

MATH 601 (DUE 10/23)

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CONTENTS

1. Field Extension

1

1. FIELD EXTENSION

Exercise. (Problem 1) Let p be a prime number. Let $K = \mathbb{Z}/p\mathbb{Z}(t)$ be the fraction field of $\mathbb{Z}/p\mathbb{Z}[t]$.

- (i) What is the characteristic of K ?
- (ii) What is the characteristic of any extension field of K ?
- (iii) Show that the Frobenius endomorphism, $F : K \rightarrow K$ is not a ring isomorphism.
- (iv) Let $f(x) = x^p - t \in K[x]$. Prove that $f(x)$ is irreducible.
- (v) Prove that $f(x)$ is not a separable polynomial.
- (vi) Construct an explicit field extension $K \subset L$ such that $f(x) \in L[x]$ has a factor of positive degree $< p$.
- (vii) With f and L above find all the roots of $f(x)$ in L and determine their multiplicities.

Proof.

- (i) We will write $k \cdot 1$ to denote $1 + 1 + \cdots + 1$ (k times). Since $p \cdot 1 = 0$ in K , the characteristic of K is at most p . Let k denote the characteristic of K . Let $i : \mathbb{Z}/p\mathbb{Z} \rightarrow (\mathbb{Z}/p\mathbb{Z})[t], i' : \mathbb{Z}/p\mathbb{Z}[t] \rightarrow K$ be inclusions. Then $i' \circ i : \mathbb{Z}/p\mathbb{Z} \rightarrow K$ is an injective ring homomorphism. $k \cdot 1 \neq 0$ in $\mathbb{Z}/p\mathbb{Z}$. Thus $(i' \circ i)(k \cdot 1) = k \cdot (i' \circ i)(1) = k' \cdot 1 = 0$. Since $i' \circ i$ is injective, this implies $k \cdot 1 = 0$. Therefore, $k \geq p$, so k must be equal to p .

□

Exercise. (Problem 2) Let F be a field of characteristic 0. Let $f(x) \in F[x]$ be an irreducible polynomial. Then $f(x)$ is separable.

Proof. Since $f(x)$ is irreducible, $f(x)$ is not a unit. Since F is a field, all polynomials of degree 0 are units. Thus $\deg(f(x)) \geq 1$.

Finish the proof. Check Notability for a sketch.

□