MATH 611 (DUE 10/23)

HIDENORI SHINOHARA

1. Simplicial and Singular Homology

Exercise. (Problem 2) Show that the Δ-complex obtained from Δ^3 by performing the edge identifications $[v_0, v_1] \sim [v_1, v_3]$ and $[v_0, v_2] \sim [v_2, v_3]$ deformation retracts onto a Klein bottle. Find other pairs of identifications of edges that produce Δ-complexes deformation retracting onto a torus, a 2-sphere, and $\mathbb{R}\mathbf{P}^2$.

Proof. The deformation retraction of Δ^3 onto a Klein bottle is described in 1. We will start by "pushing" Δ^3 from edge (v_1, v_2). This will leave the surface that consists of the triangles $[v_0, v_1, v_3]$ and $[v_0, v_2, v_3]$. (In other words, a diamond shape consisting of the vertices $[v_0, v_1, v_3, v_2]$.) Step 2 in Figure 1 is what Δ^3 should look like after the deformation retract. Step 3 through 6 show why this is a Klein bottle.

Figure 2 shows the identification of edges for a torus, 2-sphere, and $\mathbb{R}\mathbf{P}^2$. $\square$

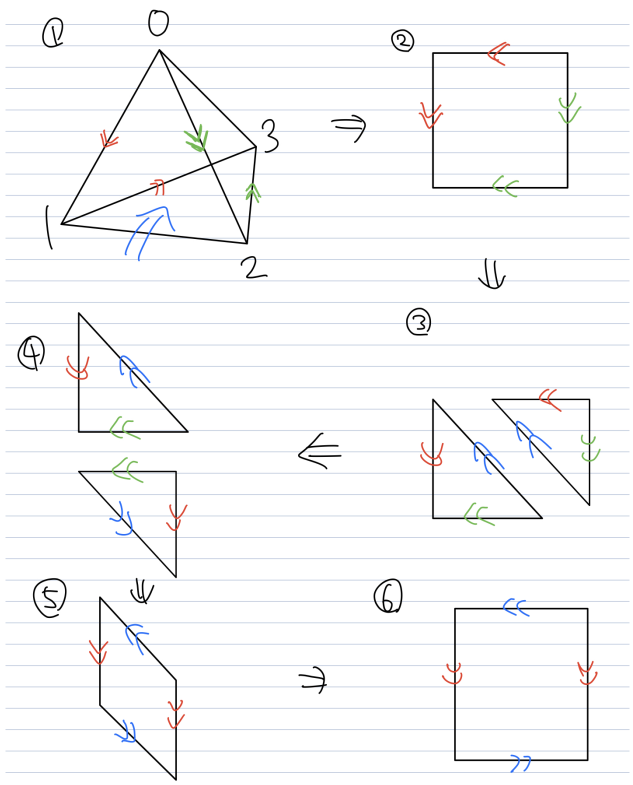

Figure 1. Problem 2(Klein Bottle)

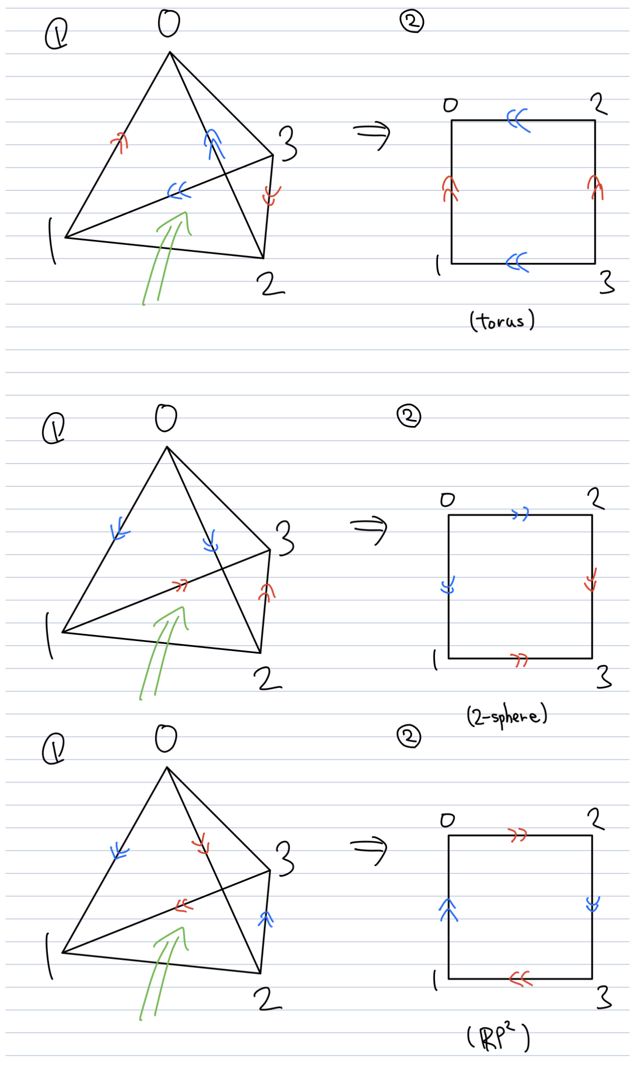

FIGURE 2. Problem 2(Torus, 2-Sphere, $\mathbb{R}\mathbf{P}^2$)

Exercise. (Problem 4) Compute the simplicial homology groups of the triangular parachute obtained from Δ^2 by identifying its three vertices to a single point.

Proof. Let v_0 denote the only vertex, e_1, e_2, e_3 denote the three edges of the parachute, and σ denote the face of the parachute as in Figure 3. $C_k = 0$ for $k \geq 3$ because Δ^2 with the vertices identified does not contain any k-dimensional simplicies for $k \geq 3$. $C_2 = \langle \sigma \rangle, C_1 = \langle e_1, e_2, e_3 \rangle, C_0 = \langle v_0 \rangle$. For each n, ∂_n is defined such that $\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha | [v_0, \cdots, \hat{v}_i, \cdots, v_n]$.

- $\partial_2(\sigma) = e_3 - e_2 + e_1$.
- $\partial_1(e_1) = v - v = 0$. Similarly, $\partial_1(e_2) = \partial_1(e_3) = 0$.
- ∂_0 and ∂_3 are both the zero map.

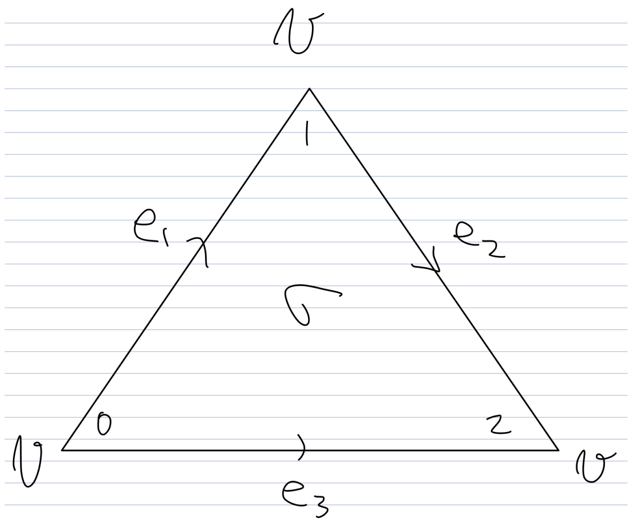

FIGURE 3. Problem 4

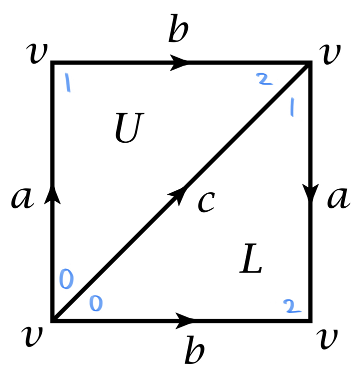

FIGURE 4. Problem 5

Thus

$$H_n = \begin{cases} \{0\} & (n \geq 3) \\ \ker(\partial_2)/\operatorname{Im}(\partial_3) = 0/0 \cong 0 & (n = 2) \\ \ker(\partial_1)/\operatorname{Im}(\partial_2) = \langle e_1, e_2, e_3 \rangle / \langle e_3 - e_2 + e_1 \rangle \cong \langle e_1, e_2, -e_2 + e_1 \rangle \cong \mathbb{Z}^2 & (n = 1) \\ \ker(\partial_0)/\operatorname{Im}(\partial_1) = \langle v \rangle / 0 \cong \mathbb{Z} & (n = 0). \end{cases}$$

□

Exercise. (Problem 5) Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure described at the beginning of this section.

Proof. We will use the notations in Figure 4.

$$C_n = \begin{cases} 0 & (n \geq 3) \\ \langle U, L \rangle & (n = 2) \\ \langle a, b, c \rangle & (n = 1) \\ \langle v \rangle & (n = 0). \end{cases}$$

$\partial_n = 0$ for $n \geq 3$ and $n = 0$.

$$\begin{aligned} \partial_2(U) &= \sum_{i=0}^{2} (-1)^i \sigma|[0,1,2] \\ &= \sigma|[1,2] - \sigma|[0,2] + \sigma|[0,1] \\ &= b - c + a. \\ \partial_2(L) &= \sum_{i=0}^{2} (-1)^i \sigma|[0,1,2] \\ &= \sigma|[1,2] - \sigma|[0,2] + \sigma|[0,1] \\ &= a - b + c. \end{aligned}$$

$\partial_1(a) = 0$ since $\partial_1(a) = \sigma|[1] - \sigma|[0] = v - v = 0$. Similarly, $\partial_1(b) = \partial_1(c) = 0$. Thus $H_n = \{0\}$ if $(n \geq 3)$. $H_2 = \ker(\partial_2)/\operatorname{Im}(\partial_3) = 0/0 \cong 0$.

$$\begin{aligned} H_1 &= \ker(\partial_1)/\operatorname{Im}(\partial_2) \\ &= \langle a, b, c \rangle \,/\, \langle b - c + a, a - b + c \rangle \\ &\cong \langle a, b, a + b \mid a - b + (a + b) \rangle \\ &\cong \langle a, b \mid 2a \rangle \\ &\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}. \end{aligned}$$

$H_0 = \ker(\partial_0)/\operatorname{Im}(\partial_1) = \langle v \rangle \,/0 \cong \mathbb{Z}$. □

Exercise. (Problem 7) Find a way of identifying pairs of faces of Δ^3 to produce a Δ-complex structure on S^3 having a single 3-simplex, and compute the simplicial homology groups of this Δ-complex.

Proof. We will identify $[0,2,3] \sim [1,2,3]$ and $[0,1,2] \sim [0,1,3]$ of the tetrahedra T in Figure 5. Then we have

$$\begin{aligned} C_3 &= \{T\} \\ C_2 &= \{f_1, f_2\} \\ C_1 &= \{e_1, e_2, e_3\} \\ C_0 &= \{v_1, v_2\}. \end{aligned}$$

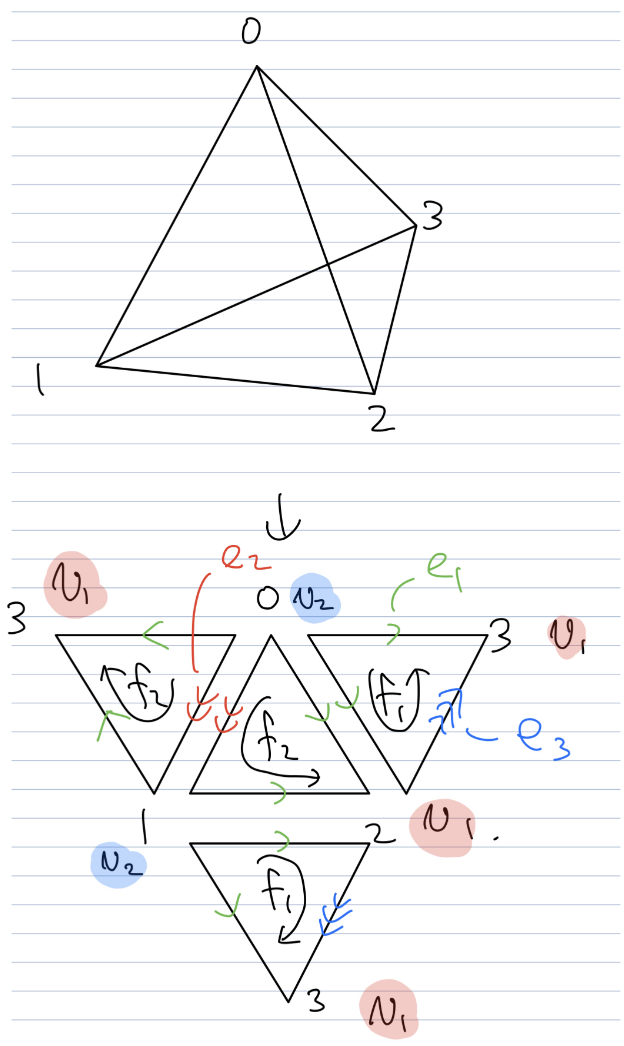

Figure 5. Problem 7

We will examine ∂.

$$\begin{aligned}
\partial_3(T) &= [1,2,3]-[0,2,3]+[0,1,3]-[0,1,2] = f_1 - f_1 + f_2 - f_2 = 0.\\
\partial_2(f_1) &= [2,3]-[0,3]+[0,2] = e_3 - e_1 + e_1 = e_3.\\
\partial_2(f_2) &= [1,2]-[0,2]+[0,1] = e_1 - e_1 + e_2 = e_2.\\
\partial_1(e_1) &= [3]-[0] = v_1 - v_2.\\
\partial_1(e_2) &= [1]-[0] = v_2 - v_2 = 0.\\
\partial_1(e_3) &= [3]-[2] = v_1 - v_1 = 0.
\end{aligned}$$

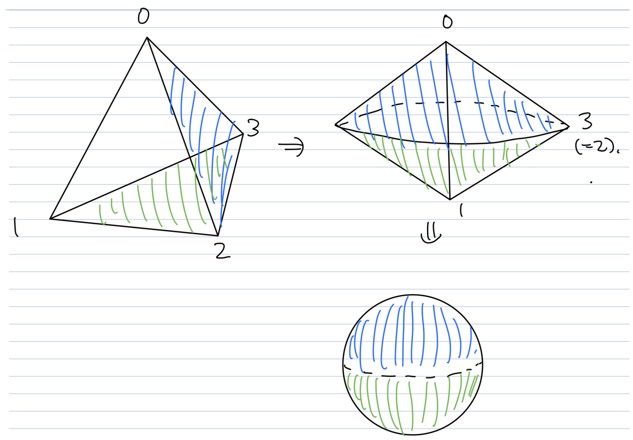

FIGURE 6. Problem 7(S^3)

Therefore,

$$\begin{aligned}
H_3 &= \langle T \rangle / 0 = \mathbb{Z}. \\
H_2 &= 0/0 = 0. \\
H_1 &= \langle e_1, e_3 \rangle / \langle e_2, e_3 \rangle = 0. \\
H_1 &= \langle v_1, v_2 \rangle / \langle v_1 - v_2 \rangle = \mathbb{Z}.
\end{aligned}$$

As shown in Figure 6, it is isomorphic to a 3-ball where the boundary of the northern hemisphere is identified with the boundary of the southern hemisphere by the reflection along the equator. Therefore, this figure is indeed an S^3.

If I have time, describe this more carefully.

□

Exercise. (Problem 8) Construct a 3 dimensional Δ-complex X from n tetrahedra $T_1, \cdots, T_n$ by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure, so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1}, subscripts being taken mod n. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show the simplicial homology groups of X in dimensions 0, 1, 2, 3 are $\mathbb{Z}, \mathbb{Z}_n, 0, \mathbb{Z}$, respectively.

Proof. Let $T_0, \cdots, T_{n-1}$ denote the n tetrahedra. Let $v_0, v_1, e_0, \cdots, e_{n+1}, f_0, \cdots, f_{2n-1}$ denote the vertices and edges as in Figure 7. (It has 4 tetrahedra, but they all represent the same T_i. I wrote four of them only because the figure would be too complicated if I denoted all the vertices, edges, faces in one picture.)

Then we have

- $C_3 = \{T_0, \cdots, T_{n-1}\}$.
- $C_2 = \{f_0, \cdots, f_{2n-1}\}$.
- $C_1 = \{e_0, \cdots, e_{n+1}\}$.
- $C_0 = \{v_0, v_1\}$.

Now we will examine ∂.

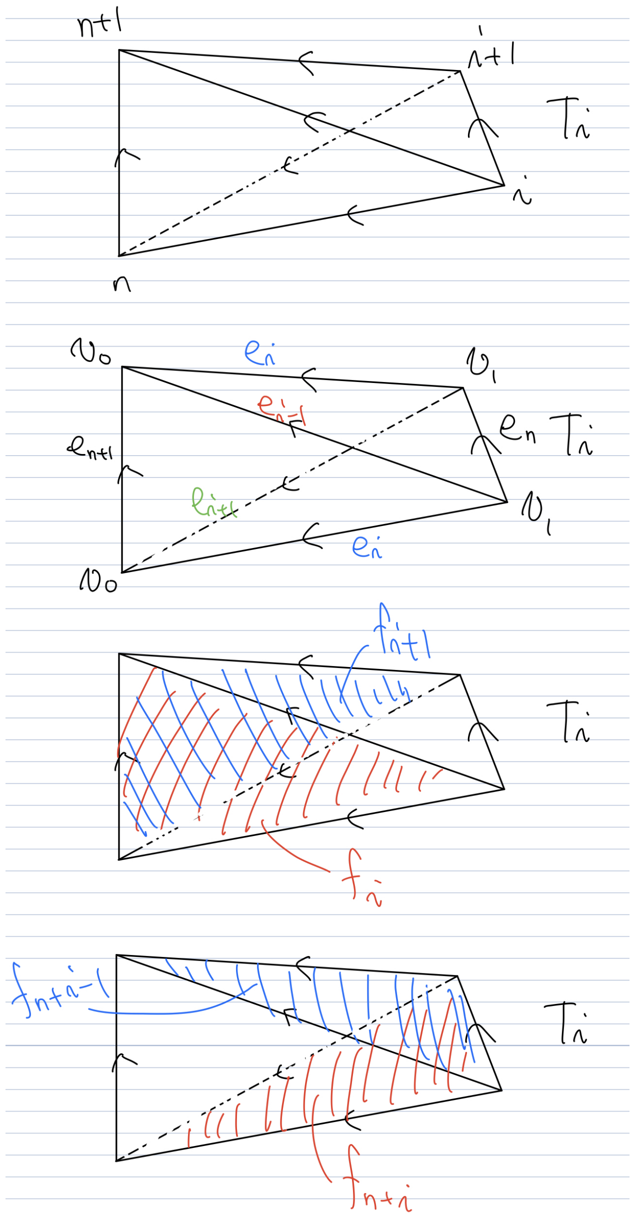

FIGURE 7. Problem 8

$$
\begin{aligned}
\partial_3(T_i) &= [i+1, n, n+1] - [i, n, n+1] + [i, i+1, n+1] - [i, i+1, n] \\
&= f_{i+1} - f_i + f_{n+i-1} - f_{n+i}. \\
\partial_2(f_i) &= [n, n+1] - [i, n+1] + [i, n] = e_{n+1} - e_{i-1} + e_i. \\
\partial_2(f_{n+i}) &= [i+1, n] - [i, n] + [i, i+1] = e_{i+1} - e_i + e_n. \\
\partial_1(e_i) &= \begin{cases} v_0 - v_1 & (0 \le i \le n-1) \\ 0 & (i = n, n+1). \end{cases}
\end{aligned}
$$

Therefore,

$$\partial_3(\sum a_i T_i) = \sum (a_{i-1} - a_i) f_i + \sum (a_{i+1} - a_i) f_{n+i}.$$
$$\partial_2(\sum a_i f_i + \sum b_i f_{n+i}) = e_{n+1} \sum a_i + e_n \sum b_i + \sum (a_i - a_{i+1} + b_{i-1} - b_i) e_i.$$
$$\partial_1(\sum a_i e_i) = (\sum_{i=0}^{n-1} a_i) v_0 - (\sum_{i=0}^{n-1} a_i) v_1.$$

Hence,

- $H_3 = \langle T_0 + \cdots + T_{n-1} \rangle / 0 = \mathbb{Z}$.
- Let $\sum a_i f_i + \sum b_i f_{n+i} \in \ker(\partial_2)$. Then $\sum a_i = \sum b_i = 0$, and $b_{i-1} + a_i = b_i + a_{i+1}$ for each i. By induction, $b_{i-1} + a_i = b_{i+j} + a_{i+j}$ for each i, j. $\sum a_i = \sum b_i$ implies that $\sum (a_i + b_{i-1}) = 0$. Thus for each i, $n(a_i + b_{i-1}) = 0$. This implies that $a_i + b_{i-1} = 0$ for each i. In other words, $b_i = -a_{i+1}$ for each i. Moreover, if $\sum a_i = \sum b_i = 0$ and $b_i = -a_{i+1}$ for each i, then $\partial(\sum a_i f_i + \sum b_i f_{n+i}) = 0$. Therefore, $\ker(\partial_2) = \{\sum a_i f_i + \sum -ai + 1 f_{n+i} \mid \sum a_i = 0\}$.

  On the other hand, $\mathrm{Im}(\partial_3) = \{\sum (a_{i-1} - a_i) f_i + \sum (a_{i+1} - a_i) f_{n+i}\}$. Let $A_i = a_{i-1} - a_i$ for each i. Then $\mathrm{Im}(\partial_3) = \{\sum A_i f_i + \sum (-A_{i+1}) f_{n+i} \mid \sum A_i = 0\}$.

  It is clear that $\ker(\partial_2) = \mathrm{Im}(\partial_3)$. Thus $H_2 = 0$.
- $\ker(\partial_1) = \left\langle e_n, e_{n+1}, e_i - e_{i+1} \,|_{i=0}^{n-2} \right\rangle = \left\langle e_n, e_{n+1}, e_{n+1} - e_i + e_{i+1} \,|_{i=0}^{n-2} \right\rangle$.

  And $\mathrm{Im}(\partial_2) = \langle e_{n+1} - e_{i-1} + e_i, e_i - e_{i-1} + e_n \rangle = \langle e_{n+1} - e_{i-1} + e_i, e_n - e_{n+1} \rangle$ where $i = 0, \cdots, n-1$. Moreover, $(e_{n+1} - e_0 + e_1) + \cdots + (e_{n+1} - e_n + e_0) = ne_{n+1}$, $\mathrm{Im}(\partial_2) = \left\langle ne_{n+1}, e_n - e_{n+1}, e_{n+1} - e_{i-1} + e_i \,|_{i=1}^{n-1} \right\rangle$. Therefore,

  $$H_1 = \left\langle e_n, e_{n+1}, e_{n+1} - e_i + e_{i+1} \,|_{i=0}^{n-2} \right\rangle / \left\langle ne_{n+1}, e_n - e_{n+1}, e_{n+1} - e_{i-1} + e_i \,|_{i=1}^{n-1} \right\rangle.$$

  Thus $H_1 = \mathbb{Z}/n$.
- $H_0 = \langle v_0, v_1 \rangle / \langle v_0 - v_1 \rangle = \mathbb{Z}$.

$\square$