MATH 601 HOMEWORK (DUE 9/11)

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Exercise. (1) Show that 2×2 matrices give a functor, M_2, from the category of rings to itself, $R \mapsto M_2(R)$.

Proof. Let R, R' be rings and $\phi \in \mathrm{Hom}(R, R')$. Let $M_2(\phi) : M_2(R) \to M_2(R')$ be defined such that

$$(M_2(\phi))\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{bmatrix}.$$

We claim that M_2 is indeed a functor.

- Claim 1: For any $\phi \in \mathrm{Hom}(R, R')$, $M_2(\phi) \in \mathrm{Hom}(M_2(R), M_2(R'))$. In other words, we want to show that $M_2(\phi)$ is a ring homomorphism for any ϕ.

$$\begin{aligned}
(M_2(\phi))\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) &= (M_2(\phi))\left(\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}\right) \\
&= \begin{bmatrix} \phi(a+e) & \phi(b+f) \\ \phi(c+g) & \phi(d+h) \end{bmatrix} \\
&= \begin{bmatrix} \phi(a)+\phi(e) & \phi(b)+\phi(f) \\ \phi(c)+\phi(g) & \phi(d)+\phi(h) \end{bmatrix} \\
&= \begin{bmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{bmatrix} + \begin{bmatrix} \phi(e) & \phi(f) \\ \phi(g) & \phi(h) \end{bmatrix} \\
&= (M_2(\phi))\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (M_2(\phi))\begin{bmatrix} e & f \\ g & h \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&(M_2(\phi))\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) \\
&= (M_2(\phi))\left(\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}\right) \\
&= \begin{bmatrix} \phi(ae+bg) & \phi(af+bh) \\ \phi(ce+dg) & \phi(cf+dh) \end{bmatrix} \\
&= \begin{bmatrix} \phi(a)\phi(e)+\phi(b)\phi(g) & \phi(a)\phi(f)+\phi(b)\phi(h) \\ \phi(c)\phi(e)+\phi(d)\phi(g) & \phi(c)\phi(f)+\phi(d)\phi(h)) \end{bmatrix} \\
&= \begin{bmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{bmatrix}\begin{bmatrix} \phi(e) & \phi(f) \\ \phi(g) & \phi(h) \end{bmatrix} \\
&= (M_2(\phi))\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)(M_2(\phi))\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)
\end{aligned}$$

Therefore, $M_2(\phi)$ is indeed a ring homomorphism.

- For any ring R and the identity function Id_R, $M_2(\mathrm{Id}_R)$ is the identity map on $M_2(R)$ because it maps each element in a given matrix to itself.
- Let $f \in \mathrm{Hom}(A, B), g \in \mathrm{Hom}(B, C)$.

$$\begin{aligned}
(M_2(f \circ g))\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= \begin{bmatrix} (f\circ g)(a) & (f\circ g)(b) \\ (f\circ g)(c) & (f\circ g)(d) \end{bmatrix} \\
&= \begin{bmatrix} f(g(a)) & f(g(b)) \\ f(g(c)) & f(g(d)) \end{bmatrix} \\
&= M_2(f)\left(\begin{bmatrix} g(a) & g(b) \\ g(c) & g(d) \end{bmatrix}\right) \\
&= M_2(f)\left(M_2(g)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right) \\
&= (M_2(f)\circ M_2(g))\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).
\end{aligned}$$

Therefore, M_2 is indeed a functor. $\square$

Exercise. (Problem 8 from More exercises) Consider the subgroup, $D_5 = \langle (12345), (14)(23) \rangle \subset S_5$.

(1) Set $a = (12345)$ and compute a^{-1}.
(2) Set $b = (14)(23)$ and compute aba^{-1}.
(3) Show that every element in D_5 may be written in the form $a^i b^j$ for some $i, j \in \mathbb{Z}$.
(4) Compute $|D_5|$.

Proof.

(1) a sends 1 to 2, 2 to 3, $\cdots$. We want a^{-1} to do the opposite. Thus $a^{-1} = (15432)$. Since $(12345)(15432) = (15432)(12345) = (1)$, (15432) is indeed a^{-1}.
(2) $aba^{-1} = (a(1)a(4))(a(2)a(3)) = (25)(34)$.
(3) $ba = (14)(23)(12345) = (13)(45)$, and $a^{-1}b = (15432)(14)(23) = (13)(45)$. Therefore, $ba = a^{-1}b$. We claim that $ba^n = a^{-n}b$ for every $n \in \mathbb{N}$. Suppose $ba^n = a^{-n}b$ for some $n \in \mathbb{N}$. Then $ba^{n+1} = (ba^n)a = (a^{-n}b)a = a^{-n}(ba) = a^{-n}a^{-1}b = a^{-n-1}b$. By mathematical induction, $ba^n = a^{-n}b$ for every $n \in \mathbb{N}$.

    For any $n \in \mathbb{N}$, $ba^n = a^{-n}b$, so $a^nba^n = b$, and thus $a^nb = ba^{-n}$. Therefore, we have $ba^k = a^{-k}b$ for every $k \in \mathbb{Z}$.

    We claim that for any $i, j \in \mathbb{Z}$, b^ja^i can be written in the desired form. Since $b^2 = e$, we consider two cases based on the parity of j. If j is even, then $b^j = e$, so $b^ja^i = a^i$. If j is odd, then $b^j = b$, so $b^ja^i = ba^i = a^{-i}b$, which is in the desired form.

    TODO
(4) TODO

$\square$