MATH 611 PROBLEM SET 1 (DUE 9/4)

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Exercise 0.1. (Exercise 4, Chapter 0) A deformation retraction in the weak sense of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \mathrm{Id}, f_1(X) \subset A$, and $f_t(A) \subset A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \to X$ is a homotopy equivalence.

Proof. Let $i : A \to X$ denote the inclusion. Let $F : X \times I \to X$ denote the associated map $(x, t) \to f_t(x)$. Then F is a continuous function by the definition of a homotopy.

Let $f : X \to A$ be defined by $f(x) = F(x, 1) = f_1(x)$. This definition makes sense because $f_1(X) \subset A$. We claim that $f_1 \circ i \simeq \mathrm{Id}_A$ and $i \circ f_1 \simeq \mathrm{Id}_X$.

Consider $G : A \times I \to A$ such that $G(a, t) = F(a, t)$ for all $(a, t) \in A \times I$. This definition makes sense because $f_t(A) \subset A$ for all t.

Then G is a homotopy in A between $f \circ i$ and Id_A because:

- G is a restriction of F, so G is continuous.
- $\forall a \in A, G(a, 0) = F(a, 0) = f_0(a) = \mathrm{Id}_X(a) = \mathrm{Id}_A(a)$.
- $\forall a \in A, G(a, 1) = F(a, 1) = f(a) = f(i(a)) = (f \circ i)(a)$.

Therefore, $f \circ i \simeq \mathrm{Id}_A$.

F is a homotopy between f_0 and f_1.

- We are given that $f_0 = \mathrm{Id}_X$.
- For any $x \in X$, $(i \circ f)(x) = i(f(x)) = f(x) = f_1(x)$, so $i \circ f = f_1$.

Therefore, F is a homotopy between Id_X and $i \circ F$, so $i \circ f \simeq \mathrm{Id}_X$.

In conclusion, i is indeed a homotopy equivalence. $\square$