QUALIFYING EXAM PREP

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Abstract. In order to prepare for the qualifying exam, I decided to solve problems from Hatcher and Dummit and Foote.

Contents

1. Algebra

1.1. **Groups.** The topics to cover: Elementary concepts (homomorphism, subgroup, coset, normal subgroup), solvable groups, commutator subgroup, Sylow theorems, structure of finitely generated Abelian groups. Symmetric, alternating, dihedral, and general linear groups.

1.1.1. *Structure of finitely generated Abelian groups.*

Exercise. (Problem 5, Section 5.2) Let G be a finite abelian group of type $(n_1, n_2, \cdots, n_t)$. Prove that G contains an element of order m if and only if $m \mid n_1$. Deduce that G is of exponent n_1.

Proof. Let $\phi : G \to \mathbb{Z}^{n_1} \oplus \cdots \oplus \mathbb{Z}^{n_t}$ be an isomorphism. Let $x \in G$ and m be the order of x. Let $\phi(x) = (a_1, \cdots, a_t)$. Then $\phi(n_1 x) = (n_1 a_1, \cdots, n_1 a_t) = 0$ because $n_i \mid n_1$ for each i. Thus $m \mid n_1$.

On the other hand, if $m \mid n_1$, then $\phi^{-1}(n_1/m, 0, \cdots, 0)$ is an element of order m in G.

Therefore, every element's order is a divisor of n_1 and there exists an element of order n_1, so G is of exponent n_1. □

1.2. **Rings.** The topics to cover: Commutative rings and ideals (principal, prime, maximal). Integral domains, principal ideal domains, polynomial rings, Eisenstein's irreducibility criterion. Structure of finitely generated modules over a principal ideal domain.

1.2.1. *Chinese remainder theorem.*

Exercise. (Problem 1, Section 7.6) Let R be a ring with identity $1 \neq 0$. An element $e \in R$ is called an idempotent if $e^2 = e$. Assume e is an idempotent in R and $er = re$ for all $r \in R$. Prove that Re and $R(1-e)$ are two-sided ideals of R and that $R \cong Re \times R(1-e)$. Show that e and $1-e$ are identities for the subrings Re and $R(1-e)$ respectively.

Proof. Re is clearly nonempty and $re + r'e = (r + r')e \in Re$ for all $re, r'e \in Re$. For all $r' \in R$ and $re \in Re$, $r'(re) = (r'r)e \in Re$ and $(re)r' = r(er') = r(r'e) = (rr')e \in Re$. Thus Re is a two-sided ideal of R. $(1-e)^2 = 1 - e - e + e^2 = 1 - e$, and, for every $r \in R$, $r(1-e) = r - re = r - er = (1-e)r$. Thus $R(1-e)$ is a two-sided ideal of R. Finally, $\phi : R \to R/Re \times R/R(1-e)$ defined by $x \mapsto (x + Re, x + R(1-e))$ is a ring homomorphism with $\ker(\phi) = Re \cap R(1-e)$ by the Chinese Remainder Theorem. Let $r(1-e) \in \ker(\phi) = Re \cap R(1-e)$. Then $r(1-e)e = r(1-e)$ since $r(1-e) \in Re$. However, this implies $r(1-e)e = r(e-e^2) = r0 = 0$. Thus $\ker(\phi) = 0$, so $R \cong R/Re \times R/R(1-e)$. ☐

1.2.2. *Euclidean domains.*

Exercise. (Exercise 3, Section 8.1) Let R be a Euclidean domain. Let m be the minimum integer in the set of norms of nonzero elements of R. Prove that every nonzero element of R of norm m is a unit. Deduce that a nonzero element of norm zero (if such an element exists) is a unit.

Proof. Let x be an element of norm m. Since R is a Euclidean domain, $1 = ax + b$ for some $a, b \in R$ with $b = 0$ or $N(b) < N(m)$. Since $N(x) = m$, $b = 0$. Thus a is the multiplicative inverse of x, so x is a unit. Since the norm has to be nonnegative, the last part of the problem is trivial. ☐

1.3. **Fields Extensions.** Finite, algebraic, separable, inseparable, transcendental, splitting field of a polynomial, primitive element theorem, algebraic closure. Finite fields.

1.4. **Galois Theory.** Finite Galois extensions and the Galois correspondence between subgroups of the Galois group and sub-extensions. Solvable extensions and solving equations by radicals.

Exercise. (Exercise 4 (Chapter 14)) Prove that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic.

Proof. Suppose they are and let ϕ be a ring isomorphism. $\phi(\sqrt{2}) = a + b\sqrt{3}$ for some $a, b \in \mathbb{Q}$. This implies $\phi(2) = (a^2 + 3b^2) + 2ab\sqrt{3}$. On the other hand, $\phi(2) = \phi(1) + \phi(1) = 1 + 1 = 2$. Thus $2ab = 0$. If $a = 0$, then $3b^2 = 2$, but $\sqrt{2/3}$ is not rational. If $b = 0$, then $a^2 = 2$, but $\sqrt{2}$ is not rational. This is a contradiction, so such a homomorphism does not exist. ☐

2. Algebraic topology

2.1. **Fundamental group.** Computation of the fundamental group, van Kampen's theorem, covering spaces.

Exercise. (Exercise 8, Section 1.1) Does the Borsuk-Ulam theorem hold for the torus?

Proof. No. Consider the natural projection map of $S^1 \times S^1$ into $\mathbb{R}^2$. From Figure 1, it is clear that $f(x, y) = -f(-x, -y)$. However, $f(x, y) \neq 0$ for any $(x, y) \in S^1 \times S^1$. Thus $f(x, y) \neq f(-x, -y)$ for all (x, y). ☐

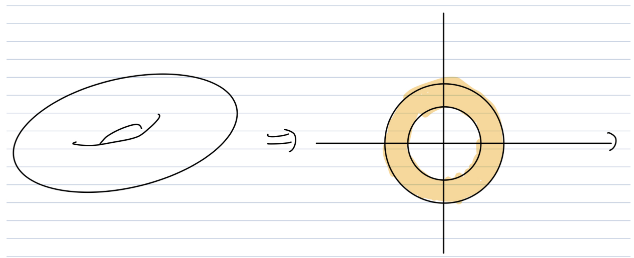

FIGURE 1. Ex 1-1-8

Exercise. (Exercise 2, Section 1.2) Let $X \subset \mathbb{R}^m$ be the union of convex open sets $X_1, \cdots, X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all i, j, k. Show that X is simply-connected.

Proof. The case that $n = 1$ is trivial. Suppose we have shown this for $n \in \mathbb{N}$ and let $X_1, \cdots, X_{n+1}$ be a set of convex open sets such that the intersection of any three is nonempty. Let $Y = X_1 \cup \cdots \cup X_n$. By the inductive hypothesis, Y is simply connected. We have

- Y, X_{n+1} are both path-connected open sets such that the intersection is nonempty.
- We claim that $Y \cap X_{n+1}$ is path connected. Let $a, b \in Y \cap X_{n+1}$. Then $a \in X_i \cap X_{n+1}$ and $b \in X_j \cap X_{n+1}$ for some $i, j \in \{1, \cdots, n\}$. Then $X_i \cap X_j \cap X_{n+1}$ is nonempty, and choose a point y in the intersection. Then a and y can be joined by a path because $X_i \cap X_{n+1}$ is convex. Similarly, b and y can be joined by a path. Therefore, a and b can be joined by a path, so $Y \cap X_{n+1}$ is path connected.

By the van Kampen theorem, $\pi_1(X) \cong \pi_1(Y) * \pi_1(X_{n+1})/N$ for some normal subgroup N. However, it does not matter what N is because $\pi_1(Y)$ and $\pi_1(X_{n+1})$ are both trivial. Thus $\pi_1(X) = 1$. □

Exercise. (Exercise 1, Section 1.3) For a covering space $p : \tilde{X} \to X$ and a subspace $A \subset X$, let $\tilde{A} = p^{-1}(A)$. Show that the restriction $p : \tilde{A} \to A$ is a covering space.

Proof. Let $x \in A$. Then x has a neighborhood in X such that $p^{-1}(U)$ is a union of disjoint open sets $\{V_\alpha\}$ each of which is mapped homeomorphically onto U by p.

$U \cap A$ is a neighborhood of x in A. Then $p^{-1}(U \cap A) = p^{-1}(U) \cap p^{-1}(A) = \coprod (V_\alpha \cap \tilde{A})$. Each $V_\alpha \cap \tilde{A}$ is open in $\tilde{A}$. Moreover, $V_\alpha \cap \tilde{A}$ is mapped homeomorphically onto $U \cap A$ by p because

- p is injective when restricted to V_α. Restricting it further to $V_\alpha \cap \tilde{A}$ gives an injection.
- Let $y \in U \cap A$. Then $y \in U$, so there exists exactly one point $\tilde{y}$ in V_α such that $p(\tilde{y}) = y$.
- A restriction of a continuous map is continuous.

Therefore, $p : \tilde{A} \to A$ is a covering map. □

2.2. **Homology.** Singular chains, chain complexes, homotopy invariance. Relationship between the first homology and the fundamental group, relative homology. The long exact sequence of relative homology. The Mayer-Vietoris sequence.

Exercise. (Exercise 1, Section 2.1) What familiar space is the quotient Δ-complex of a 2-simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$, preserving the ordering of vertices?

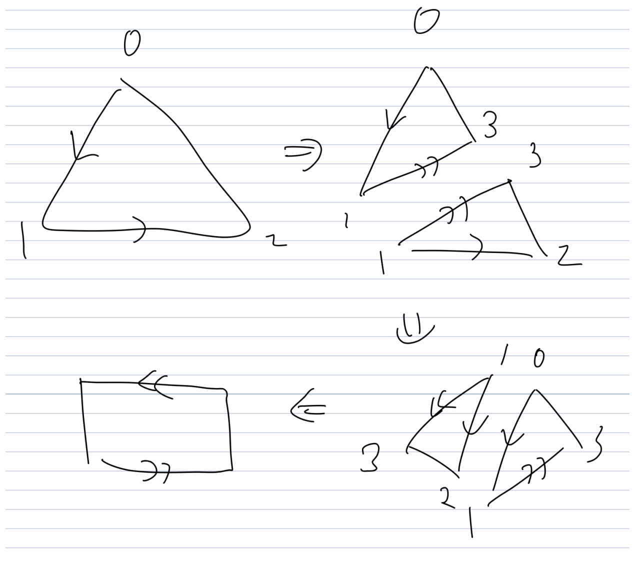

FIGURE 2. Ex 2-1-1

Proof. This can be seen easily if we cut the triangle in half and rotate one of the pieces as shown in 2. This space is a Mobius band. □