MATH 612(HOMEWORK 5)

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Exercise. (2.2.7) Let $f(x_1, \cdots, x_n) = (-x_1, x_2, x_3, \cdots, x_n)$. Then

$$\begin{array}{ccc} \mathbb{R}^n \setminus \{0\} & \xrightarrow{f} & \mathbb{R}^n \setminus \{0\} \\ \downarrow r & & \downarrow r \\ S^{n-1} & \xrightarrow{\text{reflection}} & S^{n-1} \end{array}$$

where r is the obvious deformation retraction. By (e) on P.134, the reflection map induces -1 on $H^{n-1}(S^{n-1})$. By naturality, f_* is -1.

Similarly, let $f(x_1, \cdots, x_n) = (cx_1, x_2, x_3, \cdots, x_n)$ with $c > 0$. Then

$$\begin{array}{ccc} \mathbb{R}^n \setminus \{0\} & \xrightarrow{f} & \mathbb{R}^n \setminus \{0\} \\ \downarrow r & & \downarrow r \\ S^{n-1} & \xrightarrow{g} & S^{n-1} \end{array}$$

where r is the obvious deformation retraction. Then g is a function that is homotopy equivalent to the identity map on S^{n-1}. By (e) on P.134, g induces the identity map on $H^{n-1}(S^{n-1})$. By naturality, f_* is 1.

Using the exact same argument, $(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) \mapsto (x_1, \cdots, x_j, \cdots, x_i, \cdots, x_n)$ induces -1 because a reflection is -1 and $(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) \mapsto (x_1, \cdots, x_i, \cdots, x_j + x_i, \cdots, x_n)$ induces 1 because homotopy equivalent maps induce the same map. Therefore, we have shown that elementary matrices induce 1 or -1 based on the sign of their determinants. Any invertible linear operation can be written as a product of elementary matrices and since $(fg)_* = f_* g_*$ the given invertible linear operation induces 1 or -1 based on the sign of their determinants.

Exercise. (3.3.1) Let A, B be two copies of $\mathbb{R}_{\geq 0}$. Consider the space X obtained from $A \cup B$ and the relation $a \sim b$ whenever $a = b = 0$ or $a = b > 1$. Then such a space is clearly second countable and locally homeomorphic to $\mathbb{R}$. For each point $x \in X$, $H_1(\mathbb{R}^1, \mathbb{R}^1 - \{x\}) = H_0(S^0) = \{[-1], [1]\}$ because S^0 consists of two points -1, 1. Therefore, for every point $x \in X \setminus \{0\}$, there are exactly two choices of generators, each of which corresponds to a number larger than x or smaller than x, which we will refer to "positive" and "negative" for convenience. Suppose X is orientable. The 0.1 in the blue neighborhood which was originally in A has an orientation that is either positive or negative. Without loss of generality, it has the positive orientation. This implies that 1 in A has the positive orientation, which in turn implies that all numbers > 1 sufficiently close to 1 have the positive orientation.

Then the 0.1 in the blue neighborhood which was originally in B has an orientation that is negative. Therefore, the orientation at 1 which was originally in A has the positive orientation, and 1 which was originally in B has the negative orientation. This implies that

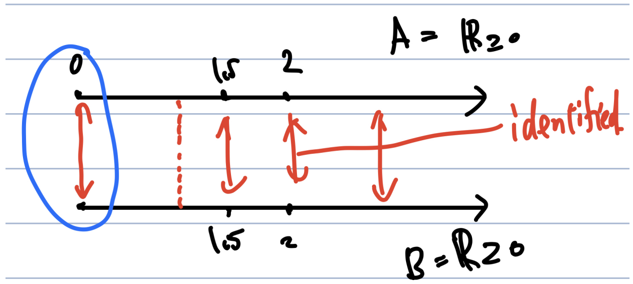

FIGURE 1. Orientability

1 in B has the positive orientation, which in turn implies that all numbers > 1 sufficiently close to 1 have the negative orientation.

This is a contradiction, so X is not orientable.

Exercise. (3.3.2) It suffices to consider the case when M is connected because orientations can be chosen for each connected component. By Proposition 3.25, M is orientable if and only if an orientable two-sheeted covering space $\tilde{M}$ has two components. Each component has exactly one lift of x. Since the covering map maps some neighborhood of each lift to a neighborhood of x homeomorphically, each component in $\tilde{M}$ is connected after removing the two lifts of x. Thus $\tilde{M} \setminus \{\tilde{x}_1, \tilde{x}_2\}$ is an orientable two-sheeted covering space of $M \setminus \{x\}$ with two components. By Proposition 3.25, $M \setminus \{x\}$ is orientable.

Exercise. (3.3.3) Let $\tilde{M}$ be a covering space of an orientable manifold M with a covering map q. Let $\tilde{x} \in \tilde{M}$ be given. Then $H_n(\tilde{M} \mid \tilde{x}) \cong H_n(\tilde{U} \mid \tilde{x})$ where $\tilde{U}$ is a neighborhood of $\tilde{x}$ that is homeomorphically mapped to an open subset $U \subset M$ by q. Since M is orientable, we have a local orientation $\mu_x \in H_n(U \mid x)$. q induces an isomorphism between $H_n(U \mid x)$ and $H_n(\tilde{U} \mid \tilde{x})$. Let $\mu_{\tilde{x}} \in H_n(\tilde{M} \mid \tilde{x})$ denote the element corresponding to μ_x by the isomorphisms. This assignment satisfies the local consistency because for each $\tilde{x} \in \tilde{M}$, we pick a small neighborhood $\tilde{B}$ around $\tilde{x}$ that is mapped homeomorphically to a neighborhood B around x by q where B satisfies the local consistency condition around x.

Exercise. (Extra) We know that $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1})$ where $|\alpha| = 2$. Let f be a homeomorphism from $\mathbb{C}\mathrm{P}^n$ to itself. Suppose n is even. Then f induces a homomorphism on $H^n(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ which is generated by $\alpha^{n/2}$. Similarly, f induces a homomorphism on $H^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ which is generated by α^n. By naturality, $f^*(\alpha^n) = (f^*(\alpha^{n/2}))^2$. Clearly, $(f^*(\alpha^{n/2}))^2 \neq -\alpha^n$.