

MATH 601 HOMEWORK (DUE 9/11)

HIDENORI SHINOHARA

Exercise. (1) Show that 2×2 matrices give a functor, M_2 , from the category of rings to itself, $R \mapsto M_2(R)$.

Proof. Let R, R' be rings and $\phi \in \text{Hom}(R, R')$. Let $M_2(\phi) : M_2(R) \rightarrow M_2(R')$ be defined such that

$$(M_2(\phi)) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{bmatrix}.$$

We claim that M_2 is indeed a functor.

- Claim 1: For any $\phi \in \text{Hom}(R, R')$, $M_2(\phi) \in \text{Hom}(M_2(R), M_2(R'))$.
In other words, we want to show that $M_2(\phi)$ is a ring homomorphism for any ϕ .

$$\begin{aligned} (M_2(\phi)) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) &= (M_2(\phi)) \left(\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \right) \\ &= \begin{bmatrix} \phi(a+e) & \phi(b+f) \\ \phi(c+g) & \phi(d+h) \end{bmatrix} \\ &= \begin{bmatrix} \phi(a) + \phi(e) & \phi(b) + \phi(f) \\ \phi(c) + \phi(g) & \phi(d) + \phi(h) \end{bmatrix} \\ &= \begin{bmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{bmatrix} + \begin{bmatrix} \phi(e) & \phi(f) \\ \phi(g) & \phi(h) \end{bmatrix} \\ &= (M_2(\phi)) \begin{bmatrix} a & b \\ c & d \end{bmatrix} + (M_2(\phi)) \begin{bmatrix} e & f \\ g & h \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
& (M_2(\phi)) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) \\
&= (M_2(\phi)) \left(\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} \right) \\
&= \begin{bmatrix} \phi(ae + bg) & \phi(af + bh) \\ \phi(ce + dg) & \phi(cf + dh) \end{bmatrix} \\
&= \begin{bmatrix} \phi(a)\phi(e) + \phi(b)\phi(g) & \phi(a)\phi(f) + \phi(b)\phi(h) \\ \phi(c)\phi(e) + \phi(d)\phi(g) & \phi(c)\phi(f) + \phi(d)\phi(h) \end{bmatrix} \\
&= \begin{bmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{bmatrix} \begin{bmatrix} \phi(e) & \phi(f) \\ \phi(g) & \phi(h) \end{bmatrix} \\
&= (M_2(\phi)) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) (M_2(\phi)) \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} \right)
\end{aligned}$$

Therefore, $M_2(\phi)$ is indeed a ring homomorphism.

- For any ring R and the identity function Id_R , $M_2(\text{Id}_R)$ is the identity map on $M_2(R)$ because it maps each element in a given matrix to itself.
- Let $f \in \text{Hom}(A, B)$, $g \in \text{Hom}(B, C)$.

$$\begin{aligned}
(M_2(f \circ g)) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) &= \begin{bmatrix} (f \circ g)(a) & (f \circ g)(b) \\ (f \circ g)(c) & (f \circ g)(d) \end{bmatrix} \\
&= \begin{bmatrix} f(g(a)) & f(g(b)) \\ f(g(c)) & f(g(d)) \end{bmatrix} \\
&= M_2(f) \left(\begin{bmatrix} g(a) & g(b) \\ g(c) & g(d) \end{bmatrix} \right) \\
&= M_2(f) \left(M_2(g) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \\
&= (M_2(f) \circ M_2(g)) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right).
\end{aligned}$$

Therefore, M_2 is indeed a functor. \square

Exercise. (Problem 8 from More exercises) Consider the subgroup, $D_5 = \langle (12345), (14)(23) \rangle \subset S_5$.

- (1) Set $a = (12345)$ and compute a^{-1} .
- (2) Set $b = (14)(23)$ and compute aba^{-1} .
- (3) Show that every element in D_5 may be written in the form $a^i b^j$ for some $i, j \in \mathbb{Z}$.
- (4) Compute $|D_5|$.

Proof.

- (1) a sends 1 to 2, 2 to 3, \dots . We want a^{-1} to do the opposite. Thus $a^{-1} = (15432)$. Since $(12345)(15432) = (15432)(12345) = (1)$, (15432) is indeed a^{-1} .
- (2) $aba^{-1} = (a(1)a(4))(a(2)a(3)) = (25)(34)$.
- (3) $ba = (14)(23)(12345) = (13)(45)$, and $a^{-1}b = (15432)(14)(23) = (13)(45)$. Therefore, $ba = a^{-1}b$. We claim that $ba^n = a^{-n}b$ for every $n \in \mathbb{N}$. Suppose $ba^n = a^{-n}b$ for some $n \in \mathbb{N}$. Then $ba^{n+1} = (ba^n)a = (a^{-n}b)a = a^{-n}(ba) = a^{-n}a^{-1}b = a^{-(n+1)}b$. By mathematical induction, $ba^n = a^{-n}b$ for every $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$, $ba^n = a^{-n}b$, so $a^nba^n = b$, and thus $a^n b = ba^{-n}$. Therefore, we have $ba^k = a^{-k}b$ for every $k \in \mathbb{Z}$.

We claim that for any $i, j \in \mathbb{Z}$, $b^j a^i$ can be written in the desired form. Since $b^2 = e$, we consider two cases based on the parity of j . If j is even, then $b^j = e$, so $b^j a^i = a^i$. If j is odd, then $b^j = b$, so $b^j a^i = ba^i = a^{-i}b$ as shown above.

We will prove the general case. By the argument above, it suffices to show that every element in D_5 can be represented as a word of length ≤ 2 . Let $x_1^{i_1} \cdots x_k^{i_k} \in D_5$ be given where $i_1, \dots, i_k \in \mathbb{Z}$ and each x_i is either a or b . Since D_5 is generated by a, b , every element can be represented in this form. We will show that every element in D_5 can be represented as a word of length ≤ 2 by using strong induction. If $k \leq 2$, then we are done. Suppose that we can represent every element in D_5 of length $\leq k$ as a word of length ≤ 2 for some $k \geq 2$. Let $x = x_1^{i_1} \cdots x_{k+1}^{i_{k+1}} \in D_5$. If $x_1 = x_2$, then $x = x_2^{i_1+i_2} x_3^{i_3} \cdots x_{k+1}^{i_{k+1}}$, so by the inductive hypothesis, this can be represented as a word of length ≤ 2 . If $x_2 = x_3$, then $x = x_1^{i_1} x_2^{i_2+i_3} x_4^{i_4} \cdots x_{k+1}^{i_{k+1}}$, so by the inductive hypothesis, this can be represented as a word of length ≤ 2 . Suppose $x_1 \neq x_2$ and $x_2 \neq x_3$. Then there are two cases:

- Case 1: $(x_1, x_2, x_3) = (a, b, a)$. By the argument above, $b^{i_2} a^{i_3}$ can be represented as $a^i b^j$ for some $i, j \in \mathbb{Z}$. Therefore, $a^{i_1} (b^{i_2} a^{i_3}) = a^{i_1} (a^i b^j) = a^{i_1+i} b^j$, so x can be represented as a word of length k . By the inductive hypothesis, x can be represented as a word of length ≤ 2 .
 - Case 2: $(x_1, x_2, x_3) = (b, a, b)$. By the argument above, $b^{i_1} a^{i_2}$ can be represented as $a^i b^j$ for some $i, j \in \mathbb{Z}$. Therefore, $(b^{i_1} a^{i_2}) b^{i_3} = (a^i b^j) b^{i_3} = a^i b^{j+i_3}$. By the inductive hypothesis, x can be represented as a word of length ≤ 2 .
- (4)
 - $a^1 = a \neq (1)$.
 - $a^2 = (13524) \neq (1)$.

- $a^3 = (14253) \neq (1)$.
- $a^4 = (15432) \neq (1)$.
- $a^5 = (1)$.

Therefore, the order of a is 5. Since $b \neq (1)$ and $b^2 = (1)$, the order of b is 2. We claim that there are exactly 10 elements in D_5 .

- Claim 1: $|D_5| \leq 10$. Let $x \in D_5$. Then there exist $i, j \in \mathbb{Z}$ such that $x = a^i b^j$. Since the order of a is 5 and the order of b is 2, we can assume that $0 \leq i \leq 4$ and $0 \leq j \leq 1$. Therefore, $D_5 \subset \{a^i b^j \mid 0 \leq i \leq 4, 0 \leq j \leq 1\}$. Thus there are at most 10 elements in D_5 .
- Claim 2: $|D_5| \leq 10$. Let $a^i b^j, a^{i'} b^{j'} \in \{a^i b^j \mid 0 \leq i \leq 4, 0 \leq j \leq 1\}$. Suppose $a^i b^j = a^{i'} b^{j'}$. Then $a^{i-i'} = b^{j'-j}$. We have calculated all the powers of a above, and none of them is equal to b . Therefore, $i - i' \equiv 0 \pmod{5}$ and $j - j' \equiv 0 \pmod{2}$. Since $0 \leq i, i' \leq 4, 0 \leq j, j' \leq 1$, $i = i'$ and $j = j'$. This implies that the set $\{a^i b^j \mid 0 \leq i \leq 4, 0 \leq j \leq 1\}$ contains exactly 10 elements. Since the set is a subset of D_5 , D_5 contains at least 10 elements.

Therefore, D_5 contains exactly 10 elements.

□