INTRODUCTION TO SMOOTH MANIFOLDS

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Abstract. I am not good at smooth manifolds, so I decided to read and solve problems from Introduction to Smooth Manifolds.

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Exercise 1.1. Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to *any* open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

Proof. It is clear that a "manifold" satisfying the open-ball or $\mathbb{R}^n$ definition satisfies the open-subset definition. Let M be a manifold satisfying the open-subset definition. Let $x \in M$ be given and let $U, \hat{U}, \phi$ be given according to the definition. Since $\hat{U}$ is open, there exists an open ball B such that $\phi(x) \in B \subset \hat{U}$. Restrict ϕ to $\phi^{-1}(B)$. Then $\phi^{-1}(B)$ is an open subset of M containing x, and $\phi\mid_{\phi^{-1}(B)}$ is a homeomorphism between $\phi^{-1}(B)$ and B. Thus M satisfies the open-ball definition.

$B(x,r) \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ by the map $(x_1 + a_1, \cdots, x_n + a_n) \mapsto (\frac{a_1}{r-a_1}, \cdots, \frac{a_n}{r-a_n})$ where $x = (x_1, \cdots, x_n)$ is the center of $B(x,r)$ and r is the radius. Since the composition of two homeomorphisms gives a homeomorphism, M also satisfies the $\mathbb{R}^n$ definition as well. $\square$