MATH 601 HOMEWORK (DUE 9/4)

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Exercise. (2.1) Show that the function $g : \mathbb{R} \to S^1$, $g(r) = \exp(2\pi i r)$, where $i^2 = -1$, satisfies the property that $g(r) = g(r')$ if and only if $r \sim r'$. Use this to explicitly construct a bijective map from the orbit space of the action to S^1, g : $\mathbb{R}/\sim = \mathbb{Z}\backslash\mathbb{R} \to S^1$.

Proof.

- Let $r, r' \in \mathbb{R}$ such that $r \sim r'$. Let $k \in \mathbb{Z}$ such that $k * r' = r$. Therefore, $k + r' = r$.

$$\begin{aligned} g(r) &= \exp(2\pi i r) \\ &= \exp(2\pi i(k + r')) \\ &= \exp(2\pi i k + 2\pi i r') \\ &= \exp(2\pi i k)\exp(2\pi i r') \\ &= \exp(2\pi i r') \\ &= g(r'). \end{aligned}$$

- Let $r, r' \in \mathbb{R}$ such that $g(r) = g(r')$.

$$\begin{aligned} \exp(2\pi i r) = \exp(2\pi i r') &\implies \exp(2\pi i(r - r')) = 1 \\ &\implies \cos(2\pi(r - r')) + i\sin(2\pi(r - r')) = 1 \\ &\implies \sin(2\pi(r - r')) = 0 \\ &\implies r - r' \in \mathbb{Z} \\ &\implies \exists k \in \mathbb{Z}, r = k * r' \\ &\implies r \sim r'. \end{aligned}$$

Let g : $\mathbb{Z}\backslash\mathbb{R} \to S^1$ be defined such that g($[r]$) $= g(r)$ for each $[r] \in \mathbb{Z}\backslash\mathbb{R}$.

- Well-defined? Let $[r] = [r'] \in \mathbb{Z}\backslash\mathbb{R}$. Then $r \sim r'$. We showed that $g(r) = g(r')$ if $r \sim r'$ earlier. Therefore, g is indeed well-defined.
- Injective? Let $[r], [r'] \in Z\backslash\mathbb{R}$. Suppose g($[r]$) $=$ g($[r']$). Then $g(r) = g(r')$. We showed earlier that this implies $r \sim r'$. In other words, $[r] = [r']$. Therefore, g is injective.

- Surjective? Let $z \in S^1$. Express z as $re^{i\theta}$ where $r, \theta \in \mathbb{R}$. Since $|z| = 1$, we can assume that $r = 1$ without loss of generality. (If $r = -1$, then $e^{i\pi} = -1$, so θ can be redefined as $\theta + \pi$.)
  Then $[\theta/2\pi]$ is an element in $\mathbb{Z}\backslash\mathbb{R}$, and g$([\theta/2\pi]) = g(\theta/2\pi) = \exp(2\pi i \cdot \theta/2\pi) = \exp(i\theta) = z$. Therefore, g is indeed surjective.

$\square$

Exercise. (2.2) Let $* : G \times S \to S$ be a left action of G. Show that $s \star g = g^{-1} * s$ defines a right action of G on S.

Proof. Let $s \in S, g, h \in G$ be given.

$$\begin{aligned}(s \star g) \star h &= h^{-1} * (s \star g)\\ &= h^{-1} * (g^{-1} * s)\\ &= (h^{-1}g^{-1}) * s\\ &= (gh)^{-1} * s\\ &= s \star (gh).\end{aligned}$$

Let $e \in G$ denote the identity element and let $s \in S$ be given.

$$\begin{aligned}s \star e &= e^{-1} * s\\ &= e * s\\ &= s.\end{aligned}$$

Therefore, $\star$ is indeed a right action of G on S. $\square$

Exercise. (2.3)

(1) Let $h, h' \in G$ lie in the same conjugacy class. Show that h and h' have the same order.
(2) Give an example of a group and two elements of the same order which do not line in the same conjugacy class.

Proof. (1) Since h and h' lie in the same conjugacy class, there must exist an element $g \in G$ such that $h = g * h'$. In other words, $h = g \cdot h' \cdot g^{-1}$. We will show that $h^n = g \cdot (h')^n \cdot g^{-1}$ for all $n \in \mathbb{N}$ using mathematical induction.
- When $n = 1$, the statement is true.

- Suppose $h^n = g \cdot (h')^n \cdot g^{-1}$ for some $n \in \mathbb{N}$.

$$\begin{aligned}
h^{n+1} &= h^n \cdot h \\
&= (g \cdot (h')^n \cdot g^{-1}) \cdot (g \cdot h' \cdot g^{-1}) \\
&= g \cdot (h')^n \cdot (g^{-1} \cdot g) \cdot h' \cdot g^{-1} \\
&= g \cdot (h')^n \cdot h' \cdot g^{-1} \\
&= g \cdot (h')^{n+1} \cdot g^{-1}.
\end{aligned}$$

Therefore, $h^n = g \cdot (h')^n \cdot g^{-1}$ for all $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$, if $h^n = e$, then $g \cdot (h')^n \cdot g^{-1} = e$, so $(h')^n = g^{-1}g = e$. For any $n \in \mathbb{N}$, If $(h')^n = e$, then $h^n = geg^{-1} = e$. Therefore, $\forall n \in \mathbb{N}, h^n = e \iff (h')^n = e$.

This implies that if the order of one of h or h' is infinite, the other has to be infinite as well. On the other hand, if the order of one of h or h' is finite, the other has to be finite as well. Suppose that the orders of h and h' are finite and let n denote the order of h. Then $h^n = e$ and $h^m \neq e$ for each natural number $m < n$. Then $(h')^n = e$ and $(h')^m \neq e$ for each natural number $m < n$. Therefore, the order of h' is n as well.

We showed that, regardless of whether the order is finite, h and h' have the same order.

(2) We will consider the Klein 4-group $K =$
e, a, b, c. Since $a^2 = b^2 = e$, a and b have the order 2. Suppose that a and b lie in the same conjugacy class. Then there must exist a $g \in K$ such that $a = gbg^{-1}$. Since K is abelian, $a = gbg^{-1} = gg^{-1}b = eb = b$. This is a contradiction, so there a and b do not lie in the same conjugacy class. Thus we found two elements of the same order which do not lie in the same conjugacy class.

$\square$