INTRODUCTION TO SMOOTH MANIFOLDS

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Contents

1. Chapter 1: Smooth Manifolds

1.1. Exercises.

Exercise 1.1. Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to *any* open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

Proof. It is clear that a "manifold" satisfying the open-ball or $\mathbb{R}^n$ definition satisfies the open-subset definition. Let M be a manifold satisfying the open-subset definition. Let $x \in M$ be given and let $U, \hat{U}, \phi$ be given according to the definition. Since $\hat{U}$ is open, there exists an open ball B such that $\phi(x) \in B \subset \hat{U}$. Restrict ϕ to $\phi^{-1}(B)$. Then $\phi^{-1}(B)$ is an open subset of M containing x, and $\phi\mid_{\phi^{-1}(B)}$ is a homeomorphism between $\phi^{-1}(B)$ and B. Thus M satisfies the open-ball definition.

$B(x, r) \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ by the map $(x_1 + a_1, \cdots, x_n + a_n) \mapsto (\frac{a_1}{r-a_1}, \cdots, \frac{a_n}{r-a_n})$ where $x = (x_1, \cdots, x_n)$ is the center of $B(x, r)$ and r is the radius. Since the composition of two homeomorphisms gives a homeomorphism, M also satisfies the $\mathbb{R}^n$ definition as well. □

Exercise 1.6. Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.

Proof. From the definition of π, it is easy to see that $\pi(B(x, r))$ is open in $\mathbb{RP}^n$ where $x \in S^n$ and $0 < r < 1$.

Let $[x], [y] \in \mathbb{RP}^n$ be given. Without loss of generality, assume $x, y \in S^n$. Let $r = \min\{|x - y|, |x + y|, 1\}/2$. Then $U_x = \pi(B(x, r)), U_y = \pi(B(y, r))$ contain $[x], [y]$, respectively. $\pi^{-1}(U_x), \pi^{-1}(U_y)$ are both open in $\mathbb{R}^{n+1} \setminus \{0\}$ which can be seen easily by writing down exactly which points belong to them, so U_x, U_y are both open in $\mathbb{RP}^n$. Then $\pi^{-1}(U_x \cap U_y) = \pi^{-1}(U_x) \cap \pi^{-1}(U_y) = \emptyset$, so $U_x \cap U_y = \emptyset$. Therefore, $\mathbb{RP}^n$ is Hausdorff.

Let $\mathcal{B} = \{\pi(B(x, 1/k)) \mid x \in \mathbb{Q}^{n+1} \cap S^n, k \in \{2, 3, 4, \cdots\}\}$. Then $\mathcal{B}$ is a countable collection of open sets whose union is $\mathbb{RP}^n$. Let $U \subset \mathbb{RP}^n$ be a nonempty open set. Let $[x] \in U$. Since π is a quotient map, $\pi^{-1}(U)$ is open. Moreover, $x \in \pi^{-1}(U)$. Without loss of generality, $x \in S^n$. Then $x \in B(x', 1/k) \subset \pi^{-1}(U)$ for some $B(x', 1/k) \in \mathcal{B}$. Then $[x] = \pi(x) \in \pi(B(x', 1/k)) \subset \pi(\pi^{-1}(U)) = U$. Therefore, $\mathcal{B}$ is a countable basis of $\mathbb{RP}^n$. ☐

Exercise 1.7. Show that $\mathbb{RP}^n$ is compact.

Proof. $\pi(S^n) = \mathbb{RP}^n$ and S^n is compact because it is a closed, bounded subset of $\mathbb{R}^{n+1}$. (Heine-Borel) Moreover, the image of a compact set under a continuous map is compact. (See A.45(a)) Thus $\mathbb{RP}^n$ is compact. ☐

Exercise 1.14. Suppose $\mathcal{X}$ is a locally finite collection of subsets of a topological space M.

(a) The collection $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.
(b) $\overline{\bigcup_{X\in\mathcal{X}} X} = \bigcup_{X\in\mathcal{X}} \overline{X}$.

Proof.

(a) Let $p \in M$. Then there exists an open set U containing x such that there are only finitely many $X \in \mathcal{X}$ such that $U \cap X \neq \emptyset$. Let $X \in \mathcal{X}$.
  - If $U \cap X \neq \emptyset$, then $U \cap \overline{X} \supset U \cap X \neq \emptyset$.
  - If $U \cap X = \emptyset$, then U^c is closed, so $\overline{X} \subset U^c$. In other words, $U \cap \overline{X} = \emptyset$.

  This shows that the number of $X \in \mathcal{X}$ that intersects U and the number of $\overline{X} \in \mathcal{X}$ that intersects U are the same. Therefore, $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.
(b) Since the closure of a set is defined to be the intersection of all closed sets containing it, $\bigcup_{X\in\mathcal{X}} \overline{X} \subset \overline{\bigcup_{X\in\mathcal{X}} X}$. Let $x \notin \bigcup_{X\in\mathcal{X}} \overline{X}$. Then there exists a neighborhood U of x such that U intersects only finitely many $X \in \mathcal{X}$. Let $X_1, \cdots, X_n$ denote them. By the same argument as part (a), $\overline{X_1}, \cdots, \overline{X_n}$ are the only elements in $\{\overline{X} \mid X \in \mathcal{X}\}$ that U intersects. Since $x \notin \overline{X_i}$ for each $i = 1, \cdots, n$, $U^c \cup \overline{X_1} \cup \cdots \cup \overline{X_n}$ is a closed set which contains all $X \in \mathcal{X}$ but does not contain x. In other words, $x \notin \overline{\bigcup_{X\in\mathcal{X}} X}$.

☐

Exercise 1.18. Let M be a topological manifold. Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Proof. Let $\mathcal{A}, \mathcal{A}'$ be two smooth atlases.

Suppose that they determine the same smooth structure $\mathcal{B}$. Then $\mathcal{A} \cup \mathcal{A}' \subset \mathcal{B}$, so $\mathcal{A} \cup \mathcal{A}'$ must be a smooth atlas. By Proposition 1.17(a), $\mathcal{A} \cup \mathcal{A}'$ determines a unique smooth structure, but it must be $\mathcal{B}$ because $\mathcal{B}$ contains the union.

On the other hand, suppose that their union is a smooth atlas. Let $\mathcal{B}$ be the smooth structure that the union determines. Such $\mathcal{B}$ must exist by Proposition 1.17(a). By the same proposition, $\mathcal{A}, \mathcal{A}'$ must determine the unique smooth structures. However, they must be $\mathcal{B}$ because $\mathcal{B}$ contains both $\mathcal{A}$ and $\mathcal{A}'$. ☐

Exercise 1.20. Every smooth manifold has a countable basis of regular coordinate balls.

Proof. Let M be an n-dimensional smooth manifold. We consider the special case that there exists a single chart (ϕ, U) with $U = M$. Let $x \in \hat{U}$ with rational coordinates. Then there exists $s > 0$ such that $B(x, s) \subset \hat{U}$. For each rational number $r \in (0, s)$, we consider the chart $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r)))$.

Let $\mathcal{B}$ be the collection of all such charts for each $x \in \hat{U}$ and r. We claim that $\mathcal{B}$ is a smooth atlas.

- Let $p \in M$. Then $\phi(p) \in \hat{U}$. Since $\hat{U}$ is open, $\phi(p) \in B(x, r) \subset \hat{U}$ for some x with rational coordinates and a positive rational number r. Then $p \in \phi^{-1}(B(x, r))$, so the union of coordinate domains covers M. In other words, $\mathcal{B}$ is an atlas.
- Let $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r))), (p \mapsto \phi(p) - x', \phi^{-1}(B(x', r'))) \in \mathcal{B}$ be given. Suppose $\phi^{-1}(B(x, r)) \cap \phi^{-1}(B(x', r')) \neq \emptyset$. Let ψ, ψ' denote the coordinate maps. Then $\psi' \circ \psi^{-1}$ is a composition of ϕ, ϕ^{-1} and translation maps, so it is smooth.

Therefore, $\mathcal{B}$ is a smooth atlas.

Since $\mathcal{B}$ is a smooth atlas, there exists a smooth structure $\mathcal{A}$ on M containing $\mathcal{B}$ by Proposition 1.17(a). We claim that $\mathcal{B}$, a subset of the smooth structure $\mathcal{A}$, is a countable basis of regular coordinate balls.

- $\mathcal{B}$ is a countable collection because $x \in \mathbb{Q}^n$ and $r \in \mathbb{Q}$.
- Let $(p \mapsto \phi(p)-x, \phi^{-1}(B(x,r))) \in \mathcal{B}$ be given. Then there exists a chart $(p \mapsto \phi(p)-x, \phi^{-1}(B(x,r')))$ in $\mathcal{B}$ with $r' > r$. Let $B = \phi^{-1}(B(x,r)), B' = \phi^{-1}(B(x,r'))$. Let ψ denote the map $p \mapsto \phi(p) - x$. Then $\psi(B) = B(0,r)$ and $\psi(B') = B(0,r')$, respectively. Moreover, $\psi(\overline{B}) = \overline{B(0,r)}$ because ψ is a homeomorphism.

Now let M be an arbitrary smooth n-manifold. By definition, each point of M is in the domain of a chart. By Proposition A.16, M is covered by countably many charts $\{(U_i, \phi_i)\}$. By the previous argument, each U_i has a countable basis of regular coordinate balls. Each regular coordinate ball in U_i is indeed a regular coordinate ball in M because $\overline{B}$ is a compact subset of M, which is Hausdorff, so $\overline{B}$ is closed. In other words, the closure of B in U_i is the same as the closure of B in M. □

Exercise 1.39. Let M be a topological n-manifold with boundary.

(a) $\operatorname{Int} M$ is an open subset of M and a topological n-manifold without boundary.
(b) ∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.
(c) M is a topological manifold if and only if $\partial M = \emptyset$.
(d) If $n = 0$, then $\partial M = \emptyset$ and M is a 0-manifold.

Proof.

(a) Let $x \in \operatorname{Int} M$. Let (ϕ, U) be an interior chart for x. Then $x \in U \subset \operatorname{Int} M$ because every point in U is in an interior chart (ϕ, U). A subspace of M must be Hausdorff and second-countable by Proposition A.17(g, i), so $\operatorname{Int} M$ is a second-countable, Hausdorff space in which every point has a neighborhood homeomorphic to an open subset in $\mathbb{R}^n$. Thus $\operatorname{Int} M$ is an n-manifold without boundary.
(b) Since $\partial M = M \setminus \operatorname{Int} M$ and $\operatorname{Int} M$ is open in M, ∂M is closed in M. Let $x \in \partial M$. Let (ϕ, U) be a boundary chart of x. If a point $y \in U$ gets mapped into $\operatorname{Int} \mathbb{H}^n$, then it is certainly an interior point. Thus $\phi(U \cap \partial M) \subset \partial \mathbb{H}^n$. Then $\pi_{n-1} \circ \phi$ is a homeomorphism that maps $U \cap \partial M$ into an open subset of $\mathbb{R}^{n-1}$ where $\pi_{n-1} : (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_{n-1})$.
(c) If ∂M is empty, then $M = \operatorname{Int} M$, so (a) implies that M is an n-dimensional manifold. If M is a topological manifold, every point is an interior point. Since a point cannot be both an interior point and a boundary point, ∂M is empty.
(d) If $n = 0$, then $\partial \mathbb{H}^0 = \emptyset$. Thus, the condition that $\phi(U) \cap \partial \mathbb{H}^n \neq \emptyset$ can never be satisfied, so there cannot be any boundary point.

□

Exercise 1.41. Let M be a topological manifold with boundary.

(a) M has a countable basis of precompact coordinate balls and half-balls.
(b) M is locally compact.
(c) M is paracompact.
(d) M is locally path-connected.
(e) M has countably many components, each of which is an open subset of M and a connected topological manifold with boundary.
(f) The fundamental group of M is countable.

Proof.

(a)
(b)
(c)
(d) Let $U \subset M$ be a nonempty open subset and choose $x \in U$. Then there exists a chart (V, ϕ) such that $x \in V$. Since $\phi(x)$ is a point in an open set $\phi(U \cap V)$, there exists $r > 0$ such that $B(\phi(x), r) \subset \phi(V)$. Then $N(x, U) = \phi^{-1}(B(\phi(x), r))$ is a path-connected neighborhood of x that is contained in $U \cap V \subset U$. Therefore, $\{N(x, U) \mid \text{open } U \subset M, x \in U\}$ forms a basis of M consisting of path-connected sets.

(e)
(f)

□

Exercise 1.44. Suppose M is a smooth n-manifold with boundary and U is an open subset of M. Prove the following statements:

(a) U is a topological n-manifold with boundary, and the atlas consisting of all smooth charts (V, ϕ) for M such that $V \subset U$ defines a smooth structure on U. With this topology and smooth structure, U is called an ***open submanifold with boundary***.
(b) If $U \subset \operatorname{Int} M$, then U is actually a smooth manifold (without boundary); in this case we call it an ***open submanifold of M***.
(c) $\operatorname{Int} M$ is an open submanifold of M (without boundary).

Proof. Let $\mathcal{T}$ denote the topology of M and $\mathcal{A}$ denote the smooth structure of M.

(a) The subspace topology on U is equivalent to $\mathcal{T}_U = \{V \in \mathcal{T} \mid V \subset U\}$ because U is open. By Proposition A.17(A.18(Proof of Proposition A.17)), U is Hausdorff and second-countable. For every point $p \in U$, there exists a $V \in \mathcal{T}$ with a homeomorphism $\phi : V \to \hat{V}$ where $\hat{V}$ is an open subset of $\mathbb{R}^n$ (or $\mathbb{H}^n$) Since $U \cap V$ is an open subset of V, ϕ restricted to $U \cap V$ is a homeomorphism between $U \cap V$ and $\phi(U \cap V)$, which is an open subset of $\mathbb{R}^n$ (or $\mathbb{H}^n$). Therefore, U is a topological n-manifold with boundary.
Let $\mathcal{A}_U = \{(\phi, V) \in \mathcal{A} \mid V \subset U\}$. Then $\mathcal{A}_U$ is clearly a collection of charts on U whose union covers U. Moreover, any two charts in $\mathcal{A}_U$ are clearly smoothly compatible. Let (ϕ, V) be a chart on U that is smoothly compatible with every chart in $\mathcal{A}_U$. Let $(\psi, W) \in \mathcal{A}$. Then $(\psi_{W\cap U}, W \cap U)$ is a chart on M and it must be smoothly compatible with every chart in $\mathcal{A}$. Therefore, $(\psi_{W\cap U}, W \cap U) \in \mathcal{A}$, so it must belong to $\mathcal{A}_U$. This implies that (ϕ, V) and $(\psi_{W\cap U}, W \cap U)$ are smoothly compatible. Since $V \subset W \cap U$, this implies that (ϕ, V) and (ψ, W) are smoothly compatible.
Thus (ϕ, V) is smoothly compatible with every chart in $\mathcal{A}$, so $(\phi, V) \in \mathcal{A}$. This implies that (ϕ, V) is in $\mathcal{A}_U$, so $\mathcal{A}_U$ is indeed a maximal smooth atlas.
(b) Let $p \in U$. Then $p \in \operatorname{Int} M$, so there exists $(\phi, V) \in \mathcal{A}$ such that $p \in V$ and $\phi(V)$ is open in $\mathbb{R}^n$. Then $(\phi|_{V\cap U}, V \cap U)$ is a chart that is smoothly compatible with every chart in $\mathcal{A}$, so $(\phi|_{V\cap U}, V \cap U) \in \mathcal{A}$. Thus it must be in $\mathcal{A}_U$, so $p \in U$ is an interior point of U. Therefore, U is a manifold without boundary.
(c) By 1.39, $\operatorname{Int} M$ is an open subset of M. By (b), $\operatorname{Int} M$ is an open submanifold of M without boundary.

□

1.2. **Problems.**

Problem 1-2. Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

Proof. Let I denote an uncountable index set and $X = \coprod_{\alpha\in I} \mathbb{R}$. Let $(x, \alpha_0) \in X$. Define $U = \coprod_{\alpha\in I} U_\alpha$ where $U_{\alpha_0} = \mathbb{R}$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_0$. Then U is an open neighborhood of (x, α_0) that is clearly homeomorhpic to $\mathbb{R}$. Thus X is locally Euclidean.

Let $(x_1, \alpha_1) \neq (x_2, \alpha_2) \in X$. If $\alpha_1 \neq \alpha_2$, then open neighborhoods of x_1 and x_2 formed in the same way as above separate the two points. Suppose $\alpha_1 = \alpha_2$. Without loss of generality, $x_1 < x_2$. Define $U = \coprod_{\alpha\in I} U_\alpha$ where $U_{\alpha_1} = (-\infty, (x_1 + x_2)/2)$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_1$. Similarly, define $V = \coprod_{\alpha\in I} U_\alpha$ where $U_{\alpha_1} = ((x_1 + x_2)/2, \infty)$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_2$. Then such U and V separate the two points. Therefore, X is Hausdorff.

Let $\mathcal{B}$ be a basis of X. For each $\alpha_0 \in I$, let $U_{\alpha_0} = \coprod_{\alpha\in I} U_\alpha$ where $U_{\alpha_0} = \mathbb{R}$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_0$. Then for each α_0, there must exist $B_{\alpha_0} \in \mathcal{B}$ such that $(0, \alpha_0) \in B_{\alpha_0} \subset U_{\alpha_0}$. Clearly, $B_\alpha \neq B_\beta$ if $\alpha \neq \beta$. Therefore, the cardinality of $\mathcal{B}$ is greater than or equal to that of I. Hence, X is not second-countable. □

Problem 1-7. Let N denote the **north pole** $(0, \cdots, 0, 1) \in S^n \subset \mathbb{R}^{n+1}$, and let S denote the **south pole** $(0, \cdots, 0, -1)$. Define the **stereographic projection** $\sigma : S^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1, \cdots, x^{n+1}) = \frac{(x^1, \cdots, x^n)}{1 - x^{n+1}}$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus \{S\}$.

(a) For any $x \in S^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$. Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace.

(b) Show that σ is bijective, and

$$\sigma^{-1}(u^1, \cdots, u^n) = \frac{(2u^1, \cdots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

(c) Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(S^n \setminus \{N\}, \sigma)$ and $(S^n \setminus \{S\}, \tilde{\sigma})$ defines a smooth structure on S^n.

(d) Show that this smooth structure is the same as the one defined in Example 1.31.

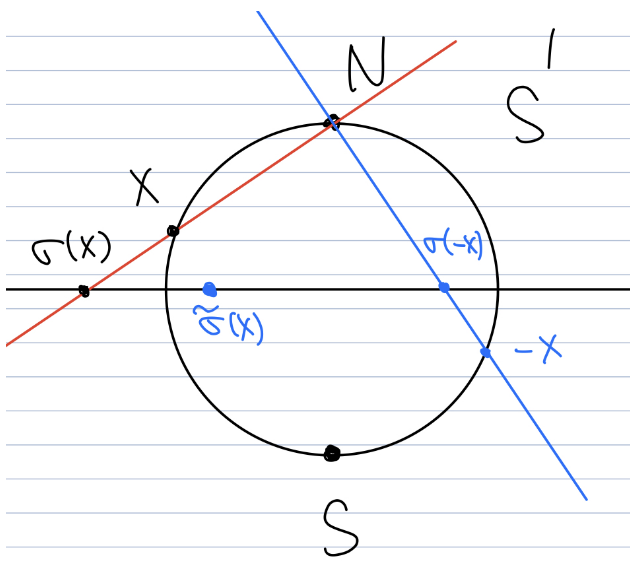

FIGURE 1. Problem 1-7

Proof.

(a) This is trivial from a basic trigonometry argument using the triangles $N, (0, \cdots, 0, x^{n+1}), (x^1, \cdots, x^{n+1})$ and $N, (0, \cdots, 0), \sigma(x^1, \cdots, x^{n+1})$.

(b) $\sigma \circ \sigma^{-1}$ and $\sigma^{-1} \circ \sigma$ are both the identity maps, so σ is bijective and σ^{-1} is its inverse.

(c) Computation shows that $\tilde{\sigma} \circ \sigma^{-1} : S^n \setminus \{N, S\} \to S^n \setminus \{N, S\}$ sends $(u^1, \cdots, u^n)$ to $(u^1, \cdots, u^n)/|u|^2$. As $|u| \neq 0$ in the domain, this map is well-defined and clearly smooth. By Proposition 1.17(a), these two charts determine a unique smooth structure.

(d) $\phi_i, \sigma, \tilde{\sigma}$ are all smooth functions of subsets of Euclidean spaces, so transition maps are always smooth. By Proposition 1.17(b), the smooth structure determined by $\sigma, \tilde{\sigma}$ is the same as the one defined in Example 1.31.

$\square$

Problem 1-8. By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle S^1 as a subset of the complex plane. An angle function on a subset $U \subset S^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$.

Show that there exists an angle function θ on an open subset $U \subset S^1$ if and only if $U \neq S^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for S^1 with its standard smooth structure.

Proof. First, we will consider the special case when $U = S^1 \backslash \{e^{it}\}$ for some $t \in \mathbb{R}$. The map $\phi : (t, t+2\pi) \to U$ defined by $\theta \mapsto e^{i\theta}$ is a bijective function. Therefore, by taking the inverse of ϕ, which is clearly continuous, we obtain a desired angle function. The case of an arbitrary proper open subset of U is the same as this special case because we simply need to restrict the domain of the map obtained above. On the other hand, suppose $U = S^1$. Suppose there exists an angle function f on U. Define $g : S^1 \to \mathbb{R}$ by $g(z) = f(z) - f(-z)$.

- $g(1) \neq 0$ because $g(1) \neq 0 \implies f(1) = f(-1)$, which is clearly impossible.
- $g(1) > 0$ implies that $g(-1) < 0$. By the intermediate value theorem, $g(z) = 0$ for some $z \in S^1$. This is a contradiction.
- If $g(1) < 0$, $g(-1) > 0$, and we obtain a contradiction in the same manner.

Therefore, such an f cannot exist. Hence, an angle function exists if and only if U is an proper open subset of S^1.

Let $(U_i^\pm, \phi_i^\pm)$ and (U, ϕ) be given where ϕ maps U into $(t, t+2\pi)$ for some $t \in \mathbb{R}$. We will show that they are smoothly compatible. Let $V = U \cap U_i^\pm$. The map $\phi_i^\pm \circ \phi^{-1} : \phi(V) \to \phi_i^\pm(V)$ is $\phi_i^\pm \circ \exp$. Since it is a composition of a projection map with a smooth map, this is smooth. Therefore, (U, ϕ) is indeed a coordinate chart for S^1 with its standard smooth structure. □

Problem 1-12(Proof of Proposition 1.45). Suppose $M_1, \cdots, M_k$ are smooth manifolds and N is a smooth manifold with boundary. Then $M_1 \times \cdots \times M_k \times N$ is a smooth manifold with boundary, and $\partial(M_1 \times \cdots \times M_k \times N) = M_1 \times \cdots \times M_k \times \partial N$.

Proof. By Example 1.34, $M_1 \times \cdots \times M_k$ is a smooth manifold. Thus it suffices to show that $M \times N$ is a smooth manifold with boundary if M is a smooth manifold and N is a smooth manifold with boundary. Let m, n be the dimensions of M, N.

First, we show that $M \times N$ is a topological manifold with boundary and $\partial(M \times N) = M \times \partial N$. Let $(p, q) \in M \times N$. Then $p \in M$, so there exists a chart (U, ϕ) such that $p \in U$ and $\hat{U} = \phi(U) \subset \mathbb{R}^m$.

- Suppose $q \in \operatorname{Int} N$. Then there exists a chart (V, ψ) such that $\hat{V} = \psi(V) \subset \mathbb{R}^n$. $\phi \times \psi$ is a homeomorphism between $U \times V$ and $\hat{U} \times \hat{V} \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. Thus $(U \times V, \phi \times \psi)$ is a chart for (p, q).
- Suppose $q \in \operatorname{bd} N$. Then there exists a chart (V, ψ) such that $\hat{V} = \psi(V) \subset \mathbb{H}^n$ and $\psi(q) \in \partial\mathbb{H}^n$. $\phi \times \psi$ is a homeomorphism between $U \times V$ and $\hat{U} \times \hat{V} \subset \mathbb{R}^m \times \mathbb{H}^n = \mathbb{H}^{m+n}$. Moreover, $(\phi \times \psi)(p, q) = (\phi(p), \psi(q)) \in \mathbb{R}^m \times \mathbb{H}^n = \mathbb{H}^{m+n}$. Thus $(U \times V, \phi \times \psi)$ is a boundary chart for (p, q).

Therefore, $M \times N$ is a topological manifold with boundary and $\partial(M \times N) = M \times (\partial N)$.

Let $\mathcal{A}_M, \mathcal{A}_N$ be the smooth structures of M, N. Define $\mathcal{A}_{M \times N} = \{(U \times V, \phi \times \psi) \mid (U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N\}$. Then $\mathcal{A}_{M \times N}$ is an atlas because we showed earlier that each $(U \times V, \phi \times \psi)$ is a chart. Let $(U_1 \times V_1, \phi_1 \times \psi_1), (U_2 \times V_2, \phi_2 \times \psi_2) \in \mathcal{A}_{M \times N}$. Then $(\phi_2 \times \psi_2) \circ (\phi_1 \times \psi_1)^{-1} = (\phi_2 \circ \phi_1^{-1}) \times (\psi_2 \circ \psi_1^{-1})$ is a smooth map from $(\phi_1 \times \psi_1)(U_1 \times V_1)$ into $(\phi_2 \times \psi_2)(U_2 \times V_2)$. Thus every pair of charts in $\mathcal{A}_{M \times N}$ is smoothly compatible. In other words, $\mathcal{A}_{M \times N}$ is a smooth atlas.

On the other hand, $\mathcal{A}_{M \times N}$ must be maximal because the restriction of any smoothly compatible chart to M, N gives a smoothly compatible chart, which must belong to $\mathcal{A}_M, \mathcal{A}_N$, respectively. Thus $M \times N$ is a smooth manifold with boundary. □

2. Chapter 2: Smooth Maps

2.1. Exercises.

Exercise 2.1. Let M be a smooth manifold with or without boundary. Show that pointwise multiplication turns $C^\infty(M)$ into a commutative ring and a commutative and associative algebra over $\mathbb{R}$.

Proof.

- The constant map $f(p) = 0$ is clearly in $C^\infty(M)$ and it is the additive identity.
- The constant map $f(p) = 1$ is clearly in $C^\infty(M)$ and it is the multiplicative identity.

- Let $f \in C^\infty(M), g \in C^\infty(M)$. Let $p \in M$ and (ϕ, U) be a smooth chart for p. Then $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are both smooth(Exercise 2.3), real-valued maps defined on an open subset of $\mathbb{R}^n$. Thus $f+g$ is in $C^\infty(M)$ Moreover, $f+g = g+f$ because addition in $\mathbb{R}$ is commutative.
- Let $f, g, h \in C^\infty(M)$. Let $p \in M$ and (ϕ, U) be a smooth chart for p. Then $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are both smooth(Exercise 2.3), real-valued maps defined on an open subset of $\mathbb{R}^n$. Therefore, fg is in $C^\infty(M)$ Moreover, $fg = gf$ and $(fg)h = f(gh)$ because multiplication in $\mathbb{R}$ is commutative and associative.
- Let $c \in \mathbb{R}, f \in C^\infty(M)$. Then cf can be seen as fg where g is the constant function whose value is c. As shown above, $cf \in C^\infty(M)$.

□

Exercise 2.2. Let U be an open submanifold of $\mathbb{R}^n$ with its standard smooth manifold structure. Show that a function $f : U \to \mathbb{R}^k$ is smooth in the sense just defined if and only if it is smooth in the sense of ordinary calculus. Do the same for an open submanifold with boundary in $\mathbb{H}^n$.

Proof. f is smooth in the sense just defined if and only if $f \circ \mathrm{Id}^{-1}$ is smooth in the sense of ordinary calculus. Since $f \circ \mathrm{Id}^{-1} = f$, $f \circ \mathrm{Id}^{-1}$ is smooth in the sense of ordinary calculus if and only if f is smooth in the sense of ordinary calculus. □

Exercise 2.3. Let M be a smooth manifold with or without boundary, and suppose $f : M \to \mathbb{R}^k$ is a smooth function. Show that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$ is smooth for every smooth chart (U, ϕ) for M.

Proof. Let $\phi(x) \in \phi(U)$. Since f is smooth, there exists (V, ψ) such that $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}^k$ is smooth and $x \in V$. Let $W = U \cap V$. Then $f \circ \psi^{-1} : \psi(W) \to \mathbb{R}^k$ is smooth and $\psi \circ \phi^{-1} : \phi(W) \to \psi(W)$ is a diffeomorphism where $\phi(W)$ is a neighborhood of W. Then the restriction of $f \circ \psi^{-1}$ to $\phi(W)$ is identical to $(f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})$. Since he composition of a smooth function is smooth, $f \circ \psi^{-1}$ is smooth. □

Exercise 2.7(Prove Proposition 2.5). Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a map. Then F is smooth if and only if either of the following conditions is satisfied:

(a) For every $p \in M$, there exist smooth charts (U, ϕ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \phi^{-1}$ is smooth from $\phi(U \cap F^{-1}(V))$ to $\psi(V)$.

(b) F is continuous and there exist smooth atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is a smooth map from $\phi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

Proof. Let $\mathcal{A}_M$ and $\mathcal{A}_N$ be smooth structures of M and N. Suppose F is smooth. By Proposition 2.4, F is continuous. For every $p \in M$ there exist coordinate charts (U_p, ϕ_p) containing p and (V_p, ψ_p) containing $F(p)$ such that $F(U_p) \subset V_p$ and $\psi_p \circ F_p \circ \phi_p^{-1}$ is smooth from $\phi_p(U_p)$ to $\psi_p(V_p)$. Then $\{(U_p, \phi_p) \mid p \in M\} \subset \mathcal{A}_M$ and $A_n\{(V_p, \psi_p) \mid p \in M\} \subset \mathcal{A}_N$ are smooth atlases. Moreover, for every (U_p, ϕ_p) and (V_q, ψ_q), $\psi_q \circ F \circ \phi_p^{-1}$ is a smooth map from $\phi_p(U_p \cap F^{-1}(V_q))$ to $\psi_q(V_q)$ because $\psi_q \circ F \circ \phi_p^{-1} = (\psi_q \circ \psi_p^{-1}) \circ (\psi_p \circ F \circ \phi_p^{-1})$ where $\psi_q \circ \psi_q^{-1}$ and $\psi_p \circ F \circ \phi_p^{-1}$ are smooth. Therefore, the definition implies (b).

(b) implies (a) because if F is continuous, $F^{-1}(V_\beta)$ is open in M for every β, so $U \cap F^{-1}(V)$ is open in M.

Finally, we show that (a) implies the definition. Suppose F satisfies (a). Let $p \in M$. Let $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ be smooth charts satisfying the properties described in (a). Let $U' = U \cap F^{-1}(V)$ and consider $(U', \phi\mid_{U'})$. Then $(U', \phi\mid_{U'}) \in \mathcal{A}_M$ because it must be smoothly compatible with any other smooth coordinate chart in $\mathcal{A}_M$. Moreover, $F(U') \subset V$ and $\psi \circ F \circ (\phi\mid_{U'})^{-1} : \phi(U') \to \psi(V)$ is smooth. Therefore, (a) implies the definition.

Hence, (a), (b) and the definition are all equivalent. □

Exercise 2.7(Proof of Proposition 2.6). Let M and N be smooth manifolds with or without boundary, and let $F : M \to N$ be a map.

(a) If every point $p \in M$ has a neighborhood U such that the restriction $F|_U$ is smooth, then F is smooth.

(b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

Proof. Let $\mathcal{A}_M, \mathcal{A}_N$ be smooth structures of M, N, respectively.

(a) Let $p \in M$. Let U be a neighborhood of p such that $F|_U$ is smooth. By 1.44, U is a smooth manifold with the induced smooth structure $\mathcal{A}_U = \{(V, \phi) \in \mathcal{A}_M \mid V \subset U\}$. Since $F|_U$ is smooth, there exist $(V, \phi) \in \mathcal{A}_U$ and $(W, \psi) \in \mathcal{A}_N$ such that:
- $F|_U(V) \subset W$.
- $\psi \circ F|_U \circ \phi^{-1} : \phi(V) \to \psi(W)$ is smooth.

Since $V \subset U$, $F(V) \subset W, \psi \circ F \circ \phi^{-1} : \phi(V) \to \psi(W)$ is smooth, and $(V, \phi) \in \mathcal{A}$. Therefore, F is smooth.

(b) Let $U \subset M$ be an open subset. By 1.44, U is a smooth manifold with the induced smooth structure $\mathcal{A}_U = \{(V, \phi) \in \mathcal{A}_M \mid V \subset U\}$. Let $p \in U$. Then $p \in F$, so there exist $(V, \phi) \in \mathcal{A}_M, (W, \psi) \in \mathcal{A}_N$ such that $F(V) \subset W$ and $\psi \circ F \circ \phi^{-1} : \phi(V) \to \psi(W)$ is smooth. Then $(V \cap U, \phi|_{V \cap U})$ is a chart that is smoothly compatible with every chart in $\mathcal{A}_M$. Therefore, $(V \cap U, \phi|_{V \cap U}) \in \mathcal{A}_M$. Moreover, $\phi|_{V \cap U}(V \cap U) \subset \phi(V) \subset W$ and $\psi \circ F \circ (\phi|_{V \cap U}(V \cap))^{-1}$ is clearly smooth. Therefore, $F|_U$ is smooth.

□

Exercise 2.9. Suppose $F : M \to N$ is a smooth map between smooth manifolds with or without boundary. Show that the coordinate representation of F with respect to *every* pair of smooth charts for M and N is smooth.

Proof. Let $(M, \mathcal{A}_M), (N, \mathcal{A}_N)$ be smooth manifolds with or without boundary. Let $F : M \to N$ be a smooth map. Let $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$ be given. We must show that $\hat{F} = \psi \circ F \circ \phi^{-1}$ is a smooth function from $\phi(U \cap F^{-1}(V))$ to $\psi(V)$. Let $\phi(p) \in \phi(U \cap F^{-1}(V))$. Then $p \in M$, so there exist $(U_0, \phi_0) \in \mathcal{A}_M$ and $(V_0, \psi_0) \in \mathcal{A}_N$ such that
- $p \in U_0 \subset U \cap F^{-1}(V)$;
- $\phi_0(U_0) \subset V_0$;
- $\psi_0 \circ F \circ \phi_0^{-1} : \phi_0(U_0) \to \psi(V_0)$ is smooth.

Then $\psi \circ F \circ \phi^{-1}|_{\phi(U_0)} = (\psi \circ \psi_0^{-1}) \circ (\psi_0 \circ F \circ \phi_0^{-1}) \circ (\phi_0 \circ \phi)$. Since the composition of smooth functions in Euclidean spaces is smooth, $\hat{F}$ is smooth. □

Exercise 2.11(Proof of Proposition 2.10). Let M, N and P be smooth manifolds with or without boundary.

(a) Every constant map $c : M \to N$ is smooth.
(b) The identity map of M is smooth.
(c) If $U \subset M$ is an open submanifold with or without boundary, then the inclusion map $U \to M$ is smooth.

Proof. Let $\mathcal{A}_M, \mathcal{A}_N, \mathcal{A}_P$ be smooth structures of M, N, P, respectively.

(a) F is clearly continuous. Moreover, for every $(U_\alpha, \phi_\alpha) \in \mathcal{A}_M, (V_\beta, \psi_\beta) \in \mathcal{A}_N$, $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is a constant map, so it is smooth. By (2.7(Prove Proposition 2.5)), F is smooth.
(b) Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}_M$ such that $p \in U$. Then $F(U) \subset U$ and $\phi \circ F \circ \phi^{-1} = \mathrm{Id}_U$, so it is smooth. Therefore, F is smooth.
(c) By 1.44, $\mathcal{A}_U = \{(V, \phi) \mid V \subset U\}$ is a smooth structure of U. Let $p \in U$. Then $p \in V$ for some $(V, \phi) \in \mathcal{A}_U$. Then $(V, \phi) \in \mathcal{A}_M$, trivially. Since $F(V) \subset V$ and $\phi \circ F \circ \phi^{-1}$ is simply the identity map on V, F is smooth.

□

Exercise 2.16(Proof of Proposition 2.15).

(a) Every composition of diffeomorphisms is a diffeomorphism.
(b) Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.
(c) Every diffeomorphism is a homeomorphism and an open map.
(d) The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.
(e) "Diffeomorphic" is an equivalence relation on the class of all smooth manifolds with or without boundary.

Exercise 2.16(Proof of Proposition 2.15). Let $(M, \mathcal{A}_M), (N, \mathcal{A}_N), (P, \mathcal{A}_P)$ be smooth manifolds with or without boundary, and let $F : M \to N, G : N \to P$ be diffeomorphisms.

(a) By Proposition 2.10(d), $G \circ F$ and $F^{-1} \circ G^{-1}$ are smooth. Then $(G \circ F) \circ (F^{-1} \circ G^{-1})$ and $(F^{-1} \circ G^{-1}) \circ (G \circ F)$ are both the identity map on the corresponding space, so $F^{-1} \circ G^{-1}$ is the smooth inverse of $G \circ F$. Therefore, $G \circ F$ is a diffeomorphism.

(b) By Example 1.34, we know that $M_1 \times \cdots \times M_k$ and $N_1 \times \cdots \times N_k$ are both smooth manifolds. Let $\mathcal{A}_{M_i}, \mathcal{A}_{N_i}, \mathcal{A}_M$ and $\mathcal{A}_N$ denote the smooth manifold structures of $M_i, N_i, M_1 \times \cdots \times M_k, N_1 \times \cdots \times N_k$, respectively. Let a smooth map $F_i : M_i \to N_i$ be given for each i. Let $(p_1, \cdots, p_k) \in M_1 \times \cdots M_k$ be given. Then there exist $(U_i, \phi_i) \in \mathcal{A}_{M_i}$ and $(V_i, \psi_i) \in \mathcal{A}_{N_i}$ such that $p_i \in U_i, F_i(U_i) \subset V_i, \psi_i \circ F_i \circ \phi_i^{-1} : \phi_i(U_i) \to \psi_i(V_i)$ is smooth for each i. This implies that $(\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots (\psi_k \circ F_k \circ \phi_k^{-1}) = (\psi_1 \times \cdots \times \psi_k) \circ (F_1 \times \cdots \times F_k) \circ (\phi_1 \times \cdots \times \phi_k)^{-1}$ is smooth.

Therefore, $F_1 \times \cdots \times F_k$ is smooth. Using the exact same argument, we can conclude that $F_1^{-1} \times \cdots \times F_k^{-1}$ is smooth. Since $(F_1 \times \cdots \times F_k)^{-1} = F_1^{-1} \times \cdots \times F_k^{-1}$, $F_1 \times \cdots \times F_k$ is a diffeomorphism.

(c) Proposition 2.4 states that every smooth map is continuous. Thus F and F^{-1} are both continuous. Therefore, F is a homeomorphism and also an open map.

(d) Let $U \subset M$ be an open subset. By (2.7(Proof of Proposition 2.6)), $F|_U$ is smooth. Since F is a homeomorphism as shown in (c), $F(U)$ is an open subset of N. Therefore, $F^{-1}|_{F(U)}$ is smooth by (2.7(Proof of Proposition 2.6)). Clearly, $F|_U$ and $F^{-1}|_{F(U)}$ are the inverse of each other. Therefore, $F|_U$ is a diffeomorphism.

(e) By (2.11(Proof of Proposition 2.10)), the identity map on M is a diffeomorphism, so the reflexive property is satisfied. Moreover, $(F^{-1})^{-1} = F$, so the symmetric property is satisfied. By (a), the composition of two diffeomorphisms is a diffeomorphism, so the transitive property is satisfied. Therefore, "diffeomorphic" is an equivalence relation.

Exercise 2.19(Proof of Theorem 2.18). Suppose M and N are smooth manifolds with boundary and $F : M \to N$ is a diffeomorphism. Then $F(\partial M) = \partial N$, and F restricts to a diffeomorphism from $\operatorname{Int} M$ to $\operatorname{Int} N$.

Proof. Let $\mathcal{A}_M, \mathcal{A}_N$ denote the smooth structures of M, N, respectively. Let $p \in \partial M$. Then there exists a chart containing p that sends p to $\partial \mathbb{H}^n$. By Theorem 1.46, every chart containing p sends p to $\partial \mathbb{H}^n$.

Since F is smooth, there exist $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$ such that $F(U) \subset V$ and $\psi \circ F \circ \phi^{-1}$ is a smooth map from $\phi(U)$ to $\psi(V)$. F^{-1} is a homeomorphism by (2.16(Proof of Proposition 2.15)). Then $(\phi^{-1} \circ F^{-1}, F(U))$ is a coordinate chart around $F(p)$ because we obtain a homeomorphism by restricting the composition of two injective continuous maps to its image. Moreover, we claim that $(\phi^{-1} \circ F^{-1}, F(U))$ is smoothly compatible with every chart in $\mathcal{A}_N$. Let $(\psi_1, V_1) \in \mathcal{A}_N$ be given. Then $(\phi^{-1} \circ F^{-1}) \circ \psi_1^{-1} = (\phi^{-1} \circ F^{-1} \circ \psi^{-1}) \circ (\psi \circ \psi_1^{-1})$, and the composition of two smooth maps is smooth. Therefore, $(\phi^{-1} \circ F^{-1}, F(U)) \in \mathcal{A}_N$, and this chart contains $F(p)$ and sends $F(p)$ to $\partial \mathbb{H}^n$. In other words, $F(p) \in \partial N$.

Since F^{-1} is also smooth, $F^{-1}(\partial N) \subset \partial M$. $F^{-1}(\partial N) \subset \partial M \implies F(F^{-1}(\partial N)) \subset F(\partial M) \subset \partial N$. Since F is a bijection, $F(F^{-1}(\partial N)) = \partial N$. Therefore, $F(\partial M) = \partial N$.

This implies that $F(\operatorname{Int} M) = \operatorname{Int} N$. By (1.44(c)) and (2.16(Proof of Proposition 2.15)(d)), F is a diffeomorphism between $\operatorname{Int} M$ and $\operatorname{Int} N$. □

Problem 2-27. Give a counterexample to show that the conclusion of the extension lemma can be false if A is not closed.

Proof. Let $M = \mathbb{R}, A = (0, 1), f(x) = 1/x$. Then f is smooth on A, but $\lim_{x \to 0} f = \infty$, so f cannot be extended continuously. □

2.2. **Problems.**

Problem 2-1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, ϕ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \phi^{-1}$ is smooth as a map from $\phi(U \cap f^{-1}(V))$ to $\psi(V)$, but f is not smooth in the sense we defined in this chapter.

Proof. $\phi = \psi = \mathrm{Id}$ in this solution.

If $x \geq 0$, then let $U = \mathbb{R}, V = (0, \infty)$. Then $\phi(U \cap f^{-1}(V)) = [0, \infty)$. Thus $\psi \circ f \circ \phi^{-1} : [0, \infty) \to (0, \infty)$ is the constant map that sends every number to 1. Therefore, it is smooth.

If $x < 0$, then let $U = \mathbb{R}, V = (-\infty, 1)$. Then $\phi(U \cap f^{-1}(V)) = (-\infty, 0)$. Thus $\psi \circ f \circ \phi^{-1} : (-\infty, 0) \to (-\infty, 1)$ is the constant map that sends every number to 0. Therefore, it is smooth.

It might seem that we can apply (2.7(Prove Proposition 2.5)) to show that f is smooth, but (2.7(Prove Proposition 2.5)) requires that $U \cap f^{-1}(V)$ be open in M.

f maps the interval $(-1, 1)$ to $\{0, 1\}$. Since the image of a connected set under a continuous map must be connected, f cannot be continuous. By Proposition 2.4, f cannot be smooth. □

Problem 2-2(Proof of Proposition 2.12). Suppose $M_1, \cdots, M_k$ and N are smooth manifolds with or without boundary, such that at most one of $M_1, \cdots, M_k$ has nonempty boundary. For each i, let $\pi_i : M_1 \times \cdots \times M_k \to M_i$ denote the projection onto the M_i factor. A map $F : N \to M_1 \times \cdots \times M_k$ is smooth if and only if each of the component maps $F_i = \pi_i \circ F : N \to M_i$ is smooth.

Proof. Let $\mathcal{A}_{M_1}, \cdots, \mathcal{A}_{M_k}, \mathcal{A}_N$ be the smooth structures of $M_1, \cdots, M_k, N$. Let $d_1, \cdots, d_k$ denote the dimensions of $M_1, \cdots, M_n$, respectively. Let $d = \sum d_i$.

First, suppose that F is smooth. By (2.11(Proof of Proposition 2.10)), the composition of smooth maps is smooth. Thus it suffices to show that $\pi_i : M_1 \times \cdots \times M_k \to M_i$ is smooth for each i. We show that π_1 is smooth and the other cases can be shown similarly.

Let $(x_1, \cdots, x_k) \in M_1 \times \cdots \times M_k$. Then for each i, there exist $(U_i, \phi_i) \in \mathcal{A}_{M_i}$ and $(V_i, \psi_i) \in \mathcal{A}_{M_i}$ such that $x_i \in U_i$ and $\phi_i(U_i) \subset V_i$. Then we have $(\phi_1 \times \cdots \times \phi_k)(U_1 \times \cdots \times U_k) \subset V_1 \times \cdots \times V_k$ and the composition $\phi_i \circ \pi_1 \circ (\phi_1 \times \cdots \times \phi_k)^{-1}$ is the projection of the first d_1 coordinates from $\mathbb{R}^n$ onto $\mathbb{R}^{d_1}$. Therefore, it is clearly smooth, so π_1 is smooth.

Suppose each $F_i = \pi_i \circ F : N \to M_i$ is smooth. Let $p \in N$. Then for each i, there exist $(U_i, \phi_i) \in \mathcal{A}_N$ and $(V_i, \psi_i) \in \mathcal{A}_{M_i}$ such that $p \in U_i, F_i(U_i) \subset V_i$ and $\psi_k \circ F_i \circ \phi_i^{-1}$. Let $U = U_1 \cap \cdots \cap U_k$. U is a neighborhood of p and the restriction of ϕ_1 to U is a homeomorphism. Then we claim that $(\phi_1, U) \in \mathcal{A}_N$ and $(\psi_1 \times \cdots \times \psi_k, V_1 \times \cdots \times V_k) \in \mathcal{A}_{M_1 \times \cdots \times M_k}$ are charts that satisfy the necessary properties.

- $F(U) \subset V_1 \times \cdots \times V_k$.
- For each i, $\psi_i \circ F_i \circ \phi_1^{-1} = (\psi_i \circ F_i \circ \phi_i^{-1}) \circ (\phi_i \circ \phi_1^{-1}) : \phi_1(U) \to \psi_i(V_i)$ is smooth because the composition of two smooth maps is smooth. Thus $(\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots \times (\psi_k \circ F_k \circ \phi_1^{-1}) : \phi_1(U) \to \psi_1(V_1) \times \cdots \times \psi_k(V_k)$ is smooth. Moreover, $(\psi_1 \times \cdots \times \psi_k) \circ F \circ \phi_1^{-1} = (\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots \times (\psi_k \circ F_k \circ \phi_1^{-1})$.

Therefore, F is smooth. □

Problem 2-3. For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

(a) $p_n : S^1 \to S^1$ is the nth power map for $n \in \mathbb{Z}$, given in complex notation by $p_n(z) = z^n$.
(b) $\alpha : S^n \to S^n$ is the antipodal map $\alpha(x) = -x$.
(c) $F : S^3 \to S^2$ is given by $F(w, z) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w})$ where we think of S^3 as the subset $\{(w, z) : |w|^2 + |z|^2 = 1\}$ of $\mathbb{C}^2$.

Proof.

(a) Example 1.31 shows the existence of a smooth structure of S^1 and let $\mathcal{A}$ denote it. Let $p \in S^1$. Then there exist $(U_i^\pm, \phi_i^\pm), (U_j^\pm, \phi_j^\pm) \in \mathcal{A}$ around $p, p_n(p)$, respectively. Then the composition $\phi_j^\pm \circ f \circ (\phi_i^\pm)^{-1}$ is equal to one of $\cos(n(\arccos(x))), \sin(n(\arcsin(x))), \cos(n(\arcsin(x))), \sin(n(\arccos(x)))$, all of which are clearly smooth. By Proposition 2.5(a), p_n is smooth.
(b) Example 1.31 shows the existence of a smooth structure of S^n and let $\mathcal{A}$ denote it. Let $p \in S^1$. Then there exists a chart $(U_i^\pm, \phi_i^\pm)$ in $\mathcal{A}$ around p. Then $(U_i^\mp, \phi_i^\mp)$ is a chart containing $\alpha(p)$ with $\alpha(U_i^\pm) \subset U_i^\mp$. Then $\phi_i^\mp \circ \alpha \circ \phi_i^\pm$ is the map $x \mapsto -x$, which is clearly smooth.

(c) Let $z = a + bi, w = c + di$. $z\overline{w} = ac + bd + i(bc - ad)$ and $w\overline{z} = (ac + bd) - i(bc - ad)$. Then $z\overline{w} + w\overline{z} = 2(ac + bd) = 2\operatorname{Re}(z\overline{w})$ and $i(w\overline{z} - z\overline{w}) = 2\operatorname{Im}(z\overline{w})$.

$$\begin{aligned}(2\operatorname{Re}(z\overline{w}))^2 + (2\operatorname{Im}(z\overline{w}))^2 + (|z|^2 - |w|^2)^2 &= 4|z\overline{w}|^2 + (|z|^2 - |w|^2)^2 \\ &= 4|z|^2|\overline{w}|^2 + (|z|^2 - |w|^2)^2 \\ &= (|z|^2 + |w|^2)^2 \\ &= 1.\end{aligned}$$

Therefore, F indeed maps S^3 into S^2. Moreover, this map is continuous. Let $(z = a+bi, w = c+di) \in S^3$ be given. Suppose that (U_4^+, ϕ_4^+) and (V_3^+, ψ_3^+) are charts containing (z, w) and $F(z, w)$. Then $\psi_3^+ \circ F \circ \phi_4^+ : (a, b, c) \mapsto (2u, 2v)$ where $u + iv = (a + bi)(c - \sqrt{1 - a^2 - b^2 - c^2}i)$ which is a smooth map from $\phi_4^+(U_4^+) \subset \mathbb{R}^3$ into $\mathbb{R}^2$. Other cases are similar, and thus F is smooth by Proposition 2.5(b).

□

Problem 2-5. Let $\mathbb{R}$ be the real line with its standard smooth structure, and let $\tilde{R}$ denote the same topological manifold with the smooth structure defined in Example 1.23. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is smooth in the usual sense.

(a) Show that f is also smooth as a map from $\mathbb{R}$ to $\tilde{\mathbb{R}}$.
(b) Show that f is smooth as a map from $\tilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3.

Proof.

(a) The "$\psi \circ f \circ \phi^{-1}$" is simply f^3, which is a smooth map from $\mathbb{R}$ to $\mathbb{R}$. Thus $f : \tilde{\mathbb{R}} \to \mathbb{R}$ is smooth.
(b) Solve this!

□

Problem 2-6. Let $P : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Show that the map $\tilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\tilde{P}([x]) = [P(x)]$ is well defined and smooth.

Proof. Let $P_1, \cdots, P_{k+1}$ denote the component functions of P.

Suppose $[x_1 : \cdots : x_{n+1}] = [y_1 : \cdots : y_{n+1}]$. Then there exists $\lambda \neq 0$ such that $(y_1, \cdots, y_{n+1}) = (\lambda x_1, \cdots, \lambda x_{n+1})$. $P(y_1, \cdots, y_{n+1}) = P(\lambda x_1, \cdots, \lambda x_{n+1}) = \lambda^d P(x_1, \cdots, x_{n+1})$. Since $\lambda^d \neq 0$, $[P(y_1, \cdots, y_{n+1})] = [P(x_1, \cdots, x_{n+1})]$. Therefore, $\tilde{P}$ is well-defined.

Let $\tilde{p} = [p_1 : \cdots : p_{n+1}] \in \mathbb{RP}^n$ be given. Without loss of generality, assume $p_{n+1} \neq 0$. Consider the chart (U, ψ_{n+1}) with $U = \{[x_1 : \cdots : x_{n+1}] \mid x_{n+1} \neq 0\}$. Let $q_i = P_i(p_1, \cdots, p_{n+1})$. Without loss of generality, assume $q_{k+1} \neq 0$. Then $\tilde{P}(\tilde{p})$ is contained in $V = \{[y_1 : \cdots : y_{k+1}] \mid y_{k+1} \neq 0\}$. Since P is smooth, there exists $0 < \delta < |x_{n+1}|$ such that $|(x_1, \cdots, x_{n+1}) - (p_1, \cdots, p_{n+1})| < \delta$ implies $P_{k+1}(x_1, \cdots, x_{n+1}) \neq 0$. Then $[p_1 : \cdots : p_{n+1}] \in \pi(B(p_1, \cdots, p_{n+1})) \subset U \cap F^{-1}(V)$. Therefore, $U \cap F^{-1}(V)$ is open in $\mathbb{RP}^n$.

Finally the composition map $\psi_{k+1} \cdot \tilde{P} \cdot \phi_{n+1}^{-1}$ sends $(x_1/x_{n+1}, \cdots, x_n/x_{n+1})$ to $(y_1/y_{k+1}, \cdots, y_k/y_{k+1})$ where $y_i = P_i(x_1, \cdots, x_{n+1})$. In other words, $(x_1, \cdots, x_n) \mapsto (y_1/y_{k+1}, \cdots, y_k/y_{k+1})$ where $y_i = P_i(x_1, \cdots, x_n, 1)$. Since each P_i is smooth, this map must be smooth as well. By (2.7(Prove Proposition 2.5)), $\tilde{P}$ is smooth. □

Problem 2-7. Let M be a nonempty smooth n-manifold with or without boundary, and suppose $n \geq 1$. Show that the vector space $C^\infty(M)$ is infinite-dimensional.

Proof. Let $k \in \mathbb{N}$ be given. Let $p \in M$ be chosen arbitrarily. Let (U, ϕ) be a smooth chart containing p. Then $\hat{U} = \phi(U)$ is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. In each case, we can pick k distinct points $x_1, \cdots, x_k \in \hat{U}$ because $\hat{U}$ is a nonempty open subset and $n \geq 1$. Since $\hat{U}$ is open, there exist open $U_1, \cdots, U_k$ such that $x_i \in U_i \subset \hat{U}$ and $U_i \cap U_j$ whenever $i \neq j$. Moreover, $\{x_i\}$ is a closed subset. By Proposition 2.25, we obtain k bump functions f_i for $\{x_i\}$ supported in U_i. Extend each f_i by setting $f_i(q) = 0$ for any $q \notin U$. Then each f_i lives in $C^\infty(M)$. Clearly, $\sum c_i f_i = 0$ implies $c_i = 0$, so $\{f_1, \cdots, f_k\}$ is linearly independent. Therefore, $C^\infty(M)$ is infinite-dimensional. □

Problem 2-14. Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \le f(x) \le 1$ for all $x \in M$, $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

Proof. By Theorem 2.29, there exist $\alpha, \beta \in C^\infty(M)$ such that $\alpha^{-1}(0) = A$ and $\beta^{-1}(0) = B$. Then $f(x) = \alpha(x)/(\alpha(x) + \beta(x))$ is a desired map. ☐

3. Chapter 3: Tangent Vectors

3.1. Exercises.

Proposition 3.2. Let $a \in \mathbb{R}^n$.

(a) For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map $D_v|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$D_v|_a f = D_v f(a) = \frac{d}{dt}\Big|_{t=0} f(a + tv)$$

is a derivation at a.

(b) The map $v_a \mapsto D_v|_a$ is an isomorphism from $\mathbb{R}^n_a$ onto $T_a\mathbb{R}^n$.

Proof.

(a) $D_v|_a$ is linear because $D_v|_a(f + cg) = D_v(f + cg)(a) = D_v(f)(a) + cD_v g(a) = D_v|_a(f) + cD_v|_a(g)$ because directional derivatives are linear. Moreover, the product rule is satisfied because directional derivatives satisfy that. Therefore, $D_v|_a$ is a linear map that satisfies directional derivatives, so it is a derivation.

(b) Let $\phi : \mathbb{R}^n_a \to T_a\mathbb{R}^n$ be defined such that $v_a \mapsto D_v|_a$. We first claim that ϕ is linear.

$$\begin{aligned}
\phi(v_a + cw_a)(f) &= \phi((v + cw)_a)(f) \\
&= D_{v+cw} f(a) \\
&= D_v f(a) + cD_w f(a) \\
&= D_v|_a(f) + cD_w|_a(f) \\
&= \phi(v_a)(f) + c\phi(w_a)(f) \\
&= (\phi(v_a) + c\phi(w_a))(f).
\end{aligned}$$

Next, we claim that $\ker(\phi) = 0$. Let $v_a \in \ker(\phi) \subset \mathbb{R}^n_a$. Let $v_1, \cdots, v_n \in \mathbb{R}^n$ be chosen such that $v_a = \sum_{i=1}^n v_i e_i|_a$. For each j, let $x^j : \mathbb{R}^n \to \mathbb{R}$ denote the projection map of the jth coordinate. Then $0 = D_v|_a(x^j) = \frac{d}{dt}|_{t=0} x^j(a + tv) = v_j$ for each j. Therefore, $v_1 = \cdots = v_n = 0$, so $\ker(\phi) = 0$. Since ϕ is linear, ϕ must be injective.

Lastly, we claim that ϕ is surjective. Let $w \in T_a\mathbb{R}^n$ be given. For each j, let $v_j = w(x^j)$. Let $v = (v_1, \cdots, v_n)$. We claim that $\phi(v_a) = w$. Let $f \in C^\infty(\mathbb{R}^n)$. By Theorem C.15, we can write

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + \sum_{i,j=1}^n (x^i - a^i)(x^j - a^j) \int_{0,1} F(t)dt$$

where $F(t)$ is some function. Since $(x^i - a^i)$ and $(x^j - a^j)\int_{0,1} F(t)dt$ vanish at $x = a$, $w((x^i - a^i)(x^j - a^j)\int_{0,1} F(t)dt) = 0$ for any i, j. Therefore,

$$\begin{aligned}
w(f) &= w(f(a)) + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)(w(x^i) - w(a^i)) \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a) w(x^i) \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a) v_i \\
&= \phi(v_a)(f)
\end{aligned}$$

which proves that ϕ is surjective.

$\square$

Exercise 3.5(Proof of Lemma 3.4). Suppose M is a smooth manifold with or without boundary, $p \in M, v \in T_pM$, and $f, g \in C^\infty(M)$.

(a) If f is a constant function, then $vf = 0$.
(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

Proof. This is similar to Lemma 3.1.

(a) Let h be the constant function that always takes the value 1. Then $v(h) = v(h^2) = h(p)v(h) + h(p)v(h) = 2v(h)$, so $v(h) = 0$. Since $f(p) = ch(p)$ for some $c \in \mathbb{R}$ and v is linear, this implies $0 = cv(h) = v(ch) = v(f)$.
(b) $v(fg) = f(p)vg + g(p)vf = 0 + 0 = 0$.

$\square$

Exercise 3.7(Proof of Proposition 3.6). Let M, N, and P be smooth manifolds with or without boundary, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.

(a) $dF_p : T_pM \to T_{F(p)}N$ is linear.
(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}P$.
(c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.
(d) If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Proof. (a) $\forall v, w \in T_pM, \forall c \in \mathbb{R}, \forall f \in C^\infty(N)$,

$$\begin{aligned}
dF_p(cv + w)(f) &= (cv + w)(f \circ F) \\
&= (cv)(f \circ F) + w(f \circ F) \\
&= c(v(f \circ F)) + w(f \circ F) \\
&= c(dF_p(v)(f)) + dF_p(w)(f) \\
&= (cdF_p(v))(f) + dF_p(w)(f) \\
&= (cdF_p(v) + dF_p(w))(f).
\end{aligned}$$

Therefore, $dF_p(cv + w) = cdF_p(v) + dF_p(w)$.

(b) $\forall v \in T_pM, f \in C^\infty(P)$,

$$\begin{aligned}
d(G \circ F)_p(v)(f) &= v(f \circ (G \circ F)) \\
&= v((f \circ G) \circ F) \\
&= (dF_p(v))(f \circ G) \\
&= (dG_{F(p)}(dF_p(v)))(f) \\
&= ((dG_{F(p)} \circ dF_p)(v))(f)
\end{aligned}$$

Therefore, $d(G \circ F)_p = dG_{F(p)} \circ dF_p$.

(c) $\forall v \in T_p(M), \forall f \in C^\infty(M)$,

$$\begin{aligned}
d(\mathrm{Id}_M)_p(v)(f) &= v(f \circ \mathrm{Id}_M) \\
&= v(f).
\end{aligned}$$

Therefore, $d(\mathrm{Id}_M)_p(v) = v$, so $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$.

(d) F^{-1} exists and it is a smooth map since F is a diffeomorphism. By combining (b) and (c), we obtain dF_p and $dF^{-1}_{F(p)}$ are the inverse of each other. Therefore, dF_p is an isomorphism.

$\square$

Proposition 3.10. If M is an n-dimensional smooth manifold, then for each $p \in M$, the tangent space T_pM is an n-dimensional vector space.

Proof. Let $\mathcal{A}$ denote the smooth structure of M and let $p \in M$ be given. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Then

$$T_pM \overset{di_p}{\cong} T_pU \overset{d\phi_p}{\cong} T_{\phi(p)}\hat{U} \overset{di_{\phi(p)}}{\cong} T_{\phi(p)}\mathbb{R}^n$$

where di_p is induced by the inclusion map $i : U \to M$ and $di_{\phi(p)}$ is induced by the inclusion map $: \hat{U} \to \mathbb{R}^n$. $di_p, d\phi_p, di_{\phi(p)}$ are all isomorphisms by (3.7(Proof of Proposition 3.6)(d)) and Proposition 3.9. Therefore, $\dim(T_pM) = n$. □

Proposition 3.15. Let M be a smooth n-manifold with or without boundary, and let $p \in M$. Then T_pM is an n-dimensional vector space, and for any smooth chart $(U, (x^i))$ containing p, the coordinate vectors $\partial/\partial x^1|_p, \cdots, \partial/\partial x^n|_p$ form a basis for T_pM.

Proof. By Proposition 3.12, T_pM is an n-dimensional vector space. By Corollary 3.3, the $\partial/\partial x^i|_{\phi(p)}$ form a basis for $T_{\phi(p)}\mathbb{R}^n$. By Proposition 3.6(d), $d\phi_p : T_pM \to T_{\phi(p)}\mathbb{R}^n$ is an isomorphism. Since $d\phi_p$ is an isomorphism between vector spaces, $d\phi_p$ sends a basis to a basis. In other words, the $\partial/\partial x^i|_p = (d\phi_p)^{-1}(\partial/\partial x^i|_{\phi(p)})$ form a basis. □

Remark. *The discussion on PP.61-62 shows the connection between differentials and Jacobian matrices. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map and let $p \in \mathbb{R}^n$ be given.*

$$dF_p\Big(\frac{\partial}{\partial x^i}\Big|_p\Big)(f) = \frac{\partial}{\partial x^i}\Big|_p(f \circ F) \qquad \textit{(definition of d)} \tag{3.1}$$

$$= \frac{\partial(f \circ F)}{\partial x^i}(p) \qquad \textit{(Just a partial derivative of } f \circ F\textit{)} \tag{3.2}$$

$$= \sum_{j=1}^{m} \frac{\partial f}{\partial y^j}(F(p))\frac{\partial F^j}{\partial x^i}(p) \tag{3.3}$$

$$= \sum_{j=1}^{m} \frac{\partial F^j}{\partial x^i}(p)\frac{\partial f}{\partial y^j}(F(p)) \qquad \textit{(Multiplication is commutative in } \mathbb{R}\textit{)} \tag{3.4}$$

$$= \sum_{j=1}^{m} \frac{\partial F^j}{\partial x^i}(p)\frac{\partial}{\partial y^j}\Big|_{F(p)}(f). \tag{3.5}$$

Therefore, we obtain that $dF_p(\frac{\partial}{\partial x^i}\big|_p) = \sum_{j=1}^m \frac{\partial F^j}{\partial x^i}(p)\frac{\partial}{\partial y^j}\big|_{F(p)}$. $\{\partial/\partial x^i\}$ and $\{\partial/\partial y^j\}$ form bases for $T_p\mathbb{R}^n$ and $T_{F(p)}\mathbb{R}^m$, respectively, so it makes sense to put dF_p is a matrix form. Then we obtain

$$\begin{bmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{bmatrix}$$

which is identical to the Jacobian matrix of F at p. Two things to note:

- *It makes sense to discuss the Jacobian matrix of F because F is a map from $\mathbb{R}^m$ to $\mathbb{R}^n$.*
- *The same calculation applies if $F : U \to V$ where U, V are open subsets of $\mathbb{R}^n, \mathbb{R}^m$ or where U, V are open subsets of $\mathbb{H}^n, \mathbb{H}^m$.*

We now consider a more general case when $F : M \to N$ is a smooth map between two smooth manifolds with or without boundary. Let $p \in M$ be given. Let $(U, \phi), (V, \psi)$ be smooth charts of M, N that contain $p, F(p)$, respectively. Let $\hat{F} = \psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(V)$ and $\hat{p} = \phi(p)$. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} U \cap F^{-1}(V) & \xrightarrow{F} & F \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \psi} \\ \phi(U \cap F^{-1}(V)) & \xrightarrow{\hat{F}} & \hat{V} \end{array}$$

We compute

$$
\begin{aligned}
dF_p(\frac{\partial}{\partial x^i}\Big|_p) &= dF_p(d(\phi^{-1})_{\hat{p}}(\frac{\partial}{\partial x^i}\Big|_{\hat{p}})) && \textit{(Definition of a coordinate vector)}\\
&= (dF_p \circ d(\phi^{-1})_{\hat{p}})(\frac{\partial}{\partial x^i}\Big|_{\hat{p}})\\
&= (d(F\circ\phi^{-1})_{\hat{p}})(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}) && (3.7(Proof of Proposition 3.6))\\
&= d(\psi^{-1}\circ\hat{F})_{\hat{p}}(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}) && \textit{(See the diagram above)}\\
&= (d(\psi^{-1})_{\hat{F}(\hat{p})}\circ d\hat{F}_{\hat{p}})(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}) && (3.7(Proof of Proposition 3.6))\\
&= d(\psi^{-1})_{\hat{F}(\hat{p})}(d\hat{F}_{\hat{p}}(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}))\\
&= d(\psi^{-1})_{\hat{F}(\hat{p})}(\sum_{j=1}^{m}\frac{\partial\hat{F}^j}{\partial x^i}(\hat{p})\frac{\partial}{\partial y^j}\Big|_{\hat{F}(\hat{p})}) && (\textit{Discussion above})\\
&= \sum_{j=1}^{m}\frac{\partial\hat{F}^j}{\partial x^i}(\hat{p})d(\psi^{-1})_{\hat{F}(\hat{p})}(\frac{\partial}{\partial y^j}\Big|_{\hat{F}(\hat{p})}) && (\textit{Linearity of a differential})\\
&= \sum_{j=1}^{m}\frac{\partial\hat{F}^j}{\partial x^i}(\hat{p})d(\psi^{-1})_{\psi(F(p))}(\frac{\partial}{\partial y^j}\Big|_{\psi(F(p))}) && (\textit{Diagram above})\\
&= \sum_{j=1}^{m}\frac{\partial\hat{F}^j}{\partial x^i}(\hat{p})\frac{\partial}{\partial y^j}\Big|_{F(p)} && (\textit{Definition of a coordinate vector}).
\end{aligned}
$$

Therefore, even in this general case, dF_p is represented in coordinate bases by the Jacobian matrix of $\hat{F}$.

Remark. *The notation on P.63-64 is not easy to understand.*

Let M be an n-dimensional smooth manifold. Let $(U,\phi=(x^i)),(V,\psi=(\tilde{x}^i))$ be two smooth charts on M and $p\in U\cap V$. The textbook denotes the transition map $\psi\circ\phi^{-1}:\phi(U\cap v)\to\psi(U\cap V)$ by

$$\psi\circ\phi^{-1}(x)=(\tilde{x}^1(x),\cdots,\tilde{x}^n(x)).$$

What (I think) this really means is

$$(\psi\circ\phi^{-1})(x^1(p),\cdots,x^n(p))=(\tilde{x}^1(p),\cdots,\tilde{x}^n(p))$$

for each $p\in U\cap V$. The idea is that $\phi=(x^i)$ is a diffeomorphism, so the textbook decides to denote each point in $\phi(U\cap V)$ by x because every point in $\phi(U\cap V)$ can be denoted by $(x^1(p),\cdots,x^n(p))$ for a unique $p\in U\cap V$.

Moreover, the second part of this discussion (after "By (3.9), the differential $d(\psi\circ\phi^{-1})_{\phi(p)}$ can be written") is even more confusing because:

- *The textbook simply uses x^i and $\tilde{x}^i$ to represent the coordinates of $\hat{U}$ and $\hat{V}$ instead of the coordinate functions of ϕ and ψ.*
- *$\hat{U}$ and $\hat{V}$ both live in $\mathbb{R}^n$, so it might seem unnecessary to use both x^i and $\tilde{x}^i$. It is actually necessary because we want to use $\partial/\partial x^i$ to talk about the coordinate vectors induced by ϕ and $\partial/\partial\tilde{x}^i$ to talk about the coordinate vectors induced by ψ.*

Finally, (3.12) in the textbook can be derived as following:

$$\begin{aligned}
v &= \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_p \\
&= \sum_{i=1}^n v^i (\sum_{j=1}^n \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial \tilde{x}^j}\Big|_p) && \textit{((3.11) in the textbook)} \\
&= \sum_{i=1}^n (\sum_{j=1}^n v^i \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial \tilde{x}^j}\Big|_p) \\
&= \sum_{j=1}^n \big[\sum_{i=1}^n v^i \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p})\big] \frac{\partial}{\partial \tilde{x}^j}\Big|_p.
\end{aligned}$$

Exercise 3.17. Let (x, y) denote the standard coordinates on $\mathbb{R}^2$. Verify that $(\tilde{x}, \tilde{y})$ are global smooth coordinates on $\mathbb{R}^2$, where

$$\tilde{x} = x, \quad \tilde{y} = y + x^3.$$

Let p be the point $(1, 0) \in \mathbb{R}^2$ (in standard coordinates), and show that

$$\frac{\partial}{\partial x}\Big|_p \neq \frac{\partial}{\partial \tilde{x}}\Big|_p,$$

even though the coordinate function x and $\tilde{x}$ are identically equal.

Proof. The map $(x, y) \mapsto (x, y + x^3)$ is a smooth automorphism on $\mathbb{R}^2$.

$$\begin{aligned}
\frac{\partial}{\partial x}\Big|_p &= \frac{\partial \tilde{x}}{\partial x}(1,0)\frac{\partial}{\partial \tilde{x}}\Big|_p + \frac{\partial \tilde{y}}{\partial x}(1,0)\frac{\partial}{\partial \tilde{y}}\Big|_p && \text{((3.11) in the textbook)} \\
&= \frac{\partial}{\partial \tilde{x}}\Big|_p + 3\frac{\partial}{\partial \tilde{y}}\Big|_p && (\frac{\partial \tilde{y}}{\partial x} = 3x^2) \\
&\neq \frac{\partial}{\partial \tilde{x}}\Big|_p.
\end{aligned}$$

□

Exercise 3.19. Suppose M is a smooth manifold with boundary. Show that TM has a natural topology and smooth structure making it into a smooth manifold with boundary, such that if $(U, (x^i))$ is any smooth boundary chart for M, then rearranging the coordinates in the natural chart $(\pi^{-1}(U), (x^i, v^i))$ for TM yields a boundary chart $(\pi^{-1}(U), (v^i, x^i))$.

Proof. The proof is similar to that of Proposition 3.18. We begin by defining the maps that will become our smooth charts. Given any smooth (possibly boundary) chart (U, ϕ) for M, note that $\pi^{-1}(U) \subset TM$ is the set of all tangent vectors to M at all points of U. Let $(x^1, \cdots, x^n)$ denote the coordinate functions of ϕ, and define a map $\tilde{\phi}$ that maps $\pi^{-1}(U)$ into $\mathbb{H}^{2n}$ or $\mathbb{R}^{2n}$ by

$$\tilde{\phi}(v^i \frac{\partial}{\partial x^i}\Big|_p) = (v^1, \cdots, v^n, x^1(p), \cdots, x^n(p)).$$

In case (U, ϕ) is a boundary chart, $\tilde{(\phi)}$ indeed maps $\pi^{-1}(U)$ into $\mathbb{H}^{2n}$ because $x^n(p) \geq 0$. Its image set is $\mathbb{R}^n \times \phi(U)$, which is an open subset of $\mathbb{R}^{2n}$ or $\mathbb{H}^{2n}$. Now suppose we are given two smooth charts (U, ϕ) and (V, ψ) for M, and let $(\pi^{-1}(U), \tilde{\phi}), (\pi^{-1}(V), \tilde{\psi})$ be the corresponding charts on TM. The sets

$$\begin{aligned}
&\tilde{\phi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \mathbb{R}^n \times \phi(U \cap V) && \text{and} \\
&\tilde{\psi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \mathbb{R}^n \times \psi(U \cap V)
\end{aligned}$$

are open in $\mathbb{R}^{2n}$ or $\mathbb{H}^{2n}$, and the transition map $\tilde{\psi} \circ \tilde{\phi}$ can be written explicitly using (3.1) as

$$\begin{aligned}
&(\tilde{\psi} \circ \tilde{\phi}^{-1})(x^1, \cdots, x^n, v^1, \cdots, v^n) \\
&= (\frac{\partial \tilde{x}^1}{\partial x^j}(x)v^j, \cdots, \frac{\partial \tilde{x}^n}{\partial x^j}(x)v^j, \tilde{x}^1(x), \cdots, \tilde{x}^n(x)).
\end{aligned}$$

The rest of the proof is identical to that of Proposition 3.18. $\square$

Proposition 3.21. If $F : M \to N$ is a smooth map, then its global differential $dF : TM \to TN$ is a smooth map.

Proof. Let $(p_0, v_0) \in TM$ be given. It suffices to show that F is smooth in an neighborhood around (p_0, v_0). Let $(U, \phi), (V, \psi)$ be given such that $p_0 \in U, F(U) \subset V$ and $F(p_0) \in V$. The set $\pi^{-1}(U)$ is open in TM because that is how we give TM a topology in Proposition 3.18. Moreover, $\pi^{-1}(U)$ is a neighborhood of (p_0, v_0) in TM. We will consider the charts $(\pi^{-1}, \tilde{\phi})$ and $(\pi^{-1}(V), \tilde{\psi})$ as defined in Proposition 3.18. Let $(x^1, \cdots, x^m, v^1, \cdots, v^m) \in \tilde{\phi}(\pi^{-1}(U))$. Let $x = (x^1, \cdots, x^m)$ and $p = \phi^{-1}(x)$. Then

$$dF_p(\frac{\partial}{\partial x^i}\Big|_p) = \frac{\partial F^j}{\partial x^i}(p)\frac{\partial}{\partial y^j}\Big|_{F(p)}$$

by (3.9) in the textbook. Then

$$(\tilde{\psi} \circ dF \circ \tilde{\phi}^{-1})(x^1, \cdots, x^m, v^1, \cdots, v^m) = ((\psi^1 \circ F \circ \phi^{-1})(x), \cdots, (\psi^n \circ F \circ \phi^{-1})(x), \frac{\partial F^1}{\partial x^i}(p)v^i, \cdots, \frac{\partial F^n}{\partial x^i}(p)v^i)$$

Since each component function is smooth, $\tilde{\psi} \circ dF \circ \tilde{\phi}^{-1}$ is smooth. Therefore, dF is a smooth map from TM to TN. $\square$

Proposition 3.23. Suppose M is a smooth manifold with or without boundary and $p \in M$. Every $v \in T_pM$ is the velocity of some smooth curve in M.

Proof. First, suppose that $p \in \operatorname{Int} M$. Let (U, ϕ) be a smooth coordinate chart centered at p, and write $v = v^i \partial/\partial x^i|_p$ in terms of the coordinate basis. Without loss of generality, $\phi(p) = 0 \in \mathbb{R}^n$. Now, define $\hat{\gamma} : (-\epsilon, \epsilon) \to \hat{U}$ by $\hat{\gamma}(t) = (tv_1, \cdots, tv_n)$ for sufficiently small $\epsilon > 0$. Let $\gamma : (-\epsilon, \epsilon) \to U$ be defined by $\gamma = \phi^{-1} \circ \hat{\gamma}$. Then γ is actually a smooth map from a 1-manifold $(-\epsilon, \epsilon)$ to an n-manifold M (with or without boundary). We will use the formula derived in (3.1) and obtain

$$\begin{aligned}
\gamma'(0) &= d\gamma(\frac{d}{dt}\Big|_0) \\
&= \sum_{j=1}^n \frac{\partial \hat{\gamma}^j}{\partial t}(0)\frac{\partial}{\partial x^j}\Big|_p \\
&= \sum_{j=1}^n v^j \frac{\partial}{\partial x^j}\Big|_p \\
&= v.
\end{aligned}$$

If $p \in \partial M$, then we do the exact same thing as above except that the domain will be $[0, \epsilon)$ if $v_n > 0$ and $(-\epsilon, 0]$ if $v_n \leq 0$. $\square$

Proposition 3.24. Let $F : M \to N$ be a smooth map, and let $\gamma : J \to M$ be a smooth curve. For any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma : J \to N$ is given by

$$(F \circ \gamma)'(t_0) = dF(\gamma'(t_0)).$$

Proof.

$$\begin{aligned}
(F \circ \gamma)'(t_0) &= d(F \circ \gamma)(\frac{d}{dt}\Big|_{t_0}) && \text{(definition of the velocity)} \\
&= (dF \circ d\gamma)(\frac{d}{dt}\Big|_{t_0}) && \text{(Corollary 3.22(a))} \\
&= dF(d\gamma(\frac{d}{dt}\Big|_{t_0})) \\
&= dF(\gamma'(t_0)) && \text{(definition of the velocity).}
\end{aligned}$$

$\square$

Proposition 3.25. Suppose $F : M \to N$ is a smooth map, $p \in M$, and $v \in T_pM$. Then

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma : J \to M$ such that $0 \in J, \gamma(0) = p$, and $\gamma'(0) = v$.

Proof. This is a special case of (3.24) where $t_0 = 0$. $\square$

3.2. **Problems.**

Problem 3-1. Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Show that $dF_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

Proof. Suppose $dF_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$. It suffices to show that for every $p \in M$, there exists a neighborhood of p on which F is constant. Let $p \in M$ and $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$ be given such that $p \in U$ and $F(U) \subset V$. Without loss of generality, we assume $\hat{U} = \phi(U)$ is an open ball in $\mathbb{R}^m$. Then for any i, j and for any $q \in \hat{U}$,

$$\begin{aligned} dF_q(\frac{\partial}{\partial x^i}|_q)(\pi_j \circ \psi) = 0 &\implies (\frac{\partial}{\partial x^i}|_q)(\pi_j \circ \psi \circ F) = 0 \\ &\implies (\frac{\partial}{\partial x^i}|_{\phi(q)})(\pi_j \circ \psi \circ F \circ \phi^{-1}) = 0. \end{aligned}$$

Fix j. Then every partial derivative of $\pi_j \circ \psi \circ F \circ \phi^{-1}$ at every point in $\hat{U}$ is 0. The intermediate value theorem implies that $\pi_j \circ \psi \circ F \circ \phi^{-1}$ is constant on $\hat{U}$ because $\hat{U}$ is an open ball. In other words, $(\pi_j \circ \psi \circ F \circ \phi^{-1})(\hat{U}) = \{y_j\}$ for some $y_j \in \mathbb{R}$. Since this is true for every j and π_j is the projection of the jth coordinate, $(\psi \circ F \circ \phi^{-1})(\hat{U}) = \{y\}$ where $y = (y_1, \cdots, y_n)$. Then $(F \circ \phi^{-1})(\hat{U}) = F(U) = \psi^{-1}(y)$. Since ψ is a homeomorphism, there exists exactly one point in $\psi^{-1}(U)$. In other words, F is constant on U. Therefore, F is constant on each path component.

Suppose F is constant on each component of M. Let $p \in M$. Choose a chart $(U, \phi) \in \mathcal{A}_M$ such that $p \in U$. Then $F \circ \phi^{-1}$ is constant in a neighborhood around $\phi(p)$. For any i,

$$\begin{aligned} dF_p(\frac{\partial}{\partial x^i}|_p)(f) &= \frac{\partial}{\partial x^i}|_p(f \circ F) \\ &= \frac{\partial}{\partial x^i}|_{\phi(p)}(f \circ F \circ \phi^{-1}) \\ &= 0 \end{aligned}$$

because $f \circ F \circ \phi^{-1}$ is constant in a neighborhood around $\phi(p)$. By Proposition 3.15, $\partial/\partial x^i|_p$ form a basis for T_pM. Since dF_p sends each basis element to 0, $dF_p = 0$. $\square$

Problem 3-2(Proof of Proposition 3.14). Let $M_1, \cdots, M_k$ be smooth manifolds, and for each j, let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection onto the M_j factor. For any point $p = (p_1, \cdots, p_k) \in M_1 \times \cdots \times M_k$, the map

$$\alpha : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \cdots, d(\pi_k)_p(v))$$

is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.

Proof. It suffices to show this for the case that $k = 2$ because the results extend to arbitrary k by induction. Let $\mathcal{A}_{M_1}, \mathcal{A}_{M_2}, \mathcal{A}_{M_1 \times M_2}$ be the smooth structures of $M_1, M_2, M_1 \times M_2$.

We first define a lot of notations.

- Let d_1, d_2 denote the dimensions of M_1, M_2 and let $d = d_1 + d_2$ denote the dimension of $M_1 \times M_2$.
- Let $p = (p_1, p_2) \in M_1 \times M_2$ be given. Choose $(U, \phi = (x^i)) \in \mathcal{A}_{M_1}, (V, \psi = (y^i)) \in \mathcal{A}_{M_2}$ with $p_1 \in U$ and $p_2 \in V$. Let $q_1 = \phi(p_1), q_2 = \psi(p_2), q = q_1 \times q_2$.
- $(U \times V, (z^i)) \in \mathcal{A}_{M_1 \times M_2}$ and $(p_1, p_2) \in U \times V$ where $(z^i) = \phi \times \psi$. More specifically, $z^i = x^i \circ \pi_1$ for $1 \le i \le d_1$ and $z^i = y^i \circ \pi_2$ for $d_1 + 1 \le i \le d_1 + d_2$.

Note that we use x^i, y^i, z^i, π_1 to mean two different things in this solution:

- x^i is either the ith coordinate function of ϕ or the ith projection map $\mathbb{R}^{d_1} \to \mathbb{R}$.
- y^i is either the ith coordinate function of ψ or the ith projection map $\mathbb{R}^{d_2} \to \mathbb{R}$.
- z^i is either the ith coordinate function of $\phi \times \psi$ or the ith projection map $\mathbb{R}^{d_1+d_2} \to \mathbb{R}$.
- π_1 is either the projection map $M_1 \times M_2 \to M_1$ or the projection map $\mathbb{R}^{d_1+d_2} \to \mathbb{R}^{d_1}$.
- π_2 is either the projection map $M_1 \times M_2 \to M_2$ or the projection map $\mathbb{R}^{d_1+d_2} \to \mathbb{R}^{d_2}$.

By Proposition 3.15, $\{\partial/\partial x^1|_{p_1}, \cdots, \partial/\partial x^{d_1}|_{p_1}\}, \{\partial/\partial y^1|_{p_2}, \cdots, \partial/\partial y^{d_2}|_{p_2}\}, \{\partial/\partial z^1|_p, \cdots, \partial/\partial z^{d_1+d_2}|_p\}$ form bases for $T_{p_1}M_1, T_{p_2}M_2, T_p(M_1 \times M_2)$.

$\alpha(\partial/\partial z^1|_p) = (d(\pi_1)_p(\partial/\partial z^1|_p), d(\pi_2)_p(\partial/\partial z^1|_p))$. We claim that $d(\pi_1)_p(\partial/\partial z^1|_p) = \partial/\partial x^1|p_1$.

$$\begin{aligned}
d(\pi_1)_p(\partial/\partial z^1|_p)(f) &= d(\pi_1)_p(d(\phi^{-1} \times \psi^{-1})_q)(\frac{\partial}{\partial z^1}|_q)(f) \\
&= (d(\pi_1)_p \circ d(\phi^{-1} \times \psi^{-1})_q)(\frac{\partial}{\partial z^1}|_q)(f) \\
&= d(\pi_1 \circ (\phi^{-1} \times \psi^{-1})_q)(\frac{\partial}{\partial z^1}|_q)(f) \\
&= \lim_{h \to 0} \frac{(f \circ \pi_1 \circ (\phi^{-1} \times \psi^{-1}))(q + e_1 h) - (f \circ \pi_1 \circ (\phi^{-1} \times \psi^{-1}))(q)}{h} \\
&= \lim_{h \to 0} \frac{(f \circ \pi_1)(\phi^{-1}(q_1 + e_1 h), p_2) - (f \circ \pi_1)(p)}{h} \\
&= \lim_{h \to 0} \frac{f(\phi^{-1}(q_1 + e_1 h)) - f(p_1)}{h} \\
&= \lim_{h \to 0} \frac{f(\phi^{-1}(q_1 + e_1 h)) - f(\phi^{-1}(q_1))}{h} \\
&= (\frac{\partial}{\partial x^1}|_{q_1})(f \circ \phi^{-1}) \\
&= d(\phi^{-1})_{q_1}(\frac{\partial}{\partial x^1}|_{q_1})(f) \\
&= (\frac{\partial}{\partial x^1}|_{p_1})(f).
\end{aligned}$$

The same result can be shown for the other combinations of π_1, π_2 and $z^1, \cdots, z^{d_1+d_2}$. For any $c_1, \cdots, c_{d_1+d_2} \in \mathbb{R}$,

$$\begin{aligned}
\alpha(\sum_{i=1}^{d_1+d_2} c_i \frac{\partial}{\partial z^i}|_p) &= \sum_{i=1}^{d_1+d_2} c_i \alpha(\frac{\partial}{\partial z^i}|_p) \\
&= \sum_{i=1}^{d_1+d_2} c_i (d(\pi_1)_p \frac{\partial}{\partial z^i}|_p, d(\pi_2)_p \frac{\partial}{\partial z^i}|_p) \\
&= \sum_{i=1}^{d_1} c_i (d(\pi_1)_p \frac{\partial}{\partial z^i}|_p, d(\pi_2)_p \frac{\partial}{\partial z^i}|_p) + \sum_{i=d_1+1}^{d_2} c_i (d(\pi_1)_p \frac{\partial}{\partial z^i}|_p, d(\pi_2)_p \frac{\partial}{\partial z^i}|_p) \\
&= \sum_{i=1}^{d_1} c_i (\frac{\partial}{\partial x^i}|_{p_1}, 0) + \sum_{i=1}^{d_2} c_{d_1+i} (0, \frac{\partial}{\partial y^i}|_{p_2}) \\
&= (c_1 \frac{\partial}{\partial x^1}|_{p_1} + \cdots + c_{d_1} \frac{\partial}{\partial x^{d_1}}|_{p_1}, c_{d_1+1} \frac{\partial}{\partial y^1}|_{p_2} + \cdots + c_{d_1+d_2} \frac{\partial}{\partial y^{d_2}}|_{p_2}).
\end{aligned}$$

Therefore, α is bijective. $\square$

Problem 3-3. Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.

Proof. Let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the obvious projections of the corresponding coordinates. π_1, π_2 are clearly smooth, so Proposition 3.21 shows that $d\pi_1 : T(M \times N) \to TM$ and $d\pi_2 : T(M \times N) \to TN$ are both smooth. By (2.16(Proof of Proposition 2.15)(b)), $d\pi_1 \times d\pi_2$ is a smooth map.

By (3-2(Proof of Proposition 3.14)), $d\pi_1 \times d\pi_2$ is a bijection between $T_{(p,q)}(M \times N)$ and $T_p(M) \times T_q(N)$. Since $d\pi_1 \times d\pi_2$ sends $((p,q), \sigma)$ to $(p, d\pi_1(\sigma)) \times (q, d\pi_2(\sigma))$, we conclude that $d\pi_1 \times d\pi_2$ is bijective. ☐

4. Chapter 4: Submersions, Immersions, and Embeddings

Proposition 4.1. Suppose $F : M \to N$ is a smooth map and $p \in M$. If dF_p is surjective, then p has a neighborhood U such that $F|_U$ is a submersion. If dF_p is injective, then p has a neighborhood U such that $F|_U$ is an immersion.

Proof. Let (U, ϕ) be a chart containing p and (V, ψ) be a chart containing $F(p)$. We may assume $F(U) \subset V$. It suffices to show that if the Jacobian of F with respect to (U, ϕ) is full rank at p, then it is full rank in some neighborhood of p contained in U. Example 1.28 in the textbook shows that the set of full rank matrices is an open subset of $M(m \times n, \mathbb{R})$. We will use the notation $J|_q$ to denote the Jacobian of F with respect to (U, ϕ) at $q \in U$. Then $J|_p$ is an element of an open subset of $M(m \times n, \mathbb{R})$. Each entry of $J|_q$ is of the from $\frac{\partial}{\partial x^i}(\psi^j \circ F \circ \phi)(\phi(q))$ where each $(\frac{\partial}{\partial x^i}(\psi^j \circ F \circ \phi)) \circ \phi$ is a smooth function. Therefore, there exists a neighborhood of p such that the Jacobian matrix of F with respect to (U, ϕ) is full rank. ☐

Exercise 4.3(Verification of Example 4.2). Verify the following claims:

(a) Suppose $M_1, \cdots, M_k$ are smooth manifolds. Then each of the projection maps $\pi_i : M_1 \times \cdots \times M_k \to M_i$ is a smooth submersion.
(b) If $\gamma : J \to M$ is a smooth curve in a smooth manifold M with or without boundary, then γ is a smooth immersion if and only if $\gamma'(t) \neq 0$ for all $t \in J$.

Proof.

(a) Let $d_1, \cdots, d_k$ denote the dimensions of $M_1, \cdots, M_k$, respectively. Let $M = M_1 \times \cdots \times M_k$.

(2-2(Proof of Proposition 2.12)) implies that π_i is smooth for each i by setting $F = \mathrm{Id} : M \to M$. Let $p = (p_1, \cdots, p_k) \in M$. Thus it suffices to show that the dimension of $d(\pi_i)_p(T_p(M))$ is the same as the dimension of $T_{p_i}(M_i)$.

By Proposition 3.12, $\dim(T_p(M)) = \sum d_i$. Since the α defined in (3-2(Proof of Proposition 3.14)) is an isomorphism,

$$\dim(d(\pi_1)_p(T_p(M)) \oplus \cdots \oplus d(\pi_k)_p(T_p(M))) = \dim(T_p(M)) = \sum d_i. \tag{4.1}$$

However, for each i, $d(\pi_i)_p(T_p(M)) \subset T_{p_i}M_i$. Thus $\dim(d(\pi_i)_p(T_p(M))) \leq \dim(T_{p_i}M_i) = d_i$. By (4.1), $\dim(d(\pi_i)_p(T_p(M))) = \dim(T_{p_i}M_i)$.

(b) γ is a smooth immersion if and only if $d\gamma_t : T_tJ \to T_{\gamma(t)}M$ is injective for each $t \in J$. Since each T_tJ is a 1-dimensional vector space spanned by $d/dt|_t$, $d\gamma_t$ is injective if and only if $d\gamma_t$ sends the basis element to a nonzero element. Finally, $\gamma'(t) = d\gamma(d/dt|_t)$. Therefore, γ is a smooth immersion if and only if $\gamma'(t) \neq 0$ for all $t \in J$.

☐

Exercise 4.4. Show that a composition of smooth submersions is a smooth submersion, and a composition of smooth immersions is a smooth immersion. Give a counterexample to show that a composition of maps of constant rank need not have constant rank.

Proof. Let M, N, L be smooth manifolds with or without boundary, and $F : M \to N, G : N \to L$ be given. If F, G are submersions, dF_p and $dG_{F(p)}$ are surjective for each p. Then $d(G \circ F)_p = dG_{F(p)} \circ dF_p$ is surjective for each p by (3.7(Proof of Proposition 3.6)). Thus a composition of smooth submersions is a smooth submersion. By the exact same argument, a composition of smooth immersions is a smooth immersion.

Counterexample?

☐

Proposition 4.5. Suppose M and N are smooth manifolds, and $F : M \to N$ is a smooth map. If $p \in M$ is a point such that dF_p is invertible, then there are connected neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.

Proof. Since dF_p is invertible, $\dim(T_pM) = \dim(T_{F(p)}N)$. Let $n = \dim(T_pM)$. By (3.10), n is the dimension of M and N. Let $(U, \phi), (V, \psi)$ be smooth charts containing $p, F(p)$, respectively, such that $\phi(p) = \psi(F(p)) = 0 \in \mathbb{R}^n$ and $F(U) \subset V$. Let $\hat{F} = \psi \circ F \circ \phi^{-1}$. Then $\hat{F}$ is a smooth map from an open subset $\hat{U} \subset \mathbb{R}^n$ into an open subset $\hat{V} \subset \mathbb{R}^n$. Then $d\hat{F}|_0 = d\psi_{F(p)} \circ dF_p \circ d\phi_0^{-1}$. Each function on the right hand side is bijective, so $d\hat{F}|_0$ is bijective. Since the differential of a smooth map between Euclidean spaces coincides with the total derivative of the map, we may apply the ordinary inverse function theorem. Thus there exist connected open subsets $\hat{U}_0 \subset \hat{U}$ and $\hat{V}_0 \subset \hat{V}$ both containing 0 such that $\hat{F}$ is a diffeomorphism from $\hat{U}_0$ to $\hat{V}_0$. Since ϕ and ψ are homeomorphisms, U_0 and V_0 are connected neighborhoods of p and $F(p)$ respectively. Finally, since $F = \psi^{-1} \circ \hat{F} \circ \phi$, F is a diffeomorphism from U_0 to V_0. □

Proposition 4.6.

(a) Every composition of local diffeomorphisms is a local diffeomorphism.
(b) Every finite product of local diffeomorphisms between smooth manifolds is a local diffeomorphism.
(c) Every local diffeomorphism is a local homeomorphism and an open map.
(d) The restriction of a local diffeomorphism to an open subsmanifold with or without boundary is a local diffeomorphism.
(e) Every diffeomorphism is a local diffeomorphism.
(f) Every bijective local diffeomorphism is a diffeomorphism.

Proof.

(a) Let L, M, N be manifolds with or without boundary. Let $F : L \to M$ and $G : M \to N$ be local diffeomorphisms. Let $p \in L$. Then there exist open sets U, V containing $p, F(p)$, respectively, such that $F(U), G(V)$ are open, and $F|_U, G|_V$ are diffeomorphisms. Let $W = F^{-1}(F(U) \cap V)$. Then W is a neighborhood of p such that $G(F(W)) = G(F(U) \cap V) = G(F(U)) \cap G(V)$, which is open in N. Moreover, $(G \circ F)|_W$ is clearly a diffeomorphism because a restriction of a diffeomorphism is a diffeomorphism and the composition of diffeomorphisms is a diffeomorphism.
(b) Let $M_1, \cdots, M_n, N_1, \cdots, N_n$ be $2n$ smooth manifolds and $F_i : M_i \to N_i$ be a local diffeomorphism for each $i = 1, \cdots, n$. Let $M = M_1 \times \cdots \times M_n, N = N_1 \times \cdots \times N_n$ and $F = F_1 \times \cdots \times F_n$. Let $p = (p_1, \cdots, p_n) \in M$ be given. Since each F_i is a local diffeomorphism, there exists an open set U_i containing p_i such that $F_i(U_i)$ is open in N_i and $F_i|_{U_i}$ is a diffeomorphism for each i.

    Then $U = U_1 \times \cdots \times U_n$ is an open subset of M containing p and $F(U) = F_1(U_1) \times \cdots \times F_n(U_n)$ is open in N. Since $F|_U = F_1|_{U_1} \times \cdots \times F_n|_{U_n}$, $F|_U$ is a diffeomorphism by (2.16(Proof of Proposition 2.15)(b)).
(c) A diffeomorphism is a homeomorphism, so a local diffeomorphism is a local homeomorphism. Let $F : M \to N$ be a local diffeomorphism and an open set $U \subset M$ be given. For every $p \in U$, there exists a neighborhood U_p of p such that $F(U_p)$ is open and $F|_{U_p}$ is a diffeomorphism. $U_p \cap U$ is open in M. Since $F|_{U_p}$ is a diffeomorphism, $F|_{U_p}(U_p \cap U) = F(U_p \cap U)$ is open in $F(U_p)$. Since $F(U_p)$ is open, $F(U_p \cap U)$ is open in N. Then $F(U_p \cap U) = F(U_p) \cap F(U)$ is open in N. Since $F(U) = \cup_{p \in U}(F(U_p) \cap F(U))$, $F(U)$ is open in N.
(d) Let $F : M \to N$ be a local diffeomorphism. Let $U \subset M$ be an open submanifold with or without boundary. For every $p \in U$, there exists a neighborhood U_p of p in M such that $F(U_p)$ is open in N and $F|_{U_p}$ is a diffeomorphism. Since $U_p \cap U$ is open in M, $F(U_p \cap U)$ is open in N. Moreover, $F|_{U_p \cap U}$ is a diffeomorphism. Thus $F|_U$ is a local diffeomorphism.
(e) Let $F : M \to N$ be a diffeomorphism. For every point $p \in M$, the "restriction" of F to M satisfies the definition.
(f) A local diffeomorphism is smooth, so a bijective local diffeomorphism is a diffeomorphism.

□

Exercise 4.8. Suppose M and N are smooth manifolds (without boundary), and $F : M \to N$ is a map.

(a) F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.

(b) If $\dim M = \dim N$ and F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.

Proof. Suppose that F is a local diffeomorphism. Let $p \in M$. Then p has a neighborhood U such that $F|_U$ is a diffeomorphism. Then $d(F|_U)_p$ is an isomorphism by (3.7(Proof of Proposition 3.6)). Clearly, $dF_p = d(F|_U)_p$. Therefore, dF_p is an isomorphism for each p. In other words, F is both a smooth immersion and submersion.

Suppose that F is both a smooth immersion and submersion. Then dF_p is injective and surjective for each $p \in M$. Therefore, dF_p is invertible for each $p \in M$. By (4.5), there exist open sets U, V containing $p, F(p)$ such that $F : U \to V$ is a diffeomorphism. This is exactly the definition of a local diffeomorphism.

Since $\dim M = \dim N$, either the injectivity or surjectivity of dF_p implies that dF_p is an isomorphism. Then (b) follows from (a). $\square$

Proposition 4.13. Let M and N be smooth manifolds, let $F : M \to N$ be a smooth map, and suppose M is connected. Then the following are equivalent:

(a) For each $p \in M$ there exist smooth charts containing p and $F(p)$ in which the coordinate representation of F is linear.

(b) F has constant rank.

Proof. Suppose (a). Let $p \in M$. Then the coordinate representation of dF is linear in some neighborhood U of p. This implies that the rank of dF is constant in U. Since M is connected, this implies that the rank of dF is constant throughout M.

On the other hand, suppose (b). Let $p \in M$. Then the rank theorem guarantees the existence of smooth charts (U, ϕ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subset V$ in which F has a coordinate representation of the form $\hat{F}(x^1, \cdots, x^r, x^{r+1}, \cdots, x^m) = (x^1, \cdots, x^r, 0, \cdots, 0)$. $\hat{F}$ is clearly linear because

$$\begin{aligned}
\hat{F}(c(x^1, \cdots, x^m) + (y^1, \cdots, y^m)) &= \hat{F}(cx^1 + y^1, \cdots, cx^m + y^m) \\
&= (cx^1 + y^1, \cdots, cx^r + y^r, 0, \cdots, 0) \\
&= c(x^1, \cdots, x^r, 0, \cdots, 0) + (y^1, \cdots, y^r, 0, \cdots, 0).
\end{aligned}$$

$\square$

Exercise (4.16). Show that every composition of smooth embeddings is a smooth embedding.

Proof. We showed that a composition of smooth immersions is a smooth immersion in (4.4). Every composition of topological embeddings is a topological embedding. Therefore, every composition of smooth embeddings is a smooth embedding. $\square$

Proposition (4.17).

(a) If M is a smooth manifold with or without boundary and $U \subset M$ is an open submanifold, the inclusion map $U \mapsto M$ is a smooth embedding.

(b) If $M_1, \cdots, M_k$ are smooth manifolds and $p_i \in M_i$ are arbitrarily chosen points, each of the maps $i_j : M_j \to M_1 \times \cdots \times M_k$ given by

$$i_j(q) = (p_1, \cdots, p_{j-1}, q, p_{j+1}, \cdots, p_k)$$

is a smooth embedding. In particular, the inclusion map $\mathbb{R}^n \to \mathbb{R}^{n+k}$ given by sending $(x^1, \cdots, x^n)$ to $(x^1, \cdots, x^n, 0, \cdots, 0)$ is a smooth embedding.

Proof.

(a) i is clearly a topological embedding. By Proposition 3.9, $di_p : T_pU \to T_pM$ is an isomorphism for each $p \in U$. Therefore, i is a smooth immersion, which means i is a smooth embedding.

(b) Since a cross product of finitely many smooth manifolds is a smooth manifold, it suffices to show that $i : M \to M \times N$ is a smooth embedding. i is clearly a topological embedding. Moreover, for any $p \in M$, $d(\pi_1)_p \circ di_p = d(\pi_1 \circ i)_p = d\,\mathrm{Id}_p = \mathrm{Id}$, so di_p must be injective. In other words, i is a smooth immersion. Therefore, i is a smooth embedding.

$\square$

5. Appendix A: Review of Topology

Exercise A.18(Proof of Proposition A.17). Let X be a topological space and let S be a subspace of X.

(a)
(b)
(c)
(d)
(e)
(f) If $\mathcal{B}$ is a basis for the topology of X, then $\mathcal{B}_S = \{B \cap S \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on S.
(g) If X is Hausdorff, then so is S.
(h) If X is first-countable, then so is S.
(i) If X is second-countable, then so is S.

Proof.

(a)
(b)
(c)
(d)
(e)
(f) The union of $B \cap S$ is S. Let $U \cap S$ be an open subset of S where U is open in X, and $x \in U \cap S$. Then there exists $B \in \mathcal{B}$ such that $x \in B \subset U$ since $\mathcal{B}$ is a basis. Therefore, $x \in B \cap S \subset U \cap S$ with $B \cap S \in \mathcal{B}_S$.
(g) Let $x \neq y \in S$. There exist two disjoint open sets U, V of X containing x, y, respectively. Then $U \cap S$ and $V \cap S$ are disjoint open sets of X containing x, y, respectively.
(h)
(i) Let $\mathcal{B}$ be a countable basis of X. Then $\{B \cap S \mid B \in \mathcal{B}\}$ is a countable basis of S by (f).

□

Exercise A.24(Proof of Proposition A.23). Suppose $X_1, \cdots, X_k$ are topological spaces, and let $X_1 \times \cdots \times X_k$ be their product space.

(a) CHARACTERISTIC PROPERTY: If B is a topological space, a map $F : B \to X_1 \times \cdots \times X_k$ is continuous if and only if each of its component functions $F_i = \pi_i \circ F : B \to X_i$ is continuous.

Proof.

(a) Suppose F is continuous. Since π_i is continuous by (c) and the composition of continuous functions is continuous, $\pi_1 \circ F$ is continuous. Suppose each component function is continuous. Let $B_1 \times \cdots \times B_k$ be a basis element of $X_1 \times \cdots \times X_k$.

$$\begin{aligned} F^{-1}(B_1 \times \cdots \times B_k) &= F^{-1}(\cap_{i=1}^k \pi_i^{-1}(B_1 \times \cdots \times B_k)) \\ &= \cap_{i=1}^k F^{-1}(\pi_i^{-1}(B_1 \times \cdots \times B_k)) \\ &= \cap_{i=1}^k (\pi_i \circ F)^{-1}(B_1 \times \cdots \times B_k). \end{aligned}$$

Since the intersection of finitely many open sets is open, F is continuous.

□

6. Appendix B: Review of Linear Algebra

Exercise B.49. Two norms $|\cdot|_1$ and $|\cdot|_2$ on a vector space V are said to be equivalent if there are positive constants c, C such that

$$c|v|_1 \leq |v|_2 \leq C|v|_1$$

for all $v \in V$. Show that equivalent norms determine the same topology.

Proof. Such a relation is symmetric for $c|v|_1 \leq |v|_2 \leq C|v|_1$ implies $(1/C)|v|_2 \leq |v|_1 \leq (1/c)|v|_2$. Let $\mathcal{T}_1, \mathcal{T}_2$ be the topologies induced by $|\cdot|_1, |\cdot|_2$. It suffices to show that $\forall v \in V, \forall U \in \mathcal{T}_2, (v \in U \implies \exists r > 0, B_1(v,r) \subset U)$ where $B_1(v,r)$ is the open ball centered at v with the radius r using the $|\cdot|_1$. Since $v \in U$ and U is open, $\exists r > 0$ such that $B_2(v,r) \subset U$. Then for any $w \in V$, $|v-w|_1 \leq |v-w|_2/c$, so $B_1(v, r/c) \subset B_2(v,r)$. ☐

7. Appendix C: Review of Calculus

Exercise C.1. Suppose that $F : U \to W$ is differentiable at $a \in U$. Show that the linear map satisfying

$$\lim_{v \to 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} = 0$$

is unique.

Proof. Let L, L' be two such linear maps.

$$\begin{aligned}
\lim_{v\to 0} \frac{|Lv - L'v|}{|v|} &= \lim_{v \to 0} \frac{|(F(a+v) - F(a) - L'v) - (F(a+v) - F(a) - Lv)|}{|v|} \\
&= \lim_{v\to 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} + \lim_{v\to 0} \frac{|F(a+v) - F(a) - L'v|}{|v|} \\
&= 0 + 0 = 0.
\end{aligned}$$

If $L \neq L'$, $(L - L')v_0 \neq 0$ for some v_0. Then $\lim_{v\to 0} \frac{|Lv - L'v|}{|v|} = \lim_{h \to 0} \frac{|L(hv_0) - L'(hv_0)|}{|hv_0|} = \frac{|(L-L')v_0|}{|v_0|} \neq 0$. This is a contradiction, so $L = L'$. ☐

8. Dictionary

8.1. Topological Manifolds.

Definition 8.1 (Topological Manifold)**.** A *topological n-manifold* is a Hausdorff, second-countable topological space each point of which has a neighborhood that is homeomorphic to an open subset $\mathbb{R}^n$.

Definition 8.2 (Coordinates)**.** Let M be a topological n-manifold. Let U be an open subset of M, $\hat{U}$ be an open subset of $\mathbb{R}^n$, $\phi : U \to \hat{U}$ be a homeomorphism.

- The pair (U, ϕ) is called a *coordinate chart* or a *chart*.
- U is called a *coordinate domain* or a *coordinate neighborhood* and ϕ is called a *coordinate map*.
- If $\phi(U)$ is an open ball in $\mathbb{R}^n$, U is called a *coordinate ball*.
- If $\phi(U)$ is an open cube in $\mathbb{R}^n$, U is called a *coordinate cube*.
- The coordinate functions of ϕ are often denoted as $(x^1, \cdots, x^n)$. Thus a chart is sometimes denoted by $(U, (x^1, \cdots, x^n))$ or $(U, (x^i))$.

Definition 8.3 (Atlas)**.** Let M be a topological n-manifold. An *atlas for M* is a collection of charts (U_α, ϕ_α) such that $M = \bigcup_\alpha U_\alpha$.

Definition 8.4 (Transition Map)**.** Let M be a topological n-manifold and $(U, \phi), (V, \psi)$ be coordinate charts such that $U \cap V \neq \emptyset$. $\psi \circ \phi^{-1} : \phi(U \cap V) \mapsto \psi(U \cap V)$ is called a *transition map* from ϕ to ψ.

Definition 8.5 (Closed Upper Half-Space)**.** $\mathbb{H}^n = \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$, and $\partial\mathbb{H}^n = \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid x^n = 0\}$.

Definition 8.6 (Manifold With Boundary)**.** Let M be a second-countable Hausdorff space and fix n. Suppose that for every $p \in M$, one of the following conditions is satisfied:

(1) There exists a neighborhood U of p and a homeomorphism $\phi : U \to \hat{U}$ where $\hat{U}$ is an open subset of $\mathbb{R}^n$. p is called an *interior point* and (U, ϕ) is called an *interior chart*.
(2) There exists a neighborhood U of p and a homeomorphism $\phi : U \to \hat{U}$ where $\hat{U}$ is an open subset of $\mathbb{H}^n$ with $\phi(p) \in \partial\mathbb{H}^n$. p is called a *boundary point*.

Then M is called an *n-dimensional topological manifold with boundary*. Note that every topological manifold is a topological manifold with boundary.

Definition 8.7 (Support)**.** If f is any real-valued or vector-valued function on a topological space M, the *support of* f, denoted by $\operatorname{supp} f$, is the closure of the set of points where f is nonzero:

$$\operatorname{supp} f = \overline{\{p \in M : f(p) \neq 0\}}.$$

Definition 8.8 (Bump Function)**.** If M is a topological space, $A \subset M$ is a closed subset, and $U \subset M$ is an open subset containing A, a continuous function $\psi : M \to \mathbb{R}$ is called a *bump function for* A *supported in* U if $0 \leq \psi \leq 1$ on M, $\psi \equiv 1$ on A, and $\operatorname{supp} \psi \subset U$.

8.2. **Smooth Manifolds.**

Definition 8.9 (Smoothly Compatible)**.** Let M be a topological n-manifold. Two coordinate charts $(U, \phi), (V, \psi)$ are called *smoothly compatible* if $U \cap V = \emptyset$ or the transition map $\psi \circ \phi^{-1}$ is a diffeomorphism.

Definition 8.10 (Smooth Atlas)**.** Let M be a topological n-manifold. A *smooth atlas* is an atlas $\mathcal{A}$ such that any two charts in $\mathcal{A}$ are smoothly compatible with each other.

Definition 8.11 (Smooth Structure)**.** If M is a topological n-manifold, an atlas $\mathcal{A}$ that is not properly contained in any larger smooth atlas is called *maximal* or a *smooth structure on* M

Definition 8.12 (Smooth Manifold)**.** A *smooth manifold* is a topological manifold equipped with a smooth structure.

Definition 8.13. Suppose $(M, \mathcal{A})$ is a smooth manifold.

- Any chart $(U, \phi) \in \mathcal{A}$ is called a *smooth chart.*
- Given a smooth chart (U, ϕ), U is called a smooth coordinate domain and ϕ is called a *smooth coordinate map.*
- Given a smooth chart (U, ϕ), U is called a *smooth coordinate ball* if it is a coordinate ball.

Remark. *One must define a smooth structure on a topological manifold before talking about a smooth chart.*

Definition 8.14 (Smooth Maps)**.** Let M, N be smooth manifolds with or without boundary and $F : M \to N$ be a map. F is a *smooth map* if for every $p \in M$, there exist smooth charts (U, ϕ) containing p and (V, ψ) containing $F(p)$ such that

- $F(U) \subset V$;
- $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth.

Definition 8.15 (Coordinate Representatin of a Smooth Map)**.** Let $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ be smooth manifolds. Let $F : M \to N$ be a smooth map and $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ be given. Then $\hat{F} = \psi \circ F \circ \phi^{-1}$ is called the coordinate representation of F with respect to (U, ϕ) and (V, ψ).

Definition 8.16 (Diffeomorphism)**.** Let M, N be smooth manifolds with or without boundary. A diffeomorphism is a smooth map $F : M \to N$ with a smooth inverse.

Definition 8.17 (Smooth on a subset)**.** Let M, N be smooth manifolds with or without boundary and $A \subset M$ be an arbitrary subset. A map $F : A \to N$ is said to be *smooth on* A if every $p \in A$ has an open neighborhood $W \subset M$ such that there exists a smooth map $\tilde{F} : W \to N$ with $\tilde{F}_{W \cap A} = F$.

8.3. **Tangent Vectors.**

Definition 8.18 (Derivation)**.** Let M be a smooth manifold with or without boundary. A derivation at $p \in M$ is a linear map $v : C^\infty(M) \to \mathbb{R}$ such that

$$v(fg) = f(p)vg + g(p)vf$$

for all $f, g \in C^\infty(M)$.

This corresponds to “arrows that are tangent to M and whose basepoints are attached to M at p” even though it may not be easy to see that from this definition.

Definition 8.19 (Tangent Space)**.** The tangent space T_pM to M at p is the vector space of all derivations of $C^\infty(M)$ at p.

Derivation of $C^\infty(M)$	Geometric tangent vector on M
Differential of a smooth map between manifolds	Total derivative of a map between Euclidean spaces

Definition 8.20 (Differential)**.** M, N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. The *differential of F at p* is the linear map $dF_p : T_pM \to T_{F(p)}N$ defined by

$$dF_p(v) := f \mapsto v(f \circ F)$$

Equivalently, $\forall v \in T_pM, \forall f \in C^\infty(N), dF_p(v)(f) = v(f \circ F)$. This corresponds to "the directional derivative of F at p in the direction of the arrow v."

Definition 8.21 (Coordinate Vectors)**.** Let $(M, \mathcal{A})$ be a smooth manifold without boundary. Let $p \in M$ and choose a chart $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Then the *coordinate vectors at p*, denoted by $\frac{\partial}{\partial x^i}|_p$, are derivations $C^\infty(U) \to \mathbb{R}$ such that

$$\frac{\partial}{\partial x^i}\Big|_p := f \mapsto \frac{\partial}{\partial x^i}\Big|_{\phi(p)}(f \circ \phi^{-1}).$$

Definition 8.22 (Tangent Bundle)**.** Let M be a smooth manifold with or without boundary. The tangent bundle of M, denoted by TM, is the disjoint union $\coprod_{p \in M} T_pM$.

Definition 8.23 (Projection Map)**.** Let M be a smooth manifold with or without boundary. The projection map $\pi : TM \to M$ is the map defined by $(p, v) \mapsto p$.

Definition 8.24 (Curve)**.** If M is a manifold with or without boundary, we define a *curve in M* to be a continuous map $\gamma : J \to M$ where $J \subset \mathbb{R}$ is an interval.

Definition 8.25 (Velocity of a curve)**.** Let $\gamma : J \to M$ and $t_0 \in J$ be given. The *velocity of γ at t_0*, denoted by $\gamma'(t_0)$ is the vector

$$\gamma'(t_0) = d\gamma(\frac{d}{dt}\Big|_{t_0}) \in T_{\gamma(t_0)}M,$$

where $d/dt|_{t_0}$ is the standard coordinate basis vector in $T_{t_0}\mathbb{R}$.

8.4. **Submersions, Immersions, and Embeddings.**

Definition 8.26 (Rank)**.** Let M, N be smooth manifolds with or without boundary and let $F : M \to N$ be a smooth map. Then the rank of F at $p \in M$ is:

- The rank of the linear map $dF_p : T_pM \to T_{F(p)}N$.
- The dimension of the subspace $dF_p(T_pM)$ in the vector space $T_{F(P)}N$.

It is easy to see that the two definitions above are always equivalent.

Definition 8.27 (Submersions and Immersions)**.** Let M, N be smooth manifolds with or without boundary and let $F : M \to N$ be a smooth map.

- If F has the same rank at every point $p \in M$, then F is said to have *constant rank*, and the rank is denoted by $\operatorname{rank} F$.
- If the rank of F at $p \in M$ is equal to $\max\{\dim M, \dim N\}$, then F is said to have *full rank at p*.
- If F has full rank everywhere, then F is said to have *full rank*.
- If F has constant rank and $\operatorname{rank} F = \dim N$, F is called a *smooth submersion*.
- If F has constant rank and $\operatorname{rank} F = \dim M$, F is called a *smooth immersion*.

Definition 8.28 (Local Diffeomorphisms)**.** Let M, N be smooth manifolds with or without boundary, a map $F : M \to N$ is called a local diffeomorphism if every point $p \in M$ has a neighbhorhood U such that $F(U)$ is open in N and $F|_U : U \to F(U)$ is a diffeomorphism.