

# MATH 611 (DUE 10/23)

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## 1. SIMPLICIAL AND SINGULAR HOMOLOGY

**Exercise.** (Problem 2) Show that the  $\Delta$ -complex obtained from  $\Delta^3$  by performing the edge identifications  $[v_0, v_1] \sim [v_1, v_3]$  and  $[v_0, v_2] \sim [v_2, v_3]$  deformation retracts onto a Klein bottle. Find other pairs of identifications of edges that produce  $\Delta$ -complexes deformation retracting onto a torus, a 2-sphere, and  $\mathbb{RP}^2$ .

*Proof.* The deformation retraction of  $\Delta^3$  onto a Klein bottle is described in 1. We will start by “pushing”  $\Delta^3$  from edge  $(v_1, v_2)$ . This will leave the surface that consists of the triangles  $[v_0, v_1, v_3]$  and  $[v_0, v_2, v_3]$ . (In other words, a diamond shape consisting of the vertices  $[v_0, v_1, v_3, v_2]$ .) Step 2 in Figure 1 is what  $\Delta^3$  should look like after the deformation retract. Step 3 through 6 show why this is a Klein bottle.

Figure 2 shows the identification of edges for a torus, 2-sphere, and  $\mathbb{RP}^2$ .

□

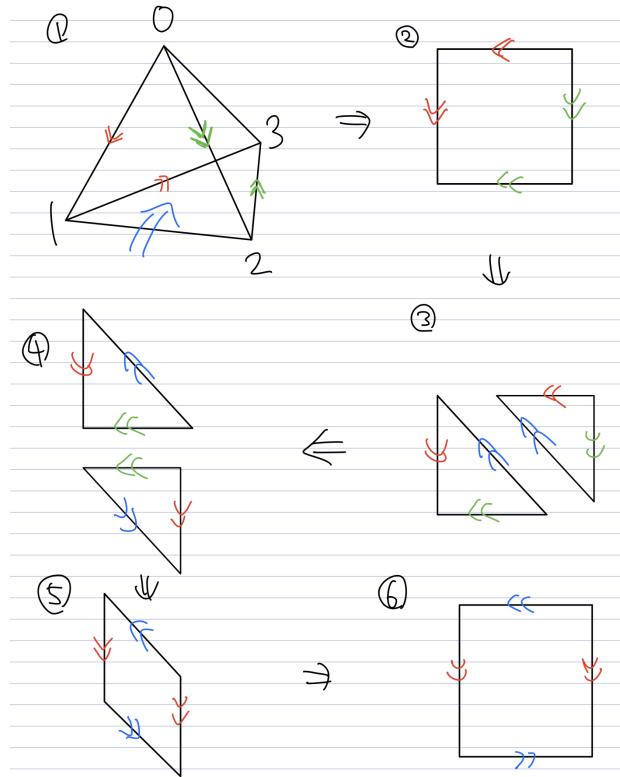


FIGURE 1. Problem 2(Klein Bottle)

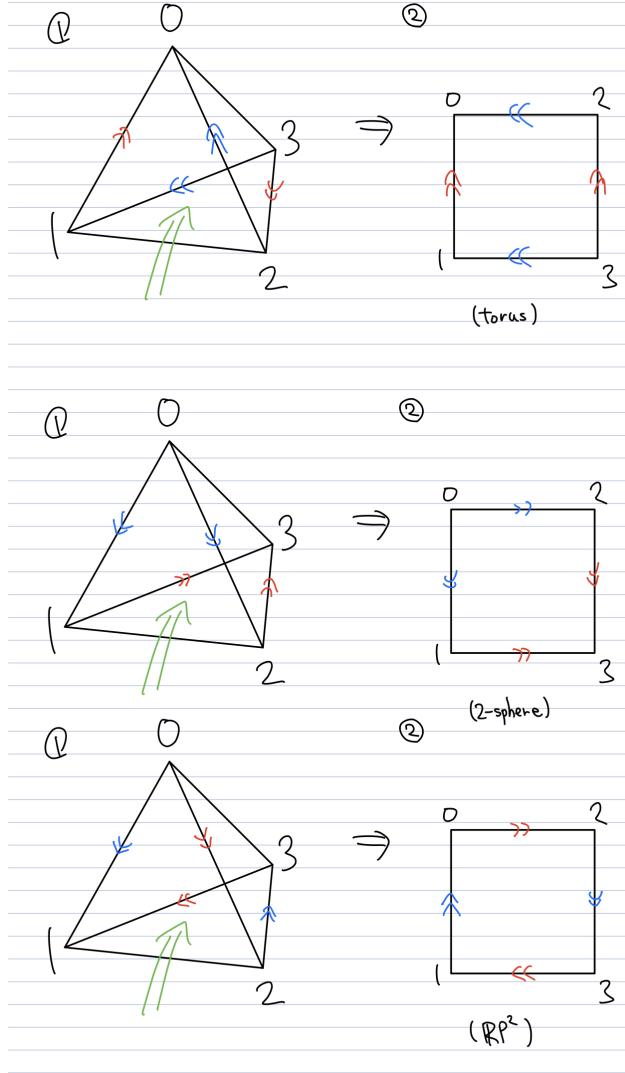


FIGURE 2. Problem 2(Torus, 2-Sphere,  $\mathbb{RP}^2$ )

**Exercise.** (Problem 4) Compute the simplicial homology groups of the triangular parachute obtained from  $\Delta^2$  by identifying its three vertices to a single point.

*Proof.* Let  $v_0$  denote the only vertex,  $e_1, e_2, e_3$  denote the three edges of the parachute, and  $\sigma$  denote the face of the parachute as in Figure 3.  $C_k = 0$  for  $k \geq 3$  because  $\Delta^2$  with the vertices identified does not contain any  $k$ -dimensional simplices for  $k \geq 3$ .  $C_2 = \langle \sigma \rangle, C_1 = \langle e_1, e_2, e_3 \rangle, C_0 = \langle v_0 \rangle$ . For each  $n$ ,  $\partial_n$  is defined such that  $\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha | [v_0, \dots, \hat{v}_i, \dots, v_n]$ .

- $\partial_2(\sigma) = e_3 - e_2 + e_1$ .
- $\partial_1(e_1) = v - v = 0$ . Similarly,  $\partial_1(e_2) = \partial_1(e_3) = 0$ .
- $\partial_0$  and  $\partial_3$  are both the zero map.

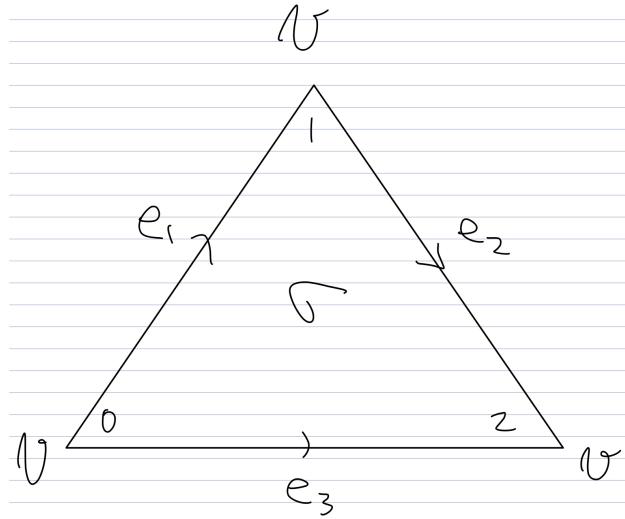


FIGURE 3. Problem 4

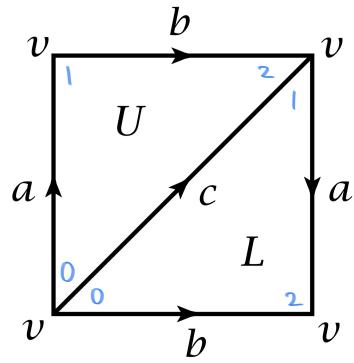


FIGURE 4. Problem 5

Thus

$$H_n = \begin{cases} \{0\} & (n \geq 3) \\ \ker(\partial_2) / \text{Im}(\partial_3) = 0/0 \cong 0 & (n = 2) \\ \ker(\partial_1) / \text{Im}(\partial_2) = \langle e_1, e_2, e_3 \rangle / \langle e_3 - e_2 + e_1 \rangle \cong \langle e_1, e_2, -e_2 + e_1 \rangle \cong \mathbb{Z}^2 & (n = 1) \\ \ker(\partial_0) / \text{Im}(\partial_1) = \langle v \rangle / 0 \cong \mathbb{Z} & (n = 0). \end{cases}$$

□

**Exercise.** (Problem 5) Compute the simplicial homology groups of the Klein bottle using the  $\Delta$ -complex structure described at the beginning of this section.

*Proof.* We will use the notations in Figure 4.

$$C_n = \begin{cases} 0 & (n \geq 3) \\ \langle U, L \rangle & (n = 2) \\ \langle a, b, c \rangle & (n = 1) \\ \langle v \rangle & (n = 0). \end{cases}$$

$\partial_n = 0$  for  $n \geq 3$  and  $n = 0$ .

$$\begin{aligned} \partial_2(U) &= \sum_{i=0}^2 (-1)^i \sigma |[0, 1, 2]| \\ &= \sigma |[1, 2] - \sigma |[0, 2] + \sigma |[0, 1]| \\ &= b - c + a. \\ \partial_2(L) &= \sum_{i=0}^2 (-1)^i \sigma |[0, 1, 2]| \\ &= \sigma |[1, 2] - \sigma |[0, 2] + \sigma |[0, 1]| \\ &= a - b + c. \end{aligned}$$

$\partial_1(a) = 0$  since  $\partial_1(a) = \sigma |[1] - \sigma |[0]| = v - v = 0$ . Similarly,  $\partial_1(b) = \partial_1(c) = 0$ . Thus  $H_n = \{0\}$  if  $(n \geq 3)$ .  $H_2 = \ker(\partial_2)/\text{Im}(\partial_3) = 0/0 \cong 0$ .

$$\begin{aligned} H_1 &= \ker(\partial_1)/\text{Im}(\partial_2) \\ &= \langle a, b, c \rangle / \langle b - c + a, a - b + c \rangle \\ &\cong \langle a, b, a + b \mid a - b + (a + b) \rangle \\ &\cong \langle a, b \mid 2a \rangle \\ &\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}. \end{aligned}$$

$$H_0 = \ker(\partial_0)/\text{Im}(\partial_1) = \langle v \rangle / 0 \cong \mathbb{Z}.$$

□

**Exercise.** (Problem 7) Find a way of identifying pairs of faces of  $\Delta^3$  to produce a  $\Delta$ -complex structure on  $S^3$  having a single 3-simplex, and compute the simplicial homology groups of this  $\Delta$ -complex.

*Proof.* We will identify  $[0, 2, 3] \sim [1, 2, 3]$  and  $[0, 1, 2] \sim [0, 1, 3]$  of the tetrahedra  $T$  in Figure ???. Then we have

$$\begin{aligned} C_3 &= \{T\} \\ C_2 &= \{f_1, f_2\} \\ C_1 &= \{e_1, e_2, e_3\} \\ C_0 &= \{v_1, v_2\}. \end{aligned}$$

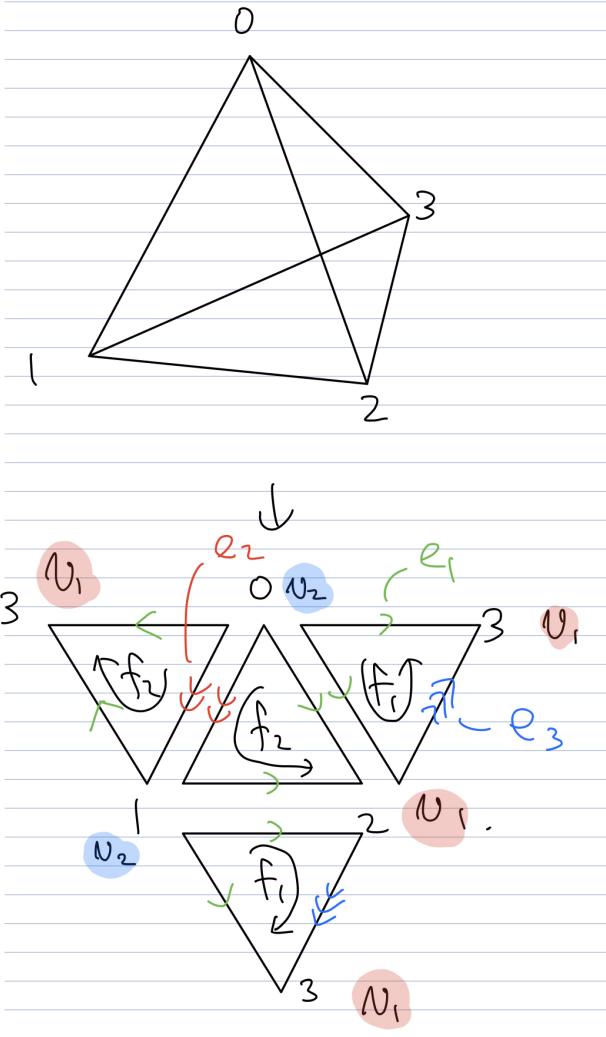


FIGURE 5. Problem 7

We will examine  $\partial$ .

$$\partial_3(T) = [1, 2, 3] - [0, 2, 3] + [0, 1, 3] - [0, 1, 2] = f_1 - f_1 + f_2 - f_2 = 0.$$

$$\partial_2(f_1) = [2, 3] - [0, 3] + [0, 2] = e_3 - e_1 + e_1 = e_3.$$

$$\partial_2(f_2) = [1, 2] - [0, 2] + [0, 1] = e_1 - e_1 + e_2 = e_2.$$

$$\partial_1(e_1) = [3] - [0] = v_1 - v_2.$$

$$\partial_1(e_2) = [1] - [0] = v_2 - v_2 = 0.$$

$$\partial_1(e_3) = [3] - [2] = v_1 - v_1 = 0.$$

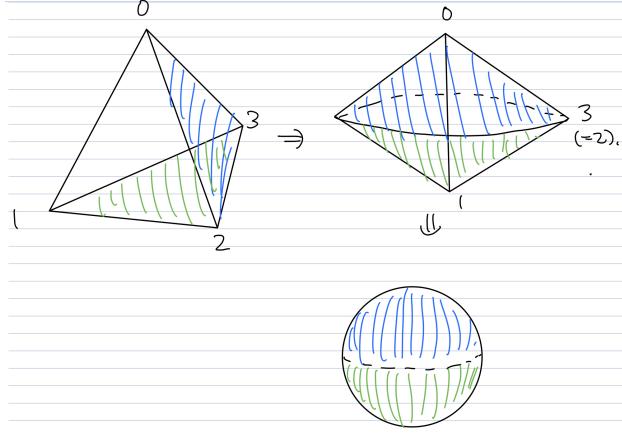


FIGURE 6. Problem 7( $S^3$ )

Therefore,

$$H_3 = \langle T \rangle / 0 = \mathbb{Z}.$$

$$H_2 = 0/0 = 0.$$

$$H_1 = \langle e_1, e_3 \rangle / \langle e_2, e_3 \rangle = 0.$$

$$H_1 = \langle v_1, v_2 \rangle / \langle v_1 - v_2 \rangle = \mathbb{Z}.$$

As shown in Figure 6, it is isomorphic to a 3-ball where the boundary of the northern hemisphere is identified with the boundary of the southern hemisphere by the reflection along the equator. Therefore, this figure is indeed an  $S^3$ .

If I have time, describe this more carefully.

□

**Exercise.** (Problem 8) Construct a 3 dimensional  $\Delta$ -complex  $X$  from  $n$  tetrahedra  $T_1, \dots, T_n$  by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure, so that each  $T_i$  shares a common vertical face with its two neighbors  $T_{i-1}$  and  $T_{i+1}$ , subscripts being taken mod  $n$ . Then identify the bottom face of  $T_i$  with the top face of  $T_{i+1}$  for each  $i$ . Show the simplicial homology groups of  $X$  in dimensions 0, 1, 2, 3 are  $\mathbb{Z}, \mathbb{Z}_n, 0, \mathbb{Z}$ , respectively.

*Proof.* Let  $T_0, \dots, T_{n-1}$  denote the  $n$  tetrahedra. Let  $v_0, v_1, e_0, \dots, e_{n+1}, f_0, \dots, f_{2n-1}$  denote the vertices and edges as in Figure 7. (It has 4 tetrahedra, but they all represent the same  $T_i$ . I wrote four of them only because the figure would be too complicated if I denoted all the vertices, edges, faces in one picture.)

Then we have

- $C_3 = \{T_0, \dots, T_{n-1}\}$ .
- $C_2 = \{f_0, \dots, f_{2n-1}\}$ .
- $C_1 = \{e_0, \dots, e_{n+1}\}$ .
- $C_0 = \{v_0, v_1\}$ .

Now we will examine  $\partial$ .

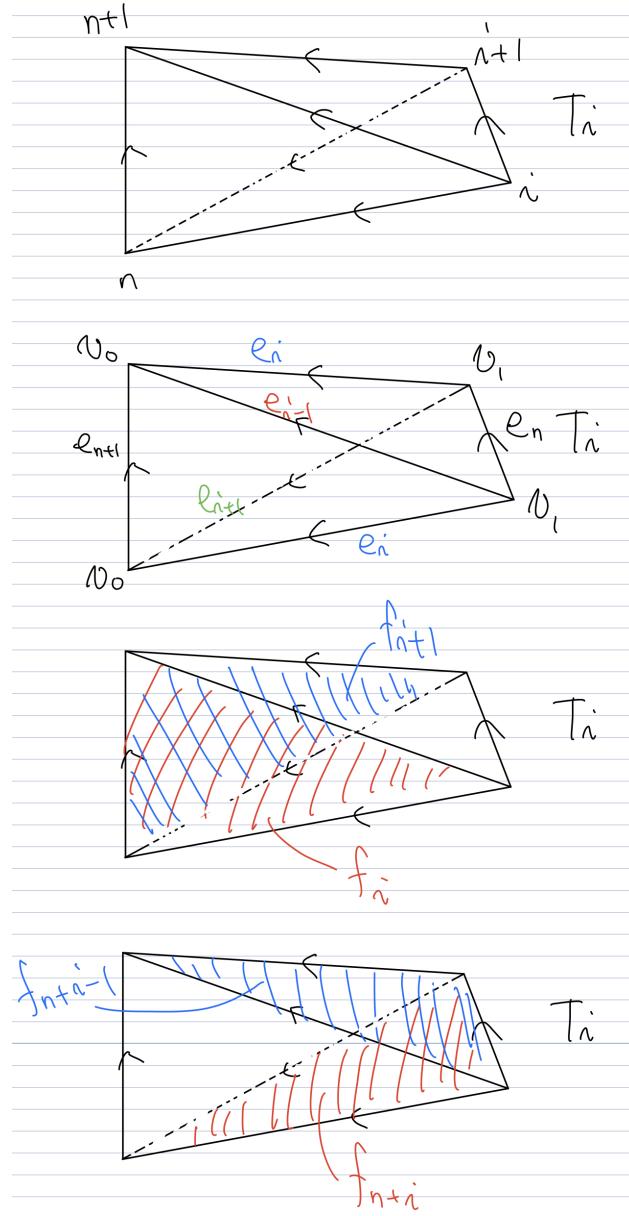


FIGURE 7. Problem 8

$$\begin{aligned}\partial_3(T_i) &= [i+1, n, n+1] - [i, n, n+1] + [i, i+1, n+1] - [i, i+1, n] \\ &= f_{i+1} - f_i + f_{n+i-1} - f_{n+i}.\end{aligned}$$

$$\begin{aligned}\partial_2(f_i) &= [n, n+1] - [i, n+1] + [i, n] \\ &= e_{n+1} - e_{i-1} + e_i.\end{aligned}$$

$$\begin{aligned}\partial_2(f_{n+i}) &= [i+1, n] - [i, n] + [i, i+1] \\ &= e_{i+1} - e_{i-1} + e_n.\end{aligned}$$

$$\partial_1(e_i) = \begin{cases} v_0 - v_1 & (0 \leq i \leq n-1) \\ 0 & (i = n, n+1) \end{cases}$$

Therefore,

$$\begin{aligned}H_3 &= \langle T_0 + \cdots + T_{n-1} \rangle / 0 = \mathbb{Z}. \\H_2 &=? / \langle f_{i+1} - f_i + f_{n+i-1} - f_{n+i} \rangle = 0 \\H_1 &= \langle e_n, e_{n+1}, e_i - e_j \rangle / \langle e_{n+1} + e_i - e_{i-1}, e_n + e_{i+1} - e_i \rangle ? \mathbb{Z}^n \\H_0 &= \langle v_0, v_1 \rangle / \langle v_0 - v_1 \rangle = \mathbb{Z}.\end{aligned}$$

Calculate the homology groups above!

□