

# MATH 633 MIDTERM

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## 1. GOURSAT, CAUCHY ON THE DISC, AND THE PROOFS IN SECTION 5 OF CHAPTER 3.

**Proposition 1.1** (Goursat's Theorem). *If  $\Omega$  is an open set in  $\mathbb{C}$ , and  $T \subset \Omega$  is a triangle whose interior is also contained in  $\Omega$ , then*

$$\int_T f(z)dz = 0$$

*whenever  $f$  is holomorphic in  $\Omega$ .*

*Proof.*

- Let  $T^0 = T$ . Having created  $T^i$ , create 4 triangles from  $T^i$  as shown in the textbook with the natural orientation. Then one of the 4 triangles, denoted by  $T^{i+1}$ , must satisfy  $|\int_{T^i} f(z)dz| \leq 4|\int_{T^{i+1}} f(z)dz|$ . Since  $\{T_i\}$  is a sequence of nonempty compact sets whose diameter diminishes, there must exist a unique point  $z_0$  that belongs to all  $T^i$ .
- Since  $f$  is holomorphic at  $z_0$ ,  $f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$  where  $\psi(z) \rightarrow 0$  as  $z \rightarrow z_0$ .
- Since  $f(z_0) + f'(z_0)(z - z_0)$  has a primitive,  $\int_{T^n} f(z)dz = \int_{T^n} \psi(z)(z - z_0)dz$  for any  $n$ .  $|\int_{T^n} \psi(z)(z - z_0)dz| \leq \epsilon_n dp/4^n$  where  $\epsilon_n = \sup_{z \in T^n} |\psi(z)|$ ,  $d$  the diameter of  $T$ , and  $p$  the perimeter of  $T$ .  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , so  $|\int_T f(z)dz| \leq \epsilon_n dp = 0$  as  $n \rightarrow \infty$ . □

**Proposition 1.2** (Cauchy's Theorem for a Disk). *Suppose  $f$  is holomorphic in an open set containing the circle  $C$  and its interior. Then*

$$\int_C f(z)dz = 0.$$

*Proof.* Since  $f$  has a primitive, the integral over a closed curve is 0.

Do I need more than this?

□

**Proposition 1.3** (Theorem 5.1). *If  $f$  is holomorphic in  $\Omega$ , then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

*whenever the two curves  $\gamma_0$  and  $\gamma_1$  are homotopic in  $\Omega$ .*

*Proof.*

- Let  $F : (s, t) \mapsto \gamma_s(t)$  be a homotopy between  $\gamma_0$  and  $\gamma_1$ . Let  $\epsilon > 0$  be chosen such that  $B(F(s, t), 3\epsilon) \subset \Omega$  for all  $s, t$ . Such an  $\epsilon$  must exist because  $F([0, 1]^2)$  is compact.

- Choose  $\delta > 0$  such that  $\sup_{t \in [0,1]} |\gamma_{s_1}(t) - \gamma_{s_2}(t)| < \epsilon$  whenever  $|s_1 - s_2| < \delta$ . Such a  $\delta$  must exist because  $F$  is uniformly continuous.
- Pick  $|s_1 - s_2| < \delta$ . Choose discs  $D_0, \dots, D_n$  of radius  $2\epsilon$  and points  $\{z_0, \dots, z_{n+1}\}, \{w_0, \dots, w_{n+1}\}$  on  $\gamma_{s_1}, \gamma_{s_2}$ , respectively such that  $z_i, z_{i+1}, w_i, w_{i+1} \in D_i$ . Let  $F_i$  denote the primitive of  $f$  on  $D_i$ . Then  $F_{i+1}(z_{i+1}) - F_i(z_{i+1}) = F_{i+1}(w_{i+1}) - F_i(w_{i+1})$ .

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$$\begin{aligned}
\int_{\gamma_{s_1}} f - \int_{\gamma_{s_2}} f &= \sum_{i=0}^n [F_i(z_{i+1}) - F_i(z_i)] - \sum_{i=0}^n [F_i(w_{i+1}) - F_i(w_i)] \\
&= \sum_{i=0}^n [F_i(z_{i+1}) - F_i(z_i) - F_i(w_{i+1}) + F_i(w_i)] \\
&= F_n(z_{n+1}) - F_n(w_{n+1}) - (F_0(z_0) - F_0(w_0)) \\
&= 0.
\end{aligned}$$

□

**Proposition 1.4** (Theorem 5.2). *Any holomorphic function in a simply connected domain has a primitive.*

*Proof.*

- Fix a point  $z_0$  in  $\Omega$  and define  $F(z) = \int_{\gamma} f(w)dw$  where  $\gamma$  is a path from  $z_0$  to  $z$ . Then  $F(z+h) - F(z) = \int_{\eta} f(w)dw$  where  $\eta$  is the path from  $z$  to  $z+h$ .
- Since  $f$  is continuous at  $z$ ,  $f(w) = f(z) + \psi(w)$  where  $\psi(w) \rightarrow 0$  as  $w \rightarrow z$ . Therefore,  $F(z+h) - F(z) = f(z) \int_{\eta} dw + \int_{\eta} \psi(w)dw = f(z)h + \int_{\eta} \psi(w)dw$ . Since  $\left| (\int_{\eta} \psi(w)dw)/h \right| \leq \sup_{w \in \eta} |\psi(w)| = 0$  as  $h \rightarrow 0$ . Thus  $\lim_{h \rightarrow 0} (F(z+h) - F(z))/h = f(z)$ .

□