# MATH 611 (DUE 11/20)

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**Exercise.** (Problem 1)

**Exercise.** (Problem 28 (a)) Let $A, B$ be the Mobius strip and a torus with a small neighborhood around them so the strip and torus are contained in $A$ and $B$. For any $n \geq 3$, the exact sequence $H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B)$ implies that $H_n(X) \cong H_n(A) \oplus H_n(B) = 0 \oplus 0 = 0$ because the intersection $A \cap B$ is homotopic to $S^1$, so $H_n(A \cap B) = H_{n-1}(A \cap B) = 0$. $H_0(X) = \mathbb{Z}$ because $X$ has only one path component.

We will examine the LES

$$\tilde{H}_2(A \cap B) \to \tilde{H}_2(A) \oplus \tilde{H}_2(B) \xrightarrow{f_1} \tilde{H}_2(X) \xrightarrow{f_2} \tilde{H}_1(A \cap B) \xrightarrow{f_3} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{f_4} \tilde{H}_1(X) \to \tilde{H}_0(A \cap B).$$

- Sine $\tilde{H}_2(A \cap B) = 0$, so $f_1$ is injective.
- $\tilde{H}_1(A \cap B) = \mathbb{Z}$, and $f_3(1) = (2, (1, 0))$ because the intersection goes around the mobius strip twice while it only goes around the torus once. Then $f_3$ is injective, so $\mathrm{Im}(f_2) = \ker(f_3) = 0$. This implies that $\mathrm{Im}(f_1) = \ker(f_2) = H_2$, so $f_1$ is surjective.

Therefore, $f_1$ is bijective, so $H_2(X) = \tilde{H}_2(X) = \tilde{H}_2(A) \oplus \tilde{H}_2(B) = 0 \oplus \mathbb{Z} = \mathbb{Z}$.

Finally, $f_4$'s surjectivity implies that

$$\begin{aligned} \tilde{H}_1(X) &\cong \tilde{H}_1(A) \oplus \tilde{H}_1(B) / \ker(f_4) \\ &= \mathbb{Z} \oplus \mathbb{Z}^2 / \langle (2, (1, 0)) \rangle \\ &\cong \langle a, b, c \rangle \,/\, \langle 2a + b \rangle \\ &\cong \langle a, b, c \mid 2a + b \rangle \\ &\cong \langle a, -2a, c \rangle \\ &\cong \langle a, c \rangle = \mathbb{Z} \oplus \mathbb{Z}. \end{aligned}$$

Thus $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$.

**Exercise.** (Problem 28 (b)) Let $A, B$ be the Mobius strip and $\mathbb{R}P^2$ with a small neighborhood around them so the strip and $\mathbb{R}P^2$ are contained in $A$ and $B$. For any $n \geq 3$, the exact sequence $H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B)$ implies that $H_n(X) \cong H_n(A) \oplus H_n(B) = 0 \oplus 0 = 0$ because the intersection $A \cap B$ is homotopic to $S^1$, so $H_n(A \cap B) = H_{n-1}(A \cap B) = 0$. Since $X = A \cup B$ has one path component, $H_0(X) = \mathbb{Z}$. We will consider the LES

$$\tilde{H}_2(A) \oplus \tilde{H}_2(B) \xrightarrow{f_1} \tilde{H}_2(X) \xrightarrow{f_2} \tilde{H}_1(A \cap B) \xrightarrow{f_3} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{f_4} \tilde{H}_1(X) \to \tilde{H}_0(A \cap B).$$

$\tilde{H}_1(A \cap B) = \mathbb{Z}$, and $f_3$ maps 1 to $(2, 1)$ because the generator wraps around the Mobius strip twice and the $\mathbb{R}P^2$ once. Then $f_3$ is injective, so $f_2$ is the zero map. In other words, $\ker(f_2) = \tilde{H}_2(X)$, so $f_1$ is surjective. Since $\tilde{H}_2(A) \oplus \tilde{H}_2(B) = 0$, $\tilde{H}_2(X) = 0$. Thus $H_2(X) = 0$.

By the first isomorphism theorem and exactness,

$$
\begin{aligned}
\tilde{H}_1(X) &= \tilde{H}_1(A) \oplus \tilde{H}_1(B) / \ker(f_4) \\
&= (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) / \langle (2,1) \rangle \\
&\cong \langle a, b \mid 2b \rangle / \langle 2a + b \rangle \\
&= \langle a, b \mid 2b, 2a + b \rangle \\
&= \langle a, -2a \mid 2(-2a) \rangle \\
&= \langle a \mid 4a \rangle \\
&= \mathbb{Z}_4.
\end{aligned}
$$

Therefore, $H_1(X) = \mathbb{Z}_4$.

**Exercise.** (Problem 29) As shown earlier,

$$
H_n(M_g) = \begin{cases} \mathbb{Z}^{2g} & (n = 1) \\ \mathbb{Z} & (n = 0, 2) \\ 0 & (n \geq 3). \end{cases}
$$

Let $R_1, R_2$ be the first and second $R$ with a small neighborhood around them. Then $X = R_1 \cup R_2$ and $R_1 \cap R_2$ is homotopy equivalent to $M_g$. Let $n \geq 3$. Consider the sequence

$$
H_n(R_1) \oplus H_n(R_2) \to H_n(X) \to H_{n-1}(R_1 \cap R_2) \to H_{n-1}(R_1) \oplus H_{n-1}(R_2).
$$

A solid $g$-torus deformation retracts to the wedge sum of $g$ $S^1$'s. $H_n(R_1) = H_n(R_2) = \oplus_{i=1}^{g} H_n(S^1) = 0$ for $n \geq 2$. By the exactness, we have $H_n(X) = H_{n-1}(R_1 \cap R_2) = H_{n-1}(M_g)$. Therefore, $H_n(X) = 0$ for $n \geq 4$, and $H_3(X) = \mathbb{Z}$. $H_0(X) = \mathbb{Z}$ because $X$ contains only one path component.

Consider the sequence

$$
\tilde{H}_2(R_1) \oplus \tilde{H}_2(R_2) \to \tilde{H}_2(X) \xrightarrow{\alpha} \tilde{H}_1(R_1 \cap R_2) \xrightarrow{\beta} \tilde{H}_1(R_1) \oplus \tilde{H}_1(R_2) \xrightarrow{\gamma} \tilde{H}_1(X) \to \tilde{H}_0(R_1 \cap R_2).
$$

Then this is equivalent to

$$
0 \to \tilde{H}_2(X) \xrightarrow{\alpha} \tilde{H}_1(R_1 \cap R_2) \xrightarrow{\beta} \tilde{H}_1(R_1) \oplus \tilde{H}_1(R_2) \xrightarrow{\gamma} \tilde{H}_1(X) \to 0.
$$

By the exactness, $\alpha$ is injective and $\gamma$ is surjective. Let $a_1, \cdots, a_g, b_1, \cdots, b_g$ be generators of $\tilde{H}_1(R_1 \cap R_2)$ where $a_i$ wraps around the $i$th "arm"(or "handle") and $b_i$ wraps around the $i$th "hole". Then $\beta(a_i) = (0, 0)$ because in $R_1$ and $R_2$, each of which is a solid torus, the "arm" gets filled in. On the other hand, $\beta(b_i) = (b_i, b_i)$ for each $i$.

$$
\begin{aligned}
H_1(X) &= \tilde{H}_1(X) \\
&= \mathrm{Im}(\gamma) \\
&= \tilde{H}_1(R_1) \oplus \tilde{H}_1(R_2) / \ker(\gamma) \\
&= \tilde{H}_1(R_1) \oplus \tilde{H}_1(R_2) / \mathrm{Im}(\beta) \\
&= \left\langle b_1, \cdots, b_g, b_1', \cdots, b_g' \right\rangle / \left\langle b_1 + b_1', \cdots, b_g + b_g' \right\rangle \\
&= \langle b_1, \cdots, b_g \rangle \\
&= \mathbb{Z}^g.
\end{aligned}
$$

Since $\alpha$ is injective, $\mathrm{Im}(\alpha)$ is isomorphic to $\tilde{H}_2(X)$. Thus $H_2(X) = \tilde{H}_2(X) = \mathrm{Im}(\alpha) = \ker(\beta) = \langle a_1, \cdots, a_g \rangle = \mathbb{Z}^g$.

- For $n \geq 4$, we have $H_n(R) \to H_n(R, M_g) \to H_{n-1}(M_g)$. As shown earlier, $H_n(R) = H_{n-1}(M_g) = 0$, so the exactness implies that $H_n(R, M_g) = 0$.
- We will consider $H_3(R) \to H_3(R, M_g) \to H_2(M_g) \to H_2(R)$. $H_3(R) = H_2(R) = 0$, so $H_3(R, M_g) = H_2(M_g)$ by the exactness. Thus $H_3(R, M_g) = \mathbb{Z}$.

- We will consider $0 \to \tilde{H}_2(R, M_g) \xrightarrow{\alpha} \tilde{H}_1(M_g) \xrightarrow{\beta} \tilde{H}_1(R) \xrightarrow{\gamma} \tilde{H}_1(R, M_g) \to 0$. (We have 0 on both ends because $\tilde{H}_2(R) = \tilde{H}_0(M_g) = 0$. Let $a_i, b_i$ be generators of $\tilde{H}_1 M_g$ such that $a_i$'s wrap around the handles and $b_i$'s wrap around the holes. Using the same discussion as above, $a_i \mapsto 0$ and $b_i \mapsto b_i$ by $\beta$.
    - By the exactness, $\alpha$ is injective. Thus $\tilde{H}_2(R, M_g) = \mathrm{Im}(\alpha) = \ker(\beta) = \langle a_1, \cdots, a_g \rangle$. Therefore, $\tilde{H}_2(R, M_g) = \mathbb{Z}^g$.
    - By the exactness, $\gamma$ is surjective. $\tilde{H}_1(R, M_g) = \mathrm{Im}(\gamma) = \tilde{H}_1(R)/\ker(\gamma) = \tilde{H}_1(R)/\mathrm{Im}(\beta)$. $\tilde{H}_1(R)$ is generated by $b_1, \cdots, b_g$ as it deformation retracts to $S^1 \vee \cdots \vee S^1$, so $\beta$ is surjective. Therefore, $\tilde{H}_1(R, M_g) = 0$.
- $0 = H_1(R, M_g) \to H_0(M_g) \xrightarrow{f} H_0(R) \to H_0(R, M_g)$ is exact. Moreover, $f$ must be an isomorphism because both $M_g$ and $R$ consist of one path component. Therefore, the exactness implies $H_0(R, M_g) = 0$.