# QUALIFYING EXAM PREP

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Abstract. In order to prepare for the qualifying exam, I decided to solve problems from Hatcher and Dummit and Foote.

Contents



## 1. Algebra

1.1. **Groups.** The topics to cover: Elementary concepts (homomorphism, subgroup, coset, normal subgroup), solvable groups, commutator subgroup, Sylow theorems, structure of finitely generated Abelian groups. Symmetric, alternating, dihedral, and general linear groups.

1.2. **Rings.** The topics to cover: Commutative rings and ideals (principal, prime, maximal). Integral domains, Euclidean domains, principal ideal domains, polynomial rings, Eisenstein's irreducibility criterion, Chinese remainder theorem. Structure of finitely generated modules over a principal ideal domain.

1.2.1. *Chinese remainder theorem.*

**Exercise.** (Problem 1, Section 7.6) Let $R$ be a ring with identity $1 \neq 0$. An element $e \in R$ is called an idempotent if $e^2 = e$. Assume $e$ is an idempotent in $R$ and $er = re$ for all $r \in R$. Prove that $Re$ and $R(1-e)$ are two-sided ideals of $R$ and that $R \cong Re \times R(1-e)$. Show that $e$ and $1-e$ are identities for the subrings $Re$ and $R(1-e)$ respectively.

*Proof.* $Re$ is clearly nonempty and $re + r'e = (r + r')e \in Re$ for all $re, r'e \in Re$. For all $r' \in R$ and $re \in Re$, $r'(re) = (r'r)e \in Re$ and $(re)r' = r(er') = r(r'e) = (rr')e \in Re$. Thus $Re$ is a two-sided ideal of $R$. $(1-e)^2 = 1 - e - e + e^2 = 1 - e$, and, for every $r \in R$, $r(1-e) = r - re = r - er = (1-e)r$. Thus $R(1-e)$ is a two-sided ideal of $R$. Finally, $\phi : R \to R/Re \times R/R(1-e)$ defined by $x \mapsto (x + Re, x + R(1-e))$ is a ring homomorphism with $\ker(\phi) = Re \cap R(1-e)$ by the Chinese Remainder Theorem. Let

$r(1-e) \in \ker(\phi) = Re \cap R(1-e)$. Then $r(1-e)e = r(1-e)$ since $r(1-e) \in Re$. However, this implies $r(1-e)e = r(e-e^2) = r0 = 0$. Thus $\ker(\phi) = 0$, so $R \cong R/Re \times R/R(1-e)$. □

1.3. **Fields Extensions.** Finite, algebraic, separable, inseparable, transcendental, splitting field of a polynomial, primitive element theorem, algebraic closure. Finite fields.

1.4. **Galois Theory.** Finite Galois extensions and the Galois correspondence between subgroups of the Galois group and sub-extensions. Solvable extensions and solving equations by radicals.

**Exercise.** (Exercise 4 (Chapter 14)) Prove that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic.

*Proof.* Suppose they are and let $\phi$ be a ring isomorphism. $\phi(\sqrt{2}) = a + b\sqrt{3}$ for some $a, b \in \mathbb{Q}$. This implies $\phi(2) = (a^2 + 3b^2) + 2ab\sqrt{3}$. On the other hand, $\phi(2) = \phi(1) + \phi(1) = 1 + 1 = 2$. Thus $2ab = 0$. If $a = 0$, then $3b^2 = 2$, but $\sqrt{2/3}$ is not rational. If $b = 0$, then $a^2 = 2$, but $\sqrt{2}$ is not rational. This is a contradiction, so such a homomorphism does not exist. □

## 2. Algebraic topology

2.1. **Fundamental group.** Computation of the fundamental group, van Kampen's theorem, covering spaces.

**Exercise.** (Exercise 8, Section 1.1) Does the Borsuk-Ulam theorem hold for the torus?

*Proof.* No. Consider the natural projection map of $S^1 \times S^1$ into $\mathbb{R}^2$. From Figure 1, it is

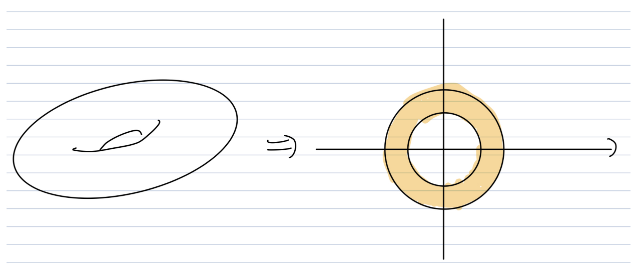

Figure 1. Ex 1-1-8

clear that $f(x, y) = -f(-x, -y)$. However, $f(x, y) \neq 0$ for any $(x, y) \in S^1 \times S^1$. Thus $f(x, y) \neq f(-x, -y)$ for all $(x, y)$. □

2.2. **Homology.** Singular chains, chain complexes, homotopy invariance. Relationship between the first homology and the fundamental group, relative homology. The long exact sequence of relative homology. The Mayer-Vietoris sequence.

**Exercise.** (Exercise 1, Section 2.1) What familiar space is the quotient $\Delta$-complex of a 2-simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$, preserving the ordering of vertices?

*Proof.* This can be seen easily if we cut the triangle in half and rotate one of the pieces as shown in 2. This space is a Mobius band. □

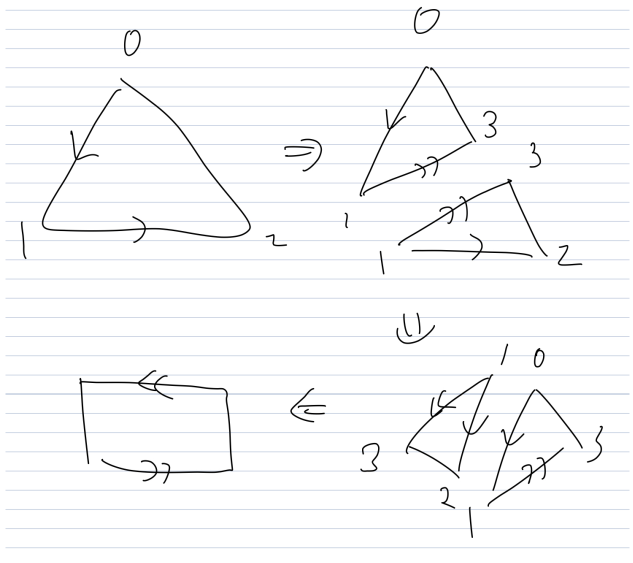


FIGURE 2. Ex 2-1-1