MATH 620 HOMEWORK (DUE 9/10)

HIDENORI SHINOHARA

Exercise. Show that $F_*: T_p\mathbb{R}^n \to T_q\mathbb{R}^m$.

Proof. Let $v_1, v_2 \in T_pU, c \in \mathbb{R}$. Then $v_1 = c_1^j \frac{\partial}{\partial x^j} \mid_p, v_2 = c_2^j \frac{\partial}{\partial x^j} \mid_p$ where $c_i^j \in \mathbb{R}$. Let $\gamma_1(t) = p + t(c_1^1, \cdots, c_1^n), \gamma_2(t) = p + t(c_2^1, \cdots, c_2^n), \gamma = c\gamma_1 + \gamma_2$. Then there exist unique $b_1^1, \cdots, b_1^m, b_2^1, \cdots, b_2^m, b^1, \cdots, b^m \in \mathbb{R}$ such that

- $F_*(v_1) = b_1^s \frac{\partial}{\partial y^s}$.
- $F_*(v_2) = b_2^s \frac{\partial}{\partial y^s}$.
- $F_*(cv_1 + v_2) = b^s \frac{\partial}{\partial y^s}$.

For each s,

$$\begin{aligned}
b_s &= (F_*(cv_1 + v_2))(y^s) \\
&= \frac{d}{dt} y^s \circ F \circ \gamma(t)\Big|_{t=0} \\
&= \frac{d}{dt} F^s \circ \gamma(t)\Big|_{t=0} && \text{(Let } F^s = y^s \circ F.\text{)} \\
&= \frac{\partial F^s}{\partial x^j}\Big|_p (cc_1^j + c_2^j) \\
&= c\frac{\partial F^s}{\partial x^j}\Big|_p c_1^j + \frac{\partial F^s}{\partial x^j}\Big|_p c_2^j \\
&= c\frac{d}{dt} F^s \circ \gamma_1(t)\Big|_p c_1^j + \frac{d}{dt} F^s \circ \gamma_2(t)\Big|_p c_2^j \\
&= c(F_*v_1)(y^s) + (F_*v_2)(y^s) \\
&= cb_1^s + b_2^s.
\end{aligned}$$

Therefore, $F_*(cv_1 + v_2) = cF_*(v_1) + F_*(v_2)$. $\square$

Exercise. Prove that if $f_i \in \mathscr{C}^\infty$, then $f_I dx^I \in \mathcal{A}^k$.

Proof. Let $\eta = f_I dx^I$ and let $\zeta = dx^I$. Let $X_1, \cdots, X_k \in \mathfrak{X}(\mathbb{R}^n)$. We must show that $F: \mathbb{R}^n \to \mathbb{R}$ defined by $F(p) = \eta_p(X_{1,p}, \cdots, X_{k,p})$ is smooth. For any $p \in \mathbb{R}^n$,

$$\begin{aligned}
F(p) &= \eta_p(X_{1,p}, \cdots, X_{k,p}) \\
&= (f_{i_1}(p)dx^{i_1}|_p \wedge \cdots \wedge f_{i_k}(p)dx^{i_k}|_p)(X_{1,p}, \cdots, X_{k,p}) \\
&= \sum_{\sigma \in S_k} (f_{i_{\sigma_1}}(p)dx^{i_{\sigma_1}}|_p)(X_{1,p}) \cdots (f_{i_{\sigma_k}}(p)dx^{i_{\sigma_k}}|_p)(X_{k,p}) \\
&= (f_1(p) \cdots f_k(p)) \sum_{\sigma \in S_k} (dx^{i_{\sigma_1}}|_p)(X_{1,p}) \cdots (dx^{i_{\sigma_k}}|_p)(X_{k,p}) \\
&= (f_1(p) \cdots f_k(p))\zeta_p(X_{1,p}, \cdots, X_{k,p}).
\end{aligned}$$

As discussed in the lecture, $\zeta = dx^I \in \mathcal{A}^k(\mathbb{R}^n)$. Thus the mapping $p \mapsto \zeta_p(X_{1,p}, \cdots, X_{k,p})$ must be smooth. Since each f_i is smooth and the product of smooth functions is smooth, $p \mapsto f_1(p) \cdots f_k(p) \zeta_p(X_{1,p}, \cdots, X_{k,p})$ is smooth. Therefore, F is smooth, so $\eta = f_I dx^I \in \mathcal{A}^k$. $\square$