INTRODUCTION TO SMOOTH MANIFOLDS

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Contents

1. Chapter 1: Smooth Manifolds

1.1. Exercises.

Exercise 1.1. Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to *any* open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

Proof. It is clear that a "manifold" satisfying the open-ball or $\mathbb{R}^n$ definition satisfies the open-subset definition. Let M be a manifold satisfying the open-subset definition. Let $x \in M$ be given and let $U, \hat{U}, \phi$ be given according to the definition. Since $\hat{U}$ is open, there exists an open ball B such that $\phi(x) \in B \subset \hat{U}$. Restrict ϕ to $\phi^{-1}(B)$. Then $\phi^{-1}(B)$ is an open subset of M containing x, and $\phi\mid_{\phi^{-1}(B)}$ is a homeomorphism between $\phi^{-1}(B)$ and B. Thus M satisfies the open-ball definition.

$B(x, r) \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ by the map $(x_1 + a_1, \cdots, x_n + a_n) \mapsto (\frac{a_1}{r-a_1}, \cdots, \frac{a_n}{r-a_n})$ where $x = (x_1, \cdots, x_n)$ is the center of $B(x, r)$ and r is the radius. Since the composition of two homeomorphisms gives a homeomorphism, M also satisfies the $\mathbb{R}^n$ definition as well. □

Exercise 1.6. Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.

Proof. From the definition of π, it is easy to see that $\pi(B(x, r))$ is open in $\mathbb{RP}^n$ where $x \in S^n$ and $0 < r < 1$.

Let $[x], [y] \in \mathbb{RP}^n$ be given. Without loss of generality, assume $x, y \in S^n$. Let $r = \min\{|x - y|, |x + y|, 1\}/2$. Then $U_x = \pi(B(x, r)), U_y = \pi(B(y, r))$ contain $[x], [y]$, respectively. $\pi^{-1}(U_x), \pi^{-1}(U_y)$ are both open in $\mathbb{R}^{n+1} \setminus \{0\}$ which can be seen easily by writing down exactly which points belong to them, so U_x, U_y are both open in $\mathbb{RP}^n$. Then $\pi^{-1}(U_x \cap U_y) = \pi^{-1}(U_x) \cap \pi^{-1}(U_y) = \emptyset$, so $U_x \cap U_y = \emptyset$. Therefore, $\mathbb{RP}^n$ is Hausdorff.

Let $\mathcal{B} = \{\pi(B(x, 1/k)) \mid x \in \mathbb{Q}^{n+1} \cap S^n, k \in \{2, 3, 4, \cdots\}\}$. Then $\mathcal{B}$ is a countable collection of open sets whose union is $\mathbb{RP}^n$. Let $U \subset \mathbb{RP}^n$ be a nonempty open set. Let $[x] \in U$. Since π is a quotient map, $\pi^{-1}(U)$ is open. Moreover, $x \in \pi^{-1}(U)$. Without loss of generality, $x \in S^n$. Then $x \in B(x', 1/k) \subset \pi^{-1}(U)$ for some

$B(x', 1/k) \in \mathcal{B}$. Then $[x] = \pi(x) \in \pi(B(x', 1/k)) \subset \pi(\pi^{-1}(U)) = U$. Therefore, $\mathcal{B}$ is a countable basis of $\mathbb{RP}^n$. $\square$

Exercise 1.7. Show that $\mathbb{RP}^n$ is compact.

Proof. $\pi(S^n) = \mathbb{RP}^n$ and S^n is compact because it is a closed, bounded subset of $\mathbb{R}^{n+1}$. (Heine-Borel) Moreover, the image of a compact set under a continuous map is compact. (See A.45(a)) Thus $\mathbb{RP}^n$ is compact. $\square$

Exercise 1.14. Suppose $\mathcal{X}$ is a locally finite collection of subsets of a topological space M.

(a) The collection $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.
(b) $\overline{\bigcup_{X \in \mathcal{X}} X} = \bigcup_{X \in \mathcal{X}} \overline{X}$.

Proof.

(a) Let $p \in M$. Then there exists an open set U containing x such that there are only finitely many $X \in \mathcal{X}$ such that $U \cap X \neq \emptyset$. Let $X \in \mathcal{X}$.
- If $U \cap X \neq \emptyset$, then $U \cap \overline{X} \supset U \cap X \neq \emptyset$.
- If $U \cap X = \emptyset$, then U^c is closed, so $\overline{X} \subset U^c$. In other words, $U \cap \overline{X} = \emptyset$.

This shows that the number of $X \in \mathcal{X}$ that intersects U and the number of $\overline{X} \in \mathcal{X}$ that intersects U are the same. Therefore, $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.

(b) Since the closure of a set is defined to be the intersection of all closed sets containing it, $\bigcup_{X \in \mathcal{X}} \overline{X} \subset \overline{\bigcup_{X \in \mathcal{X}} X}$. Let $x \notin \bigcup_{X \in \mathcal{X}} \overline{X}$. Then there exists a neighborhood U of x such that U intersects only finitely many $X \in \mathcal{X}$. Let $X_1, \cdots, X_n$ denote them. By the same argument as part (a), $\overline{X_1}, \cdots, \overline{X_n}$ are the only elements in $\{\overline{X} \mid X \in \mathcal{X}\}$ that U intersects. Since $x \notin \overline{X_i}$ for each $i = 1, \cdots, n$, $U^c \cup \overline{X_1} \cup \cdots \cup \overline{X_n}$ is a closed set which contains all $X \in \mathcal{X}$ but does not contain x. In other words, $x \notin \overline{\bigcup_{X \in \mathcal{X}} X}$.

$\square$

Exercise 1.18. Let M be a topological manifold. Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Proof. Let $\mathcal{A}, \mathcal{A}'$ be two smooth atlases.

Suppose that they determine the same smooth structure $\mathcal{B}$. Then $\mathcal{A} \cup \mathcal{A}' \subset \mathcal{B}$, so $\mathcal{A} \cup \mathcal{A}'$ must be a smooth atlas. By Proposition 1.17(a), $\mathcal{A} \cup \mathcal{A}'$ determines a unique smooth structure, but it must be $\mathcal{B}$ because $\mathcal{B}$ contains the union.

On the other hand, suppose that their union is a smooth atlas. Let $\mathcal{B}$ be the smooth structure that the union determines. Such $\mathcal{B}$ must exist by Proposition 1.17(a). By the same proposition, $\mathcal{A}, \mathcal{A}'$ must determine the unique smooth structures. However, they must be $\mathcal{B}$ because $\mathcal{B}$ contains both $\mathcal{A}$ and $\mathcal{A}'$. $\square$

Exercise 1.20. Every smooth manifold has a countable basis of regular coordinate balls.

Proof. Let M be an n-dimensional smooth manifold. We consider the special case that there exists a single chart (ϕ, U) with $U = M$. Let $x \in \hat{U}$ with rational coordinates. Then there exists $s > 0$ such that $B(x, s) \subset \hat{U}$. For each rational number $r \in (0, s)$, we consider the chart $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r)))$.

Let $\mathcal{B}$ be the collection of all such charts for each $x \in \hat{U}$ and r. We claim that $\mathcal{B}$ is a smooth atlas.

- Let $p \in M$. Then $\phi(p) \in \hat{U}$. Since $\hat{U}$ is open, $\phi(p) \in B(x, r) \subset \hat{U}$ for some x with rational coordinates and a positive rational number r. Then $p \in \phi^{-1}(B(x, r))$, so the union of coordinate domains covers M. In other words, $\mathcal{B}$ is an atlas.
- Let $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r))), (p \mapsto \phi(p) - x', \phi^{-1}(B(x', r'))) \in \mathcal{B}$ be given. Suppose $\phi^{-1}(B(x, r)) \cap \phi^{-1}(B(x', r')) \neq \emptyset$. Let ψ, ψ' denote the coordinate maps. Then $\psi' \circ \psi^{-1}$ is a composition of ϕ, ϕ^{-1} and translation maps, so it is smooth.

Therefore, $\mathcal{B}$ is a smooth atlas.

Since $\mathcal{B}$ is a smooth atlas, there exists a smooth structure $\mathcal{A}$ on M containing $\mathcal{B}$ by Proposition 1.17(a). We claim that $\mathcal{B}$, a subset of the smooth structure $\mathcal{A}$, is a countable basis of regular coordinate balls.

- $\mathcal{B}$ is a countable collection because $x \in \mathbb{Q}^n$ and $r \in \mathbb{Q}$.

- Let $(p \mapsto \phi(p) - x, \phi^{-1}(B(x,r))) \in \mathcal{B}$ be given. Then there exists a chart $(p \mapsto \phi(p) - x, \phi^{-1}(B(x,r')))$ in $\mathcal{B}$ with $r' > r$. Let $B = \phi^{-1}(B(x,r)), B' = \phi^{-1}(B(x,r'))$. Let ψ denote the map $p \mapsto \phi(p) - x$. Then $\psi(B) = B(0,r)$ and $\psi(B') = B(0,r')$, respectively. Moreover, $\psi(\overline{B}) = \overline{B(0,r)}$ because ψ is a homeomorphism.

Work on the case when there is no chart that covers the entire manifold.

$\square$

Exercise 1.39. Let M be a topological n-manifold with boundary.

(a) $\operatorname{Int} M$ is an open subset of M and a topological n-manifold without boundary.
(b) ∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.
(c) M is a topological manifold if and only if $\partial M = \emptyset$.
(d) If $n = 0$, then $\partial M = \emptyset$ and M is a 0-manifold.

Proof.

(a) Let $x \in \operatorname{Int} M$. Let (ϕ, U) be an interior chart for x. Then $x \in U \subset \operatorname{Int} M$ because every point in U is in an interior chart (ϕ, U). A subspace of M must be Hausdorff and second-countable by Proposition A.17(g, i), so $\operatorname{Int} M$ is a second-countable, Hausdorff space in which every point has a neighborhood homeomorphic to an open subset in $\mathbb{R}^n$. Thus $\operatorname{Int} M$ is an n-manifold without boundary.
(b) Since $\partial M = M \setminus \operatorname{Int} M$ and $\operatorname{Int} M$ is open in M, ∂M is closed in M. Let $x \in \partial M$. Let (ϕ, U) be a boundary chart of x. If a point $y \in U$ gets mapped into $\operatorname{Int} \mathbb{H}^n$, then it is certainly an interior point. Thus $\phi(U \cap \partial M) \subset \partial \mathbb{H}^n$. Then $\pi_{n-1} \circ \phi$ is a homeomorphism that maps $U \cap \partial M$ into an open subset of $\mathbb{R}^{n-1}$ where $\pi_{n-1} : (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_{n-1})$.
(c) If ∂M is empty, then $M = \operatorname{Int} M$, so (a) implies that M is an n-dimensional manifold. If M is a topological manifold, every point is an interior point. Since a point cannot be both an interior point and a boundary point, ∂M is empty.
(d) If $n = 0$, then $\partial \mathbb{H}^0 = \emptyset$. Thus, the condition that $\phi(U) \cap \partial \mathbb{H}^n \neq \emptyset$ can never be satisfied, so there cannot be any boundary point.

$\square$

Exercise 1.41. Let M be a topological manifold with boundary.

(a) M has a countable basis of precompact coordinate balls and half-balls.
(b) M is locally compact.
(c) M is paracompact.
(d) M is locally path-connected.
(e) M has countably many components, each of which is an open subset of M and a connected topological manifold with boundary.
(f) The fundamental group of M is countable.

Proof.

(a)
(b)
(c)
(d) Let $U \subset M$ be a nonempty open subset and choose $x \in U$. Then there exists a chart (V, ϕ) such that $x \in V$. Since $\phi(x)$ is a point in an open set $\phi(U \cap V)$, there exists $r > 0$ such that $B(\phi(x), r) \subset \phi(V)$. Then $N(x, U) = \phi^{-1}(B(\phi(x), r))$ is a path-connected neighborhood of x that is contained in $U \cap V \subset U$. Therefore, $\{N(x, U) \mid \text{open } U \subset M, x \in U\}$ forms a basis of M consisting of path-connected sets.
(e)
(f)

$\square$

Exercise 1.44. Suppose M is a smooth n-manifold with boundary and U is an open subset of M. Prove the following statements:

(a) U is a topological n-manifold with boundary, and the atlas consisting of all smooth charts (V, ϕ) for M such that $V \subset U$ defines a smooth structure on U. With this topology and smooth structure, U is called an ***open submanifold with boundary***.
(b) If $U \subset \operatorname{Int} M$, then U is actually a smooth manifold (without boundary); in this case we call it an ***open submanifold of M***.
(c) $\operatorname{Int} M$ is an open submanifold of M (without boundary).

Proof. Let $\mathcal{T}$ denote the topology of M and $\mathcal{A}$ denote the smooth structure of M.

(a) The subspace topology on U is equivalent to $\mathcal{T}_U = \{V \in \mathcal{T} \mid V \subset U\}$ because U is open. By Proposition A.17(A.18(Proof of Proposition A.17)), U is Hausdorff and second-countable. For every point $p \in U$, there exists a $V \in \mathcal{T}$ with a homeomorphism $\phi : V \to \hat{V}$ where $\hat{V}$ is an open subset of $\mathbb{R}^n$ (or $\mathbb{H}^n$) Since $U \cap V$ is an open subset of V, ϕ restricted to $U \cap V$ is a homeomorphism between $U \cap V$ and $\phi(U \cap V)$, which is an open subset of $\mathbb{R}^n$ (or $\mathbb{H}^n$). Therefore, U is a topological n-manifold with boundary.

Let $\mathcal{A}_U = \{(\phi, V) \in \mathcal{A} \mid V \subset U\}$. Then $\mathcal{A}_U$ is clearly a collection of charts on U whose union covers U. Moreover, any two charts in $\mathcal{A}_U$ are clearly smoothly compatible. Let (ϕ, V) be a chart on U that is smoothly compatible with every chart in $\mathcal{A}_U$. Let $(\psi, W) \in \mathcal{A}$. Then $(\psi_{W \cap U}, W \cap U)$ is a chart on M and it must be smoothly compatible with every chart in $\mathcal{A}$. Therefore, $(\psi_{W \cap U}, W \cap U) \in \mathcal{A}$, so it must belong to $\mathcal{A}_U$. This implies that (ϕ, V) and $(\psi_{W \cap U}, W \cap U)$ are smoothly compatible. Since $V \subset W \cap U$, this implies that (ϕ, V) and (ψ, W) are smoothly compatible.

Thus (ϕ, V) is smoothly compatible with every chart in $\mathcal{A}$, so $(\phi, V) \in \mathcal{A}$. This implies that (ϕ, V) is in $\mathcal{A}_U$, so $\mathcal{A}_U$ is indeed a maximal smooth atlas.

(b) Let $p \in U$. Then $p \in \operatorname{Int} M$, so there exists $(\phi, V) \in \mathcal{A}$ such that $p \in V$ and $\phi(V)$ is open in $\mathbb{R}^n$. Then $(\phi|_{V \cap U}, V \cap U)$ is a chart that is smoothly compatible with every chart in $\mathcal{A}$, so $(\phi|_{V \cap U}, V \cap U) \in \mathcal{A}$. Thus it must be in $\mathcal{A}_U$, so $p \in U$ is an interior point of U. Therefore, U is a manifold without boundary.
(c) By 1.39, $\operatorname{Int} M$ is an open subset of M. By (b), $\operatorname{Int} M$ is an open submanifold of M without boundary.

$\square$

1.2. **Problems.**

Problem 1-2. Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

Proof. Let I denote an uncountable index set and $X = \coprod_{\alpha \in I} \mathbb{R}$. Let $(x, \alpha_0) \in X$. Define $U = \coprod_{\alpha \in I} U_\alpha$ where $U_{\alpha_0} = \mathbb{R}$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_0$. Then U is an open neighborhood of (x, α_0) that is clearly homeomorhpic to $\mathbb{R}$. Thus X is locally Euclidean.

Let $(x_1, \alpha_1) \neq (x_2, \alpha_2) \in X$. If $\alpha_1 \neq \alpha_2$, then open neighborhoods of x_1 and x_2 formed in the same way as above separate the two points. Suppose $\alpha_1 = \alpha_2$. Without loss of generality, $x_1 < x_2$. Define $U = \coprod_{\alpha \in I} U_\alpha$ where $U_{\alpha_1} = (-\infty, (x_1 + x_2)/2)$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_1$. Similarly, define $V = \coprod_{\alpha \in I} U_\alpha$ where $U_{\alpha_1} = ((x_1 + x_2)/2, \infty)$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_2$. Then such U and V separate the two points. Therefore, X is Hausdorff.

Let $\mathcal{B}$ be a basis of X. For each $\alpha_0 \in I$, let $U_{\alpha_0} = \coprod_{\alpha \in I} U_\alpha$ where $U_{\alpha_0} = \mathbb{R}$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_0$. Then for each α_0, there must exist $B_{\alpha_0} \in \mathcal{B}$ such that $(0, \alpha_0) \in B_{\alpha_0} \subset U_{\alpha_0}$. Clearly, $B_\alpha \neq B_\beta$ if $\alpha \neq \beta$. Therefore, the cardinality of $\mathcal{B}$ is greater than or equal to that of I. Hence, X is not second-countable. $\square$

Problem 1-12(Proof of Proposition 1.45). Suppose $M_1, \cdots, M_k$ are smooth manifolds and N is a smooth manifold with boundary. Then $M_1 \times \cdots \times M_k \times N$ is a smooth manifold with boundary, and $\partial(M_1 \times \cdots \times M_k \times N) = M_1 \times \cdots \times M_k \times \partial N$.

Proof. By Example 1.34, $M_1 \times \cdots \times M_k$ is a smooth manifold. Thus it suffices to show that $M \times N$ is a smooth manifold with boundary if M is a smooth manifold and N is a smooth manifold with boundary. Let m, n be the dimensions of M, N.

First, we show that $M \times N$ is a topological manifold with boundary and $\partial(M \times N) = M \times \partial N$. Let $(p, q) \in M \times N$. Then $p \in M$, so there exists a chart (U, ϕ) such that $p \in U$ and $\hat{U} = \phi(U) \subset \mathbb{R}^m$.

- Suppose $q \in \operatorname{Int} N$. Then there exists a chart (V, ψ) such that $\hat{V} = \psi(V) \subset \mathbb{R}^n$. $\phi \times \psi$ is a homeomorphism between $U \times V$ and $\hat{U} \times \hat{V} \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. Thus $(U \times V, \phi \times \psi)$ is a chart for (p, q).
- Suppose $q \in \operatorname{bd} N$. Then there exists a chart (V, ψ) such that $\hat{V} = \psi(V) \subset \mathbb{H}^n$ and $\psi(q) \in \partial\mathbb{H}^n$. $\phi \times \psi$ is a homeomorphism between $U \times V$ and $\hat{U} \times \hat{V} \subset \mathbb{R}^m \times \mathbb{H}^n = \mathbb{H}^{m+n}$. Moreover, $(\phi \times \psi)(p, q) = (\phi(p), \psi(q)) \in \mathbb{R}^m \times \mathbb{H}^n = \mathbb{H}^{m+n}$. Thus $(U \times V, \phi \times \psi)$ is a boundary chart for (p, q).

Therefore, $M \times N$ is a topological manifold with boundary and $\partial(M \times N) = M \times (\partial N)$.

Let $\mathcal{A}_M, \mathcal{A}_N$ be the smooth structures of M, N. Define $\mathcal{A}_{M \times N} = \{(U \times V, \phi \times \psi) \mid (U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N\}$. Then $\mathcal{A}_{M \times N}$ is an atlas because we showed earlier that each $(U \times V, \phi \times \psi)$ is a chart. Let $(U_1 \times V_1, \phi_1 \times \psi_1), (U_2 \times V_2, \phi_2 \times \psi_2) \in \mathcal{A}_{M \times N}$. Then $(\phi_2 \times \psi_2) \circ (\phi_1 \times \psi_1)^{-1} = (\phi_2 \circ \phi_1^{-1}) \times (\psi_2 \circ \psi_1^{-1})$ is a smooth map from $(\phi_1 \times \psi_1)(U_1 \times V_1)$ into $(\phi_2 \times \psi_2)(U_2 \times V_2)$. Thus every pair of charts in $\mathcal{A}_{M \times N}$ is smoothly compatible. In other words, $\mathcal{A}_{M \times N}$ is a smooth atlas.

On the other hand, $\mathcal{A}_{M \times N}$ must be maximal because the restriction of any smoothly compatible chart to M, N gives a smoothly compatible chart, which must belong to $\mathcal{A}_M, \mathcal{A}_N$, respectively. Thus $M \times N$ is a smooth manifold with boundary. □

2. Chapter 2: Smooth Maps

2.1. Exercises.

Exercise 2.1. Let M be a smooth manifold with or without boundary. Show that pointwise multiplication turns $C^\infty(M)$ into a commutative ring and a commutative and associative algebra over $\mathbb{R}$.

Proof.

- The constant map $f(p) = 0$ is clearly in $C^\infty(M)$ and it is the additive identity.
- The constant map $f(p) = 1$ is clearly in $C^\infty(M)$ and it is the multiplicative identity.
- Let $f \in C^\infty(M), g \in C^\infty(M)$. Let $p \in M$ and (ϕ, U) be a smooth chart for p. Then $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are both smooth(Exercise 2.3), real-valued maps defined on an open subset of $\mathbb{R}^n$. Thus $f + g$ is in $C^\infty(M)$ Moreover, $f + g = g + f$ because addition in $\mathbb{R}$ is commutative.
- Let $f, g, h \in C^\infty(M)$. Let $p \in M$ and (ϕ, U) be a smooth chart for p. Then $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are both smooth(Exercise 2.3), real-valued maps defined on an open subset of $\mathbb{R}^n$. Therefore, fg is in $C^\infty(M)$ Moreover, $fg = gf$ and $(fg)h = f(gh)$ because multiplication in $\mathbb{R}$ is commutative and associative.
- Let $c \in \mathbb{R}, f \in C^\infty(M)$. Then cf can be seen as fg where g is the constant function whose value is c. As shown above, $cf \in C^\infty(M)$.

□

Exercise 2.2. Let U be an open submanifold of $\mathbb{R}^n$ with its standard smooth manifold structure. Show that a function $f : U \to \mathbb{R}^k$ is smooth in the sense just defined if and only if it is smooth in the sense of ordinary calculus. Do the same for an open submanifold with boundary in $\mathbb{H}^n$.

Proof. f is smooth in the sense just defined if and only if $f \circ \mathrm{Id}^{-1}$ is smooth in the sense of ordinary calculus. Since $f \circ \mathrm{Id}^{-1} = f$, $f \circ \mathrm{Id}^{-1}$ is smooth in the sense of ordinary calculus if and only if f is smooth in the sense of ordinary calculus. □

Exercise 2.3. Let M be a smooth manifold with or without boundary, and suppose $f : M \to \mathbb{R}^k$ is a smooth function. Show that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$ is smooth for every smooth chart (U, ϕ) for M.

Proof. Let $\phi(x) \in \phi(U)$. Since f is smooth, there exists (V, ψ) such that $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}^k$ is smooth and $x \in V$. Let $W = U \cap V$. Then $f \circ \psi^{-1} : \psi(W) \to \mathbb{R}^k$ is smooth and $\psi \circ \phi^{-1} : \phi(W) \to \psi(W)$ is a diffeomorphism where $\phi(W)$ is a neighborhood of W. Then the restriction of $f \circ \psi^{-1}$ to $\phi(W)$ is identical to $(f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})$. Since he composition of a smooth function is smooth, $f \circ \psi^{-1}$ is smooth. □

Exercise 2.7(Prove Proposition 2.5). Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a map. Then F is smooth if and only if either of the following conditions is satisfied:

(a) For every $p \in M$, there exist smooth charts (U, ϕ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \phi^{-1}$ is smooth from $\phi(U \cap F^{-1}(V))$ to $\psi(V)$.

(b) F is continuous and there exist smooth atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is a smooth map from $\phi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

Proof. Let $\mathcal{A}_M$ and $\mathcal{A}_N$ be smooth structures of M and N. Suppose F is smooth. By Proposition 2.4, F is continuous. For every $p \in M$ there exist coordinate charts (U_p, ϕ_p) containing p and (V_p, ψ_p) containing $F(p)$ such that $F(U_p) \subset V_p$ and $\psi_p \circ F_p \circ \phi_p^{-1}$ is smooth from $\phi_p(U_p)$ to $\psi_p(V_p)$. Then $\{(U_p, \phi_p) \mid p \in M\} \subset \mathcal{A}_M$ and $A_n\{(V_p, \psi_p) \mid p \in M\} \subset \mathcal{A}_N$ are smooth atlases. Moreover, for every (U_p, ϕ_p) and (V_q, ψ_q), $\psi_q \circ F \circ \phi_p^{-1}$ is a smooth map from $\phi_p(U_p \cap F^{-1}(V_q))$ to $\psi_q(V_q)$ because $\psi_q \circ F \circ \phi_p^{-1} = (\psi_q \circ \psi_p^{-1}) \circ (\psi_p \circ F \circ \phi_p^{-1})$ where $\psi_q \circ \psi_q^{-1}$ and $\psi_p \circ F \circ \phi_p^{-1}$ are smooth. Therefore, the definition implies (b).

(b) implies (a) because if F is continuous, $F^{-1}(V_\beta)$ is open in M for every β, so $U \cap F^{-1}(V)$ is open in M.

Finally, we show that (a) implies the definition. Suppose F satisfies (a). Let $p \in M$. Let $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ be smooth charts satisfying the properties described in (a). Let $U' = U \cap F^{-1}(V)$ and consider $(U', \phi\mid_{U'})$. Then $(U', \phi\mid_{U'}) \in \mathcal{A}_M$ because it must be smoothly compatible with any other smooth coordinate chart in $\mathcal{A}_M$. Moreover, $F(U') \subset V$ and $\psi \circ F \circ (\phi\mid_{U'})^{-1} : \phi(U') \to \psi(V)$ is smooth. Therefore, (a) implies the definition.

Hence, (a), (b) and the definition are all equivalent. □

Exercise 2.7(Proof of Proposition 2.6). Let M and N be smooth manifolds with or without boundary, and let $F : M \to N$ be a map.

(a) If every point $p \in M$ has a neighborhood U such that the restriction $F|_U$ is smooth, then F is smooth.

(b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

Proof. Let $\mathcal{A}_M, \mathcal{A}_N$ be smooth structures of M, N, respectively.

(a) Let $p \in M$. Let U be a neighborhood of p such that $F|_U$ is smooth. By 1.44, U is a smooth manifold with the induced smooth structure $\mathcal{A}_U = \{(V, \phi) \in \mathcal{A}_M \mid V \subset U\}$. Since $F|_U$ is smooth, there exist $(V, \phi) \in \mathcal{A}_U$ and $(W, \psi) \in \mathcal{A}_N$ such that:
   - $F|_U(V) \subset W$.
   - $\psi \circ F|_U \circ \phi^{-1} : \phi(V) \to \psi(W)$ is smooth.

   Since $V \subset U$, $F(V) \subset W, \psi \circ F \circ \phi^{-1} : \phi(V) \to \psi(W)$ is smooth, and $(V, \phi) \in \mathcal{A}$. Therefore, F is smooth.

(b) Let $U \subset M$ be an open subset. By 1.44, U is a smooth manifold with the induced smooth structure $\mathcal{A}_U = \{(V, \phi) \in \mathcal{A}_M \mid V \subset U\}$. Let $p \in U$. Then $p \in F$, so there exist $(V, \phi) \in \mathcal{A}_M, (W, \psi) \in \mathcal{A}_N$ such that $F(V) \subset W$ and $\psi \circ F \circ \phi^{-1} : \phi(V) \to \psi(W)$ is smooth. Then $(V \cap U, \phi|_{V \cap U})$ is a chart that is smoothly compatible with every chart in $\mathcal{A}_M$. Therefore, $(V \cap U, \phi|_{V \cap U}) \in \mathcal{A}_M$. Moreover, $\phi|_{V \cap U}(V \cap U) \subset \phi(V) \subset W$ and $\psi \circ F \circ (\phi|_{V \cap U}(V \cap))^{-1}$ is clearly smooth. Therefore, $F|_U$ is smooth.

□

Exercise 2.9. Suppose $F : M \to N$ is a smooth map between smooth manifolds with or without boundary. Show that the coordinate representation of F with respect to *every* pair of smooth charts for M and N is smooth.

Proof. Let $(M, \mathcal{A}_M), (N, \mathcal{A}_N)$ be smooth manifolds with or without boundary. Let $F : M \to N$ be a smooth map. Let $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$ be given. We must show that $\hat{F} = \psi \circ F \circ \phi^{-1}$ is a smooth function from $\phi(U \cap F^{-1}(V))$ to $\psi(V)$. Let $\phi(p) \in \phi(U \cap F^{-1}(V))$. Then $p \in M$, so there exist $(U_0, \phi_0) \in \mathcal{A}_M$ and $(V_0, \psi_0) \in \mathcal{A}_N$ such that

- $p \in U_0 \subset U \cap F^{-1}(V)$;
- $\phi_0(U_0) \subset V_0$;
- $\psi_0 \circ F \circ \phi_0^{-1} : \phi_0(U_0) \to \psi(V_0)$ is smooth.

Then $\psi \circ F \circ \phi^{-1}|_{\phi(U_0)} = (\psi \circ \psi_0^{-1}) \circ (\psi_0 \circ F \circ \phi_0^{-1}) \circ (\phi_0 \circ \phi)$. Since the composition of smooth functions in Euclidean spaces is smooth, $\hat{F}$ is smooth. □

Exercise 2.11(Proof of Proposition 2.10). Let M, N and P be smooth manifolds with or without boundary.

(a) Every constant map $c : M \to N$ is smooth.
(b) The identity map of M is smooth.
(c) If $U \subset M$ is an open submanifold with or without boundary, then the inclusion map $U \to M$ is smooth.

Proof. Let $\mathcal{A}_M, \mathcal{A}_N, \mathcal{A}_P$ be smooth structures of M, N, P, respectively.

(a) F is clearly continuous. Moreover, for every $(U_\alpha, \phi_\alpha) \in \mathcal{A}_M, (V_\beta, \psi_\beta) \in \mathcal{A}_N$, $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is a constant map, so it is smooth. By (2.7(Prove Proposition 2.5)), F is smooth.
(b) Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}_M$ such that $p \in U$. Then $F(U) \subset U$ and $\phi \circ F \circ \phi^{-1} = \mathrm{Id}_U$, so it is smooth. Therefore, F is smooth.
(c) By 1.44, $\mathcal{A}_U = \{(V, \phi) \mid V \subset U\}$ is a smooth structure of U. Let $p \in U$. Then $p \in V$ for some $(V, \phi) \in \mathcal{A}_U$. Then $(V, \phi) \in \mathcal{A}_M$, trivially. Since $F(V) \subset V$ and $\phi \circ F \circ \phi^{-1}$ is simply the identity map on V, F is smooth.

□

Exercise 2.16(Proof of Proposition 2.15).

(a) Every composition of diffeomorphisms is a diffeomorphism.
(b) Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.
(c) Every diffeomorphism is a homeomorphism and an open map.
(d) The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.
(e) "Diffeomorphic" is an equivalence relation on the class of all smooth manifolds with or without boundary.

Exercise 2.16(Proof of Proposition 2.15). Let $(M, \mathcal{A}_M), (N, \mathcal{A}_N), (P, \mathcal{A}_P)$ be smooth manifolds with or without boundary, and let $F : M \to N, G : N \to P$ be diffeomorphisms.

(a) By Proposition 2.10(d), $G \circ F$ and $F^{-1} \circ G^{-1}$ are smooth. Then $(G \circ F) \circ (F^{-1} \circ G^{-1})$ and $(F^{-1} \circ G^{-1}) \circ (G \circ F)$ are both the identity map on the corresponding space, so $F^{-1} \circ G^{-1}$ is the smooth inverse of $G \circ F$. Therefore, $G \circ F$ is a diffeomorphism.
(b) By Example 1.34, we know that $M_1 \times \cdots \times M_k$ and $N_1 \times \cdots \times N_k$ are both smooth manifolds. Let $\mathcal{A}_{M_i}, \mathcal{A}_{N_i}, \mathcal{A}_M$ and $\mathcal{A}_N$ denote the smooth manifold structures of $M_i, N_i, M_1 \times \cdots \times M_k, N_1 \times \cdots \times N_k$, respectively. Let a smooth map $F_i : M_i \to N_i$ be given for each i. Let $(p_1, \cdots, p_k) \in M_1 \times \cdots M_k$ be given. Then there exist $(U_i, \phi_i) \in \mathcal{A}_{M_i}$ and $(V_i, \psi_i) \in \mathcal{A}_{N_i}$ such that $p_i \in U_i, F_i(U_i) \subset V_i, \psi_i \circ F_i \circ \phi_i^{-1} : \phi_i(U_i) \to \psi_i(V_i)$ is smooth for each i. This implies that $(\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots (\psi_k \circ F_k \circ \phi_k^{-1}) = (\psi_1 \times \cdots \times \psi_k) \circ (F_1 \times \cdots \times F_k) \circ (\phi_1 \times \cdots \times \phi_k)^{-1}$ is smooth.
Therefore, $F_1 \times \cdots \times F_k$ is smooth. Using the exact same argument, we can conclude that $F_1^{-1} \times \cdots \times F_k^{-1}$ is smooth. Since $(F_1 \times \cdots \times F_k)^{-1} = F_1^{-1} \times \cdots \times F_k^{-1}$, $F_1 \times \cdots \times F_k$ is a diffeomorphism.
(c) Proposition 2.4 states that every smooth map is continuous. Thus F and F^{-1} are both continuous. Therefore, F is a homeomorphism and also an open map.
(d) Let $U \subset M$ be an open subset. By (2.7(Proof of Proposition 2.6)), $F|_U$ is smooth. Since F is a homeomorphism as shown in (c), $F(U)$ is an open subset of N. Therefore, $F^{-1}|_{F(U)}$ is smooth by (2.7(Proof of Proposition 2.6)). Clearly, $F|_U$ and $F^{-1}|_{F(U)}$ are the inverse of each other. Therefore, $F|_U$ is a diffeomorphism.
(e) By (2.11(Proof of Proposition 2.10)), the identity map on M is a diffeomorphism, so the reflexive property is satisfied. Moreover, $(F^{-1})^{-1} = F$, so the symmetric property is satisfied. By (a), the composition of two diffeomorphisms is a diffeomorphism, so the transitive property is satisfied. Therefore, "diffeomorphic" is an equivalence relation.

Exercise 2.19(Proof of Theorem 2.18). Suppose M and N are smooth manifolds with boundary and $F : M \to N$ is a diffeomorphism. Then $F(\partial M) = \partial N$, and F restricts to a diffeomorphism from $\operatorname{Int} M$ to $\operatorname{Int} N$.

Proof. Let $\mathcal{A}_M, \mathcal{A}_N$ denote the smooth structures of M, N, respectively. Let $p \in \partial M$. Then there exists a chart containing p that sends p to $\partial\mathbb{H}^n$. By Theorem 1.46, every chart containing p sends p to $\partial\mathbb{H}^n$.

Since F is smooth, there exist $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$ such that $F(U) \subset V$ and $\psi \circ F \circ \phi^{-1}$ is a smooth map from $\phi(U)$ to $\psi(V)$. F^{-1} is a homeomorphism by (2.16(Proof of Proposition 2.15)). Then $(\phi^{-1} \circ F^{-1}, F(U))$ is a coordinate chart around $F(p)$ because we obtain a homeomorphism by restricting the composition of two injective continuous maps to its image. Moreover, we claim that $(\phi^{-1} \circ F^{-1}, F(U))$ is smoothly compatible with every chart in $\mathcal{A}_N$. Let $(\psi_1, V_1) \in \mathcal{A}_N$ be given. Then $(\phi^{-1} \circ F^{-1}) \circ \psi_1^{-1} = (\phi^{-1} \circ F^{-1} \circ \psi^{-1}) \circ (\psi \circ \psi_1^{-1})$, and the composition of two smooth maps is smooth. Therefore, $(\phi^{-1} \circ F^{-1}, F(U)) \in \mathcal{A}_N$, and this chart contains $F(p)$ and sends $F(p)$ to $\partial\mathbb{H}^n$. In other words, $F(p) \in \partial N$.

Since F^{-1} is also smooth, $F^{-1}(\partial N) \subset \partial M$. $F^{-1}(\partial N) \subset \partial M \implies F(F^{-1}(\partial N)) \subset F(\partial M) \subset \partial N$. Since F is a bijection, $F(F^{-1}(\partial N)) = \partial N$. Therefore, $F(\partial M) = \partial N$.

This implies that $F(\operatorname{Int} M) = \operatorname{Int} N$. By (1.44(c)) and (2.16(Proof of Proposition 2.15)(d)), F is a diffeomorphism between $\operatorname{Int} M$ and $\operatorname{Int} N$. ☐

2.2. **Problems.**

Problem 2-1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, ϕ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \phi^{-1}$ is smooth as a map from $\phi(U \cap f^{-1}(V))$ to $\psi(V)$, but f is not smooth in the sense we defined in this chapter.

Proof. $\phi = \psi = \operatorname{Id}$ in this solution.

If $x \geq 0$, then let $U = \mathbb{R}, V = (0, \infty)$. Then $\phi(U \cap f^{-1}(V)) = [0, \infty)$. Thus $\psi \circ f \circ \phi^{-1} : [0, \infty) \to (0, \infty)$ is the constant map that sends every number to 1. Therefore, it is smooth.

If $x < 0$, then let $U = \mathbb{R}, V = (-\infty, 1)$. Then $\phi(U \cap f^{-1}(V)) = (-\infty, 0)$. Thus $\psi \circ f \circ \phi^{-1} : (-\infty, 0) \to (-\infty, 1)$ is the constant map that sends every number to 0. Therefore, it is smooth.

It might seem that we can apply (2.7(Prove Proposition 2.5)) to show that f is smooth, but (2.7(Prove Proposition 2.5)) requires that $U \cap f^{-1}(V)$ be open in M.

f maps the interval $(-1, 1)$ to $\{0, 1\}$. Since the image of a connected set under a continuous map must be connected, f cannot be continuous. By Proposition 2.4, f cannot be smooth. ☐

Problem 2-2(Proof of Proposition 2.12). Suppose $M_1, \cdots, M_k$ and N are smooth manifolds with or without boundary, such that at most one of $M_1, \cdots, M_k$ has nonempty boundary. For each i, let $\pi_i : M_1 \times \cdots \times M_k \to M_i$ denote the projection onto the M_i factor. A map $F : N \to M_1 \times \cdots \times M_k$ is smooth if and only if each of the component maps $F_i = \pi_i \circ F : N \to M_i$ is smooth.

Proof. Let $\mathcal{A}_{M_1}, \cdots, \mathcal{A}_{M_k}, \mathcal{A}_N$ be the smooth structures of $M_1, \cdots, M_k, N$. Let $d_1, \cdots, d_k$ denote the dimensions of $M_1, \cdots, M_n$, respectively. Let $d = \sum d_i$.

First, suppose that F is smooth. By (2.11(Proof of Proposition 2.10)), the composition of smooth maps is smooth. Thus it suffices to show that $\pi_i : M_1 \times \cdots \times M_k \to M_i$ is smooth for each i. We show that π_1 is smooth and the other cases can be shown similarly.

Let $(x_1, \cdots, x_k) \in M_1 \times \cdots \times M_k$. Then for each i, there exist $(U_i, \phi_i) \in \mathcal{A}_{M_i}$ and $(V_i, \psi_i) \in \mathcal{A}_{M_i}$ such that $x_i \in U_i$ and $\phi_i(U_i) \subset V_i$. Then we have $(\phi_1 \times \cdots \times \phi_k)(U_1 \times \cdots \times U_k) \subset V_1 \times \cdots \times V_k$ and the composition $\phi_i \circ \pi_1 \circ (\phi_1 \times \cdots \times \phi_k)^{-1}$ is the projection of the first d_1 coordinates from $\mathbb{R}^n$ onto $\mathbb{R}^{d_1}$. Therefore, it is clearly smooth, so π_1 is smooth.

Suppose each $F_i = \pi_i \circ F : N \to M_i$ is smooth. Let $p \in N$. Then for each i, there exist $(U_i, \phi_i) \in \mathcal{A}_N$ and $(V_i, \psi_i) \in \mathcal{A}_{M_i}$ such that $p \in U_i, F_i(U_i) \subset V_i$ and $\psi_k \circ F_i \circ \phi_i^{-1}$. Let $U = U_1 \cap \cdots \cap U_k$. U is a neighborhood of p and the restriction of ϕ_1 to U is a homeomorphism. Then we claim that $(\phi_1, U) \in \mathcal{A}_N$ and $(\psi_1 \times \cdots \times \psi_k, V_1 \times \cdots \times V_k) \in \mathcal{A}_{M_1 \times \cdots \times M_k}$ are charts that satisfy the necessary properties.

- $F(U) \subset V_1 \times \cdots \times V_k$.
- For each i, $\psi_i \circ F_i \circ \phi_1^{-1} = (\psi_i \circ F_i \circ \phi_i^{-1}) \circ (\phi_i \circ \phi_1^{-1}) : \phi_1(U) \to \psi_i(V_i)$ is smooth because the composition of two smooth maps is smooth. Thus $(\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots \times (\psi_k \circ F_k \circ \phi_1^{-1}) : \phi_1(U) \to \psi_1(V_1) \times \cdots \times \psi_k(V_k)$ is smooth. Moreover, $(\psi_1 \times \cdots \times \psi_k) \circ F \circ \phi_1^{-1} = (\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots \times (\psi_k \circ F_k \circ \phi_1^{-1})$.

Therefore, F is smooth. □

3. Chapter 3: Tangent Vectors

Exercise 3.5(Proof of Lemma 3.4). Suppose M is a smooth manifold with or without boundary, $p \in M, v \in T_pM$, and $f, g \in C^\infty(M)$.

(a) If f is a constant function, then $vf = 0$.
(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

Proof.

(a) Let h be the constant function that always takes the value 1. Then $f(p) = ch(p)$ for some $c \in \mathbb{R}$. Then $v(ff) = f(p)vf + f(p)vf$, so $c^2v(h) = c^2v(h) + c^2v(h)$. Therefore, $c^2v(h) = 0$, so $cv(h) = 0$. Since v is linear, this implies $0 = v(ch) = v(f)$, so $v(f) = 0$.
(b) $v(fg) = f(p)vg + g(p)vf = 0 + 0 = 0$.

□

Exercise 3.7(Proof of Proposition 3.6). Let M, N, and P be smooth manifolds with or without boundary, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.

(a) $dF_p : T_pM \to T_{F(p)}N$ is linear.
(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G \circ F(p)}P$.
(c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.
(d) If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Proof. (a) $\forall v, w \in T_pM, \forall c \in \mathbb{R}, \forall f \in C^\infty(N)$,

$$\begin{aligned} dF_p(cv + w)(f) &= (cv + w)(f \circ F) \\ &= (cv)(f \circ F) + w(f \circ F) \\ &= c(v(f \circ F)) + w(f \circ F) \\ &= c(dF_p(v)(f)) + dF_p(w)(f) \\ &= (cdF_p(v))(f) + dF_p(w)(f) \\ &= (cdF_p(v) + dF_p(w))(f). \end{aligned}$$

Therefore, $dF_p(cv + w) = cdF_p(v) + dF_p(w)$.

(b) $\forall v \in T_pM, f \in C^\infty(P)$,

$$\begin{aligned} d(G \circ F)_p(v)(f) &= v(f \circ (G \circ F)) \\ &= v((f \circ G) \circ F) \\ &= (dF_p(v))(f \circ G) \\ &= (dG_{F(p)}(dF_p(v)))(f) \\ &= ((dG_{F(p)} \circ dF_p)(v))(f) \end{aligned}$$

Therefore, $d(G \circ F)_p = dG_{F(p)} \circ dF_p$.

(c) $\forall v \in T_p(M), \forall f \in C^\infty(M)$,

$$\begin{aligned} d(\mathrm{Id}_M)_p(v)(f) &= v(f \circ \mathrm{Id}_M) \\ &= v(f). \end{aligned}$$

Therefore, $d(\mathrm{Id}_M)_p(v) = v$, so $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$.

(d) F^{-1} exists and it is a smooth map since F is a diffeomorphism. By combining (b) and (c), we obtain dF_p and $dF^{-1}_{F(p)}$ are the inverse of each other. Therefore, dF_p is an isomorphism.

□

4. Appendix A: Review of Topology

Exercise A.18(Proof of Proposition A.17). Let X be a topological space and let S be a subspace of X.

(a)
(b)
(c)
(d)
(e)
(f) If $\mathcal{B}$ is a basis for the topology of X, then $\mathcal{B}_S = \{B \cap S \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on S.
(g) If X is Hausdorff, then so is S.
(h) If X is first-countable, then so is S.
(i) If X is second-countable, then so is S.

Proof.

(a)
(b)
(c)
(d)
(e)
(f) The union of $B \cap S$ is S. Let $U \cap S$ be an open subset of S where U is open in X, and $x \in U \cap S$. Then there exists $B \in \mathcal{B}$ such that $x \in B \subset U$ since $\mathcal{B}$ is a basis. Therefore, $x \in B \cap S \subset U \cap S$ with $B \cap S \in \mathcal{B}_S$.
(g) Let $x \neq y \in S$. There exist two disjoint open sets U, V of X containing x, y, respectively. Then $U \cap S$ and $V \cap S$ are disjoint open sets of X containing x, y, respectively.
(h)
(i) Let $\mathcal{B}$ be a countable basis of X. Then $\{B \cap S \mid B \in \mathcal{B}\}$ is a countable basis of S by (f).

□

Exercise A.24(Proof of Proposition A.23). Suppose $X_1, \cdots, X_k$ are topological spaces, and let $X_1 \times \cdots \times X_k$ be their product space.

(a) CHARACTERISTIC PROPERTY: If B is a topological space, a map $F : B \to X_1 \times \cdots \times X_k$ is continuous if and only if each of its component functions $F_i = \pi_i \circ F : B \to X_i$ is continuous.

Proof.

(a) Suppose F is continuous. Since π_i is continuous by (c) and the composition of continuous functions is continuous, $\pi_1 \circ F$ is continuous. Suppose each component function is continuous. Let $B_1 \times \cdots \times B_k$ be a basis element of $X_1 \times \cdots \times X_k$.

$$\begin{aligned} F^{-1}(B_1 \times \cdots \times B_k) &= F^{-1}(\cap_{i=1}^k \pi_i^{-1}(B_1 \times \cdots \times B_k)) \\ &= \cap_{i=1}^k F^{-1}(\pi_i^{-1}(B_1 \times \cdots \times B_k)) \\ &= \cap_{i=1}^k (\pi_i \circ F)^{-1}(B_1 \times \cdots \times B_k). \end{aligned}$$

Since the intersection of finitely many open sets is open, F is continuous.

□

5. Appendix C: Review of Calculus

Exercise C.1. Suppose that $F : U \to W$ is differentiable at $a \in U$. Show that the linear map satisfying

$$\lim_{v\to 0}\frac{|F(a+v)-F(a)-Lv|}{|v|}=0$$

is unique.

Proof. Let L, L' be two such linear maps.

$$\begin{aligned}\lim_{v\to 0}\frac{|Lv-L'v|}{|v|}&=\lim_{v\to 0}\frac{|(F(a+v)-F(a)-L'v)-(F(a+v)-F(a)-Lv)|}{|v|}\\&=\lim_{v\to 0}\frac{|F(a+v)-F(a)-Lv|}{|v|}+\lim_{v\to 0}\frac{|F(a+v)-F(a)-L'v|}{|v|}\\&=0+0=0.\end{aligned}$$

If $L\neq L'$, $(L-L')v_0\neq 0$ for some v_0. Then $\lim_{v\to 0}\frac{|Lv-L'v|}{|v|}=\lim_{h\to 0}\frac{|L(hv_0)-L'(hv_0)|}{|hv_0|}=\frac{|(L-L')v_0|}{|v_0|}\neq 0$. This is a contradiction, so $L=L'$. □

6. Dictionary

6.1. Topological Manifolds.

Definition 6.1 (Topological Manifold)**.** A *topological n-manifold* is a Hausdorff, second-countable topological space each point of which has a neighborhood that is homeomorphic to an open subset $\mathbb{R}^n$.

Definition 6.2 (Coordinates)**.** Let M be a topological n-manifold. Let U be an open subset of M, $\hat{U}$ be an open subset of $\mathbb{R}^n$, $\phi : U \to \hat{U}$ be a homeomorphism.

- The pair (U, ϕ) is called a *coordinate chart* or a *chart*.
- U is called a *coordinate domain* or a *coordinate neighborhood* and ϕ is called a *coordinate map*.
- If $\phi(U)$ is an open ball in $\mathbb{R}^n$, U is called a *coordinate ball*.
- If $\phi(U)$ is an open cube in $\mathbb{R}^n$, U is called a *coordinate cube*.
- The coordinate functions of ϕ are often denoted as $(x^1, \cdots, x^n)$. Thus a chart is sometimes denoted by $(U, (x^1, \cdots, x^n))$ or $(U, (x^i))$.

Definition 6.3 (Atlas)**.** Let M be a topological n-manifold. An *atlas for M* is a collection of charts (U_α, ϕ_α) such that $M = \bigcup_\alpha U_\alpha$.

Definition 6.4 (Transition Map)**.** Let M be a topological n-manifold and $(U, \phi), (V, \psi)$ be coordinate charts such that $U \cap V \neq \emptyset$. $\psi \circ \phi^{-1} : \phi(U \cap V) \mapsto \psi(U \cap V)$ is called a *transition map* from ϕ to ψ.

Definition 6.5 (Closed Upper Half-Space)**.** $\mathbb{H}^n = \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$, and $\partial\mathbb{H}^n = \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid x^n = 0\}$.

Definition 6.6 (Manifold With Boundary)**.** Let M be a second-countable Hausdorff space and fix n. Suppose that for every $p \in M$, one of the following conditions is satisfied:

(1) There exists a neighborhood U of p and a homeomorphism $\phi : U \to \hat{U}$ where $\hat{U}$ is an open subset of $\mathbb{R}^n$. p is called an *interior point* and (U, ϕ) is called an *interior chart*.
(2) There exists a neighborhood U of p and a homeomorphism $\phi : U \to \hat{U}$ where $\hat{U}$ is an open subset of $\mathbb{H}^n$ with $\phi(p) \in \partial\mathbb{H}^n$. p is called a *boundary point*.

Then M is called an *n-dimensional topological manifold with boundary*. Note that every topological manifold is a topological manifold with boundary.

6.2. Smooth Manifolds.

Definition 6.7 (Smoothly Compatible)**.** Let M be a topological n-manifold. Two coordinate charts $(U, \phi), (V, \psi)$ are called *smoothly compatible* if $U \cap V = \emptyset$ or the transition map $\psi \circ \phi^{-1}$ is a diffeomorphism.

Definition 6.8 (Smooth Atlas)**.** Let M be a topological n-manifold. A *smooth atlas* is an atlas $\mathcal{A}$ such that any two charts in $\mathcal{A}$ are smoothly compatible with each other.

Definition 6.9 (Smooth Structure)**.** If M is a topological n-manifold, an atlas $\mathcal{A}$ that is not properly contained in any larger smooth atlas is called *maximal* or a *smooth structure on M*

Definition 6.10 (Smooth Manifold)**.** A *smooth manifold* is a topological manifold equipped with a smooth structure.

Definition 6.11. Suppose $(M, \mathcal{A})$ is a smooth manifold.

- Any chart $(U, \phi) \in \mathcal{A}$ is called a *smooth chart.*
- Given a smooth chart (U, ϕ), U is called a smooth coordinate domain and ϕ is called a *smooth coordinate map.*
- Given a smooth chart (U, ϕ), U is called a *smooth coordinate ball* if it is a coordinate ball.

Remark 6.12. One must define a smooth structure on a topological manifold before talking about a smooth chart.

Definition 6.13 (Smooth Maps)**.** Let M, N be smooth manifolds with or without boundary and $F : M \to N$ be a map. F is a *smooth map* if for every $p \in M$, there exist smooth charts (U, ϕ) containing p and (V, ψ) containing $F(p)$ such that

- $F(U) \subset V$;
- $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth.

Definition 6.14 (Coordinate Representatin of a Smooth Map)**.** Let $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ be smooth manifolds. Let $F : M \to N$ be a smooth map and $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ be given. Then $\hat{F} = \psi \circ F \circ \phi^{-1}$ is called the coordinate representation of F with respect to (U, ϕ) and (V, ψ).

Definition 6.15 (Diffeomorphism)**.** Let M, N be smooth manifolds with or without boundary. A diffeomorphism is a smooth map $F : M \to N$ with a smooth inverse.

6.3. **Tangent Vectors.**

Definition 6.16 (Derivation)**.** Let M be a smooth manifold with or without boundary. A derivation at $p \in M$ is a linear map $v : C^\infty(M) \to \mathbb{R}$ such that

$$v(fg) = f(p)vg + g(p)vf$$

for all $f, g \in C^\infty(M)$.

Definition 6.17 (Tangent Space)**.** The tangent space T_pM to M at p is the vector space of all derivations of $C^\infty(M)$ at p.

Definition 6.18 (Differential)**.** M, N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. The *differential of F at p* is the linear map $dF_p : T_pM \to T_{F(p)}N$ defined by

$$dF_p(v) := f \mapsto v(f \circ F)$$

Equivalently, $\forall v \in T_pM, \forall f \in C^\infty(N), dF_p(v)(f) = v(f \circ F)$.

Definition 6.19 (Coordinate Vectors)**.** Let $(M, \mathcal{A})$ be a smooth manifold without boundary. Let $p \in M$ and choose a chart $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Then the *coordinate vectors at p*, denoted by $\frac{\partial}{\partial x^i}|_p$, are derivations $C^\infty(U) \to \mathbb{R}$ such that

$$\frac{\partial}{\partial x^i}\Big|_p := f \mapsto \frac{\partial}{\partial x^i}\Big|_{\phi(p)} (f \circ \phi^{-1}).$$

Definition 6.20 (Tangent Bundle)**.** Let M be a smooth manifold with or without boundary. The tangent bundle of M, denoted by TM, is the disjoint union $\coprod_{p \in M} T_pM$.

Definition 6.21 (Projection Map)**.** Let M be a smooth manifold with or without boundary. The projection map $\pi : TM \to M$ is the map defined by $(p, v) \mapsto p$.

6.4. **Submersions, Immersions, and Embeddings.**

Definition 6.22 (Rank)**.** Let M, N be smooth manifolds with or without boundary and let $F : M \to N$ be a smooth map. Then the rank of F at $p \in M$ is:

- The rank of the linear map $dF_p : T_pM \to T_{F(p)}N$.
- The dimension of the subspace $dF_p(T_pM)$ in the vector space $T_{F(P)}N$.

It is easy to see that the two definitions above are always equivalent.

Definition 6.23 (Submersions and Immersions)**.** Let M, N be smooth manifolds with or without boundary and let $F : M \to N$ be a smooth map.

- If F has the same rank at every point $p \in M$, then F is said to have *constant rank*, and the rank is denoted by $\operatorname{rank} F$.
- If the rank of F at $p \in M$ is equal to $\max\{\dim M, \dim N\}$, then F is said to have *full rank at* p.
- If F has full rank everywhere, then F is said to have *full rank*.
- If F has constant rank and $\operatorname{rank} F = \dim N$, F is called a *smooth submersion*.
- If F has constant rank and $\operatorname{rank} F = \dim M$, F is called a *smooth immersion*.