MATH 601 (DUE 10/9)

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Contents

1. Rings of Fractions

Exercise. (Problem 3) Let $T \subset R$ be the subset consisting of all non zero divisors.

- Show that T is a multiplicative set.
- Let $s \in T$ and let $S = \{1, s, s^2, s^3, \cdots\} \subset T$. Show that the following rings are isomorphic: $S^{-1}R$, the subring $R[1/s] \subset T^{-1}R$, and the quotient ring $R[x]/(sx-1)$.

Proof.

- – Let $a, b \in T$. Let $c \in R$ be given. If $(ab)c = 0$, then $a(bc) = 0$. Since a is a non zero divisor, $bc = 0$. Since b is a non zero divisor, $c = 0$. Since R is a commutative ring throughout this handout, there is no need to check the case that $c(ab) = 0$. Thus ab is a non zero divisor, so T is closed under multiplication.
  – $1 \in T$ since $\forall c \in R, c \cdot 1 = 0 \implies c = 0$.

  Therefore, T is indeed a multiplicative set.
- $S^{-1}R$ and $R[1/s]$ are isomorphic because:
  – They are the same set. They both contain all equivalence classes $[(r, s)]$ for $r \in R$ and $s \in S$ with the same equivalence relation.
  – They have the same addition and multiplication.

  Let $i : R \to S^{-1}R$ be the inclusion map, and choose $1/s \in S^{-1}R$. Consider the homomorphism $\phi : R[x] \to S^{-1}R$ associated to i and $1/s$ by the mapping property of polynomials. Then $\phi(\sum_{i=0}^{n} r_i x^i) = \sum_{i=0}^{n} \frac{r_i}{s^i}$. For any $r/s^k \in S^{-1}R$, $\phi(rx^k) = r/s^k$, so ϕ is surjective. We claim that $\ker(\phi) = (sx - 1)$.
  – $\phi(sx-1) = \frac{s}{s} - 1 = 0$. Thus $(sx - 1) \subset \ker(\phi)$.
  – We will prove that every polynomial in $\ker(\phi)$ is a product of $sx - 1$ and a polynomial in $R[x]$ by induction on the degree. For any $r \in R$, $\phi(r) = r$. Thus the only constant polynomial in $\ker(\phi)$ is 0, which is indeed a product of $sx-1$ and $0 \in R[x]$. Suppose that every polynomial in $\ker(\phi)$ of degree $< n \in \mathbb{N}$ is a product of $sx - 1$ and a polynomial in $R[x]$. Let $f(x) = \sum_{i=0}^{n} a_i x^i$. Then $0 = \phi(f(x)) = \sum_{i=0}^{n} \frac{a_i}{s^i}$. Thus $\sum_{i=0}^{n} a_i s^{n-i} = 0$, so $a_n + s\sum_{i=0}^{n-1} a_i s^{n-i-1} = 0$. Then $g(x) = f(x) - (\sum_{i=0}^{n-1} a_i s^{n-i-1})(sx - 1)$ is a polynomial of degree $\leq n - 1$.

Moreover, $\phi(g(x)) = \phi(f(x)) - \sum_{i=0}^{n-1} a_i s^{n-i-1}\phi(sx-1) = 0$, so $g(x) \in \ker(\phi)$. By the inductive hypothesis, every polynomial in $\ker(\phi)$ of degree $< n$ is a product of $sx - 1$ and a polynomial in $R[x]$. Therefore, $g(x) \in (sx - 1)$. Since $\sum_{i=0}^{n-1} a_i s^{n-i-1}\phi(sx-1) \in (sx-1)$ as well, $f(x) \in (sx - 1)$. By mathematical induction, $\ker(\phi) \subset (sx - 1)$.

By Lemma 8 (iii) of the Commutative Ring handout, $\bar{\phi}$ induced by ϕ is a ring isomorphism from $R[x]/(sx - 1)$ to $\phi(R[x]) = S^{-1}R$.

□

2. Modules

Exercise. (Problem 1) For each of the $\mathbb{Z}$-modules listed in the handout, answer the questions in the handout.

Proof.

(a) $M = \mathbb{Z}^3 \times \mathbb{Z}/86\mathbb{Z}$.
  (i) M is finitely generated.
  (ii) M is finitely presented.
  (iii) 4.
  (iv) Yes.
  (v) Yes.
  (vi) No.

(b) $M = \prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z}$.
  (i) M is not finitely generated.
  (ii) M is not finitely presented.
  (iii) Infinite.
  (iv) No.
  (v) No.
  (vi) Yes.

(c) $M = \mathbb{Z}[1/p] \subset \mathbb{Q}$.
  (i) M is not finitely generated.
  (ii) M is not finitely presented.
  (iii) 1.
  (iv) No.
  (v) No.
  (vi) Yes.

(d) $M = \mathbb{Q}/\mathbb{Z}_{(p)}$.
  (i) M is not finitely generated.
  (ii) M is not finitely presented.
  (iii) 1.
  (iv) No.
  (v) No.
  (vi) Yes.

□

3. The Quadratic Equation

Exercise. (Problem 20) Construct ring isomorphisms $\mathbb{Z}[x]/(x^2-2) \to \mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}/(p)[x]/(x^2-2) \to \mathbb{Z}[\sqrt{2}]/(p)$.

Proof. Let $i : \mathbb{Z} \to \mathbb{Z}[\sqrt{2}]$ be the inclusion and $s = \sqrt{2} \in \mathbb{Z}[\sqrt{2}]$. By the mapping property of polynomials, there exists a ring homomorphism $\phi : \mathbb{Z}[x] \to \mathbb{Z}[\sqrt{2}]$ such that $\phi(\sum_{i=0}^n r_i x^i) = \sum_{i=0}^n i(r_i)s^i$. In other words, ϕ maps $f(x)$ into $f(\sqrt{2})$. For each $a+b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$, $\phi(a+bx) = a + b\sqrt{2}$, so ϕ is surjective. We claim that $\ker(\phi) = (x^2 - 2)$.

- Since $\sqrt{2}^2 - 2 = 2 - 2 = 0$, $x^2 - 2 \in \ker(\phi)$. Moreover, $(x^2 - 2) \subset \ker(\phi)$.
- Let $f(x) \in \ker(\phi)$. Since $\mathbb{Z}[x]$ is a Euclidean domain, $f(x) = q(x)(x^2 - 2) + ax + b$ for some $q(x) \in \mathbb{Z}[x], a, b \in \mathbb{Z}$. Since $ax + b = f(x) - q(x)(x^2 - 2)$, $a\sqrt{2} + b = 0$. Since a, b are integers, $a = b = 0$. This implies $f(x) \in (x^2 - 2)$.

Therefore, $\ker(\phi) = (x^2 - 2)$. By Lemma 8 of the Commutative Ring handout, $\tilde{\phi} : \mathbb{Z}[x]/(x^2 - 2) \to \mathbb{Z}[\sqrt{2}]$ induced by ϕ is an isomorphism.

We will solve the second part using the same approach. We will assume that p is a prime. Consider the inclusion $\mathbb{Z}/(p) \hookrightarrow \mathbb{Z}[\sqrt{2}]/(p)$ and the element $\sqrt{2} + (p) \in \mathbb{Z}[\sqrt{2}]/(p)$. Let $\Phi : \mathbb{Z}/(p)[x] \to \mathbb{Z}[\sqrt{2}]/(p)$ be a ring homomorphism associated to the inclusion $\mathbb{Z}/(p) \to \mathbb{Z}[\sqrt{2}]/(p)$ and element $\sqrt{2} + (p)$. We will examine how Φ behaves.

$$\begin{aligned}
\Phi(\sum_{i=0}^n (a_i + (p))x^i) &= \sum_{i=0}^n (a_i + (p))(\sqrt{2} + (p))^i \\
&= \sum_{i=0}^n (a_i + (p))(\sqrt{2}^i + (p)) \\
&= \sum_{i=0}^n (a_i \sqrt{2}^i + (p)) \\
&= (\sum_{i=0}^n a_i \sqrt{2}^i) + (p).
\end{aligned}$$

For any $a + b\sqrt{2} + (p) \in \mathbb{Z}[\sqrt{2}]/(p)$, $\Phi((a + (p)) + (b + (p))x) = a + b\sqrt{2} + (p)$, so Φ is surjective. We claim that $\ker(\Phi) = (x^2 - 2)$. Here, by $x^2 - 2$, we mean $(1 + (p))x^2 - (2 + (p))$.

- Since $\sqrt{2}^2 - 2 = 0$, $(x^2 - 2) \in \ker(\Phi)$.
- Let $f(x) \in \ker(\Phi) \subset \mathbb{Z}/(p)[x]$. Since p is a prime, $\mathbb{Z}/(p)$ is a field. Thus $\mathbb{Z}/(p)[x]$ is a Euclidean domain. Choose $q(x) \in \mathbb{Z}/(p)[x]$ and $\overline{a} = a + (p), \overline{b} = b + (p) \in \mathbb{Z}/(p)$ such that $f(x) = (x^2 - 2)q(x) + (\overline{a}x + \overline{b})$. Then $0 = \Phi(f(x)) = \Phi(x^2 - 2)\Phi(q(x)) + \Phi(\overline{a}x + \overline{b}) = 0 + \Phi(\overline{a}x + \overline{b})$. Thus $0 = \Phi(\overline{a}x + \overline{b}) = (a\sqrt{2} + b) + (p)$. Therefore, $a\sqrt{2} + b \in (p)$. Since $a, b \in \mathbb{Z}$, this is possible only if $a = 0$ and $b \in (p)$. In other words, this is possible only if $\overline{a} = \overline{b} = 0$. Therefore, $f(x) = (x^2 - 2)q(x) \in (x^2 - 2)$.

Therefore, $\ker(\Phi) = (x^2 - 2)$, so the homomorphism $\tilde{\Phi}$ induced by Φ is an isomorphism from $\mathbb{Z}/(p)[x]/(x^2 - 2) \to \mathbb{Z}[\sqrt{2}]/(p)$ by Lemma 8 of the Commutative Ring handout. $\square$

Exercise. (Problem 21) Let $p \in \mathbb{Z}$ be an odd prime. Show that $\mathbb{Z}[\sqrt{2}]/(p)$ is an integral domain if and only if $(x^2 - 2)$ is an irreducible element of $\mathbb{Z}/(p)[x]$. Show that this occurs if and only if 2 is not a square in $\mathbb{Z}/(p)$.

Proof. By Problem 20, $\mathbb{Z}[\sqrt{2}]/(p)$ is isomorphic to $\mathbb{Z}/(p)[x]/(x^2-2)$. Thus it suffices to show that $\mathbb{Z}/(p)[x]/(x^2 - 2)$ is an integral domain if and only if $x^2 - 2$ is an irreducible element of $\mathbb{Z}/(p)[x]$. By Corollary 4 on P.300 (Dummit and Foote), since $\mathbb{Z}/(p)$ is a field, $\mathbb{Z}/(p)[x]$ is a UFD. By Proposition 12 on P.286, a nonzero element generates a prime ideal if and only if it is irreducible. By Proposition 13 on P.255, $(x^2 - 2)$ is a prime ideal if and only if $\mathbb{Z}/(p)[x]/(x^2 - 2)$ is an integral domain. Therefore, $\mathbb{Z}/(p)[x]/(x^2 - 2)$ is an integral domain if and only if $x^2 - 2$ is an irreducible element.

We will show that $\mathbb{Z}[\sqrt{2}]/(p)$ is not an integral domain if and only if 2 is a square in $\mathbb{Z}/(p)$.

- For any $a + (p) \in \mathbb{Z}/(p)$, $(a + \sqrt{2} + (p))(a - \sqrt{2} + (p)) = (a^2 - 2) + (p)$ in $\mathbb{Z}[\sqrt{2}]/(p)$. If $(a + (p))^2 = 2 + (p)$ for some $a + (p) \in \mathbb{Z}/(p)$, then $(a + \sqrt{2} + (p))(a - \sqrt{2} + (p)) = (2 - 2) + (p) = 0$. Thus $\mathbb{Z}[\sqrt{2}]/(p)$ is not an integral domain.
- Suppose that $\mathbb{Z}[\sqrt{2}]/(p)$ is not an integral domain. Then $x^2 - 2$ is not irreducible. Since the degree of $x^2 - 2$ is 2 and every nonzero constant polynomial in $\mathbb{Z}/(p)[x]$ is a unit, $x^2 - 2$ must have a factor of degree 1. Choose $a, b \in \mathbb{Z}/(p)$ such that $(x+a)(x+b) = x^2-2$. We can assume that the leading coefficients of the factors are 1 because $\mathbb{Z}/(p)$ is a field. This implies $x^2 + (a + b)x + ab = x^2 - 2$. Thus $a + b = 0$, so $b = -a$, and thus $-2 = -a^2$. 2 is a square in $\mathbb{Z}/(p)$.

Therefore, $\mathbb{Z}[\sqrt{2}]/(p)$ is an integral domain if and only if 2 is not a square in $\mathbb{Z}/(p)$. □

Exercise. (Problem 22) Use your answers to 21 and 19 to determine for which of the following values of p, $x^2 - 2y^2 = p$ has a solution: $p = 3, 5, 7, 11, 13, 17$.

Proof. By Problem 19, $x^2 - 2y^2 = p$ has a solution if and only if p is irreducible in $\mathbb{Z}[\sqrt{2}]$. Since $\mathbb{Z}[\sqrt{2}]$ is a UFD by Problem 14, by Proposition 12 on P.286, p generates a prime ideal if and only if p is irreducible. By Proposition 13 on P.255, (p) is a prime ideal if and only if $\mathbb{Z}[\sqrt{2}]/(p)$ is an integral domain. By Problem 21, 2 is not a square in $\mathbb{Z}/(p)$ if and only if $\mathbb{Z}[\sqrt{2}]/(p)$ is an integral domain.

Therefore, $x^2 - 2y^2 = p$ has a solution if and only if 2 is not a square in $\mathbb{Z}/(p)$.

- (Modulo 3) $2^2 \equiv 1$.
- (Modulo 5) $2^2 \equiv 4, 3^2 \equiv 4, 4^2 \equiv 1$.
- (Modulo 7) $2^2 \equiv 4, 3^2 \equiv 2$.
- (Modulo 11) $2^2 \equiv 4, 3^2 \equiv 9, 4^2 \equiv 5, 5^2 \equiv 3, 6^2 \equiv 3, 7^2 \equiv 5, 8^2 \equiv 9, 9^2 \equiv 4, 10^2 \equiv 1$.
- (Modulo 13) $2^2 \equiv 4, 3^2 \equiv 9, 4^2 \equiv 3, 5^2 \equiv 12, 6^2 \equiv 10, 7^2 \equiv 10, 8^2 \equiv 12, 9^2 \equiv 3, 10^2 \equiv 9, 11^2 \equiv 4, 12^2 \equiv 1$.
- (Modulo 17) $2^2 \equiv 4, 3^2 \equiv 9, 4^2 \equiv 16, 5^2 \equiv 8, 6^2 \equiv 2$.

Therefore, $x^2 - 2y^2 = p$ has a solution if $p = 7, 17$ and it does not if $p = 3, 5, 11, 13$. □

4. Factorization in Integral Domains

Exercise. (Problem 5)

- Let k be a field and let $a \in k$. Construct a k-algebra isomorphism, $k[x, y]/(x - a) \to k[y]$. Justify your answer.

- Let $f(x, y) \in k[x, y]$. What is the image of $f(x, y)$ under the above isomorphism?

Proof.

- Let ϕ be defined such that $\phi(f(x, y) + (x - a)) = f(a, y)$.
  - Well-defined? Let $f(x, y) + (x - a) = g(x, y) + (x - a)$. Then $g(x, y) = f(x, y) + h(x, y)(x - a)$.
$$\begin{aligned}\phi(g(x, y) + (x - a)) &= \phi((f(x, y) + h(x, y)(x - a)) + (x - a)) \\ &= f(a, y) + h(a, y)(a - a) \\ &= f(a, y) \\ &= \phi(f(x, y)).\end{aligned}$$
  - k-algebra homomorphism? Let $c \in k, f, g \in k[x, y]$ be given.
$$\begin{aligned}\phi(c(f + (x - a)) &= \phi(cf + (x - a)) \\ &= cf(a, y) \\ &= c\phi(f + (x - a)). \\ \phi((f + g) + (x - a)) &= (f + g)(a, y) \\ &= f(a, y) + g(a, y) \\ &= \phi(f + (x - a)) + \phi(g + (x - a)). \\ \phi((fg) + (x - a)) &= (fg)(a, y) \\ &= f(a, y)g(a, y) \\ &= \phi(f + (x - a))\phi(g + (x - a)).\end{aligned}$$
- $\phi(f(x, y) + (x - a)) = f(a, y)$.

□

Exercise. (Problem 6)

- Give an example of a field k, an element $a \in k$ and a reducible polynomial $f(x, y) \in k[x, y]$ of degree n in y such that $f(a, y) \in k[y]$ is irreducible and has degree n.
- Suppose given a polynomial $f \in k[x, y]$ which when viewed as an element of $k(x)[y]$ has degree n (in y) and content 1. Suppose there is some $a \in k$ such that $f(a, y) \in k[y]$ is irreducible and has degree n. Show that $f(x, y) \in k[x, y]$ is irreducible.
- Give an example of a field k, an element, $a \in k$, and a reducible polynomial $f(x, y) \in k[x, y]$, which when viewed as an element of $k(x)[y]$ has degree n and content 1 such that $f(a, y) \in k[y]$ is irreducible.

Proof.

- Let $k = \mathbb{Q}, a = 1, f(x, y) = xy$. Then the degree of $f(x, y)$ in y is 1. $f(x, y) = xy \in k[x, y]$ is reducible since x and y are not units in $k[x, y]$. However, $f(a, y) = 1y = y$ is irreducible in $k[y]$.
- Choose $f_1, \cdots, f_n \in k[x]$ such that $f(x, y) = f_n(x)y^n + \cdots + f_1(x)y^1 + f_0(x)$. Then $f(a, y) = f_n(a)y^n + \cdots + f_1(a)y^1 + f_0(a)$. Let $h_1(x, y), h_2(x, y) \in k[x]$ be given such that $f(x, y) = h_1(x, y)h_2(x, y)$. Then $f(a, y) = h_1(a, y)h_2(a, y)$. Then $h_1(a, y)$ or $h_2(a, y)$ is a unit in $k[y]$ since $f(a, y)$ is irreducible in $k[y]$. Without loss of generality, we will assume $h_1(a, y)$ is a unit in $k[y]$.

It is given that $\deg_y(f(a,y))$, the degree of $f(a,y)$ in y, is n. Thus $\deg_y(h_1(a,y)) + \deg_y(h_2(a,y)) = n$. Since $\deg_y(h_1(a,y)) = 0$, $\deg_y(h_2(a,y)) = n$. Therefore, $\deg_y(h_2(x,y)) \geq n$.

On the other hand, $\deg_y(f(x,y)) = \deg_y(h_1(x,y)) + \deg_y(h_2(x,y))$, so $\deg_y(h_2(x,y)) \leq n$. Thus $\deg_y(h_2(x,y)) = n$. Let $g_1(x), \cdots, g_n(x) \in k[x]$ such that $h_2(x,y) = g_n(x)y^n + \cdots + g_1(x)y^1 + g_0(x)$. Then $f(x,y) = h_1(x,y)h_2(x,y) = (h_1(x,y)g_n(x))y^n + \cdots + (h_1(x,y)g_1(x))y^1 + h_1(x,y)g_0(x)$.

Since $\deg_y(h_2(x,y)) = n$, $\deg_y(h_1(x,y)) = 0$. Thus, $h_1(x,y) \in k[x]$, so $h_1(x,y)g_i(x) \in k[x]$ for each i. Therefore, $h_1(x,y)g_i(x) = f_i(x)$ for each i.

Let $p \in k[x]$ be an irreducible element. If $p \mid h_1(x,y)$, then $p \mid f_i(x) = h_1(x,y)g_i(x)$ for each i, so $\mathrm{ord}_p(f_i) \geq 1$ for each i. Therefore, $o_p(f(x,y)) \geq 1$, and thus $p \mid \mathrm{cont}(f(x,y))$. However, since $\mathrm{cont}(f(x,y)) = 1$, $p \nmid h_1(x,y)$. Thus $h_1(x,y)$ is a unit in $k[x]$ since it cannot be divided by any irreducible elements. Since $h_1(x,y)$ is a unit in $k[x]$ and $k[y]$, it must consist only of a constant term, which is a unit in k. Hence, $h_1(x,y)$ is a unit in $k[x,y]$.

We have shown that for any $h_1(x,y), h_2(x,y) \in k[x,y]$, $h_1h_2 = f$ implies one of h_1 or h_2 is a unit. Therefore, $f(x,y)$ is an irreducible element in $k[x,y]$.

- Let $k = \mathbb{Q}, a = 1, f(x,y) = (x-1)y^2 + y$. Then $f(x,y)$, which when viewed as an element of $k(x)[y]$ has degree 1.
  - The coefficient of y is 1, and $\mathrm{ord}_p(1) = 0$ for any p because $1 \in k[x]^*$.
  - The coefficient of y^2, when $f(x,y)$ is viewed as an element of $k(x)[y]$ is $x-1$. Thus for any irreducible element $p \in k[x]$, $\mathrm{ord}_p(x-1) \geq 0$.

  Therefore, $o_p(f(x)) = 0$ for any irreducible element $p \in k[x]$. Thus $\mathrm{cont}(f(x,y)) = 1$.

  $f(a,y) = y \in k[y]$, and this is irreducible.

$\square$