

# MATH 602 (HOMEWORK 5)

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**Exercise.** (1) This can be proved using induction. The base case  $m = 1$  is trivial. Suppose that the proposition has been shown for some  $m \in \mathbb{N}$ . We will show the  $(m + 1)$  case. By the definition of a determinant,

$$\Delta = \sum_{k=1}^{m+1} (-1)^{k+1} \det(M_{k,1})$$

where  $M_{k,1}$  is the matrix obtained by deleting the  $k$ th row and 1st column. We can apply the inductive hypothesis to each  $M_{k,1}$  because, for instance, when  $k = 1$ ,

$$\begin{aligned} \det(M_{1,1}) &= \det \begin{bmatrix} \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^m \\ & \ddots & & \\ \alpha_{m+1} & \alpha_{m+1}^2 & \cdots & \alpha_{m+1}^m \end{bmatrix} \\ &= \alpha_2 \cdots \alpha_{m+1} \det \begin{bmatrix} 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{m-1} \\ & \ddots & & & \\ 1 & \alpha_{m+1} & \alpha_{m+1}^2 & \cdots & \alpha_{m+1}^{m-1} \end{bmatrix} \\ &= \alpha_2 \cdots \alpha_{m+1} \prod_{2 \leq i < j \leq m} (\alpha_j - \alpha_i). \end{aligned}$$

A similar argument can be applied to other cases and we obtain

$$\Delta = \sum_{k=1}^{m+1} (-1)^{k+1} (\alpha_1 \cdots \hat{\alpha}_k \cdots \alpha_m) \prod_{i < j, i \neq k, j \neq k} (\alpha_j - \alpha_i).$$

It can be observed that, for each  $k = 1, \dots, m+1$ , the  $k$ th term  $(\alpha_1 \cdots \hat{\alpha}_k \cdots \alpha_m) \prod_{i < j, i \neq k, j \neq k} (\alpha_j - \alpha_i)$  does not contain any  $\alpha_k$ . On the other hand, for any  $l \neq k$ , every term that we obtain when expanding the  $l$ th term contains  $\alpha_k$ . Therefore, it suffices to show that, for each  $k$ , the sum of all the terms in  $\prod_{1 \leq i < j \leq m+1} (\alpha_j - \alpha_i)$  that do not contain  $\alpha_k$  is equal to the  $k$ th term in the above expression.

$$\begin{aligned} \prod_{1 \leq i < j \leq m+1} (\alpha_j - \alpha_i) &= \prod_{k+1 \leq j} (\alpha_j - \alpha_k) \prod_{j \leq k-1} (\alpha_k - \alpha_j) \prod_{1 \leq i < j \leq m+1, i \neq k, j \neq k} (\alpha_j - \alpha_i) \\ &= (-1)^{k-1} \prod_{j \neq k} (\alpha_j - \alpha_k) \prod_{1 \leq i < j \leq m+1, i \neq k, j \neq k} (\alpha_j - \alpha_i) \\ &= (-1)^{k-1} (\alpha_1 \cdots \hat{\alpha}_k \cdots \alpha_{m+1}) \prod_{1 \leq i < j \leq m+1, i \neq k, j \neq k} (\alpha_j - \alpha_i) + \alpha_k F(\alpha_1, \dots, \alpha_{m+1}) \\ &= (-1)^{k+1} (\alpha_1 \cdots \hat{\alpha}_k \cdots \alpha_{m+1}) \prod_{1 \leq i < j \leq m+1, i \neq k, j \neq k} (\alpha_j - \alpha_i) + \alpha_k F(\alpha_1, \dots, \alpha_{m+1}) \end{aligned}$$

for some polynomial  $F$ .

$\Delta^2 \neq \prod_{i \neq j} (\alpha_j - \alpha_i)$  in general. Let  $\alpha_1 = 0, \alpha_2 = 1$ . Then  $\det(A)^2 = \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^2 = 1$ . On the other hand,  $\prod_{i \neq j} (\alpha_j - \alpha_i) = (0 - 1)(1 - 0) = -1$ .

**Exercise.** (3) Let  $x_1, \dots, x_m$  be generators of  $C$  as an  $A$ -algebra, and let  $y_1, \dots, y_n$  be generators of  $C$  as a  $B$ -module. Since  $y_1, \dots, y_n$  generate  $C$  as a  $B$ -module, every element in  $C$  can be expressed as a linear combination of  $y_i$ 's over  $B$ . Specifically,  $x_i = \sum b_{ij} y_j$  and  $y_i y_j = \sum b_{ijk} y_k$  for some  $b_{ij}, b_{ijk} \in B$ . Let  $B_0$  be the  $A$ -algebra generated by  $b_{ij}$  and  $b_{ijk}$ . Clearly,  $A \subset B_0 \subset B$ . Since  $A$  is Noetherian,  $B_0$  is Noetherian.

Every element of  $C$  is a finite sum of monomials consisting of  $x_i$ 's with coefficients in  $A$ . Since each  $x_i$  can be written as a linear combination of  $y_i$ 's over  $B_0$ , every element in  $C$  can be written as a finite sum of monomials of  $y_i$ 's with coefficients in  $B_0$ . Since every  $y_i y_j$  can be written as a linear combination of  $y_i$ 's over  $B_0$ , every element in  $C$  can be written as a linear combination of  $y_i$ 's over  $B_0$ . Therefore,  $C$  is finitely generated as a  $B_0$ -module.  $B_0$  is Noetherian and  $B$  is a submodule of  $C$ ,  $B$  is finitely generated as a  $B_0$ -module. Since  $B_0$  is finitely generated as an  $A$ -algebra, it follows that  $B$  is finitely generated as an  $A$ -algebra.

**Exercise.** (4) Let  $K$  denote the field of fractions of  $A$ . Let  $a/b \in K$  be an element integral over  $A$ . Since  $A$  is a UFD, we assume that there is no irreducible element  $q$  that divides both  $a$  and  $b$ . Since  $a/b$  is integral over  $A$ ,  $(a/b)^n + c_{n-1}(a/b)^{n-1} + \dots + c_0 = 0$  for some  $c_0, \dots, c_{n-1} \in A$ . This implies  $a^n + b(c_{n-1}a^{n-1} + c_{n-1}ba^{n-2} + \dots + c_0b^{n-1}) = 0$ . Then every irreducible element that divides  $b$  divides  $a^n$ , so every irreducible element that divides  $b$  divides  $a$ . Since there exists no irreducible element that divides both  $a$  and  $b$ ,  $b$  must be a unit element. In other words,  $a/b \in A$ .

**Exercise.** (5) Since  $R$  is Noetherian,  $\sqrt{I}$  is generated by finitely many elements. Let  $g_1, \dots, g_n$  denote a set of generators of  $\sqrt{I}$ .

For each  $i$ , there exists  $m_i \geq 1$  such that  $g_i^{m_i} \in I$ . Let  $N = \sum m_i$ . Then  $(\sqrt{I})^N = \sqrt{I} \cdots \sqrt{I}$  consists of elements of the form  $(\sum_{i=1}^n x_{1,i} g_i) \cdots (\sum_{i=1}^n x_{N,i} g_i)$ . Each term that we obtain by expanding it is of the form  $x g_1^{k_1} \cdots g_n^{k_n}$  for some  $k_1, \dots, k_n$  with  $k_1 + \dots + k_n = N$ . This implies that for at least one  $i$ ,  $m_i \geq k_i$ , so each term in the expansion belongs to  $I$ . Therefore, every element in  $(\sqrt{I})^N$  is in  $I$ .

**Exercise.** (6) Let  $ab \in \sqrt{q}$ . Then  $a^n b^n \in q$  for some  $n \in \mathbb{N}$ . Then  $a^n \in q$  or  $(b^n)^m \in q$  for some  $m \in \mathbb{N}$ . If  $a^n \in q$ , then  $a \in \sqrt{q}$ . If  $b^{nm} \in q$ , then  $b \in \sqrt{q}$ . Therefore,  $\sqrt{q}$  is prime.

Let  $f : A \rightarrow B$  be given and  $q$  be a primary ideal of  $B$ . Let  $ab \in f^{-1}(q)$ . Then  $f(a)f(b) \in q$ , so  $f(a) \in q$  or  $(f(b))^m \in q$  for some  $m \geq 1$ . If  $f(a) \in q$ , then  $a \in f^{-1}(q)$ . If  $f(b^m) \in q$ , then  $b^m \in f^{-1}(q)$ . Therefore,  $f^{-1}(q)$  is primary.

**Exercise.** (7) Since  $\sqrt{I}$  is maximal,  $I \neq R$ .

Let  $x + I, y + I \in A/I$  be two nonzero elements such that  $(x + I)(y + I) = 0$ . In other words,  $xy \in I$ . Since  $I \subset \sqrt{I}$ ,  $(x + \sqrt{I})(y + \sqrt{I}) = 0$ . Since  $\sqrt{I}$  is maximal,  $A/\sqrt{I}$  is a field. Therefore,  $x + \sqrt{I} = 0$  or  $y + \sqrt{I} = 0$ . In other words,  $x \in \sqrt{I}$  or  $y \in \sqrt{I}$ . If  $x \in \sqrt{I}$ , then  $x + I$  is nilpotent in  $A/I$ . Suppose  $x \notin \sqrt{I}$ . Since  $\sqrt{I}$  is maximal,  $(x) + \sqrt{I} = (1)$ . Therefore,  $ax + b = 1$  for some  $a \in R$  and  $b \in \sqrt{I}$ . Since  $b \in \sqrt{I}$ ,  $b^n \in I$  for some  $n \geq 1$ . Therefore  $1 = ((ax + b) + I)^n = (ax + b)^n + I = xc + I$  for some element  $c$  since  $b^n + I = 0$ .

However, this implies  $0 = (x + I)(y + I)(c + I) = y + I$ , which is a contradiction. Therefore,  $x + I$  must be nilpotent in  $A + I$ . By symmetry,  $y + I$  must be nilpotent in  $A + I$ .

We have shown that every zero divisor in  $A/I$  is nilpotent, which is precisely the definition of a primary ideal.

**Exercise.** (8) Let  $F = \{\text{ann}(x) \mid 0 \neq x \in A\}$ . Since  $A$  is Noetherian,  $F$  has a maximal element. We claim that every maximal element  $\text{ann}(x)$  in  $F$  is a prime ideal. Let  $\text{ann}(x)$  be a maximal element in  $F$ . Suppose  $ab \in \text{ann}(x)$  and  $b \notin \text{ann}(x)$ . Since  $\text{ann}(x) \subset \text{ann}(bx)$  and  $\text{ann}(x)$  is a maximal element,  $\text{ann}(x) = \text{ann}(bx)$ . Since  $ab \in \text{ann}(x)$ ,  $abx = 0$ , so  $a \in \text{ann}(bx)$ . Therefore,  $a \in \text{ann}(x)$ .

Let  $a$  be a zero divisor of  $A$ . Then  $ay = 0$  for some  $y \neq 0$  in  $A/(0) = A$ . In other words,  $a \in \text{ann}(y) \in F$ . By the argument above,  $a \in \text{ann}(x)$  for some associated prime of  $(0)$  containing  $\text{ann}(y)$ . The other direction is trivial from the definition of an associated prime.

**Exercise.** (9) Let  $x \in (q : b)$ . Then  $xb \in q$ . Since  $b \notin q$ ,  $x^n \in q$  for some  $n \geq 1$ . However, this implies  $x \in p$ . Since  $(q : b) \subset p$ ,  $\sqrt{(q : b)} \subset \sqrt{p} = p$ . Clearly,  $q \subset (q : b)$ , so  $p = \sqrt{q} \subset \sqrt{(q : b)}$ . Therefore,  $p = \sqrt{(q : b)}$ .

We will now show that  $\sqrt{(q : b)}$  is primary. Let  $x, y$  be chosen such that  $xy \in (q : b)$ . If  $y^n \in (q : b)$  for some  $n \geq 1$ , we are done. In other words, if  $y \in \sqrt{(q : b)} = p$ , then we are done. Suppose otherwise. Then  $xyb \in q$ , so  $(xb)y \in q$ . This implies  $xb \in q$  because  $y \notin \sqrt{q}$ . This implies  $x \in (q : b)$ , and we are done.