

# MATH 612 (HOMEWORK 1)

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**Exercise.** (Exercise 1(a)) The case of  $G = \mathbb{Z}$  is discussed in Example 2.42.

$$H_k(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0 \text{ and for } k = n \text{ odd} \\ \mathbb{Z}_2 & \text{for } k \text{ odd, } 0 < k < n \\ 0 & \text{otherwise.} \end{cases}$$

Suppose  $n$  is even. For any abelian group  $G$ , we obtain the cellular chain complex

$$0 \rightarrow G \xrightarrow{2} G \xrightarrow{0} \cdots \xrightarrow{2} G \xrightarrow{0} G \rightarrow 0.$$

If  $n$  is odd, we obtain

$$0 \rightarrow G \xrightarrow{0} G \xrightarrow{2} \cdots \xrightarrow{2} G \xrightarrow{0} G \rightarrow 0.$$

- Suppose  $k$  is even and  $2 \leq k \leq n$ . The homology at  $\xrightarrow{0} G \xrightarrow{2}$  is
  - 0 if  $G = \mathbb{Q}, \mathbb{Z}/p^l\mathbb{Z}$  with  $p \neq 2$ .
  - $\mathbb{Z}/2\mathbb{Z}$  if  $G = \mathbb{Z}/2^l$ .
- Suppose  $k$  is odd and  $1 \leq k \leq n-1$ . The homology at  $\xrightarrow{2} G \xrightarrow{0}$  is
  - $G/2G \cong 0$  if  $G = \mathbb{Q}, \mathbb{Z}/p^l\mathbb{Z}$  with  $p \neq 2$  because multiplication by 2 is an isomorphism.
  - $\mathbb{Z}/2\mathbb{Z}$  if  $G = \mathbb{Z}/2^l$ .
- Suppose  $k = n$  and  $n$  is odd, or  $k = 0$ . The homology at  $\xrightarrow{0} G \xrightarrow{0}$  is  $G$ .

When  $G = \mathbb{Q}$ , the universal coefficient theorem gives an isomorphism  $H_k(X) \otimes Q \cong H_k(X; \mathbb{Q})$  since  $Q$  is torsion free.  $\mathbb{Z} \otimes \mathbb{Q} \cong \mathbb{Q}$  and  $\mathbb{Z}/2 \otimes \mathbb{Q} = 0$  because 2 is invertible in  $\mathbb{Q}$ . This agrees with the results above.

When  $G = \mathbb{Z}/2^l$ , we have  $0 \rightarrow H_k(X) \otimes G \rightarrow H_k(X; G) \rightarrow \text{Tor}(H_{k-1}(X), G) \rightarrow 0$ . If  $k = n$  and  $k$  is odd,  $H_k(X) = \mathbb{Z}$ , so  $\mathbb{Z}/2^l \cong H_k(X; \mathbb{Z}/2^l)$ . If  $k-1 = n$  and  $k-1$  is odd, we obtain  $0 \rightarrow 0 \rightarrow H_k(X; \mathbb{Z}/2^l) \rightarrow \text{Tor}(\mathbb{Z}, \mathbb{Z}/2^l) \rightarrow 0$ , so  $H_k(X; \mathbb{Z}/2^l) = 0$ . If  $k$  is odd and  $0 < k < n$ ,  $0 \rightarrow \mathbb{Z}/2 \otimes \mathbb{Z}/2^l \rightarrow H_k(X; \mathbb{Z}/2^l) \rightarrow \text{Tor}(H_{k-1}(X), \mathbb{Z}/2^l) \rightarrow 0$ . The Tor is 0 because if  $k = 0$ ,  $H_{k-1}(X) = \mathbb{Z}$  and  $H_{k-1}(X) = 0$  otherwise. Thus  $H_k(X; \mathbb{Z}/2^l) = \mathbb{Z}/2 \otimes \mathbb{Z}/2^l = \mathbb{Z}/2$ . In any other cases, the universal coefficient theorem gives the SES  $0 \rightarrow 0 \rightarrow H_n(X; G) \rightarrow 0 \rightarrow 0$ . This agrees with the results above.

Suppose  $G = \mathbb{Z}/p^l$ . Then the case that  $k = n$  and  $k$  is odd and the case that  $k-1 = n$  and  $k$  is odd can be handled in the same way as above. Suppose  $k$  is odd and  $0 < k < n$ . Then  $\mathbb{Z}/2 \otimes \mathbb{Z}/p^l = 0$ . Moreover,  $\text{Tor}(H_{k-1}(X), \mathbb{Z}) = 0$  as discussed above. Thus  $H_k(X) = 0$ . In any other cases, the universal coefficient theorem gives the SES  $0 \rightarrow 0 \rightarrow H_n(X; G) \rightarrow 0 \rightarrow 0$ . This agrees with the results above.

**Exercise.** (Exercise 1(b)) As discussed in Example 2.37,  $H_2(N_g; \mathbb{Z}) = 0$ ,  $H_1(N_g; \mathbb{Z}) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$ , and  $H_0(N_g; \mathbb{Z}) = \mathbb{Z}$ . For an abelian group  $G$ , the cellular chain complex is

$$0 \rightarrow G \xrightarrow{d_2} G^g \xrightarrow{d_1} G \rightarrow 0.$$

As discussed in Example 2.37,  $d_2(1) = (2, 2, \dots, 2)$  and  $d_1 = 0$ . If  $G = \mathbb{Z}/p^l$  with  $p \neq 2$  or  $G = \mathbb{Q}$ , then  $H_2(X; G) = 0$ ,  $H_1(X; G) = G^g / \langle (1, \dots, 1) \rangle = G^{g-1}$  and  $H_0(X; G) = G$  because  $2^{-1}$  exists. Suppose  $G = \mathbb{Z}/2^l$ . Then  $H_2(X; G) = \mathbb{Z}/2$  because the kernel is an index-2 subgroup.  $H_1(X; G) = G^g / \langle (2a, \dots, 2a) \rangle = G^{g-1} \otimes \mathbb{Z}/2$ , and  $H_0(X; G) = G$ .

We will verify the results using the universal coefficient theorem.

Suppose  $G = \mathbb{Q}$ . Then  $\text{Tor}(H_{n-1}(C), G) = 0$  for any  $n$ . Thus  $H_0(X; G) = \mathbb{Z} \otimes \mathbb{Q} = \mathbb{Q}$  and  $H_1(X; G) = (\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2) \otimes \mathbb{Q} = (\mathbb{Z} \otimes \mathbb{Q})^{g-1} \oplus (\mathbb{Z}_2 \otimes \mathbb{Q}) = \mathbb{Q}^{g-1}$ .

Suppose  $G = \mathbb{Z}/p^l$  with  $p \neq 2$ . When  $n = 1$ ,  $H_{n-1}(C) = \mathbb{Z}$ , so  $\text{Tor}(H_{n-1}(C), G) = 0$ . Thus  $H_1(C; G) = (\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2) \otimes \mathbb{Z}/p^l = (\mathbb{Z}/p^l)^{g-1}$ . When  $n = 2$ ,  $H_n(C) = 0$  and  $\text{Tor}(H_{n-1}(C), \mathbb{Z}/p^l) = 0$  because multiplication by  $p^l$  does not kill any element in  $\mathbb{Z}^{g-1} \oplus \mathbb{Z}/2$ .

Suppose  $G = \mathbb{Z}/2^l$ . When  $n = 1$ ,  $\text{Tor}(H_{n-1}(C), G) = \text{Tor}(\mathbb{Z}, G) = 0$ . Thus  $H_n(C; G) = H_n(C) \otimes G = (\mathbb{Z}/2^l)^{g-1} \oplus \mathbb{Z}/2$ . When  $n = 2$ ,  $H_n(C) = 0$  and  $\text{Tor}(H_{n-1}(C), \mathbb{Z}/2^l) = \ker((\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2) \xrightarrow{2^l} (\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2)) = \mathbb{Z}/2$ . Thus  $H_2(C; G) = \mathbb{Z}/2$ .

**Exercise.** (Exercise 1(c)) For a  $\mathbb{Z}$ -module  $R$ , we have

$$0 \rightarrow R \xrightarrow{0} R \xrightarrow{a} R \xrightarrow{0} R \rightarrow 0.$$

When  $R = \mathbb{Z}$ , we obtain

$$H_k(X; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0, 2n - 1 \\ \mathbb{Z}_m & \text{for } k \text{ odd, } 0 < k < 2n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

When  $R$  is an abelian group such that  $1 + 1 + \dots + 1 = 0$  ( $a$  times),  $H_i(X; R) = R$  if  $i = 0, 1, 2, 3$ . Otherwise,  $H_3(X; R) = H_0(X; R) = R$  and all other homology groups are 0.