

QUALIFYING EXAM PREP

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ABSTRACT. In order to prepare for the qualifying exam, I decided to solve problems from Hatcher and Dummit and Foote.

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1. ALGEBRA

1.1. **Groups.** The topics to cover: Elementary concepts (homomorphism, subgroup, coset, normal subgroup), solvable groups, commutator subgroup, Sylow theorems, structure of finitely generated Abelian groups. Symmetric, alternating, dihedral, and general linear groups.

1.2. **Rings.** The topics to cover: Commutative rings and ideals (principal, prime, maximal). Integral domains, Euclidean domains, principal ideal domains, polynomial rings, Eisenstein's irreducibility criterion, Chinese remainder theorem. Structure of finitely generated modules over a principal ideal domain.

1.2.1. *Chinese remainder theorem.*

Exercise. (Problem 1, Section 7.6) Let R be a ring with identity $1 \neq 0$. An element $e \in R$ is called an idempotent if $e^2 = e$. Assume e is an idempotent in R and $er = re$ for all $r \in R$. Prove that Re and $R(1 - e)$ are two-sided ideals of R and that $R \cong Re \times R(1 - e)$. Show that e and $1 - e$ are identities for the subrings Re and $R(1 - e)$ respectively.

Proof. Re is clearly nonempty and $re + r'e = (r + r')e \in Re$ for all $re, r'e \in Re$. For all $r' \in R$ and $re \in Re$, $r'(re) = (r'r)e \in Re$ and $(re)r' = r(er') = r(r'e) = (rr')e \in Re$. Thus Re is a two-sided ideal of R . $(1 - e)^2 = 1 - e - e + e^2 = 1 - e$, and, for every $r \in R$, $r(1 - e) = r - re = r - er = (1 - e)r$. Thus $R(1 - e)$ is a two-sided ideal of R . Finally, $\phi : R \rightarrow R/Re \times R/R(1 - e)$ defined by $x \mapsto (x + Re, x + R(1 - e))$ is a ring homomorphism with $\ker(\phi) = Re \cap R(1 - e)$ by the Chinese Remainder Theorem. Let

$r(1-e) \in \ker(\phi) = Re \cap R(1-e)$. Then $r(1-e)e = r(1-e)$ since $r(1-e) \in Re$. However, this implies $r(1-e)e = r(e-e^2) = r0 = 0$. Thus $\ker(\phi) = 0$, so $R \cong R/Re \times R/R(1-e)$. \square

1.3. Fields Extensions. Finite, algebraic, separable, inseparable, transcendental, splitting field of a polynomial, primitive element theorem, algebraic closure. Finite fields.

1.4. Galois Theory. Finite Galois extensions and the Galois correspondence between subgroups of the Galois group and sub-extensions. Solvable extensions and solving equations by radicals.

Exercise. (Exercise 4 (Chapter 14)) Prove that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic.

Proof. Suppose they are and let ϕ be a ring isomorphism. $\phi(\sqrt{2}) = a + b\sqrt{3}$ for some $a, b \in \mathbb{Q}$. This implies $\phi(2) = (a^2 + 3b^2) + 2ab\sqrt{3}$. On the other hand, $\phi(2) = \phi(1) + \phi(1) = 1 + 1 = 2$. Thus $2ab = 0$. If $a = 0$, then $3b^2 = 2$, but $\sqrt{2}/3$ is not rational. If $b = 0$, then $a^2 = 2$, but $\sqrt{2}$ is not rational. This is a contradiction, so such a homomorphism does not exist. \square

2. ALGEBRAIC TOPOLOGY

2.1. Fundamental group. Computation of the fundamental group, van Kampen's theorem, covering spaces.

Exercise. (Exercise 8, Section 1.1) Does the Borsuk-Ulam theorem hold for the torus?

Proof. No. Consider the natural projection map of $S^1 \times S^1$ into \mathbb{R}^2 . From Figure 1, it is

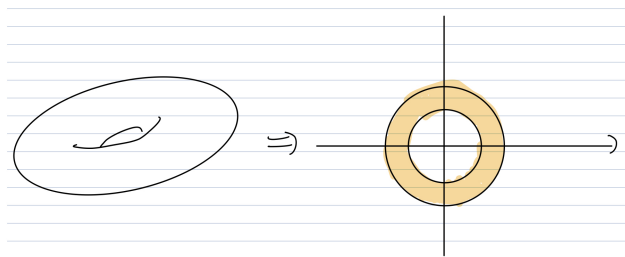


FIGURE 1. Ex 1-1-8

clear that $f(x, y) = -f(-x, -y)$. However, $f(x, y) \neq 0$ for any $(x, y) \in S^1 \times S^1$. Thus $f(x, y) \neq f(-x, -y)$ for all (x, y) . \square

Exercise. (Exercise 2, Section 1.2) Let $X \subset \mathbb{R}^m$ be the union of convex open sets X_1, \dots, X_n such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all i, j, k . Show that X is simply-connected.

Proof. The case that $n = 1$ is trivial. Suppose we have shown this for $n \in \mathbb{N}$ and let X_1, \dots, X_{n+1} be a set of convex open sets such that the intersection of any three is nonempty. Let $Y = X_1 \cup \dots \cup X_n$. By the inductive hypothesis, Y is simply connected. We have

- Y, X_{n+1} are both path-connected open sets such that the intersection is nonempty.
- We claim that $Y \cap X_{n+1}$ is path connected. Let $a, b \in Y \cap X_{n+1}$. Then $a \in X_i \cap X_{n+1}$ and $b \in X_j \cap X_{n+1}$ for some $i, j \in \{1, \dots, n\}$. Then $X_i \cap X_j \cap X_{n+1}$ is nonempty, and choose a point y in the intersection. Then a and y can be joined by a path because $X_i \cap X_{n+1}$ is convex. Similarly, b and y can be joined by a path. Therefore, a and b can be joined by a path, so $Y \cap X_{n+1}$ is path connected.

By the van Kampen theorem, $\pi_1(X) \cong \pi_1(Y) * \pi_1(X_{n+1})/N$ for some normal subgroup N . However, it does not matter what N is because $\pi_1(Y)$ and $\pi_1(X_{n+1})$ are both trivial. Thus $\pi_1(X) = 1$. \square

2.2. Homology. Singular chains, chain complexes, homotopy invariance. Relationship between the first homology and the fundamental group, relative homology. The long exact sequence of relative homology. The Mayer-Vietoris sequence.

Exercise. (Exercise 1, Section 2.1) What familiar space is the quotient Δ -complex of a 2-simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$, preserving the ordering of vertices?

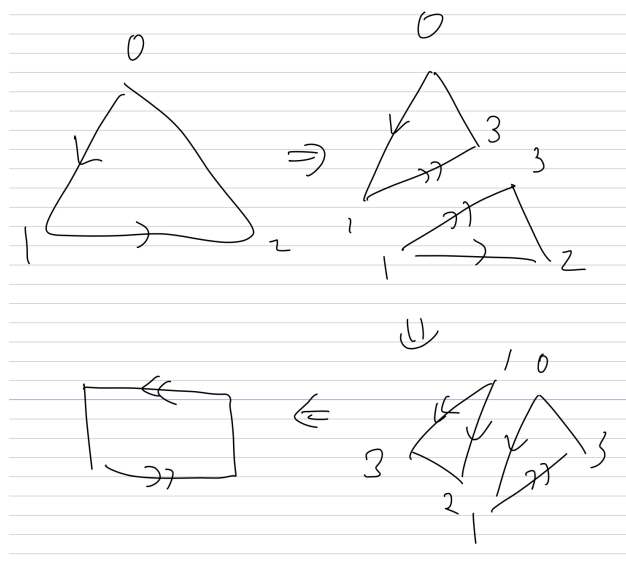


FIGURE 2. Ex 2-1-1

Proof. This can be seen easily if we cut the triangle in half and rotate one of the pieces as shown in 2. This space is a Möbius band. \square