

# MATH 602 HOMEWORK 4

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**Exercise.** (1) Let  $a/s \in S^{-1}\sqrt{I}$ . Then  $a^n \in I$  and  $s \in S$  for some  $n \in \mathbb{N}$ . This implies  $(a/s)^n \in S^{-1}I$ , so  $a/s \in \sqrt{S^{-1}I}$ .

Let  $a/s \in \sqrt{S^{-1}I}$ . Then  $a^n/s^n \in S^{-1}I$  for some  $n \in \mathbb{N}$ . Then  $a^n \in I$ , so  $a \in \sqrt{I}$ . Since  $s \in S$ ,  $a/s \in S^{-1}\sqrt{I}$ .

**Exercise.** (2) Let  $\{V_\alpha\}$  be an open cover of  $\text{Spec}(R)$ . For each  $\alpha$ ,  $\text{Spec}(R) \setminus V_\alpha = V(a_\alpha)$  for some ideal  $a_\alpha$  of  $R$ .  $\text{Spec}(R) = \cup_{\alpha \in I} V_\alpha = \cup_{\alpha \in I} (\text{Spec}(R) \setminus V(a_\alpha)) = \text{Spec}(R) \setminus V(\cup_{\alpha \in I} a_\alpha) = \text{Spec}(R) \setminus V(\sum a_\alpha)$ . In other words,  $V(\sum a_\alpha) = \emptyset$ . Since every proper ideal is contained in a maximal ideal,  $\sum a_\alpha = (1)$ . This implies  $1 = x_{\alpha_1} + \cdots + x_{\alpha_n}$  for some  $\alpha_1, \dots, \alpha_n \in I$  and  $x_{\alpha_i} \in a_{\alpha_i}$ . Then  $\cup V_{\alpha_i} = \text{Spec}(R) \setminus V(\cup a_{\alpha_i}) = \text{Spec}(R) \setminus V(1) = \text{Spec}(R)$ . Thus  $\text{Spec}(R)$  is indeed compact.

**Exercise.** (3) Suppose that  $I$  is generated by one element  $x$ . Then  $ax = 0 \implies a = 0$  because  $A$  is an integral domain. Therefore,  $I$  is a free module with a basis  $\{x\}$ .

On the other hand, suppose that  $I$  is a free module with a basis  $\{x_\alpha\}$ . Since it is a basis, each  $x_\alpha \neq 0$ . Moreover, if the basis contains more than 2 elements,  $(-x_{\alpha'})x_\alpha + x_\alpha x_{\alpha'} = 0$ , so it is not linearly independent. Therefore, the basis must contain exactly one element.

**Exercise.** (4a) If  $m = 0$  in each  $M_{f_i}$ , then, for each  $i$ ,  $f_i^{k_i}m = 0$  for some  $k_i \geq 0$ . Let  $k = \max\{k_1, \dots, k_n\}$ . Since  $1 \in \langle f_1, \dots, f_n \rangle$ ,  $1$  can be expressed as a linear combination of  $N$  monomials consisting of  $f_i$ 's. Then  $m = 1m = 1^{Nk}m = 0$  because each monomial in the  $Nk$ th power of such a linear combination of  $N$  monomials contains at least  $k$  appearances of one monomial, which kills  $m$ .

**Exercise.** (6a)  $(M : N)$  is nonempty. For any  $a, b \in (M : N)$ ,  $(a - b)N = aN + (-b)N = aN + bN \subset M$ , so  $a - b \in (M : N)$ . Finally, for any  $a \in (M : N)$ ,  $x \in R$ ,  $(xa)N = a(xN) \subset aN \subset M$ ,  $ax \in (M : N)$ .

**Exercise.** (6b)

$$\begin{aligned} a \in \text{Ann}((M + N)/M) &\iff a((M + N)/M) = 0 \\ &\iff \forall (m + n) + M \in (M + N)/M, a((m + n) + M) = 0 \\ &\iff \forall (m + n) + M \in (M + N)/M, am + an \in M \\ &\iff \forall n \in N, an \in M \\ &\iff aN \subset M \\ &\iff a \in (M : N). \end{aligned}$$

**Exercise.** (6c) First, we assume that  $J$  is generated by a single element  $x$ . Then  $Rx = R/\text{Ann}(x)$ . Then  $S^{-1}(Rx) = S^{-1}R/S^{-1}\text{Ann}(x)$ . On the other hand,  $S^{-1}(Rx)$  is an ideal of  $S^{-1}R$  generated by  $x$ , so  $(S^{-1}R)x \cong S^{-1}R/\text{Ann}(S^{-1}Rx)$ . Therefore,  $S^{-1}\text{Ann}(x) = \text{Ann}(S^{-1}Rx)$ . In other words,  $S^{-1}\text{Ann}(J) = \text{Ann}(S^{-1}J)$ .

Moreover, if  $J_1, J_2$  are generated by single elements,

$$\begin{aligned}
S^{-1} \operatorname{Ann}(J_1 + J_2) &= S^{-1}(\operatorname{Ann}(J_1) \cap \operatorname{Ann}(J_2)) \\
&= S^{-1} \operatorname{Ann}(J_1) \cap S^{-1} \operatorname{Ann}(J_2) \\
&= \operatorname{Ann}(S^{-1}J_1) \cap \operatorname{Ann}(S^{-1}J_2) \\
&= \operatorname{Ann}(S^{-1}J_1 + S^{-1}J_2) \\
&= \operatorname{Ann}(S^{-1}(J_1 + J_2)).
\end{aligned}$$

By induction,  $S^{-1} \operatorname{Ann}(J) = \operatorname{Ann}(S^{-1}J)$  for any finitely generated ideal. Then

$$\begin{aligned}
S^{-1}(I : J) &= S^{-1} \operatorname{Ann}((I + J)/I) \\
&= \operatorname{Ann}(S^{-1}(I + J)/S^{-1}I) \\
&= \operatorname{Ann}((S^{-1}I + S^{-1}J)/S^{-1}I) \\
&= (S^{-1}I : S^{-1}J).
\end{aligned}$$

**Exercise.** (7) Let  $q \in V(p)$ . Suppose  $M_q = 0$ . Let  $m/s \in (A - q)^{-1}M$ . Then  $tm = 0$  for some  $t \in A - q$ . In other words, for each  $m \in M$ , there exists  $t \in A - q$  such that  $tm = 0$ .

Since  $p \subset q$ , for each  $m$ , the  $t$  must live in  $A - p$ . Therefore,  $M_p = 0$ . However, this is a contradiction because  $p \in \operatorname{Supp}(M)$ . Thus  $q \in \operatorname{Supp}(M)$ .

**Exercise.** (8) Let  $b/s \in S^{-1}B$ . Then  $b \in B$ , so  $b^n + a_{n-1}b^{n-1} + \cdots + a_1b + a_0 = 0$  where  $a_i \in A$ . This implies that  $(b/s)^n + (a_{n-1}/s)(b/s)^{n-1} + \cdots + (a_1/s^{n-1})(b/s) + a_0/s^n = 0$ , thus  $b/s$  is integral over  $S^{-1}A$ .