MATH 612 (HOMEWORK 3)

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Exercise. (3.1.11) Using the cellular homology, we obtain

$$\tilde{H}_i(X) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & (i = n) \\ 0 & (i \neq n). \end{cases}$$

$$\tilde{H}^i(X) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & (i = n+1) \\ 0 & (i \neq n+1). \end{cases}$$

From previous homework,

$$\tilde{H}^i(X/S^n) = \tilde{H}_i(S^{n+1}) = \begin{cases} \mathbb{Z} & (i = n+1) \\ 0 & (i \neq n+1). \end{cases}$$

Thus the map on $\tilde{H}_i(-;\mathbb{Z})$ is the zero map for each i. On the other hand, the long exact sequence of a pair gives us $\tilde{H}^{n+1}(X, S^n; \mathbb{Z}) \xrightarrow{q^*} \tilde{H}^{n+1}(X; \mathbb{Z}) \to \tilde{H}^{n+1}(S^n; \mathbb{Z})$ where $\tilde{H}^{n+1}(S^n; \mathbb{Z}) = 0$, so q^* is surjective. Therefore, it is nontrivial because $\tilde{H}^{n+1}(X; \mathbb{Z}) \neq 0$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Ext}(H_n(X); \mathbb{Z}) & \longrightarrow & H^{n+1}(X; \mathbb{Z}) & \longrightarrow & \text{Hom}(H_{n+1}(X); \mathbb{Z}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Ext}(H_n(X/S^n); \mathbb{Z}) & \longrightarrow & H^{n+1}(X/S^n; \mathbb{Z}) & \longrightarrow & \text{Hom}(H_{n+1}(X/S^n); \mathbb{Z}) & \longrightarrow & 0
\end{array}
$$

is

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_m & \longrightarrow & \mathbb{Z}_m & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0.
\end{array}
$$

This splitting is not natural because the middle term in the first sequence is isomorphic to $\mathbb{Z}_m \oplus 0$ and the second one is $0 \oplus \mathbb{Z}$.

The long exact sequence of a pair gives us $\tilde{H}_n(S^n; \mathbb{Z}) \to \tilde{H}_n(X; \mathbb{Z}) \to \tilde{H}_n(X, S^n; \mathbb{Z}) = \tilde{H}_n(S^{n+1}; \mathbb{Z}) = 0$ which implies the surjectivity of the induced map. Since $\tilde{H}_n(X; \mathbb{Z}) \neq 0$, the induced map is nonzero.

The map induced on $\tilde{H}^i(-;\mathbb{Z})$ is the zero map for any i because at least one of $\tilde{H}^i(S^n; \mathbb{Z})$ or $\tilde{H}^i(X; \mathbb{Z})$ is 0 for each i.

Exercise. (3.1.13)

Exercise. (3.2.1)

Exercise. (3.2.2) Suppose X is the union of contractible open sets $A_1, \cdots, A_n$. Since each A_i is contractible, $H^k(X, A_i; R) = H^k(X; R)$ for all $k \geq 1$.

$$H^{k_1}(X, A_1; R) \times \cdots \times H^{k_n}(X, A_n; R) \longrightarrow H^{k_1+\cdots+k_n}(X, A_1 \cup \cdots \cup A_n; R)$$

$$\downarrow \cong \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$H^{k_1}(X; R) \times \cdots \times H^{k_n}(X; R) \xrightarrow{\quad f \quad} H^{k_1+\cdots+k_n}(X; R).$$

This diagram commutes by the naturality of a cup product. $H^{k_1+\cdots+k_n}(X, \bigcup_i A_i; R) = H^{k_1+\cdots+k_n}(X, X; R) = 0$ for all $k + l \geq 1$. By the commutativity of this diagram, the function f must be 0.

Exercise. (3.2.3(a)) Suppose otherwise. Let $f : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m$ be such a function. Then f induces a map on $f^* : H^*(\mathbb{R}\mathrm{P}^m) \to H^*(\mathbb{R}\mathrm{P}^n)$. In other words, $f^* : \mathbb{Z}_m[\alpha]/(\alpha^{m+1}) \to \mathbb{Z}_n[\beta]/(\beta^{n+1})$ where α, β are generators of $H^1(\mathbb{R}\mathrm{P}^m)$ and $H^1(\mathbb{R}\mathrm{P}^n)$. $H^1(\mathbb{R}\mathrm{P}^m; \mathbb{Z}_2) = \mathbb{Z}_2 = \{0, \alpha\}$ and $H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) = \mathbb{Z}_2 = \{0, \beta\}$. Since f induces a nontrivial map $H^1(\mathbb{R}\mathrm{P}^m; \mathbb{Z}_2) \to H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$, $f^*(\alpha) = \beta$. However, $f^*(0) = f^*(\alpha^{m+1}) = (f^*(\alpha))^{m+1} = \beta^{m+1} \neq 0$ because $m < n$. This is a contradiction, so such a function does not exist.

CP?

Exercise. (3.2.3(b))

Exercise. (3.2.6)

Exercise. (3.2.7)