MATH 612 (HOMEWORK 2)

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Exercise. (Exercise 1)

Work on the first part.

Fix H and let $f : G \to G'$ be given. Let $0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_2} H \to 0$ be a free resolution of H. We obtain two cochain complexes where f_* is a chain map from the top one to the bottom one.

$$0 \leftarrow \mathrm{Hom}(F_1, G) \xleftarrow{f_1^*} \mathrm{Hom}(F_0, G) \xleftarrow{f_2^*} \mathrm{Hom}(H, G) \leftarrow 0$$

$$0 \leftarrow \mathrm{Hom}(F_1, G') \xleftarrow{f_1^*} \mathrm{Hom}(F_0, G') \xleftarrow{f_2^*} \mathrm{Hom}(H, G') \leftarrow 0.$$

f_* indeed makes the diagram commute because For any $\sigma \in \mathrm{Hom}(H, G)$,

$$\begin{aligned} f_*(f_2^*(\sigma)) &= f_*(\sigma \circ f_2) \\ &= f \circ (\sigma \circ f_2) \\ &= (f \circ \sigma) \circ f_2 \\ &= f_2^*(f \circ \sigma) \\ &= f_2^*(f_*(\sigma)). \end{aligned}$$

Similarly, $f_*(f_1^*(\sigma)) = f_1^*(f_*(\sigma))$ for every $\sigma \in \mathrm{Hom}(F_0, G)$. Since a chain map induces a homomorphism on cohomology groups, f induces a map from $\mathrm{Ext}(H, G) \to \mathrm{Ext}(H, G')$.