

# MATH 602 (HOMEWORK 5)

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**Exercise.** (1) This can be proved using induction. The base case  $m = 1$  is trivial. Suppose that the proposition has been shown for some  $m \in \mathbb{N}$ . We will show the  $(m + 1)$  case. By the definition of a determinant,

$$\Delta = \sum_{k=1}^{m+1} (-1)^{k+1} \det(M_{k,1})$$

where  $M_{k,1}$  is the matrix obtained by deleting the  $k$ th row and 1st column. We can apply the inductive hypothesis to each  $M_{k,1}$  because, for instance, when  $k = 1$ ,

$$\begin{aligned} \det(M_{1,1}) &= \det \begin{bmatrix} \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^m \\ & \ddots & & \\ \alpha_{m+1} & \alpha_{m+1}^2 & \cdots & \alpha_{m+1}^m \end{bmatrix} \\ &= \alpha_2 \cdots \alpha_{m+1} \det \begin{bmatrix} 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{m-1} \\ & \ddots & & & \\ 1 & \alpha_{m+1} & \alpha_{m+1}^2 & \cdots & \alpha_{m+1}^{m-1} \end{bmatrix} \\ &= \alpha_2 \cdots \alpha_{m+1} \prod_{2 \leq i < j \leq m} (\alpha_j - \alpha_i). \end{aligned}$$

A similar argument can be applied to other cases and we obtain

$$\Delta = \sum_{k=1}^{m+1} (-1)^{k+1} (\alpha_1 \cdots \hat{\alpha}_k \cdots \alpha_m) \prod_{i < j, i \neq k, j \neq k} (\alpha_j - \alpha_i).$$

It can be observed that, for each  $k = 1, \dots, m+1$ , the  $k$ th term  $(\alpha_1 \cdots \hat{\alpha}_k \cdots \alpha_m) \prod_{i < j, i \neq k, j \neq k} (\alpha_j - \alpha_i)$  does not contain any  $\alpha_k$ . On the other hand, for any  $l \neq k$ , every term that we obtain when expanding the  $l$ th term contains  $\alpha_k$ . Therefore, it suffices to show that, for each  $k$ , the sum of all the terms in  $\prod_{1 \leq i < j \leq m+1} (\alpha_j - \alpha_i)$  that do not contain  $\alpha_k$  is equal to the  $k$ th term in the above expression.

$$\begin{aligned} \prod_{1 \leq i < j \leq m+1} (\alpha_j - \alpha_i) &= \prod_{k+1 \leq j} (\alpha_j - \alpha_k) \prod_{j \leq k-1} (\alpha_k - \alpha_j) \prod_{1 \leq i < j \leq m+1, i \neq k, j \neq k} (\alpha_j - \alpha_i) \\ &= (-1)^{k-1} \prod_{j \neq k} (\alpha_j - \alpha_k) \prod_{1 \leq i < j \leq m+1, i \neq k, j \neq k} (\alpha_j - \alpha_i) \\ &= (-1)^{k-1} (\alpha_1 \cdots \hat{\alpha}_k \cdots \alpha_{m+1}) \prod_{1 \leq i < j \leq m+1, i \neq k, j \neq k} (\alpha_j - \alpha_i) + \alpha_k F(\alpha_1, \dots, \alpha_{m+1}) \\ &= (-1)^{k+1} (\alpha_1 \cdots \hat{\alpha}_k \cdots \alpha_{m+1}) \prod_{1 \leq i < j \leq m+1, i \neq k, j \neq k} (\alpha_j - \alpha_i) + \alpha_k F(\alpha_1, \dots, \alpha_{m+1}) \end{aligned}$$

for some polynomial  $F$ .

$\Delta^2 \neq \prod_{i \neq j} (\alpha_j - \alpha_i)$  in general. Let  $\alpha_1 = 0, \alpha_2 = 1$ . Then  $\det(A)^2 = \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^2 = 1$ . On the other hand,  $\prod_{i \neq j} (\alpha_j - \alpha_i) = (0 - 1)(1 - 0) = -1$ .