MATH 620 HOMEWORK (DUE 9/10)

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Exercise. Show that $F_* : T_p\mathbb{R}^n \to T_q\mathbb{R}^m$.

Proof. Let $v_1, v_2 \in T_pU, c \in \mathbb{R}$. Then $v_1 = c_1^j \frac{\partial}{\partial x^j} \mid_p, v_2 = c_2^j \frac{\partial}{\partial x^j} \mid_p$ where $c_i^j \in \mathbb{R}$. Let $\gamma_1(t) = p + t(c_1^1, \cdots, c_1^n), \gamma_2(t) = p + t(c_2^1, \cdots, c_2^n), \gamma = c\gamma_1 + \gamma_2$. Then there exist unique $b_1^1, \cdots, b_1^m, b_2^1, \cdots, b_2^m, b^1, \cdots, b^m \in \mathbb{R}$ such that

- $F_*(v_1) = b_1^s \frac{\partial}{\partial y^s}$.
- $F_*(v_2) = b_2^s \frac{\partial}{\partial y^s}$.
- $F_*(cv_1 + v_2) = b^s \frac{\partial}{\partial y^s}$.

For each s,

$$\begin{aligned}
b_s &= (F_*(cv_1 + v_2))(y^s) \\
&= \frac{d}{dt} y^s \circ F \circ \gamma(t)\Big|_{t=0} \\
&= \frac{d}{dt} F^s \circ \gamma(t)\Big|_{t=0} && \text{(Let } F^s = y^s \circ F.\text{)} \\
&= \frac{\partial F^s}{\partial x^j}\Big|_p (cc_1^j + c_2^j) \\
&= c\frac{\partial F^s}{\partial x^j}\Big|_p c_1^j + \frac{\partial F^s}{\partial x^j}\Big|_p c_2^j \\
&= c\frac{d}{dt} F^s \circ \gamma_1(t)\Big|_p c_1^j + \frac{d}{dt} F^s \circ \gamma_2(t)\Big|_p c_2^j \\
&= c(F_* v_1)(y^s) + (F_* v_2)(y^s) \\
&= cb_1^s + b_2^s.
\end{aligned}$$

Therefore, $F_*(cv_1 + v_2) = cF_*(v_1) + F_*(v_2)$. □

Exercise. Prove that if $f_I \in \mathscr{C}^\infty$, then $f_I dx^I \in \mathcal{A}^k$.

Proof. Let $\eta = \sum_I f_I dx^I$. Let $X_1, \cdots, X_k \in \mathfrak{X}(\mathbb{R}^n)$. We must show that $F : \mathbb{R}^n \to \mathbb{R}$ defined by $F(p) = \eta_p(X_{1,p}, \cdots, X_{k,p})$ is smooth. For any $p \in \mathbb{R}^n$,

$$\begin{aligned}
F(p) &= \sum_I \eta_p(X_{1,p}, \cdots, X_{k,p}) \\
&= \sum_I f_I(p)(dx^{i_1}|_p \wedge \cdots \wedge dx^{i_k}|_p)(X_{1,p}, \cdots, X_{k,p}) \\
&= \sum_I f_I(p) \sum_{\sigma \in S_k} (dx^{i_{\sigma_1}}|_p)(X_{1,p}) \cdots (dx^{i_{\sigma_k}}|_p(X_{k,p})).
\end{aligned}$$

Since products and sums of smooth functions are smooth, it suffices to show $p \mapsto dx^i|_p(X_{j,p})$ is smooth for each i, j. Then $dx^i|_p(X_{j,p}) = X_{j,p}(x^i)$, which is smooth because $\mathfrak{X}$ is defined to be the collection of all smooth vector fields. $\square$

Exercise. Given $\eta \in \mathscr{A}^k(V), \omega \in \mathscr{A}^l(V)$, prove that $F^*(\eta \wedge \omega) = (F^*\eta) \wedge (F^*\omega)$.

Proof. Let $p \in V, v_1, \cdots, v_{k+l} \in V$.

$$\begin{aligned}
(F^*(\eta \wedge \omega))_p(v_1, \cdots, v_{k+l}) &= (\eta \wedge \omega)_p(F_*v_1, \cdots, F_*v_{k+l}) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \eta_p(F_*v_{\sigma_1}, \cdots, F_*v_{\sigma_k}) \omega_p(F_*v_{\sigma_{k+1}}, \cdots, F_*v_{\sigma_{k+l}}) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) (F^*\eta)_p(v_{\sigma_1}, \cdots, v_{\sigma_k}) (F^*\omega)_p(v_{\sigma_{k+1}}, \cdots, v_{\sigma_{k+l}}) \\
&= ((F^*\eta) \wedge (F^*\omega))_p(v_1, \cdots, v_{k+l}).
\end{aligned}$$

$\square$

Exercise. Define $F : \mathbb{R}^2_{(s,t)} \to \mathbb{R}^3_{(x,y,z)}$ such that $F(s,t) = (s^2, st, t^2)$. Compute the following:

(1) $F^*(xyz)$.
(2) $F^*(xydz + yzdx + zxdy)$.
(3) $F^*(dx \wedge dy - zdx \wedge dz + y^2 dy \wedge dz)$.
(4) $F^*(dx \wedge dy \wedge dz)$.

Proof. We have

- $F^*x = s^2$,
- $F^*y = st$,
- $F^*z = t^2$.

Therefore,

- $F^*dx = 2sds$,
- $F^*dy = tds + sdt$,
- $F^*dz = 2tdt$.

(1) $F^*(xyz) = (s^2)(st)(t^2) = (st)^3$.
(2)

$$\begin{aligned}
F^*(xydz + yzdx + zxdy) &= s^2(st)(2tdt) + (st)t^2(2sds) + t^2s^2(tds + sdt) \\
&= 3t^2s^3dt + 3s^2t^3ds.
\end{aligned}$$

(3)

$$\begin{aligned}
&F^*(dx \wedge dy - zdx \wedge dz + y^2 dy \wedge dz) \\
&= F^*(dx) \wedge F^*(dy) - F^*(zdx) \wedge F^*(dz) + F^*(y^2 dy) \wedge F^*(dz) \\
&= 2sds \wedge (tds + sdt) - (2st^2 ds) \wedge 2tdt + (st)^2(tds + sdt) \wedge 2tdt.
\end{aligned}$$

(4) $F^*(dx \wedge dy \wedge dz) = F^*(dx) \wedge F^*(dy) \wedge F^*(dz) = 2sds \wedge (tds + sdt) \wedge 2tdt = 0$ because the dimension of the vector space is 2 and that is smaller than the number of variables, 3.

$\square$