

MATH 611 (DUE 11/6)

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1. SIMPLICIAL AND SINGULAR HOMOLOGY

Exercise. (Problem 14) Determine whether there exists a short exact sequence $0 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_8 \oplus \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow 0$. More generally, determine which abelian groups A fit into a short exact sequence $0 \rightarrow \mathbb{Z}_{p^m} \rightarrow A \rightarrow \mathbb{Z}_{p^n} \rightarrow 0$ with p prime. What about the case of short exact sequences $0 \rightarrow A \rightarrow \mathbb{Z}_n \rightarrow 0$?

Proof. Let $\phi_1 : \mathbb{Z}_4 \rightarrow \mathbb{Z}_8 \oplus \mathbb{Z}_2, \phi_2 : \mathbb{Z}_8 \oplus \mathbb{Z}_2 \rightarrow \mathbb{Z}_4$ be defined such that $\phi_1(a) = (2a, a)$ and $\phi_2(a, b) = a + 2b$. Then $\ker(\phi_1) = 0, \text{Im}(\phi_1) = \ker(\phi_2) = \{(2k, k) \mid 0 \leq k \leq 3\}$ and $\text{Im}(\phi_2) = \mathbb{Z}_4$. Thus this is indeed an exact sequence.

We claim that $A = \bigoplus_{i=1}^k \mathbb{Z}_{p^{a_i}}$ where $a_1 \geq \max\{m, n\}, a_i \geq a_{i+1}, \sum a_i = m + n$. It is clear that $\sum a_i = m + n$ since A modulo the image of \mathbb{Z}_{p^m} is \mathbb{Z}_{p^n} .

- Suppose that $A = \mathbb{Z}_{p^a} \oplus \mathbb{Z}_{p^b} \oplus \mathbb{Z}_{p^c}$ where $a \geq b \geq c \geq 1$ and $a + b + c = m + n$.

We have $0 \rightarrow \mathbb{Z}_{p^m} \xrightarrow{\alpha} A \xrightarrow{\beta} \mathbb{Z}_{p^n} \rightarrow 0$. Then α is injective, so $a \geq m$, and β is surjective, so $a \geq n$. Therefore, $a \geq \max\{m, n\}$. This implies that $b < m$ and $b < n$. (Otherwise, $a + b \geq m + n = a + b + c$, which is a contradiction.) Therefore, the first coordinate of $\alpha(1)$ has to be an element in \mathbb{Z}_{p^a} of order p^m . Without loss of generality, we will assume $\alpha(1) = (p^{a-m}, x, y)$ where x, y are some integers. Then $p^b \alpha(1) = (p^{a+b-m}, 0, 0)$. Since $a + b - m < a$, $p^b(\alpha(1)) \neq 0$. Moreover, $p^b(\alpha(1)) \in \text{Im}(\alpha) = \ker(\beta)$.

Why is this a problem?

□

Exercise. (Problem 15) For an exact sequence $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E$ show that $C = 0$ if and only if the map $A \rightarrow B$ is surjective and $D \rightarrow E$ is injective. Hence, for a pair of spaces (X, A) , the inclusion $A \rightarrow X$ induces isomorphisms on all homology groups if and only if $H_n(X, A) = 0$ for all n .

Proof. Suppose $C = 0$. $\text{Im}(\phi_{AB}) = \ker(\phi_{BC}) = B$, so ϕ_{AB} is surjective. $\ker(\phi_{DE}) = \text{Im}(\phi_{CD}) = \{0\}$, so ϕ_{DE} is injective.

On the other hand, suppose ϕ_{AB} is surjective and ϕ_{DE} is injective. $\text{Im}(\phi_{CD}) = \ker(\phi_{DE}) = \{0\}$, so ϕ_{CD} is the zero map. Therefore, $\ker(\phi_{CD}) = C$. $\ker(\phi_{BC}) = \text{Im}(\phi_{AB}) = B$, so ϕ_{BC} is the zero map. Therefore, $\text{Im}(\phi_{BC}) = 0$. Hence, $C = \ker(\phi_{CD}) = \text{Im}(\phi_{BC}) = 0$.

By Theorem 2.16 and the discussion at the bottom of P.117(Hatcher), we have a long exact sequence of homology groups

$$(1.1) \quad H_n(A) \xrightarrow{i_*} H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X)$$

for $n \geq 1$. Suppose the inclusion induces isomorphisms on all homology groups. Then $H_n(X, A) = 0$ for all $n \geq 1$ by the first part. Moreover, we have $H_1(X, A) \rightarrow H_0(A) \rightarrow$

$H_0(X) \rightarrow H_0(X, A) \rightarrow 0$. Since $H_1(X, A) = 0$, by the first part, $H_0(X) = 0$. In order for $0 \rightarrow H_0(X, A) \rightarrow 0$ to be exact, $H_0(X, A)$ must be 0. Therefore, $H_n(X, A) = 0$ for all $n \geq 0$.

Suppose that $H_n(X, A) = 0$ for all $n \geq 0$. By exact sequence 1.1 above, $i_* : H_n(A) \rightarrow H_n(X)$ is surjective for $n \geq 1$ and injective for $n \geq 0$. Thus i_* is bijective for all $n \geq 1$. We have $H_1(X, A) \rightarrow H_0(A) \rightarrow H_0(X) \rightarrow H_0(X, A)$. Since $H_1(X, A) = H_0(X, A) = 0$, i_* must be bijective by the exactness. Therefore, the inclusion induces isomorphisms for all n . \square

Exercise. (Problem 16)

- Show that $H_0(X, A) = 0$ if and only if A meets each path-component of X .
- Show that $H_1(X, A) = 0$ if and only if $H_1(A) \rightarrow H_1(X)$ is surjective and each path-component of X contains at most one path-component of A .

Proof.

- Let $\gamma_x + C_0(A) \in C_0(X)/C_0(A)$. Since A meets each path-component of X , there exists a path $\gamma : I \rightarrow X$ that joins a point $a \in A$ and the image of γ_x . Then γ can be seen as an element of $C_1(X)$ since γ maps a 1-simplex into X . Moreover, $\partial\gamma = \gamma_x - \gamma_a$ where $\gamma_a \in C_0(A)$ with $\text{Im}(\gamma_a) = a$. Therefore, $\partial(\gamma + C_1(A)) = \gamma_x + C_0(A)$, so $\gamma_x + C_0(A) \in \text{Im}(\partial)$. Hence, $H_0(X, A) = \ker(\partial_0)/\text{Im}(\partial_1) = (C_0(X)/C_0(A))/(C_0(X)/C_1(A)) = 0$.

On other hand, suppose that A does not meet each path component of X . Let $x \in X$ be a point in a path component that A does not intersect. Let $\gamma_x : \Delta^0 \rightarrow X$ such that $\text{Im}(\gamma_x) = \{x\}$. Then $\gamma_x \in \ker(\partial_0) = C_0(X, A)$. Let $\gamma + C_1(A) \in C_1(X, A)$. Then $\partial_1(\gamma + C_1(A)) = \partial_1(\gamma) + C_0(A)$. Let $\gamma_{x_1}, \gamma_{x_2} \in C_0(X)$ such that $\partial_1(\gamma) = \gamma_{x_1} - \gamma_{x_2}$. $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$ if and only if $\gamma_{x_1} - \gamma_{x_2} - \gamma_x \in C_0(A)$.

– If γ lies in the same path component as x , then so do x_1 and x_2 . Suppose $x = x_1$.

Since $-\gamma_{x_2} \notin C_0(A)$, $\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A)$. The case when $x \neq x_1$ and $x = x_2$ and the case when $x \neq x_1$ and $x \neq x_2$ can be proven in a similar way.

– If γ lies in a different path component, then $\gamma_x \neq \gamma_{x_1}$ and $\gamma_x \neq \gamma_{x_2}$. Therefore,

$$\gamma_{x_1} - \gamma_{x_2} + C_0(A) \neq \gamma_x + C_0(A).$$

Therefore, $\gamma_x \notin \text{Im}(\partial_1)$. Thus $H_0(X, A) = C_0(X, A)/\text{Im}(\partial_1)$ is not 0.

- Suppose $H_1(X, A) = 0$. By the exact sequence $H_1(A) \xrightarrow{\phi} H_1(X) \rightarrow H_1(X, A) \rightarrow H_0(A) \xrightarrow{\psi} H_1(X)$, we know that ϕ is surjective and ψ is injective. Suppose that there is a path component of X that contains two or more path components of A . Let a, b be points in two distinct path components of A that are contained in a path component of X . We will regard a, b as functions $\Delta^0 \rightarrow A$. Then $a, b \in C_0(A)$ and $[a] \neq [b]$ in $H_0(A)$ because a, b are in different path components of A , so $a - b \notin \text{Im}(\partial_1)$. However, a, b live in the same path component of X , $a - b \in \text{Im}(\partial_1) \subset C_0(X)$. Therefore, $\phi([a]) = \phi([b])$ where $[a] \neq [b]$. This is a contradiction because ϕ is injective. Therefore, each path component of X contains at most one path component of A .

Show the other direction.

\square

Exercise. (Problem 17)

- Compute the homology groups $H_n(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X .

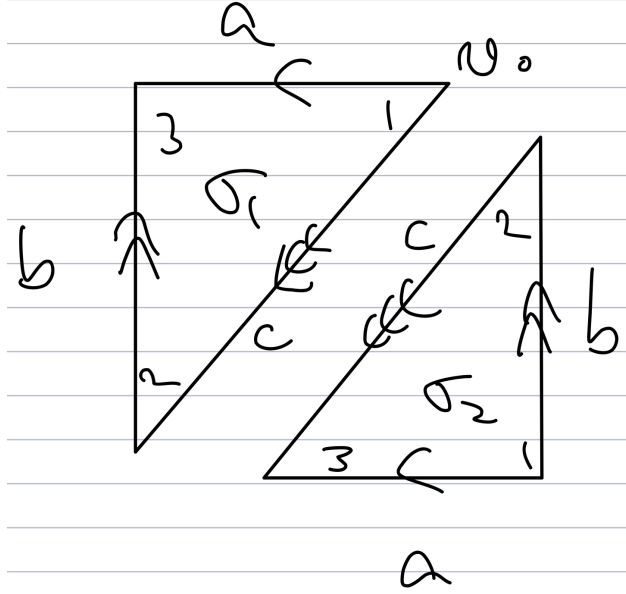


FIGURE 1. Problem 17

- Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus two with A and B the circles shown.

Proof.

- We will apply Theorem 2.16 to get the exact sequence with $H_n(A)$, $H_n(X)$, $H_n(X, A)$.
 - When $n \geq 3$, $H_n(S^2) \rightarrow H_n(S^2, A) \rightarrow H_{n-1}(A)$ shows that $H_n(S^2, A)$ is 0 by the exactness since $H_n(S^2) = H_{n-1}(A) = 0$.
 - When $n = 2$, $H_n(A) \rightarrow H_n(S^2) \xrightarrow{\phi} H_n(S^2, A) \rightarrow H_{n-1}(A)$ shows that $H_n(S^2, A) = H_n(S^2) = \mathbb{Z}$. This is because $H_n(A) = H_{n-1}(A) = 0$ so ϕ is an isomorphism by the exactness.
 - By Problem 16, $H_0(X, A) = 0$. By the exact sequence $0 \rightarrow H_1(X, A) \rightarrow H_0(A) \rightarrow H_0(X)$ where $H_0(A) = \mathbb{Z}^{|A|}$ and $H_0(X) = \mathbb{Z}$, we have $H_1(X, A) = \mathbb{Z}^{|A|-1}$.

We will first compute the homology groups of a torus using Figure 1. $C_2 = \{\sigma_1, \sigma_2\}$, $C_1 = \{a, b, c\}$, $C_0 = \{v_0\}$.

- $H_2 = \ker(\partial_2) / \text{Im}(\partial_3) = \langle \sigma_1 - \sigma_2 \rangle / 0 = \mathbb{Z}$.
- $H_1 = \ker(\partial_1) / \text{Im}(\partial_2) = \langle a, b, c \rangle / \langle b - a + c, c - a + b \rangle = \mathbb{Z}^2$ because $b - a + c = c - a + b$.
- $H_0 = \ker(\partial_0) / \text{Im}(\partial_1) = \langle v_0 \rangle / 0 = \mathbb{Z}$.

Again, we will apply Theorem 2.16 to get the exact sequence with $H_n(A)$, $H_n(X)$, and $H_n(X, A)$.

- When $n \geq 3$, $H_n(S^1 \times S^1) \rightarrow H_n(S^1 \times S^1, A) \rightarrow H_{n-1}(A)$ shows that $H_n(S^1 \times S^1, A)$ is 0 by the exactness since $H_n(S^1 \times S^1) = H_{n-1}(A) = 0$.
- When $n = 2$, $H_n(A) \rightarrow H_n(S^1 \times S^1) \xrightarrow{\phi} H_n(S^1 \times S^1, A) \rightarrow H_{n-1}(A)$ shows that $H_n(S^1 \times S^1, A) = H_n(S^1 \times S^1) = \mathbb{Z}$. This is because $H_n(A) = H_{n-1}(A) = 0$ so ϕ is an isomorphism by the exactness.

- By Problem 16, $H_0(X, A) = 0$. By the exact sequence $H_1(X, A) \rightarrow H_0(A) \rightarrow H_0(X)$ where $H_0(A) = \mathbb{Z}^{|A|}$ and $H_0(X) = \mathbb{Z}$, we have $H_1(X, A) = \mathbb{Z}^{|A|-1}$.
- (X, A) is a good pair because A is a nonempty closed subspace that is a deformation retract of some neighborhood in X . By Proposition 2.22, $H_n(X, A) = \tilde{H}_n(X/A)$ for all n . The quotient space X/A is $T^2 \vee T^2$ where T^2 is a torus. By Corollary 2.25, $\tilde{H}_n(X/A) = \tilde{H}_n(T^2) \oplus \tilde{H}_n(T^2)$. $((T^2, p)$ is clearly a good pair for a point $p \in T^2$.) We calculate above that

$$H_n(T^2) = \begin{cases} \mathbb{Z} & (n = 0, 2) \\ \mathbb{Z}^2 & (n = 1) \\ 0 & (n \geq 3). \end{cases}$$

Therefore,

$$H_n(X, A) = \tilde{H}_n(X/A) = \tilde{H}_n(T^2) \oplus \tilde{H}_n(T^2) = \begin{cases} \mathbb{Z}^2 & (n = 2) \\ \mathbb{Z}^4 & (n = 1) \\ 0 & (n = 0, n \geq 3). \end{cases}$$

Similarly, X/B is $T^2 \vee S^1$ because (X, B) is a good pair. Moreover, (S^1, p) is a good pair for a point $p \in S^1$, so it suffices to check $\tilde{H}_n(T^2) \oplus \tilde{H}_n(S^1)$.

Therefore,

$$H_n(X, B) = \tilde{H}_n(X/B) = \tilde{H}_n(T^2) \oplus \tilde{H}_n(S^1) = \begin{cases} \mathbb{Z} & (n = 2) \\ \mathbb{Z}^3 & (n = 1) \\ 0 & (n = 0, n \geq 3). \end{cases}$$

□

Exercise. (Problem 26) Show that $H_1(X, A)$ is not isomorphic to $\tilde{H}_1(X/A)$ if $X = [0, 1]$ and A is the sequence $1, 1/2, 1/3, \dots$ together with its limit 0.

Proof. We will show that $H_1(X, A)$ is countable, and $\tilde{H}_1(X/A) = H_q(X/A)$ is uncountable. We have an exact sequence $\tilde{H}_1(X) \rightarrow \tilde{H}_1(X, A) \xrightarrow{\phi} \tilde{H}_0(A) \rightarrow \tilde{H}_0(X)$. Since $H_1(X, A) = \tilde{H}_1(X, A) = \tilde{H}_0(X) = 0$, ϕ is an isomorphism. Thus $\tilde{H}_1(X, A) = \tilde{H}_0(A) = \ker(\partial_1)/\text{Im}(\partial_2)$. Since A is a disjoint union of points, $\text{Im}(\partial_2) = 0$. $\ker(\partial_1) = \{\sum n_i \alpha_i \mid n_i \in \mathbb{Z}, \sum n_i = 0\}$ where α_i is the point $1/i$ by the definition of a reduced homology. Then this is generated by $\{\alpha_1 - \alpha_2, \alpha_1 - \alpha_3, \alpha_1 - \alpha_4, \dots\}$, so $\tilde{H}_1(X, A)$ is countable.

We will show the existence of an injective map ζ from the direct product $\prod_{i=1}^{\infty} \mathbb{Z}$ to $H_1(X/A)$, which is homeomorphic to the Hawaiian earring. We will refer to the n th ring C_n as in Example 1.25. Let $(k_1, \dots) \in \prod_{i=1}^{\infty} \mathbb{Z}$ be given. Construct the map $f : I \rightarrow X/A$ that wraps k_n times around C_n in the time interval $[1 - 1/n, 1 - 1/(n+1)]$. This infinite composition of loops is certainly continuous at each time less than 1, and it is continuous at time 1 since every neighborhood of the basepoint in X/A contains all but finitely many of the circles C_n . This shows that $f \in C_1(X/A)$. Moreover, $\partial(f) = v_0 - v_0 = 0$ where v_0 is the origin of the Hawaiian earring. Therefore, $[f] \in H_1(X/A)$. We define $\zeta(k_1, \dots) = [f]$.

Let $(k_1, \dots) \neq (l_1, \dots) \in \prod_{i=1}^{\infty} \mathbb{Z}$ be given. Let $\zeta(k_1, \dots) = f, \zeta(l_1, \dots) = g$ as described above. Let i be an index such that $k_i \neq l_i$. Let $F : X/A \rightarrow S^1$ be a continuous map that maps C_n onto S_1 and C_i to -1 for all i where S_1 is seen as a subset of \mathbb{C} . Then F induces

a group homomorphism $F_* : H_1(X/A) \rightarrow H_1(S^1)$ where $F([f]) = k_n$ and $F([g]) = l_n$. Since $F([f]) \neq F([g])$, $[f] \neq [g]$. This shows the injectivity of ζ and hence $H_1(X/A)$ must be uncountable.

Therefore, $H_1(X, A)$ is not isomorphic to $H_1(X/A)$. □