QUALIFYING EXAM PREP

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Abstract. In order to prepare for the qualifying exam, I decided to solve problems from Hatcher and Dummit and Foote.

Contents

1. Algebra

1.1. **Groups.** The topics to cover: Elementary concepts (homomorphism, subgroup, coset, normal subgroup), solvable groups, commutator subgroup, Sylow theorems, structure of finitely generated Abelian groups. Symmetric, alternating, dihedral, and general linear groups.

1.2. **Rings.** The topics to cover: Commutative rings and ideals (principal, prime, maximal). Integral domains, Euclidean domains, principal ideal domains, polynomial rings, Eisenstein's irreducibility criterion, Chinese remainder theorem. Structure of finitely generated modules over a principal ideal domain.

1.2.1. *Chinese remainder theorem.*

Exercise. (Problem 1, Section 7.6) Let R be a ring with identity $1 \neq 0$. An element $e \in R$ is called an idempotent if $e^2 = e$. Assume e is an idempotent in R and $er = re$ for all $r \in R$. Prove that Re and $R(1-e)$ are two-sided ideals of R and that $R \cong Re \times R(1-e)$. Show that e and $1-e$ are identities for the subrings Re and $R(1-e)$ respectively.

Proof. Re is clearly nonempty and $re + r'e = (r + r')e \in Re$ for all $re, r'e \in Re$. For all $r' \in R$ and $re \in Re$, $r'(re) = (r'r)e \in Re$ and $(re)r' = r(er') = r(r'e) = (rr')e \in Re$. Thus Re is a two-sided ideal of R. $(1-e)^2 = 1 - e - e + e^2 = 1 - e$, and, for every $r \in R$, $r(1-e) = r - re = r - er = (1-e)r$. Thus $R(1-e)$ is a two-sided ideal of R. Finally, $\phi : R \to R/Re \times R/R(1-e)$ defined by $x \mapsto (x + Re, x + R(1-e))$ is a ring homomorphism with $\ker(\phi) = Re \cap R(1-e)$ by the Chinese Remainder Theorem. Let

$r(1-e) \in \ker(\phi) = Re \cap R(1-e)$. Then $r(1-e)e = r(1-e)$ since $r(1-e) \in Re$. However, this implies $r(1-e)e = r(e-e^2) = r0 = 0$. Thus $\ker(\phi) = 0$, so $R \cong R/Re \times R/R(1-e)$. □

1.3. **Fields Extensions.** Finite, algebraic, separable, inseparable, transcendental, splitting field of a polynomial, primitive element theorem, algebraic closure. Finite fields.

1.4. **Galois Theory.** Finite Galois extensions and the Galois correspondence between subgroups of the Galois group and sub-extensions. Solvable extensions and solving equations by radicals.

2. Algebraic topology

2.1. **Fundamental group.** Computation of the fundamental group, van Kampen's theorem, covering spaces.

Exercise. (Exercise 8, Section 1.1) Does the Borsuk-Ulam theorem hold for the torus?

Proof. No. Consider the natural projection map of $S^1 \times S^1$ into $\mathbb{R}^2$. From Figure 1, it is

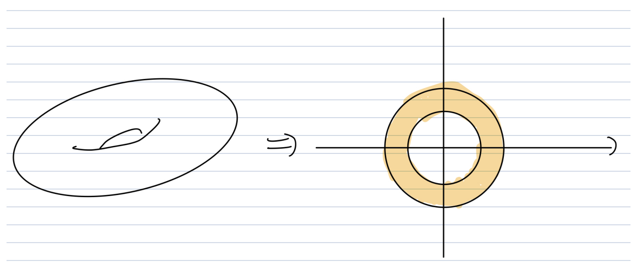

Figure 1. Ex 1-1-8

clear that $f(x, y) = -f(-x, -y)$. However, $f(x, y) \neq 0$ for any $(x, y) \in S^1 \times S^1$. Thus $f(x, y) \neq f(-x, -y)$ for all (x, y). □

2.2. **Homology.** Singular chains, chain complexes, homotopy invariance. Relationship between the first homology and the fundamental group, relative homology. The long exact sequence of relative homology. The Mayer-Vietoris sequence.