MATH 611 HOMEWORK (DUE 9/25)

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Exercise. (Problem 4, Chapter 1.3) Construct a simply-connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle intersecting it in two points.

Proof. We claim that $X = X \setminus \{0\}$ is a simply-connected covering space of X. (See Figure 1.)

- Simply-connected? $\tilde{X}$ deformation retracts onto S^2. By Proposition 1.17, $\pi_1(\tilde{X}) = \pi_1(S^2)$. Proposition 1.14 shows that $\pi_1(S^2) = 0$. Therefore, $\tilde{X}$ is simply connected.
- Covering space? Let $p : \tilde{X} \to X$ be defined such that
  - $p \mid_{S^2} = \mathrm{Id}_{S^2}$. In other words, p maps every point on the spherical part of $\tilde{X}$ to the same place on the spherical part of X.

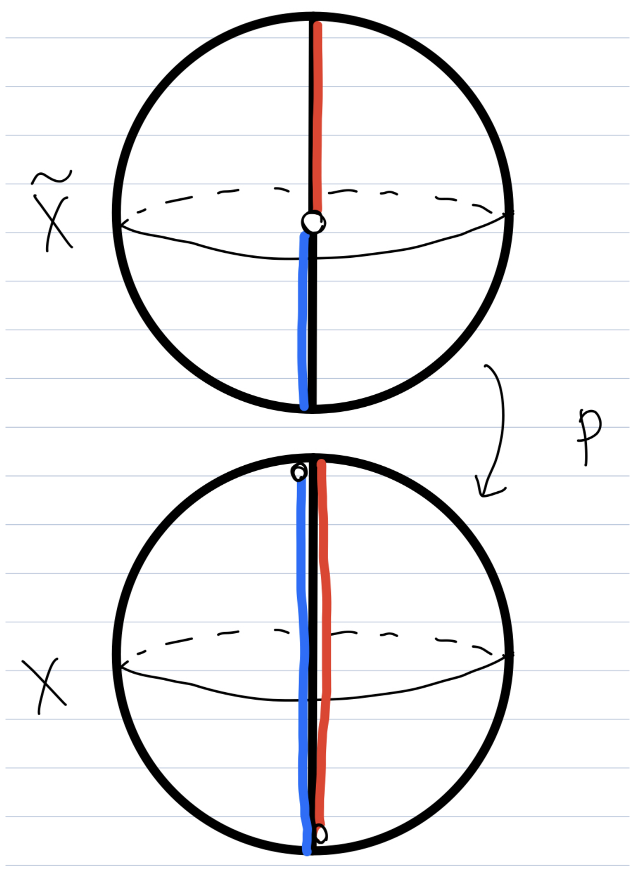

Figure 1. Problem 4 (Part 1)

- p extends the red line to the diameter (the top half of the diameter, in other words, the radius in the northern hemisphere in Figure 1). In other words, p maps $(0, 1)$ on the Y axis into $(-1, 1)$.
- p extends the blue line to the diameter (the bottom half of the diameter, in other words, the radius in the southern hemisphere in Figure 1). In other words, p maps $(-1, 0)$ on the Y axis into $(-1, 1)$.

Then p is clearly continuous and surjective. We will show that every point has a neighborhood that is evenly covered. Let $x \in X$ be given.

- Case 1: x is on the sphere and is disjoint from the diameter. Then a neighborhood of x that is disjoint from the diameter is clearly evenly covered.
- Case 2: x is on the diameter and is disjoint from the sphere. Let U be a neighborhood of x that is disjoint from the sphere. Then U is contained in both the image of the red line under p and the image of the blue line under p. Thus $p^{-1}(U)$ is a union of an open subset of the red line and open subset of the blue line in $\tilde{X}$. Therefore, U is evenly covered.
- Case 3: x is the north pole or south pole.

  I started to feel like this solution doesn't work because of this case...

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Exercise. (Problem 5, Chapter 1.3) Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0, 1] \times [0, 1]$ together with the segments of the vertical lines $x = 1/2, 1/3, 1/4, \cdots$ inside the square. Show that for every covering space $X \to X$ there is some neighborhood of the left edge of X that lifts homeomorphically to $\tilde{X}$. Deduce that X has no simply-connected covering space.

Proof. Idea: Open cover of the left edge by evenly covered open sets. Find a finite subcover. By the tube lemma, there exists $U \times I$ that covers the left edge and a partition $0 = t_0 < \cdots < t_n = 1$ such that $U \times [t_i, t_{i+1}]$ is contained in an evenly covered neighborhood. Inductively, show $U \times [0, t_i]$ is in an evenly covered neighborhood. Lift a loop in X with a vertical line $x = 1/n$ for some large n. Then the element maps back to itself by p. In other words, $p_*(\pi_1(\tilde{X})) \neq 0$.

□

Exercise. (Problem 7, Chapter 1.3) Let Y be the quasi-circle in the figure in the textbook. Collapsing the segment of Y in the y-axis to a point gives a quotient map $f : Y \to S^1$. Show that f does not lift to the covering space $\mathbb{R} \to S^1$, even though $\pi_1(Y) = 0$. Thus local path-connectedness of Y is a necessary hypothesis in the lifting criterion.

Proof. The lifting criterion is Proposition 1.33. The only property that Y is missing is the local path connectedness. But I'm not sure how to make use of it.

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