

# INTRODUCTION TO SMOOTH MANIFOLDS

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## 1. DICTIONARY

### 1.1. Topological Manifolds.

**Definition 1.1** (Topological Manifold). A *topological  $n$ -manifold* is a Hausdorff, second-countable topological space each point of which has a neighborhood that is homeomorphic to an open subset  $\mathbb{R}^n$ .

**Definition 1.2** (Coordinates). Let  $M$  be a topological  $n$ -manifold. Let  $U$  be an open subset of  $M$ ,  $\hat{U}$  be an open subset of  $\mathbb{R}^n$ ,  $\phi : U \rightarrow \hat{U}$  be a homeomorphism.

- The pair  $(U, \phi)$  is called a *coordinate chart* or a *chart*.
- $U$  is called a *coordinate domain* or a *coordinate neighborhood* and  $\phi$  is called a *coordinate map*.
- If  $\phi(U)$  is an open ball in  $\mathbb{R}^n$ ,  $U$  is called a *coordinate ball*.
- If  $\phi(U)$  is an open cube in  $\mathbb{R}^n$ ,  $U$  is called a *coordinate cube*.

**Definition 1.3** (Atlas). Let  $M$  be a topological  $n$ -manifold. An *atlas for  $M$*  is a collection of charts  $(U_\alpha, \phi_\alpha)$  such that  $M = \bigcup_\alpha U_\alpha$ .

**Definition 1.4** (Transition Map). Let  $M$  be a topological  $n$ -manifold and  $(U, \phi), (V, \psi)$  be coordinate charts such that  $U \cap V \neq \emptyset$ .  $\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$  is called a *transition map* from  $\phi$  to  $\psi$ .

### 1.2. Smooth Manifolds.

**Definition 1.5** (Smoothly Compatible). Let  $M$  be a topological  $n$ -manifold. Two coordinate charts  $(U, \phi), (V, \psi)$  are called *smoothly compatible* if  $U \cap V = \emptyset$  or the transition map  $\psi \circ \phi^{-1}$  is a diffeomorphism.

**Definition 1.6** (Smooth Atlas). Let  $M$  be a topological  $n$ -manifold. A *smooth atlas* is an atlas  $\mathcal{A}$  such that any two charts in  $\mathcal{A}$  are smoothly compatible with each other.

**Definition 1.7** (Smooth Structure). If  $M$  is a topological  $n$ -manifold, an atlas  $\mathcal{A}$  that is not properly contained in any larger smooth atlas is called *maximal* or a *smooth structure on  $M$* .

**Definition 1.8** (Smooth Manifold). A *smooth manifold* is a topological manifold equipped with a smooth structure.

**Definition 1.9.** Suppose  $(M, \mathcal{A})$  is a smooth manifold.

- Any chart  $(U, \phi) \in \mathcal{A}$  is called a *smooth chart*.
- Given a smooth chart  $(U, \phi)$ ,  $U$  is called a *smooth coordinate domain* and  $\phi$  is called a *smooth coordinate map*.

- Given a smooth chart  $(U, \phi)$ ,  $U$  is called a *smooth coordinate ball* if it is a coordinate ball.

*Remark 1.10.* One must define a smooth structure on a topological manifold before talking about a smooth chart.

## 2. CHAPTER 1: SMOOTH MANIFOLDS

**Exercise 1.1.** Show that equivalent definitions of manifolds are obtained if instead of allowing  $U$  to be homeomorphic to *any* open subset of  $\mathbb{R}^n$ , we require it to be homeomorphic to an open ball in  $\mathbb{R}^n$ , or to  $\mathbb{R}^n$  itself.

*Proof.* It is clear that a “manifold” satisfying the open-ball or  $\mathbb{R}^n$  definition satisfies the open-subset definition. Let  $M$  be a manifold satisfying the open-subset definition. Let  $x \in M$  be given and let  $U, \hat{U}, \phi$  be given according to the definition. Since  $\hat{U}$  is open, there exists an open ball  $B$  such that  $\phi(x) \in B \subset \hat{U}$ . Restrict  $\phi$  to  $\phi^{-1}(B)$ . Then  $\phi^{-1}(B)$  is an open subset of  $M$  containing  $x$ , and  $\phi|_{\phi^{-1}(B)}$  is a homeomorphism between  $\phi^{-1}(B)$  and  $B$ . Thus  $M$  satisfies the open-ball definition.

$B(x, r) \subset \mathbb{R}^n$  is homeomorphic to  $\mathbb{R}^n$  by the map  $(x_1 + a_1, \dots, x_n + a_n) \mapsto (\frac{a_1}{r-a_1}, \dots, \frac{a_n}{r-a_n})$  where  $x = (x_1, \dots, x_n)$  is the center of  $B(x, r)$  and  $r$  is the radius. Since the composition of two homeomorphisms gives a homeomorphism,  $M$  also satisfies the  $\mathbb{R}^n$  definition as well.  $\square$

**Exercise 1.6.** Show that  $\mathbb{RP}^n$  is Hausdorff and second-countable, and is therefore a topological  $n$ -manifold.

*Proof.* From the definition of  $\pi$ , it is easy to see that  $\pi(B(x, r))$  is open in  $\mathbb{RP}^n$  where  $x \in S^n$  and  $0 < r < 1$ .

Let  $[x], [y] \in \mathbb{RP}^n$  be given. Without loss of generality, assume  $x, y \in S^n$ . Let  $r = \min\{|x - y|, |x + y|, 1\}/2$ . Then  $U_x = \pi(B(x, r)), U_y = \pi(B(y, r))$  contain  $[x], [y]$ , respectively.  $\pi^{-1}(U_x), \pi^{-1}(U_y)$  are both open in  $\mathbb{R}^{n+1} \setminus \{0\}$  which can be seen easily by writing down exactly which points belong to them, so  $U_x, U_y$  are both open in  $\mathbb{RP}^n$ . Then  $\pi^{-1}(U_x \cap U_y) = \pi^{-1}(U_x) \cap \pi^{-1}(U_y) = \emptyset$ , so  $U_x \cap U_y = \emptyset$ . Therefore,  $\mathbb{RP}^n$  is Hausdorff.

Let  $\mathcal{B} = \{\pi(B(x, 1/k)) \mid x \in \mathbb{Q}^{n+1} \cap S^n, k \in \{2, 3, 4, \dots\}\}$ . Then  $\mathcal{B}$  is a countable collection of open sets whose union is  $\mathbb{RP}^n$ . Let  $U \subset \mathbb{RP}^n$  be a nonempty open set. Let  $[x] \in U$ . Since  $\pi$  is a quotient map,  $\pi^{-1}(U)$  is open. Moreover,  $x \in \pi^{-1}(U)$ . Without loss of generality,  $x \in S^n$ . Then  $x \in B(x', 1/k) \subset \pi^{-1}(U)$  for some  $B(x', 1/k) \in \mathcal{B}$ . Then  $[x] = \pi(x) \in \pi(B(x', 1/k)) \subset \pi(\pi^{-1}(U)) = U$ . Therefore,  $\mathcal{B}$  is a countable basis of  $\mathbb{RP}^n$ .  $\square$

**Exercise 1.7.** Show that  $\mathbb{RP}^n$  is compact.

*Proof.*  $\pi(S^n) = \mathbb{RP}^n$  and  $S^n$  is compact because it is a closed, bounded subset of  $\mathbb{R}^{n+1}$ . (Heine-Borel) Moreover, the image of a compact set under a continuous map is compact. (See A.45(a)) Thus  $\mathbb{RP}^n$  is compact.  $\square$

**Exercise 1.14.** Suppose  $\mathcal{X}$  is a locally finite collection of subsets of a topological space  $M$ .

- The collection  $\{\overline{X} : X \in \mathcal{X}\}$  is also locally finite.
- $\overline{\bigcup_{X \in \mathcal{X}} X} = \bigcup_{X \in \mathcal{X}} \overline{X}$ .

*Proof.*

- Let  $p \in M$ . Then there exists an open set  $U$  containing  $p$  such that there are only finitely many  $X \in \mathcal{X}$  such that  $U \cap X \neq \emptyset$ . Let  $X \in \mathcal{X}$ .
  - If  $U \cap X \neq \emptyset$ , then  $U \cap \overline{X} \supset U \cap X \neq \emptyset$ .
  - If  $U \cap X = \emptyset$ , then  $U^c$  is closed, so  $\overline{X} \subset U^c$ . In other words,  $U \cap \overline{X} = \emptyset$ .

This shows that the number of  $X \in \mathcal{X}$  that intersects  $U$  and the number of  $\overline{X} \in \mathcal{X}$  that intersects  $U$  are the same. Therefore,  $\{\overline{X} : X \in \mathcal{X}\}$  is also locally finite.

- Since the closure of a set is defined to be the intersection of all closed sets containing it,  $\bigcup_{X \in \mathcal{X}} \overline{X} \subset \overline{\bigcup_{X \in \mathcal{X}} X}$ . Let  $x \notin \bigcup_{X \in \mathcal{X}} \overline{X}$ . Then there exists a neighborhood  $U$  of  $x$  such that  $U$  intersects only finitely many  $X \in \mathcal{X}$ . Let  $X_1, \dots, X_n$  denote them. By the same argument as part (a),  $\overline{X_1}, \dots, \overline{X_n}$  are the only elements in  $\{\overline{X} \mid X \in \mathcal{X}\}$  that  $U$  intersects. Since  $x \notin \overline{X_i}$  for each  $i = 1, \dots, n$ ,  $U^c \cup \overline{X_1} \cup \dots \cup \overline{X_n}$  is a closed set which contains all  $X \in \mathcal{X}$  but does not contain  $x$ . In other words,  $x \notin \overline{\bigcup_{X \in \mathcal{X}} X}$ .  $\square$

**Exercise 1.18.** Let  $M$  be a topological manifold. Two smooth atlases for  $M$  determine the same smooth structure if and only if their union is a smooth atlas.

*Proof.* Let  $\mathcal{A}, \mathcal{A}'$  be two smooth atlases.

Suppose that they determine the same smooth structure  $\mathcal{B}$ . Then  $\mathcal{A} \cup \mathcal{A}' \subset \mathcal{B}$ , so  $\mathcal{A} \cup \mathcal{A}'$  must be a smooth atlas. By Proposition 1.17(a),  $\mathcal{A} \cup \mathcal{A}'$  determines a unique smooth structure, but it must be  $\mathcal{B}$  because  $\mathcal{B}$  contains the union.

On the other hand, suppose that their union is a smooth atlas. Let  $\mathcal{B}$  be the smooth structure that the union determines. Such  $\mathcal{B}$  must exist by Proposition 1.17(a). By the same proposition,  $\mathcal{A}, \mathcal{A}'$  must determine the unique smooth structures. However, they must be  $\mathcal{B}$  because  $\mathcal{B}$  contains both  $\mathcal{A}$  and  $\mathcal{A}'$ .  $\square$

**Exercise 1.20.** Every smooth manifold has a countable basis of regular coordinate balls.

*Proof.* Let  $M$  be an  $n$ -dimensional smooth manifold. We consider the special case that there exists a single chart  $(\phi, U)$  with  $U = M$ . Let  $x \in \hat{U}$  with rational coordinates. Then there exists  $s > 0$  such that  $B(x, s) \subset \hat{U}$ . For each rational number  $r \in (0, s)$ , we consider the chart  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r)))$ .

Let  $\mathcal{B}$  be the collection of all such charts for each  $x \in \hat{U}$  and  $r$ . We claim that  $\mathcal{B}$  is a smooth atlas.

- Let  $p \in M$ . Then  $\phi(p) \in \hat{U}$ . Since  $\hat{U}$  is open,  $\phi(p) \in B(x, r) \subset \hat{U}$  for some  $x$  with rational coordinates and a positive rational number  $r$ . Then  $p \in \phi^{-1}(B(x, r))$ , so the union of coordinate domains covers  $M$ . In other words,  $\mathcal{B}$  is an atlas.
- Let  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r)))$ ,  $(p \mapsto \phi(p) - x', \phi^{-1}(B(x', r')))) \in \mathcal{B}$  be given. Suppose  $\phi^{-1}(B(x, r)) \cap \phi^{-1}(B(x', r')) \neq \emptyset$ . Let  $\psi, \psi'$  denote the coordinate maps. Then  $\psi' \circ \psi^{-1}$  is a composition of  $\phi, \phi^{-1}$  and translation maps, so it is smooth.

Therefore,  $\mathcal{B}$  is a smooth atlas.

Since  $\mathcal{B}$  is a smooth atlas, there exists a smooth structure  $\mathcal{A}$  on  $M$  containing  $\mathcal{B}$  by Proposition 1.17(a). We claim that  $\mathcal{B}$ , a subset of the smooth structure  $\mathcal{A}$ , is a countable basis of regular coordinate balls.

- $\mathcal{B}$  is a countable collection because  $x \in \mathbb{Q}^n$  and  $r \in \mathbb{Q}$ .
- Let  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r))) \in \mathcal{B}$  be given. Then there exists a chart  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r'))))$  in  $\mathcal{B}$  with  $r' > r$ . Let  $B = \phi^{-1}(B(x, r))$ ,  $B' = \phi^{-1}(B(x, r'))$ . Let  $\psi$  denote the map  $p \mapsto \phi(p) - x$ . Then  $\psi(B) = B(0, r)$  and  $\psi(B') = B(0, r')$ , respectively. Moreover,  $\psi(\overline{B}) = \overline{B(0, r)}$  because  $\psi$  is a homeomorphism.

Work on the case when there is no chart that covers the entire manifold.

$\square$

**Exercise 1.39.** Let  $M$  be a topological  $n$ -manifold with boundary.

- Int  $M$  is an open subset of  $M$  and a topological  $n$ -manifold without boundary.
- $\partial M$  is a closed subset of  $M$  and a topological  $(n - 1)$ -manifold without boundary.
- $M$  is a topological manifold if and only if  $\partial M = \emptyset$ .
- If  $n = 0$ , then  $\partial M = \emptyset$  and  $M$  is a 0-manifold.

*Proof.*

- Let  $x \in \text{Int } M$ . Let  $(\phi, U)$  be an interior chart for  $x$ . Then  $x \in U \subset \text{Int } M$  because every point in  $U$  is in an interior chart  $(\phi, U)$ . A subspace of  $M$  must be Hausdorff and second-countable by Proposition A.17(g, i), so Int  $M$  is a second-countable, Hausdorff space in which every point has a neighborhood homeomorphic to an open subset in  $\mathbb{R}^n$ . Thus Int  $M$  is an  $n$ -manifold without boundary.
- Since  $\partial M = M \setminus \text{Int } M$  and Int  $M$  is open in  $M$ ,  $\partial M$  is closed in  $M$ . Let  $x \in \partial M$ . Let  $(\phi, U)$  be a boundary chart of  $x$ . If a point  $y \in U$  gets mapped into Int  $\mathbb{H}^n$ , then it is certainly an interior point. Thus  $\phi(U \cap \partial M) \subset \partial \mathbb{H}^n$ . Then  $\pi_{n-1} \circ \phi$  is a homeomorphism that maps  $U \cap \partial M$  into an open subset of  $\mathbb{R}^{n-1}$  where  $\pi_{n-1} : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$ .
- If  $\partial M$  is empty, then  $M = \text{Int } M$ , so (a) implies that  $M$  is an  $n$ -dimensional manifold. If  $M$  is a topological manifold, every point is an interior point. Since a point cannot be both an interior point and a boundary point,  $\partial M$  is empty.
- If  $n = 0$ , then  $\partial \mathbb{H}^0 = \emptyset$ . Thus, the condition that  $\phi(U) \cap \partial \mathbb{H}^n \neq \emptyset$  can never be satisfied, so there cannot be any boundary point.

□

## 3. CHAPTER 2: SMOOTH MAPS

**Exercise 2.1.** Let  $M$  be a smooth manifold with or without boundary. Show that pointwise multiplication turns  $C^\infty(M)$  into a commutative ring and a commutative and associative algebra over  $\mathbb{R}$ .

*Proof.*

- The constant map  $f(p) = 0$  is clearly in  $C^\infty(M)$  and it is the additive identity.
- The constant map  $f(p) = 1$  is clearly in  $C^\infty(M)$  and it is the multiplicative identity.
- Let  $f, g \in C^\infty(M)$ . Let  $p \in M$  and  $(\phi, U)$  be a smooth chart for  $p$ . Then  $f \circ \phi^{-1}$  and  $g \circ \phi^{-1}$  are both smooth (Exercise 2.3), real-valued maps defined on an open subset of  $\mathbb{R}^n$ . Thus  $f + g$  is in  $C^\infty(M)$ . Moreover,  $f + g = g + f$  because addition in  $\mathbb{R}$  is commutative.
- Let  $f, g, h \in C^\infty(M)$ . Let  $p \in M$  and  $(\phi, U)$  be a smooth chart for  $p$ . Then  $f \circ \phi^{-1}$  and  $g \circ \phi^{-1}$  are both smooth (Exercise 2.3), real-valued maps defined on an open subset of  $\mathbb{R}^n$ . Therefore,  $fg$  is in  $C^\infty(M)$ . Moreover,  $fg = gf$  and  $(fg)h = f(gh)$  because multiplication in  $\mathbb{R}$  is commutative and associative.
- Let  $c \in \mathbb{R}, f \in C^\infty(M)$ . Then  $cf$  can be seen as  $fg$  where  $g$  is the constant function whose value is  $c$ . As shown above,  $cf \in C^\infty(M)$ .

□

**Exercise 2.2.** Let  $U$  be an open submanifold of  $\mathbb{R}^n$  with its standard smooth manifold structure. Show that a function  $f : U \rightarrow \mathbb{R}^k$  is smooth in the sense just defined if and only if it is smooth in the sense of ordinary calculus. Do the same for an open submanifold with boundary in  $\mathbb{H}^n$ .

*Proof.*  $f$  is smooth in the sense just defined if and only if  $f \circ \text{Id}^{-1}$  is smooth in the sense of ordinary calculus. Since  $f \circ \text{Id}^{-1} = f$ ,  $f \circ \text{Id}^{-1}$  is smooth in the sense of ordinary calculus if and only if  $f$  is smooth in the sense of ordinary calculus. □

**Exercise 2.3.** Let  $M$  be a smooth manifold with or without boundary, and suppose  $f : M \rightarrow \mathbb{R}^k$  is a smooth function. Show that  $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}^k$  is smooth for every smooth chart  $(U, \phi)$  for  $M$ .

*Proof.* Let  $\phi(x) \in \phi(U)$ . Since  $f$  is smooth, there exists  $(V, \psi)$  such that  $f \circ \psi^{-1} : \psi(V) \rightarrow \mathbb{R}^k$  is smooth and  $x \in V$ . Let  $W = U \cap V$ . Then  $f \circ \psi^{-1} : \psi(W) \rightarrow \mathbb{R}^k$  is smooth and  $\psi \circ \phi^{-1} : \phi(W) \rightarrow \psi(W)$  is a diffeomorphism where  $\phi(W)$  is a neighborhood of  $W$ . Then the restriction of  $f \circ \psi^{-1}$  to  $\phi(W)$  is identical to  $(f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})$ . Since the composition of a smooth function is smooth,  $f \circ \psi^{-1}$  is smooth. □

## 4. APPENDIX C: REVIEW OF CALCULUS

**Exercise C.1.** Suppose that  $F : U \rightarrow W$  is differentiable at  $a \in U$ . Show that the linear map satisfying

$$\lim_{v \rightarrow 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} = 0$$

is unique.

*Proof.* Let  $L, L'$  be two such linear maps.

$$\begin{aligned} \lim_{v \rightarrow 0} \frac{|Lv - L'v|}{|v|} &= \lim_{v \rightarrow 0} \frac{|(F(a+v) - F(a) - L'v) - (F(a+v) - F(a) - Lv)|}{|v|} \\ &= \lim_{v \rightarrow 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} + \lim_{v \rightarrow 0} \frac{|F(a+v) - F(a) - L'v|}{|v|} \\ &= 0 + 0 = 0. \end{aligned}$$

If  $L \neq L'$ ,  $(L - L')v_0 \neq 0$  for some  $v_0$ . Then  $\lim_{v \rightarrow 0} \frac{|Lv - L'v|}{|v|} = \lim_{h \rightarrow 0} \frac{|L(hv_0) - L'(hv_0)|}{|hv_0|} = \frac{|(L - L')v_0|}{|v_0|} \neq 0$ . This is a contradiction, so  $L = L'$ . □