

INTRODUCTION TO SMOOTH MANIFOLDS

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Exercise 1.1. Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to *any* open subset of \mathbb{R}^n , we require it to be homeomorphic to an open ball in \mathbb{R}^n , or to \mathbb{R}^n itself.

Proof. It is clear that a “manifold” satisfying the open-ball or \mathbb{R}^n definition satisfies the open-subset definition. Let M be a manifold satisfying the open-subset definition. Let $x \in M$ be given and let U, \hat{U}, ϕ be given according to the definition. Since \hat{U} is open, there exists an open ball B such that $\phi(x) \in B \subset \hat{U}$. Restrict ϕ to $\phi^{-1}(B)$. Then $\phi^{-1}(B)$ is an open subset of M containing x , and $\phi|_{\phi^{-1}(B)}$ is a homeomorphism between $\phi^{-1}(B)$ and B . Thus M satisfies the open-ball definition.

$B(x, r) \subset \mathbb{R}^n$ is homeomorphic to \mathbb{R}^n by the map $(x_1 + a_1, \dots, x_n + a_n) \mapsto (\frac{a_1}{r-a_1}, \dots, \frac{a_n}{r-a_n})$ where $x = (x_1, \dots, x_n)$ is the center of $B(x, r)$ and r is the radius. Since the composition of two homeomorphisms gives a homeomorphism, M also satisfies the \mathbb{R}^n definition as well. \square

Exercise 1.6. Show that \mathbb{RP}^n is Hausdorff and second-countable, and is therefore a topological n -manifold.

Proof. From the definition of π , it is easy to see that $\pi(B(x, r))$ is open in \mathbb{RP}^n where $x \in S^n$ and $0 < r < 1$.

Let $[x], [y] \in \mathbb{RP}^n$ be given. Without loss of generality, assume $x, y \in S^n$. Let $r = \min\{|x - y|, |x + y|, 1\}/2$. Then $U_x = \pi(B(x, r)), U_y = \pi(B(y, r))$ contain $[x], [y]$, respectively. $\pi^{-1}(U_x), \pi^{-1}(U_y)$ are both open in $\mathbb{R}^{n+1} \setminus \{0\}$ which can be seen easily by writing down exactly which points belong to them, so U_x, U_y are both open in \mathbb{RP}^n . Then $\pi^{-1}(U_x \cap U_y) = \pi^{-1}(U_x) \cap \pi^{-1}(U_y) = \emptyset$, so $U_x \cap U_y = \emptyset$. Therefore, \mathbb{RP}^n is Hausdorff.

Let $\mathcal{B} = \{\pi(B(x, 1/k)) \mid x \in \mathbb{Q}^{n+1} \cap S^n, k \in \{2, 3, 4, \dots\}\}$. Then \mathcal{B} is a countable collection of open sets whose union is \mathbb{RP}^n . Let $U \subset \mathbb{RP}^n$ be a nonempty open set. Let $[x] \in U$. Since π is a quotient map, $\pi^{-1}(U)$ is open. Moreover, $x \in \pi^{-1}(U)$. Without loss of generality, $x \in S^n$. Then $x \in B(x', 1/k) \subset \pi^{-1}(U)$ for some $B(x', 1/k) \in \mathcal{B}$. Then $[x] = \pi(x) \in \pi(B(x', 1/k)) \subset \pi(\pi^{-1}(U)) = U$. Therefore, \mathcal{B} is a countable basis of \mathbb{RP}^n . \square

Exercise 1.7. Show that \mathbb{RP}^n is compact.

Proof. $\pi(S^n) = \mathbb{RP}^n$ and S^n is compact because it is a closed, bounded subset of \mathbb{R}^{n+1} . (Heine-Borel) Moreover, the image of a compact set under a continuous map is compact. (See A.45(a)) Thus \mathbb{RP}^n is compact. \square

Exercise 1.14. Suppose \mathcal{X} is a locally finite collection of subsets of a topological space M .

- (a) The collection $\{\bar{X} : X \in \mathcal{X}\}$ is also locally finite.
- (b) $\overline{\bigcup_{X \in \mathcal{X}} X} = \bigcup_{X \in \mathcal{X}} \bar{X}$.

Proof.

- (a) Let $p \in M$. Then there exists an open set U containing p such that there are only finitely many $X \in \mathcal{X}$ such that $U \cap X \neq \emptyset$. Let $X \in \mathcal{X}$.
 - If $U \cap X \neq \emptyset$, then $U \cap \bar{X} \supset U \cap X \neq \emptyset$.

- If $U \cap X = \emptyset$, then U^c is closed, so $\overline{X} \subset U^c$. In other words, $U \cap \overline{X} = \emptyset$.

This shows that the number of $X \in \mathcal{X}$ that intersects U and the number of $\overline{X} \in \mathcal{X}$ that intersects U are the same. Therefore, $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.

- (b) Since the closure of a set is defined to be the intersection of all closed sets containing it, $\bigcup_{X \in \mathcal{X}} \overline{X} \subset \overline{\bigcup_{X \in \mathcal{X}} X}$. Let $x \notin \bigcup_{X \in \mathcal{X}} \overline{X}$. Then there exists a neighborhood U of x such that U intersects only finitely many $X \in \mathcal{X}$. Let X_1, \dots, X_n denote them. By the same argument as part (a), $\overline{X_1}, \dots, \overline{X_n}$ are the only elements in $\{\overline{X} \mid X \in \mathcal{X}\}$ that U intersects. Since $x \notin \overline{X_i}$ for each $i = 1, \dots, n$, $U^c \cup \overline{X_1} \cup \dots \cup \overline{X_n}$ is a closed set which contains all $X \in \mathcal{X}$ but does not contain x . In other words, $x \notin \bigcup_{X \in \mathcal{X}} \overline{X}$.

□

Exercise 1.18. Let M be a topological manifold. Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Proof. Let $\mathcal{A}, \mathcal{A}'$ be two smooth atlases.

Suppose that they determine the same smooth structure \mathcal{B} . Then $\mathcal{A} \cup \mathcal{A}' \subset \mathcal{B}$, so $\mathcal{A} \cup \mathcal{A}'$ must be a smooth atlas. By Proposition 1.17(a), $\mathcal{A} \cup \mathcal{A}'$ determines a unique smooth structure, but it must be \mathcal{B} because \mathcal{B} contains the union.

On the other hand, suppose that their union is a smooth atlas. Let \mathcal{B} be the smooth structure that the union determines. Such \mathcal{B} must exist by Proposition 1.17(a). By the same proposition, $\mathcal{A}, \mathcal{A}'$ must determine the unique smooth structures. However, they must be \mathcal{B} because \mathcal{B} contains both \mathcal{A} and \mathcal{A}' .

□

Exercise 1.20. Every smooth manifold has a countable basis of regular coordinate balls.

I thought I had solved this, but I don't think I did. I don't think I understand the definition of regular coordinate balls very well, and that makes it hard for me to solve this problem. In particular, does the B' have to be in the same atlas? Why does the center of $\phi(B)$ have to be 0? On top of that, I can't find much information on regular coordinate balls online, which makes me think that this is not a very useful concept.

Proof.

□

Exercise 1.39. Let M be a topological n -manifold with boundary.

- $\text{Int } M$ is an open subset of M and a topological n -manifold without boundary.
- ∂M is a closed subset of M and a topological $(n-1)$ -manifold without boundary.
- M is a topological manifold if and only if $\partial M = \emptyset$.
- If $n = 0$, then $\partial M = \emptyset$ and M is a 0-manifold.

Proof.

- Let $x \in \text{Int } M$. Let (ϕ, U) be an interior chart for x . Then $x \in U \subset \text{Int } M$ because every point in U is in an interior chart (ϕ, U) . A subspace of M must be Hausdorff and second-countable by Proposition A.17(g, i), so $\text{Int } M$ is a second-countable, Hausdorff space in which every point has a neighborhood homeomorphic to an open subset in \mathbb{R}^n . Thus $\text{Int } M$ is an n -manifold without boundary.
- Since $\partial M = M \setminus \text{Int } M$ and $\text{Int } M$ is open in M , ∂M is closed in M . Let $x \in \partial M$. Let (ϕ, U) be a boundary chart of x . If a point $y \in U$ gets mapped into $\text{Int } \mathbb{H}^n$, then it is certainly an interior point. Thus $\phi(U \cap \partial M) \subset \partial \mathbb{H}^n$. Then $\pi_{n-1} \circ \phi$ is a homeomorphism that maps $U \cap M$ into an open subset of \mathbb{R}^{n-1} where $\pi_{n-1} : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$.
- If ∂M is empty, then $M = \text{Int } M$, so (a) implies that M is an n -dimensional manifold. If M is a topological manifold, every point is an interior point. Since a point cannot be both an interior point and a boundary point, ∂M is empty.

□