

MATH 601 (DUE 10/23)

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CONTENTS

1. Field Extension	1
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1. FIELD EXTENSION

Exercise. (Problem 1) Let p be a prime number. Let $K = \mathbb{Z}/p\mathbb{Z}(t)$ be the fraction field of $\mathbb{Z}/p\mathbb{Z}[t]$.

- (i) What is the characteristic of K ?
- (ii) What is the characteristic of any extension field of K ?
- (iii) Show that the Frobenius endomorphism, $F : K \rightarrow K$ is not a ring isomorphism.
- (iv) Let $f(x) = x^p - t \in K[x]$. Prove that $f(x)$ is irreducible.
- (v) Prove that $f(x)$ is not a separable polynomial.
- (vi) Construct an explicit field extension $K \subset L$ such that $f(x) \in L[x]$ has a factor of positive degree $< p$.
- (vii) With f and L above find all the roots of $f(x)$ in L and determine their multiplicities.

Proof.

- (i) We will write $k \cdot 1$ to denote $1 + 1 + \cdots + 1$ (k times). Since $p \cdot 1 = 0$ in K , the characteristic of K is at most p . Let k denote the characteristic of K . Let $i : \mathbb{Z}/p\mathbb{Z} \rightarrow (\mathbb{Z}/p\mathbb{Z})[t]$, $i' : \mathbb{Z}/p\mathbb{Z}[t] \rightarrow K$ be inclusions. Then $i' \circ i : \mathbb{Z}/p\mathbb{Z} \rightarrow K$ is an injective ring homomorphism. $k \cdot 1 \neq 0$ in $\mathbb{Z}/p\mathbb{Z}$. Thus $(i' \circ i)(k \cdot 1) = k \cdot (i' \circ i)(1) = k' \cdot 1 = 0$. Since $i' \circ i$ is injective, this implies $k \cdot 1 = 0$. Therefore, $k \geq p$, so k must be equal to p .

□

Exercise. (Problem 2) Let F be a field of characteristic 0. Let $f(x) \in F[x]$ be an irreducible polynomial. Then $f(x)$ is separable.

Proof. Since $f(x)$ is irreducible, $f(x)$ is not a unit. Since F is a field, all polynomials of degree 0 are units. Thus $\deg(f(x)) \geq 1$.

Finish the proof. Check Notability for a sketch.

□

Exercise. (Problem 3) Let F be a field. Let $f(x) \in F[x]$ be an irreducible polynomial which is not separable. Show that $f'(x) = 0 \in F[x]$.

Proof. Suppose $f(x)$ is irreducible. Then $f(x) \neq 0$ and $f(x)$ is not a unit by definition. Thus $\deg(f(x)) \geq 1$.

Since $f(x)$ is not separable, there exists a non-unit $g(x) \in F[x]$ such that $g(x) \mid f(x)$ and $g(x) \mid f'(x)$ by Lemma 3.2 from the Field Extension handout. Since $f(x)$ is irreducible and $g(x)$ is not a unit, $f(x)$ is the product of $g(x)$ and a unit. This implies that $\deg(f(x)) = \deg(g(x))$.

Since $g(x) \mid f'(x)$, $f'(x) = h(x)g(x)$. If $f'(x) = 0$, we are done. Suppose otherwise. Then $\deg(f'(x)) = \deg(h(x)) + \deg(g(x)) = \deg(h(x)) + \deg(f(x)) \geq \deg(f(x))$. However, by the definition of the $'$ operator, $\deg(f'(x)) < \deg(f(x))$. This is a contradiction, so $f'(x) = 0$. \square