MATH 633 HOMEWORK 3

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Exercise. (Problem 1) A simply connected space is clearly piecewise smooth simply connected. Let Ω be piecewise smooth simply connected and $\gamma_1, \gamma_2 : [0,1] \to \Omega$ be two continuous curves with the same end points. Since Ω is open, $\gamma_1(t)$ has an open ball around it that is contained in Ω for each $t \in [0,1]$. Since $[0,1]$ is compact and γ_1 is continuous, $\gamma_1([0,1])$ is compact. Hence, there is a finite partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that $\gamma_1([t_i, t_{i+1}])$ is contained in an open ball $\subset \Omega$ for each i. Then γ_1 is homotopic to the curve $\gamma_{1'}$ that consists of n straight lines, ith of which is the line between $\gamma_1(t_i)$ and $\gamma_1(t_{i-1})$ where $i = 1, \cdots, n$. This can be shown by the "straight-line" homotopy because $\gamma_1([t_{i-1}, t_i])$ and the ith straight line are in an open ball contained in Ω.

A similar argument can be applied to show that γ_2 is homotopic to a curve $\gamma_{2'}$ that consists of finitely many straight lines. A curve consisting of finitely many straight lines is clearly piecewise smooth.

Therefore, $\gamma_1 \sim \gamma_{1'} \sim \gamma_{2'} \sim \gamma_2$. Thus Ω is simply connected.

Exercise. (Problem 2) Define $T(x,y) = x + iy$.

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\begin{aligned}
\int_S f dz &= \int_0^1 f(t)dt + \int_0^1 f(it)(it)'dt + \int_0^1 f(1+it)(1+it)'dt + \int_0^1 f(t+i)(t+i)'dt \\
&= \int_0^1 f(t) + f(t+i)dt + i\int_0^1 f(it) + f(1+it)dt \\
&= \int_0^1 f(T(x,0)) + f(T(x,1))dx + i\int_0^1 f(T(0,y)) + f(T(1,y))dy \\
&= \int_0^1 u(T(x,0)) + u(T(x,1))dx + i\int_0^1 u(T(0,y)) + u(T(1,y))dy \\
&+ i\int_0^1 v(T(x,0)) + v(T(x,1))dx - \int_0^1 v(T(0,y)) + v(T(1,y))dy \\
&= \int_0^1 u(T(x,0)) + u(T(x,1))dx - \int_0^1 v(T(0,y)) + v(T(1,y))dy \\
&+ i(\int_0^1 u(T(0,y)) + u(T(1,y))dy + \int_0^1 v(T(x,0)) + v(T(x,1))dx) \\
&= \int_S u \circ T dx + \int_S -v \circ T dy + i(\int_S v \circ T dx + \int_S u \circ T) \\
&= \int_{\operatorname{int} S} -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + i(\int_{\operatorname{int} S} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) \\
&= 0.
\end{aligned}
$$

Exercise. (Problem 3) Let $\gamma_1, \gamma_2, \gamma_3$ denote the real part of the contour, the arc, and the rest, respectively. $\int_{\gamma_1+\gamma_2+\gamma_3} e^{z^2} = 0$ because e^{z^2} is entire.

Let $C = \int_0^\infty \cos(x^2)dx, S = \int_0^\infty \sin(x^2)dx$.

- By setting $\gamma_1(t) = t$ where $t \in [0, R]$, we obtain $\lim_{R\to\infty} \int_0^R e^{-t^2}dt = \sqrt{\pi}/2$. Thus the integration over γ_1 is $\sqrt{\pi}/2$.
- Let $\gamma_2(t) = Re^{it}$ with $t \in [0, \pi/4]$. Then $\left|\int_0^{\pi/4} e^{-\gamma_2^2}\gamma_2' dt\right| \leq \int_0^{\pi/4} \left|e^{-\gamma_2^2}\gamma_2'\right| dt = R\int_0^{\pi/4} \left|e^{-R^2e^{2it}}\right| dt$. Since $|e^z| = e^{\mathrm{Re}(z)}$, it suffices to calculate $R\int_0^{\pi/4} e^{-R^2\cos(2t)}dt$. $\cos(2t) \geq 1 - \frac{4t}{\pi} \geq 0$ on $[0, \pi/4]$, where the $1 - \frac{4t}{\pi}$ denotes the straight-line approximation of $\cos(2t)$ over the given interval. Thus the integral is bounded by $R\int_0^{\pi/4} e^{-R^2(1-\frac{4t}{\pi})}dt$.

$$\begin{aligned} R\int_0^{\pi/4} e^{-R^2(1-\frac{4t}{\pi})}dt &= \frac{R}{e^{R^2}}\Big(\frac{\pi}{4R^2}e^{\frac{4R^2t}{\pi}}\Big|_0^{\pi/4}\Big) \\ &= \frac{\pi}{4Re^{R^2}}(e^{R^2} - 1) \\ &= \frac{\pi}{4R}(1 - \frac{1}{e^{R^2}}). \end{aligned}$$

When $R \to 0$, it is clear that the limit approaches 0. Therefore, $\int_\gamma e^{-z^2}dz = 0$.

- Let $\gamma_3(t) = te^{i\pi/4}$ with $0 \leq t \leq R$. To simplify the calculation , γ_3 is oriented in the opposite way, so we will multiply -1 in the end.

$$\begin{aligned} \lim_{R\to\infty}\int_0^R e^{-\gamma_3^2(t)}\gamma_3'(t)dt &= \frac{1+i}{\sqrt{2}}\lim_{R\to\infty}\int_0^R e^{-t^2e^{\pi i/2}}dt \\ &= \frac{1+i}{\sqrt{2}}\lim_{R\to\infty}\int_0^R e^{-t^2 i}dt \\ &= \frac{1+i}{\sqrt{2}}\lim_{R\to\infty}\int_0^R \cos\big(-t^2\big) + i\sin\big(-t^2\big)dt \\ &= \frac{1+i}{\sqrt{2}}[\lim_{R\to\infty}\int_0^R \cos\big(-t^2\big) + \lim_{R\to\infty}\int_0^R i\sin\big(-t^2\big)dt] \\ &= \frac{1+i}{\sqrt{2}}[\lim_{R\to\infty}\int_0^R \cos\big(t^2\big) - \lim_{R\to\infty}\int_0^R i\sin\big(t^2\big)dt] \\ &= \frac{1+i}{\sqrt{2}}[C - iS]. \end{aligned}$$

By putting these together and comparing the real and imaginary parts, we obtain a system of equations:

$$\begin{aligned} 0 &= \frac{\pi}{2} - \frac{C}{\sqrt{2}} - \frac{S}{\sqrt{2}} \\ 0 &= -C + S. \end{aligned}$$

By solving it, we obtain $S = C = \sqrt{2\pi}/4$.

Exercise. (Problem 4)

- Ω_1 is simply connected because any two continuous curves with the same end points are joined by the straight-line homotopy.
- Ω_2 is not simply connected because Ω_2 is homeomorphic to S^1 which has a nontrivial fundamental group. In other words, $\phi : \theta \mapsto (a+b)e^{2\pi i\theta}/2$ is a continuous curve in Ω that is not homotopic to the constant curve at $(a+b)/2$.
- Ω_3 is not simply connected because it is not connected.