MATH 611 HOMEWORK (DUE 9/18)

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Exercise. (Problem 12, Chapter 1.2) The Klein bottle is usually pictured as a subspace of $\mathbb{R}^3$ like the subspace $X \subset \mathbb{R}^3$ shown in the first figure at the right. If one wanted a model that could actually function as a bottle, one would delete the open disk bounded by the circle of self-intersection of X, producing a subspace $Y \subset X$. Show that $\pi_1(X) \approx \mathbb{Z} * \mathbb{Z}$ and that $\pi_1(Y)$ has the presentation $\langle a, b, c \mid aba^{-1}b^{-1}cb^{\epsilon}c^{-1} \rangle$ for $\epsilon = \pm 1$. Show also that $\pi_1(Y)$ is isomorphic to $\pi_1(\mathbb{R}^3 \setminus Z)$ for Z the graph shown in the figure.

Proof. We will construct X from the 1-skeleton in Figure 1. The 1-skeleton has three loops a, b, c, so the fundamental group is $\langle a, b, c \mid \rangle$. The main difference between X and the "proper" Klein bottle is that the loop a actually gets glued on the surface. Thus we will glue the first 2-cell to a, and another 2-cell on the loop $c^{-1}acbab^{-1}$. Therefore, we end up with the fundamental group $\langle a, b, c \mid a, c^{-1}aca^{-1}bab^{-1} \rangle$. Then $\langle a, b, c \mid a, c^{-1}acabab^{-1} \rangle \approx \langle b, c \mid \rangle \approx \mathbb{Z} * \mathbb{Z}$ since the relation $c^{-1}aca^{-1}bab^{-1}$ is trivial by the relation a.

In order to calculate the fundamental group of Y, it suffices to repeat the following step without attaching a 2-cell to a. Thus the fundamental group is $G = \langle a, b, c \mid c^{-1}aca^{-1}bab^{-1} \rangle$.

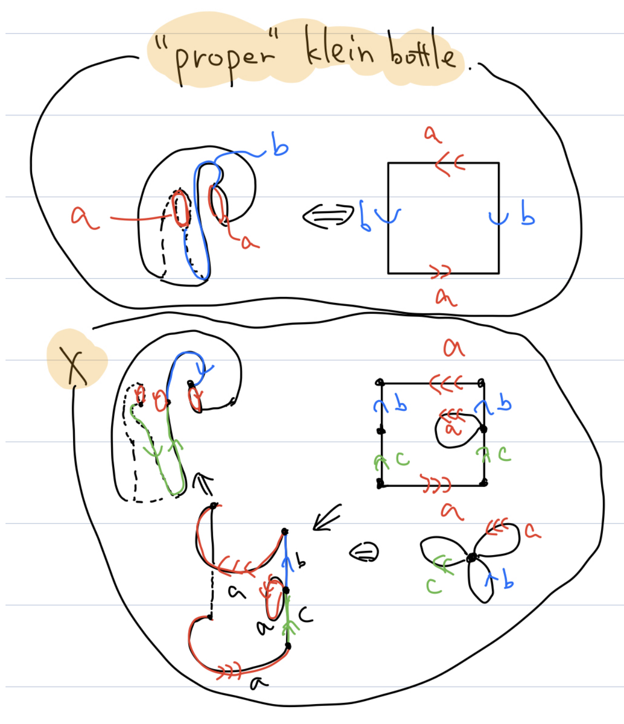

FIGURE 1. Fundamental Group of X

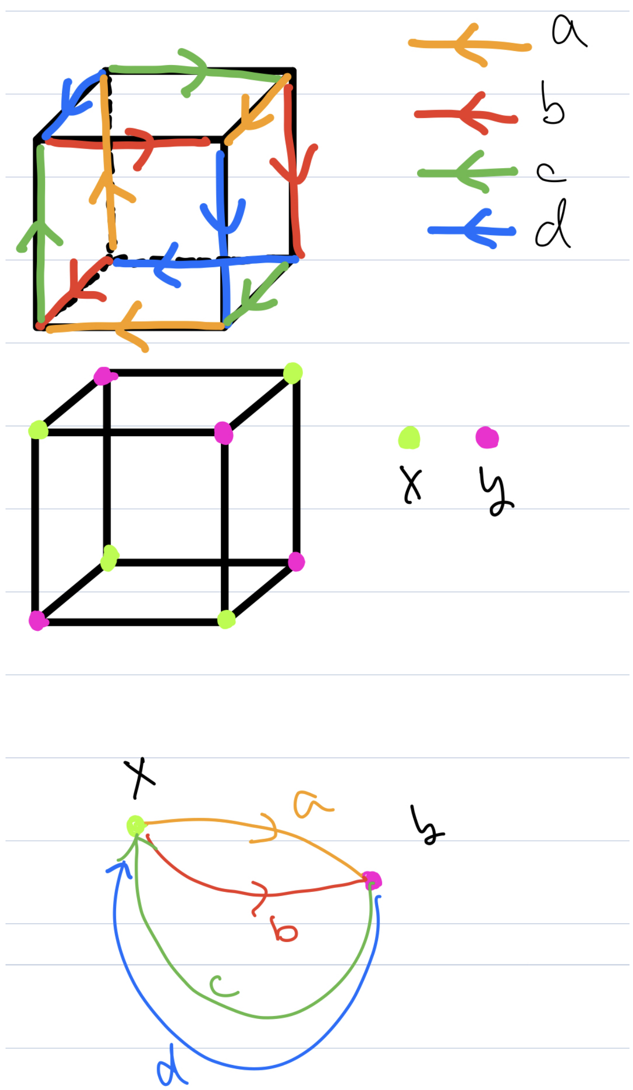

FIGURE 2. Problem 14

This is isomorphic to the group given in the textbook, $H = \langle a, b, c \mid aba^{-1}b^{-1}cbc^{-1} \rangle$ by $\phi : G \to H$ that maps a to b, b to c, and c to a^{-1}. $\square$

Exercise. (Problem 14, Chapter 1.2) Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ of order eight.

Proof. The vertices and edges get identified as in Figure 2. Thus we have two 0-cells and four 1-cells. Since the opposite faces are identified and the cube has 6 faces, we need to glue three 2-cells to the cube. Lastly, we need a 3-cell glued to the three faces. By Proposition 1.26, the fundamental group of a 2-skeleton is the same as the fundamental group of a space obtained by attaching 3-cells, so it suffices to consider the fundamental group we obtain by attaching the three 2-cells to the graph. As in Figure 2, the graph has 4 edges between two vertices. The fundamental group of this is $\langle ab^{-1}, ac, ad\rangle$ because by "shrinking" a we obtain the graph consisting of one vertex and three loops. By attaching a 2-cell to each of the top-bottom pair, left-right pair, and the front-back pair, we obtain

$$\langle ac, ab^{-1}, ad \mid ab^{-1}d^{-1}c, adc^{-1}b^{-1}, acbd\rangle.$$

Thus this is the fundamental group of the given space. We claim that $(ac)^2 = (ab^{-1})^2 = (ad)^2 = (ac)(ab^{-1})(ad)$.

- $(ac)^2 = (ab^{-1})^2$?

$$\begin{aligned} ac = d^{-1}b^{-1} &\implies ab^{-1}bc = d^{-1}b^{-1} \\ &\implies ab^{-1}ad = d^{-1}b^{-1} \\ &\implies ab^{-1}a = d^{-1}b^{-1}d^{-1} \\ &\implies ab^{-1}ab^{-1} = d^{-1}b^{-1}d^{-1}b^{-1} \\ &\implies (ab^{-1})^2 = (d^{-1}b^{-1})^2 \\ &\implies (ab^{-1})^2 = (ac)^2. \end{aligned}$$

- $(ac)^2 = (ad)^2$?

$$\begin{aligned} ab^{-1} = c^{-1}d &\implies cab^{-1} = d \\ &\implies ca = db \\ &\implies cac = dbc \\ &\implies cac = dad \\ &\implies acac = adad \\ &\implies (ac)^2 = (ad)^2. \end{aligned}$$

- $(ad)^2 = (ac)(ab^{-1})(ad)$? $(ac)(ab^{-1}) = acc^{-1}d = ad$, so $(ac)(ab^{-1})(ad) = (ad)^2$.

Moreover, we claim that $(ac)^2 \neq e$ and $(ac)^4 = e$.

- $(ac)^2 \neq e$.

  Prove this!

- $(ac)^4 = e$.

  Prove this!

□

Exercise. (Problem 22, Chapter 1.2)

- Show that $\pi_1(\mathbb{R}^3 - K)$ has a presentation with one generator x_i for each strip R_i and one relation of the form $x_i x_j x_i^{-1} = x_k$ for each square S_l, where the indices are as in the figures above.

Proof.

- We will construct the 2-dimensional complex X by first attaching R_i's. We will attach R_i one by one. We begin with a plane $\mathbb{R}^2$ whose fundamental group is 0. A rectangular strip R_i has a fundamental group isomorphic to $\mathbb{Z}$ since it is homotopy

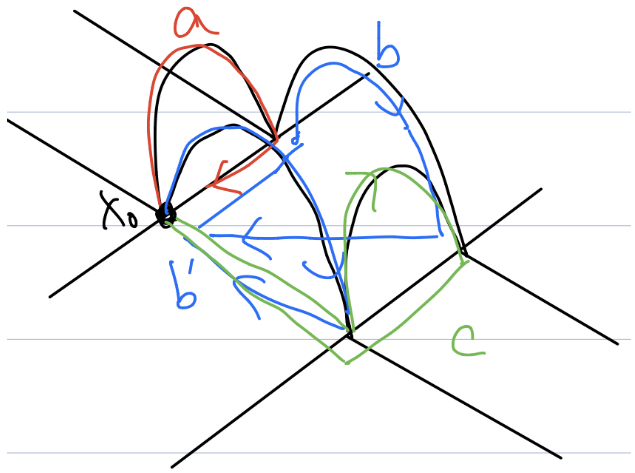

FIGURE 3. Wirtinger presentation

equivalent to S^1. Thus it is a free group with one generator. We will calculate the fundamental group of a space we obtain after attaching T to R_i using Van Kampen. The intersection is a rectangle, so the intersection is simply connected. Thus the fundamental group of the new space is simply the free product of T and R_i. Therefore, the fundamental group of the space we obtain by attaching all the R_i's is $\langle x_1, \cdots, x_n \mid \rangle$ where n is the number of R_i's and each x_i corresponds to R_i.

Now, we will attach S_l's and we will do so one by one. The fundamental group of each S_l is 0 since each S_l is simply connected. Thus attaching S_l's does not add any new generators to the fundamental group. Figure 3 shows the intersection between an S_l and the current space X. a, b, b', c denote loops based at x_0, and $[b] = [b']$. Moreover, $[a], [b], [c]$ are exactly the generator of the corresponding rectangular strip. We will consider the intersection between S_l and X.

- The loop that goes through the intersection is path homotopic to $abc^{-1}b^{-1}$ in X.
- The loop that goes through the intersection is nulhomotopic in S_l since S_l is simply connected.

By Van Kampen, the new group is $\pi_1(X) * \pi_1(S_l)/(i_X(g)i_{S_l}(g)^{-1})$ where g is any loop in the intersection. Since $\pi_1(S_l) = 0$, $i_{S_l}(g) = e$ for any g. Then $(i_X(g)) = ([abc^{-1}b^{-1}])$ since the intersection is homeomorphic to S^1 and $[abc^{-1}b^{-1}]$ is a generator. Since $\pi_1(S_l) = 0$, we have $\pi_1(X)/([a][b][c^{-1}][b^{-1}])$.

After attaching all the S_l's we will end up with $\langle x_1, \cdots, x_n \mid [a_l][b_l][c_l^{-1}][b_l^{-1}]\rangle$ where

- For each S_l, we add a relation $[a_l][b_l][c_l^{-1}][b_l^{-1}]$. Note that this means $[a_l][b_l][c_l^{-1}][b_l^{-1}] = e$, so $[a_l] = [b_l][c_l][b_l^{-1}]$, and this is exactly the desired relation.
- Each x_i corresponds to a rectangular strip R_i. These are the only generators because S_l's are all simply connected.

- The abelianization of $\pi_1(\mathbb{R}^3 - K)$ turns a relation $x_i x_j x_i^{-1} = x_k$ into $x_j = x_k$. In other words, this implies that, at each square S_l, the generators for the two strips that are "separated" by the middle strip are identified. Let x_i, x_j be two distinct generators. Since K is a knot, there exists a finite sequence $x_i = x_{i_0}, \cdots, x_{i_k} = x_j$ of generators such that the corresponding strips $R_{i_0}, \cdots, R_{i_k}$ are next to each other. (See Figure

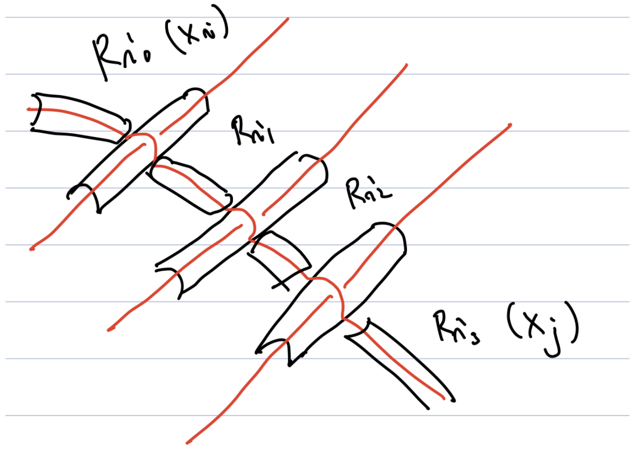

FIGURE 4. Problem 22 (b)

4) Since each intersection has a square, $x_{i_t} = x_{i_{t+1}}$ for each t. (For instance, in Figure 4, $x_{i_0} = x_{i_1}$ because of the intersection between R_{i_0} and R_{i_1}. Similarly, $x_{i_1} = x_{i_2}$ and $x_{i_2} = x_{i_3}$.) Therefore, $x_i = x_{i_0} = x_{i_1} = \cdots = x_{i_k} = x_j$.

This implies that any two generators are identified after the abelianization. Hence, $\pi_1(\mathbb{R}^3 - K)$ is a free group with one generator and no relations, so it is isomorphic to $(\mathbb{Z}, +)$.

$\square$