

MATH 611 (DUE 11/13)

HIDENORI SHINOHARA

1. SIMPLICIAL AND SINGULAR HOMOLOGY

Exercise. (Problem 20) Show that $H(X) = H(SX)$ for all n , where SX is the suspension of X . More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified.

Proof. It suffices to only prove the second part. Let $(CX)^k$ denote the union of k cones with their bases identified. Specifically, $(CX)^2 = SX$. We claim that $\tilde{H}_n(X) \cong \tilde{H}_{n+1}((CX)^{k+1})$ for all $k \geq 1$.

Let $i \geq 1$. We have $\tilde{H}_i(CX) \rightarrow \tilde{H}_i((CX)^{k+1}) \rightarrow \tilde{H}_i((CX)^{k+1}, CX) \rightarrow \tilde{H}_{i-1}(CX)$. Since CX is contractible by the deformation retract $f_t : (x, s) \mapsto (x, s(1-t))$, $\tilde{H}_i(CX) = \tilde{H}_{i-1}(CX) = 0$. By the exactness, $\tilde{H}_i((CX)^{k+1}) = \tilde{H}_i((CX)^{k+1}, CX)$. Let x be the north pole of CX . We will consider the inclusion $((CX)^{k+1} - x, CX - x) \rightarrow ((CX)^{k+1}, CX)$. By the excision theorem, $\tilde{H}_i((CX)^{k+1} - x, CX - x) = \tilde{H}_i((CX)^{k+1}, CX)$.

We will consider the exact sequence $\tilde{H}_i((CX)^{k+1} - x) \rightarrow \tilde{H}_i((CX)^{k+1} - x, CX - x) \rightarrow \tilde{H}_{i-1}(CX - x) \rightarrow \tilde{H}_{i-1}((CX)^{k+1} - x)$. Since $(CX)^{k+1} - x$ is contractible by the same argument as CX , $\tilde{H}_i((CX)^{k+1} - x) = \tilde{H}_{i-1}((CX)^{k+1} - x) = 0$. By the exactness, we have $\tilde{H}_i((CX)^{k+1} - x, CX - x) = \tilde{H}_{i-1}(CX - x)$. Since $CX - x$ is homeomorphic to X , $\tilde{H}_{i-1}(CX - x) = \tilde{H}_{i-1}(X)$. Hence, $\tilde{H}_i((CX)^{k+1}) = \tilde{H}_{i-1}(X)$. \square

Exercise. (Problem 22) Prove by induction on dimension the following facts about the homology of a finite-dimensional CW complex X , using the observation that X_n/X_{n-1} is a wedge sum of n spheres:

- If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.
- $H_n(X)$ is free with basis in bijective correspondence with the n cells if there are no cells of dimension $n-1$ or $n+1$.
- If X has k n -cells, then $H_n(X)$ is generated by at most k elements.

Proof.

- X^0 is a set of points, so it is clear that $H_i(X) = 0$ for $i > 0$. Let $k \geq 0$. Suppose that $H_i(X) = 0$ for $i > k$. Let $n = k+1$. Then we have an exact sequence $H_i(X^{n-1}) \rightarrow H_i(X^n) \rightarrow H_i(X^n, X^{n-1})$ for any $i > n$. Since (X^n, X^{n-1}) is a good pair, $H_{n+1}(X^n, X^{n-1}) = H_{n+1}(X^n/X^{n-1}) = H_{n+1}(\vee_\alpha S^n) = \oplus_\alpha 0 = 0$. By the inductive hypothesis, $H_i(X^{n-1}) = 0$. Therefore, the exactness of $0 \rightarrow H_i(X^n) \rightarrow 0$ implies that $H_i(X^n) = 0$ for all $i > n$. \square

Exercise. (Problem 27) Let $f : (X, A) \rightarrow (Y, B)$ be a map such that both $f : X \rightarrow Y, f : A \rightarrow B$ are homotopy equivalences.

- Show that $f_* : H_n(X, A) \rightarrow H_n(Y, B)$ is an isomorphism for all n .
- For the case of the inclusion $f : (D^n, S^{n-1}) \rightarrow (D^n, D^n \setminus \{0\})$, show that f is not a homotopy equivalence of pairs - there is no $g : (D^n, D^n \setminus \{0\}) \rightarrow (D^n, S^{n-1})$ such that fg and gf are homotopic to the identity through maps of pairs.

Proof.

- For each $n \geq 1$, we have an exact sequence $H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A) \rightarrow H_{n-1}(X)$ and another one with X, A replaced with Y, B . Moreover, they are connected by homomorphisms $f_* : H_n(A) \rightarrow H_n(B), f_* : H_n(X) \rightarrow H_n(Y), f_* : H_n(X, A) \rightarrow H_n(Y, B)$ such that the diagram commutes. (naturality) Since $f : X \rightarrow Y$ and $f : A \rightarrow B$ are both homotopy equivalences, $f_* : H_n(X) \rightarrow H_n(Y), f_* : H_n(A) \rightarrow H_n(B)$ are isomorphisms. By the Five lemma, $f_* : H_n(X, A) \rightarrow H_n(X, B)$ is an isomorphism.

The exact sequence $H_1(A) \rightarrow H_1(X) \rightarrow H_1(X, A) \rightarrow 0$ can be extended to $H_1(A) \rightarrow H_1(X) \rightarrow H_1(X, A) \rightarrow 0 \rightarrow 0$ by appending 0 at the end. Using the same argument as above, $f_* : H_1(X, A) \rightarrow H_1(Y, B)$ is an isomorphism.

- Suppose $f : (D^n, S^{n-1}) \rightarrow (D^n, D^n - \{0\})$ is a homotopy equivalence. Then there exists a $g : (D^n, D^n - \{0\}) \rightarrow (D^n, S^{n-1})$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity maps in corresponding domains. Since g is continuous, $g(\overline{D^n - \{0\}}) = \overline{g(D^n - \{0\})} \subset \overline{S^{n-1}} = S^{n-1}$. Therefore, g maps D^n into S^{n-1} . Since f maps S^{n-1} into D^n , $g \circ f$ maps S^{n-1} into S^{n-1} . We know this is homotopic to the identity map from the problem statement. Similarly, $f \circ g$ maps D^n into D^n and we know this is homotopic to the identity map from the problem statement. Therefore, this implies that D^n and S^{n-1} are homotopy equivalent. However, this is false because D^n is contractible but S^{n-1} is not.

Hence, f cannot be homotopy equivalent.

□