and the equality $p_A' = p_A' \circ \mathrm{Id}_{P'}$, we can conclude that $h_2' = \mathrm{Id}_{P'}$.

$$\begin{aligned} p_A &= p'_A \circ h'_1 \\ &= (p_A \circ h_2) \circ h'_1 \\ &= p_A \circ (h_2 \circ h'_1) \end{aligned}$$

We showed earlier that Id_P is the only map that satisfies $p_A = p_A \circ \mathrm{Id}_P$, so $h_2 \circ h'_1 = \mathrm{Id}_P$.

$$\begin{aligned} p'_A &= p_A \circ h_2 \\ &= (p'_A \circ h'_1) \circ h_2 \\ &= p'_A \circ (h'_1 \circ h_2). \end{aligned}$$

Similarly, $\mathrm{Id}_{P'}$ is the only map such that $p'_A = p'_A \circ \mathrm{Id}_{P'}$, so $h'_1 \circ h_2 = \mathrm{Id}_{P'}$.

We showed that $h'_1 \in \mathrm{Hom}_{\mathbf{C}}(P, P')$ and $h_2 \in \mathrm{Hom}_{\mathbf{C}}(P,' P)$ satisfy $h_2 \circ h'_1 = \mathrm{Id}_P$ and $h'_1 \circ h_2 = \mathrm{Id}_{P'}$. In addition, we showed that $p_A = p'_A \circ h'_1$ and $p_B = p'_B \circ h'_1$. Therefore, h'_1 is the desired isomorphism. $\square$