INTRODUCTION TO SMOOTH MANIFOLDS

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Contents

1. Chapter 1: Smooth Manifolds

Exercise 1.1. Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to *any* open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

Proof. It is clear that a "manifold" satisfying the open-ball or $\mathbb{R}^n$ definition satisfies the open-subset definition. Let M be a manifold satisfying the open-subset definition. Let $x \in M$ be given and let $U, \hat{U}, \phi$ be given according to the definition. Since $\hat{U}$ is open, there exists an open ball B such that $\phi(x) \in B \subset \hat{U}$. Restrict ϕ to $\phi^{-1}(B)$. Then $\phi^{-1}(B)$ is an open subset of M containing x, and $\phi\mid_{\phi^{-1}(B)}$ is a homeomorphism between $\phi^{-1}(B)$ and B. Thus M satisfies the open-ball definition.

$B(x, r) \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ by the map $(x_1 + a_1, \cdots, x_n + a_n) \mapsto (\frac{a_1}{r-a_1}, \cdots, \frac{a_n}{r-a_n})$ where $x = (x_1, \cdots, x_n)$ is the center of $B(x, r)$ and r is the radius. Since the composition of two homeomorphisms gives a homeomorphism, M also satisfies the $\mathbb{R}^n$ definition as well. □

Exercise 1.6. Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.

Proof. From the definition of π, it is easy to see that $\pi(B(x, r))$ is open in $\mathbb{RP}^n$ where $x \in S^n$ and $0 < r < 1$.

Let $[x], [y] \in \mathbb{RP}^n$ be given. Without loss of generality, assume $x, y \in S^n$. Let $r = \min\{|x - y|, |x + y|, 1\}/2$. Then $U_x = \pi(B(x, r)), U_y = \pi(B(y, r))$ contain $[x], [y]$, respectively. $\pi^{-1}(U_x), \pi^{-1}(U_y)$ are both open in $\mathbb{R}^{n+1} \setminus \{0\}$ which can be seen easily by writing down exactly which points belong to them, so U_x, U_y are both open in $\mathbb{RP}^n$. Then $\pi^{-1}(U_x \cap U_y) = \pi^{-1}(U_x) \cap \pi^{-1}(U_y) = \emptyset$, so $U_x \cap U_y = \emptyset$. Therefore, $\mathbb{RP}^n$ is Hausdorff.

Let $\mathcal{B} = \{\pi(B(x, 1/k)) \mid x \in \mathbb{Q}^{n+1} \cap S^n, k \in \{2, 3, 4, \cdots\}\}$. Then $\mathcal{B}$ is a countable collection of open sets whose union is $\mathbb{RP}^n$. Let $U \subset \mathbb{RP}^n$ be a nonempty open set. Let $[x] \in U$. Since π is a quotient map, $\pi^{-1}(U)$ is open. Moreover, $x \in \pi^{-1}(U)$. Without loss of generality, $x \in S^n$. Then $x \in B(x', 1/k) \subset \pi^{-1}(U)$ for some $B(x', 1/k) \in \mathcal{B}$. Then $[x] = \pi(x) \in \pi(B(x', 1/k)) \subset \pi(\pi^{-1}(U)) = U$. Therefore, $\mathcal{B}$ is a countable basis of $\mathbb{RP}^n$. □

Exercise 1.7. Show that $\mathbb{RP}^n$ is compact.

Proof. $\pi(S^n) = \mathbb{RP}^n$ and S^n is compact because it is a closed, bounded subset of $\mathbb{R}^{n+1}$. (Heine-Borel) Moreover, the image of a compact set under a continuous map is compact. (See A.45(a)) Thus $\mathbb{RP}^n$ is compact. □

Exercise 1.14. Suppose $\mathcal{X}$ is a locally finite collection of subsets of a topological space M.

(a) The collection $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.
(b) $\overline{\bigcup_{X \in \mathcal{X}} X} = \bigcup_{X \in \mathcal{X}} \overline{X}$.

Proof.

(a) Let $p \in M$. Then there exists an open set U containing x such that there are only finitely many $X \in \mathcal{X}$ such that $U \cap X \neq \emptyset$. Let $X \in \mathcal{X}$.
  - If $U \cap X \neq \emptyset$, then $U \cap \overline{X} \supset U \cap X \neq \emptyset$.
  - If $U \cap X = \emptyset$, then U^c is closed, so $\overline{X} \subset U^c$. In other words, $U \cap \overline{X} = \emptyset$.

  This shows that the number of $X \in \mathcal{X}$ that intersects U and the number of $\overline{X} \in \mathcal{X}$ that intersects U are the same. Therefore, $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.

(b) Since the closure of a set is defined to be the intersection of all closed sets containing it, $\bigcup_{X \in \mathcal{X}} \overline{X} \subset \overline{\bigcup_{X \in \mathcal{X}} X}$. Let $x \notin \bigcup_{X \in \mathcal{X}} \overline{X}$. Then there exists a neighborhood U of x such that U intersects only finitely many $X \in \mathcal{X}$. Let $X_1, \cdots, X_n$ denote them. By the same argument as part (a), $\overline{X_1}, \cdots, \overline{X_n}$ are the only elements in $\{\overline{X} \mid X \in \mathcal{X}\}$ that U intersects. Since $x \notin \overline{X_i}$ for each $i = 1, \cdots, n$, $U^c \cup \overline{X_1} \cup \cdots \cup \overline{X_n}$ is a closed set which contains all $X \in \mathcal{X}$ but does not contain x. In other words, $x \notin \overline{\bigcup_{X \in \mathcal{X}} X}$.

□

Exercise 1.18. Let M be a topological manifold. Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Proof. Let $\mathcal{A}, \mathcal{A}'$ be two smooth atlases.

Suppose that they determine the same smooth structure $\mathcal{B}$. Then $\mathcal{A} \cup \mathcal{A}' \subset \mathcal{B}$, so $\mathcal{A} \cup \mathcal{A}'$ must be a smooth atlas. By Proposition 1.17(a), $\mathcal{A} \cup \mathcal{A}'$ determines a unique smooth structure, but it must be $\mathcal{B}$ because $\mathcal{B}$ contains the union.

On the other hand, suppose that their union is a smooth atlas. Let $\mathcal{B}$ be the smooth structure that the union determines. Such $\mathcal{B}$ must exist by Proposition 1.17(a). By the same proposition, $\mathcal{A}, \mathcal{A}'$ must determine the unique smooth structures. However, they must be $\mathcal{B}$ because $\mathcal{B}$ contains both $\mathcal{A}$ and $\mathcal{A}'$. □

Exercise 1.20. Every smooth manifold has a countable basis of regular coordinate balls.

I thought I had solved this, but I don't think I did. I don't think I understand the definition of regular coordinate balls very well, and that makes it hard for me to solve this problem. In particular, does the B' have to be in the same atlas? Why does the center of $\phi(B)$ have to be 0? On top of that, I can't find much information on regular coordinate balls online, which makes me think that this is not a very useful concept.

Proof.

□

Exercise 1.39. Let M be a topological n-manifold with boundary.

(a) $\operatorname{Int} M$ is an open subset of M and a topological n-manifold without boundary.
(b) ∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.
(c) M is a topological manifold if and only if $\partial M = \emptyset$.
(d) If $n = 0$, then $\partial M = \emptyset$ and M is a 0-manifold.

Proof.

(a) Let $x \in \operatorname{Int} M$. Let (ϕ, U) be an interior chart for x. Then $x \in U \subset \operatorname{Int} M$ because every point in U is in an interior chart (ϕ, U). A subspace of M must be Hausdorff and second-countable by Proposition A.17(g, i), so $\operatorname{Int} M$ is a second-countable, Hausdorff space in which every point has a neighborhood homeomorphic to an open subset in $\mathbb{R}^n$. Thus $\operatorname{Int} M$ is an n-manifold without boundary.

(b) Since $\partial M = M \setminus \operatorname{Int} M$ and $\operatorname{Int} M$ is open in M, ∂M is closed in M. Let $x \in \partial M$. Let (ϕ, U) be a boundary chart of x. If a point $y \in U$ gets mapped into $\operatorname{Int} \mathbb{H}^n$, then it is certainly an interior point. Thus $\phi(U \cap \partial M) \subset \partial \mathbb{H}^n$. Then $\pi_{n-1} \circ \phi$ is a homeomorphism that maps $U \cap \partial M$ into an open subset of $\mathbb{R}^{n-1}$ where $\pi_{n-1} : (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_{n-1})$.

(c) If ∂M is empty, then $M = \operatorname{Int} M$, so (a) implies that M is an n-dimensional manifold. If M is a topological manifold, every point is an interior point. Since a point cannot be both an interior point and a boundary point, ∂M is empty.

(d) If $n = 0$, then $\partial \mathbb{H}^0 = \emptyset$. Thus, the condition that $\phi(U) \cap \partial \mathbb{H}^n \neq \emptyset$ can never be satisfied, so there cannot be any boundary point.

□

2. Chapter 2: Smooth Maps

Exercise 2.1. Let M be a smooth manifold with or without boundary. Show that pointwise multiplication turns $C^\infty(M)$ into a commutative ring and a commutative and associative algebra over $\mathbb{R}$.

Proof.

- The constant map $f(p) = 0$ is clearly in $C^\infty(M)$ and it is the additive identity.
- The constant map $f(p) = 1$ is clearly in $C^\infty(M)$ and it is the multiplicative identity.
- Let $f \in C^\infty(M), g \in C^\infty(M)$. Let $p \in M$ and (ϕ, U) be a smooth chart for p. Then $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are both smooth(Exercise 2.3), real-valued maps defined on an open subset of $\mathbb{R}^n$. Thus $f+g$ is in $C^\infty(M)$ Moreover, $f+g = g+f$ because addition in $\mathbb{R}$ is commutative.
- Let $f, g, h \in C^\infty(M)$. Let $p \in M$ and (ϕ, U) be a smooth chart for p. Then $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are both smooth(Exercise 2.3), real-valued maps defined on an open subset of $\mathbb{R}^n$. Therefore, fg is in $C^\infty(M)$ Moreover, $fg = gf$ and $(fg)h = f(gh)$ because multiplication in $\mathbb{R}$ is commutative and associative.
- Let $c \in \mathbb{R}, f \in C^\infty(M)$. Then cf can be seen as fg where g is the constant function whose value is c. As shown above, $cf \in C^\infty(M)$.

$\square$

Exercise 2.2. Let U be an open submanifold of $\mathbb{R}^n$ with its standard smooth manifold structure. Show that a function $f : U \to \mathbb{R}^k$ is smooth in the sense just defined if and only if it is smooth in the sense of ordinary calculus. Do the same for an open submanifold with boundary in $\mathbb{H}^n$.

Proof. f is smooth in the sense just defined if and only if $f^{-1} \circ \mathrm{Id}^{-1}$ is smooth in the sense of ordinary calculus. Since $f^{-1} \circ \mathrm{Id}^{-1} = f^{-1}$, $f^{-1} \circ \mathrm{Id}^{-1}$ is smooth in the sense of ordinary calculus if and only if f^{-1} is smooth in the sense of ordinary calculus. $\square$

Exercise 2.3. Let M be a smooth manifold with or without boundary, and suppose $f : M \to \mathbb{R}^k$ is a smooth function. Show that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$ is smooth for every smooth chart (U, ϕ) for M.

Proof. Let $\phi(x) \in \phi(U)$. Since f is smooth, there exists (V, ψ) such that $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}^k$ is smooth and $x \in V$. Let $W = U \cap V$. Then $f \circ \psi^{-1} : \psi(W) \to \mathbb{R}^k$ is smooth and $\psi \circ \phi^{-1} : \phi(W) \to \psi(W)$ is a diffeomorphism where $\phi(W)$ is a neighborhood of W. Then the restriction of $f \circ \psi^{-1}$ to $\phi(W)$ is identical to $(f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})$. Since he composition of a smooth function is smooth, $f \circ \psi^{-1}$ is smooth. $\square$

3. Appendix C: Review of Calculus

Exercise C.1. Suppose that $F : U \to W$ is differentiable at $a \in U$. Show that the linear map satisfying

$$\lim_{v \to 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} = 0$$

is unique.

Proof. Let L, L' be two such linear maps.

$$\begin{aligned}
\lim_{v \to 0} \frac{|Lv - L'v|}{|v|} &= \lim_{v \to 0} \frac{|(F(a+v) - F(a) - L'v) - (F(a+v) - F(a) - Lv)|}{|v|} \\
&= \lim_{v \to 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} + \lim_{v \to 0} \frac{|F(a+v) - F(a) - L'v|}{|v|} \\
&= 0 + 0 = 0.
\end{aligned}$$

If $L \neq L'$, $(L - L')v_0 \neq 0$ for some v_0. Then $\lim_{v \to 0} \frac{|Lv - L'v|}{|v|} = \lim_{h \to 0} \frac{|L(hv_0) - L'(hv_0)|}{|hv_0|} = \frac{|(L-L')v_0|}{|v_0|} \neq 0$. This is a contradiction, so $L = L'$. $\square$