MATH 601 (DUE 10/9)

HIDENORI SHINOHARA

Contents

1. Modules

Exercise. (Problem 1) For each of the $\mathbb{Z}$-modules listed in the handout, answer the questions in the handout.

Proof.

(a) $M = \mathbb{Z}^3 \times \mathbb{Z}/86\mathbb{Z}$.

Solve this problem!

(b) $M = \prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z}$.

Solve this problem!

(c) $M = \mathbb{Z}[1/p] \subset \mathbb{Q}$.

Solve this problem!

(d) $M = \mathbb{Q}/\mathbb{Z}_{(p)}$.

Solve this problem!

$\square$

2. Rings of Fractions

Exercise. (Problem 3) Let $T \subset R$ be the subset consisting of all nonzero divisors.

- Show that T is a multiplicative set.
- Let $s \in T$ and let $S = \{1, s, s^2, s^3, \cdots\} \subset T$. Show that the following rings are isomorphic: $S^{-1}R$, the subring $R[1/s] \subset T^{-1}R$, and the quotient ring $R[x]/(sx - 1)$.

Proof.

- Prove this!
- Prove this!

$\square$

3. The Quadratic Equation

Exercise. (Problem 20)

Exercise. (Problem 21)

Exercise. (Problem 22)

4. Factorization in Integral Domains

Exercise. (Problem 5)

- Let k be a field and let $a \in k$. Construct a k-algebra isomorphism, $k[x, y]/(x - a) \to k[y]$. Justify your answer.
- Let $f(x, y) \in k[x, y]$. What is the image of $f(x, y)$ under the above isomorphism?

Proof.

Ask Professor Schoen after class on Friday.

- Let ϕ be defined such that $\phi(f(x, y) + (x - a)) = f(a, y)$.
  - Well-defined? Let $f(x, y) + (x - a) = g(x, y) + (x - a)$. Then $g(x, y) = f(x, y) + h(x, y)(x - a)$.
  $$\begin{aligned} \phi(g(x, y) + (x - a)) &= \phi((f(x, y) + h(x, y)(x - a)) + (x - a)) \\ &= f(a, y) + h(a, y)(a - a) \\ &= f(a, y) \\ &= \phi(f(x, y)). \end{aligned}$$
  - k-algebra homomorphism? Let $c \in k$, $f, g \in k[x, y]$ be given.
  $$\begin{aligned} \phi(c(f + (x - a)) &= \phi(cf + (x - a)) \\ &= cf(a, y) \\ &= c\phi(f + (x - a)). \\ \phi((f + g) + (x - a)) &= (f + g)(a, y) \\ &= f(a, y) + g(a, y) \\ &= \phi(f + (x - a)) + \phi(g + (x - a)). \\ \phi((fg) + (x - a)) &= (fg)(a, y) \\ &= f(a, y)g(a, y) \\ &= \phi(f + (x - a))\phi(g + (x - a)). \end{aligned}$$
- $\phi(f(x, y) + (x - a)) = f(a, y)$.

□

Exercise. (Problem 6)

- Give an example of a field k, an element $a \in k$ and a reducible polynomial $f(x, y) \in k[x, y]$ of degree n in y such that $f(a, y) \in k[y]$ is irreducible and has degree n.

- [redacted]
- Give an example of a field k, an element, $a \in k$, and a reducible polynomial $f(x, y) \in k[x, y]$, which when viewed as an element of $k(x)[y]$ has degree n and content 1 such that $f(a, y) \in k[y]$ is irreducible.

Proof.

- Let $k = \mathbb{Q}, a = 1, f(x, y) = xy$. Then the degree of $f(x, y)$ in y is 1. $f(x, y) = xy \in k[x, y]$ is reducible since x and y are not units in $k[x, y]$. However, $f(a, y) = 1y = y$ is irreducible in $k[y]$.
- [redacted]
- Let $k = \mathbb{Q}, a = 1, f(x, y) = (x - 1)y^2 + y$. Then $f(x, y)$, which when viewed as an element of $k(x)[y]$ has degree 1. The content is 1 since
  - The coefficient of y is 1, and $\text{ord}_p(1) = 0$ for any p.
  - $x - 1$ is a prime in $k(x)[y]$ since
    [redacted]

  [redacted]

□