MATH 611 (DUE 10/2)

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Exercise. (Problem 10, Chapter 1.3) Find all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$, up to isomorphisms of covering spaces without base points.

Proof. Let $X = S^1 \vee S^1$. By the discussion on P.70 of the textbook, we know that n-sheeted covering spaces of X are classified by equivalence classes of homomorphisms $\pi_1(X, x_0) \to S_n$. Let a, b denote paths in X as in Figure 1. We can identify each homomorphism ϕ by checking what ϕ maps a and b to. (Strictly speaking, $\pi_1(X, x_0)$ is generated by $[a], [b]$, but we will abuse notations by writing a and b instead of $[a], [b]$.)

The following are all the cases. Figure 2 shows the corresponding graphs.

- Case 1: $\phi_1(a) = \phi_1(b) = (1)$. The space that corresponds to this homomorphism is disconnected.
- Case 2: $\phi_2(a) = (12), \phi_2(b) = (1)$. This generates a connected covering space.
- Case 3: $\phi_3(a) = (1), \phi_3(b) = (12)$. This generates a connected covering space.
- Case 4: $\phi_4(a) = (12), \phi_4(b) = (12)$. This generates a connected covering space.

$\phi_1 \neq \phi_2$ and $(12)\phi_1(12) \neq \phi_2$, so ϕ_1 and ϕ_2 are not conjugates of each other. Similarly, ϕ_2 and ϕ_3 are not conjugates of each other, and neither are ϕ_1 and ϕ_3.

Thus the three graphs corresponding to Case 2, 3 and 4 in Figure 2 are all the 2-sheeted covering spaces of X.

We will take the exact same approach for the case of 3. If a certain vertex is fixed in both $\phi(a)$ and $\phi(b)$, then such a vertex is disjoint from the rest of the graph. We will use that property to reduce the possibilities.

- Case 1: $\phi_1 : a \mapsto (1), b \mapsto (1)$ The following maps are conjugates of ϕ_1
  - $a \mapsto (1), b \mapsto (1)$

  This graph is not connected because every vertex is fixed.
- Case 2: $\phi_2 : a \mapsto (12), b \mapsto (1)$ The following maps are conjugates of ϕ_2
  - $a \mapsto (23), b \mapsto (1)$
  - $a \mapsto (13), b \mapsto (1)$
  - $a \mapsto (12), b \mapsto (1)$

  This graph is not connected because vertex 3 is fixed.
- Case 3: $\phi_3 : a \mapsto (1), b \mapsto (12)$ The following maps are conjugates of ϕ_3

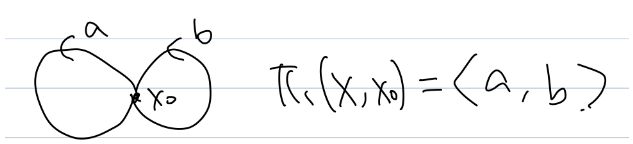

FIGURE 1. Problem 10 ($X = S^1 \vee S^1$)

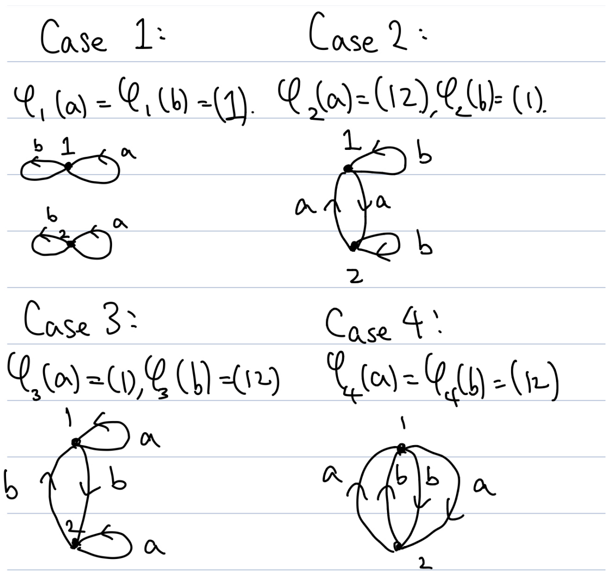

FIGURE 2. Problem 10 (2-sheeted covers)

- $a \mapsto (1), b \mapsto (12)$
- $a \mapsto (1), b \mapsto (23)$
- $a \mapsto (1), b \mapsto (13)$

This is the same as Case 2.

- Case 4: $\phi_4 : a \mapsto (12), b \mapsto (13)$ The following maps are conjugates of ϕ_4
  - $a \mapsto (13), b \mapsto (12)$
  - $a \mapsto (12), b \mapsto (23)$
  - $a \mapsto (12), b \mapsto (13)$
  - $a \mapsto (13), b \mapsto (23)$
  - $a \mapsto (23), b \mapsto (12)$
  - $a \mapsto (23), b \mapsto (13)$

  See Figure 3.
- Case 5: $\phi_5 : a \mapsto (12), b \mapsto (123)$ The following maps are conjugates of ϕ_5
  - $a \mapsto (23), b \mapsto (123)$
  - $a \mapsto (12), b \mapsto (123)$
  - $a \mapsto (12), b \mapsto (132)$
  - $a \mapsto (13), b \mapsto (132)$
  - $a \mapsto (13), b \mapsto (123)$
  - $a \mapsto (23), b \mapsto (132)$

  See Figure 3.
- Case 6: $\phi_6 : a \mapsto (123), b \mapsto (12)$ The following maps are conjugates of ϕ_6
  - $a \mapsto (123), b \mapsto (13)$
  - $a \mapsto (132), b \mapsto (12)$
  - $a \mapsto (132), b \mapsto (23)$
  - $a \mapsto (132), b \mapsto (13)$
  - $a \mapsto (123), b \mapsto (12)$
  - $a \mapsto (123), b \mapsto (23)$

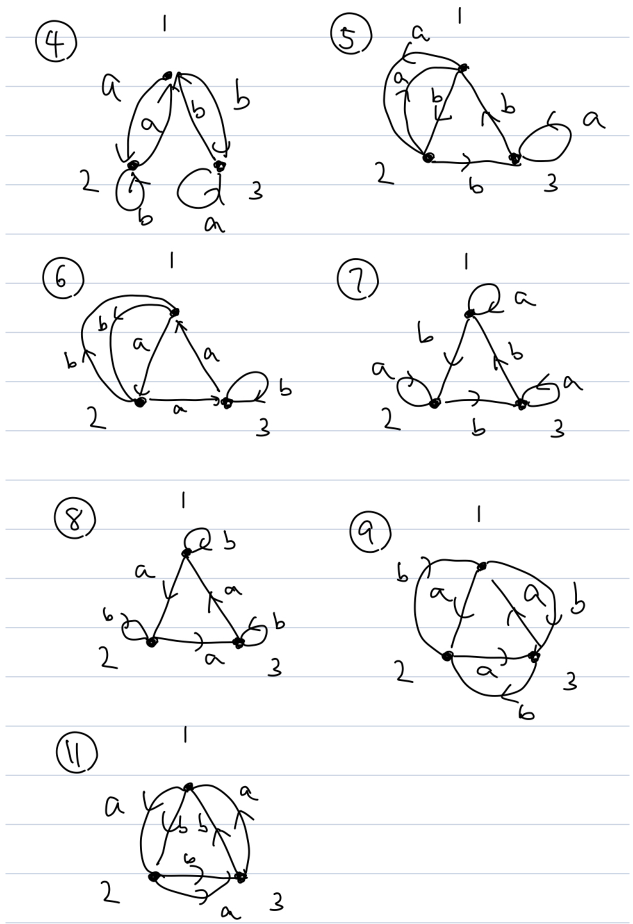

FIGURE 3. Problem 10 (3-sheeted)

See Figure 3.

- Case 7: $\phi_7 : a \mapsto (1), b \mapsto (123)$ The following maps are conjugates of ϕ_7
  - $a \mapsto (1), b \mapsto (132)$
  - $a \mapsto (1), b \mapsto (123)$

  See Figure 3.
- Case 8: $\phi_8 : a \mapsto (123), b \mapsto (1)$ The following maps are conjugates of ϕ_8
  - $a \mapsto (132), b \mapsto (1)$
  - $a \mapsto (123), b \mapsto (1)$

  See Figure 3.
- Case 9: $\phi_9 : a \mapsto (123), b \mapsto (132)$ The following maps are conjugates of ϕ_9
  - $a \mapsto (123), b \mapsto (132)$
  - $a \mapsto (132), b \mapsto (123)$

  See Figure 3.
- Case 10: $\phi_{10} : a \mapsto (23), b \mapsto (23)$ The following maps are conjugates of ϕ_{10}
  - $a \mapsto (12), b \mapsto (12)$
  - $a \mapsto (23), b \mapsto (23)$
  - $a \mapsto (13), b \mapsto (13)$

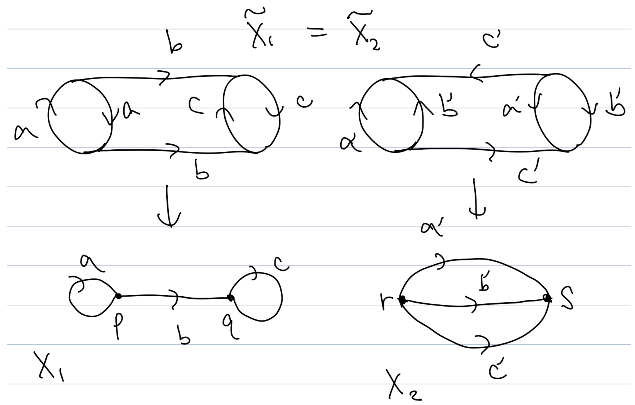

Figure 4. Problem 11

Vertex 1 is disconnected from the rest of the graph since it is fixed.

- Case 11: $\phi_{11} : a \mapsto (123), b \mapsto (123)$ The following maps are conjugates of ϕ_{11}
  - $a \mapsto (132), b \mapsto (132)$
  - $a \mapsto (123), b \mapsto (123)$

  See Figure 3.

Since there are 6 elements in S_3, there are 36 possible homomorphisms. The list above contains all of them. Therefore, Figure 3 lists all the possible 3-sheeted covers. □

Exercise. (Problem 11, Chapter 1.3) Construct finite graphs X_1 and X_2 having a common finite-sheeted covering space $\tilde{X}_1 = \tilde{X}_2$, but such that there is no space having both X_1 and X_2 as covering spaces.

Proof. Figure 4 shows X_1, X_2 and $\tilde{X}_1 = \tilde{X}_2$.

We claim that there exists no space having both X_1 and X_2 as covering spaces. On the contrary, suppose there exists such a space X with covering maps $p_1 : X_1 \to X, p_2 : X_2 \to X$. Then every point in X must have a neighborhood that homeomorphic to an open subset of X_1. Since X_1 is a graph, that means X is locally a line and a vertex with edges. In other words, X must be a graph.

There must exist a neighborhood of $p_1(p)$ and a neighborhood of p such that they are homeomorphic. Since p is a vertex of degree 3, $p_1(p)$ must be a vertex of degree 3 as well. Similarly, $p_1(q)$ must be a vertex of degree 3 as well.

Since p, q are the only vertices of X_1, X contains at most two vertices and their degrees must be 3. Since the sum of degrees of all vertices must be even from elementary graph theory, X must contain two vertices of degree 3.

If X only consists of loops, then the degree of each vertex will be even. Thus the two vertices must be joined by at least one edge. Then if one vertex has a loop, the other must have a loop as well in order to have degree 3. If there exists another edge joining the two vertices, there must be a third one in order for the two vertices to have degree 3. Therefore, X_1, X_2 are the only graphs with two vertices of degree 3.

Suppose that X_1 is a covering space of X_2 with a covering map $f : X_1 \to X_2$. Without loss of generality, $f(p) = r, f(q) = s$. Consider the path a' in X_2. Lifting a' to X_1 will result in a path from p to q. This implies that f maps points on the path b into points on a path a'.

Now consider the path b' in X_2. Lifting b' to X_1 will again result in a path from p to q. This implies that f maps points on the path b into points on a path b'.

This implies that every point on the path b must be mapped to r or s. This is a contradiction because f is continuous and $\{b(t) \mid t \in [0,1]\}$ is connected, but $\{r,s\}$ is disconnected.

Thus X_1 is not a covering space of X_2.

Similarly, suppose that X_2 is a covering space of X_1 with a covering map $g : X_2 \to X_1$. Without loss of generality, $g(r) = p, g(s) = q$. This implies $g^{-1}(p) = \{r\}$, so the number of sheets is 1. In other words, g is injective. Consider the path a in X_1. Lifting a to X_2 results into a loop based at r. Since $a : I \to X_1$ is injective, $\tilde{a} : I \to X_2$ is injective since $g \circ \tilde{a} = a$. Then $\tilde{a}(t) = s$ for some $t \in [0,1]$, so $a(t) = g(\tilde{a}(t)) = g(s) = q$. However, q is not a point on a. This is a contradiction, so X_2 is not a covering space of X_1.

Hence, there exists no space that has both X_1 and X_2 as covering spaces. ☐

Exercise. (Problem 14, Chapter 1.3) Find all the connected covering spaces of $\mathbb{R}\mathbf{P}^2 \vee \mathbb{R}\mathbf{P}^2$.

Proof. Let $X = \mathbb{R}\mathbf{P}^2 \vee \mathbb{R}\mathbf{P}^2$. By Theorem 1.38 of the textbook, it suffices to check all the conjugacy classes of subgroups of $\pi_1(X, x_0)$.

Since $\pi_1(\mathbb{R}\mathbf{P}^2) = \langle a \mid a^2 \rangle$, $\pi_1(X, x_0) = \langle a, b \mid a^2 = b^2 = e \rangle$ by Van Kampen. Since $a^2 = b^2 = e$, we can express each element in $\pi_1(X, x_0)$ uniquely as a word which alternates a, b.

Here are all the conjugacy classes of subgroups:

(1) Conjugacy class represented by $\langle e \rangle$.
(2) Conjugacy class represented by $\langle a \rangle$. This conjugacy class contains $\langle bab \rangle, \langle ababa \rangle, \langle bababab \rangle, \cdots$.
(3) Conjugacy class represented by $\langle b \rangle$. This conjugacy class contains $\langle aba \rangle, \langle babab \rangle, \langle abababa \rangle, \cdots$.
(4) Conjugacy class represented by $\left\langle (ab)^k \right\rangle$ for each $k \in \mathbb{N}$. There are no other elements in these conjugacy classes.
(5) Conjugacy class represented by $\langle a, w \rangle$ for each word w that starts and ends with b, For each w, $\langle bab, bwb \rangle, \langle ababa, abwba \rangle, \cdots$ are the elements in the conjugacy class of $\langle a, w \rangle$. Each conjugacy class of this type contains finitely many elements. For instance, when $w = bababab$, $\langle a, bababab \rangle, \langle bab, ababa \rangle, \langle ababa, bab \rangle, \langle bababab, a \rangle$ are the only elements in this class.

Figure 5 shows covering spaces corresponding to each conjugacy class.

We will prove that we have listed all the conjugacy classes, and that there are exactly 5 classes.

- All classes?
- Different?

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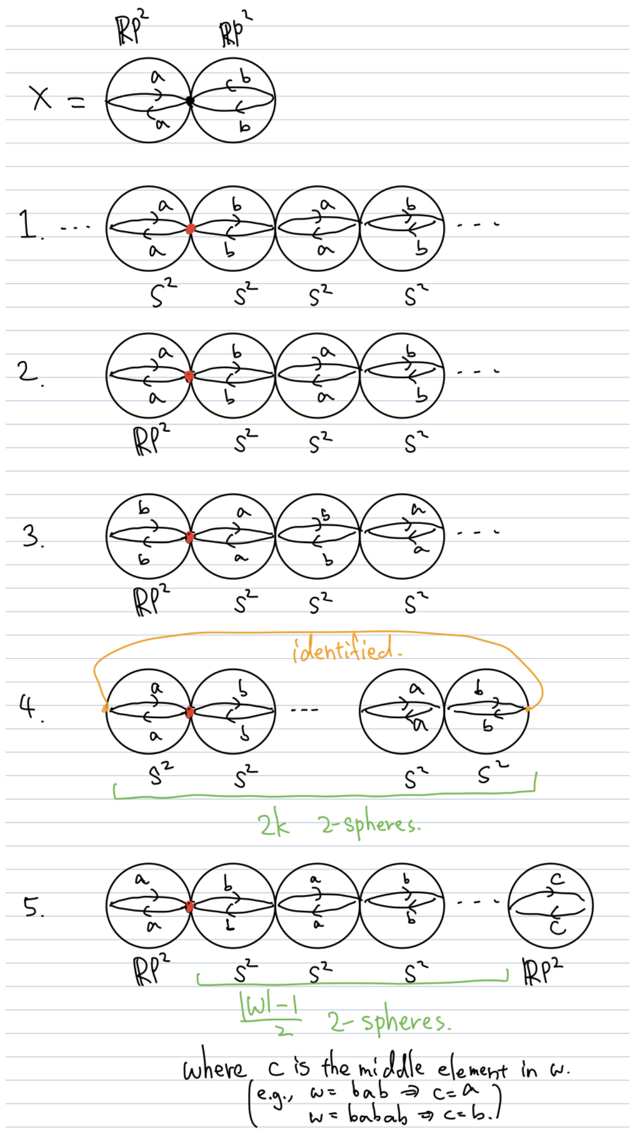

FIGURE 5. Problem 14