

MATH 601 (DUE 11/22)

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1. THE THEOREM ON SYMMETRIC POLYNOMIALS

Exercise. (Problem 1) By substituting $u_4 = 0$, we get $u_1^2 u_2 u_3 + u_1 u_2^2 u_3 + u_1 u_2 u_3^2 = s_3 s_1$. $s_3 s_1$ with 4 variables expands to $u_1^2 u_2 u_3 + u_1^2 u_2 u_4 + u_1^2 u_3 u_4 + u_1 u_2^2 u_3 + u_1 u_2^2 u_4 + u_1 u_2 u_3^2 + 4u_1 u_2 u_3 u_4 + u_1 u_2 u_4^2 + u_1 u_3 u_4^2 + u_1 u_3 u_4^2 + u_2^2 u_3 u_4 + u_2 u_3^2 u_4 + u_2 u_3 u_4^2$. Then $s_3 s_1 - f$ where f is the original polynomial gives us $4u_1 u_2 u_3 u_4 = 4s_4$. Therefore, $f = s_3 s_1 - 4s_4$.

Exercise. (Problem 2) We are given that $|M - xI| = x^3 - ax^2 + bx - c$. This implies that $|M - (-x)I| = -x^3 - ax^2 - bx - c$. Since the determinant function preserves multiplication, $|M - xI||M - (-x)I| = |M^2 - x^2I|$. This implies $|M^2 - x^2I| = -x^6 + (a^2 - 2b)x^4 + (b^2 + 2ac)x^2 + c^2$. Therefore, the characteristic polynomial of M is $-x^3 + (a^2 - 2b)x^2 + (b^2 + 2ac)x + c^2$.

2. GALOIS THEORY VI

Exercise. (Problem 3)

- $\{(123), (132), e\}$ is clearly a subgroup of the stabilizer group S_v of v . Since $(12) \notin S_v$, $3 \leq |S_v| \leq 5$. By Lagrange's Theorem, $S_v = \langle (123) \rangle$.
- By (i), $S_3 v$ contains only $[S_3 : S_v] = 2$ elements. Thus $v' = (12) \cdot v = u_2 u_1^2 + u_1 u_3^2 + u_3 u_2^2$.
- By substituting $u_3 = 0$ for $v + v'$, we get $u_1 u_2^2 + u_2 u_1^2 = s_1 s_2$. Then $v + v' - s_1 s_2 = -3u_1 u_2 u_3 = -3s_3$. Therefore, $v + v' = s_1 s_2 + 3s_3$.
- We will use the fundamental theorem of Galois Theory. $F(v) = K^{\langle (123) \rangle}$, so $|\langle (123) \rangle| = 3 = [K : F(v)]$. Moreover, $|\langle \text{Gal}(K/F) \rangle| = [K : F]$. Therefore, $[F(v) : F] = [K : F] / [K : F(v)] = |\langle \text{Gal}(K/F) \rangle| / 3$.
- Calculation shows that $vv' = 9s_3^2 + s_3 s_1^3 - 6s_3 s_1 s_2 + s_2^3$. By substituting $s_1 = 0, s_2 = p, s_3 = q$, we get $9q^2 + p^3$.

Exercise. (Problem 4)

- The discriminant can be expressed as $-4s_1^3 s_3 + s_1^2 s_2^2 + 18s_1 s_2 s_3 - 4s_2^3 - 27s_3^2$. By substituting $s_1 = 1, s_2 = -2, s_3 = -1$, we get 49.

```
from sympy.polys.polyfuncs import symmetrize
from sympy import *
```

```

u1, u2, u3 = symbols('u1_u2_u3')

u = [u1, u2, u3]

discriminant = 1
for i in range(3):
    for j in range(i + 1, 3):
        discriminant *= (u[i] - u[j]) * (u[i] - u[j])

print(latex(symmetrize(discriminant, formal = True)[0]))

```