

# MATH 601 (DUE 11/22)

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### 1. THE THEOREM ON SYMMETRIC POLYNOMIALS

**Exercise.** (Problem 1) By substituting  $u_4 = 0$ , we get  $u_1^2 u_2 u_3 + u_1 u_2^2 u_3 + u_1 u_2 u_3^2 = s_3 s_1$ .  $s_3 s_1$  with 4 variables expands to  $u_1^2 u_2 u_3 + u_1^2 u_2 u_4 + u_1^2 u_3 u_4 + u_1 u_2^2 u_3 + u_1 u_2^2 u_4 + u_1 u_2 u_3^2 + 4u_1 u_2 u_3 u_4 + u_1 u_2 u_4^2 + u_1 u_3^2 u_4 + u_1 u_3 u_4^2 + u_2^2 u_3 u_4 + u_2 u_3^2 u_4 + u_2 u_3 u_4^2$ . Then  $s_3 s_1 - f$  where  $f$  is the original polynomial gives us  $4u_1 u_2 u_3 u_4 = 4s_4$ . Therefore,  $f = s_3 s_1 - 4s_4$ .

**Exercise.** (Problem 2) We are given that  $|M - xI| = x^3 - ax^2 + bx - c$ . This implies that  $|M - (-x)I| = -x^3 - ax^2 - bx - c$ . Since the determinant function preserves multiplication,  $|M - xI||M - (-x)I| = |M^2 - x^2 I|$ . This implies  $|M^2 - x^2 I| = -x^6 + (a^2 - 2b)x^4 + (b^2 + 2ac)x^2 + c^2$ . Therefore, the characteristic polynomial of  $M$  is  $-x^3 + (a^2 - 2b)x^2 + (b^2 + 2ac)x + c^2$ .

### 2. GALOIS THEORY VI

**Exercise.** (Problem 3)

- $\{(123), (132), e\}$  is clearly a subgroup of the stabilizer group  $S_v$  of  $v$ . Since  $(12) \notin S_v$ ,  $3 \leq |S_v| \leq 5$ . By Lagrange's Theorem,  $S_v = \langle (123) \rangle$ .
- By (i),  $S_3 v$  contains only  $[S_3 : S_v] = 2$  elements. Thus  $v' = (12) \cdot v = u_2 u_1^2 + u_1 u_3^2 + u_3 u_2^2$ .
- By substituting  $u_3 = 0$  for  $v + v'$ , we get  $u_1 u_2^2 + u_2 u_1^2 = s_1 s_2$ . Then  $v + v' - s_1 s_2 = -3u_1 u_2 u_3 = -3s_3$ . Therefore,  $v + v' = s_1 s_2 + 3s_3$ .
- We will use the fundamental theorem of Galois Theory.  $F(v) = K^{\langle (123) \rangle}$ , so  $|\langle (123) \rangle| = 3 = [K : F(v)]$ . Moreover,  $|\langle \text{Gal}(K/F) \rangle| = [K : F]$ . Therefore,  $[F(v) : F] = [K : F]/[K : F(v)] = |\langle \text{Gal}(K/F) \rangle|/3$ .
- Calculation shows that  $vv' = 9s_3^2 + s_3 s_1^3 - 6s_3 s_1 s_2 + s_2^3$ . By substituting  $s_1 = 0, s_2 = p, s_3 = q$ , we get  $9q^2 + p^3$ .
- Since  $A_3$  is the only proper transitive subgroup of  $S_3$ ,  $\text{Gal}(K/F) = S_3$  if and only if  $\sigma \in \text{Gal}(K/F)$  where  $\sigma$  corresponds to the permutation  $(12)$ . (i.e.,  $u_1 \mapsto u_2, u_2 \mapsto u_1$ .)  $v, v'$  are not fixed by  $\sigma$ , so  $v, v' \notin F$  if  $\text{Gal}(K/F) = S_3$ .  $v, v'$  are fixed by every permutation if  $\text{Gal}(K/F) = A_3$  because it is generated by  $\sigma'$  that corresponds to  $(123)$ . Therefore, we can conclude that  $\text{Gal}(K/F) \neq S_3$  if and only if  $v, v' \in F$ .

$v, v' \in F$  if and only if  $(y - v)(y - v')$  factors in  $F$ . Therefore,  $h(y) = y^2 - (v + v')y + vv' = y^2 - 3qy + (9q^2 + p^3)$  is the desired polynomial.

- (g) The discriminant is  $(-3q)^2 - 4(9q^2 + p^3) = 9q^2 - 36q^2 - 4p^3 = -27q^2 - 4p^3$ .

**Exercise.** (Problem 4)

- (a) The discriminant can be expressed as  $-4s_1^3s_3 + s_1^2s_2^2 + 18s_1s_2s_3 - 4s_2^3 - 27s_3^2$ . By substituting  $s_1 = 1, s_2 = -2, s_3 = -1$ , we get 49.

```
from sympy.polys.polyfuncs import symmetrize
from sympy import *
```

```
u1, u2, u3 = symbols('u1_u2_u3')
```

```
u = [u1, u2, u3]
```

```
discriminant = 1
```

```
for i in range(3):
```

```
    for j in range(i + 1, 3):
```

```
        discriminant *= (u[i] - u[j]) * (u[i] + u[j])
```

```
print(latex(symmetrize(discriminant, formal = True)[0]))
```

**Exercise.** (Problem 5)

- (a) 
- (b)  $x^4 + x + 1$  is irreducible because

- It does not have a linear factor by the rational root theorem.
- If it factors into two rational quadratic polynomials, they will factor into two monic integer quadratic polynomials, namely,  $x^2 + ax + b$  and  $x^2 - ax + 1/b$  based on the coefficients. This implies  $b = \pm 1$ . Since the coefficient of  $x$  is 1,  $-ab + a/b = 1$ , but this implies  $b \neq \pm 1$ .

We will use the discussion presented in the Galois Theory IV handout. By (i), the discriminant is 229, so  $h(y) = y^2 - 229$ . Also,  $g(y) = y^3 - 4y - 1$  since  $a = b = 0, c = -1, d = 1$ . Therefore, both  $h(y)$  and  $g(y)$  are irreducible, so the Galois group is  $S_4$ .

- (c) It does not have a linear factor by the rational root theorem. Based on coefficients, if it factors into quadratic polynomials, it will be  $(x^2 + ax + b)(x^2 - ax + c)$  for some  $a, b, c \in \mathbb{Z}$  by Gauss' lemma. This gives  $bc = 12$  and  $-ab + ac = -8$ , so  $a(c - 12/c) = -8$ . This is a quadratic polynomial in  $c$  with the discriminant  $64 - 48a$ . This must be a square for  $c$  to exist. By checking each possible value of  $a$ , we get  $64 - 48 \cdot -8 = 448, 64 - 48 \cdot -4 = 256, 64 - 48 \cdot -2 = 160, 64 - 48 \cdot -1 = 112, 64 - 48 \cdot 1 = 16$ . (For other  $a$ ,  $64 - 48a < 0$ .) Thus the only two possible values are  $a = 1, -4$ .  $a = 1$  gives  $c - b = -8$  and  $bc = 12$ , which we can confirm to be impossible by examining the divisors of 12. Similarly,  $a = -4$  gives  $c - b = 2$  and  $bc = 12$  and this is impossible to satisfy. Therefore,  $x^4 - 8x + 12$  is irreducible over  $\mathbb{Q}$ .
- (d) Again, we will use the discussion presented in the Galois Theory IV handout. By calculating the discriminant, we have  $h(y) = y^2 - 331776$  and  $g(y) = y^3 -$

$48y - 64$ .  $h(y)$  factors as  $576^2 = 331776$ .  $g(y)$  does not factor by the rational root theorem. Therefore, the Galois group is  $A_4$ .


### 3. GALOIS THEORY V(FURTHER EXERCISES)

#### 4. GALOIS THEORY V

**Exercise.** (Problem 1)

- (a)  $D_4$  is a subgroup of  $S_4$  and the Galois Theory V handout states that  $S_4$  is solvable and any subgroup of a finite solvable group is solvable.
- (b) If  $S_5$  is solvable,  $A_5 \leq S_5$  is solvable. However, as stated in the handout,  $A_5$  is not solvable.

**Exercise.** (Problem 2)

- (a) 
- (b)  $G$  has a subgroup  $G_1 = \{(1), (12)(35), (12345), (13)(45), (13524), (14)(23), (14253), (15)(24), (15432), (25)(34)\}$  of index 2.  $G_1$  has a subgroup  $G_2 = \{(1), (12345), (13524), (14253), (15432)\}$  of index 2.  $G_2$  is abelian, so we can pick  $G_3 = \{(1)\}$ . Then  $G_3 \subset G_2 \subset G_1 \subset G$  is a filtration.