MATH 633 (HOMEWORK 1)

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Exercise. (Problem 1)

- From basic topology, the closure of A is equal to the intersection of all closed sets containing A. The intersection of closed sets is closed, so $\overline{A}$ is closed.
- Let $z \in \overline{A} \setminus A$. z is a limit point of A and $A \subset B$, so z is a limit point of B. Since B is closed, $z \in B$.

Exercise. (Problem 2)

- Not open, not closed, not compact. The boundary is $\{x + iy | |x| = |y| = 1\}$.
- Not open. Closed. Compact. The boundary is A.
- Not open. Closed. Not compact. The boundary is the real line.
- Open. Not closed. Not compact. The boundary is $\{0\}$.

Exercise. (Problem 3)

- $f'(z) = -1/z^2$.
- $|z|^2 \cdot (1/z) = \overline{z}$, which is not differentiable anywhere on $\mathbb{C}$. Since $1/z$ is differentiable everywhere on $z \neq 0$, $|z|^2$ is not differentiable anywhere on $z \neq 0$. Thus $|z|$ is not differentiable anywhere on $z \neq 0$.

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

does not exist. This is because the limit is 1 with $h_n = 1/n$, but the limit is -1 with $h_n = -1/n$. Therefore, $|z|$ is nowhere differentiable.

Exercise. (Problem 4) TODO

Exercise. (Problem 5(i)(ii))

$$\lim_{h \to 0} \frac{(f + g)(z_0 + h) - (f + g)(z_0)}{h}$$
$$= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} + \lim_{h \to 0} \frac{g(z_0 + h) - g(z_0)}{h}$$
$$= f'(z_0) + g'(z_0).$$

$$\lim_{h \to 0} \frac{(fg)(z_0 + h) - (fg)(z_0)}{h}$$
$$= \lim_{h \to 0} \frac{f(z_0 + h)g(z_0 + h) - f(z_0 + h)g(z_0) + f(z_0 + h)g(z_0) - f(z_0)g(z_0)}{h}$$
$$= \lim_{h \to 0} f(z_0 + h)\frac{g(z_0 + h) - g(z_0)}{h} + g(z_0) \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$
$$= f(z_0)g'(z_0) + g(z_0)f'(z_0).$$

Exercise. (Problem 5(iii))

$$\begin{aligned}
&\lim_{h\to 0}\frac{1/g(z_0+h)-1/g(z_0)}{h}\\
&=\lim_{h\to 0}\frac{1}{g(z_0)g(z_0+h)}\frac{g(z_0)-g(z_0+h)}{h}\\
&=\frac{g'(z_0)}{g^2(z_0)}.
\end{aligned}$$

By applying Problem 5(ii), we obtain the quotient rule.