QUALIFYING EXAM PREP

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ABSTRACT. In order to prepare for the qualifying exam, I decided to solve problems from Hatcher and Dummit and Foote.

CONTENTS

1. ALGEBRA

1.1. Groups. The topics to cover: Elementary concepts (homomorphism, subgroup, coset, normal subgroup), solvable groups, commutator subgroup, Sylow theorems, structure of finitely generated Abelian groups. Symmetric, alternating, dihedral, and general linear groups.

1.1.1. *Structure of finitely generated Abelian groups.*

Exercise. (Problem 5, Section 5.2) Let G be a finite abelian group of type $(n_1, n_2, \cdots, n_t)$. Prove that G contains an element of order m if and only if $m \mid n_1$. Deduce that G is of exponent n_1.

Proof. Let $\phi : G \to \mathbb{Z}^{n_1} \oplus \cdots \oplus \mathbb{Z}^{n_t}$ be an isomorphism. Let $x \in G$ and m be the order of x. Let $\phi(x) = (a_1, \cdots, a_t)$. Then $\phi(n_1 x) = (n_1 a_1, \cdots, n_1 a_t) = 0$ because $n_i \mid n_1$ for each i. Thus $m \mid n_1$.

On the other hand, if $m \mid n_1$, then $\phi^{-1}(n_1/m, 0, \cdots, 0)$ is an element of order m in G.

Therefore, every element's order is a divisor of n_1 and there exists an element of order n_1, so G is of exponent n_1. $\qquad\square$

1.2. Rings. The topics to cover: Commutative rings and ideals (principal, prime, maximal). Integral domains, Euclidean domains, principal ideal domains, polynomial rings, Eisenstein's irreducibility criterion, Chinese remainder theorem. Structure of finitely generated modules over a principal ideal domain.

1.2.1. *Chinese remainder theorem.*

Exercise. (Problem 1, Section 7.6) Let R be a ring with identity $1 \neq 0$. An element $e \in R$ is called an idempotent if $e^2 = e$. Assume e is an idempotent in R and $er = re$ for all $r \in R$. Prove that Re and $R(1-e)$ are two-sided ideals of R and that $R \cong Re \times R(1-e)$. Show that e and $1-e$ are identities for the subrings Re and $R(1-e)$ respectively.

Proof. Re is clearly nonempty and $re + r'e = (r + r')e \in Re$ for all $re, r'e \in Re$. For all $r' \in R$ and $re \in Re$, $r'(re) = (r'r)e \in Re$ and $(re)r' = r(er') = r(r'e) = (rr')e \in Re$. Thus Re is a two-sided ideal of R. $(1-e)^2 = 1 - e - e + e^2 = 1 - e$, and, for every $r \in R$, $r(1-e) = r - re = r - er = (1-e)r$. Thus $R(1-e)$ is a two-sided ideal of R. Finally, $\phi : R \to R/Re \times R/R(1-e)$ defined by $x \mapsto (x + Re, x + R(1-e))$ is a ring homomorphism with $\ker(\phi) = Re \cap R(1-e)$ by the Chinese Remainder Theorem. Let $r(1-e) \in \ker(\phi) = Re \cap R(1-e)$. Then $r(1-e)e = r(1-e)$ since $r(1-e) \in Re$. However, this implies $r(1-e)e = r(e-e^2) = r0 = 0$. Thus $\ker(\phi) = 0$, so $R \cong R/Re \times R/R(1-e)$. $\square$

1.3. **Fields Extensions.** Finite, algebraic, separable, inseparable, transcendental, splitting field of a polynomial, primitive element theorem, algebraic closure. Finite fields.

1.4. **Galois Theory.** Finite Galois extensions and the Galois correspondence between subgroups of the Galois group and sub-extensions. Solvable extensions and solving equations by radicals.

Exercise. (Exercise 4 (Chapter 14)) Prove that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic.

Proof. Suppose they are and let ϕ be a ring isomorphism. $\phi(\sqrt{2}) = a + b\sqrt{3}$ for some $a, b \in \mathbb{Q}$. This implies $\phi(2) = (a^2 + 3b^2) + 2ab\sqrt{3}$. On the other hand, $\phi(2) = \phi(1) + \phi(1) = 1 + 1 = 2$. Thus $2ab = 0$. If $a = 0$, then $3b^2 = 2$, but $\sqrt{2}/3$ is not rational. If $b = 0$, then $a^2 = 2$, but $\sqrt{2}$ is not rational. This is a contradiction, so such a homomorphism does not exist. $\square$

2. ALGEBRAIC TOPOLOGY

2.1. **Fundamental group.** Computation of the fundamental group, van Kampen's theorem, covering spaces.

Exercise. (Exercise 8, Section 1.1) Does the Borsuk-Ulam theorem hold for the torus?

Proof. No. Consider the natural projection map of $S^1 \times S^1$ into $\mathbb{R}^2$. From Figure 1, it is
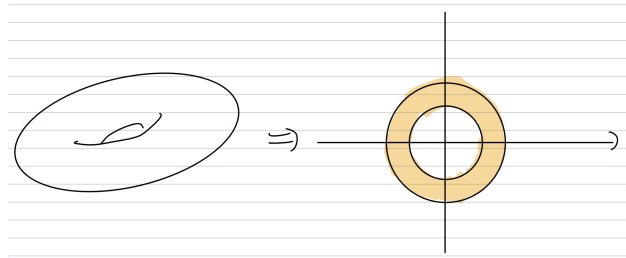

FIGURE 1. Ex 1-1-8

clear that $f(x, y) = -f(-x, -y)$. However, $f(x, y) \neq 0$ for any $(x, y) \in S^1 \times S^1$. Thus $f(x, y) \neq f(-x, -y)$ for all (x, y). $\square$

Exercise. (Exercise 2, Section 1.2) Let $X \subset \mathbb{R}^m$ be the union of convex open sets $X_1, \cdots, X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all i, j, k. Show that X is simply-connected.

Proof. The case that $n = 1$ is trivial. Suppose we have shown this for $n \in \mathbb{N}$ and let $X_1, \cdots, X_{n+1}$ be a set of convex open sets such that the intersection of any three is nonempty. Let $Y = X_1 \cup \cdots \cup X_n$. By the inductive hypothesis, Y is simply connected. We have

- Y, X_{n+1} are both path-connected open sets such that the intersection is nonempty.
- We claim that $Y \cap X_{n+1}$ is path connected. Let $a, b \in Y \cap X_{n+1}$. Then $a \in X_i \cap X_{n+1}$ and $b \in X_j \cap X_{n+1}$ for some $i, j \in \{1, \cdots, n\}$. Then $X_i \cap X_j \cap X_{n+1}$ is nonempty, and choose a point y in the intersection. Then a and y can be joined by a path because $X_i \cap X_{n+1}$ is convex. Similarly, b and y can be joined by a path. Therefore, a and b can be joined by a path, so $Y \cap X_{n+1}$ is path connected.

By the van Kampen theorem, $\pi_1(X) \cong \pi_1(Y) * \pi_1(X_{n+1})/N$ for some normal subgroup N. However, it does not matter what N is because $\pi_1(Y)$ and $\pi_1(X_{n+1})$ are both trivial. Thus $\pi_1(X) = 1$. $\square$

Exercise. (Exercise 1, Section 1.3) For a covering space $p : \tilde{X} \to X$ and a subspace $A \subset X$, let $\tilde{A} = p^{-1}(A)$. Show that the restriction $p : \tilde{A} \to A$ is a covering space.

Proof. Let $x \in A$. Then x has a neighborhood in X such that $p^{-1}(U)$ is a union of disjoint open sets $\{V_\alpha\}$ each of which is mapped homeomorphically onto U by p.

$U \cap A$ is a neighborhood of x in A. Then $p^{-1}(U \cap A) = p^{-1}(U) \cap p^{-1}(A) = \coprod(V_\alpha \cap \tilde{A})$. Each $V_\alpha \cap \tilde{A}$ is open in $\tilde{A}$. Moreover, $V_\alpha \cap \tilde{A}$ is mapped homeomorphically onto $U \cap A$ by p because

- p is injective when restricted to V_α. Restricting it further to $V_\alpha \cap \tilde{A}$ gives an injection.
- Let $y \in U \cap A$. Then $y \in U$, so there exists exactly one point $\tilde{y}$ in V_α such that $p(\tilde{y}) = y$.
- A restriction of a continuous map is continuous.

Therefore, $p : \tilde{A} \to A$ is a covering map. $\square$

2.2. **Homology.** Singular chains, chain complexes, homotopy invariance. Relationship between the first homology and the fundamental group, relative homology. The long exact sequence of relative homology. The Mayer-Vietoris sequence.

Exercise. (Exercise 1, Section 2.1) What familiar space is the quotient Δ-complex of a 2-simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$, preserving the ordering of vertices?

Proof. This can be seen easily if we cut the triangle in half and rotate one of the pieces as shown in 2. This space is a Mobius band. $\square$
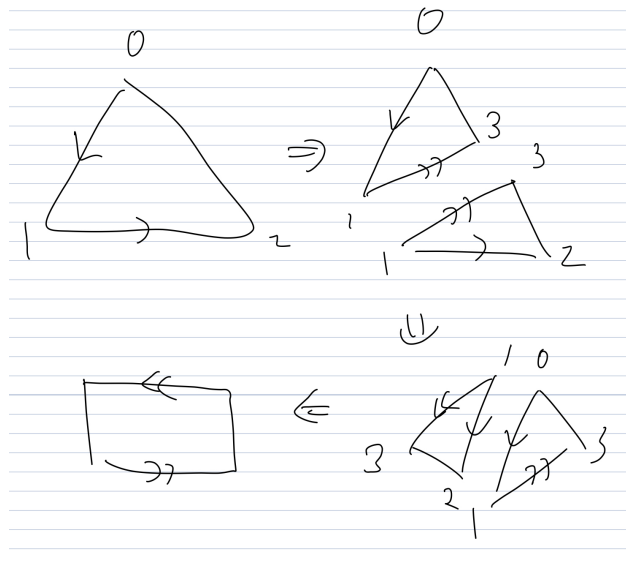
FIGURE 2. Ex 2-1-1