MATH 601 HOMEWORK (DUE 9/11)

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Exercise. (1) Show that 2×2 matrices give a functor, M_2, from the category of rings to itself, $R \mapsto M_2(R)$.

Proof. Let R, R' be rings and $\phi \in \mathrm{Hom}(R, R')$. Let $M_2(\phi) : M_2(R) \to M_2(R')$ be defined such that

$$(M_2(\phi))\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{bmatrix}.$$

We claim that M_2 is indeed a functor.

- Claim 1: For any $\phi \in \mathrm{Hom}(R, R')$, $M_2(\phi) \in \mathrm{Hom}(M_2(R), M_2(R'))$. In other words, we want to show that $M_2(\phi)$ is a ring homomorphism for any ϕ.

$$\begin{aligned}
(M_2(\phi))\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) &= (M_2(\phi))\left(\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}\right) \\
&= \begin{bmatrix} \phi(a+e) & \phi(b+f) \\ \phi(c+g) & \phi(d+h) \end{bmatrix} \\
&= \begin{bmatrix} \phi(a)+\phi(e) & \phi(b)+\phi(f) \\ \phi(c)+\phi(g) & \phi(d)+\phi(h) \end{bmatrix} \\
&= \begin{bmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{bmatrix} + \begin{bmatrix} \phi(e) & \phi(f) \\ \phi(g) & \phi(h) \end{bmatrix} \\
&= (M_2(\phi)) \begin{bmatrix} a & b \\ c & d \end{bmatrix} + (M_2(\phi)) \begin{bmatrix} e & f \\ g & h \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&(M_2(\phi))\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) \\
&= (M_2(\phi))\left(\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}\right) \\
&= \begin{bmatrix} \phi(ae+bg) & \phi(af+bh) \\ \phi(ce+dg) & \phi(cf+dh) \end{bmatrix} \\
&= \begin{bmatrix} \phi(a)\phi(e)+\phi(b)\phi(g) & \phi(a)\phi(f)+\phi(b)\phi(h) \\ \phi(c)\phi(e)+\phi(d)\phi(g) & \phi(c)\phi(f)+\phi(d)\phi(h)) \end{bmatrix} \\
&= \begin{bmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{bmatrix}\begin{bmatrix} \phi(e) & \phi(f) \\ \phi(g) & \phi(h) \end{bmatrix} \\
&= (M_2(\phi))\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)(M_2(\phi))\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)
\end{aligned}$$

Therefore, $M_2(\phi)$ is indeed a ring homomorphism.

- For any ring R and the identity function Id_R, $M_2(\mathrm{Id}_R)$ is the identity map on $M_2(R)$ because it maps each element in a given matrix to itself.
- Let $f \in \mathrm{Hom}(A, B), g \in \mathrm{Hom}(B, C)$.

$$\begin{aligned}
(M_2(f \circ g))\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= \begin{bmatrix} (f\circ g)(a) & (f\circ g)(b) \\ (f\circ g)(c) & (f\circ g)(d) \end{bmatrix} \\
&= \begin{bmatrix} f(g(a)) & f(g(b)) \\ f(g(c)) & f(g(d)) \end{bmatrix} \\
&= M_2(f)\left(\begin{bmatrix} g(a) & g(b) \\ g(c) & g(d) \end{bmatrix}\right) \\
&= M_2(f)\left(M_2(g)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right) \\
&= (M_2(f) \circ M_2(g))\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).
\end{aligned}$$

Therefore, M_2 is indeed a functor. □

Exercise. (Problem 4 from More exercises)

(1) If F is a functor from category C to a category C' and G is a functor from a category C' to a category C'', under what conditions is a composite functor, $G \circ F : C \to C''$ defined?
(2) For a ring R write $GL_2(R)$ for the set of all invertible 2×2 matrices with entries in R. List the exercises above and the sections of the handouts which combine to give a proof that GL_2 is a functor from rings to groups.
(3) For a commutative ring R let $SL_2(R)$ denote the set of all 2×2 matrices with entries in R and determinant 1. Is SL_2 a functor from commutative rings to groups?
(4) Let k be a field. There is a natural right action of $GL_2(k)$ on $\mathbb{P}^1(k)$. Write down how an element of $GL_2(k)$ acts on an element of $\mathbb{P}^1(k)$ using homogeneous coordinates.
(5) Determine the subroup of $GL_2(k)$ which acts as the identity on $\mathbb{P}^1(k)$.

Proof.

(1) A composition of two functors is always a functor.
(2) Exercise 1 from More exercises shows that M_2 is a functor from the category of rings to itself. From "Units as a functor" in the handout from the first lecture, we know that passing from rings to units is a functor from the category of rings to the category of groups. Then by composing M_2 with the operation to take units, we get GL_2. Exercise 4(a) from More exercises shows that a composition of two functors is a functor. Thus GL_2 is a functor.
(3) Yes, it is.
(4)
$$(x_0 : x_1) \star \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (ax_0 + cx_1 : bx_0 + dx_1).$$
(5) We claim that the subgroup $\{tI \mid t \in k^\star\}$ acts as the identity on $\mathbb{P}^1(k)$. $(x_0 : x_1) \star tI = (tx_0 : tx_1) = (x_0 : x_1)$. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P}^1(k)$. Suppose A acts as the identity on $\mathbb{P}^1(k)$.
- Case 1: $b \neq 0$. Then $(1 : 0) \star \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a : b)$. Since $b \neq 0$, $(t \cdot 1, t \cdot 0) \neq (a, b)$ for any $t \in k^\times$. Therefore, A does not act as the identity on $\mathbb{P}^1(k)$.
- Case 2: $d \neq 0$. Then $(0 : 1) \star \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (c : d)$. Since $d \neq 0$, $(t \cdot 0, t \cdot 1) \neq (c, d)$ for any $t \in k^\times$. Therefore, A does not act as the identity on $\mathbb{P}^1(k)$.
- Case 3: $b = d = 0$ and $a \neq d$. Then $(1 : 1) \star \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a : d)$. Since $a \neq d$, $(t \cdot 1, t \cdot 1) \neq (a, d)$ for any $t \in k^\times$. Therefore, A does not act as the identity on $\mathbb{P}^1(k)$.

This means that $b = d = 0$ and $a = d$. Since A is invertible, $a \neq 0$. Thus A is indeed an element of $\{tI \mid t \in k^\times\}$. Therefore, $\{tI \mid t \in k^\times\}$ is exactly the set of elements that act as the identity on $\mathbb{P}^1(k)$.

□

Exercise. (Problem 8 from More exercises) Consider the subgroup, $D_5 = \langle (12345), (14)(23) \rangle \subset S_5$.
(1) Set $a = (12345)$ and compute a^{-1}.
(2) Set $b = (14)(23)$ and compute aba^{-1}.
(3) Show that every element in D_5 may be written in the form $a^i b^j$ for some $i, j \in \mathbb{Z}$.
(4) Compute $|D_5|$.
(5) Draw a regular pentagon with vertices labeled successively 1, 2, 3, 4, 5. Show that D_5 acts on the pentagon by describing he action in geometric terms.
(6) Recall that a group acts on its subgroups by conjugation, $H \subset G, H \mapsto gHg^{-1}$. The orbits of this action are called conjugacy classes of subgroups. Determine all the conjugacy classes of subgroups of D_5.

Proof.
(1) a sends 1 to 2, 2 to 3, $\cdots$. We want a^{-1} to do the opposite. Thus $a^{-1} = (15432)$. Since $(12345)(15432) = (15432)(12345) = (1)$, (15432) is indeed a^{-1}.

(2) $aba^{-1} = (a(1)a(4))(a(2)a(3)) = (25)(34)$.

(3) $ba = (14)(23)(12345) = (13)(45)$, and $a^{-1}b = (15432)(14)(23) = (13)(45)$. Therefore, $ba = a^{-1}b$. We claim that $ba^n = a^{-n}b$ for every $n \in \mathbb{N}$. Suppose $ba^n = a^{-n}b$ for some $n \in \mathbb{N}$. Then $ba^{n+1} = (ba^n)a = (a^{-n}b)a = a^{-n}(ba) = a^{-n}a^{-1}b = a^{-n-1}b$. By mathematical induction, $ba^n = a^{-n}b$ for every $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$, $ba^n = a^{-n}b$, so $a^n ba^n = b$, and thus $a^n b = ba^{-n}$. Therefore, we have $ba^k = a^{-k}b$ for every $k \in \mathbb{Z}$.

We claim that for any $i, j \in \mathbb{Z}$, $b^j a^i$ can be written in the desired form. Since $b^2 = e$, we consider two cases based on the parity of j. If j is even, then $b^j = e$, so $b^j a^i = a^i$. If j is odd, then $b^j = b$, so $b^j a^i = ba^i = a^{-i}b$ as shown above.

We will prove the general case. By the argument above, it suffices to show that every element in D_5 can be represented as a word of length ≤ 2. Let $x_1^{i_1} \cdots x_k^{i_k} \in D_5$ be given where $i_1, \cdots, i_k \in \mathbb{Z}$ and each x_i is either a or b. Since D_5 is generated by a, b, every element can be represented in this form. We will show that every element in D_5 can be represented as a word of length ≤ 2 by using strong induction. If $k \leq 2$, then we are done. Suppose that we can represent every element in D_5 of length $\leq k$ as a word of length ≤ 2 for some $k \geq 2$. Let $x = x_1^{i_1} \cdots x_{k+1}^{i_{k+1}} \in D_5$. If $x_1 = x_2$, then $x = x_2^{i_1+i_2} x_3^{i_3} \cdots x_{k+1}^{i_{k+1}}$, so by the inductive hypothesis, this can be represented as a word of length ≤ 2. If $x_2 = x_3$, then $x = x_1^{i_1} x_2^{i_2+i_3} x_4^{i_4} \cdots x_{k+1}^{i_{k+1}}$, so by the inductive hypothesis, this can be represented as a word of length ≤ 2. Suppose $x_1 \neq x_2$ and $x_2 \neq x_3$. Then there are two cases:

- Case 1: $(x_1, x_2, x_3) = (a, b, a)$. By the argument above, $b^{i_2} a^{i_3}$ can be represented as $a^i b^j$ for some $i, j \in \mathbb{Z}$. Therefore, $a^{i_1}(b^{i_2} a^{i_3}) = a^{i_1}(a^i b^j) = a^{i_1+i} b^j$, so x can be represented as a word of length k. By the inductive hypothesis, x can be represented as a word of length ≤ 2.
- Case 2: $(x_1, x_2, x_3) = (b, a, b)$. By the argument above, $b^{i_1} a^{i_2}$ can be represented as $a^i b^j$ for some $i, j \in \mathbb{Z}$. Therefore, $(b^{i_1} a^{i_2}) b^{i_3} = (a^i b^j) b^{i_3} = a^i b^{j+i_3}$. By the inductive hypothesis, x can be represented as a word of length ≤ 2.

(4)
- $a^1 = a \neq (1)$.
- $a^2 = (13524) \neq (1)$.
- $a^3 = (14253) \neq (1)$.
- $a^4 = (15432) \neq (1)$.
- $a^5 = (1)$.

Therefore, the order of a is 5. Since $b \neq (1)$ and $b^2 = (1)$, the order of b is 2. We claim that there are exactly 10 elements in D_5.

- Claim 1: $|D_5| \leq 10$. Let $x \in D_5$. Then there exist $i, j \in \mathbb{Z}$ such that $x = a^i b^j$. Since the order of a is 5 and the order of b is 2, we can assume that $0 \leq i \leq 4$ and $0 \leq j \leq 1$. Therefore, $D_5 \subset \{a^i b^j \mid 0 \leq i \leq 4, 0 \leq j \leq 1\}$. Thus there are at most 10 elements in D_5.
- Claim 2: $|D_5| \leq 10$. Let $a^i b^j, a^{i'} b^{j'} \in \{a^i b^j \mid 0 \leq i \leq 4, 0 \leq j \leq 1\}$. Suppose $a^i b^j = a^{i'} b^{j'}$. Then $a^{i-i'} = b^{j'-j}$. We have calculated all the powers of a above, and none of them is equal to b. Therefore, $i - i' \equiv 0 \pmod 5$ and $j - j' \equiv 0 \pmod 2$. Since $0 \leq i, i' \leq 4, 0 \leq j, j' \leq 1$, $i = i'$ and $j = j'$. This implies that the set $\{a^i b^j \mid 0 \leq i \leq 4, 0 \leq j \leq 1\}$ contains exactly 10 elements. Since the set is a subset of D_5, D_5 contains at least 10 elements.

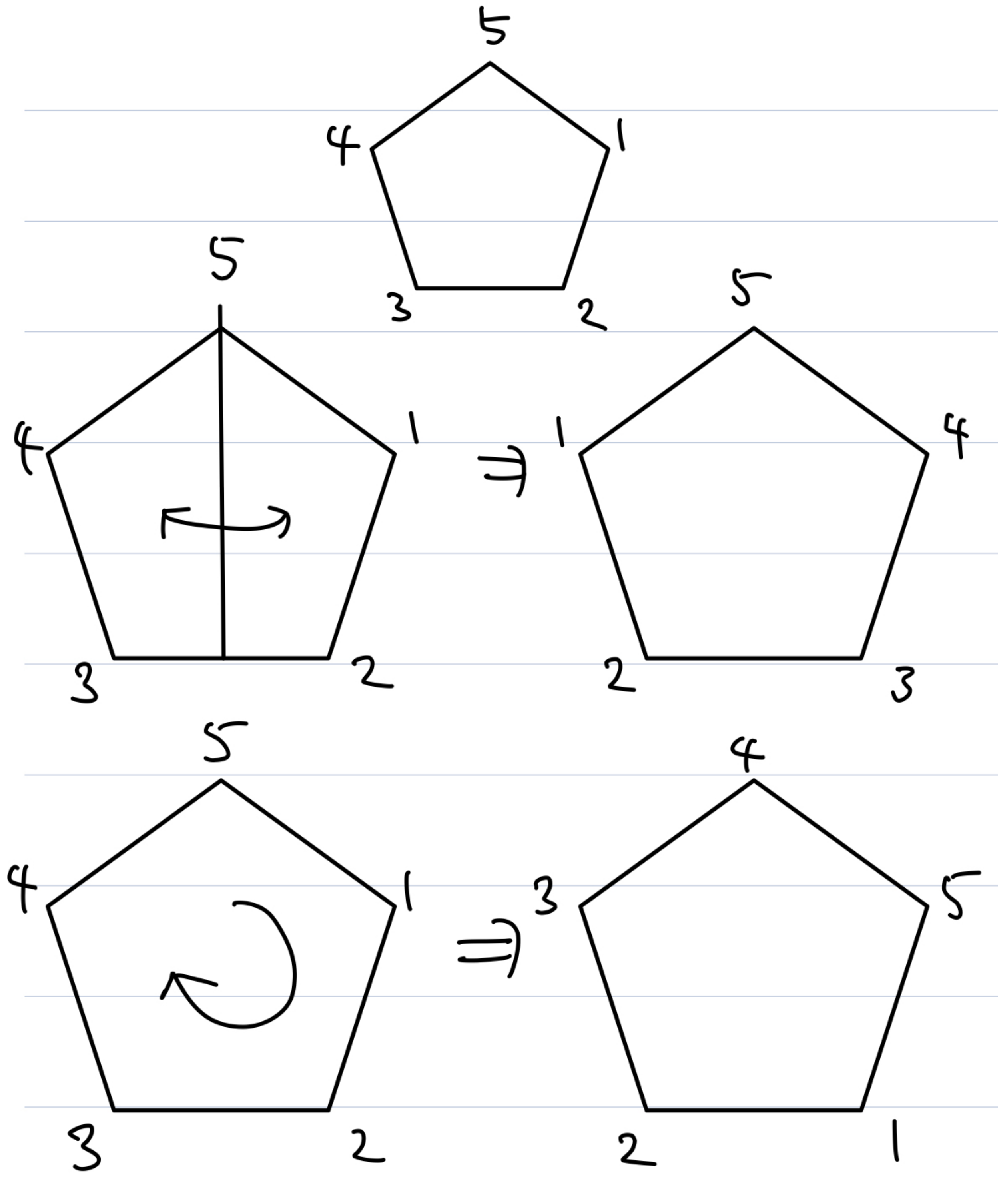

FIGURE 1. Interpretate D_5 geometrically

Therefore, D_5 contains exactly 10 elements.

(5) a corresponds to a reflection, and b corresponds to a rotation as in the figure.

(6) We will first identify all the subgroups of D_5. By Lagrange's Theorem, a subgroup must have exactly 1, 2, 5, or 10 elements. Since the case when the order is 1 or 10 is trivial, we will consider order 2 and 5.

- Subgroups of order 2. They are cyclic groups generated by elements of order 2. a has order 5, so a does not form a subgroup of order 2. The order of a^2, a^3, a^4 must divide a^5 by Lagrange's theorem since $\langle a^i \rangle$ is a subset of $\langle a \rangle$. Since 5 is prime, the order of a^2, a^3, a^4 must be 5. Thus none of a, a^2, a^3, a^4 generate a subgroup of order 2. Moreover, $a^5 = e$ does not form a subgroup of order 2. The remaining elements are b, ab, a^2b, a^3b, a^4b.
  - $b = (14)(23)$, and $b^2 = (1)$.
  - $ab = (12345)(14)(23) = (15)(24)$, and $(ab)^2 = (1)$.
  - $a^2b = (12345)(15)(24) = (25)(34)$, and $(a^2b)^2 = (1)$.
  - $a^3b = (12345)(25)(34) = (12)(35)$, and $(a^3b)^2 = (1)$.
  - $a^4b = (12345)(12)(35) = (13)(45)$, and $(a^4b)^2 = (1)$.

  Thus $\langle b \rangle, \langle ab \rangle, \langle a^2b \rangle, \langle a^3b \rangle, \langle a^4b \rangle$ are all the distinct subgroups of order 2.
- Subgroups of order 5. Since 5 is prime, they are cyclic groups generated by elements of order 5. As shown above, the only elements of order 5 are a, a^2, a^3, a^4, and they all generate the same subgroup. Thus $\langle a \rangle$ is the only subgroup of order 5.

Now, we will determine all the conjugacy classes of subgroups of D_5. Since $|H| = |gHg^{-1}|$ for each subgroup H and $g \in G$, it suffices to compare subgroups of the same order.

- Subgroups of order 1. The only subgroup of order 1 is the trivial group, and it is the only subgroup in its conjugacy class.
- Subgroups of order 2. The set of all the subgroups of order 2 are $\{\langle a^i b \rangle \mid 0 \leq i \leq 4\}$. Let $0 \leq i \leq 4$ be given. Then $a^3(a^ib)a^{-3} = a^{i+3}(ba^{-3}) = a^{i+3}a^3b = a^{i+6}b = a^{i+1}b$. Therefore, $\langle a^ib \rangle \sim \langle a^{i+1}b \rangle$ for each $0 \leq i \leq 4$. In other words, the set of all the subgroups of order 2 is an equivalence class.
- Subgroups of order 5. The only subgroup of order 5 is $\langle a \rangle$, and it is the only subgroup in its conjugacy class.
- Subgroups of order 10. The only subgroup of order 10 is itself, and it is the only subgroup in its conjugacy class.

Therefore, there are 4 conjugacy classes, $\{\langle e \rangle\}, \{\langle a^ib \rangle \mid 0 \leq i \leq 4\}, \{\langle a \rangle\}, \{D_5\}$.

$\square$