

MATH 602 HOMEWORK 2

HIDENORI SHINOHARA

Exercise. (Problem 1) We will assume that the problem meant to say “ su with $s \in S \setminus \{0\}$ ” because it would be trivial otherwise. Choose $a_{n-1}, \dots, a_0 \in R$ such that $u^n + a_{n-1}u^{n-1} + \dots + a_1u + a_0 = 0$. If $a_0 = 0$, then $u(u^{n-1} + a_{n-1}u^{n-2} + \dots + a_1) = 0$. Since we are dealing with integral domains, this implies $u^{n-1} + a_{n-1}u^{n-2} + \dots + a_1 = 0$. By repeating this process, we obtain a monic polynomial with coefficients in R and a nonzero constant term that u satisfies.

Therefore, we may assume $a_0 \neq 0$. Then $u(a_1 + a_2u + \dots + a_{n-1}u^{n-2} + u^{n-1}) = -a_0 \in R$. Since $a_0 \neq 0$, $a_1 + a_2u + \dots + a_{n-1}u^{n-2} + u^{n-1}$ is a nonzero element in S . Hence, we showed that some multiple of u lives in R .

Exercise. (Problem 2) Let $R = \mathbb{Z}$ and $S = 2\mathbb{Z}$. $R \setminus S$ is not even an ideal because $0 \notin R \setminus S$. Thus $R \setminus S$ is not a prime ideal.

Exercise. (Problem 3)

Solve this!

Exercise. (Problem 4) Let $p \in \text{Spec}(R)$ such that $I \subset p$. Define $p/I = \{x + I \mid x \in p\} \subset R/I$. By the third isomorphism theorem of rings, p/I is an ideal of R/I . Let $x + I, y + I \in R/I$ and suppose $(x + I)(y + I) \in p/I$. Then $xy + I = z + I$ for some $z \in p$. Thus $xy - z \in I \subset p$ and $z \in p$. Thus $xy \in p$, so $x \in p$ or $y \in p$. This implies $x + I \in p/I$ or $y + I \in p/I$, so $p/I \in \text{Spec}(R/I)$.

On the other hand, let $A/I \subset \text{Spec}(R/I)$ be given. By the third isomorphism theorem of rings, every ideal of R/I must be of the form A/I where A is an ideal of R that contains I . Let $x, y \in R$ and suppose $xy \in A$. Then $xy + I \in A/I$, so $(x + I)(y + I) \in A/I$. Without loss of generality, $x + I \in A/I$. Then $x + I = a + I$ for some $a \in A$. Thus $x - a \in I \subset A$ and $a \in A$, so $x \in A$. Therefore, A is a prime ideal of R containing I .

Exercise. (Problem 5) By the second isomorphism of rings, $R \cap q$ is an ideal in R . Let $x, y \in R$. Suppose $xy \in R \cap q$. Then $xy \in q$, so $x \in q$ or $y \in q$. Without loss of generality, $x \in q$. Then $x \in R \cap q$. Thus $R \cap q$ is a prime ideal of R .