INTRODUCTION TO SMOOTH MANIFOLDS

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Contents

1. Chapter 1: Smooth Manifolds

Exercise 1.1. Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to *any* open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

Proof. It is clear that a "manifold" satisfying the open-ball or $\mathbb{R}^n$ definition satisfies the open-subset definition. Let M be a manifold satisfying the open-subset definition. Let $x \in M$ be given and let $U, \hat{U}, \phi$ be given according to the definition. Since $\hat{U}$ is open, there exists an open ball B such that $\phi(x) \in B \subset \hat{U}$. Restrict ϕ to $\phi^{-1}(B)$. Then $\phi^{-1}(B)$ is an open subset of M containing x, and $\phi|_{\phi^{-1}(B)}$ is a homeomorphism between $\phi^{-1}(B)$ and B. Thus M satisfies the open-ball definition.

$B(x, r) \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ by the map $(x_1 + a_1, \cdots, x_n + a_n) \mapsto (\frac{a_1}{r-a_1}, \cdots, \frac{a_n}{r-a_n})$ where $x = (x_1, \cdots, x_n)$ is the center of $B(x, r)$ and r is the radius. Since the composition of two homeomorphisms gives a homeomorphism, M also satisfies the $\mathbb{R}^n$ definition as well. $\square$

Exercise 1.6. Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.

Proof. From the definition of π, it is easy to see that $\pi(B(x, r))$ is open in $\mathbb{RP}^n$ where $x \in S^n$ and $0 < r < 1$.

Let $[x], [y] \in \mathbb{RP}^n$ be given. Without loss of generality, assume $x, y \in S^n$. Let $r = \min\{|x-y|, |x+y|, 1\}/2$. Then $U_x = \pi(B(x, r)), U_y = \pi(B(y, r))$ contain $[x], [y]$, respectively. $\pi^{-1}(U_x), \pi^{-1}(U_y)$ are both open in $\mathbb{R}^{n+1} \setminus \{0\}$ which can be seen easily by writing down exactly which points belong to them, so U_x, U_y are both open in $\mathbb{RP}^n$. Then $\pi^{-1}(U_x \cap U_y) = \pi^{-1}(U_x) \cap \pi^{-1}(U_y) = \emptyset$, so $U_x \cap U_y = \emptyset$. Therefore, $\mathbb{RP}^n$ is Hausdorff.

Let $\mathcal{B} = \{\pi(B(x, 1/k)) \mid x \in \mathbb{Q}^{n+1} \cap S^n, k \in \{2, 3, 4, \cdots\}\}$. Then $\mathcal{B}$ is a countable collection of open sets whose union is $\mathbb{RP}^n$. Let $U \subset \mathbb{RP}^n$ be a nonempty open set. Let $[x] \in U$. Since π is a quotient map, $\pi^{-1}(U)$ is open. Moreover, $x \in \pi^{-1}(U)$. Without loss of generality, $x \in S^n$. Then $x \in B(x', 1/k) \subset \pi^{-1}(U)$ for some $B(x', 1/k) \in \mathcal{B}$. Then $[x] = \pi(x) \in \pi(B(x', 1/k)) \subset \pi(\pi^{-1}(U)) = U$. Therefore, $\mathcal{B}$ is a countable basis of $\mathbb{RP}^n$. $\square$

Exercise 1.7. Show that $\mathbb{RP}^n$ is compact.

Proof. $\pi(S^n) = \mathbb{RP}^n$ and S^n is compact because it is a closed, bounded subset of $\mathbb{R}^{n+1}$. (Heine-Borel) Moreover, the image of a compact set under a continuous map is compact. (See A.45(a)) Thus $\mathbb{RP}^n$ is compact. $\square$

Exercise 1.14. Suppose $\mathcal{X}$ is a locally finite collection of subsets of a topological space M.

(a) The collection $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.
(b) $\overline{\bigcup_{X\in\mathcal{X}} X} = \bigcup_{X\in\mathcal{X}} \overline{X}$.

Proof.

(a) Let $p \in M$. Then there exists an open set U containing x such that there are only finitely many $X \in \mathcal{X}$ such that $U \cap X \neq \emptyset$. Let $X \in \mathcal{X}$.
- If $U \cap X \neq \emptyset$, then $U \cap \overline{X} \supset U \cap X \neq \emptyset$.
- If $U \cap X = \emptyset$, then U^c is closed, so $\overline{X} \subset U^c$. In other words, $U \cap \overline{X} = \emptyset$.

This shows that the number of $X \in \mathcal{X}$ that intersects U and the number of $\overline{X} \in \mathcal{X}$ that intersects U are the same. Therefore, $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.

(b) Since the closure of a set is defined to be the intersection of all closed sets containing it, $\bigcup_{X\in\mathcal{X}} \overline{X} \subset \overline{\bigcup_{X\in\mathcal{X}} X}$. Let $x \notin \bigcup_{X\in\mathcal{X}} \overline{X}$. Then there exists a neighborhood U of x such that U intersects only finitely many $X \in \mathcal{X}$. Let $X_1, \cdots, X_n$ denote them. By the same argument as part (a), $\overline{X_1}, \cdots, \overline{X_n}$ are the only elements in $\{\overline{X} \mid X \in \mathcal{X}\}$ that U intersects. Since $x \notin \overline{X_i}$ for each $i = 1, \cdots, n$, $U^c \cup \overline{X_1} \cup \cdots \cup \overline{X_n}$ is a closed set which contains all $X \in \mathcal{X}$ but does not contain x. In other words, $x \notin \overline{\bigcup_{X\in\mathcal{X}} X}$.

□

Exercise 1.18. Let M be a topological manifold. Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Proof. Let $\mathcal{A}, \mathcal{A}'$ be two smooth atlases.

Suppose that they determine the same smooth structure $\mathcal{B}$. Then $\mathcal{A} \cup \mathcal{A}' \subset \mathcal{B}$, so $\mathcal{A} \cup \mathcal{A}'$ must be a smooth atlas. By Proposition 1.17(a), $\mathcal{A} \cup \mathcal{A}'$ determines a unique smooth structure, but it must be $\mathcal{B}$ because $\mathcal{B}$ contains the union.

On the other hand, suppose that their union is a smooth atlas. Let $\mathcal{B}$ be the smooth structure that the union determines. Such $\mathcal{B}$ must exist by Proposition 1.17(a). By the same proposition, $\mathcal{A}, \mathcal{A}'$ must determine the unique smooth structures. However, they must be $\mathcal{B}$ because $\mathcal{B}$ contains both $\mathcal{A}$ and $\mathcal{A}'$. □

Exercise 1.20. Every smooth manifold has a countable basis of regular coordinate balls.

Proof. Let M be an n-dimensional smooth manifold. We consider the special case that there exists a single chart (ϕ, U) with $U = M$. Let $x \in \hat{U}$ with rational coordinates. Then there exists $s > 0$ such that $B(x, s) \subset \hat{U}$. For each rational number $r \in (0, s)$, we consider the chart $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r)))$.

Let $\mathcal{B}$ be the collection of all such charts for each $x \in \hat{U}$ and r. We claim that $\mathcal{B}$ is a smooth atlas.

- Let $p \in M$. Then $\phi(p) \in \hat{U}$. Since $\hat{U}$ is open, $\phi(p) \in B(x, r) \subset \hat{U}$ for some x with rational coordinates and a positive rational number r. Then $p \in \phi^{-1}(B(x, r))$, so the union of coordinate domains covers M. In other words, $\mathcal{B}$ is an atlas.
- Let $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r))), (p \mapsto \phi(p) - x', \phi^{-1}(B(x', r'))) \in \mathcal{B}$ be given. Suppose $\phi^{-1}(B(x, r)) \cap \phi^{-1}(B(x', r')) \neq \emptyset$. Let ψ, ψ' denote the coordinate maps. Then $\psi' \circ \psi^{-1}$ is a composition of ϕ, ϕ^{-1} and translation maps, so it is smooth.

Therefore, $\mathcal{B}$ is a smooth atlas.

Since $\mathcal{B}$ is a smooth atlas, there exists a smooth structure $\mathcal{A}$ on M containing $\mathcal{B}$ by Proposition 1.17(a). We claim that $\mathcal{B}$, a subset of the smooth structure $\mathcal{A}$, is a countable basis of regular coordinate balls.

- $\mathcal{B}$ is a countable collection because $x \in \mathbb{Q}^n$ and $r \in \mathbb{Q}$.
- Let $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r))) \in \mathcal{B}$ be given. Then there exists a chart $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r')))$ in $\mathcal{B}$ with $r' > r$. Let $B = \phi^{-1}(B(x, r)), B' = \phi^{-1}(B(x, r'))$. Let ψ denote the map $p \mapsto \phi(p) - x$. Then $\psi(B) = B(0, r)$ and $\psi(B') = B(0, r')$, respectively. Moreover, $\psi(\overline{B}) = \overline{B(0, r)}$ because ψ is a homeomorphism.

Work on the case when there is no chart that covers the entire manifold.

□

Exercise 1.39. Let M be a topological n-manifold with boundary.

(a) $\operatorname{Int} M$ is an open subset of M and a topological n-manifold without boundary.
(b) ∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.
(c) M is a topological manifold if and only if $\partial M = \emptyset$.
(d) If $n = 0$, then $\partial M = \emptyset$ and M is a 0-manifold.

Proof.

(a) Let $x \in \operatorname{Int} M$. Let (ϕ, U) be an interior chart for x. Then $x \in U \subset \operatorname{Int} M$ because every point in U is in an interior chart (ϕ, U). A subspace of M must be Hausdorff and second-countable by Proposition A.17(g, i), so $\operatorname{Int} M$ is a second-countable, Hausdorff space in which every point has a neighborhood homeomorphic to an open subset in $\mathbb{R}^n$. Thus $\operatorname{Int} M$ is an n-manifold without boundary.
(b) Since $\partial M = M \setminus \operatorname{Int} M$ and $\operatorname{Int} M$ is open in M, ∂M is closed in M. Let $x \in \partial M$. Let (ϕ, U) be a boundary chart of x. If a point $y \in U$ gets mapped into $\operatorname{Int} \mathbb{H}^n$, then it is certainly an interior point. Thus $\phi(U \cap \partial M) \subset \partial \mathbb{H}^n$. Then $\pi_{n-1} \circ \phi$ is a homeomorphism that maps $U \cap \partial M$ into an open subset of $\mathbb{R}^{n-1}$ where $\pi_{n-1} : (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_{n-1})$.
(c) If ∂M is empty, then $M = \operatorname{Int} M$, so (a) implies that M is an n-dimensional manifold. If M is a topological manifold, every point is an interior point. Since a point cannot be both an interior point and a boundary point, ∂M is empty.
(d) If $n = 0$, then $\partial \mathbb{H}^0 = \emptyset$. Thus, the condition that $\phi(U) \cap \partial \mathbb{H}^n \neq \emptyset$ can never be satisfied, so there cannot be any boundary point.

$\square$

Exercise 1.44. Suppose M is a smooth n-manifold with boundary and U is an open subset of M. Prove the following statements:

(a) U is a topological n-manifold with boundary, and the atlas consisting of all smooth charts (V, ϕ) for M such that $V \subset U$ defines a smooth structure on U. With this topology and smooth structure, U is called an ***open submanifold with boundary***.
(b) If $U \subset \operatorname{Int} M$, then U is actually a smooth manifold (without boundary); in this case we call it an ***open submanifold of M***.
(c) $\operatorname{Int} M$ is an open submanifold of M (without boundary).

Proof. Let $\mathcal{T}$ denote the topology of M and $\mathcal{A}$ denote the smooth structure of M.

(a) The subspace topology on U is equivalent to $\mathcal{T}_U = \{V \in \mathcal{T} \mid V \subset U\}$ because U is open. By Proposition A.17(A.18(Proof of Proposition A.17)), U is Hausdorff and second-countable. For every point $p \in U$, there exists a $V \in \mathcal{T}$ with a homeomorphism $\phi : V \to \hat{V}$ where $\hat{V}$ is an open subset of $\mathbb{R}^n$ (or $\mathbb{H}^n$) Since $U \cap V$ is an open subset of V, ϕ restricted to $U \cap V$ is a homeomorphism between $U \cap V$ and $\phi(U \cap V)$, which is an open subset of $\mathbb{R}^n$ (or $\mathbb{H}^n$). Therefore, U is a topological n-manifold with boundary.

Let $\mathcal{A}_U = \{(\phi, V) \in \mathcal{A} \mid V \subset U\}$. Then $\mathcal{A}_U$ is clearly a collection of charts on U whose union covers U. Moreover, any two charts in $\mathcal{A}_U$ are clearly smoothly compatible. Let (ϕ, V) be a chart on U that is smoothly compatible with every chart in $\mathcal{A}_U$. Let $(\psi, W) \in \mathcal{A}$. Then $(\psi_{W \cap U}, W \cap U)$ is a chart on M and it must be smoothly compatible with every chart in $\mathcal{A}$. Therefore, $(\psi_{W \cap U}, W \cap U) \in \mathcal{A}$, so it must belong to $\mathcal{A}_U$. This implies that (ϕ, V) and $(\psi_{W \cap U}, W \cap U)$ are smoothly compatible. Since $V \subset W \cap U$, this implies that (ϕ, V) and (ψ, W) are smoothly compatible.

Thus (ϕ, V) is smoothly compatible with every chart in $\mathcal{A}$, so $(\phi, V) \in \mathcal{A}$. This implies that (ϕ, V) is in $\mathcal{A}_U$, so $\mathcal{A}_U$ is indeed a maximal smooth atlas.
(b) Let $p \in U$. Then $p \in \operatorname{Int} M$, so there exists $(\phi, V) \in \mathcal{A}$ such that $p \in V$ and $\phi(V)$ is open in $\mathbb{R}^n$. Then $(\phi|_{V \cap U}, V \cap U)$ is a chart that is smoothly compatible with every chart in $\mathcal{A}$, so $(\phi|_{V \cap U}, V \cap U) \in \mathcal{A}$. Thus it must be in $\mathcal{A}_U$, so $p \in U$ is an interior point of U. Therefore, U is a manifold without boundary.
(c) By 1.39, $\operatorname{Int} M$ is an open subset of M. By (b), $\operatorname{Int} M$ is an open submanifold of M without boundary.

$\square$

2. Chapter 2: Smooth Maps

Exercise 2.1. Let M be a smooth manifold with or without boundary. Show that pointwise multiplication turns $C^\infty(M)$ into a commutative ring and a commutative and associative algebra over $\mathbb{R}$.

Proof.

- The constant map $f(p) = 0$ is clearly in $C^\infty(M)$ and it is the additive identity.
- The constant map $f(p) = 1$ is clearly in $C^\infty(M)$ and it is the multiplicative identity.
- Let $f \in C^\infty(M), g \in C^\infty(M)$. Let $p \in M$ and (ϕ, U) be a smooth chart for p. Then $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are both smooth(Exercise 2.3), real-valued maps defined on an open subset of $\mathbb{R}^n$. Thus $f+g$ is in $C^\infty(M)$ Moreover, $f+g = g+f$ because addition in $\mathbb{R}$ is commutative.
- Let $f, g, h \in C^\infty(M)$. Let $p \in M$ and (ϕ, U) be a smooth chart for p. Then $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are both smooth(Exercise 2.3), real-valued maps defined on an open subset of $\mathbb{R}^n$. Therefore, fg is in $C^\infty(M)$ Moreover, $fg = gf$ and $(fg)h = f(gh)$ because multiplication in $\mathbb{R}$ is commutative and associative.
- Let $c \in \mathbb{R}, f \in C^\infty(M)$. Then cf can be seen as fg where g is the constant function whose value is c. As shown above, $cf \in C^\infty(M)$.

□

Exercise 2.2. Let U be an open submanifold of $\mathbb{R}^n$ with its standard smooth manifold structure. Show that a function $f : U \to \mathbb{R}^k$ is smooth in the sense just defined if and only if it is smooth in the sense of ordinary calculus. Do the same for an open submanifold with boundary in $\mathbb{H}^n$.

Proof. f is smooth in the sense just defined if and only if $f \circ \mathrm{Id}^{-1}$ is smooth in the sense of ordinary calculus. Since $f \circ \mathrm{Id}^{-1} = f$, $f \circ \mathrm{Id}^{-1}$ is smooth in the sense of ordinary calculus if and only if f is smooth in the sense of ordinary calculus. □

Exercise 2.3. Let M be a smooth manifold with or without boundary, and suppose $f : M \to \mathbb{R}^k$ is a smooth function. Show that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$ is smooth for every smooth chart (U, ϕ) for M.

Proof. Let $\phi(x) \in \phi(U)$. Since f is smooth, there exists (V, ψ) such that $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}^k$ is smooth and $x \in V$. Let $W = U \cap V$. Then $f \circ \psi^{-1} : \psi(W) \to \mathbb{R}^k$ is smooth and $\psi \circ \phi^{-1} : \phi(W) \to \psi(W)$ is a diffeomorphism where $\phi(W)$ is a neighborhood of W. Then the restriction of $f \circ \psi^{-1}$ to $\phi(W)$ is identical to $(f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})$. Since he composition of a smooth function is smooth, $f \circ \psi^{-1}$ is smooth. □

Exercise 2.7(Prove Proposition 2.5). Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a map. Then F is smooth if and only if either of the following conditions is satisfied:

(a) For every $p \in M$, there exist smooth charts (U, ϕ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \phi^{-1}$ is smooth from $\phi(U \cap F^{-1}(V))$ to $\psi(V)$.

(b) F is continuous and there exist smooth atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is a smooth map from $\phi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

Proof. Let $\mathcal{A}_M$ and $\mathcal{A}_N$ be smooth structures of M and N. Suppose F is smooth. By Proposition 2.4, F is continuous. For every $p \in M$ there exist coordinate charts (U_p, ϕ_p) containing p and (V_p, ψ_p) containing $F(p)$ such that $F(U_p) \subset V_p$ and $\psi_p \circ F_p \circ \phi_p^{-1}$ is smooth from $\phi_p(U_p)$ to $\psi_p(V_p)$. Then $\{(U_p, \phi_p) \mid p \in M\} \subset \mathcal{A}_M$ and $A_n\{(V_p, \psi_p) \mid p \in M\} \subset \mathcal{A}_N$ are smooth atlases. Moreover, for every (U_p, ϕ_p) and (V_q, ψ_q), $\psi_q \circ F \circ \phi_p^{-1}$ is a smooth map from $\phi_p(U_p \cap F^{-1}(V_q))$ to $\psi_q(V_q)$ because $\psi_q \circ F \circ \phi_p^{-1} = (\psi_q \circ \psi_p^{-1}) \circ (\psi_p \circ F \circ \phi_p^{-1})$ where $\psi_q \circ \psi_q^{-1}$ and $\psi_p \circ F \circ \phi_p^{-1}$ are smooth. Therefore, the definition implies (b).

(b) implies (a) because if F is continuous, $F^{-1}(V_\beta)$ is open in M for every β, so $U \cap F^{-1}(V)$ is open in M.

Finally, we show that (a) implies the definition. Suppose F satisfies (a). Let $p \in M$. Let $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ be smooth charts satisfying the properties described in (a). Let $U' = U \cap F^{-1}(V)$ and consider $(U', \phi \mid_{U'})$. Then $(U', \phi \mid_{U'}) \in \mathcal{A}_M$ because it must be smoothly compatible with any other smooth coordinate chart in $\mathcal{A}_M$. Moreover, $F(U') \subset V$ and $\psi \circ F \circ (\phi \mid_{U'})^{-1} : \phi(U') \to \psi(V)$ is smooth. Therefore, (a) implies the definition.

Hence, (a), (b) and the definition are all equivalent. □

Exercise 2.7(Proof of Proposition 2.6). Let M and N be smooth manifolds with or without boundary, and let $F : M \to N$ be a map.

(a) If every point $p \in M$ has a neighborhood U such that the restriction $F|_U$ is smooth, then F is smooth.
(b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

Proof. Let $\mathcal{A}_M, \mathcal{A}_N$ be smooth structures of M, N, respectively.

(a) Let $p \in M$. Let U be a neighborhood of p such that $F|_U$ is smooth. By 1.44, U is a smooth manifold with the induced smooth structure $\mathcal{A}_U = \{(V, \phi) \in \mathcal{A}_M \mid V \subset U\}$. Since $F|_U$ is smooth, there exist $(V, \phi) \in \mathcal{A}_U$ and $(W, \psi) \in \mathcal{A}_N$ such that:
   - $F|_U(V) \subset W$.
   - $\psi \circ F|_U \circ \phi^{-1} : \phi(V) \to \psi(W)$ is smooth.

   Since $V \subset U$, $F(V) \subset W, \psi \circ F \circ \phi^{-1} : \phi(V) \to \psi(W)$ is smooth, and $(V, \phi) \in \mathcal{A}$. Therefore, F is smooth.
(b) Let $U \subset M$ be an open subset. By 1.44, U is a smooth manifold with the induced smooth structure $\mathcal{A}_U = \{(V, \phi) \in \mathcal{A}_M \mid V \subset U\}$. Let $p \in U$. Then $p \in F$, so there exist $(V, \phi) \in \mathcal{A}_M, (W, \psi)\imath\mathcal{A}_N$ such that $F(V) \subset W$ and $\psi \circ F \circ \phi^{-1} : \phi(V) \to \psi(W)$ is smooth. Then $(V \cap U, \phi|_{V \cap U})$ is a chart that is smoothly compatible with every chart in $\mathcal{A}_M$. Therefore, $(V \cap U, \phi|_{V \cap U}) \in \mathcal{A}_M$. Moreover, $\phi|_{V \cap U}(V \cap U) \subset \phi(V) \subset W$ and $\psi \circ F \circ (\phi|_{V \cap U}(V \cap))^{-1}$ is clearly smooth. Therefore, $F|_U$ is smooth.

□

Exercise 2.9. Suppose $F : M \to N$ is a smooth map between smooth manifolds with or without boundary. Show that the coordinate representation of F with respect to *every* pair of smooth charts for M and N is smooth.

Proof. Let $(M, \mathcal{A}_M), (N, \mathcal{A}_N)$ be smooth manifolds with or without boundary. Let $F : M \to N$ be a smooth map. Let $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$ be given. We must show that $\hat{F} = \psi \circ F \circ \phi^{-1}$ is a smooth function from $\phi(U \cap F^{-1}(V))$ to $\psi(V)$. Let $\phi(p) \in \phi(U \cap F^{-1}(V))$. Then $p \in M$, so there exist $(U_0, \phi_0) \in \mathcal{A}_M$ and $(V_0, \psi_0) \in \mathcal{A}_N$ such that

- $p \in U_0 \subset U \cap F^{-1}(V)$;
- $\phi_0(U_0) \subset V_0$;
- $\psi_0 \circ F \circ \phi_0^{-1} : \phi_0(U_0) \to \psi(V_0)$ is smooth.

Then $\psi \circ F \circ \phi^{-1}|_{\phi(U_0)} = (\psi \circ \psi_0^{-1}) \circ (\psi_0 \circ F \circ \phi_0^{-1}) \circ (\phi_0 \circ \phi)$. Since the composition of smooth functions in Euclidean spaces is smooth, $\hat{F}$ is smooth. □

Exercise 2.11(Proof of Proposition 2.10). Let M, N and P be smooth manifolds with or without boundary.

(a) Every constant map $c : M \to N$ is smooth.
(b) The identity map of M is smooth.
(c) If $U \subset M$ is an open submanifold with or without boundary, then the inclusion map $U \to M$ is smooth.

Proof. Let $\mathcal{A}_M, \mathcal{A}_N, \mathcal{A}_P$ be smooth structures of M, N, P, respectively.

(a) F is clearly continuous. Moreover, for every $(U_\alpha, \phi_\alpha) \in \mathcal{A}_M, (V_\beta, \psi_\beta) \in \mathcal{A}_N$, $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is a constant map, so it is smooth. By 2.7(Proof of Proposition 2.6), F is smooth.
(b) Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}_M$ such that $p \in U$. Then $F(U) \subset U$ and $\phi \circ F \circ \phi^{-1} = \mathrm{Id}_U$, so it is smooth. Therefore, F is smooth.
(c) By 1.44, $\mathcal{A}_U = \{(V, \phi) \mid V \subset U\}$ is a smooth structure of U. Let $p \in U$. Then $p \in V$ for some $(V, \phi) \in \mathcal{A}_U$. Then $(V, \phi) \in \mathcal{A}_M$, trivially. Since $F(V) \subset V$ and $\phi \circ F \circ \phi^{-1}$ is simply the identity map on V, F is smooth.

□

Exercise 2.16(Proof of Proposition 2.15).

(a) Every composition of diffeomorphisms is a diffeomorphism.

(b) Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.
(c) Every diffeomorphism is a homeomorphism and an open map.
(d) The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.
(e) "Diffeomorphic" is an equivalence relation on the class of all smooth manifolds with or without boundary.

Exercise 2.16(Proof of Proposition 2.15). Let $(M, \mathcal{A}_M), (N, \mathcal{A}_N), (P, \mathcal{A}_P)$ be smooth manifolds with or without boundary, and let $F : M \to N, G : N \to P$ be diffeomorphisms.

(a) By Proposition 2.10(d), $G \circ F$ and $F^{-1} \circ G^{-1}$ are smooth. Then $(G \circ F) \circ (F^{-1} \circ G^{-1})$ and $(F^{-1} \circ G^{-1}) \circ (G \circ F)$ are both the identity map on the corresponding space, so $F^{-1} \circ G^{-1}$ is the smooth inverse of $G \circ F$. Therefore, $G \circ F$ is a diffeomorphism.
(b)
(c) Proposition 2.4 states that every smooth map is continuous. Thus F and F^{-1} are both continuous. Therefore, F is a homeomorphism and also an open map.
(d) Let $U \subset M$ be an open subset. By (2.7(Proof of Proposition 2.6)), $F|_U$ is smooth. Since F is a homeomorphism as shown in (c), $F(U)$ is an open subset of N. Therefore, $F^{-1}|_{F(U)}$ is smooth by (2.7(Proof of Proposition 2.6)). Clearly, $F|_U$ and $F^{-1}|_{F(U)}$ are the inverse of each other. Therefore, $F|_U$ is a diffeomorphism.
(e) By (2.11(Proof of Proposition 2.10)), the identity map on M is a diffeomorphism, so the reflexive property is satisfied. Moreover, $(F^{-1})^{-1} = F$, so the symmetric property is satisfied. By (a), the composition of two diffeomorphisms is a diffeomorphism, so the transitive property is satisfied. Therefore, "diffeomorphic" is an equivalence relation.

3. Chapter 3: Tangent Vectors

Exercise 3.5(Proof of Lemma 3.4). Suppose M is a smooth manifold with or without boundary, $p \in M, v \in T_pM$, and $f, g \in C^\infty(M)$.

(a) If f is a constant function, then $vf = 0$.
(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

Proof.

(a) Let h be the constant function that always takes the value 1. Then $f(p) = ch(p)$ for some $c \in \mathbb{R}$. Then $v(ff) = f(p)vf + f(p)vf$, so $c^2v(h) = c^2v(h) + c^2v(h)$. Therefore, $c^2v(h) = 0$, so $cv(h) = 0$. Since v is linear, this implies $0 = v(ch) = v(f)$, so $v(f) = 0$.
(b) $v(fg) = f(p)vg + g(p)vf = 0 + 0 = 0$.

□

Exercise 3.7(Proof of Proposition 3.6). Let M, N, and P be smooth manifolds with or without boundary, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.

(a) $dF_p : T_pM \to T_{F(p)}N$ is linear.
(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}P$.
(c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.
(d) If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Proof. (a) $\forall v, w \in T_pM, \forall c \in \mathbb{R}, \forall f \in C^\infty(N)$,

$$\begin{aligned}
dF_p(cv + w)(f) &= (cv + w)(f \circ F) \\
&= (cv)(f \circ F) + w(f \circ F) \\
&= c(v(f \circ F)) + w(f \circ F) \\
&= c(dF_p(v)(f)) + dF_p(w)(f) \\
&= (cdF_p(v))(f) + dF_p(w)(f) \\
&= (cdF_p(v) + dF_p(w))(f).
\end{aligned}$$

Therefore, $dF_p(cv + w) = cdF_p(v) + dF_p(w)$.

(b) $\forall v \in T_pM, f \in C^\infty(P)$,

$$\begin{aligned} d(G\circ F)_p(v)(f) &= v(f\circ(G\circ F)) \\ &= v((f\circ G)\circ F) \\ &= (dF_p(v))(f\circ G) \\ &= (dG_{F(p)}(dF_p(v)))(f) \\ &= ((dG_{F(p)}\circ dF_p)(v))(f) \end{aligned}$$

Therefore, $d(G\circ F)_p = dG_{F(p)} \circ dF_p$.

(c) $\forall v \in T_p(M), \forall f \in C^\infty(M)$,

$$\begin{aligned} d(\mathrm{Id}_M)_p(v)(f) &= v(f\circ \mathrm{Id}_M) \\ &= v(f). \end{aligned}$$

Therefore, $d(\mathrm{Id}_M)_p(v) = v$, so $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$.

(d) F^{-1} exists and it is a smooth map since F is a diffeomorphism. By combining (b) and (c), we obtain dF_p and $dF^{-1}_{F(p)}$ are the inverse of each other. Therefore, dF_p is an isomorphism.

□

4. Appendix A: Review of Topology

Exercise A.18(Proof of Proposition A.17). Let X be a topological space and let S be a subspace of X.

(a)
(b)
(c)
(d)
(e)
(f) If $\mathcal{B}$ is a basis for the topology of X, then $\mathcal{B}_S = \{B\cap S \mid B\in\mathcal{B}\}$ is a basis for the subspace topology on S.
(g) If X is Hausdorff, then so is S.
(h) If X is first-countable, then so is S.
(i) If X is second-countable, then so is S.

Proof.

(a)
(b)
(c)
(d)
(e)
(f) The union of $B\cap S$ is S. Let $U\cap S$ be an open subset of S where U is open in X, and $x\in U\cap S$. Then there exists $B\in\mathcal{B}$ such that $x\in B\subset U$ since $\mathcal{B}$ is a basis. Therefore, $x\in B\cap S\subset U\cap S$ with $B\cap S\in\mathcal{B}_S$.
(g) Let $x\neq y\in S$. There exist two disjoint open sets U,V of X containing x,y, respectively. Then $U\cap S$ and $V\cap S$ are disjoint open sets of X containing x,y, respectively.
(h)
(i) Let $\mathcal{B}$ be a countable basis of X. Then $\{B\cap S\mid B\in\mathcal{B}\}$ is a countable basis of S by (f).

□

Exercise A.24(Proof of Proposition A.23). Suppose $X_1,\cdots,X_k$ are topological spaces, and let $X_1\times\cdots\times X_k$ be their product space.

(a) CHARACTERISTIC PROPERTY: If B is a topological space, a map $F: B\to X_1\times\cdots\times X_k$ is continuous if and only if each of its component functions $F_i=\pi_i\circ F: B\to X_i$ is continuous.

Proof.

(a) Suppose F is continuous. Since π_i is continuous by (c) and the composition of continuous functions is continuous, $\pi_1 \circ F$ is continuous. Suppose each component function is continuous. Let $B_1 \times \cdots \times B_k$ be a basis element of $X_1 \times \cdots \times X_k$.

$$\begin{aligned} F^{-1}(B_1 \times \cdots \times B_k) &= F^{-1}(\cap_{i=1}^k \pi_i^{-1}(B_1 \times \cdots \times B_k)) \\ &= \cap_{i=1}^k F^{-1}(\pi_i^{-1}(B_1 \times \cdots \times B_k)) \\ &= \cap_{i=1}^k (\pi_i \circ F)^{-1}(B_1 \times \cdots \times B_k). \end{aligned}$$

Since the intersection of finitely many open sets is open, F is continuous.

□

5. Appendix C: Review of Calculus

Exercise C.1. Suppose that $F : U \to W$ is differentiable at $a \in U$. Show that the linear map satisfying

$$\lim_{v \to 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} = 0$$

is unique.

Proof. Let L, L' be two such linear maps.

$$\begin{aligned} \lim_{v \to 0} \frac{|Lv - L'v|}{|v|} &= \lim_{v \to 0} \frac{|(F(a+v) - F(a) - L'v) - (F(a+v) - F(a) - Lv)|}{|v|} \\ &= \lim_{v \to 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} + \lim_{v \to 0} \frac{|F(a+v) - F(a) - L'v|}{|v|} \\ &= 0 + 0 = 0. \end{aligned}$$

If $L \neq L'$, $(L - L')v_0 \neq 0$ for some v_0. Then $\lim_{v \to 0} \frac{|Lv - L'v|}{|v|} = \lim_{h \to 0} \frac{|L(hv_0) - L'(hv_0)|}{|hv_0|} = \frac{|(L-L')v_0|}{|v_0|} \neq 0$. This is a contradiction, so $L = L'$. □

6. Dictionary

6.1. Topological Manifolds.

Definition 6.1 (Topological Manifold)**.** A *topological n-manifold* is a Hausdorff, second-countable topological space each point of which has a neighborhood that is homeomorphic to an open subset $\mathbb{R}^n$.

Definition 6.2 (Coordinates)**.** Let M be a topological n-manifold. Let U be an open subset of M, $\hat{U}$ be an open subset of $\mathbb{R}^n$, $\phi : U \to \hat{U}$ be a homeomorphism.

- The pair (U, ϕ) is called a *coordinate chart* or a *chart.*
- U is called a *coordinate domain* or a *coordinate neighborhood* and ϕ is called a *coordinate map.*
- If $\phi(U)$ is an open ball in $\mathbb{R}^n$, U is called a *coordinate ball.*
- If $\phi(U)$ is an open cube in $\mathbb{R}^n$, U is called a *coordinate cube.*
- The coordinate functions of ϕ are often denoted as $(x^1, \cdots, x^n)$. Thus a chart is sometimes denoted by $(U, (x^1, \cdots, x^n))$ or $(U, (x^i))$.

Definition 6.3 (Atlas)**.** Let M be a topological n-manifold. An *atlas for M* is a collection of charts (U_α, ϕ_α) such that $M = \bigcup_\alpha U_\alpha$.

Definition 6.4 (Transition Map)**.** Let M be a topological n-manifold and $(U, \phi), (V, \psi)$ be coordinate charts such that $U \cap V \neq \emptyset$. $\psi \circ \phi^{-1} : \phi(U \cap V) \mapsto \psi(U \cap V)$ is called a *transition map* from ϕ to ψ.

Definition 6.5 (Closed Upper Half-Space)**.** $\mathbb{H}^n = \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$, and $\partial\mathbb{H}^n = \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid x^n = 0\}$.

Definition 6.6 (Manifold With Boundary)**.** Let M be a second-countable Hausdorff space and fix n. Suppose that for every $p \in M$, one of the following conditions is satisfied:

(1) There exists a neighborhood U of p and a homeomorphism $\phi : U \to \hat{U}$ where $\hat{U}$ is an open subset of $\mathbb{R}^n$. p is called an *interior point* and (U, ϕ) is called an *interior chart.*
(2) There exists a neighborhood U of p and a homeomorphism $\phi : U \to \hat{U}$ where $\hat{U}$ is an open subset of $\mathbb{H}^n$ with $\hat{U} \cap \partial\mathbb{H}^n \neq \emptyset$. p is called a *boundary point* and (U, ϕ) is called a *boundary chart.*

Then M is called an *n-dimensional topological manifold with boundary.* Note that every topological manifold is a topological manifold with boundary.

6.2. **Smooth Manifolds.**

Definition 6.7 (Smoothly Compatible)**.** Let M be a topological n-manifold. Two coordinate charts $(U, \phi), (V, \psi)$ are called *smoothly compatible* if $U \cap V = \emptyset$ or the transition map $\psi \circ \phi^{-1}$ is a diffeomorphism.

Definition 6.8 (Smooth Atlas)**.** Let M be a topological n-manifold. A *smooth atlas* is an atlas $\mathcal{A}$ such that any two charts in $\mathcal{A}$ are smoothly compatible with each other.

Definition 6.9 (Smooth Structure)**.** If M is a topological n-manifold, an atlas $\mathcal{A}$ that is not properly contained in any larger smooth atlas is called *maximal* or a *smooth structure on M*

Definition 6.10 (Smooth Manifold)**.** A *smooth manifold* is a topological manifold equipped with a smooth structure.

Definition 6.11. Suppose $(M, \mathcal{A})$ is a smooth manifold.

- Any chart $(U, \phi) \in \mathcal{A}$ is called a *smooth chart.*
- Given a smooth chart (U, ϕ), U is called a smooth coordinate domain and ϕ is called a *smooth coordinate map.*
- Given a smooth chart (U, ϕ), U is called a *smooth coordinate ball* if it is a coordinate ball.

Remark 6.12. One must define a smooth structure on a topological manifold before talking about a smooth chart.

Definition 6.13 (Smooth Maps)**.** Let M, N be smooth manifolds with or without boundary and $F : M \to N$ be a map. F is a *smooth map* if for every $p \in M$, there exist smooth charts (U, ϕ) containing p and (V, ψ) containing $F(p)$ such that

- $F(U) \subset V$;
- $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth.

Definition 6.14 (Coordinate Representatin of a Smooth Map)**.** Let $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ be smooth manifolds. Let $F : M \to N$ be a smooth map and $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ be given. Then $\hat{F} = \psi \circ F \circ \phi^{-1}$ is called the coordinate representation of F with respect to (U, ϕ) and (V, ψ).

Definition 6.15 (Diffeomorphism)**.** Let M, N be smooth manifolds with or without boundary. A diffeomorphism is a smooth map $F : M \to N$ with a smooth inverse.

6.3. **Tangent Vectors.**

Definition 6.16 (Derivation)**.** Let M be a smooth manifold with or without boundary. A derivation at $p \in M$ is a linear map $v : C^\infty(M) \to \mathbb{R}$ such that

$$v(fg) = f(p)vg + g(p)vf$$

for all $f, g \in C^\infty(M)$.

Definition 6.17 (Tangent Space)**.** The tangent space T_pM to M at p is the vector space of all derivations of $C^\infty(M)$ at p.

Definition 6.18 (Differential)**.** M, N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. The *differential of F at p* is the linear map $dF_p : T_pM \to T_{F(p)}N$ defined by

$$dF_p(v) := f \mapsto v(f \circ F)$$

Equivalently, $\forall v \in T_pM, \forall f \in C^\infty(N), dF_p(v)(f) = v(f \circ F)$.

Definition 6.19 (Coordinate Vectors)**.** Let $(M, \mathcal{A})$ be a smooth manifold without boundary. Let $p \in M$ and choose a chart $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Then the *coordinate vectors at* p, denoted by $\frac{\partial}{\partial x^i}|_p$, are derivations $C^\infty(U) \to \mathbb{R}$ such that

$$\frac{\partial}{\partial x^i}\Big|_p := f \mapsto \frac{\partial}{\partial x^i}\Big|_{\phi(p)}(f \circ \phi^{-1}).$$