# MATH 611 PROBLEM SET 1 (DUE 9/4)

## HIDENORI SHINOHARA

**Exercise 0.1.** (Exercise 4, Chapter 0) A deformation retraction in the weak sense of a space $X$ to a subspace $A$ is a homotopy $f_t : X \to X$ such that $f_0 = \mathrm{Id}, f_1(X) \subset A$, and $f_t(A) \subset A$ for all $t$. Show that if $X$ deformation retracts to $A$ in this weak sense, then the inclusion $A \to X$ is a homotopy equivalence.

*Proof.* Let $i : A \to X$ denote the inclusion. Let $F : X \times I \to X$ denote the associated map $(x, t) \to f_t(x)$. Then $F$ is a continuous function by the definition of a homotopy.

Let $f : X \to A$ be defined by $f(x) = F(x, 1) = f_1(x)$. This definition makes sense because $f_1(X) \subset A$. We claim that $f_1 \circ i \simeq \mathrm{Id}_A$ and $i \circ f_1 \simeq \mathrm{Id}_X$.

Consider $G : A \times I \to A$ such that $G(a, t) = F(a, t)$ for all $(a, t) \in A \times I$. This definition makes sense because $f_t(A) \subset A$ for all $t$.

Then $G$ is a homotopy in $A$ between $f \circ i$ and $\mathrm{Id}_A$ because:

- $G$ is a restriction of $F$, so $G$ is continuous.
- $\forall a \in A, G(a, 0) = F(a, 0) = f_0(a) = \mathrm{Id}_X(a) = \mathrm{Id}_A(a)$.
- $\forall a \in A, G(a, 1) = F(a, 1) = f(a) = f(i(a)) = (f \circ i)(a)$.

Therefore, $f \circ i \simeq \mathrm{Id}_A$.

$F$ is a homotopy between $f_0$ and $f_1$.

- We are given that $f_0 = \mathrm{Id}_X$.
- For any $x \in X, (i \circ f)(x) = i(f(x)) = f(x) = f_1(x)$, so $i \circ f = f_1$.

Therefore, $F$ is a homotopy between $\mathrm{Id}_X$ and $i \circ F$, so $i \circ f \simeq \mathrm{Id}_X$.

In conclusion, $i$ is indeed a homotopy equivalence. $\square$

**Exercise 0.2.** (Exercise 5, Chapter 0) Show that if a space $X$ deformation retracts to a point $x \in X$, then for each neighborhood $U$ of $x$ in $X$ there exists a neighborhood $V \subset U$ of $x$ such that the inclusion map $V \to U$ is nullhomotopic.

*Proof.* Let $p \in X$ be a point to which $X$ deformation retracts. Since $X$ deformation retracts to $p$, there exists a map $F : X \times I \to X$ such that

(1) $\forall x \in X, F(x, 0) = x$.
(2) $\forall x \in X, F(x, 1) = p$.

(3) $\forall t \in I, F(p,t) = p$.
(4) $F$ is continuous.

Let $U$ be a neighborhood of $p$. Then $F^{-1}(U)$ is an open subset of the product space $X \times I$. By the 3rd property of $F$ mentioned above, the slice $\{p\} \times I$ is a subset of $F^{-1}(U)$. Since $I$ is compact, there must be a open subset $V$ of $X$ such that $\{p\} \times I \subset V \times I \subset F^{-1}(U)$ by the tube lemma.

We claim that this $V$ is a desired subset.

- $V$ is an open subset of $X$.
- Since $\{p\} \times I \subset V \times I$, $p \in V$.
- Since $V \times I \subset F^{-1}(U)$, $F(V \times I) \subset U$. This implies that $\forall v \in V, F(v,0) = v \in U$. Therefore, $V \subset U$.
- We claim that the inclusion map $i : V \to U$ is nullhomotopic. Let $e_p : V \to U$ denote the constant map at $p$, $G : V \times I \to U$ be defined by $G(x,t) = F(x,t)$ for all $x \in V, t \in I$.
  - $G$ indeed maps $V \times I$ into $U$ because $F(V \times I) \subset U$. Therefore, $G$ is well-defined.
  - Since $G$ is the restriction of $F$ to $V \times I$ and $F$ is continuous, $G$ is continuous.
  - $\forall x \in V, G(x,0) = F(x,0) = x = i(x)$.
  - $\forall x \in V, G(x,1) = F(x,1) = p = e_p(x)$.
  Thus $i$ is indeed nullhomotopic.

$\square$

**Lemma 0.3.** *The neighborhood $V$ that we find in Problem 5 is connected.*

*Proof.* Suppose otherwise. Let $A, B$ denote a separation of $V$. Without loss of generality, we assume $p \in A$. Let $q \in B$. ($B$ must be nonempty since $A, B$ are a separation.)

Let $F$ be the homotopy we defined in the solution for Problem 5 from the inclusion map to the constant map at $p$. Let $f : I \to V$ be defined such that $f(t) = F(q,t)$. Then $f$ is a path from $f(0) = F(q,0) = q$ to $f(1) = F(q,1) = p$ in $V$. Since $f$ is continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are open in $I$. Moreover, $I = f^{-1}(V) = f^{-1}(A) \cup f^{-1}(B)$ and $\emptyset = f^{-1}(\emptyset) = f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. Since $1 \in f^{-1}(p) \subset f^{-1}(A)$ and $0 \in f^{-1}(q) \subset f^{-1}(B)$, $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty. Therefore, $f^{-1}(A)$ and $f^{-1}(B)$ form a separation of $I$. However, this is impossible because $I$ is connected. $\square$

**Exercise 0.4.** (Exercise 6(a), Chapter 0) Let $X$ be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$ for $r$ a rational number in $[0,1]$. Show

that $X$ deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point.

*Proof.* Let $(a,0) \in [0,1] \times \{0\}$ be given. Let $F : X \times I \to X$ be defined such that

$$F((x,y),t) = \begin{cases} (x,(1-2t)y) & (0 \le t \le 1/2) \\ (x+(a-x)(2t-1),0) & (1/2 \le t \le 1). \end{cases}$$

$F$ is well defined because when $t = 1/2$:
- $(x,(1-2t)y) = (x,0)$.
- $(x+(a-x)(2t-1),0) = (x,0)$.

Moreover, by the pasting lemma, $F$ is continuous.

Then $F$ is a deformation retract of $X$ onto $(a,0)$ because

- 
$$F((a,0),t) = \begin{cases} (a,0(1-2t)) = (a,0) & (t \in [0,1/2]) \\ (a+(a-a)(2t-1),0) = (a,0) & (t \in [1/2,1]). \end{cases}$$

Therefore, $F((a,0),t) = (a,0)$ for any $t \in I$.
- $F((x,y),0) = (x,y)$ for any $(x,y) \in x$.
- $F((x,y),1) = (a,0)$ for any $(x,y) \in x$.

Therefore, $F$ is indeed a deformation retract of $X$ onto $(a,0)$.

Suppose that there exists a point $(a,b) \in X$ to which $X$ deformation retracts onto such that $b \ne 0$. Let $G : X \times I \to X$ denote such a deformation retract. Consider the open subset $U = B((a,b),b) \cap X$. Note that $U$ is disjoint from the segment $[0,1] \times \{0\}$. Then $U$ is a neighborhood of $(a,b)$, a point to which $X$ deformation retracts onto. By Problem 5 (Chapter 0), there must exist a neighborhood $V \subset U$ of $x$ such that the inclusion map $V \to U$ is nullhomotopic. By the Lemma we showed above, $V$ must be connected. Since $V$ is an open subset of $X$, there must exist an $r > 0$ such that $B((a,b),r) \cap X \subset V$. Let $c$ be an irrational number in $(a, a+r)$. Then $V \cap ((-\infty, c) \times \mathbb{R})$ and $V \cap ((c, \infty) \times \mathbb{R})$ form a separation of $V$. This is a contradiction, so our initial assumption that $X$ deformation retracts onto $(a,b)$ was wrong. Therefore, $X$ deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point. $\square$

**Exercise 0.5.** (Exercise 9, Chapter 0) Show that a retract of a contractible space is contractible.

*Proof.* Let $X$ be a contractible space. Then $\mathrm{Id}_X$ is homotopic to a constant map. This implies the existence of a fixed point $p \in X$ and a continuous function $F : X \times I \to X$ such that

- $\forall x \in X, F(x, 0) = x,$
- $\forall x \in X, F(x, 1) = p.$

Let $A \subset X$ be a retract of $X$, and let $r : X \to A$ denote a retraction. In other words, $r(X) = A$ and $r \mid_A = \mathrm{Id}_A$.

Let $G : A \times I \to A$ be the restriction of $r \circ F$ to $A \times I$. This makes sense because $F$ maps $A \times I$ into $X$, and $r$ maps $X$ into $A$. We claim that $G$ is a homotopy between $\mathrm{Id}_A$ and the constant map $e_{r(p)}$ such that $e_{r(p)}(a) = r(p)$ for all $a \in A$.

- $r \circ F$ is continuous since it is a composition of continuous functions. $G$ is a restriction of a continuous function, so $G$ is continuous.
- $G(a, 0) = r(F(a, 0)) = r(a) = a = \mathrm{Id}_A(a).$
- $G(a, 1) = r(F(a, 1)) = r(p) = e_{r(p)}(a).$

Therefore, $G$ is indeed a homotopy between $\mathrm{Id}_A$ and the constant map at $r(p)$. Since the identity map is homotopic to a constant map, $A$ is contractible. $\qquad\square$

**Exercise 0.6.** (Exercise 13, Chapter 0) Show that any two deformation retractions $r_t^0$ and $r_t^1$ of a space $X$ onto a subspace $A$ can be joined by a continuous family of deformation retractions $r_t^s$, $0 \le s \le 1$, of $X$ onto $A$, where continuity means that the map $X \times I \times I \to X$ sending $(x, s, t)$ to $r_t^s(x)$ is continuous.

*Proof.* Let $F : X \times I \times I \to X$ be defined such that

$$F(x, t, s) = \begin{cases} r_{t(1-2s)}^0(x) & (s \in [0, 1/2]) \\ r_{t(2s-1)}^1(x) & (s \in [1/2, 1]). \end{cases}$$

We claim that $F$ is well-defined and satisfies the desired properties.

- Let $s = 1/2$. $r_{t(1-2s)}^0(x) = r_0^0(x) = x$ because $r_t^0$ is a deformation retraction. Similarly, $r_{t(2s-1)}^1(x) = r_0^1(x) = x$ because $r_t^0$ is a deformation retraction. Therefore, $F$ is well defined when $s = 1/2$. Moreover, by the pasting lemma, $F$ is continuous. This is because the intersection $X \times I \times [0, 1/2] \cap X \times I \times [1/2, 1] = X \times I \times \{1/2\}$ is closed.
- $F(x, t, 0) = r_t^0(x)$ for any $x \times t \in X \times I.$
- $F(x, t, 1) = r_t^1(x)$ for any $x \times t \in X \times I.$

Therefore, $F$ maps $X \times I \times I \to X$ continuously sending $(x, s, t)$ to $r_t^s(x)$. $\qquad\square$

**Exercise 0.7.** (Exercise 7, Chapter 1.1) Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so $f$ restricts to the identity on the two

boundary circles of $S^1 \times I$. Show that $f$ is homotopic to the identity by a homotopy $f_t$ that is stationary on one of the boundary circles, but not by any homotopy $f_t$ that is stationary on both boundary circles.

*Proof.* Define $F : (S^1 \times I) \times I \rightarrow S^1 \times I$ such that $F((\theta, s), t) = t(\theta, s) + (1-t)f(\theta, s)$. Then $F$ is a homotopy between $f$ and the identity map that is stationary on $S^1 \times \{0\}$. This is because $F((\theta, 0), t) = t(\theta, 0) + (1 - t)f(\theta, 0) = (t\theta, 0) + ((1 - t)\theta, 0) = (\theta, 0)$ for any $(\theta, t) \in S^1 \times I$.

Suppose that there exists a homotopy $G : (S^1 \times I) \times I \rightarrow S^1 \times I$ between $f$ and the identity map that is stationary on both boundary circles. Let $H : I \times I \rightarrow S^1$ be defined such that $H(s, t) = \pi_1(F((0, t), s))$ where $\pi_1$ denotes the projection of the first coordinate.

- $H(s, 0) = \pi_1(G((0, 0), s)) = \pi_1(0, 0) = 0$ because $G$ is stationary on the circle $S^1 \times \{0\}$.
- $H(s, 1) = \pi_1(G((0, 1), s)) = \pi_1(0, 1) = 0$ because $G$ is stationary on the circle $S^1 \times \{1\}$.
- $H(0, t) = \pi_1(G((0, t), 0)) = \pi_1(f(0, t)) = \pi_1(2\pi t, t) = 2\pi t$.
- $H(1, t) = \pi_1(G((0, t), 1)) = \pi_1(0, t) = 0$.

Then $t \mapsto H(0, t)$ corresponds to the $\omega$ in Theorem 1.7, and $t \mapsto H(1, t)$ corresponds to a constant map. In other words, $H$ is a homotopy between $\omega$ and a constant map in $S^1$. However, this is a contradiction because Theorem 1.7 states that $\pi_1(S^1)$ is the infinite cyclic group generated by $\omega$. Therefore, such a homotopy $G$ does not exist.    $\square$

**Exercise 0.8.** (Exercise 16, Chapter 1.1) Show that there are no retractions $r : X \rightarrow A$ in the following cases:

- $X = \mathbb{R}^3$ with $A$ any subspace homeomorphic to $S^1$.
- $X = S^1 \times D^2$ with $A$ its boundary torus $S^1 \times S^1$.

*Proof.*

- Suppose that there exists a retract $r : X \rightarrow A$. In other words, $r$ is a continuous map such that $r \mid_A = \mathrm{Id}_A$ and $r(X) \subset A$. Let $\phi : S^1 \rightarrow A$ be a homeomorphism. Let $\omega$ be the loop defined in Theorem 1.7. Then $\pi_1(S^1, (1, 0))$ is the infinite cyclic group generated by $[\omega]$. Consider the following two paths:
  - $\phi \circ \omega : I \rightarrow A$.
  - $e_0 : I \rightarrow A$ such that $e_0(t) = (\phi \circ \omega)(0) = \phi(1, 0)$.
  
  They are paths in $A$, and $A$ is a subset of $\mathbb{R}^3$, so they are paths in $\mathbb{R}^3$. Since $\mathbb{R}^3$ is convex, we can define a linear homotopy between them. Let $F : I \times I \rightarrow \mathbb{R}^3$ such that $F(s, t) = t(\phi \circ \omega)(s) + (1-t)e_0(s)$. Therefore, the two paths are homotopic in $\mathbb{R}^3$.

We will consider $\phi^{-1} \circ r \circ F$ that maps $I \times I \to S$. Since it is a composition of continuous functions, it is continuous.

$$
\begin{aligned}
(\phi^{-1} \circ r \circ F)(s,0) &= \phi^{-1}(r(F(s,0))) \\
&= \phi^{-1}(r(e_0(s))) \\
&= \phi^{-1}(e_0(s)) && (\text{since } e_0(s) \in A) \\
&= (1,0). \\
(\phi^{-1} \circ r \circ F)(s,1) &= \phi^{-1}(r(F(s,1))) \\
&= \phi^{-1}(r((\phi \circ \omega)(s))) \\
&= \phi^{-1}(r(\phi(\omega(s)))) \\
&= \phi^{-1}(\phi(\omega(s))) && (\text{since } \phi(\omega(s)) \in A) \\
&= \omega(s). \\
(\phi^{-1} \circ r \circ F)(0,t) &= \phi^{-1}(r(\phi(1,0))) \\
&= \phi^{-1}(\phi(1,0)) \\
&= (1,0). \\
(\phi^{-1} \circ r \circ F)(1,t) &= \phi^{-1}(r(\phi(1,0))) \\
&= \phi^{-1}(\phi(1,0)) \\
&= (1,0).
\end{aligned}
$$

Therefore, $\phi^{-1} \circ r \circ F$ is a path homotopy between $\omega$ and the constant loop at $(1,0)$. This implies that $[w]$ is the identity element in $\pi_1(S^1)$. However, this is a contradiction because Theorem 1.7 states that $\pi_1(S^1)$ is the infinite cyclic group generated by $[w]$.
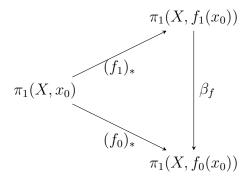
Hence, there exists no such retraction $r$.

- Suppose $X$ retracts onto $A$. By Proposition 1.17, the homomorphism $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $i : A \to X$ is injective. By Theorem 1.7, $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$. $\pi_1(D^2) = 0$ because $D^2$ is a convex subset and thus a linear homotopy connects any paths. By Proposition 1.12, $\pi_1(X) = \pi_1(S^1) \times \pi_1(D^2)$ and $\pi_1(A) = \pi_1(S^1) \times \pi_1(S^1)$. This implies the existence of a injective group homomorphism $\phi$ from $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{Z} \times 0$. This is impossible because $\phi(1) = 0 = \phi(2)$. Therefore, $X$ does not retract onto $A$.

$\square$

**Exercise 0.9.** (Exercise 20, Chapter 1.1) Suppose $f_t : X \to X$ is a homotopy such that $f_0$ and $f_1$ are each the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$.
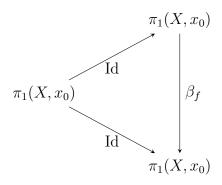
*Proof.* Let $x_0 \in X$ be given. Let $f : I \to X$ be the loop defined such that $f(t) = f_t(x_0)$.

- $f_t : X \to X$ is a homotopy.
- $f$ is a path formed by the images of the base point $x_0$.

By Lemma 1.19, the following diagram commutes.

$$
\begin{array}{ccc}
& & \pi_1(X, f_1(x_0)) \\
& \nearrow{\scriptstyle (f_1)_*} & \Big\downarrow{\scriptstyle \beta_f} \\
\pi_1(X, x_0) & & \\
& \searrow{\scriptstyle (f_0)_*} & \\
& & \pi_1(X, f_0(x_0))
\end{array}
$$

$(f_0)_* = (f_1)_* = (\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(X,x_0)}$ by a basic property of induced homomorphisms (P.34 of Hatcher). Since $f_0 = f_1 = \mathrm{Id}_X$, $f_0(x_0) = f_1(x_0) = x_0$. Therefore, the diagram above can be simplified as following:

$$
\begin{array}{ccc}
& & \pi_1(X, x_0) \\
& \nearrow{\scriptstyle \mathrm{Id}} & \Big\downarrow{\scriptstyle \beta_f} \\
\pi_1(X, x_0) & & \\
& \searrow{\scriptstyle \mathrm{Id}} & \\
& & \pi_1(X, x_0)
\end{array}
$$

Let $[g] \in \pi_1(X, x_0)$. Then by the diagram above, we have $\mathrm{Id}([g]) = \mathrm{Id}(\beta_f([g]))$. This implies $[g] = [f \cdot g \cdot \overline{f}]$. Therefore, $[g] \cdot [f] = [f] \cdot [g]$, so $[f]$ commutes with every element in $\pi_1(X, x_0)$. Hence, $[f] \in Z(\pi_1(X, x_0))$. $\square$