MATH 611 PROBLEM SET 1 (DUE 9/4)

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Exercise 0.1. (Exercise 4, Chapter 0) A deformation retraction in the weak sense of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \mathrm{Id}, f_1(X) \subset A$, and $f_t(A) \subset A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \to X$ is a homotopy equivalence.

Proof. Let $i : A \to X$ denote the inclusion. Let $F : X \times I \to X$ denote the associated map $(x, t) \to f_t(x)$. Then F is a continuous function by the definition of a homotopy.

Let $f : X \to A$ be defined by $f(x) = F(x, 1) = f_1(x)$. This definition makes sense because $f_1(X) \subset A$. We claim that $f_1 \circ i \simeq \mathrm{Id}_A$ and $i \circ f_1 \simeq \mathrm{Id}_X$.

Consider $G : A \times I \to A$ such that $G(a, t) = F(a, t)$ for all $(a, t) \in A \times I$. This definition makes sense because $f_t(A) \subset A$ for all t.

Then G is a homotopy in A between $f \circ i$ and Id_A because:

- G is a restriction of F, so G is continuous.
- $\forall a \in A, G(a, 0) = F(a, 0) = f_0(a) = \mathrm{Id}_X(a) = \mathrm{Id}_A(a)$.
- $\forall a \in A, G(a, 1) = F(a, 1) = f(a) = f(i(a)) = (f \circ i)(a)$.

Therefore, $f \circ i \simeq \mathrm{Id}_A$.

F is a homotopy between f_0 and f_1.

- We are given that $f_0 = \mathrm{Id}_X$.
- For any $x \in X$, $(i \circ f)(x) = i(f(x)) = f(x) = f_1(x)$, so $i \circ f = f_1$.

Therefore, F is a homotopy between Id_X and $i \circ F$, so $i \circ f \simeq \mathrm{Id}_X$.

In conclusion, i is indeed a homotopy equivalence. $\square$

Exercise 0.2. (Exercise 9, Chapter 0) Show that a retract of a contractible space is contractible.

Proof. Let X be a contractible space. Then Id_X is homotopic to a constant map. This implies the existence of a fixed point $p \in X$ and a continuous function $F : X \times I \to X$ such that

- $\forall x \in X, F(x, 0) = x$,
- $\forall x \in X, F(x, 1) = p$.

Let $A \subset X$ be a retract of X, and let $r : X \to A$ denote a retraction. In other words, $r(X) = A$ and $r |_A = \mathrm{Id}_A$.

Let $G : A \times I \to A$ be the restriction of $r \circ F$ to $A \times I$. This makes sense because F maps $A \times I$ into X, and r maps X into A. We claim that G is a homotopy between Id_A and the constant map $e_{r(p)}$ such that $e_{r(p)}(a) = r(p)$ for all $a \in A$.

- $r \circ F$ is continuous since it is a composition of continuous functions. G is a restriction of a continuous function, so G is continuous.
- $G(a, 0) = r(F(a, 0)) = r(a) = a = \mathrm{Id}_A(a)$.
- $G(a, 1) = r(F(a, 1)) = r(p) = e_{r(p)}(a)$.

Therefore, G is indeed a homotopy between Id_A and the constant map at $r(p)$. Since the identity map is homotopic to a constant map, A is contractible. □

Exercise 0.3. (Exercise 20, Chapter 1.1) Suppose $f_t : X \to X$ is a homotopy such that f_0 and f_1 are each the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$.

Proof. Let $x_0 \in X$ be given. Let $f : I \to X$ be the loop defined such that $f(t) = f_t(x_0)$.

- $f_t : X \to X$ is a homotopy.
- f is a path formed by the images of the base point x_0.

By Lemma 1.19, the following diagram commutes.

$$\begin{tikzcd}
 & \pi_1(X, f_1(x_0)) \arrow[dd, "\beta_f"] \\
\pi_1(X, x_0) \arrow[ur, "(f_1)_*"'] \arrow[dr, "(f_0)_*"'] & \\
 & \pi_1(X, f_0(x_0))
\end{tikzcd}$$

$(f_0)_* = (f_1)_* = (\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(X,x_0)}$ by a basic property of induced homomorphisms (P.34 of Hatcher). Since $f_0 = f_1 = \mathrm{Id}_X$, $f_0(x_0) = f_1(x_0) = x_0$. Therefore, the diagram above can be simplified as following:

$$\begin{tikzcd}
& \pi_1(X, x_0) \arrow[dd, "\beta_f"] \\
\pi_1(X, x_0) \arrow[ur, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"'] & \\
& \pi_1(X, x_0)
\end{tikzcd}$$

Let $[g] \in \pi_1(X, x_0)$. Then by the diagram above, we have $\mathrm{Id}([g]) = \mathrm{Id}(\beta_f([g]))$. This implies $[g] = [f \cdot g \cdot \overline{f}]$. Therefore, $[g] \cdot [f] = [f] \cdot [g]$, so $[f]$ commutes with every element in $\pi_1(X, x_0)$. Hence, $[f] \in Z(\pi_1(X, x_0))$. $\square$