MATH 611 (DUE 10/23)

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1. Simplicial and Singular Homology

Exercise. (Problem 2) Show that the Δ-complex obtained from Δ^3 by performing the edge identifications $[v_0, v_1] \sim [v_1, v_3]$ and $[v_0, v_2] \sim [v_2, v_3]$ deformation retracts onto a Klein bottle. Find other pairs of identifications of edges that produce Δ-complexes deformation retracting onto a torus, a 2-sphere, and $\mathbb{R}\mathbf{P}^2$.

Proof. The deformation retraction of Δ^3 onto a Klein bottle is described in 1. We will start by "pushing" Δ^3 from edge (v_1, v_2). This will leave the surface that consists of the triangles $[v_0, v_1, v_3]$ and $[v_0, v_2, v_3]$. (In other words, a diamond shape consisting of the vertices $[v_0, v_1, v_3, v_2]$.) Step 2 in Figure 1 is what Δ^3 should look like after the deformation retract. Step 3 through 6 show why this is a Klein bottle.

Figure 2 shows the identification of edges for a torus, 2-sphere, and $\mathbb{R}\mathbf{P}^2$.

□

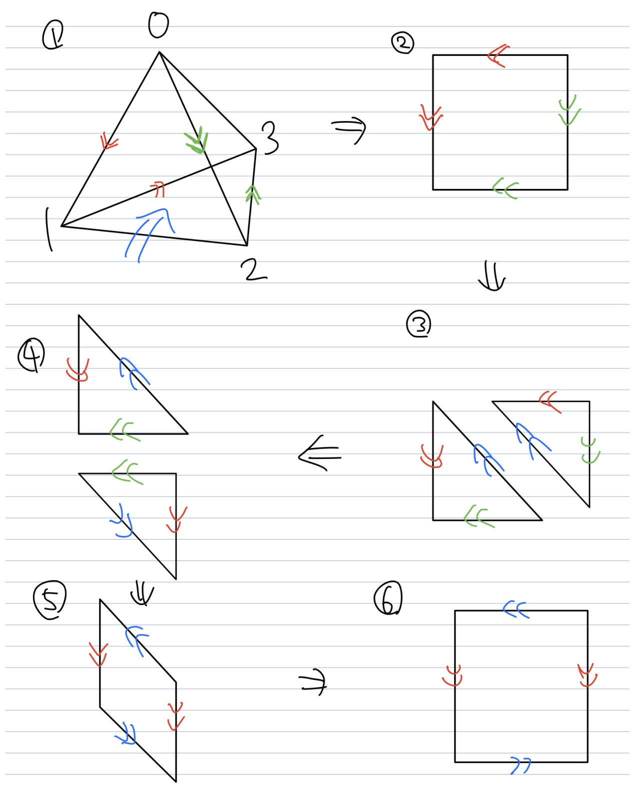

Figure 1. Problem 2(Klein Bottle)

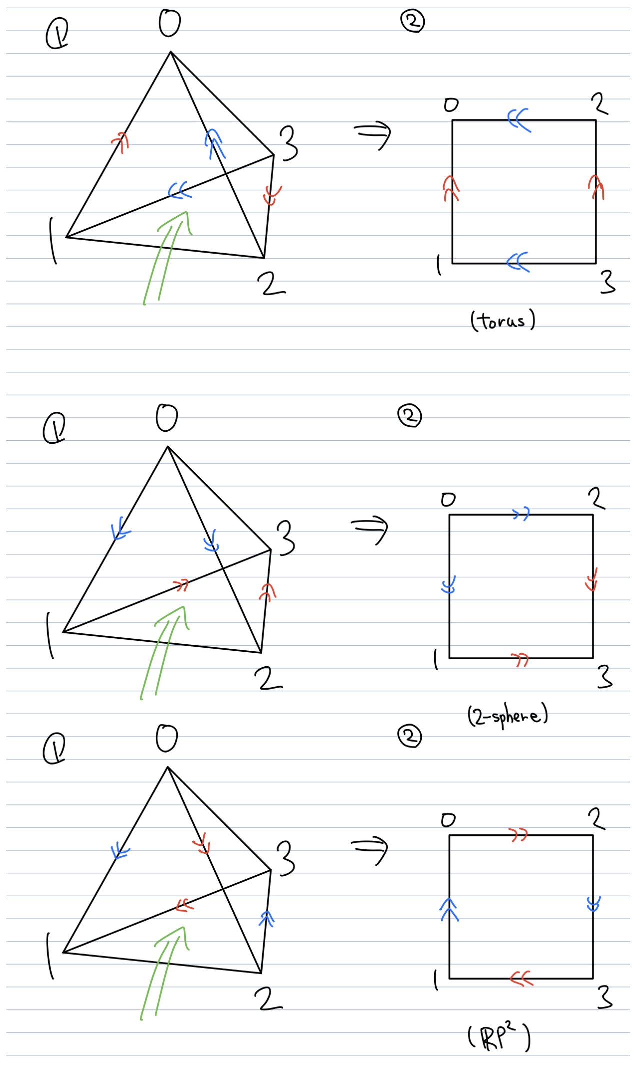

FIGURE 2. Problem 2(Torus, 2-Sphere, $\mathbb{R}\mathbf{P}^2$)

Exercise. (Problem 4) Compute the simplicial homology groups of the triangular parachute obtained from Δ^2 by identifying its three vertices to a single point.

Proof. Let v_0 denote the only vertex, e_1, e_2, e_3 denote the three edges of the parachute, and σ denote the face of the parachute as in Figure 3. $C_k = 0$ for $k \geq 3$ because Δ^2 with the vertices identified does not contain any k-dimensional simplicies for $k \geq 3$. $C_2 = \langle \sigma \rangle, C_1 = \langle e_1, e_2, e_3 \rangle, C_0 = \langle v_0 \rangle$. For each n, ∂_n is defined such that $\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha | [v_0, \cdots, \hat{v}_i, \cdots, v_n]$.

- $\partial_2(\sigma) = e_3 - e_2 + e_1$.
- $\partial_1(e_1) = v - v = 0$. Similarly, $\partial_1(e_2) = \partial_1(e_3) = 0$.
- ∂_0 and ∂_3 are both the zero map.

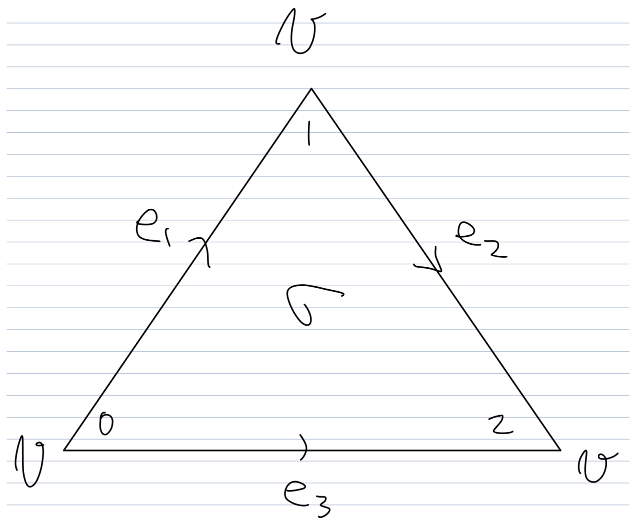

FIGURE 3. Problem 4

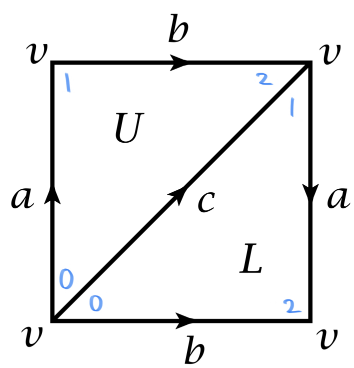

FIGURE 4. Problem 5

Thus

$$H_n = \begin{cases} \{0\} & (n \geq 3) \\ \ker(\partial_2) / \operatorname{Im}(\partial_3) = 0/0 \cong 0 & (n = 2) \\ \ker(\partial_1) / \operatorname{Im}(\partial_2) = \langle e_1, e_2, e_3 \rangle \,/\, \langle e_3 - e_2 + e_1 \rangle \cong \langle e_1, e_2, -e_2 + e_1 \rangle \cong \mathbb{Z}^2 & (n = 1) \\ \ker(\partial_0) / \operatorname{Im}(\partial_1) = \langle v \rangle \,/ 0 \cong \mathbb{Z} & (n = 0). \end{cases}$$

□

Exercise. (Problem 5) Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure described at the beginning of this section.

Proof. We will use the notations in Figure 4.

$$C_n = \begin{cases} 0 & (n \geq 3) \\ \langle U, L \rangle & (n = 2) \\ \langle a, b, c \rangle & (n = 1) \\ \langle v \rangle & (n = 0). \end{cases}$$

$\partial_n = 0$ for $n \geq 3$ and $n = 0$.

$$\begin{aligned} \partial_2(U) &= \sum_{i=0}^{2} (-1)^i \sigma|[0,1,2] \\ &= \sigma|[1,2] - \sigma|[0,2] + \sigma|[0,1] \\ &= b - c + a. \\ \partial_2(L) &= \sum_{i=0}^{2} (-1)^i \sigma|[0,1,2] \\ &= \sigma|[1,2] - \sigma|[0,2] + \sigma|[0,1] \\ &= a - b + c. \end{aligned}$$

$\partial_1(a) = 0$ since $\partial_1(a) = \sigma|[1] - \sigma|[0] = v - v = 0$. Similarly, $\partial_1(b) = \partial_1(c) = 0$. Thus $H_n = \{0\}$ if $(n \geq 3)$. $H_2 = \ker(\partial_2)/\operatorname{Im}(\partial_3) = 0/0 \cong 0$.

$$\begin{aligned} H_1 &= \ker(\partial_1)/\operatorname{Im}(\partial_2) \\ &= \langle a, b, c \rangle \,/\, \langle b - c + a, a - b + c \rangle \\ &\cong \langle a, b, a + b \mid a - b + (a + b) \rangle \\ &\cong \langle a, b \mid 2a \rangle \\ &\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}. \end{aligned}$$

$H_0 = \ker(\partial_0)/\operatorname{Im}(\partial_1) = \langle v \rangle /0 \cong \mathbb{Z}$. □

Exercise. (Problem 7) Find a way of identifying pairs of faces of Δ^3 to produce a Δ-complex structure on S^3 having a single 3-simplex, and compute the simplicial homology groups of this Δ-complex.

Proof. We will identify $[0,2,3] \sim [1,2,3]$ and $[0,1,2] \sim [0,1,3]$ of the tetrahedra T in Figure **??**. Then we have

$$\begin{aligned} C_3 &= \{T\} \\ C_2 &= \{f_1, f_2\} \\ C_1 &= \{e_1, e_2, e_3\} \\ C_0 &= \{v_1, v_2\}. \end{aligned}$$

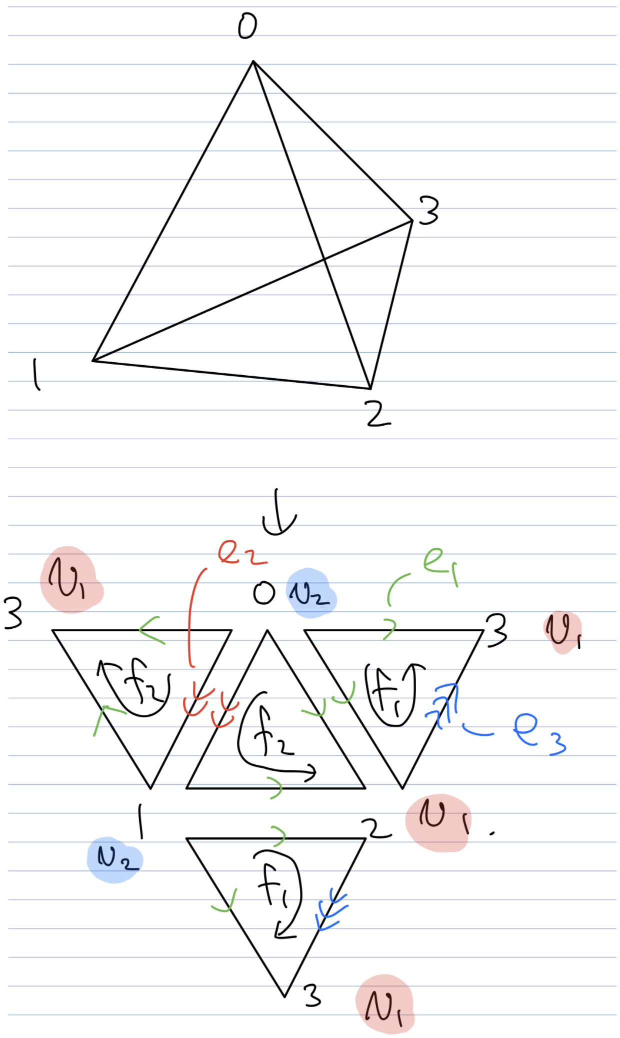

FIGURE 5. Problem 7

We will examine ∂.

$$\partial_3(T) = [1,2,3] - [0,2,3] + [0,1,3] - [0,1,2] = f_1 - f_1 + f_2 - f_2 = 0.$$
$$\partial_2(f_1) = [2,3] - [0,3] + [0,2] = e_3 - e_1 + e_1 = e_3.$$
$$\partial_2(f_2) = [1,2] - [0,2] + [0,1] = e_1 - e_1 + e_2 = e_2.$$
$$\partial_1(e_1) = [3] - [0] = v_1 - v_2.$$
$$\partial_1(e_2) = [1] - [0] = v_2 - v_2 = 0.$$
$$\partial_1(e_3) = [3] - [2] = v_1 - v_1 = 0.$$

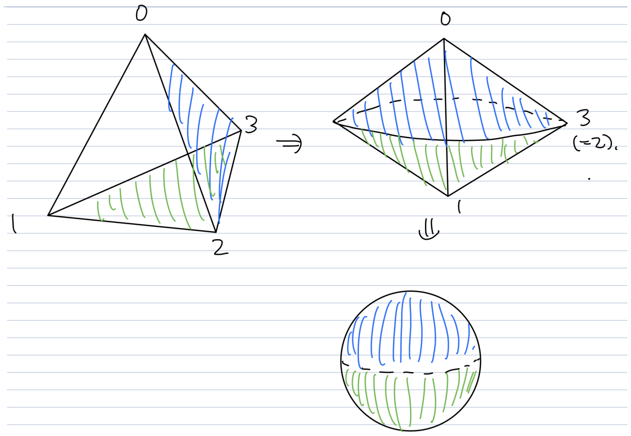

FIGURE 6. Problem 7(S^3)

Therefore,

$$\begin{aligned}
H_3 &= \langle T \rangle / 0 = \mathbb{Z}. \\
H_2 &= 0/0 = 0. \\
H_1 &= \langle e_1, e_3 \rangle / \langle e_2, e_3 \rangle = 0. \\
H_1 &= \langle v_1, v_2 \rangle / \langle v_1 - v_2 \rangle = \mathbb{Z}.
\end{aligned}$$

As shown in Figure 6, it is isomorphic to a 3-ball where the boundary of the northern hemisphere is identified with the boundary of the southern hemisphere by the reflection along the equator. Therefore, this figure is indeed an S^3.

If I have time, describe this more carefully.

$\square$

Exercise. (Problem 8) Construct a 3 dimensional Δ-complex X from n tetrahedra $T_1, \cdots, T_n$ by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure, so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1}, subscripts being taken mod n. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show the simplicial homology groups of X in dimensions 0, 1, 2, 3 are $\mathbb{Z}, \mathbb{Z}_n, 0, \mathbb{Z}$, respectively.

Proof. Let $T_0, \cdots, T_{n-1}$ denote the n tetrahedra. Let $v_0, v_1, e_0, \cdots, e_{n+1}, f_0, \cdots, f_{2n-1}$ denote the vertices and edges as in Figure 7. (It has 4 tetrahedra, but they all represent T_i.)

Then we have

- $C_3 = \{T_0, \cdots, T_{n-1}\}$.
- $C_2 = \{f_0, \cdots, f_{2n-1}\}$.
- $C_1 = \{e_0, \cdots, e_{n+1}\}$.
- $C_0 = \{v_0, v_1\}$.

Now we will examine ∂.

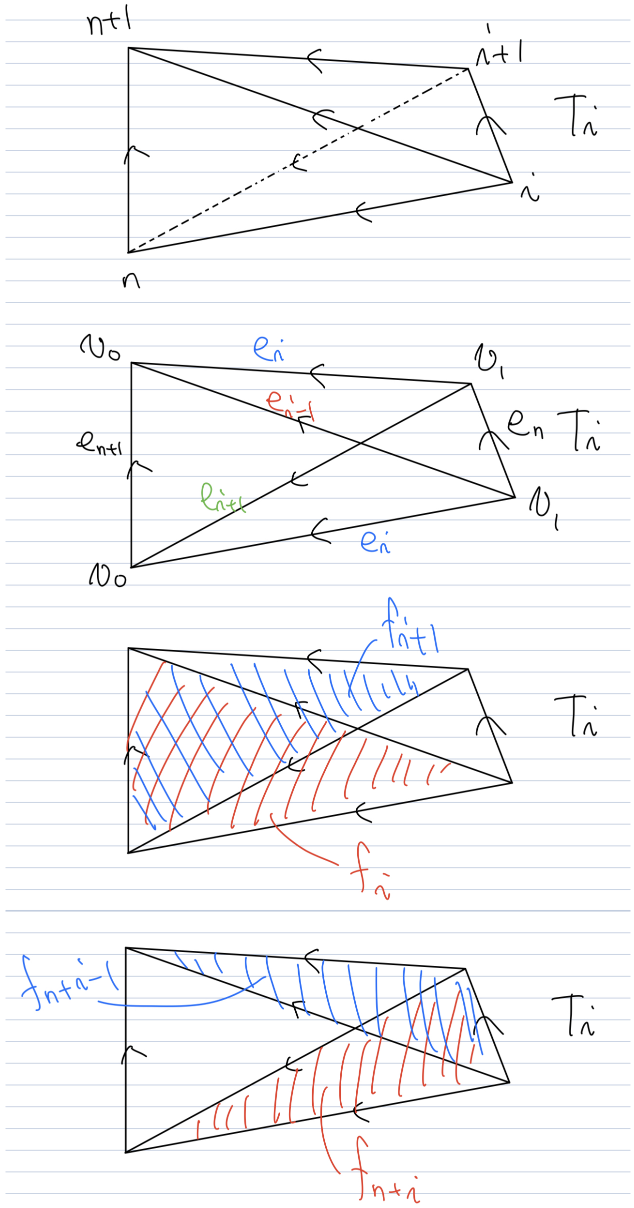

FIGURE 7. Problem 8

$$
\begin{aligned}
\partial_3(T_i) &= [i+1, n, n+1] - [i, n, n+1] + [i, i+1, n+1] - [i, i+1, n] \\
&= f_{i+1} - f_i + f_{n+i-1} - f_{n+i}. \\
\partial_2(f_i) &= [n, n+1] - [i, n+1] + [i, n] \\
&= e_{n+1} - e_{i-1} + e_i. \\
\partial_2(f_{n+i}) &= [i+1, n] - [i, n] + [i, i+1] \\
&= e_{i+1} - e_{i-1} + e_n. \\
\partial_1(e_i) &= \begin{cases} v_0 - v_1 & (0 \leq i \leq n-1) \\ 0 & (i = n, n+1). \end{cases}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
H_3 &= \langle T_0 + \cdots + T_{n-1} \rangle / 0 = \mathbb{Z}. \\
H_2 &= ? / \langle f_{i+1} - f_i + f_{n+i-1} - f_{n+i} \rangle = 0 \\
H_1 &= \langle e_n, e_{n+1}, e_i - e_j \rangle / \langle e_{n+1} + e_i - e_{i-1}, e_n + e_{i+1} - e_i \rangle ? \mathbb{Z}^n \\
H_0 &= \langle v_0, v_1 \rangle / \langle v_0 - v_1 \rangle = \mathbb{Z}.
\end{aligned}
$$

Calculate the homology groups above!

$\square$