MATH 601 (DUE 10/23)

HIDENORI SHINOHARA

Contents

1. Field Extension

Exercise. (Problem 1) Let p be a prime number. Let $K = \mathbb{Z}/p\mathbb{Z}(t)$ be the fraction field of $\mathbb{Z}/p\mathbb{Z}[t]$.

 (i) What is the characteristic of K?
 (ii) What is the characteristic of any extension field of K?
 (iii) Show that the Frobenius endormophism, $F : K \to K$ is not a ring isomorphism.
 (iv) Let $f(x) = x^p - t \in K[x]$. Prove that $f(x)$ is irreducible.
 (v) Prove that $f(x)$ is not a separable polynomial.
 (vi) Construct an explicit field extension $K \subset L$ such that $f(x) \in L[x]$ has a factor of positive degree $< p$.
 (vii) With f and L above find all the roots of $f(x)$ in L and determine their multiplicities.

Proof.
 (i) We will write $k \cdot 1$ to denote $1 + 1 + \cdots + 1$ (k times). Since $p \cdot 1 = 0$ in K, the characteristic of K is at most p. Let k denote the characteristic of K. Let $i : \mathbb{Z}/p\mathbb{Z} \to (\mathbb{Z}/p\mathbb{Z})[t], i' : \mathbb{Z}/p\mathbb{Z}[t] \to K$ be inclusions. Then $i' \circ i : \mathbb{Z}/p\mathbb{Z} \to K$ is an injective ring homomorphism. $k \cdot 1 \neq 0$ in $\mathbb{Z}/p\mathbb{Z}$. Thus $(i' \circ i)(k \cdot 1) = k \cdot (i' \circ i)(1) = k' \cdot 1 = 0$. Since $i' \circ i$ is injective, this implies $k \cdot 1 = 0$. Therefore, $k \geq p$, so k must be equal to p.

$\square$

Exercise. (Problem 2) Let F be a field of characteristic 0. Let $f(x) \in F[x]$ be an irreducible polynomial. Then $f(x)$ is separable.

Proof. Since $f(x)$ is irreducible, $f(x)$ is not a unit. Since F is a field, all polynomials of degree 0 are units. Thus $\deg(f(x)) \geq 1$.

Finish the proof. Check Notability for a sketch.

$\square$

Exercise. (Problem 3) Let F be a field. Let $f(x) \in F[x]$ be an irreducible polynomial which is not separable. Show that $f'(x) = 0 \in F[x]$.

Proof. Suppose $f(x)$ is irreducible. Then $f(x) \neq 0$ and $f(x)$ is not a unit by definition. Thus $\deg(f(x)) \geq 1$.

Since $f(x)$ is not separable, there exists a non-unit $g(x) \in F[x]$ such that $g(x) \mid f(x)$ and $g(x) \mid f'(x)$ by Lemma 3.2 from the Field Extension handout. Since $f(x)$ is irreducible and $g(x)$ is not a unit, $f(x)$ is the product of $g(x)$ and a unit. This implies that $\deg(f(x)) = \deg(g(x))$.

Since $g(x) \mid f'(x)$, $f'(x) = h(x)g(x)$. If $f'(x) = 0$, we are done. Suppose otherwise. Then $\deg(f'(x)) = \deg(h(x)) + \deg(g(x)) = \deg(h(x)) + \deg(f(x)) \geq \deg(f(x))$. However, by the definition of the $'$ operator, $\deg(f'(x)) < \deg(f(x))$. This is a contradiction, so $f'(x) = 0$. $\square$

Exercise. (Problem 5) What is the characteristic of the ring $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$?

Proof. Define $\phi : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$ such that $\phi(k) = (k, 0, 0)$. Then ϕ is injective, so the characteristic is 0. $\square$