MATH 602(HOMEWORK 1)

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Exercise. 1

- Let $p \in V(I \cap J)$. For any $\sum_{i=1}^n f_i g_i \in IJ$, we have $f_i g_i \in I \cap J$ for each i. Thus $(\sum_{i=1}^n f_i g_i)(p) = 0$, so $p \in V(IJ)$. Let $p \in V(IJ)$. Let $f \in I \cap J$. Then $f^2 \in IJ$, so $(f(p))^2 = 0$. Thus $f(p) = 0$, so $p \in V(I \cap J)$. Therefore, $V(I \cap J) = V(IJ)$.

  Let $p \in V(I) \cup V(J)$. Then either all polynomials in I vanish at p or all polynomials in J vanish at p. Thus all the polynomials in the intersection must vanish at p. Thus $V(I) \cup V(J) \subset V(I \cap J)$. On the other hand, let $p \in V(I \cap J) \setminus (V(I) \cup V(J))$. If no such element exists, we are done. Then every polynomial in the intersection vanishes at p. Let $f \in I$ and $g \in J$ be polynomials that do not vanish at p. Then $fg \in I \cap J$, so $(fg)(p) = 0$. However, this is impossible because $f(p) \neq 0$ and $g(p) \neq 0$. Therefore, $V(I) \cup V(J) = V(I \cap J)$.
- $p \in V(I + J)$ if and only if $\forall f \in I + J, f(p) = 0$ if and only if $\forall f \in I, f(p) = 0$ and $\forall f \in J, f(p) = 0$ if and only if $p \in V(I) \cap V(J)$.
- If every polynomial in J vanishes at a point, every polynomial in I must vanish at that point.
- If a polynomial vanishes in Y, then it must vanish in X.
- TODO

Exercise. 2

- $$\begin{aligned} y \in (I_1 + I_2)^e &\iff y \in f(I_1 + I_2)B \\ &\iff \exists x_1, x_2 \in I_1, I_2, b \in B, y = f(x_1 + x_2)b \\ &\iff \exists x_1, x_2 \in I_1, I_2, b \in B, y = f(x_1)b + f(x_2)b \\ &\iff y \in I_1^e + I_2^e. \end{aligned}$$
- $$\begin{aligned} y \in (I_1 \cap I_2)^e &\implies y \in f(I_1 \cap I_2)B \\ &\implies \exists x \in I_1 \cap I_2, b \in B, y = f(x)b \\ &\implies (\exists x \in I_1, b \in B, y = f(x)b) \text{ and } (\exists x \in I_2, b \in B, y = f(x)b) \\ &\implies y \in I_1^e, y \in I_2^e \\ &\implies y \in I_1^e \cap I_2^e. \end{aligned}$$
- $(I_1 I_2)^e = f(I_1 I_2)B = (f(I_1)f(I_2))B = (f(I_1)B)(f(I_2)B)$. $f(I_1)f(I_2) = f(I_1 I_2)$ because the product of two ideals consists of a finite sum of elements and f preserves finite sums.

- Let $x \in J_1^c + J_2^c$. Then $x \in f^{-1}(J_1) + f^{-1}(J_2)$. Then $x = a + b$ where $a \in f^{-1}(J_1)$ and $b \in f^{-1}(J_2)$. This implies $x = a + b$ where $f(a) \in J_1$ and $f(b) \in J_2$. Then, $f(x) = f(a+b) = f(a) + f(b) \in J_1 + J_2$, so $x \in f^{-1}(J_1 + J_2)$.
- $f^{-1}(J_1 \cap J_2) = f^{-1}(J_1) \cap f^{-1}(J_2)$ from set theory.
- Let $\sum_{i=1}^n a_i b_i \in J_1^c J_2^c$ where $a_i \in J_1^c$ and $b_i \in J_2^c$. Then $f(a_i) \in J_1$ and $f(b_i) \in J_2$. Thus $\sum f(a_i) f(b_i) \in J_1 J_2$. Since f preserves product and addition, $f(\sum a_i b_i) \in J_1 J_2$. Thus $\sum a_i b_i \in f^{-1}(J_1 J_2) = (J_1 J_2)^c$.

Exercise. 3 $(I : J)$ is nonempty because $0 \in (I : J)$. $(I : J)$ is closed under addition, and for all $x \in R$, $rJ \subset I \implies x(rJ) = r(xJ) = rJ \subset I$. Thus $(I : J)$ is an ideal.