MATH 611 (DUE 10/2)

HIDENORI SHINOHARA

Exercise. (Problem 10, Chapter 1.3) Find all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$, up to isomorphisms of covering spaces without base points.

Proof. Let $X = S^1 \vee S^1$. By the discussion on P.70 of the textbook, we know that n-sheeted covering spaces of X are classified by equivalence classes of homomorphisms $\pi_1(X, x_0) \to S_n$. Let a, b denote paths in X as in Figure 1. We can identify each homomorphism ϕ by checking what ϕ maps a and b to. (Strictly speaking, $\pi_1(X, x_0)$ is generated by $[a], [b]$, but we will abuse notations by writing a and b instead of $[a], [b]$.)

The following are all the cases. Figure 2 shows the corresponding graphs.

- Case 1: $\phi_1(a) = \phi_1(b) = (1)$. The space that corresponds to this homomorphism is disconnected.
- Case 2: $\phi_2(a) = (12), \phi_2(b) = (1)$. This generates a connected covering space.
- Case 3: $\phi_3(a) = (1), \phi_3(b) = (12)$. This generates a connected covering space.
- Case 4: $\phi_4(a) = (12), \phi_4(b) = (12)$. This generates a connected covering space.

$\phi_1 \neq \phi_2$ and $(12)\phi_1(12) \neq \phi_2$, so ϕ_1 and ϕ_2 are not conjugates of each other. Similarly, ϕ_2 and ϕ_3 are not conjugates of each other, and neither are ϕ_1 and ϕ_3.

Thus the three graphs corresponding to Case 2, 3 and 4 in Figure 2 are all the 2-sheeted covering spaces of X.

We will take the exact same approach for the case of 3. If a certain vertex is fixed in both $\phi(a)$ and $\phi(b)$, then such a vertex is disjoint from the rest of the graph. We will use that property to reduce the possibilities.

- Case 1: $\phi_1 : a \mapsto (1), b \mapsto (1)$ The following maps are conjugates of ϕ_1
  - $a \mapsto (1), b \mapsto (1)$

  This graph is not connected because every vertex is fixed.
- Case 2: $\phi_2 : a \mapsto (12), b \mapsto (1)$ The following maps are conjugates of ϕ_2
  - $a \mapsto (23), b \mapsto (1)$
  - $a \mapsto (13), b \mapsto (1)$
  - $a \mapsto (12), b \mapsto (1)$

  This graph is not connected because vertex 3 is fixed.
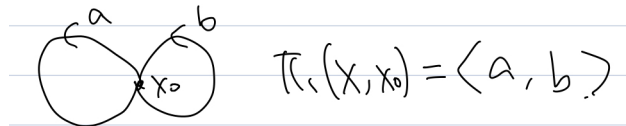- Case 3: $\phi_3 : a \mapsto (1), b \mapsto (12)$ The following maps are conjugates of ϕ_3

FIGURE 1. Problem 10 ($X = S^1 \vee S^1$)

Case 1:
$\phi_1(a) = \phi_1(b) = (1)$.

Case 2:
$\phi_2(a) = (12), \phi_2(b) = (1)$.

Case 3:
$\phi_3(a) = (1), \phi_3(b) = (12)$

Case 4:
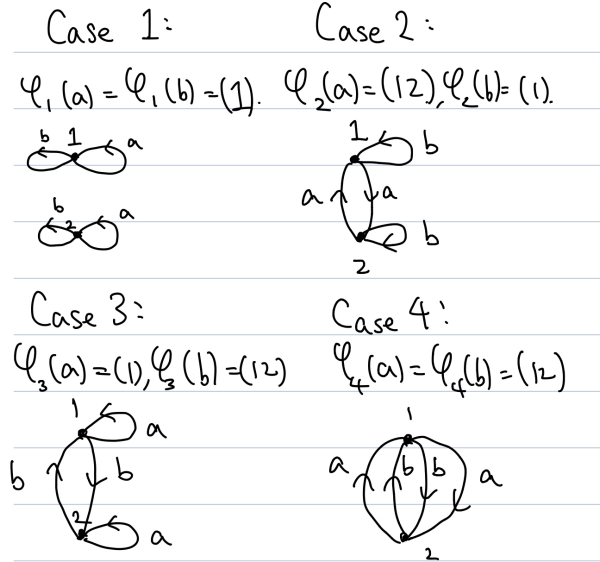$\phi_4(a) = \phi_4(b) = (12)$

FIGURE 2. Problem 10 (2-sheeted covers)

- $a \mapsto (1), b \mapsto (12)$
- $a \mapsto (1), b \mapsto (23)$
- $a \mapsto (1), b \mapsto (13)$

This is the same as Case 2.
- Case 4: $\phi_4 : a \mapsto (12), b \mapsto (13)$ The following maps are conjugates of ϕ_4
  - $a \mapsto (13), b \mapsto (12)$
  - $a \mapsto (12), b \mapsto (23)$
  - $a \mapsto (12), b \mapsto (13)$
  - $a \mapsto (13), b \mapsto (23)$
  - $a \mapsto (23), b \mapsto (12)$
  - $a \mapsto (23), b \mapsto (13)$

See Figure 3.
- Case 5: $\phi_5 : a \mapsto (12), b \mapsto (123)$ The following maps are conjugates of ϕ_5
  - $a \mapsto (23), b \mapsto (123)$
  - $a \mapsto (12), b \mapsto (123)$
  - $a \mapsto (12), b \mapsto (132)$
  - $a \mapsto (13), b \mapsto (132)$
  - $a \mapsto (13), b \mapsto (123)$
  - $a \mapsto (23), b \mapsto (132)$

See Figure 3.
- Case 6: $\phi_6 : a \mapsto (123), b \mapsto (12)$ The following maps are conjugates of ϕ_6
  - $a \mapsto (123), b \mapsto (13)$
  - $a \mapsto (132), b \mapsto (12)$
  - $a \mapsto (132), b \mapsto (23)$
  - $a \mapsto (132), b \mapsto (13)$
  - $a \mapsto (123), b \mapsto (12)$
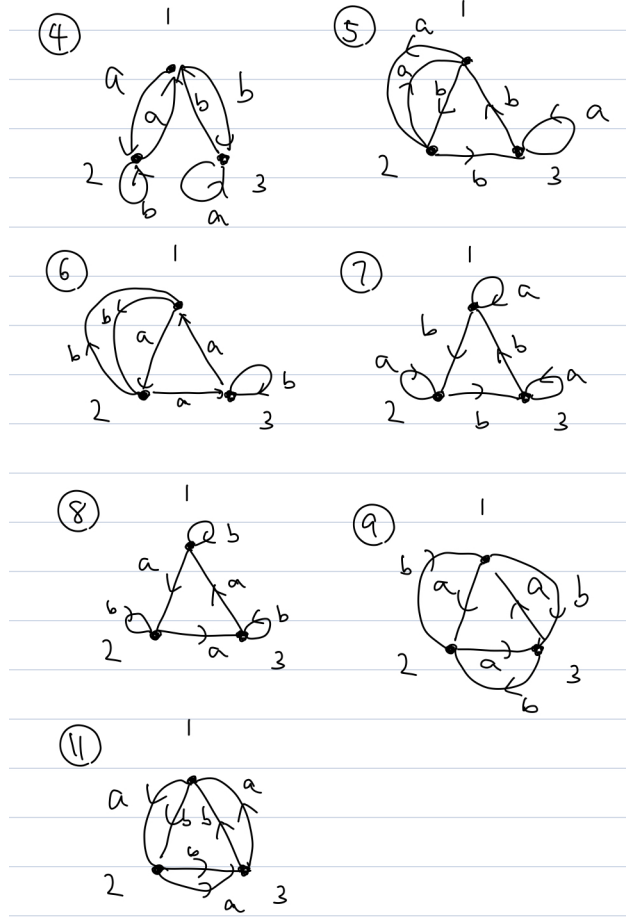 - $a \mapsto (123), b \mapsto (23)$

FIGURE 3. Problem 10 (3-sheeted)

See Figure 3.
- Case 7: $\phi_7 : a \mapsto (1), b \mapsto (123)$ The following maps are conjugates of ϕ_7
  - $a \mapsto (1), b \mapsto (132)$
  - $a \mapsto (1), b \mapsto (123)$

See Figure 3.
- Case 8: $\phi_8 : a \mapsto (123), b \mapsto (1)$ The following maps are conjugates of ϕ_8
  - $a \mapsto (132), b \mapsto (1)$
  - $a \mapsto (123), b \mapsto (1)$

See Figure 3.
- Case 9: $\phi_9 : a \mapsto (123), b \mapsto (132)$ The following maps are conjugates of ϕ_9
  - $a \mapsto (123), b \mapsto (132)$
  - $a \mapsto (132), b \mapsto (123)$

See Figure 3.
- Case 10: $\phi_{10} : a \mapsto (23), b \mapsto (23)$ The following maps are conjugates of ϕ_{10}
  - $a \mapsto (12), b \mapsto (12)$
  - $a \mapsto (23), b \mapsto (23)$
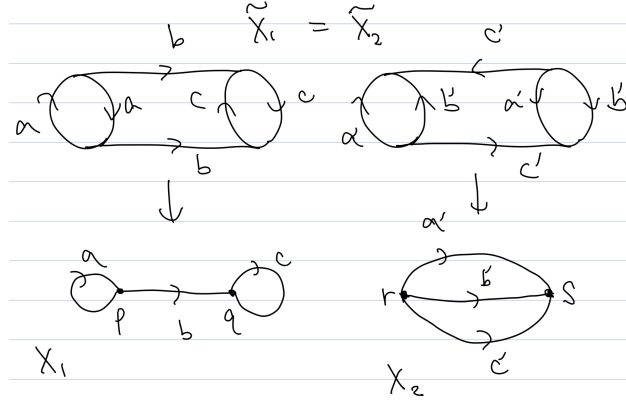 - $a \mapsto (13), b \mapsto (13)$

FIGURE 4. Problem 11

Vertex 1 is disconnected from the rest of the graph since it is fixed.
- Case 11: $\phi_{11} : a \mapsto (123), b \mapsto (123)$ The following maps are conjugates of ϕ_{11}
  - $a \mapsto (132), b \mapsto (132)$
  - $a \mapsto (123), b \mapsto (123)$

See Figure 3.

Since there are 6 elements in S_3, there are 36 possible homomorphisms. The list above contains all of them. Therefore, Figure 3 lists all the possible 3-sheeted covers. $\qquad\square$

Exercise. (Problem 11, Chapter 1.3) Construct finite graphs X_1 and X_2 having a common finite-sheeted covering space $\tilde{X}_1 = \tilde{X}_2$, but such that there is no space having both X_1 and X_2 as covering spaces.

Proof. Figure 4 shows X_1, X_2 and $\tilde{X}_1 = \tilde{X}_2$.

We claim that there exists no space having both X_1 and X_2 as covering spaces. On the contrary, suppose there exists such a space X with covering maps $p_1 : X_1 \to X, p_2 : X_2 \to X$. Then every point in X must have a neighborhood that homeomorphic to an open subset of X_1. Since X_1 is a graph, that means X is locally a line and a vertex with edges. In other words, X must be a graph.

There must exist a neighborhood of $p_1(p)$ and a neighborhood of p such that they are homeomorphic. Since p is a vertex of degree 3, $p_1(p)$ must be a vertex of degree 3 as well. Similarly, $p_1(q)$ must be a vertex of degree 3 as well.

Since p, q are the only vertices of X_1, X contains at most two vertices and their degrees must be 3. Since the sum of degrees of all vertices must be even from elementary graph theory, X must contain two vertices of degree 3.

If X only consists of loops, then the degree of each vertex will be even. Thus the two vertices must be joined by at least one edge. Then if one vertex has a loop, the other must have a loop as well in order to have degree 3. If there exists another edge joining the two vertices, there must be a third one in order for the two vertices to have degree 3. Therefore, X_1, X_2 are the only graphs with two vertices of degree 3.

Suppose that X_1 is a covering space of X_2 with a covering map $f : X_1 \to X_2$. Without loss of generality, $f(p) = r, f(q) = s$. Consider the path a' in X_2. Lifting a' to X_1 will result

in a path from p to q. This implies that f maps points on the path b into points on a path a'.

Now consider the path b' in X_2. Lifting b' to X_1 will again result in a path from p to q. This implies that f maps points on the path b into points on a path b'.

This implies that every point on the path b must be mapped to r or s. This is a contradiction because f is continuous and $\{b(t) \mid t \in [0,1]\}$ is connected, but $\{r, s\}$ is disconnected.

Thus X_1 is not a covering space of X_2.

Similarly, suppose that X_2 is a covering space of X_1 with a covering map $g : X_2 \to X_1$. Without loss of generality, $g(r) = p, g(s) = q$. This implies $g^{-1}(p) = \{r\}$, so the number of sheets is 1. In other words, g is injective. Consider the path a in X_1. Lifting a to X_2 results into a loop based at r. Since $a : I \to X_1$ is injective, $\tilde{a} : I \to X_2$ is injective since $g \circ \tilde{a} = a$. Then $\tilde{a}(t) = s$ for some $t \in [0,1]$, so $a(t) = g(\tilde{a}(t)) = g(s) = q$. However, q is not a point on a. This is a contradiction, so X_2 is not a covering space of X_1.

Hence, there exists no space that has both X_1 and X_2 as covering spaces. $\qquad\square$

Exercise. (Problem 14, Chapter 1.3) Find all the connected covering spaces of $\mathbb{R}\mathbf{P}^2 \vee \mathbb{R}\mathbf{P}^2$.

Proof. Let $X = \mathbb{R}\mathbf{P}^2 \vee \mathbb{R}\mathbf{P}^2$. By Theorem 1.38 of the textbook, it suffices to check all the conjugacy classes of subgroups of $\pi_1(X, x_0)$ and corresponding covering spaces.

Since $\pi_1(\mathbb{R}\mathbf{P}^2) = \langle a \mid a^2 \rangle$, $\pi_1(X, x_0) = \langle a, b \mid a^2 = b^2 = e \rangle$ by Van Kampen. Since $a^2 = b^2 = e$, we can express each element in $\pi_1(X, x_0)$ uniquely as a word which alternates a, b.

We will use three properties of elements in $\pi_1(X, x_0)$:

- Every word can be expressed as a finite alternating sequence of a and b.

  Maybe explain more.

- The parity of the length of a word does not depend on its representation because $a^2 = b^2 = 1$.

  Maybe explain more.

- The parity of the length of a word is invariant under conjugation because conjugating a word appends an even number of letters.

- The parity of the number of a in a word is invariant under conjugation because conjugating a word appends an even number of a.

  Maybe explain more.

- Similarly, the parity of the number of b in a word is invariant under conjugation because conjugating a word appends an even number of b.

We will call a subgroup even if the length of each element in it is even, and we will call a subgroup odd otherwise. Since the parity of the length of a word is invariant under conjugation, even subgroups and odd subgroups are not conjugates of each other. Therefore, we will classify even subgroups and odd subgroups separately.

- Let $H \subset G$ be an even subgroup. Every even length word can be expressed as $(ab)^k$ for some $k \in \mathbb{Z}$. Since $(ab)^k = e$ if and only if $k = 0$, this number k is unique to each word of odd length. Let $\phi : H \to \mathbb{Z}$ be defined such that $\phi((ab)^k) = k$. Then $\phi(H)$ is a subgroup of $(\mathbb{Z}, +)$. Since $\mathbb{Z}$ is an infinite cyclic group, $\phi(H) = (n)$ for some non-negative integer n. Thus $H = \langle (ab)^n \rangle$. For each n,

shows the covering space corresponding to $\langle (ab)^n \rangle$. Since the number of sheets is the same as the index of the subset by Proposition 1.32 of the textbook, $[\pi_1(X, x_0) : \langle (ab)^n \rangle]$ is infinity when $n = 0$ and the index is $2n$ when $n \geq 1$. Since the index of a subgroup is invariant under conjugation, $\langle (ab)^n \rangle \not\sim \langle (ab)^m \rangle$ whenever $n \neq m$.

- Let $H \subset G$ be an odd subgroup. Every odd element is an alternating sequence of a and b which starts and ends with the same letter. Thus, every odd element is conjugate to a or b. Let $w \in H$ be a word of odd length. Then w is xax^{-1} or xbx^{-1} for some x. Then $x^{-1}Hx$ contains a or b. This implies that every odd subgroup is conjugate to an odd subgroup containing either a or b. (possibly both) Since we are classifying subgroups up to conjugation, we can assume that H contains a or b without loss of generality.

We will first assume that $a \in H$ since the argument for the case $b \in H$ will be the same. Let H' be a subset of H containing only words of even length. Since $e \in H'$ and xy^{-1} is another word of even length for each $x, y \in H'$, H' is a subgroup of H. Then H' is an even subgroup of G, so by the argument above, $H' = \langle (ab)^n \rangle$ for some $n \geq 0$.

Let $w \in H$.
  - If the length of w is even, $w \in H' = \langle (ab)^n \rangle = \langle a, (ab)^n \rangle$.
  - If the length of w is odd, $aw \in H' = \langle (ab)^n \rangle = \langle a, (ab)^n \rangle$. Since $aw \in \langle a, (ab)^n \rangle$, $w \in \langle a, (ab)^n \rangle$.

Therefore, $H = \langle a, (ab)^n \rangle$.
  - When $n = 0$, $H = \langle a \rangle$.
  - When $n = 1$, $H = \langle a, ab \rangle = \langle a, b \rangle = G$.
  - When $n \geq 2$, $H = \langle a, (ab)^n \rangle$.

Using the same argument, if $b \in H$, we can conclude that H is one of the following:
  - When $n = 0$, $H = \langle b \rangle$.
  - When $n = 1$, $H = \langle b, ab \rangle = \langle a, b \rangle = G$.
  - When $n \geq 2$, $H = \langle b, (ab)^n \rangle$.

By putting these together, if H is an odd subgroup, it must be conjugate to one or more of the following:
  - $\langle a \rangle$.
  - $\langle b \rangle$.
  - $\langle a, (ab)^k \rangle$ where $k \geq 2$.
  - $\langle b, (ab)^k \rangle$ where $k \geq 2$.
  - G.

It suffices to classify these subgroups up to conjugation since this is an exhaustive list of odd subgroups up to conjugation.

First, we will use the index of each subgroup to classify them. If two subgroups have different indices, they cannot be conjugate to each other. By

, $[\pi_1(X, x_0) : \langle a \rangle] = [\pi_1(X, x_0) : \langle b \rangle] = \infty$ and $[\pi_1(X, x_0) : \langle a, (ab)^k \rangle] = [\pi_1(X, x_0) : \langle b, (ab)^k \rangle] = k$ for each $k \geq 2$. Moreover, the parity of the number of b in each word

of $\langle a \rangle$ is even since it is always 0. On the other hand, the parity of the number of b in $b \in \langle b \rangle$ is 1. Thus $\langle a \rangle$ and $\langle b \rangle$ cannot be conjugate to each other.

Finally, it remains to check if $\langle a, (ab)^k \rangle \sim \langle b, (ab)^k \rangle$ for each $k \geq 2$. For that purpose, we will consider two cases based on the parity of k.

   $- k \geq 2$ and k is odd. Let $k = 2l + 1$.

$$
\begin{aligned}
\langle a, (ab)^k \rangle &= \langle a, (ab)^{2l+1} \rangle \\
&= \langle a, (ab)^{l+1}(ab)^l \rangle \\
&= \langle a, (a(ba)^l b)(ab)^l \rangle \\
&= \langle a, a(ba)^l b(ab)^l \rangle \\
&= \langle a, a(ab)^{-l} b(ab)^l \rangle \\
&= \langle a, (ab)^{-l} b(ab)^l \rangle \\
&\sim \langle (ab)^l a(ab)^{-l}, b \rangle \\
&= \langle (ab)^l a(ab)^{-l} b, b \rangle \\
&= \langle (ab)^{2l+1}, b \rangle \\
&= \langle (ab)^k, b \rangle .
\end{aligned}
$$

Therefore, when $k \geq 2$ and k is odd, $\langle a, (ab)^k \rangle$ and $\langle b, (ab)^k \rangle$ are conjugates of each other.

   $- k \geq 2$ and k is even. The number of b in a is 0, and the number of b in $(ab)^k$ is k. In each case, the number of b is even. Since every element in $\langle a, (ab)^k \rangle$ is generated by them, the parity of the number of b in any element of $\langle a, (ab)^k \rangle$ is even. However, the number of b in $b \in \langle b, (ab)^k \rangle$ is 1, which is odd. Therefore, $\langle a, (ab)^k \rangle$ is not conjugate to $\langle b, (ab)^k \rangle$.

Hence, the following is the complete list of all the distinct subgroups of $\pi_1(X)$ up to conjugation and

Figure

has the covering spaces corresponding to each class:

- Even subgroups.
  - $\langle e \rangle$.
  - $\langle (ab)^n \rangle$ for $n \geq 1$.
- Odd subgroups.
  - $\pi_1(X)$.
  - $\langle a \rangle$.
  - $\langle a \rangle$.
  - $\langle a, (ab)^k \rangle$ for each $k \geq 2$.
  - $\langle b, (ab)^k \rangle$ for each $k \geq 2$ and k even.

$\square$