MATH 611 HOMEWORK (DUE 9/25)

HIDENORI SHINOHARA

Exercise. (Problem 4, Chapter 1.3) Construct a simply-connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle intersecting it in two points.

Proof. We claim that $\tilde{X} = X \setminus \{0\}$ is a simply-connected covering space of X. (See Figure 1.)

- Simply-connected? $\tilde{X}$ deformation retracts onto S^2. By Proposition 1.17, $\pi_1(\tilde{X}) = \pi_1(S^2)$. Proposition 1.14 shows that $\pi_1(S^2) = 0$. Therefore, $\tilde{X}$ is simply connected.
- Covering space? Let $p : \tilde{X} \to X$ be defined such that
  - $p \mid_{S^2} = \text{Id}_{S^2}$. In other words, p maps every point on the spherical part of $\tilde{X}$ to the same place on the spherical part of X.
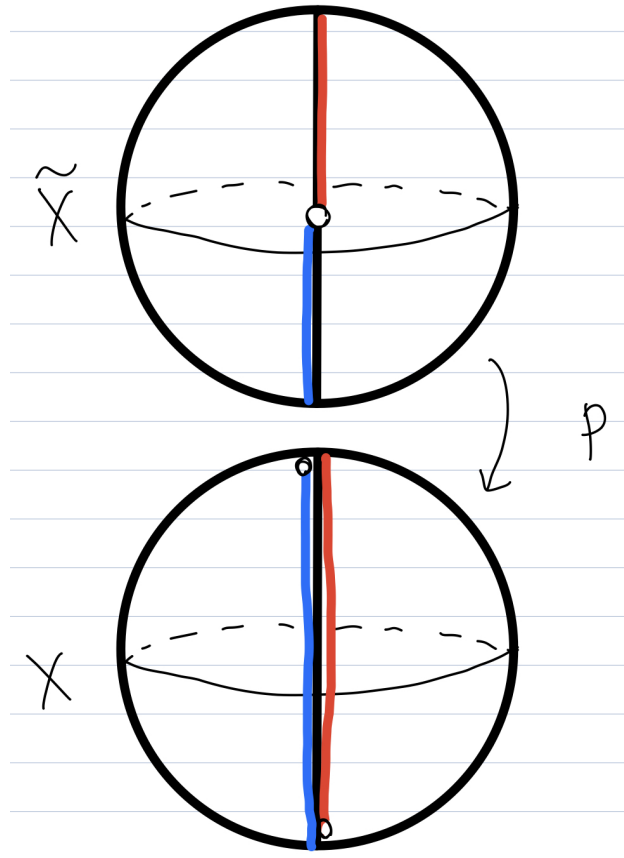

FIGURE 1. Problem 4 (Part 1)

- p extends the red line to the diameter (the top half of the diameter, in other words, the radius in the northern hemisphere in Figure 1). In other words, p maps $(0,1)$ on the Y axis into $(-1,1)$.
- p extends the blue line to the diameter (the bottom half of the diameter, in other words, the radius in the southern hemisphere in Figure 1). In other words, p maps $(-1,0)$ on the Y axis into $(-1,1)$.

Then p is clearly continuous and surjective. We will show that every point has a neighborhood that is evenly covered. Let $x \in X$ be given.

- Case 1: x is on the sphere and is disjoint from the diameter. Then a neighborhood of x that is disjoint from the diameter is clearly evenly covered.
- Case 2: x is on the diameter and is disjoint from the sphere. Let U be a neighborhood of x that is disjoint from the sphere. Then U is contained in both the image of the red line under p and the image of the blue line under p. Thus $p^{-1}(U)$ is a union of an open subset of the red line and open subset of the blue line in $\tilde{X}$. Therefore, U is evenly covered.
- Case 3: x is the north pole or south pole.

> I started to feel like this solution doesn't work because of this case...

$\square$

Exercise. (Problem 5, Chapter 1.3) Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1] \times [0,1]$ together with the segments of the vertical lines $x = 1/2, 1/3, 1/4, \cdots$ inside the square. Show that for every covering space $X \to X$ there is some neighborhood of the left edge of X that lifts homeomorphically to $\tilde{X}$. Deduce that X has no simply-connected covering space.

> Idea: Open cover of the left edge by evenly covered open sets. Find a finite subcover. By the tube lemma, there exists $U \times I$ that covers the left edge and a partition $0 = t_0 < \cdots < t_n = 1$ such that $U \times [t_i, t_{i+1}]$ is contained in an evenly covered neighborhood. Inductively, show $U \times [0, t_i]$ is in an evenly covered neighborhood. Lift a loop in X with a vertical line $x = 1/n$ for some large n. Then the element maps back to itself by p. In other words, $p_*(\pi_1(\tilde{X})) \neq 0$.

Proof.

$\square$

Exercise. (Problem 7, Chapter 1.3) Let Y be the quasi-circle in the figure in the textbook. Collapsing the segment of Y in the y-axis to a point gives a quotient map $f : Y \to S^1$. Show that f does not lift to the covering space $\mathbb{R} \to S^1$, even though $\pi_1(Y) = 0$. Thus local path-connectedness of Y is a necessary hypothesis in the lifting criterion.

> The lifting criterion is Proposition 1.33. The only property that Y is missing is the local path connectedness. But I'm not sure how to make use of it.

Proof.

$\square$

Exercise. (Problem 8, Chapter 1.3) Let $\tilde{X}$ and $\tilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces X and Y. Show that if $X \simeq Y$ then $\tilde{X} \simeq \tilde{Y}$.

Proof.

I suspected that Proposition 1.33 might be useful. But I think it's actually pretty useful for this problem because the covering spaces are simply connected, so $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is only possible if Y is simply connected. But then this means X is simply connected because they have the same homotopy type. However, this is not true in general. For instance, $X = Y = S^1$ and $\tilde{X} = \tilde{Y} = \mathbb{R}$.

$\square$