

## MATH 611 HOMEWORK (DUE 10/16)

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**Exercise.** (Problem 16) Given maps  $X \rightarrow Y \rightarrow Z$  such that both  $Y \rightarrow Z$  and the composition  $X \rightarrow Z$  are covering spaces, show that  $X \rightarrow Y$  is a covering space if  $Z$  is locally path-connected, and show that this covering space is normal if  $X \rightarrow Z$  is a normal covering space.

*Proof.* Let  $p : X \rightarrow Y, q : Y \rightarrow Z$  be given such that  $q$  and  $q \circ p$  are both covering maps. Let  $y_0 \in Y$  be given. It suffices to show that there exists a neighborhood of  $y_0$  that is evenly covered by  $p$ . (Hatcher does not require a covering map be surjective.)

Let  $z_0 = q(y_0)$ . Let  $U_{z_0}$  be a locally path-connected neighborhood of  $z_0$  contained in the intersection of the following two neighborhoods:

- A neighborhood of  $z_0$  that is evenly covered by  $q$ .
- A neighborhood of  $z_0$  that is evenly covered by  $q \circ p$ .

Those two neighborhoods of  $z_0$  must exist because  $q$  and  $q \circ p$  are covering maps. Since  $Z$  is locally path-connected, any neighborhood of  $z_0$  contains a path-connected neighborhood of  $z_0$ . Therefore, such  $U_{z_0}$  must exist. Moreover, any neighborhood contained in an evenly covered neighborhood is evenly covered. Therefore,  $U_{z_0}$  is a path-connected neighborhood of  $z_0$  that is evenly covered by both  $q$  and  $q \circ p$ .

Since  $U_{z_0}$  is evenly covered by  $q$  and  $q \circ p$ ,

- Let  $\coprod_{\alpha} U_{x_{\alpha}} = (q \circ p)^{-1}(U_{z_0})$  where  $q \circ p$  maps each  $U_{x_{\alpha}}$  into  $U_{z_0}$  homeomorphically.
- Let  $\coprod_{\beta} U_{y_{\beta}} = q^{-1}(U_{z_0})$  where  $q$  maps each  $U_{y_{\beta}}$  into  $U_{z_0}$  homeomorphically.

Draw a figure.

Since  $z_0 = q(y_0)$  and  $q$  is an covering map, there exists  $U_{y_{\beta}}$  such that  $y_0 \in U_{y_{\beta}}$ . For simplicity, we will call it  $U_{y_0}$ . In other words,  $U_{y_0}$  is a neighborhood of  $y_0$  such that  $q$  is a homeomorphism between  $U_{y_0}$  and  $U_{z_0}$ .

We claim that  $U_{y_0}$  is a neighborhood of  $y_0$  that is evenly covered by  $p$  by showing that there exists a subset  $I$  of the index set such that  $p^{-1}(U_{y_0}) = \coprod_{\alpha \in I} U_{x_{\alpha}}$ .

Show this!

Show that  $p$  is a homeomorphism between  $U_{x_{\alpha}}$  and  $U_{y_0}$ . Let  $\alpha \in I$ . Then  $U_{x_{\alpha}} \subset p^{-1}(U_{y_0})$ . Then  $p(U_{x_{\alpha}}) \subset U_{y_0}$ .

□

**Exercise.** (Problem 18) For a path-connected, locally path-connected, and semilocally simply-connected space  $X$ , call a path-connected covering space  $X \rightarrow X$  abelian if it is normal and has abelian deck transformation group. Show that  $X$  has an abelian covering space that is a covering space of every other abelian covering space of  $X$ , and that such a ‘universal’ abelian covering space is unique up to isomorphism. Describe this covering space explicitly for  $X = S^1 \vee S^1$  and  $X = S^1 \vee S^1 \vee S^1$ .

*Proof.* We will consider the commutator subgroup  $H = [\pi_1(X, x_0), \pi_1(X, x_0)] = \{[a, b] \mid a, b \in \pi_1(X, x_0)\}$  of  $\pi_1(X, x_0)$ . Since  $H$  is a subgroup of  $\pi_1(X, x_0)$  and  $X$  is path-connected, locally path connected, and semilocally simply connected, there exists a path-connected covering space  $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  such that  $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$  by Theorem 1.38.

By Proposition 1.39(b),  $G(\tilde{X})$  is isomorphic to the quotient  $N(H)/H$ .

- Since  $H$  is the commutator subgroup,  $H$  is a normal subgroup of  $\pi_1(X, x_0)$ . Thus  $N(H) = \pi_1(X, x_0)$ . Moreover, Proposition 1.39(a) asserts that  $\tilde{X}$  is normal because  $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$  is normal.
- Since  $H$  is the commutator subgroup of  $\pi_1(X, x_0) = N(H)$ ,  $N(H)/H$  is abelian.

Therefore,  $\tilde{X}$  is an abelian covering space of  $X$ .

- Show that  $\tilde{X}$  is the ‘universal’ abelian covering space.
- Show uniqueness.

- What is the hypothesis?
  - $X$  is a path-connected, locally path-connected, semilocally simply-connected space.
- What is the conclusion?
  - There exists a normal covering space of  $X$   $p : \tilde{X} \rightarrow X$  such that  $G(\tilde{X})$  is abelian.
  - $X$  has an abelian covering space that is a covering space of every other abelian covering space of  $X$ .
  - A universal abelian covering space is unique up to isomorphism.
  - Find the universal covering space of  $S^1 \vee S^1$  and  $S^1 \vee S^1 \vee S^1$ .
- Introduce suitable notations.
  - $H = p_*(\pi_1(X, x_0))$ .
- Separate the various parts of the hypothesis.
- Find the connection between the hypothesis and the conclusion.
  - “ $X$  is a path-connected, locally path-connected, semilocally simply-connected space.” This condition sounds a lot like Theorem 1.38 on P.67. By using theorem 1.38, we can associate some group to each covering map.
  - “ $\tilde{X}$  is a normal covering space of  $X$ .” By Proposition 1.39 on P.71,  $\tilde{X}$  is normal if and only if  $H$  is a normal subgroup of  $\pi_1(X, x_0)$ .
  - $G(\tilde{X})$  is abelian. By Proposition 1.39 on P.71,  $G(\tilde{X})$  is isomorphic to the quotient  $\pi_1(X, x_0)/H$  because  $\tilde{X}$  is normal. Thus  $\pi_1(X, x_0)/H$  is abelian.
- Have you seen it before?
  - This might be similar to constructing the universal covering space.
- Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.
  - Showing uniqueness up to isomorphism sounds like the universal covering space theorem.
- Keep only a part of the hypothesis, drop the other part; is the conclusion still valid?
- Could you derive something useful from the hypothesis?
- Could you think of another hypothesis from which you could easily derive the conclusion?

- Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?
- Did you use the whole hypothesis?

□

**Exercise.** (Problem 19) Use the preceding problem to show that a closed orientable surface  $M_g$  of genus  $g$  has a connected normal covering space with deck transformation group isomorphic to  $\mathbb{Z}^n$  (the product of  $n$  copies of  $\mathbb{Z}$ ) if and only if  $n \leq 2g$ . For  $n = 3$  and  $g \geq 3$ , describe such a covering space explicitly as a subspace of  $\mathbb{R}^3$  with translations of  $\mathbb{R}^3$  as deck transformations.

*Proof.* • List examples.  $n = 1, 2, g = 1$  and  $n = 1, g = 2$  are done. Try others.

- What is the hypothesis?  $M_g$  is a closed orientable surface  $M_g$  of genus  $g$ .
- What is the conclusion?  $M_g$  has a connected normal covering space with deck transformation group isomorphic to  $\mathbb{Z}^n$  if and only if  $n \leq 2g$ .
- Separate the various parts of the hypothesis.

Closed orientable surface? I don't know what to do with it. Can I just assume that this means  $M_g = (S^1 \times S^1) \vee \cdots \vee (S^1 \times S^1)$ ?

- Find the connection between the hypothesis and the conclusion.
  - The fundamental group of  $M_g$  is generated by  $2g$  elements with no relations. If we abelianize the fundamental group of  $M_g$ , we obtain  $\mathbb{Z}^{2g}$ .
- Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.
  - The previous problem shows the existence of an abelian covering space, and a normal covering space with deck transformation group isomorphic to  $\mathbb{Z}^n$  is also abelian.
- Keep only a part of the hypothesis, drop the other part; is the conclusion still valid?
- Could you derive something useful from the hypothesis?
- Could you think of another hypothesis from which you could easily derive the conclusion?
  - If  $g = 1$ , then this problem is easy. For  $n = 2$ , consider the  $xy$  plane, and for  $n = 1$ , consider the infinite chain of squares.
- Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?
- Did you use the whole hypothesis?

□

**Exercise.** (Problem 20) Construct non-normal covering spaces of the Klein bottle by a Klein bottle and by a torus.

*Proof.* Figure 1 is the idea that I have for the first part. But I don't know how to show that there exists no deck transformation with that permutation.

□

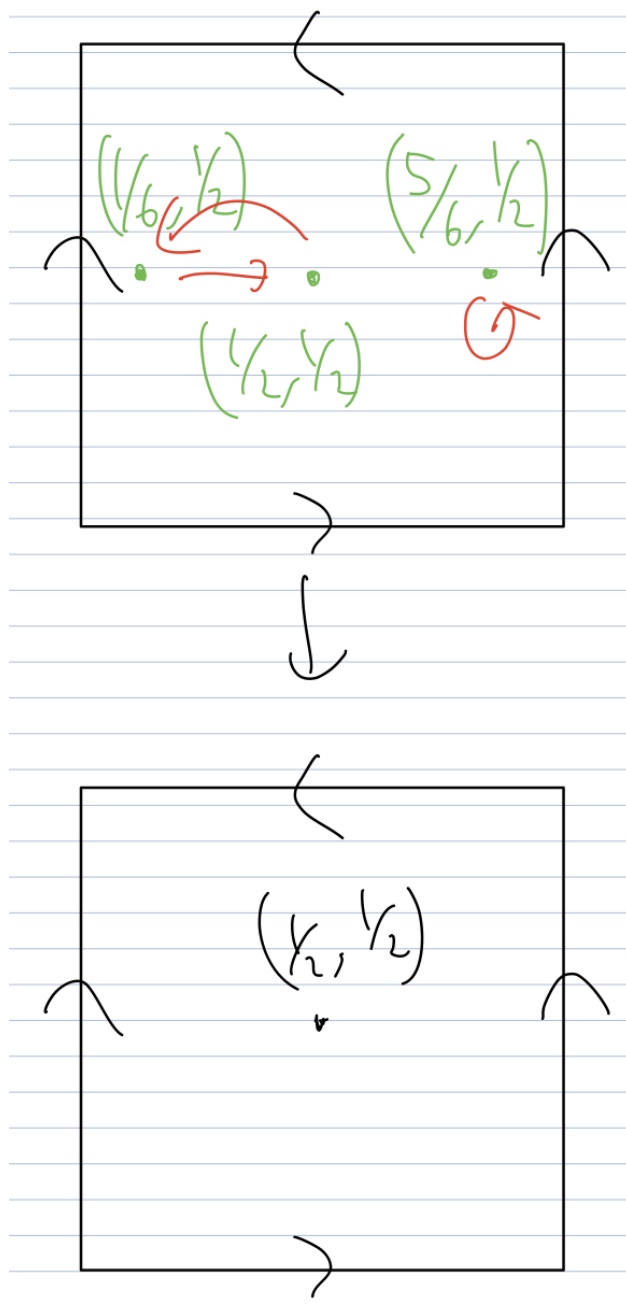


FIGURE 1. Problem 20 (Klein)

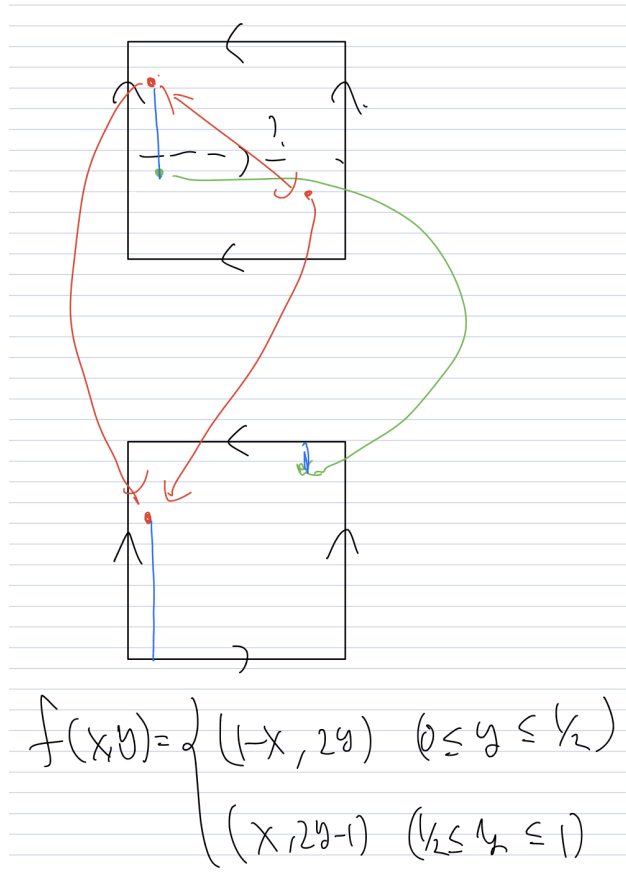


FIGURE 2. Problem 20 (Torus)