MATH 611 (DUE 11/20)

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Exercise. (Problem 1)

- As shown in Figure 1, we will let A, B denote subspaces of X such that $X = A \cup B$ and $A \cap B$ consists of two line segments. (Circled in the figure) Moreover, $X = \operatorname{int} A \cup \operatorname{int} B$.

  $H_n(A) = 0$ for all $n \geq 1$. By Proposition 2.6 (Hatcher), it suffices to consider each path component of $A \cap B$ separately. Each of them is homeomorphic to Δ^1. Thus $H_n(A \cap B) = H_n(\Delta^1) \oplus H_n(\Delta^1) = 0$ for all $n \geq 1$. Using a similar argument, $H_n(B) = H_n(\Delta^1) \oplus H_n(\Delta^1) = 0$ for all $n \geq 1$.

  By the exact sequence $H_n(A) \oplus H_n(B) \to H_n(A \cup B) \to H_{n-1}(A \cap B)$, $H_n(A \cup B) = 0$ for all $n \geq 2$.

  We will consider the exact sequence $0 \to H_1(A \cup B) \xrightarrow{\alpha} H_0(A \cap B) \xrightarrow{\beta} H_0(A) \oplus H_0(B)$. We have 0 because $H_1(A) \oplus H_1(B) = 0$. $H_0(A \cap B) = \mathbb{Z}^2, H_0(A) = \mathbb{Z}, H_0(B) = \mathbb{Z}^2$ by examining the number of path components. Let a, b be generators of $H_0(A \cap B)$. Then $\beta(a, b) = (a + b, (a, b))$ because a, b simply correspond to each path component in $A \cap B$. Therefore, β is injective. Since α is injective by the exactness, $H_1(A \cup B) = \operatorname{Im}(\alpha) = \ker(\beta) = 0$. Hence, $H_1(X) = 0$.

  By examining the number of path components, $H_0(X) = \mathbb{Z}$.
- Let A, B denote the subspaces of X as in Figure 2. Then $A \cap B$ is homotopy equivalent to S^1, and B is homotopy equivalent to the wedge sum of $2g$ S^1's.

  For any $n \geq 3$, $H_n(A) \oplus H_n(B) \to H_n(A \cup B) \to H_{n-1}(A \cap B)$ shows that $H_n(A \cup B) = 0$ because $H_n(A) = H_n(B) = 0$ and $H_{n-1}(A \cap B) = H_{n-1}(S^1) = 0$.

  We have $0 \to \tilde{H}_2(A \cup B) \xrightarrow{\alpha} \tilde{H}_1(A \cap B) \xrightarrow{\beta} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{\gamma} \tilde{H}_1(A \cup B) \to 0$. We have 0's at the end because $\tilde{H}_2(A) \oplus \tilde{H}_2(B) = \tilde{H}_0(A \cap B) = 0$. We have $\tilde{H}_1(A \cap B) = \mathbb{Z}$ and $\tilde{H}_1(A) \oplus \tilde{H}_1(B) = \vee_{i=1}^{2g} \tilde{H}_1(S^1) = \mathbb{Z}^{2g}$. β maps a generator x into $(0, 0)$ because going around $A \cap B$ once cancels out all the generators of $\tilde{H}_1(B)$. For instance, in Figure 2, $\beta(x) = a + b - a - b + \cdots = 0$. Therefore, β is the zero map.

  This implies that γ is injective. By the exactness, γ is surjective. Therefore, γ is isomorphic, and thus $H_1(A \cup B) = \tilde{H}_1(A \cup B) = \mathbb{Z}^{2g}$.

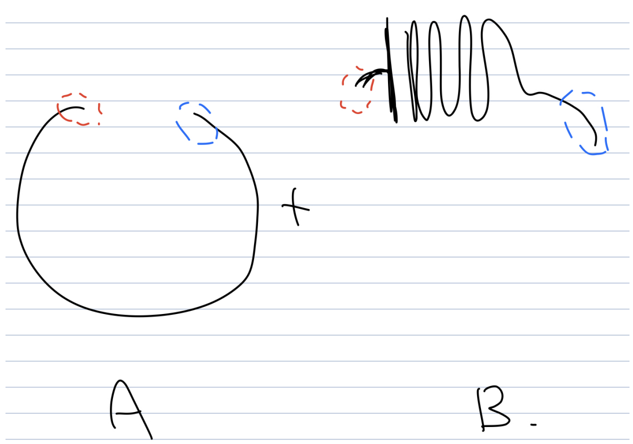

FIGURE 1. Quasi circle

FIGURE 2. Genus g surface

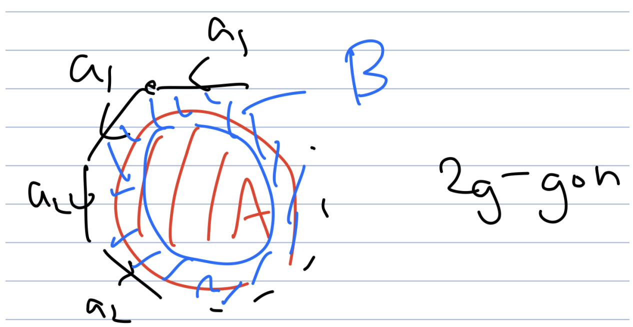

FIGURE 3. N_g

α is injective by the exactness, so $H_2(A \cup B) = \tilde{H}_2(A \cup B) = \mathrm{Im}(\alpha) = \tilde{H}_1(A \cap B) = \mathbb{Z}$. $H_0(A \cap B) = H_0(S^1) = \mathbb{Z}$.

- Let A, B denote the subspaces as in Figure 3. Then A deformation retracts onto a point, B is homotopy equivalent to $\vee_g \mathbb{R}P^1$, which is homotopy equivalent to $\vee_g S^1$. Finally, $A \cap B$ is homotopy equivalent to S^1. For any $n \geq 3$, $H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B)$ is exact, and $H_n(A) = H_n(B) = H_{n-1}(A \cap B) = 0$, so $H_n(X) = 0$. Consider $0 \to \tilde{H}_2(X) \xrightarrow{\alpha} \tilde{H}_1(A \cap B) \xrightarrow{\beta} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{\gamma} \tilde{H}_1(X) \to 0$. We have 0 at the end because $\tilde{H}_2(A) \oplus \tilde{H}_2(B) = \tilde{H}_0(A \cap B) = 0$. By exactness, α is injective and γ is surjective. Let a be a generator of $\tilde{H}_1(A \cap B) = \mathbb{Z}$. Then $\beta(a) = (0, 2(a_1 + \cdots + a_g))$ where a_i's are generators of $\tilde{H}_1(B) = \mathbb{Z}^g$.
  - Since β is injective, so $0 = \ker(\beta) = \mathrm{Im}(\alpha) = \tilde{H}_2(X) = H_2(X)$.
  - Since γ is surjective, $\tilde{H}_1(X) = \tilde{H}_1(A) \oplus \tilde{H}_1(B) / \ker(\gamma)$. Since $\ker(\gamma) = \mathrm{Im}(\beta)$, this is $\langle a_1, \cdots, a_g \mid 2(\sum a_i) \rangle$.

$$\begin{aligned} H_1(X) &= \langle a_1, \cdots, a_g \mid 2(a_1 + \cdots + a_g) \rangle \\ &= \langle a_1 + \cdots + a_g, a_2, \cdots, a_g \mid 2(a_1 + \cdots + a_g) \rangle \\ &= \langle b, a_2, \cdots, a_g \mid 2b \rangle \\ &= \mathbb{Z}^{g-1} \oplus (\mathbb{Z}/2\mathbb{Z}). \end{aligned}$$

– Since X consists of one path component, $H_0(X) = \mathbb{Z}$.

$\mathbb{R}P^3$.

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- Let $A = \mathbb{C}\mathbf{P}^n - (0 : \cdots : 0 : 1), B = \{(a_1 : \cdots : a_n : 1) \mid a_i \in \mathbb{C}\}$. Since $\{(0 : \cdots : 0 : 1)\}$ is closed, A is an open set. Moreover, $(0 : \cdots : 0 : 1)$ is an interior point of B. Therefore, $\mathrm{int}(A) \cup \mathrm{int}(B) = \mathbb{C}\mathbf{P}^n$.

  A deformation retracts onto $\mathbb{C}\mathbf{P}^{n-1}$ by $F((a_1 : \cdots : a_n : a_{n+1}), t) = (a_1 : \cdots : a_n : (1-t)a_{n+1})$. B is homeomorphic to $\mathbb{C}^n$, which is contractible. Finally, $A \cap B = B \setminus (0 : \cdots : 0 : 1)$, which is homotopy equivalent to $\mathbb{C}^n - 0$. This is homotopy equivalent to $\mathbb{R}^{2n} - 0$, so it is S^{2n-1}.

  We claim that $H_{2k}(\mathbb{C}\mathbf{P}^n) = 0$ if $k > n$ and $\mathbb{Z}$ if $k \leq n$. We will use this to calculate $H_k(\mathbb{C}\mathbf{P}^n)$ by induction on n. When $n = 0$, this is obvious because $\mathbb{C}\mathbf{P}^0$ is a point. Suppose that we have shown this for some $n-1$ where $n \in \mathbb{N}$. We will prove the case for n.
  - Let $k > 2n$. We have $H_k(A) \oplus H_k(B) \to H_k(X) \to H_{k-1}(A \cap B)$. $H_k(A) = 0$ by the inductive hypothesis. $H_k(B) = 0$ since B is contractible. $H_{k-1}(A \cap B) = 0$ since $k-1 \neq 2n-1$. Therefore, $H_k(X) = 0$ for all $k > 2n$.
  - We have the exact sequence $H_{2n}(A) \oplus H_{2n}(B) \to H_{2n}(X) \to H_{2n-1}(A \cap B) \to H_{2n-1}(A) \oplus H_{2n-1}(B)$. $H_{2n}(A) = H_{2n-1}(A) = 0$ by the inductive hypothesis since A deformation retracts onto $\mathbb{C}\mathbf{P}^{n-1}$. $H_{2n}(B) = H_{2n-1}(B) = 0$ because B is contractible. By the exactness, $H_{2n}(X) \cong H_{2n-1}(A \cap B) = \mathbb{Z}$ because $A \cap B$ is homotopy equivalent to S^{2n-1}.
  - We have the exact sequence $\tilde{H}_{2n-1}(A) \oplus \tilde{H}_{2n-1}(B) \to \tilde{H}_{2n-1}(X) \to \tilde{H}_{2n-2}(A \cap B)$. $\tilde{H}_{2n-1}(A) = 0$ by the inductive hypothesis. $\tilde{H}_{2n-1}(B) = 0$, and $\tilde{H}_{2n-2}(A \cap B) = 0$, so $H_{2n-1}(X) = \tilde{H}_{2n-1}(X) = 0$.
  - Let $1 \leq k \leq n-1$. We have the exact sequence

    $$\begin{aligned} &\tilde{H}_{2k}(A \cap B) \to \tilde{H}_{2k}(A) \oplus \tilde{H}_{2k}(B) \to \tilde{H}_{2k}(X) \to \\ &\tilde{H}_{2k-1}(A \cap B) \to \tilde{H}_{2k-1}(A) \oplus \tilde{H}_{2k-1}(B) \to \tilde{H}_{2k-1}(X) \to \\ &\tilde{H}_{2k-2}(A \cap B). \end{aligned}$$

    $\tilde{H}_{2k}(A \cap B) = \tilde{H}_{2k-2}(A \cap B) = 0$ because $2k \neq 2n-1$ and $2k-2 \neq 2n-1$ by the parity. $\tilde{H}_{2k}(B) = \tilde{H}_{2k-1}(B) = 0$. By the inductive hypothesis, $\tilde{H}_{2k}(A) = \mathbb{Z}$ and $\tilde{H}_{2k-1}(A) = 0$. By putting these together, the above sequence turns into

    $$0 \to \mathbb{Z} \xrightarrow{\alpha} \tilde{H}_{2k}(X) \to 0 \to 0 \to \tilde{H}_{2k-1}(X) \to 0.$$

    By the exactness, $H_{2k}(X) = \tilde{H}_{2k}(X) = \mathbb{Z}$ since α is an isomorphism, and $H_{2k-1}(X) = \tilde{H}_{2k-1}(X) = 0$.

  By induction, the proposition is true for all $n \in \mathbb{N}$.

Exercise. (Problem 28 (a)) Let A, B be the Mobius strip and a torus with a small neighborhood around them so the strip and torus are contained in A and B. For any $n \geq 3$, the exact sequence $H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B)$ implies that $H_n(X) \cong H_n(A) \oplus H_n(B) = 0 \oplus 0 = 0$ because the intersection $A \cap B$ is homotopic to S^1, so $H_n(A \cap B) = H_{n-1}(A \cap B) = 0$. $H_0(X) = \mathbb{Z}$ because X has only one path component.

We will examine the LES

$$\tilde{H}_2(A \cap B) \to \tilde{H}_2(A) \oplus \tilde{H}_2(B) \xrightarrow{f_1} \tilde{H}_2(X) \xrightarrow{f_2} \tilde{H}_1(A \cap B) \xrightarrow{f_3} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{f_4} \tilde{H}_1(X) \to \tilde{H}_0(A \cap B).$$

- Sine $\tilde{H}_2(A \cap B) = 0$, so f_1 is injective.
- $\tilde{H}_1(A \cap B) = \mathbb{Z}$, and $f_3(1) = (2, (1, 0))$ because the intersection goes around the mobius strip twice while it only goes around the torus once. Then f_3 is injective, so $\mathrm{Im}(f_2) = \ker(f_3) = 0$. This implies that $\mathrm{Im}(f_1) = \ker(f_2) = H_2$, so f_1 is surjective.

Therefore, f_1 is bijective, so $H_2(X) = \tilde{H}_2(X) = \tilde{H}_2(A) \oplus \tilde{H}_2(B) = 0 \oplus \mathbb{Z} = \mathbb{Z}$.

Finally, f_4's surjectivity implies that

$$\begin{aligned}
\tilde{H}_1(X) &\cong \tilde{H}_1(A) \oplus \tilde{H}_1(B)/\ker(f_4) \\
&= \mathbb{Z} \oplus \mathbb{Z}^2 / \langle (2,(1,0)) \rangle \\
&\cong \langle a, b, c \rangle / \langle 2a + b \rangle \\
&\cong \langle a, b, c \mid 2a + b \rangle \\
&\cong \langle a, -2a, c \rangle \\
&\cong \langle a, c \rangle = \mathbb{Z} \oplus \mathbb{Z}.
\end{aligned}$$

Thus $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$.

Exercise. (Problem 28 (b)) Let A, B be the Mobius strip and $\mathbb{R}P^2$ with a small neighborhood around them so the strip and $\mathbb{R}P^2$ are contained in A and B. For any $n \geq 3$, the exact sequence $H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B)$ implies that $H_n(X) \cong H_n(A) \oplus H_n(B) = 0 \oplus 0 = 0$ because the intersection $A \cap B$ is homotopic to S^1, so $H_n(A \cap B) = H_{n-1}(A \cap B) = 0$. Since $X = A \cup B$ has one path component, $H_0(X) = \mathbb{Z}$. We will consider the LES

$$\tilde{H}_2(A) \oplus \tilde{H}_2(B) \xrightarrow{f_1} \tilde{H}_2(X) \xrightarrow{f_2} \tilde{H}_1(A \cap B) \xrightarrow{f_3} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{f_4} \tilde{H}_1(X) \to \tilde{H}_0(A \cap B).$$

$\tilde{H}_1(A \cap B) = \mathbb{Z}$, and f_3 maps 1 to $(2,1)$ because the generator wraps around the Mobius strip twice and the $\mathbb{R}P^2$ once. Then f_3 is injective, so f_2 is the zero map. In other words, $\ker(f_2) = \tilde{H}_2(X)$, so f_1 is surjective. Since $\tilde{H}_2(A) \oplus \tilde{H}_2(B) = 0$, $\tilde{H}_2(X) = 0$. Thus $H_2(X) = 0$.

By the first isomorphism theorem and exactness,

$$\begin{aligned}
\tilde{H}_1(X) &= \tilde{H}_1(A) \oplus \tilde{H}_1(B)/\ker(f_4) \\
&= (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) / \langle (2,1) \rangle \\
&\cong \langle a, b \mid 2b \rangle / \langle 2a + b \rangle \\
&= \langle a, b \mid 2b, 2a + b \rangle \\
&= \langle a, -2a \mid 2(-2a) \rangle \\
&= \langle a \mid 4a \rangle \\
&= \mathbb{Z}_4.
\end{aligned}$$

Therefore, $H_1(X) = \mathbb{Z}_4$.

Exercise. (Problem 29) As shown earlier,

$$H_n(M_g) = \begin{cases} \mathbb{Z}^{2g} & (n = 1) \\ \mathbb{Z} & (n = 0, 2) \\ 0 & (n \geq 3). \end{cases}$$

Let R_1, R_2 be the first and second R with a small neighborhood around them. Then $X = R_1 \cup R_2$ and $R_1 \cap R_2$ is homotopy equivalent to M_g. Let $n \geq 3$. Consider the sequence

$$H_n(R_1) \oplus H_n(R_2) \to H_n(X) \to H_{n-1}(R_1 \cap R_2) \to H_{n-1}(R_1) \oplus H_{n-1}(R_2).$$

A solid g-torus deformation retracts to the wedge sum of g S^1's. $H_n(R_1) = H_n(R_2) = \oplus_{i=1}^{g} H_n(S^1) = 0$ for $n \geq 2$. By the exactness, we have $H_n(X) = H_{n-1}(R_1 \cap R_2) = H_{n-1}(M_g)$. Therefore, $H_n(X) = 0$ for $n \geq 4$, and $H_3(X) = \mathbb{Z}$. $H_0(X) = \mathbb{Z}$ because X contains only one path component.

Consider the sequence

$$\tilde{H}_2(R_1) \oplus \tilde{H}_2(R_2) \to \tilde{H}_2(X) \xrightarrow{\alpha} \tilde{H}_1(R_1 \cap R_2) \xrightarrow{\beta} \tilde{H}_1(R_1) \oplus \tilde{H}_1(R_2) \xrightarrow{\gamma} \tilde{H}_1(X) \to \tilde{H}_0(R_1 \cap R_2).$$

Then this is equivalent to

$$0 \to \tilde{H}_2(X) \xrightarrow{\alpha} \tilde{H}_1(R_1 \cap R_2) \xrightarrow{\beta} \tilde{H}_1(R_1) \oplus \tilde{H}_1(R_2) \xrightarrow{\gamma} \tilde{H}_1(X) \to 0.$$

By the exactness, α is injective and γ is surjective. Let $a_1, \cdots, a_g, b_1, \cdots, b_g$ be generators of $\tilde{H}_1(R_1 \cap R_2)$ where a_i wraps around the ith "arm"(or "handle") and b_i wraps around the ith "hole". Then $\beta(a_i) = (0, 0)$ because in R_1 and R_2, each of which is a solid torus, the "arm" gets filled in. On the other hand, $\beta(b_i) = (b_i, b_i)$ for each i.

$$\begin{aligned} H_1(X) &= \tilde{H}_1(X) \\ &= \mathrm{Im}(\gamma) \\ &= \tilde{H}_1(R_1) \oplus \tilde{H}_1(R_2) / \ker(\gamma) \\ &= \tilde{H}_1(R_1) \oplus \tilde{H}_1(R_2) / \mathrm{Im}(\beta) \\ &= \left\langle b_1, \cdots, b_g, b_1', \cdots, b_g' \right\rangle / \left\langle b_1 + b_1', \cdots, b_g + b_g' \right\rangle \\ &= \langle b_1, \cdots, b_g \rangle \\ &= \mathbb{Z}^g. \end{aligned}$$

Since α is injective, $\mathrm{Im}(\alpha)$ is isomorphic to $\tilde{H}_2(X)$. Thus $H_2(X) = \tilde{H}_2(X) = \mathrm{Im}(\alpha) = \ker(\beta) = \langle a_1, \cdots, a_g \rangle = \mathbb{Z}^g$.

- For $n \geq 4$, we have $H_n(R) \to H_n(R, M_g) \to H_{n-1}(M_g)$. As shown earlier, $H_n(R) = H_{n-1}(M_g) = 0$, so the exactness implies that $H_n(R, M_g) = 0$.
- We will consider $H_3(R) \to H_3(R, M_g) \to H_2(M_g) \to H_2(R)$. $H_3(R) = H_2(R) = 0$, so $H_3(R, M_g) = H_2(M_g)$ by the exactness. Thus $H_3(R, M_g) = \mathbb{Z}$.
- We will consider $0 \to \tilde{H}_2(R, M_g) \xrightarrow{\alpha} \tilde{H}_1(M_g) \xrightarrow{\beta} \tilde{H}_1(R) \xrightarrow{\gamma} \tilde{H}_1(R, M_g) \to 0$. (We have 0 on both ends because $\tilde{H}_2(R) = \tilde{H}_0(M_g) = 0$. Let a_i, b_i be generators of $\tilde{H}_1 M_g$ such that a_i's wrap around the handles and b_i's wrap around the holes. Using the same discussion as above, $a_i \mapsto 0$ and $b_i \mapsto b_i$ by β.
  - By the exactness, α is injective. Thus $\tilde{H}_2(R, M_g) = \mathrm{Im}(\alpha) = \ker(\beta) = \langle a_1, \cdots, a_g \rangle$. Therefore, $\tilde{H}_2(R, M_g) = \mathbb{Z}^g$.
  - By the exactness, γ is surjective. $\tilde{H}_1(R, M_g) = \mathrm{Im}(\gamma) = \tilde{H}_1(R) / \ker(\gamma) = \tilde{H}_1(R) / \mathrm{Im}(\beta)$. $\tilde{H}_1(R)$ is generated by $b_1, \cdots, b_g$ as it deformation retracts to $S^1 \vee \cdots \vee S^1$, so β is surjective. Therefore, $\tilde{H}_1(R, M_g) = 0$.
- $0 = H_1(R, M_g) \to H_0(M_g) \xrightarrow{f} H_0(R) \to H_0(R, M_g)$ is exact. Moreover, f must be an isomorphism because both M_g and R consist of one path component. Therefore, the exactness implies $H_0(R, M_g) = 0$.