MATH 601 (DUE 12/6)

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Contents

1. Cauchy's Theorem, Finite p-groups, The Sylow theorems

Exercise. (Problem 2) Let a prime number p be given. We will show that any group G of order p^n for some n is solvable by induction on n. When $n = 1$, $G \cong \mathbb{Z}_p$, which is abelian, so it is solvable. Suppose we have shown the proposition for some $n \in \mathbb{N}$, and let G be a group of order p^{n+1}. By Corollary 1 right above this problem statement in the handout, the center H of G is a nontrivial subgroup. Moreover, H is clearly a normal subgroup of G. Thus it makes sense to consider G/H. The order of G/H must be p^m for some $1 \leq m \leq n-1$. By the inductive hypothesis, G/H is solvable. Since every subgroup of G/H can be realized as the quotient of a subgroup of G by H, there must exist a sequence of subgroups $H = G_0 \leq G_1 \leq \cdots \leq G_l = G$ such that $G_0/H \trianglelefteq G_1/H \trianglelefteq \cdots \trianglelefteq G_l/H$ and $(G_{i+1}/H)/(G_i/H)$ is abelian for each i. By Theorem 19 [P.98, Dummit and Foote], $(G_{i+1}/H)/(G_i/H) \cong G_{i+1}/G_i$, so G_{i+1}/G_i is abelian for each i. Let i be chosen arbitrarily, and let $g \in G_{i+1}, h \in G_i$. Since $G_i/H \trianglelefteq G_{i+1}/H$, $ghg^{-1} + H = (g+H)(h+H)(g^{-1}+H) \in G_i/H$. Therefore, $ghg^{-1} \in G_i$. Thus $G_i \trianglelefteq G_{i+1}$ for each i.

We showed the existence of a sequence $H = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_l = G$ such that G_{i+1}/G_i is abelian for each i. By the inductive hypothesis, there exists a similar sequence of subgroups from $\{e\}$ to H. Therefore, G is solvable.

Exercise. (Problem 3) Let $m = 3, p = 7$. Then $|G| = 21 = pm$ with $p \nmid m$. Let t be the number of Sylow p-subgroups. By the third Sylow theorem, $t \mid m$ and $t \equiv 1 \pmod p$. The only number that satisfies this is 1, so every group of order 21 has a unique Sylow 7-subgroup.