MATH 612 (HOMEWORK 3)

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Exercise. (3.1.11) Using the cellular homology, we obtain

$$\tilde{H}_i(X) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & (i = n) \\ 0 & (i \neq n). \end{cases}$$

$$\tilde{H}^i(X) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & (i = n+1) \\ 0 & (i \neq n+1). \end{cases}$$

From previous homework,

$$\tilde{H}^i(X/S^n) = \tilde{H}_i(S^{n+1}) = \begin{cases} \mathbb{Z} & (i = n+1) \\ 0 & (i \neq n+1). \end{cases}$$

Thus the map on $\tilde{H}_i(-;\mathbb{Z})$ is the zero map for each i. On the other hand, the long exact sequence of a pair gives us $\tilde{H}^{n+1}(X, S^n;\mathbb{Z}) \xrightarrow{q^*} \tilde{H}^{n+1}(X;\mathbb{Z}) \to \tilde{H}^{n+1}(S^n;\mathbb{Z})$ where $\tilde{H}^{n+1}(S^n;\mathbb{Z}) = 0$, so q^* is surjective. Therefore, it is nontrivial because $\tilde{H}^{n+1}(X;\mathbb{Z}) \neq 0$.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}(H_n(X);\mathbb{Z}) \arrow[r] \arrow[d] & H^{n+1}(X;\mathbb{Z}) \arrow[r] \arrow[d] & \operatorname{Hom}(H_{n+1}(X);\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ext}(H_n(X/S^n);\mathbb{Z}) \arrow[r] & H^{n+1}(X/S^n;\mathbb{Z}) \arrow[r] & \operatorname{Hom}(H_{n+1}(X/S^n);\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

is

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_m \arrow[r] \arrow[d] & \mathbb{Z}_m \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

This splitting is not natural because the middle term in the first sequence is isomorphic to $\mathbb{Z}_m \oplus 0$ and the second one is $0 \oplus \mathbb{Z}$.

The long exact sequence of a pair gives us $\tilde{H}_n(S^n;\mathbb{Z}) \to \tilde{H}_n(X;\mathbb{Z}) \to \tilde{H}_n(X, S^n;\mathbb{Z}) = \tilde{H}_n(S^{n+1};\mathbb{Z}) = 0$ which implies the surjectivity of the induced map. Since $\tilde{H}_n(X;\mathbb{Z}) \neq 0$, the induced map is nonzero.

The map induced on $\tilde{H}^i(-;\mathbb{Z})$ is the zero map for any i because at least one of $\tilde{H}^i(S^n;\mathbb{Z})$ or $\tilde{H}^i(X;\mathbb{Z})$ is 0 for each i.

Exercise. (3.1.13)

Exercise. (3.2.1)

Exercise. (3.2.2) Suppose X is the union of contractible open sets $A_1, \cdots, A_n$. Since each A_i is contractible, $H^k(X, A_i; R) = H^k(X; R)$ for all $k \geq 1$.

$$\begin{tikzcd}
H^{k_1}(X,A_1;R)\times\cdots\times H^{k_n}(X,A_n;R) \arrow[r] \arrow[d, "\cong"] & H^{k_1+\cdots+k_n}(X,A_1\cup\cdots\cup A_n;R) \arrow[d] \\
H^{k_1}(X;R)\times\cdots\times H^{k_n}(X;R) \arrow[r, "f"] & H^{k_1+\cdots+k_n}(X;R).
\end{tikzcd}$$

This diagram commutes by the naturality of a cup product. $H^{k_1+\cdots+k_n}(X,\bigcup_i A_i;R) = H^{k_1+\cdots+k_n}(X,X;R) = 0$ for all $k+l \geq 1$. By the commutativity of this diagram, the function f must be 0.

Exercise. (3.2.3(a)) Suppose otherwise. Let $f : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m$ be such a function. Then f induces a map on $f^* : H^*(\mathbb{R}\mathrm{P}^m) \to H^*(\mathbb{R}\mathrm{P}^n)$. In other words, $f^* : \mathbb{Z}_m[\alpha]/(\alpha^{m+1}) \to \mathbb{Z}_n[\beta]/(\beta^{n+1})$ where α,β are generators of $H^1(\mathbb{R}\mathrm{P}^m)$ and $H^1(\mathbb{R}\mathrm{P}^n)$. $H^1(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2) = \mathbb{Z}_2 = \{0,\alpha\}$ and $H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) = \mathbb{Z}_2 = \{0,\beta\}$. Since f induces a nontrivial map $H^1(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2) \to H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2)$, $f^*(\alpha) = \beta$. However, $f^*(0) = f^*(\alpha^{m+1}) = (f^*(\alpha))^{m+1} = \beta^{m+1} \neq 0$ because $m < n$. This is a contradiction, so such a function does not exist.

$H^1(\mathbb{C}P^n;\mathbb{Z}_2) = 0$ for any n, so there exists no such nontrivial map. The case for $H^2(\mathbb{C}P^n)$ can be argued the same way as above because $H^2(\mathbb{C}P^n;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ where α is a generator of $H^2(\mathbb{C}P^n)$.

Exercise. (3.2.3(b))

Exercise. (3.2.6)

Exercise. (3.2.7) Let $f : \mathbb{R}\mathrm{P}^3 \to \mathbb{R}\mathrm{P}^2 \vee S^3$ be a homotopy equivalence. Then it induces isomorphisms.

$$\begin{tikzcd}
H^1(\mathbb{R}\mathrm{P}^3;\mathbb{Z}_2) \arrow[d, "f^*"] & \times & H^2(\mathbb{R}\mathrm{P}^3;\mathbb{Z}_2) \arrow[r] \arrow[d, "f^*"] & H^3(\mathbb{R}\mathrm{P}^3;\mathbb{Z}_2) \arrow[d, "f^*"] \\
H^1(\mathbb{R}\mathrm{P}^2 \vee S^3;\mathbb{Z}_2) & \times & H^2(\mathbb{R}\mathrm{P}^2 \vee S^3;\mathbb{Z}_2) \arrow[r] & H^3(\mathbb{R}\mathrm{P}^2 \vee S^3;\mathbb{Z}_2).
\end{tikzcd}$$

The cohomology groups of a wedge sum is the direct sum of cohomology groups of the two spaces. By rewriting the diagram above with generators, we obtain

$$\begin{tikzcd}
\{0,\alpha\} \arrow[d, "f^*"] & \times & \{0,\alpha^2\} \arrow[r] \arrow[d, "f^*"] & \{0,\alpha^3\} \arrow[d, "f^*"] \\
\{0,\beta\}\oplus\{0,\gamma\} & \times & \{0,\beta^2\}\oplus 0 \arrow[r] & 0\oplus\{0,\gamma^2\}.
\end{tikzcd}$$

This implies f^* sends α^2 to $(\beta^2,0)$ and α^3 to $(0,\gamma^2)$. However, this implies $(0,0) = (f^*(\alpha^2))^3 = (f^*(\alpha^3))^2 = (0,\gamma^4) = (0,\gamma)$. This is a contradiction because $0 \neq \gamma$.