MATH 611 HOMEWORK (DUE 10/16)

HIDENORI SHINOHARA

Exercise. (Problem 16) Given maps $X \to Y \to Z$ such that both $Y \to Z$ and the composition $X \to Z$ are covering spaces, show that $X \to Y$ is a covering space if Z is locally path-connected, and show that this covering space is normal if $X \to Z$ is a normal covering space.

Proof. Solve this!

□

Exercise. (Problem 18) For a path-connected, locally path-connected, and semilocally simply-connected space X, call a path-connected covering space $X \to X$ abelian if it is normal and has abelian deck transformation group. Show that X has an abelian covering space that is a covering space of every other abelian covering space of X, and that such a 'universal' abelian covering space is unique up to isomorphism. Describe this covering space explicitly for $X = S^1 \vee S^1$ and $X = S^1 \vee S^1 \vee S^1$.

Proof.

- What is the hypothesis?
  - X is a path-connected, locally path-connected, semilocally simply-connected space.
- What is the conclusion?
  - There exists a normal covering space of X $p : \tilde{X} \to X$ such that $G(\tilde{X})$ is abelian.
  - X has an abelian covering space that is a covering space of every other abelian covering space of X.
  - A universal abelian covering space is unique up to isomorphism.
  - Find the universal covering space of $S^1 \vee S^1$ and $S^1 \vee S^1 \vee S^1$.
- Introduce suitable notations.
  - $H = p_*(\pi_1(X, x_0))$.
- Separate the various parts of the hypothesis.
- Find the connection between the hypothesis and the conclusion.
  - "X is a path-connected, locally path-connected, semilocally simply-connected space." This condition sounds a lot like Theorem 1.38 on P.67. By using theorem 1.38, we can associate some group to each covering map.
  - "$\tilde{X}$ is a normal covering space of X." By Proposition 1.39 on P.71, $\tilde{X}$ is normal if and only if H is a normal subgroup of $\pi_1(X, x_0)$.
  - $G(\tilde{X})$ is abelian. By Proposition 1.39 on P.71, $G(\tilde{X})$ is isomorphic to the quotient $\pi_1(X, x_0)/H$ because $\tilde{X}$ is normal. Thus $\pi_1(X, x_0)/H$ is abelian.
- Have you seen it before?
  - This might be similar to constructing the universal covering space.

- Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.
  - Showing uniqueness up to isomorphism sounds like the universal covering space theorem.
- Keep only a part of the hypothesis, drop the other part; is the conclusion still valid?
- Could you derive something useful from the hypothesis?
- Could you think of another hypothesis from which you could easily derive the conclusion?
- Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?
- Did you use the whole hypothesis?

□

Exercise. (Problem 19) Use the preceding problem to show that a closed orientable surface M_g of genus g has a connected normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ (the product of N copies of $\mathbb{Z}$) if and only if $n \leq 2g$. For $n = 3$ and $g \geq 3$, describe such a covering space explicitly as a subspace of $\mathbb{R}^3$ with translations of $\mathbb{R}^3$ as deck transformations.

Proof.
- What is the hypothesis? M_g is a closed orientable surface M_g of genus g.
- What is the conclusion? M_g has a connected normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ if and only if $n \leq 2g$.
- Separate the various parts of the hypothesis.

  Closed orientable surface? I don't know what to do with it. Can I just assume that this means $M_g = (S^1 \times S^1) \vee \cdots \vee (S^1 \times S^1)$?
- Find the connection between the hypothesis and the conclusion.
  - The fundamental group of M_g is generated by $2g$ elements with no relations. If we abelianize the fundamental group of M_g, we obtain $\mathbb{Z}^{2g}$.
- Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.
  - The previous problem shows the existence of an abelian covering space, and a normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ is also abelian.
- Keep only a part of the hypothesis, drop the other part; is the conclusion still valid?
- Could you derive something useful from the hypothesis?
- Could you think of another hypothesis from which you could easily derive the conclusion?
  - If $g = 1$, then this problem is easy. For $n = 2$, consider the xy plane, and for $n = 1$, consider the infinite chain of squares.
- Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?
- Did you use the whole hypothesis?

□

Exercise. (Problem 20) Construct non-normal covering spaces of the Klein bottle by a Klein bottle and by a torus.

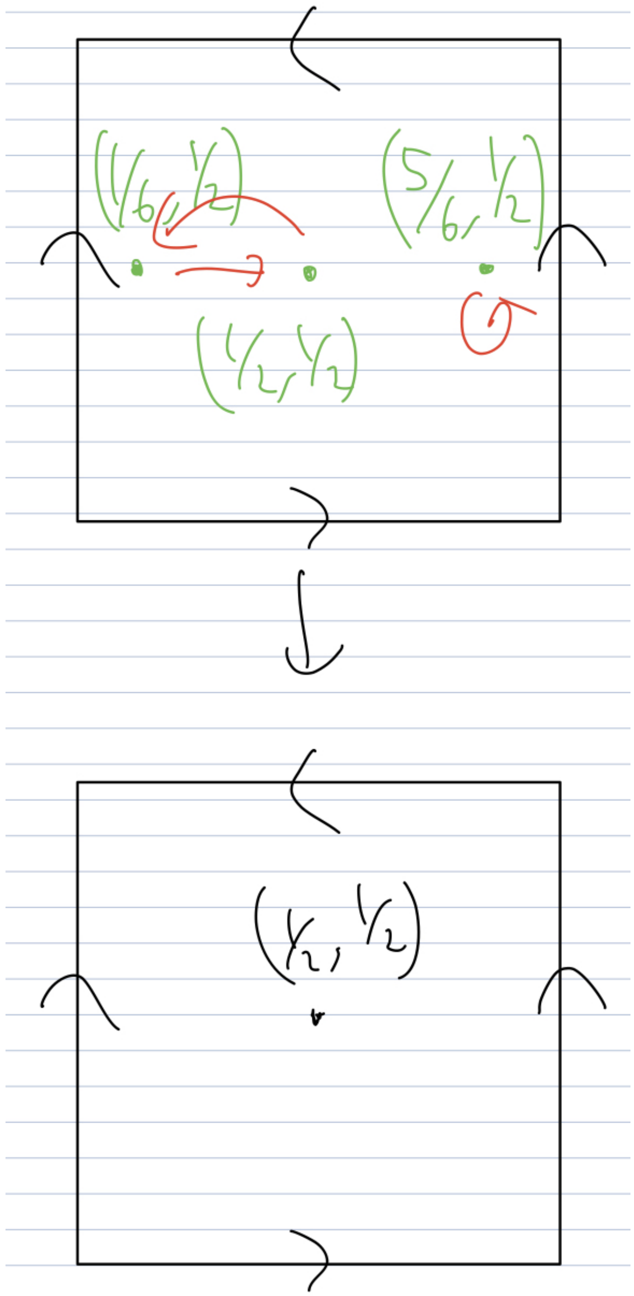
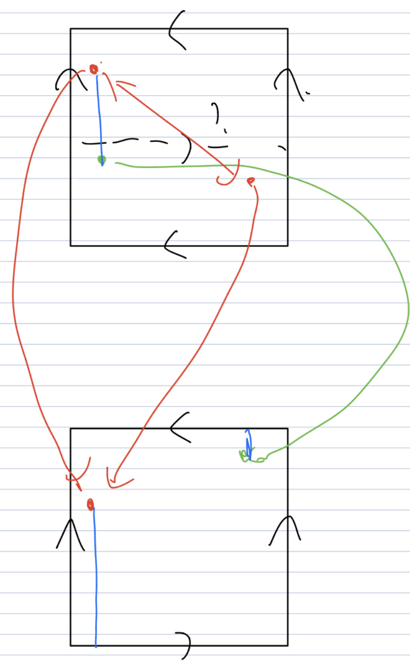


FIGURE 1. Problem 20 (Klein)

Proof. Figure 1 is the idea that I have for the first part. But I don't know how to show that there exists no deck transformation with that permutation. □



$$f(x,y) = \begin{cases} (1-x,\ 2y) & (0 \le y \le 1/2) \\ (x,\ 2y-1) & (1/2 \le y \le 1) \end{cases}$$

FIGURE 2. Problem 20 (Torus)