MATH 633 (FINAL EXAM)

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Exercise. (1) Since f is holomorphic and $f \neq 0$, $1/f$ is a non-constant, holomorphic function on the region Ω. By the maximum modulus principle, $1/f$ cannot attain a maximum value in Ω. Therefore, f cannot attain a minimum value in Ω.

Exercise. (2) It suffices to show that, for every $R > 0$, f is holomorphic on the open disk centered at 0 with radius R. Let $R > 0$ be given. Let T be a triangle inside the open disk D centered at 0 with radius R. If none of the three edges of T lies on the x or y axis, then $\int_T f(z)dz = 0$. Suppose some of the three edges of T lies on the x and/or y axis. Then $T_n = T + (1+i)/n$ lies in D for any $n \geq N$ for a sufficiently large N. Since none of the three edges of T_n lies on the x or y axis, $\int_{T_n} f = 0$ for any $n \geq N$. Then $\int_T f = \lim_{n\to\infty} \int_{T_n} f = 0$.

Exercise. (6) Let $f = 3z^2$ and $g = z^5 + 1$. Then $|f| > |g|$ on the unit circle. By Rouche's theorem, f and $f + g$ have the same number of zeros inside the unit circle. Clearly, f only has one zero with multiplicity 2. Thus $p = f + g$ has exactly two zeros inside the unit circle.

Let $f = z^5$ and $g = 3z^2 + 1$. Then $|f| > |g|$ on the circle centered at 0 with radius 2 because $|g| \leq 3 \cdot 2 \cdot 2 + 1 = 13 < 32 = |f|$. By Rouche's theorem, f and $f + g$ have the same number of zeros inside C. f clearly has one zero with multiplicity 5, so $p = f + g$ has exactly 5 zeros inside C.

Therefore, in the annulus, p has 5 - 2 = 3 zeros.