MATH 611 HOMEWORK (DUE 9/25)

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Exercise. (Problem 4, Chapter 1.3) Construct a simply-connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle intersecting it in two points.

Proof. We claim that the space described in Figure 1 is a covering space of X.

- The shape is an infinitely long chain of spheres and lines. The chain goes infinitely both ways (up and down). Let g be a loop in this space. Then g is contained in finitely many spheres and line segments. Since spheres and line segments are simply connected, a finitely long chain of them is simply connected by Van Kampen. Therefore, g is homotopic to the constant loop, thus this space is simply connected.
- We will map each sphere to the sphere of X. Each line will be mapped to the diameter up side down. Figure 1 shows how each part gets mapped.
- We claim that such a mapping is a covering map and thus this infinite chain is indeed a covering space. Let $x \in X$.
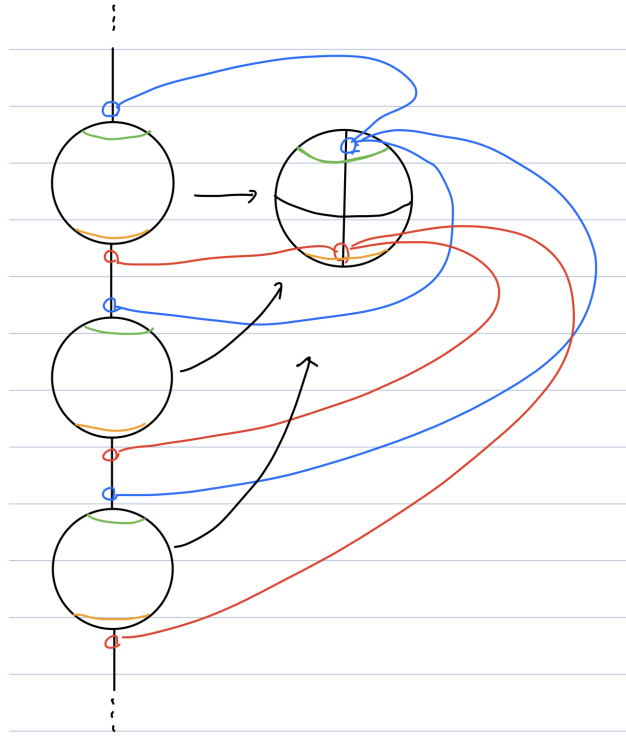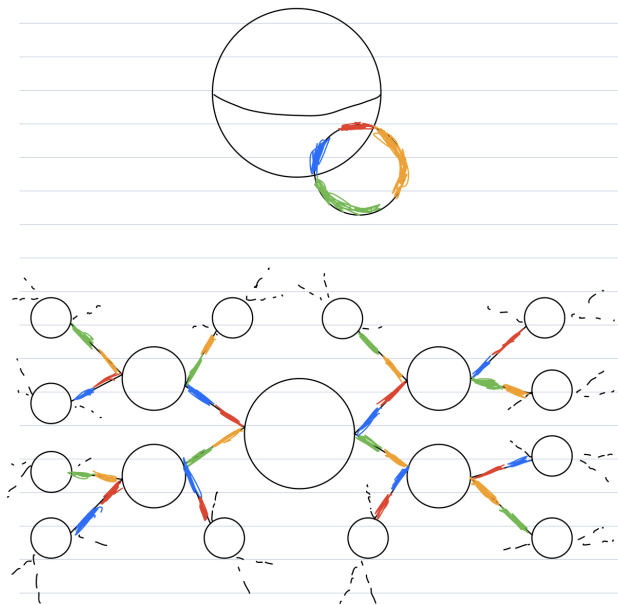

FIGURE 1. Problem 4 (Part 1)

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FIGURE 2. Problem 4 (Part 2)

- If x is on the diameter and disjoint from the sphere, a neighborhood that is disjoint from the sphere is evenly covered.
- If x is on the sphere and disjoint from the diameter, a neighborhood that is disjoint from the diameter is evenly covered.
- If x is the north pole, a neighborhood that does not contain the south pole is evenly covered.
- If x is the south pole, a neighborhood that does not contain the north pole is evenly covered.

Therefore, the space described in Figure 1 is a covering space of X.

Figure 2 has an example of a simply-connected covering space of X. This space can be constructed by first having a sphere in the middle. 4 edges should be attached at two points of the sphere. And each edge should be connected to a different sphere. We continue attaching them such that each sphere has exactly two points each of which has two edges. The edges should be mapped to the circle as described in 2 by 4 different colors. It is clear that each point has an evenly covered neighborhood. □

Exercise. (Problem 5, Chapter 1.3) Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1] \times [0,1]$ together with the segments of the vertical lines $x = 1/2, 1/3, 1/4, \cdots$ inside the square. Show that for every covering space $\tilde{X} \to X$ there is some neighborhood of the left edge of X that lifts homeomorphically to $\tilde{X}$. Deduce that X has no simply-connected covering space.

Proof. For each $y \in [0,1]$, the point $(0,y)$ has a neighborhood U_y that is evenly covered. Then there exists an open rectangle $R_y \subset \mathbb{R}^2$ such that $y \in R_y \cap Y \subset U_y$. Since any open subset of an evenly covered set is evenly covered, such $R_y \cap Y$ is evenly covered. Let V_y denote $R_y \cap Y$ for each y. $\{V_y \mid y \in [0,1]\}$ is an open cover of the segment $\{0\} \times [0,1]$. Since the segment is compact, there exists a finite subcover, $V_{y_1}, \cdots, V_{y_k}$.
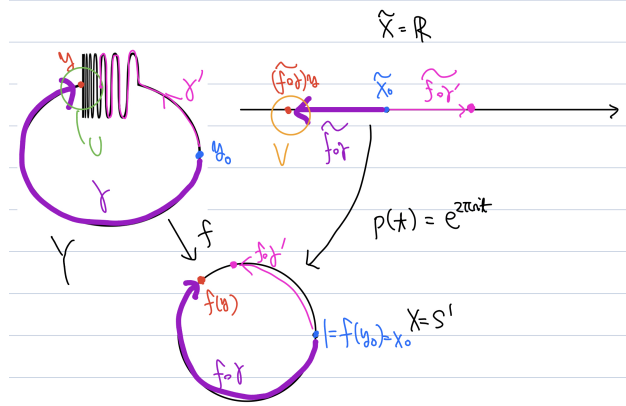
FIGURE 3. Problem 7

Since we found a finite cover where each of them is rectangular shaped, there exists an $N \in \mathbb{N}$ such that $\forall n \geq N, \{1/n\} \times [0,1] \subset \bigcup_{i=1}^{n} V_{y_i}$. Since each $V_{y_1}, \cdots, V_{y_k}$ is a subset of an open rectangle, there must exist a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ such that for all i, $[0, 1/N] \times [t_i, t_{i+1}]$ is contained in an evenly covered open subset. We will inductively show that $[0, 1/N] \times [0, t_i]$ is contained in an evenly covered open subset. □

Exercise. (Problem 7, Chapter 1.3) Let Y be the quasi-circle in the figure in the textbook. Collapsing the segment of Y in the y-axis to a point gives a quotient map $f : Y \to S^1$. Show that f does not lift to the covering space $\mathbb{R} \to S^1$, even though $\pi_1(Y) = 0$. Thus local path-connectedness of Y is a necessary hypothesis in the lifting criterion.

Proof. We will use the lifting criterion (Proposition 1.33) and its proof in the textbook. The only missing property among the required ones is that Y is not locally path connected. By the first part of the proof (up to "This shows that $\tilde{f}$ is well-defined"), we know that if there exists $\tilde{f}$, it must be the function that the first part of the proof describes. Note that the first part does not require that Y be locally path connected. We will prove that the defined mapping is not continuous.

Let $x_0 = 1 \in S^1$. Let $y_0 \in Y$ be the point that gets mapped to x_0. Let y be the point on the intersection of the $[-1, 1]$ segment and the circle, and let γ denote the path from y_0 to y. Then $\widetilde{f \circ \gamma}(y)$ is somewhere around -0.6 in $\tilde{X}$. (See Figure 3.) Let $V = (\widetilde{f \circ \gamma}(y) - 0.01, \widetilde{f \circ \gamma}(y) + 0.01)$. (A small neighborhood around $\widetilde{f \circ \gamma}(y)$ that does not include 0.)

A necessary condition for $\tilde{f}$ to be continuous is that there exists a neighborhood $U \subset Y$ of y such that $\tilde{f}(U) \subset V$. (For instance, Theorem 18.1 on P.104 from Topology by Munkres.)

Let $U \subset Y$ be a neighborhood of y. Then it certainly contains some point of $y = \sin(1/x)$. Let γ' denote the path from y_0 to such a point in Y. Then $\tilde{f}$ maps the point to somewhere around 0.3 in $\tilde{X}$. In other words, $\widetilde{f \circ \gamma}(1) \notin V$, so $\tilde{f}(U) \not\subset V$.

Therefore, $\tilde{f}$ is not continuous. By the uniqueness discussed in the first part of the proof for Proposition 1.33, we conclude that there does not exist a lift of f. □

Exercise. (Problem 11, Chapter 0) Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $fg \simeq \mathrm{Id}$ and $hf \simeq \mathrm{Id}$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.
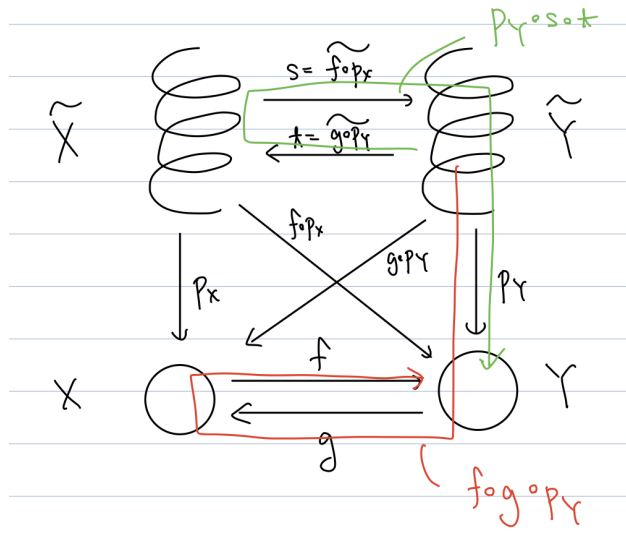
FIGURE 4. Problem 8 idea

Proof.

> First part: Let F be a homotopy between $f \circ g$ and Id, and let H be a homotopy between $h \circ f$ and Id. Let G be defined such that $G_t = h \circ F_{2t} \circ f$ for $t \in [0, 1/2]$, and $G_t = H_{2t-1}$ for $t \in [1/2, 1]$.
> Second part: Let $\phi_X : X \to X, \phi_Y : Y \to Y$ be given such that $(f \circ g) \circ \phi_Y$ is homotopic to Id_Y and $\phi_X \circ (h \circ f)$ is homotopic to Id_X. Then $f \circ (g \circ \phi_Y)$ is homotopic to Id_Y and $(\phi_X \circ h) \circ f$ is homotopic to Id_X. By the first part, f is a homotopy equivalence.

$\square$

Exercise. (Problem 8, Chapter 1.3) Let $\tilde{X}$ and $\tilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces X and Y. Show that if $X \simeq Y$ then $\tilde{X} \simeq \tilde{Y}$.

Proof.

> By Proposition 1.33, we can lift the two compositions as in Figure 5. This works because $\pi_1(\tilde{X}) = \pi_1(\tilde{Y}) = 0$.
> $f \circ g$ is homotopic to the identity map on Y. Then $f \circ g \circ p_Y$ is homotopic to p_Y. Let F be a homotopy between them such that $F_0 = st$. This is possible because st is clearly a lift of $f \circ g \circ p_Y$ if you look at the diagram. See Figure 4. We will look into what F_1 is. In the best case scenario, $F_1 = \mathrm{Id}_{\tilde{Y}}$, and st is homotopic to $\mathrm{Id}_{\tilde{Y}}$.
> However, that might not be the case. Let $r = F_1$. Let $y_0 \in Y$ and $\tilde{y}_0 \in Y$ be given such that $p_Y(\tilde{y}_0) = y_0$. Then $\tilde{y}_0, r(\tilde{y}_0)$ are both points in $\tilde{Y}$. Let k be a lift of p_Y such that $k(r(\tilde{y}_0)) = \tilde{y}_0$. $(p_Y \circ k) \circ r = p_Y \circ r = p_Y$. Therefore, $k \circ r$ is a lift of p_Y such that $(k \circ r)(\tilde{y}_0) = \tilde{y}_0$. $\mathrm{Id}_{\tilde{y}_0}$ is such a lift as well. By the uniqueness of lifts, $k \circ r = \mathrm{Id}$. Then $k \circ s \circ t \simeq k \circ r = \mathrm{Id}$. Therefore, $(k \circ s) \circ t$ is homotopic to Id, so $(k \circ s) \circ t$ is a homotopy equivalence.
> Similarly, there exists $k' : \tilde{X} \to \tilde{X}$ such that $t \circ s \circ k' \simeq r' \circ k' = \mathrm{Id}_{\tilde{X}}$. Therefore, $t \circ (s \circ k')$ is a homotopy equivalence.
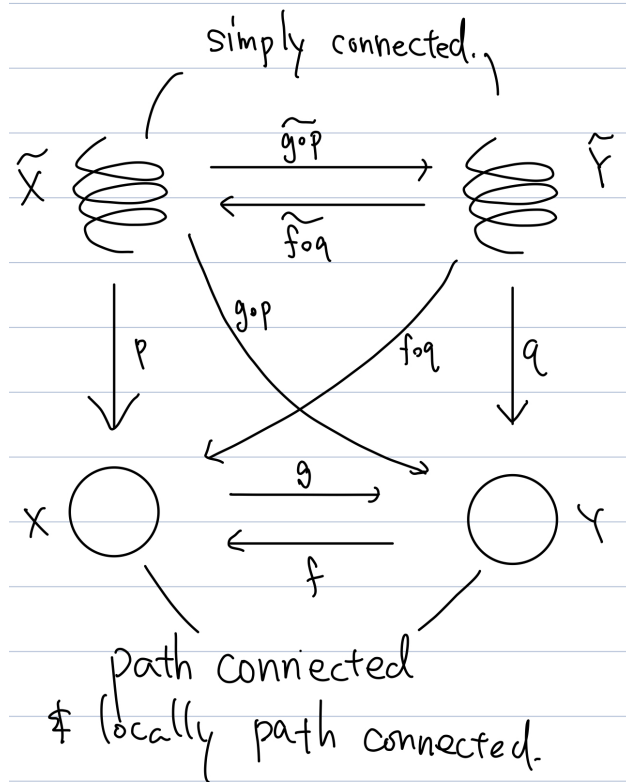> By Problem 11 in Chapter 0, t is a homotopy equivalence.

simply connected.

$\widetilde{X}$ $\xrightarrow{\widetilde{g \circ p}}$ $\widetilde{Y}$
$\xleftarrow{\widetilde{f \circ q}}$

p $g \circ p$ $f \circ q$ q

X $\xrightarrow{g}$ Y
$\xleftarrow{f}$

path connected
& locally path connected.

FIGURE 5. delete this!

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