MATH 611 FINAL

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Exercise. (Problem 2) Figure 1 shows how $K_{3,3}$ is homotopy equivalent to $S_1 \vee S_1 \vee S_1 \vee S_1$. Thus the Van Kampen theorem implies that the fundamental group is the free group generated by 4 elements $\langle a, b, c, d \rangle$ where each generator corresponds to each S_1.

Exercise. (Problem 5) Let $X = S^1 \times S^2$ and $Y = S^1 \vee S^2 \vee S^3$.

$$\begin{aligned}
\pi_1(S^1 \times S^2) &= \pi_1(S^1) \times \pi_1(S^2) && \text{(Proposition 1.12)} \\
&= \mathbb{Z} \times 0 \\
&= \mathbb{Z}. \\
\pi_1(S^1 \vee S^2 \vee S^3) &= \pi_1(S^1) * \pi_1(S^2) * \pi_1(S^3) && \text{(Van Kampen)} \\
&= \mathbb{Z} * 0 * 0 \\
&= \mathbb{Z}.
\end{aligned}$$

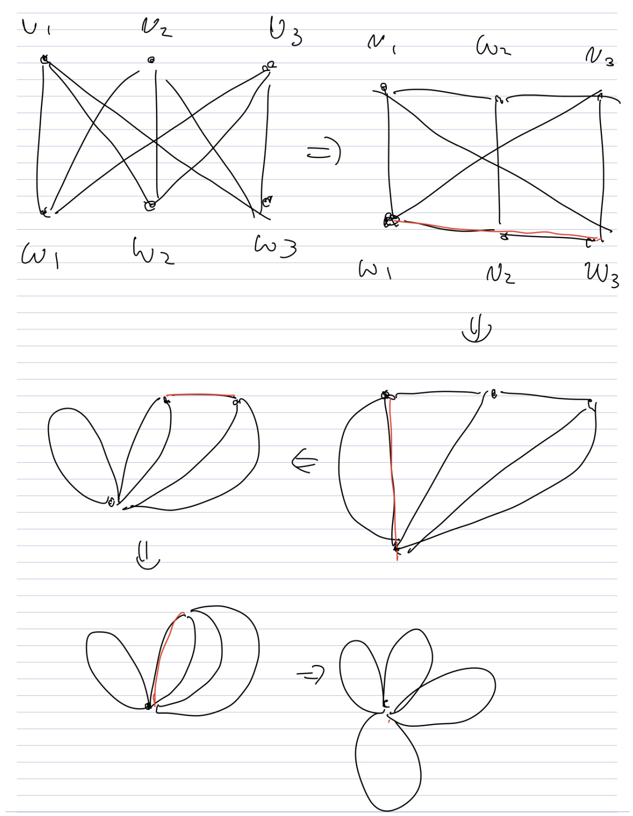

FIGURE 1. $K_{3,3}$

X and Y are both path connected, so $H_0(X) = H_0(Y) = \mathbb{Z}$.

We will consider two subspaces of X the union of whose interiors equals X. Identify each point of $X = S^1 \times S^2$ by a pair of coordinates $(\theta, (x, y, z))$ where θ is the angle in S^1 and (x, y, z) satisfies $x^2 + y^2 + z^2 = 1$. Let $A = \{(\theta, (x, y, z)) \mid -\epsilon \leq \theta \leq \pi + \epsilon\}, B = \{(\theta, (x, y, z)) \mid \pi - \epsilon \leq \theta \leq 2\pi + \epsilon\}$ where $\epsilon > 0$ is a small number. Then each A and B deformation retracts to a space homeomorphic to S^2. $A \cap B$ consists of two path components, each of which deformation retracts to a space homeomorphic to S^2. Moreover, it is clear that $\int(A) \cup \int(B) = X$. We will consider the Mayer-Vietoris sequence formed by $A, B \subset X$.

Do the Mayer Vietoris stuff.

By Corollary 2.25, $\tilde{H}_n(S^1 \vee S^2 \vee S^3) = \tilde{H}_n(S^1) \otimes \tilde{H}_n(S^2) \otimes \tilde{H}_n(S^3)$.

Therefore,

$$\tilde{H}_n(Y) = \begin{cases} \mathbb{Z} & (n = 1, 2, 3) \\ 0 & (n = 0, n \geq 4). \end{cases}$$

For $n \geq 1$, $\tilde{H}_n(Y) = H_n(Y)$, so $H_0(Y) = H_1(Y) = H_2(Y) = H_3(Y) = \mathbb{Z}$ and $H_n(Y) = 0$ for all $n \geq 4$.