MATH 601 HOMEWORK (DUE 9/18)

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Exercise. Let R be a commutative ring with one. Explain why there is a unique ring homomorphism, $\mathbb{Z} \to R$.

Proof. The existence of a ring homomorphism is clear since $\phi(n) = 1_R + \cdots + 1_R$ and $\phi(-n) = -\phi(n)$ define a homomorphism.

We will show the uniqueness of a ring homomorphism. Let $\phi_1, \phi_2 : \mathbb{Z} \to R$ be ring homomorphisms.

We claim that $\phi_1(n) = \phi_2(n)$ for each $n \in \mathbb{N}$.

- By definition, $\phi_1(1) = \phi_2(1) = 1_R$.
- Suppose $\phi_1(n) = \phi_2(n)$ for some $n \in \mathbb{N}$. Then $\phi_1(n+1) = \phi_1(n) + \phi_1(1) = \phi_2(n) + \phi_2(1) = \phi_2(n+1)$.

By mathematical induction, $\phi_1(n) = \phi_2(n)$ for each $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, $\phi_1(-n) = -\phi_1(n) = -\phi_2(n) = \phi_2(-n)$. Finally, $\phi_1(0) = \phi_1(0+0) = \phi_1(0) + \phi_1(0)$, so $\phi_1(0) = 0_R$. Similarly, $\phi_2(0) = 0_R$. Thus $\phi_1(0) = \phi_2(0)$.

Hence, we have shown that $\phi_1 = \phi_2$. □

Exercise. (Problem 2) Let $I \subset R$ be an ideal in a commutative ring. Describe a bijective correspondence between ideals in R/I and certain ideals in R.

Proof. Tried this for about 10 minutes. I think this must be related to some special ideals, so I checked the annihilator, but that doesn't really work. I suspect that this problem is fairly simple once I notice what it is, but it'll take time until I notice it.

□

Exercise. (Problem 3) Let $I, J \subset R$ be ideals in a commutative ring. Let $I + J \subset R$ denote the smallest ideal containing I and J. Observe that $I + J = \{i + j \in R : i \in I, j \in J\}$. Let $\overline{J} \subset R/I$ denote the image of J under the canonical quotient map, $R \to R/I$. Observe that $\overline{J}$ is an ideal in $S := R/I$. Use the universal mapping property of the quotient to show that $R/(I + J) \simeq S/\overline{J}$.

Proof. Tried this for 20 minutes. The problem seems complicated, but it seems that we just need some sort of category theoretical approach to solve this problem. I think I can finish it in the next 20 minutes. The universal mapping property of the quotient is proposition 6 in the handouts.

□

Exercise. (Problem 4) Let R be a commutative ring and $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ a non-zero polynomial of degree n. Suppose that $a_n \in R^\times$. Let $J = (f(x))$. Prove that every element of $R[x]/J$ may be written in exactly one way in the form $\sum_{i=0}^{n-1} r_i x^i + J$ with $r_0, r_1, \cdots, r_{n-1} \in R$.

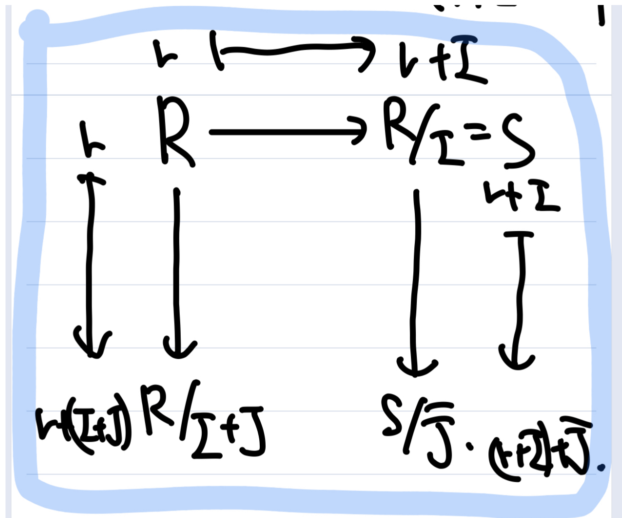

FIGURE 1. deletethis

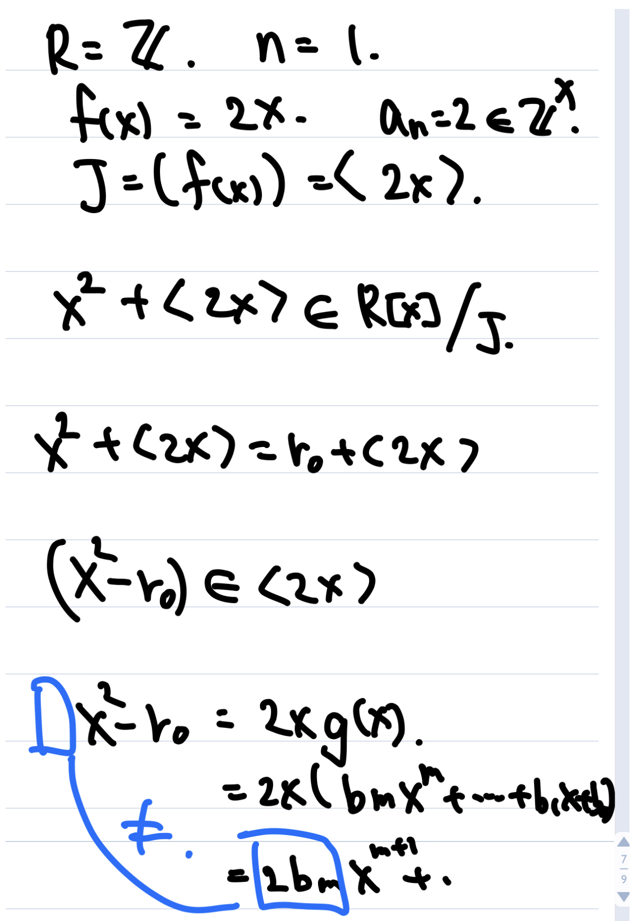

FIGURE 2. Problem 4

Proof. Tried this for 10 minutes. This problem seems wrong. See Figure 2.

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