MATH 601 (DUE 11/13)

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Contents

1. Factoring Polynomials with Coefficients in Finite Fields

Exercise. (Problem 14) For $a \in \mathbb{F}_q$, what are the possible values for $a^{(q-1)/2}$? How many different a take each value?

Proof. Let $\langle \alpha \rangle = (\mathbb{F}_q)^*$. Let $k \in \mathbb{Z}$. If k is even, then $(\alpha^k)^{(q-1)/2} = (\alpha^{k/2})^{q-1} = 1$. If $k = 2l+1$ for some l, then $(\alpha^k)^{(q-1)/2} = \alpha^{l(q-1)} \cdot \alpha^{(q-1)/2} = \alpha^{(q-1)/2} = -1$ because -1 has degree 2 and $\alpha^{(q-1)/2}$ is the only element in $\langle \alpha \rangle$ of degree 2. Therefore,

$$a^{(q-1)/2} = \begin{cases} 0 & (a = 0) \\ 1 & (\exists l \in \mathbb{Z}, a = \alpha^{2l}) \\ -1 & (\exists l \in \mathbb{Z}, a = \alpha^{2l+1}). \end{cases}$$

This is well defined because every nonzero element in $\mathbb{Z}_q$ is in $\langle \alpha \rangle$ and $2 \mid |\langle \alpha \rangle| = q-1$, so the parity of the exponent does not depend on the choice of k. Hence, 1 value gives 0, $(q-1)/2$ values give 1, and $(q-1)/2$ values give -1. $\square$

Exercise. (Problem 15) Let $f(x)$ be as in problem 13 and let $h \in \mathbb{F}_q[x]$ be a randomly chosen polynomial. What is the probability that $h^{(q^r-1)/2} = \pm 1$ in the ring $\mathbb{F}_q[x]/(f(x))$.

Proof. As shown in Problem 13 last week, there exists an isomorphism $\Phi : \mathbb{F}_q[x]/(f(x)) \to \mathbb{F}_q[x]/(f_1(x)) \times \cdots \times \mathbb{F}_q[x]/(f_m(x))$ by the Chinese Remainder Theorem. For any $h \in \mathbb{F}_q[x]$, $\Phi(h + (f)) = (h + (f_1), \cdots, h + (f_m))$. Moreover, $\Phi(h^{(q-1)/2} + (f)) = (h^{(q-1)/2} + (f_1), \cdots, h^{(q-1)/2} + (f_m))$. Therefore, $h^{(q-1)/2} + (f) = 1$ if and only if $h^{(q-1)/2} + (f_1), \cdots, h^{(q-1)/2} + (f_m)$ all equal 1.

Let $\alpha_1, \cdots, \alpha_m$ be generators of $(\mathbb{F}_q[x]/(f_1(x)))^*, \cdots, (\mathbb{F}_q[x]/(f_m(x)))^*$. For each i, $h^{(q-1)/2} + (f_i) = 1$ if and only if $h \in \langle \alpha_i^2 \rangle$ by Problem 14. Therefore, $h^{(q-1)/2} + (f) = 1$ if and only if $(h + (f_1), \cdots, h + (f_m)) \in \langle \alpha_1^2 \rangle \times \cdots \times \langle \alpha_m^2 \rangle$. There are exactly $((q^r - 1)/2)^m$ elements that satisfy that. Therefore,

$$\frac{(\frac{q^r-1}{2})^m}{(q^r)^m} = \left(\frac{q^r-1}{2q^r}\right)^m = \left(\frac{1}{2} - \frac{1}{2q^r}\right)^m.$$

is the probability that $h^{(q^r-1)/2} = 1$ in $\mathbb{F}_q[x]/(f(x))$.

Using the exact same argument, we can derive that the probability that $h^{(q^r-1)/2} = -1$ is exactly the same value. $\square$

Exercise. (Problem 16) With $f(x)$ as in problem 13, write $f(x) = g_1(x) \cdots g_m(x)$ for the factorization into irreducible factors. Express $\gcd(f(x), h^{(q^r-1)/2} - 1)$ in terms of the $g_i(x)$'s.

Proof. $\gcd(f(x), h^{(q^r-1)/2} - 1)$ is the product of $g_i(x)$'s that divide $h^{(q^r-1)/2} - 1$. It is divisible by $g_i(x)$ if and only if $h \in \langle \alpha_i^2 \rangle$ from Problem 15. $\square$