MATH 612 (HOMEWORK 2)

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Exercise. (Exercise 1) Fix G and let $\alpha : H \to H'$ be given. Let $0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0, 0 \to G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} H \to 0$ be free resolutions. By Lemma 3.1(a), we obtain two homomorphisms $\alpha_1 : F_1 \to G_1, \alpha_0 : F_0 \to G_0$ which commutes with f_i, g_i, α. Then we obtain two chain complexes

$$0 \leftarrow \mathrm{Hom}(F_1, G) \xleftarrow{f_1^*} \mathrm{Hom}(F_0, G) \xleftarrow{f_0^*} \mathrm{Hom}(H, G) \leftarrow 0$$
$$0 \leftarrow \mathrm{Hom}(F_1, G') \xleftarrow{f_1^*} \mathrm{Hom}(F_0, G') \xleftarrow{f_0^*} \mathrm{Hom}(H, G') \leftarrow 0.$$

with induced maps $\alpha_1^*, \alpha_0^*, \alpha^*$ forming a chain map from the chain complex on the bottom to the one on the top. Then α_1^* induces a map from $\mathrm{Ext}(H', G) \to \mathrm{Ext}(H, G)$.

Fix H and let $f : G \to G'$ be given. Let $0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$ be a free resolution of H. We obtain two cochain complexes where f_* is a chain map from the top one to the bottom one.

$$0 \leftarrow \mathrm{Hom}(F_1, G) \xleftarrow{f_1^*} \mathrm{Hom}(F_0, G) \xleftarrow{f_0^*} \mathrm{Hom}(H, G) \leftarrow 0$$
$$0 \leftarrow \mathrm{Hom}(F_1, G') \xleftarrow{f_1^*} \mathrm{Hom}(F_0, G') \xleftarrow{f_0^*} \mathrm{Hom}(H, G') \leftarrow 0.$$

f_* indeed makes the diagram commute because for any $\sigma \in \mathrm{Hom}(H, G)$,

$$\begin{aligned} f_*(f_0^*(\sigma)) &= f_*(\sigma \circ f_0) \\ &= f \circ (\sigma \circ f_0) \\ &= (f \circ \sigma) \circ f_0 \\ &= f_0^*(f \circ \sigma) \\ &= f_0^*(f_*(\sigma)). \end{aligned}$$

Similarly, $f_*(f_1^*(\sigma)) = f_1^*(f_*(\sigma))$ for every $\sigma \in \mathrm{Hom}(F_0, G)$. Since a chain map induces a homomorphism on cohomology groups, f induces a map from $\mathrm{Ext}(H, G) \to \mathrm{Ext}(H, G')$.

Exercise. (Exercise 1.2)

$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[r, "f_1"] \arrow[d, "\cdot n"] & F_0 \arrow[r, "f_0"] \arrow[d, "\cdot n"] & H \arrow[r] \arrow[d, "\cdot n"] & 0 \\ 0 \arrow[r] & F_1 \arrow[r, "f_1"] & F_0 \arrow[r, "f_0"] & H \arrow[r] & 0 \end{tikzcd}$$

turn into two chain complexes with a chain map

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathrm{Hom}(F_1,G) & \xleftarrow[f_1^*]{} & \mathrm{Hom}(F_0,G) & \xleftarrow[f_0^*]{} & \mathrm{Hom}(H,G) & \longleftarrow & 0 \\
& & \uparrow{\scriptstyle (\cdot n)^*} & & \uparrow{\scriptstyle (\cdot n)^*} & & \uparrow{\scriptstyle (\cdot n)^*} & & \\
0 & \longleftarrow & \mathrm{Hom}(F_1,G) & \xleftarrow[f_1^*]{} & \mathrm{Hom}(F_0,G) & \xleftarrow[f_0^*]{} & \mathrm{Hom}(H,G) & \longleftarrow & 0.
\end{array}$$

This diagram commutes because a group homomorphism for abelian groups commute with multiplication by n. Therefore, $(\cdot n)^*$ induces a homomorphism on $\mathrm{Ext}(H,G) = \mathrm{Hom}(F_1,G)/\operatorname{im}(f_1^*)$. Moreover, $\forall \phi + \operatorname{im}(f_1^*) \in \mathrm{Ext}(H,G)$,

$$(\cdot n)^*(\phi + \operatorname{im}(f_1^*)) = \phi \circ (\cdot n) + \operatorname{im}(f_1^*)$$

where $(\phi \circ (\cdot n))(x) = \phi(n(x)) = n(\phi(x)) = (n\phi)(x)$ for all $x \in F_1$. Therefore, the map induced by $(\cdot n)^*$ is simply multiplication by n.

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathrm{Hom}(F_1,G) & \xleftarrow[f_1^*]{} & \mathrm{Hom}(F_0,G) & \xleftarrow[f_0^*]{} & \mathrm{Hom}(H,G) & \longleftarrow & 0 \\
& & \downarrow{\scriptstyle (\cdot n)_*} & & \downarrow{\scriptstyle (\cdot n)_*} & & \downarrow{\scriptstyle (\cdot n)_*} & & \\
0 & \longleftarrow & \mathrm{Hom}(F_1,G) & \xleftarrow[f_1^*]{} & \mathrm{Hom}(F_0,G) & \xleftarrow[f_0^*]{} & \mathrm{Hom}(H,G) & \longleftarrow & 0.
\end{array}$$

For every $\phi \in \mathrm{Hom}(H,G)$ and $x \in F_0$,

$$\begin{aligned}
((\cdot n)_*(f_0^*(\phi)))(x) &= ((\cdot n)_*(\phi \circ f_0))(x) \\
&= n((\phi \circ f_0)(x)) \\
&= n(\phi(f_0(x))) \\
&= ((\cdot n)_*\phi)(f_0(x)) \\
&= f_0^*((\cdot n)_*\phi)(x).
\end{aligned}$$

Similarly, $(\cdot n)_*$ commutes with f_1^*, so $(\cdot n)_*$ is a chain map. For any $\phi + \operatorname{im}(f_1^*) \in \mathrm{Ext}(H,G)$, $(\cdot n)_*(\phi + \operatorname{im}(f_1^*)) = n\phi + \operatorname{im}(f_1^*)$, so it is multiplication by n.

Exercise. (Exercise 3.1.3) $\cdots \xrightarrow{d_2} \mathbb{Z}_4 \xrightarrow{d_1} \mathbb{Z}_4 \xrightarrow{d_0} \mathbb{Z}_2 \to 0$ is a free resolution where $d_0 : a \mapsto a$ and $d_i : a \mapsto 2a$ because $\ker(d_0) = \operatorname{im}(d_i) = \ker(d_i) = \{0, 2\}$ for each $i \geq 1$. Apply $\mathrm{Hom}(-, \mathbb{Z}_2)$ and replace $\mathbb{Z}_2^*$ with 0. For any $\phi \in \mathrm{Hom}(\mathbb{Z}_4, \mathbb{Z}_2)$ and $x \in \mathbb{Z}_4$, $((\cdot 2)^*(\phi))(x) = (\phi \circ (\cdot 2))(x) = \phi(2x) = \phi(0) = 0$. Thus $(\cdot 2)^*(\phi) = 0$. In other words, $d_i^* = 0$ for all $i \geq 1$, so $\mathrm{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathrm{Hom}(\mathbb{Z}_4, \mathbb{Z}_2)$ which is nontrivial because $1 \mapsto 1$ is a nontrivial group homomorphism.

Exercise. (Exercise 3.1.6(a)) The chain complex we obtain is isomorphic to $0 \to \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0$ where $\alpha(a,b) = (a+b)(1,1,-1)$. Apply $\mathrm{Hom}(-, \mathbb{Z})$, and we obtain

- $H^0(T; \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$.
- $\alpha^*(\phi) = 0$ if and only if $\phi(1,1,-1) = 0$. $(a,b,c) \mapsto a - b$ and $(a,b,c) \mapsto a + c$ form a basis for the subspace consisting of such homomorphisms. $H^1(T; \mathbb{Z}) = \ker(\alpha^*) = \mathbb{Z} \oplus \mathbb{Z}$.
- $H^2(T; \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z})/\operatorname{im}(\alpha^*) = \mathbb{Z}$ because $(a,b) \mapsto a$ and $(a,b) \mapsto a + b$ form a basis for $\mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z})$ and $\operatorname{im}(\alpha^*)$ is spanned by $(a,b) \mapsto a + b$.