MATH 611 PROBLEM SET 1 (DUE 9/4)

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Exercise 0.1. (Exercise 4, Chapter 0) A deformation retraction in the weak sense of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \text{Id}, f_1(X) \subset A$, and $f_t(A) \subset A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \to X$ is a homotopy equivalence.

Proof. Let $i : A \to X$ denote the inclusion. Let $F : X \times I \to X$ denote the associated map $(x, t) \to f_t(x)$. Then F is a continuous function by the definition of a homotopy.

Let $f : X \to A$ be defined by $f(x) = F(x, 1) = f_1(x)$. This definition makes sense because $f_1(X) \subset A$. We claim that $f_1 \circ i \simeq \text{Id}_A$ and $i \circ f_1 \simeq \text{Id}_X$.

Consider $G : A \times I \to A$ such that $G(a, t) = F(a, t)$ for all $(a, t) \in A \times I$. This definition makes sense because $f_t(A) \subset A$ for all t.

Then G is a homotopy in A between $f \circ i$ and Id_A because:

- G is a restriction of F, so G is continuous.
- $\forall a \in A, G(a, 0) = F(a, 0) = f_0(a) = \text{Id}_X(a) = \text{Id}_A(a)$.
- $\forall a \in A, G(a, 1) = F(a, 1) = f(a) = f(i(a)) = (f \circ i)(a)$.

Therefore, $f \circ i \simeq \text{Id}_A$.

F is a homotopy between f_0 and f_1.

- We are given that $f_0 = \text{Id}_X$.
- For any $x \in X$, $(i \circ f)(x) = i(f(x)) = f(x) = f_1(x)$, so $i \circ f = f_1$.

Therefore, F is a homotopy between Id_X and $i \circ F$, so $i \circ f \simeq \text{Id}_X$.

In conclusion, i is indeed a homotopy equivalence. □

Exercise 0.2. (Exercise 5, Chapter 0) Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X there exists a neighborhood $V \subset U$ of x such that the inclusion map $V \to U$ is nullhomotopic.

Proof. Let $p \in X$ be a point to which X deformation retracts. Since X deformation retracts to p, there exists a map $F : X \times I \to X$ such that

(1) $\forall x \in X, F(x, 0) = x$.
(2) $\forall x \in X, F(x, 1) = p$.

(3) $\forall t \in I, F(p,t) = p$.
(4) F is continuous.

Let U be a neighborhood of p. Then $F^{-1}(U)$ is an open subset of the product space $X \times I$. By the 3rd property of F mentioned above, the slice $\{p\} \times I$ is a subset of $F^{-1}(U)$. Since I is compact, there must be a open subset V of X such that $\{p\} \times I \subset V \times I \subset F^{-1}(U)$ by the tube lemma.

We claim that this V is a desired subset.

- V is an open subset of X.
- Since $\{p\} \times I \subset V \times I$, $p \in V$.
- Since $V \times I \subset F^{-1}(U)$, $F(V \times I) \subset U$. This implies that $\forall v \in V$, $F(v, 0) = v \in U$. Therefore, $V \subset U$.
- We claim that the inclusion map $i : V \to U$ is nullhomotopic. Let $e_p : V \to U$ denote the constant map at p, $G : V \times I \to U$ be defined by $G(x,t) = F(x,t)$ for all $x \in V, t \in I$.
  - G indeed maps $V \times I$ into U because $F(V \times I) \subset U$. Therefore, G is well-defined.
  - Since G is the restriction of F to $V \times I$ and F is continuous, G is continuous.
  - $\forall x \in V, G(x,0) = F(x,0) = x = i(x)$.
  - $\forall x \in V, G(x,1) = F(x,1) = p = e_p(x)$.

  Thus i is indeed nullhomotopic.

□

Lemma 0.3. *Let X be a topological space and $A \subset X$. If the inclusion map $A \to X$ is nullhomotopic, then A must be connected.*

Proof. TODO □

Exercise 0.4. (Exercise 6(a), Chapter 0) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point.

Proof. Let $(a, 0) \in [0,1] \times \{0\}$ be given. Let $F : X \times I \to X$ be defined such that

$$F((x,y),t) = \begin{cases} (x, (1-2t)y) & (0 \leq t \leq 1/2) \\ (x + (a-x)(2t-1), 0) & (1/2 \leq t \leq 1). \end{cases}$$

F is well defined because when $t = 1/2$:

- $(x, (1-2t)y) = (x, 0)$.

- $(x + (a - x)(2t - 1), 0) = (x, 0)$.

Moreover, by the pasting lemma, F is continuous.

Then F is a deformation retract of X onto $(a, 0)$ because

- $$F((a,0),t) = \begin{cases} (a, 0(1-2t)) = (a,0) & (t \in [0,1/2]) \\ (a + (a-a)(2t-1), 0) = (a,0) & (t \in [1/2,1]). \end{cases}$$

  Therefore, $F((a, 0), t) = (a, 0)$ for any $t \in I$.
- $F((x, y), 0) = (x, y)$ for any $(x, y) \in x$.
- $F((x, y), 1) = (a, 0)$ for any $(x, y) \in x$.

Therefore, F is indeed a deformation retract of X onto $(a, 0)$.

Suppose that there exists a point $(a, b) \in X$ to which X deformation retracts onto such that $b \neq 0$. Let $G : X \times I \to X$ denote such a deformation retract. Consider the open subset $U = B((a, b), b) \cap X$. Note that U is disjoint from the segment $[0, 1] \times \{0\}$. Then U is a neighborhood of (a, b), a point to which X deformation retracts onto. By Problem 5 (Chapter 0), there must exist a neighborhood $V \subset U$ of x such that the inclusion map $V \to U$ is nullhomotopic. By the Lemma we showed above, V must be connected. Since V is an open subset of X, there must exist an $r > 0$ such that $B((a, b), r) \cap X \subset V$. Let c be an irrational number in $(a, a + r)$. Then $V \cap ((-\infty, c) \times \mathbb{R})$ and $V \cap ((c, \infty) \times \mathbb{R})$ form a separation of V. This is a contradiction, so our initial assumption that X deformation retracts onto (a, b) was wrong. Therefore, X deformation retracts to any point in the segment $[0, 1] \times \{0\}$, but not to any other point. □

Exercise 0.5. (Exercise 9, Chapter 0) Show that a retract of a contractible space is contractible.

Proof. Let X be a contractible space. Then Id_X is homotopic to a constant map. This implies the existence of a fixed point $p \in X$ and a continuous function $F : X \times I \to X$ such that

- $\forall x \in X, F(x, 0) = x$,
- $\forall x \in X, F(x, 1) = p$.

Let $A \subset X$ be a retract of X, and let $r : X \to A$ denote a retraction. In other words, $r(X) = A$ and $r\mid_A = \mathrm{Id}_A$.

Let $G : A \times I \to A$ be the restriction of $r \circ F$ to $A \times I$. This makes sense because F maps $A \times I$ into X, and r maps X into A. We claim that G is a homotopy between Id_A and the constant map $e_{r(p)}$ such that $e_{r(p)}(a) = r(p)$ for all $a \in A$.

- $r \circ F$ is continuous since it is a composition of continuous functions. G is a restriction of a continuous function, so G is continuous.
- $G(a, 0) = r(F(a, 0)) = r(a) = a = \mathrm{Id}_A(a)$.
- $G(a, 1) = r(F(a, 1)) = r(p) = e_{r(p)}(a)$.

Therefore, G is indeed a homotopy between Id_A and the constant map at $r(p)$. Since the identity map is homotopic to a constant map, A is contractible. □

Exercise 0.6. (Exercise 13, Chapter 0) Show that any two deformation retractions r_t^0 and r_t^1 of a space X onto a subspace A can be joined by a continuous family of deformation retractions r_t^s, $0 \leq s \leq 1$, of X onto A, where continuity means that the map $X \times I \times I \to X$ sending (x, s, t) to $r_t^s(x)$ is continuous.

Proof. Let $F : X \times I \times I \to X$ be defined such that

$$F(x, t, s) = \begin{cases} r^0_{t(1-2s)}(x) & (s \in [0, 1/2]) \\ r^1_{t(2s-1)}(x) & (s \in [1/2, 1]). \end{cases}$$

We claim that F is well-defined and satisfies the desired properties.

- Let $s = 1/2$. $r^0_{t(1-2s)}(x) = r^0_0(x) = x$ because r^0_t is a deformation retraction. Similarly, $r^1_{t(2s-1)}(x) = r^1_0(x) = x$ because r^0_t is a deformation retraction. Therefore, F is well defined when $s = 1/2$. Moreover, by the pasting lemma, F is continuous. This is because the intersection $X \times I \times [0, 1/2] \cap X \times I \times [1/2, 1] = X \times I \times \{1/2\}$ is closed.
- $F(x, t, 0) = r^0_t(x)$ for any $x \times t \in X \times I$.
- $F(x, t, 1) = r^1_t(x)$ for any $x \times t \in X \times I$.

Therefore, F maps $X \times I \times I \to X$ continuously sending (x, s, t) to $r^s_t(x)$. □

Exercise 0.7. (Exercise 7, Chapter 1.1) Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles.

Proof. Define $F : (S^1 \times I) \times I \to S^1 \times I$ such that $F((\theta, s), t) = t(\theta, s) + (1-t)f(\theta, s)$. Then F is a homotopy between f and the identity map that is stationary on $S^1 \times \{0\}$. This is because $F((\theta, 0), t) = t(\theta, 0) + (1 - t)f(\theta, 0) = (t\theta, 0) + ((1 - t)\theta, 0) = (\theta, 0)$ for any $(\theta, t) \in S^1 \times I$.

Suppose that there exists a homotopy $G : (S^1 \times I) \times I \to S^1 \times I$ between f and the identity map that is stationary on both boundary circles. Let $H : I \times I \to S^1$ be defined such that $H(s,t) = \pi_1(F((0,t),s))$ where π_1 denotes the projection of the first coordinate.

- $H(s,0) = \pi_1(G((0,0),s)) = \pi_1(0,0) = 0$ because G is stationary on the circle $S^1 \times \{0\}$.
- $H(s,1) = \pi_1(G((0,1),s)) = \pi_1(0,1) = 0$ because G is stationary on the circle $S^1 \times \{1\}$.
- $H(0,t) = \pi_1(G((0,t),0)) = \pi_1(f(0,t)) = \pi_1(2\pi t, t) = 2\pi t$.
- $H(1,t) = \pi_1(G((0,t),1)) = \pi_1(0,t) = 0$.

Then $t \mapsto H(0,t)$ corresponds to the ω in Theorem 1.7, and $t \mapsto H(1,t)$ corresponds to a constant map. In other words, H is a homotopy between ω and a constant map in S^1. However, this is a contradiction because Theorem 1.7 states that $\pi_1(S^1)$ is the infinite cyclic group generated by ω. Therefore, such a homotopy G does not exist. $\square$

Exercise 0.8. (Exercise 16, Chapter 1.1) Show that there are no retractions $r : X \to A$ in the following cases:

- $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.

Proof.

- Suppose that there exists a retract $r : X \to A$. In other words, r is a continuous map such that $r|_A = \mathrm{Id}_A$ and $r(X) \subset A$. Let $\phi : S^1 \to A$ be a homeomorphism. Let ω be the loop defined in Theorem 1.7. Then $\pi_1(S^1, (1,0))$ is the infinite cyclic group generated by $[\omega]$. Consider the following two paths:
  - $\phi \circ \omega : I \to A$.
  - $e_0 : I \to A$ such that $e_0(t) = (\phi \circ \omega)(0) = \phi(1,0)$.

  They are paths in A, and A is a subset of $\mathbb{R}^3$, so they are paths in $\mathbb{R}^3$. Since $\mathbb{R}^3$ is convex, we can define a linear homotopy between them. Let $F : I \times I \to \mathbb{R}^3$ such that $F(s,t) = t(\phi \circ \omega)(s) + (1-t)e_0(s)$. Therefore, the two paths are homotopic in $\mathbb{R}^3$.

  We will consider $\phi^{-1} \circ r \circ F$ that maps $I \times I \to S$. Since it is a composition of continuous functions, it is continuous.

$$\begin{aligned}
(\phi^{-1} \circ r \circ F)(s, 0) &= \phi^{-1}(r(F(s, 0))) \\
&= \phi^{-1}(r(e_0(s))) \\
&= \phi^{-1}(e_0(s)) && (\text{since } e_0(s) \in A) \\
&= (1, 0). \\
(\phi^{-1} \circ r \circ F)(s, 1) &= \phi^{-1}(r(F(s, 1))) \\
&= \phi^{-1}(r((\phi \circ \omega)(s))) \\
&= \phi^{-1}(r(\phi(\omega(s)))) \\
&= \phi^{-1}(\phi(\omega(s))) && (\text{since } \phi(\omega(s)) \in A) \\
&= \omega(s). \\
(\phi^{-1} \circ r \circ F)(0, t) &= \phi^{-1}(r(\phi(1, 0))) \\
&= \phi^{-1}(\phi(1, 0)) \\
&= (1, 0). \\
(\phi^{-1} \circ r \circ F)(1, t) &= \phi^{-1}(r(\phi(1, 0))) \\
&= \phi^{-1}(\phi(1, 0)) \\
&= (1, 0).
\end{aligned}$$

Therefore, $\phi^{-1} \circ r \circ F$ is a path homotopy between ω and the constant loop at $(1, 0)$. This implies that $[w]$ is the identity element in $\pi_1(S^1)$. However, this is a contradiction because Theorem 1.7 states that $\pi_1(S^1)$ is the infinite cyclic group generated by $[w]$.

Hence, there exists no such retraction r.

□

Exercise 0.9. (Exercise 20, Chapter 1.1) Suppose $f_t : X \to X$ is a homotopy such that f_0 and f_1 are each the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$.

Proof. Let $x_0 \in X$ be given. Let $f : I \to X$ be the loop defined such that $f(t) = f_t(x_0)$.

- $f_t : X \to X$ is a homotopy.
- f is a path formed by the images of the base point x_0.

By Lemma 1.19, the following diagram commutes.

$$\begin{tikzcd} & \pi_1(X, f_1(x_0)) \arrow[dd, "\beta_f"] \\ \pi_1(X, x_0) \arrow[ur, "(f_1)_*"] \arrow[dr, "(f_0)_*"'] & \\ & \pi_1(X, f_0(x_0)) \end{tikzcd}$$

$(f_0)_* = (f_1)_* = (\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(X,x_0)}$ by a basic property of induced homomorphisms (P.34 of Hatcher). Since $f_0 = f_1 = \mathrm{Id}_X$, $f_0(x_0) = f_1(x_0) = x_0$. Therefore, the diagram above can be simplified as following:

$$\begin{tikzcd} & \pi_1(X, x_0) \arrow[dd, "\beta_f"] \\ \pi_1(X, x_0) \arrow[ur, "\mathrm{Id}"] \arrow[dr, "\mathrm{Id}"'] & \\ & \pi_1(X, x_0) \end{tikzcd}$$

Let $[g] \in \pi_1(X, x_0)$. Then by the diagram above, we have $\mathrm{Id}([g]) = \mathrm{Id}(\beta_f([g]))$. This implies $[g] = [f \cdot g \cdot \overline{f}]$. Therefore, $[g] \cdot [f] = [f] \cdot [g]$, so $[f]$ commutes with every element in $\pi_1(X, x_0)$. Hence, $[f] \in Z(\pi_1(X, x_0))$. $\square$