MATH 611 HOMEWORK 2 (DUE 9/11)

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Exercise. (Problem 1, Section 1.2) Show that the free product $G*H$ of nontrivial groups G and H has trivial center, and that the only elements of $G*H$ of finite order are the conjugates of finite-order elements of G and H.

Proof. Let $w \in G * H$ be given. Suppose w is not the empty word.

- Suppose the leftmost element of w is in G. Let $h \in H$ be given such that h is not the identity element of H.
  - Case 1: The rightmost element of w is an element of G. Then wh is just a concatenation, so $wh \neq hw$ because the leftmost element of wh is in G and the leftmost element of hw is in H.
  - Case 2: The rightmost element of w is an element of H, but not h^{-1}. Let h' denote the rightmost element of w and w' denote the remaining. Then $w = w'h'$, so $wh = w'(h'h)$. By the definition of a reduced word, the rightmost element of w' is an element of G, so the concatenation of w' and $h'h$ is exactly wh. The leftmost element of wh is in G and the leftmost element of hw is in H, so $wh \neq hw$.
  - Case 3: The rightmost element of w is h^{-1}. Then the rightmost element of w disappears in wh. In this case, the leftmost element of w stays the same. Therefore, the leftmost element of wh is in G and the leftmost element of hw is in H, so $wh \neq hw$.

  In each case, $wh \neq hw$.
- Suppose that the leftmost element of w is in H. Let $g \in G$ be given such that g is not the identity element of G. Using the exact same logic as above, we can conclude that $wg \neq gw$.

Therefore, w is not in the center of $G * H$, so $Z(G * H) = \{e\}$ where e denotes the empty word. $\square$