MATH 633 (HOMEWORK 5)

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Exercise. (Problem 1) Let $r > 0$ and $z \in \mathbb{C} \setminus \{0\}$ be given. Let $w(k) = \ln|z| + i(2k\pi + \operatorname{Arg}(z))$. Then for any $k \in \mathbb{Z}$, $e^{w(k)} = z$. For sufficiently large natural number k, $1/w(k) \in D_r(0) \setminus \{0\}$ and $f(1/w(k)) = z$. Thus f maps $D_r(0) \setminus \{0\}$ onto $\mathbb{C} \setminus \{0\}$.

Exercise. (Problem 2) The desired equation can be obtained by integrating $\frac{e^{iz}}{z^2+a^2}$ over the closed curve γ consisting of the interval $\gamma_1 = [-R, R]$ and the arc $\gamma_2 = Re^{\pi it}$ with $t \in [0, 1]$ as $R \to \infty$ and comparing the real part. In the following calculation, we assume R is sufficiently large.

$$\begin{aligned}
\int_\gamma \frac{e^{iz}}{z^2+a^2} &= \int_\gamma \frac{(e^{iz})/(z+ia)}{z-ia} dz \\
&= 2\pi i \frac{e^{i(ia)}}{2ia} \\
&= \pi \frac{e^{-a}}{a}. \\
\int_{\gamma_1} \frac{e^{iz}}{z^2+a^2} &= \int_{-R}^{R} \frac{e^{ix}}{x^2+a^2} dx \\
&= \int_{-R}^{R} \frac{\cos x}{x^2+a^2} dx + i \int_{-R}^{R} \frac{\sin x}{x^2+a^2} dy. \\
\left| \int_{\gamma_2} \frac{e^{iz}}{z^2+a^2} dz \right| &\le \int_0^1 \frac{\left| e^{i\gamma_2(t)} \right|}{|Re^{2\pi it} + a^2|} |\gamma_2'(t)| dz \\
&= R \int_0^1 \frac{e^{-R\sin \pi t}}{|Re^{2\pi it} + a^2|} dz \\
&\le R \int_0^1 \frac{e^{-R\sin \pi t}}{R/2} dz \\
&= 2 \int_0^1 e^{-R\sin \pi t} dz \\
&= 2 \frac{e^{-R\sin \pi t}}{-R\sin \pi} \Big|_0^1 \\
&= 0 & \text{(as } R \to \infty\text{).}
\end{aligned}$$

Exercise. (Problem 3) $p(z) = az + b$ with $a \neq 0$ are the only bijective polynomials.

By the fundamental theorem of algebra, every polynomial $p(z)$ with coefficients in $\mathbb{C}$ is of the form $a \prod_{i=1}^{n} (z - a_i)$ for $a \neq 0, a_1, \cdots, a_n \in \mathbb{C}$. If $a_i \neq a_j$ for some i, j, then p cannot be injective. Thus any bijective polynomials must be of the form $a(z - b)^n$ for some $a \neq 0$ and

$b \in \mathbb{C}$. If $n \geq 2$, then $p(\omega + b) = a\omega^n = a$ where $\omega = e^{2\pi ij/n}$ where $j = 0, \cdots, n-1$. Thus $n = 1$ if the polynomial is injective. In other words, any bijective polynomial must be linear.

On the other hand, it is clear that any non-constant linear function is bijective.

Exercise. (Problem 4)

(a) False. $e^{2k\pi i} = 1$ for any $k \in \mathbb{Z}$.

(b) False. $\Omega = \mathbb{C}$, and define

$$f(z) = \begin{cases} iz & (\mathrm{im}(z) \geq 0) \\ i\overline{z} & (\mathrm{im}(z) \leq 0). \end{cases}$$

$\overline{z}$ is not holormophic by the Cauchy-Riemann equation.

(c) True.

$$\frac{\partial f}{\partial x} = \int_{C_1(0)} \frac{g(w)}{(w-z)^2} dw$$
$$\frac{\partial f}{\partial y} = i \int_{C_1(0)} \frac{g(w)}{(w-z)^2} dw.$$

Thus this satisfies the Cauchy-Riemann equation. Moreover, f is continuous, so f is holomorphic.