

## MATH 612 (HOMEWORK 2)

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**Exercise.** (Exercise 1) Fix  $G$  and let  $\alpha : H \rightarrow H'$  be given. Let  $0 \rightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \rightarrow 0, 0 \rightarrow G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} H \rightarrow 0$  be free resolutions. By Lemma 3.1(a), we obtain two homomorphisms  $\alpha_1 : F_1 \rightarrow G_1, \alpha_0 : F_0 \rightarrow G_0$  which commutes with  $f_i, g_i, \alpha$ . Then we obtain two chain complexes

$$\begin{aligned} 0 \leftarrow \text{Hom}(F_1, G) &\xleftarrow{f_1^*} \text{Hom}(F_0, G) \xleftarrow{f_0^*} \text{Hom}(H, G) \leftarrow 0 \\ 0 \leftarrow \text{Hom}(F_1, G') &\xleftarrow{f_1^*} \text{Hom}(F_0, G') \xleftarrow{f_0^*} \text{Hom}(H, G') \leftarrow 0. \end{aligned}$$

with induced maps  $\alpha_1^*, \alpha_0^*, \alpha^*$  forming a chain map from the chain complex on the bottom to the one on the top. Then  $\alpha_1^*$  induces a map from  $\text{Ext}(H', G) \rightarrow \text{Ext}(H, G)$ .

Fix  $H$  and let  $f : G \rightarrow G'$  be given. Let  $0 \rightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \rightarrow 0$  be a free resolution of  $H$ . We obtain two cochain complexes where  $f_*$  is a chain map from the top one to the bottom one.

$$\begin{aligned} 0 \leftarrow \text{Hom}(F_1, G) &\xleftarrow{f_1^*} \text{Hom}(F_0, G) \xleftarrow{f_0^*} \text{Hom}(H, G) \leftarrow 0 \\ 0 \leftarrow \text{Hom}(F_1, G') &\xleftarrow{f_1^*} \text{Hom}(F_0, G') \xleftarrow{f_0^*} \text{Hom}(H, G') \leftarrow 0. \end{aligned}$$

$f_*$  indeed makes the diagram commute because for any  $\sigma \in \text{Hom}(H, G)$ ,

$$\begin{aligned} f_*(f_0^*(\sigma)) &= f_*(\sigma \circ f_0) \\ &= f \circ (\sigma \circ f_0) \\ &= (f \circ \sigma) \circ f_0 \\ &= f_0^*(f \circ \sigma) \\ &= f_0^*(f_*(\sigma)). \end{aligned}$$

Similarly,  $f_*(f_1^*(\sigma)) = f_1^*(f_*(\sigma))$  for every  $\sigma \in \text{Hom}(F_0, G)$ . Since a chain map induces a homomorphism on cohomology groups,  $f$  induces a map from  $\text{Ext}(H, G) \rightarrow \text{Ext}(H, G')$ .

**Exercise.** (Exercise 1.2)

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H \longrightarrow 0 \\ & & \downarrow \cdot n & & \downarrow \cdot n & & \downarrow \cdot n \\ 0 & \longrightarrow & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H \longrightarrow 0 \end{array}$$

turn into two chain complexes with a chain map

$$\begin{array}{ccccccc}
0 & \longleftarrow & \text{Hom}(F_1, G) & \xleftarrow{f_1^*} & \text{Hom}(F_0, G) & \xleftarrow{f_0^*} & \text{Hom}(H, G) \longleftarrow 0 \\
& & (\cdot n)^* \uparrow & & (\cdot n)^* \uparrow & & (\cdot n)^* \uparrow \\
0 & \longleftarrow & \text{Hom}(F_1, G) & \xleftarrow{f_1^*} & \text{Hom}(F_0, G) & \xleftarrow{f_0^*} & \text{Hom}(H, G) \longleftarrow 0.
\end{array}$$

This diagram commutes because a group homomorphism for abelian groups commute with multiplication by  $n$ . Therefore,  $(\cdot n)^*$  induces a homomorphism on  $\text{Ext}(H, G) = \text{Hom}(F_1, G)/\text{im}(f_1^*)$ . Moreover,  $\forall \phi + \text{im}(f_1^*) \in \text{Ext}(H, G)$ ,

$$(\cdot n)^*(\phi + \text{im}(f_1^*)) = \phi \circ (\cdot n) + \text{im}(f_1^*)$$

where  $(\phi \circ (\cdot n))(x) = \phi(n(x)) = n(\phi(x)) = (n\phi)(x)$  for all  $x \in F_1$ . Therefore, the map induced by  $(\cdot n)^*$  is simply multiplication by  $n$ .

$$\begin{array}{ccccccc}
0 & \longleftarrow & \text{Hom}(F_1, G) & \xleftarrow{f_1^*} & \text{Hom}(F_0, G) & \xleftarrow{f_0^*} & \text{Hom}(H, G) \longleftarrow 0 \\
& & \downarrow (\cdot n)_* & & \downarrow (\cdot n)_* & & \downarrow (\cdot n)_* \\
0 & \longleftarrow & \text{Hom}(F_1, G) & \xleftarrow{f_1^*} & \text{Hom}(F_0, G) & \xleftarrow{f_0^*} & \text{Hom}(H, G) \longleftarrow 0.
\end{array}$$

For every  $\phi \in \text{Hom}(H, G)$  and  $x \in F_0$ ,

$$\begin{aligned}
((\cdot n)_*(f_0^*(\phi)))(x) &= ((\cdot n)_*(\phi \circ f_0))(x) \\
&= n((\phi \circ f_0)(x)) \\
&= n(\phi(f_0(x))) \\
&= ((\cdot n)_*\phi)(f_0(x)) \\
&= f_0^*((\cdot n)_*\phi)(x).
\end{aligned}$$

Similarly,  $(\cdot n)_*$  commutes with  $f_1^*$ , so  $(\cdot n)_*$  is a chain map. For any  $\phi + \text{im}(f_1^*) \in \text{Ext}(H, G)$ ,  $(\cdot n)_*(\phi + \text{im}(f_1^*)) = n\phi + \text{im}(f_1^*)$ , so it is multiplication by  $n$ .

**Exercise.** (Exercise 3.1.3)  $\cdots \xrightarrow{d_2} \mathbb{Z}_4 \xrightarrow{d_1} \mathbb{Z}_4 \xrightarrow{d_0} \mathbb{Z}_2 \rightarrow 0$  is a free resolution where  $d_0 : a \mapsto a$  and  $d_i : a \mapsto 2a$  because  $\ker(d_0) = \text{im}(d_1) = \ker(d_1) = \{0, 2\}$  for each  $i \geq 1$ . Apply  $\text{Hom}(-, \mathbb{Z}_2)$  and replace  $\mathbb{Z}_2^*$  with 0. For any  $\phi \in \text{Hom}(\mathbb{Z}_4, \mathbb{Z}_2)$  and  $x \in \mathbb{Z}_4$ ,  $((\cdot 2)^*(\phi))(x) = (\phi \circ (\cdot 2))(x) = \phi(2x) = \phi(0) = 0$ . Thus  $(\cdot 2)^*(\phi) = 0$ . In other words,  $d_i^* = 0$  for all  $i \geq 1$ , so  $\text{Ext}_{\mathbb{Z}_4}^n(\mathbb{Z}_2, \mathbb{Z}_2) = \text{Hom}(\mathbb{Z}_4, \mathbb{Z}_2)$  which is nontrivial because  $1 \mapsto 1$  is a nontrivial group homomorphism.

**Exercise.** (Exercise 3.1.6(a)) The chain complex we obtain is isomorphic to  $0 \rightarrow \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \rightarrow 0$  where  $\alpha(a, b) = (a + b)(1, 1, -1)$ . If we apply  $\text{Hom}(-, \mathbb{Z})$ , we obtain

- $H^0(T; \mathbb{Z}) = \text{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$ .
- $\alpha^*(\phi) = 0$  if and only if  $\phi(1, 1, -1) = 0$ .  $(a, b, c) \mapsto a - b$  and  $(a, b, c) \mapsto a + c$  form a basis for the subspace consisting of such homomorphisms.  $H^1(T; \mathbb{Z}) = \ker(\alpha^*) = \mathbb{Z} \oplus \mathbb{Z}$ .
- $H^2(T; \mathbb{Z}) = \text{Hom}(\mathbb{Z}^2, \mathbb{Z})/\text{im}(\alpha^*) = \mathbb{Z}$  because  $(a, b) \mapsto a$  and  $(a, b) \mapsto a + b$  form a basis for  $\text{Hom}(\mathbb{Z}^2, \mathbb{Z})$  and  $\text{im}(\alpha^*)$  is spanned by  $(a, b) \mapsto a + b$ .

If we apply  $\text{Hom}(-, \mathbb{Z}_2)$ , we obtain

- $H^0(T; \mathbb{Z}_2) = \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ .
- $\alpha^*(\phi) = 0$  if and only if  $\phi(1, 1, 1) = 0$ .  $(a, b, c) \mapsto a+b$  and  $(a, b, c) \mapsto a+c$  form a basis for the subspace consisting of such homomorphisms.  $H^1(T; \mathbb{Z}_2) = \ker(\alpha^*) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ .
- $H^2(T; \mathbb{Z}_2) = \text{Hom}(\mathbb{Z}_2^2, \mathbb{Z}_2) / \text{im}(\alpha^*) = \mathbb{Z}_2$  because  $(a, b) \mapsto a$  and  $(a, b) \mapsto a+b$  form a basis for  $\text{Hom}(\mathbb{Z}_2^2, \mathbb{Z}_2)$  and  $\text{im}(\alpha^*)$  is spanned by  $(a, b) \mapsto a+b$ .

**Exercise.** (Exercise 3.1.6(b), projective plane) We obtain a chain complex  $0 \rightarrow \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^3 \xrightarrow{\beta} \mathbb{Z}^2 \rightarrow 0$  where  $\alpha(a, b) = (b-a, a-b, a+b)$  and  $\beta(a, b, c) = (a+b, -a-b)$ . By applying  $\text{Hom}(-, \mathbb{Z})$ , we obtain a cochain complex. Each  $\text{Hom}(\mathbb{Z}^k, \mathbb{Z})$  has a basis  $\{\pi_1, \pi_2, \dots, \pi_k\}$  where  $\pi_i$  is a projection on the  $i$ th coordinate. Then  $(\beta^*(\pi_1))(a, b, c) = a+b$ ,  $(\beta^*(\pi_2))(a, b, c) = -a-b$ . Thus  $\ker(\beta^*) = \langle \pi_1 + \pi_2 \rangle$  and  $\text{im}(\beta^*) = \langle \pi_1 + \pi_2 \rangle$ . The kernel and image of  $\alpha$  can be calculated similarly.

- $H^0 = \ker(\beta^*) = \mathbb{Z}$ .
- $H^1 = \ker(\alpha^*) / \text{im}(\beta^*) = \langle \pi_1 + \pi_2 \rangle / \langle \pi_1 + \pi_2 \rangle = 0$ .
- $H_2 = \ker(0) / \text{im}(\alpha^*) = \langle \pi_1, \pi_2 \rangle / \langle \pi_1 - \pi_2, \pi_1 - \pi_2 \rangle = \langle \pi_1 + \pi_2, \pi_1 \mid \pi_1 + \pi_2, 2\pi_1 \rangle = \mathbb{Z}_2$ .

Similarly, we apply  $\text{Hom}(-, \mathbb{Z}_2)$ . Each  $\text{Hom}(\mathbb{Z}^k, \mathbb{Z}_2)$  has a basis  $\{\pi_1, \pi_2, \dots, \pi_k\}$  where  $\pi_i$  is a projection on the  $i$ th coordinate. The calculation of the kernels and images are almost identical as above with the only exception  $\ker(\alpha^*)$ . This is because  $\alpha^*(\pi_i) : (a, b) \mapsto a+b$  for each  $i = 1, 2, 3$ , so the kernel is  $\langle \pi_1 + \pi_2, \pi_1 + \pi_3 \rangle$ .

- $H^0 = \ker(\beta^*) = \langle \pi_1 + \pi_2 \rangle = \mathbb{Z}_2$ .
- $H^1 = \ker(\alpha^*) / \text{im}(\beta^*) = \langle \pi_1 + \pi_2, \pi_1 + \pi_3 \rangle / \langle \pi_1 + \pi_2 \rangle = \mathbb{Z}_2$ .
- $H_2 = \ker(0) / \text{im}(\alpha^*) = \langle \pi_1, \pi_2 \rangle / \langle \pi_1 + \pi_2, \pi_1 + \pi_2 \rangle = \langle \pi_1 \rangle = \mathbb{Z}_2$ .