MATH 633 (HOMEWORK 8)

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Exercise. (Problem 13) For any $w \in D$, $\phi_w(z) = (z-w)/(1-\overline{w}z)$ is an automorphism on D that maps w to 0. Fix $w \in D$. Then $\phi_{f(w)} \circ f \circ \phi_w^{-1}$ is an automorphism that maps 0 to 0. By Lemma 2.1, $\left|(\phi_{f(w)} \circ f \circ \phi_w^{-1})(z)\right| \leq |z|$ for all $z \in D$. Fix $z \in D$. Then $\phi_w(z) \in D$, so $\left|(\phi_{f(w)} \circ f \circ \phi_w^{-1})(\phi_w(z))\right| \leq |\phi_w(z)|$. This equals to

$$\begin{aligned}
\left|(\phi_{f(w)} \circ f \circ \phi_w^{-1})(\phi_w(z))\right| \leq \left|\frac{z-w}{1-\overline{w}z}\right| &\implies \left|\phi_{f(w)}(f(z))\right| \leq \left|\frac{z-w}{1-\overline{w}z}\right| \\
&\implies \left|\frac{f(z)-f(w)}{1-\overline{f(w)}f(z)}\right| \leq \left|\frac{z-w}{1-\overline{w}z}\right| \\
&\implies \rho(f(z), f(w)) \leq \rho(z, w).
\end{aligned}$$

For any $z \in D$ and for any appropriate value of $h \neq 0$,

$$\begin{aligned}
\rho(f(z+h), f(z)) \leq \rho(z+h, z) &\implies \left|\frac{f(z+h)-f(z)}{1-\overline{f(z+h)}f(z)}\right| \leq \left|\frac{z+h-z}{1-\overline{(z+h)}z}\right| \\
&\implies \left|\frac{f(z+h)-f(z)}{h}\right| \cdot \frac{1}{\left|1-\overline{f(z+h)}f(z)\right|} \leq \left|\frac{1}{1-\overline{(z+h)}z}\right|.
\end{aligned}$$

By letting $h \to 0$, we obtain the Schwarz-Pick lemma.