# MATH 611 HOMEWORK (DUE 10/16)

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**Exercise.** (Problem 18) For a path-connected, locally path-connected, and semilocally simply-connected space $X$, call a path-connected covering space $X \to X$ abelian if it is normal and has abelian deck transformation group. Show that $X$ has an abelian covering space that is a covering space of every other abelian covering space of $X$, and that such a 'universal' abelian covering space is unique up to isomorphism. Describe this covering space explicitly for $X = S^1 \vee S^1$ and $X = S^1 \vee S^1 \vee S^1$.

*Proof.*

- What is the hypothesis?
  - $X$ is a path-connected, locally path-connected, semilocally simply-connected space.
- What is the conclusion?
  - There exists a normal covering space of $X$ $p : \tilde{X} \to X$ such that $G(\tilde{X})$ is abelian.
  - $X$ has an abelian covering space that is a covering space of every other abelian covering space of $X$.
  - A universal abelian covering space is unique up to isomorphism.
  - Find the universal covering space of $S^1 \vee S^1$ and $S^1 \vee S^1 \vee S^1$.
- Introduce suitable notations.
  - $H = p_*(\pi_1(X, x_0))$.
- Separate the various parts of the hypothesis.
- Find the connection between the hypothesis and the conclusion.
  - "$X$ is a path-connected, locally path-connected, semilocally simply-connected space." This condition sounds a lot like Theorem 1.38 on P.67. By using theorem 1.38, we can associate some group to each covering map.
  - "$\tilde{X}$ is a normal covering space of $X$." By Proposition 1.39 on P.71, $\tilde{X}$ is normal if and only if $H$ is a normal subgroup of $\pi_1(X, x_0)$.
  - $G(\tilde{X})$ is abelian. By Proposition 1.39 on P.71, $G(\tilde{X})$ is isomorphic to the quotient $\pi_1(X, x_0)/H$ because $\tilde{X}$ is normal. Thus $\pi_1(X, x_0)/H$ is abelian.
- Have you seen it before?
  - This might be similar to constructing the universal covering space.
- Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.
  - Showing uniqueness up to isomorphism sounds like the universal covering space theorem.
- Keep only a part of the hypothesis, drop the other part; is the conclusion still valid?
- Could you derive something useful from the hypothesis?
- Could you think of another hypothesis from which you could easily derive the conclusion?

- Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?
- Did you use the whole hypothesis?

□

**Exercise.** (Problem 19) Use the preceding problem to show that a closed orientable surface $M_g$ of genus $g$ has a connected normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ (the product of $N$ copies of $\mathbb{Z}$) if and only if $n \leq 2g$. For $n = 3$ and $g \geq 3$, describe such a covering space explicitly as a subspace of $\mathbb{R}^3$ with translations of $\mathbb{R}^3$ as deck transformations.

*Proof.*

- What is the hypothesis? $M_g$ is a closed orientable surface $M_g$ of genus $g$.
- What is the conclusion? $M_g$ has a connected normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ if and only if $n \leq 2g$.
- Separate the various parts of the hypothesis.

  Closed orientable surface? I don't know what to do with it. Can I just assume that this means $M_g = S^1 \wedge \cdots \wedge S^1$?
- Find the connection between the hypothesis and the conclusion.
  - The fundamental group of $M_g$ is generated by $2g$ elements with no relations. If we abelianize the fundamental group of $M_g$, we obtain $\mathbb{Z}^{2g}$.
- Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.
  - The previous problem shows the existence of an abelian covering space, and a normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ is also abelian.
- Keep only a part of the hypothesis, drop the other part; is the conclusion still valid?
- Could you derive something useful from the hypothesis?
- Could you think of another hypothesis from which you could easily derive the conclusion?
  - If $g = 1$, then this problem is easy. For $n = 2$, consider the $xy$ plane, and for $n = 1$, consider the infinite chain of squares.
- Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?
- Did you use the whole hypothesis?

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