MATH 633 HOMEWORK 9

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Exercise. (Problem 1) Let $x \in F_1$. Since Ω is bounded, there exists an $R > 0$ such that $\Omega \subset C(x, R)$. Then $F_1 \setminus C(x, R)$ and $F_2 \setminus C(x, R)$ are disjoint, closed sets whose union is $\mathbb{C} \setminus C(x, R)$, which is connected. Therefore, either $F_1 \setminus C(x, R)$ or $F_2 \setminus C(x, R)$ is empty. In other words, either $F_1 \subset C(x, R)$ or $F_2 \setminus c(x, R)$.

Exercise. (Problem 2(a)) We first assume $\omega = 0$. This is reasonable because the following argument can be extended to general cases by translating every function by ω. If r, θ are continuous, it is clear that α is continuous. Suppose α is continuous. Let $r(t) = |\alpha(t)|$. Then $r(t) : [0, 1] \to (0, \infty)$ is continuous. Moreover, $\alpha(t) = r(t)e^{i\theta(t)}$, so $r(t) = |r(t)| = \left| r(t)e^{i\theta(t)} \right| = |\alpha(t)|$, so this is the only possibility for $r(t)$.

By using the principal branch of logarithm and translation, we can find $\theta(t)$ locally. Since the logarithm function and translation function are both continuous, such local θ's are continuous. Since $[0, 1]$ is compact, we can find a finite cover of $[0, 1]$ such that we have $\theta(t)$ for each open set. Two $\theta(t)$ can be patched for any two overlapping open sets by adding $2k\pi$ for an appropriate value of k. Therefore, we can find θ that is continuous and satisfies $\alpha(t) = r(t)e^{i\theta(t)}$. Any other functions $\gamma(t)$ that satisfy the conditions must satisfy $1 = \alpha(t)/\alpha(t) = (r(t)e^{i\theta(t)})/(r(t)e^{i\gamma(t)}) = e^{i(\theta(t)-\gamma(t))}$, so $\theta(t) - \gamma(t) = 2k\pi$ for some fixed $k \in \mathbb{Z}$.

Hence, we have shown that α is continuous if and only if such continuous r, θ exist and the choice of r, θ are unique up to an additive constant for θ.