

# INTRODUCTION TO SMOOTH MANIFOLDS

HIDENORI SHINOHARA

## CONTENTS

1. Chapter 1: Smooth Manifolds	1
1.1. Exercises	1
1.2. Problems	4
2. Chapter 2: Smooth Maps	5
2.1. Exercises	5
2.2. Problems	8
3. Chapter 3: Tangent Vectors	9
3.1. Exercises	9
3.2. Problems	10
4. Chapter 4: Submersions, Immersions, and Embeddings	12
5. Appendix A: Review of Topology	13
6. Appendix C: Review of Calculus	14
7. Dictionary	14
7.1. Topological Manifolds	14
7.2. Smooth Manifolds	15
7.3. Tangent Vectors	15
7.4. Submersions, Immersions, and Embeddings	16

## 1. CHAPTER 1: SMOOTH MANIFOLDS

### 1.1. Exercises.

**Exercise 1.1.** Show that equivalent definitions of manifolds are obtained if instead of allowing  $U$  to be homeomorphic to *any* open subset of  $\mathbb{R}^n$ , we require it to be homeomorphic to an open ball in  $\mathbb{R}^n$ , or to  $\mathbb{R}^n$  itself.

*Proof.* It is clear that a “manifold” satisfying the open-ball or  $\mathbb{R}^n$  definition satisfies the open-subset definition. Let  $M$  be a manifold satisfying the open-subset definition. Let  $x \in M$  be given and let  $U, \hat{U}, \phi$  be given according to the definition. Since  $\hat{U}$  is open, there exists an open ball  $B$  such that  $\phi(x) \in B \subset \hat{U}$ . Restrict  $\phi$  to  $\phi^{-1}(B)$ . Then  $\phi^{-1}(B)$  is an open subset of  $M$  containing  $x$ , and  $\phi|_{\phi^{-1}(B)}$  is a homeomorphism between  $\phi^{-1}(B)$  and  $B$ . Thus  $M$  satisfies the open-ball definition.

$B(x, r) \subset \mathbb{R}^n$  is homeomorphic to  $\mathbb{R}^n$  by the map  $(x_1 + a_1, \dots, x_n + a_n) \mapsto (\frac{a_1}{r-a_1}, \dots, \frac{a_n}{r-a_n})$  where  $x = (x_1, \dots, x_n)$  is the center of  $B(x, r)$  and  $r$  is the radius. Since the composition of two homeomorphisms gives a homeomorphism,  $M$  also satisfies the  $\mathbb{R}^n$  definition as well.  $\square$

**Exercise 1.6.** Show that  $\mathbb{RP}^n$  is Hausdorff and second-countable, and is therefore a topological  $n$ -manifold.

*Proof.* From the definition of  $\pi$ , it is easy to see that  $\pi(B(x, r))$  is open in  $\mathbb{RP}^n$  where  $x \in S^n$  and  $0 < r < 1$ .

Let  $[x], [y] \in \mathbb{RP}^n$  be given. Without loss of generality, assume  $x, y \in S^n$ . Let  $r = \min\{|x - y|, |x + y|, 1\}/2$ . Then  $U_x = \pi(B(x, r)), U_y = \pi(B(y, r))$  contain  $[x], [y]$ , respectively.  $\pi^{-1}(U_x), \pi^{-1}(U_y)$  are both open in  $\mathbb{R}^{n+1} \setminus \{0\}$  which can be seen easily by writing down exactly which points belong to them, so  $U_x, U_y$  are both open in  $\mathbb{RP}^n$ . Then  $\pi^{-1}(U_x \cap U_y) = \pi^{-1}(U_x) \cap \pi^{-1}(U_y) = \emptyset$ , so  $U_x \cap U_y = \emptyset$ . Therefore,  $\mathbb{RP}^n$  is Hausdorff.

Let  $\mathcal{B} = \{\pi(B(x, 1/k)) \mid x \in \mathbb{Q}^{n+1} \cap S^n, k \in \{2, 3, 4, \dots\}\}$ . Then  $\mathcal{B}$  is a countable collection of open sets whose union is  $\mathbb{RP}^n$ . Let  $U \subset \mathbb{RP}^n$  be a nonempty open set. Let  $[x] \in U$ . Since  $\pi$  is a quotient map,  $\pi^{-1}(U)$  is open. Moreover,  $x \in \pi^{-1}(U)$ . Without loss of generality,  $x \in S^n$ . Then  $x \in B(x', 1/k) \subset \pi^{-1}(U)$  for some  $B(x', 1/k) \in \mathcal{B}$ . Then  $[x] = \pi(x) \in \pi(B(x', 1/k)) \subset \pi(\pi^{-1}(U)) = U$ . Therefore,  $\mathcal{B}$  is a countable basis of  $\mathbb{RP}^n$ .  $\square$

**Exercise 1.7.** Show that  $\mathbb{RP}^n$  is compact.

*Proof.*  $\pi(S^n) = \mathbb{RP}^n$  and  $S^n$  is compact because it is a closed, bounded subset of  $\mathbb{R}^{n+1}$ . (Heine-Borel) Moreover, the image of a compact set under a continuous map is compact. (See A.45(a)) Thus  $\mathbb{RP}^n$  is compact.  $\square$

**Exercise 1.14.** Suppose  $\mathcal{X}$  is a locally finite collection of subsets of a topological space  $M$ .

- (a) The collection  $\{\overline{X} : X \in \mathcal{X}\}$  is also locally finite.
- (b)  $\overline{\bigcup_{X \in \mathcal{X}} X} = \bigcup_{X \in \mathcal{X}} \overline{X}$ .

*Proof.*

- (a) Let  $p \in M$ . Then there exists an open set  $U$  containing  $x$  such that there are only finitely many  $X \in \mathcal{X}$  such that  $U \cap X \neq \emptyset$ . Let  $X \in \mathcal{X}$ .
  - If  $U \cap X \neq \emptyset$ , then  $U \cap \overline{X} \supset U \cap X \neq \emptyset$ .
  - If  $U \cap X = \emptyset$ , then  $U^c$  is closed, so  $\overline{X} \subset U^c$ . In other words,  $U \cap \overline{X} = \emptyset$ .
This shows that the number of  $X \in \mathcal{X}$  that intersects  $U$  and the number of  $\overline{X} \in \mathcal{X}$  that intersects  $U$  are the same. Therefore,  $\{\overline{X} : X \in \mathcal{X}\}$  is also locally finite.
- (b) Since the closure of a set is defined to be the intersection of all closed sets containing it,  $\bigcup_{X \in \mathcal{X}} \overline{X} \subset \overline{\bigcup_{X \in \mathcal{X}} X}$ . Let  $x \notin \bigcup_{X \in \mathcal{X}} \overline{X}$ . Then there exists a neighborhood  $U$  of  $x$  such that  $U$  intersects only finitely many  $X \in \mathcal{X}$ . Let  $X_1, \dots, X_n$  denote them. By the same argument as part (a),  $\overline{X_1}, \dots, \overline{X_n}$  are the only elements in  $\{\overline{X} \mid X \in \mathcal{X}\}$  that  $U$  intersects. Since  $x \notin \overline{X_i}$  for each  $i = 1, \dots, n$ ,  $U^c \cup \overline{X_1} \cup \dots \cup \overline{X_n}$  is a closed set which contains all  $X \in \mathcal{X}$  but does not contain  $x$ . In other words,  $x \notin \overline{\bigcup_{X \in \mathcal{X}} X}$ .  $\square$

**Exercise 1.18.** Let  $M$  be a topological manifold. Two smooth atlases for  $M$  determine the same smooth structure if and only if their union is a smooth atlas.

*Proof.* Let  $\mathcal{A}, \mathcal{A}'$  be two smooth atlases.

Suppose that they determine the same smooth structure  $\mathcal{B}$ . Then  $\mathcal{A} \cup \mathcal{A}' \subset \mathcal{B}$ , so  $\mathcal{A} \cup \mathcal{A}'$  must be a smooth atlas. By Proposition 1.17(a),  $\mathcal{A} \cup \mathcal{A}'$  determines a unique smooth structure, but it must be  $\mathcal{B}$  because  $\mathcal{B}$  contains the union.

On the other hand, suppose that their union is a smooth atlas. Let  $\mathcal{B}$  be the smooth structure that the union determines. Such  $\mathcal{B}$  must exist by Proposition 1.17(a). By the same proposition,  $\mathcal{A}, \mathcal{A}'$  must determine the unique smooth structures. However, they must be  $\mathcal{B}$  because  $\mathcal{B}$  contains both  $\mathcal{A}$  and  $\mathcal{A}'$ .  $\square$

**Exercise 1.20.** Every smooth manifold has a countable basis of regular coordinate balls.

*Proof.* Let  $M$  be an  $n$ -dimensional smooth manifold. We consider the special case that there exists a single chart  $(\phi, U)$  with  $U = M$ . Let  $x \in \hat{U}$  with rational coordinates. Then there exists  $s > 0$  such that  $B(x, s) \subset \hat{U}$ . For each rational number  $r \in (0, s)$ , we consider the chart  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r)))$ .

Let  $\mathcal{B}$  be the collection of all such charts for each  $x \in \hat{U}$  and  $r$ . We claim that  $\mathcal{B}$  is a smooth atlas.

- Let  $p \in M$ . Then  $\phi(p) \in \hat{U}$ . Since  $\hat{U}$  is open,  $\phi(p) \in B(x, r) \subset \hat{U}$  for some  $x$  with rational coordinates and a positive rational number  $r$ . Then  $p \in \phi^{-1}(B(x, r))$ , so the union of coordinate domains covers  $M$ . In other words,  $\mathcal{B}$  is an atlas.
- Let  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r))), (p \mapsto \phi(p) - x', \phi^{-1}(B(x', r')))) \in \mathcal{B}$  be given. Suppose  $\phi^{-1}(B(x, r)) \cap \phi^{-1}(B(x', r')) \neq \emptyset$ . Let  $\psi, \psi'$  denote the coordinate maps. Then  $\psi' \circ \psi^{-1}$  is a composition of  $\phi, \phi^{-1}$  and translation maps, so it is smooth.

Therefore,  $\mathcal{B}$  is a smooth atlas.

Since  $\mathcal{B}$  is a smooth atlas, there exists a smooth structure  $\mathcal{A}$  on  $M$  containing  $\mathcal{B}$  by Proposition 1.17(a). We claim that  $\mathcal{B}$ , a subset of the smooth structure  $\mathcal{A}$ , is a countable basis of regular coordinate balls.

- $\mathcal{B}$  is a countable collection because  $x \in \mathbb{Q}^n$  and  $r \in \mathbb{Q}$ .
- Let  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r))) \in \mathcal{B}$  be given. Then there exists a chart  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r')))$  in  $\mathcal{B}$  with  $r' > r$ . Let  $B = \phi^{-1}(B(x, r))$ ,  $B' = \phi^{-1}(B(x, r'))$ . Let  $\psi$  denote the map  $p \mapsto \phi(p) - x$ . Then  $\psi(B) = B(0, r)$  and  $\psi(B') = B(0, r')$ , respectively. Moreover,  $\psi(\overline{B}) = \overline{B(0, r)}$  because  $\psi$  is a homeomorphism.

Work on the case when there is no chart that covers the entire manifold.

□

**Exercise 1.39.** Let  $M$  be a topological  $n$ -manifold with boundary.

- (a)  $\text{Int } M$  is an open subset of  $M$  and a topological  $n$ -manifold without boundary.
- (b)  $\partial M$  is a closed subset of  $M$  and a topological  $(n - 1)$ -manifold without boundary.
- (c)  $M$  is a topological manifold if and only if  $\partial M = \emptyset$ .
- (d) If  $n = 0$ , then  $\partial M = \emptyset$  and  $M$  is a 0-manifold.

*Proof.*

- (a) Let  $x \in \text{Int } M$ . Let  $(\phi, U)$  be an interior chart for  $x$ . Then  $x \in U \subset \text{Int } M$  because every point in  $U$  is in an interior chart  $(\phi, U)$ . A subspace of  $M$  must be Hausdorff and second-countable by Proposition A.17(g, i), so  $\text{Int } M$  is a second-countable, Hausdorff space in which every point has a neighborhood homeomorphic to an open subset in  $\mathbb{R}^n$ . Thus  $\text{Int } M$  is an  $n$ -manifold without boundary.
- (b) Since  $\partial M = M \setminus \text{Int } M$  and  $\text{Int } M$  is open in  $M$ ,  $\partial M$  is closed in  $M$ . Let  $x \in \partial M$ . Let  $(\phi, U)$  be a boundary chart of  $x$ . If a point  $y \in U$  gets mapped into  $\text{Int } \mathbb{H}^n$ , then it is certainly an interior point. Thus  $\phi(U \cap \partial M) \subset \partial \mathbb{H}^n$ . Then  $\pi_{n-1} \circ \phi$  is a homeomorphism that maps  $U \cap \partial M$  into an open subset of  $\mathbb{R}^{n-1}$  where  $\pi_{n-1} : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$ .
- (c) If  $\partial M$  is empty, then  $M = \text{Int } M$ , so (a) implies that  $M$  is an  $n$ -dimensional manifold. If  $M$  is a topological manifold, every point is an interior point. Since a point cannot be both an interior point and a boundary point,  $\partial M$  is empty.
- (d) If  $n = 0$ , then  $\partial \mathbb{H}^0 = \emptyset$ . Thus, the condition that  $\phi(U) \cap \partial \mathbb{H}^n \neq \emptyset$  can never be satisfied, so there cannot be any boundary point.

□

**Exercise 1.41.** Let  $M$  be a topological manifold with boundary.

- (a)  $M$  has a countable basis of precompact coordinate balls and half-balls.
- (b)  $M$  is locally compact.
- (c)  $M$  is paracompact.
- (d)  $M$  is locally path-connected.
- (e)  $M$  has countably many components, each of which is an open subset of  $M$  and a connected topological manifold with boundary.
- (f) The fundamental group of  $M$  is countable.

*Proof.*

- (a)
- (b)
- (c)
- (d) Let  $U \subset M$  be a nonempty open subset and choose  $x \in U$ . Then there exists a chart  $(V, \phi)$  such that  $x \in V$ . Since  $\phi(x)$  is a point in an open set  $\phi(U \cap V)$ , there exists  $r > 0$  such that  $B(\phi(x), r) \subset \phi(V)$ . Then  $N(x, U) = \phi^{-1}(B(\phi(x), r))$  is a path-connected neighborhood of  $x$  that is contained in  $U \cap V \subset U$ . Therefore,  $\{N(x, U) \mid \text{open } U \subset M, x \in U\}$  forms a basis of  $M$  consisting of path-connected sets.
- (e)
- (f)

□

**Exercise 1.44.** Suppose  $M$  is a smooth  $n$ -manifold with boundary and  $U$  is an open subset of  $M$ . Prove the following statements:

- (a)  $U$  is a topological  $n$ -manifold with boundary, and the atlas consisting of all smooth charts  $(V, \phi)$  for  $M$  such that  $V \subset U$  defines a smooth structure on  $U$ . With this topology and smooth structure,  $U$  is called an **open submanifold with boundary**.
- (b) If  $U \subset \text{Int } M$ , then  $U$  is actually a smooth manifold (without boundary); in this case we call it an **open submanifold of  $M$** .
- (c)  $\text{Int } M$  is an open submanifold of  $M$  (without boundary).

*Proof.* Let  $\mathcal{T}$  denote the topology of  $M$  and  $\mathcal{A}$  denote the smooth structure of  $M$ .

- (a) The subspace topology on  $U$  is equivalent to  $\mathcal{T}_U = \{V \in \mathcal{T} \mid V \subset U\}$  because  $U$  is open. By Proposition A.17(A.18(Proof of Proposition A.17)),  $U$  is Hausdorff and second-countable. For every point  $p \in U$ , there exists a  $V \in \mathcal{T}$  with a homeomorphism  $\phi : V \rightarrow \hat{V}$  where  $\hat{V}$  is an open subset of  $\mathbb{R}^n$  (or  $\mathbb{H}^n$ ). Since  $U \cap V$  is an open subset of  $V$ ,  $\phi$  restricted to  $U \cap V$  is a homeomorphism between  $U \cap V$  and  $\phi(U \cap V)$ , which is an open subset of  $\mathbb{R}^n$  (or  $\mathbb{H}^n$ ). Therefore,  $U$  is a topological  $n$ -manifold with boundary.

Let  $\mathcal{A}_U = \{(\phi, V) \in \mathcal{A} \mid V \subset U\}$ . Then  $\mathcal{A}_U$  is clearly a collection of charts on  $U$  whose union covers  $U$ . Moreover, any two charts in  $\mathcal{A}_U$  are clearly smoothly compatible. Let  $(\phi, V)$  be a chart on  $U$  that is smoothly compatible with every chart in  $\mathcal{A}_U$ . Let  $(\psi, W) \in \mathcal{A}$ . Then  $(\psi|_{W \cap U}, W \cap U)$  is a chart on  $M$  and it must be smoothly compatible with every chart in  $\mathcal{A}$ . Therefore,  $(\psi|_{W \cap U}, W \cap U) \in \mathcal{A}$ , so it must belong to  $\mathcal{A}_U$ . This implies that  $(\phi, V)$  and  $(\psi|_{W \cap U}, W \cap U)$  are smoothly compatible. Since  $V \subset W \cap U$ , this implies that  $(\phi, V)$  and  $(\psi, W)$  are smoothly compatible.

Thus  $(\phi, V)$  is smoothly compatible with every chart in  $\mathcal{A}$ , so  $(\phi, V) \in \mathcal{A}$ . This implies that  $(\phi, V)$  is in  $\mathcal{A}_U$ , so  $\mathcal{A}_U$  is indeed a maximal smooth atlas.

- (b) Let  $p \in U$ . Then  $p \in \text{Int } M$ , so there exists  $(\phi, V) \in \mathcal{A}$  such that  $p \in V$  and  $\phi(V)$  is open in  $\mathbb{R}^n$ . Then  $(\phi|_{V \cap U}, V \cap U)$  is a chart that is smoothly compatible with every chart in  $\mathcal{A}$ , so  $(\phi|_{V \cap U}, V \cap U) \in \mathcal{A}$ . Thus it must be in  $\mathcal{A}_U$ , so  $p \in U$  is an interior point of  $U$ . Therefore,  $U$  is a manifold without boundary.
- (c) By 1.39,  $\text{Int } M$  is an open subset of  $M$ . By (b),  $\text{Int } M$  is an open submanifold of  $M$  without boundary.

□

## 1.2. Problems.

**Problem 1-2.** Show that a disjoint union of uncountably many copies of  $\mathbb{R}$  is locally Euclidean and Hausdorff, but not second-countable.

*Proof.* Let  $I$  denote an uncountable index set and  $X = \coprod_{\alpha \in I} \mathbb{R}$ . Let  $(x, \alpha_0) \in X$ . Define  $U = \coprod_{\alpha \in I} U_\alpha$  where  $U_{\alpha_0} = \mathbb{R}$  and  $U_\alpha = \emptyset$  when  $\alpha \neq \alpha_0$ . Then  $U$  is an open neighborhood of  $(x, \alpha_0)$  that is clearly homeomorphic to  $\mathbb{R}$ . Thus  $X$  is locally Euclidean.

Let  $(x_1, \alpha_1) \neq (x_2, \alpha_2) \in X$ . If  $\alpha_1 \neq \alpha_2$ , then open neighborhoods of  $x_1$  and  $x_2$  formed in the same way as above separate the two points. Suppose  $\alpha_1 = \alpha_2$ . Without loss of generality,  $x_1 < x_2$ . Define  $U = \coprod_{\alpha \in I} U_\alpha$  where  $U_{\alpha_1} = (-\infty, (x_1 + x_2)/2)$  and  $U_\alpha = \emptyset$  when  $\alpha \neq \alpha_1$ . Similarly, define  $V = \coprod_{\alpha \in I} U_\alpha$  where  $U_{\alpha_1} = ((x_1 + x_2)/2, \infty)$  and  $U_\alpha = \emptyset$  when  $\alpha \neq \alpha_1$ . Then such  $U$  and  $V$  separate the two points. Therefore,  $X$  is Hausdorff.

Let  $\mathcal{B}$  be a basis of  $X$ . For each  $\alpha_0 \in I$ , let  $U_{\alpha_0} = \coprod_{\alpha \in I} U_\alpha$  where  $U_{\alpha_0} = \mathbb{R}$  and  $U_\alpha = \emptyset$  when  $\alpha \neq \alpha_0$ . Then for each  $\alpha_0$ , there must exist  $B_{\alpha_0} \in \mathcal{B}$  such that  $(0, \alpha_0) \in B_{\alpha_0} \subset U_{\alpha_0}$ . Clearly,  $B_\alpha \neq B_\beta$  if  $\alpha \neq \beta$ . Therefore, the cardinality of  $\mathcal{B}$  is greater than or equal to that of  $I$ . Hence,  $X$  is not second-countable. □

**Problem 1-12(Proof of Proposition 1.45).** Suppose  $M_1, \dots, M_k$  are smooth manifolds and  $N$  is a smooth manifold with boundary. Then  $M_1 \times \dots \times M_k \times N$  is a smooth manifold with boundary, and  $\partial(M_1 \times \dots \times M_k \times N) = M_1 \times \dots \times M_k \times \partial N$ .

*Proof.* By Example 1.34,  $M_1 \times \cdots \times M_k$  is a smooth manifold. Thus it suffices to show that  $M \times N$  is a smooth manifold with boundary if  $M$  is a smooth manifold and  $N$  is a smooth manifold with boundary. Let  $m, n$  be the dimensions of  $M, N$ .

First, we show that  $M \times N$  is a topological manifold with boundary and  $\partial(M \times N) = M \times \partial N$ . Let  $(p, q) \in M \times N$ . Then  $p \in M$ , so there exists a chart  $(U, \phi)$  such that  $p \in U$  and  $\hat{U} = \phi(U) \subset \mathbb{R}^m$ .

- Suppose  $q \in \text{Int } N$ . Then there exists a chart  $(V, \psi)$  such that  $\hat{V} = \psi(V) \subset \mathbb{R}^n$ .  $\phi \times \psi$  is a homeomorphism between  $U \times V$  and  $\hat{U} \times \hat{V} \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ . Thus  $(U \times V, \phi \times \psi)$  is a chart for  $(p, q)$ .
- Suppose  $q \in \text{bd } N$ . Then there exists a chart  $(V, \psi)$  such that  $\hat{V} = \psi(V) \subset \mathbb{H}^n$  and  $\psi(q) \in \partial \mathbb{H}^n$ .  $\phi \times \psi$  is a homeomorphism between  $U \times V$  and  $\hat{U} \times \hat{V} \subset \mathbb{R}^m \times \mathbb{H}^n = \mathbb{H}^{m+n}$ . Moreover,  $(\phi \times \psi)(p, q) = (\phi(p), \psi(q)) \in \mathbb{R}^m \times \mathbb{H}^n = \mathbb{H}^{m+n}$ . Thus  $(U \times V, \phi \times \psi)$  is a boundary chart for  $(p, q)$ .

Therefore,  $M \times N$  is a topological manifold with boundary and  $\partial(M \times N) = M \times (\partial N)$ .

Let  $\mathcal{A}_M, \mathcal{A}_N$  be the smooth structures of  $M, N$ . Define  $\mathcal{A}_{M \times N} = \{(U \times V, \phi \times \psi) \mid (U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N\}$ . Then  $\mathcal{A}_{M \times N}$  is an atlas because we showed earlier that each  $(U \times V, \phi \times \psi)$  is a chart. Let  $(U_1 \times V_1, \phi_1 \times \psi_1), (U_2 \times V_2, \phi_2 \times \psi_2) \in \mathcal{A}_{M \times N}$ . Then  $(\phi_2 \times \psi_2) \circ (\phi_1 \times \psi_1)^{-1} = (\phi_2 \circ \phi_1^{-1}) \times (\psi_2 \circ \psi_1^{-1})$  is a smooth map from  $(\phi_1 \times \psi_1)(U_1 \times V_1)$  into  $(\phi_2 \times \psi_2)(U_2 \times V_2)$ . Thus every pair of charts in  $\mathcal{A}_{M \times N}$  is smoothly compatible. In other words,  $\mathcal{A}_{M \times N}$  is a smooth atlas.

On the other hand,  $\mathcal{A}_{M \times N}$  must be maximal because the restriction of any smoothly compatible chart to  $M, N$  gives a smoothly compatible chart, which must belong to  $\mathcal{A}_M, \mathcal{A}_N$ , respectively. Thus  $M \times N$  is a smooth manifold with boundary.  $\square$

## 2. CHAPTER 2: SMOOTH MAPS

### 2.1. Exercises.

**Exercise 2.1.** Let  $M$  be a smooth manifold with or without boundary. Show that pointwise multiplication turns  $C^\infty(M)$  into a commutative ring and a commutative and associative algebra over  $\mathbb{R}$ .

*Proof.*

- The constant map  $f(p) = 0$  is clearly in  $C^\infty(M)$  and it is the additive identity.
- The constant map  $f(p) = 1$  is clearly in  $C^\infty(M)$  and it is the multiplicative identity.
- Let  $f \in C^\infty(M), g \in C^\infty(M)$ . Let  $p \in M$  and  $(\phi, U)$  be a smooth chart for  $p$ . Then  $f \circ \phi^{-1}$  and  $g \circ \phi^{-1}$  are both smooth (Exercise 2.3), real-valued maps defined on an open subset of  $\mathbb{R}^n$ . Thus  $f + g$  is in  $C^\infty(M)$ . Moreover,  $f + g = g + f$  because addition in  $\mathbb{R}$  is commutative.
- Let  $f, g, h \in C^\infty(M)$ . Let  $p \in M$  and  $(\phi, U)$  be a smooth chart for  $p$ . Then  $f \circ \phi^{-1}$  and  $g \circ \phi^{-1}$  are both smooth (Exercise 2.3), real-valued maps defined on an open subset of  $\mathbb{R}^n$ . Therefore,  $fg$  is in  $C^\infty(M)$ . Moreover,  $fg = gf$  and  $(fg)h = f(gh)$  because multiplication in  $\mathbb{R}$  is commutative and associative.
- Let  $c \in \mathbb{R}, f \in C^\infty(M)$ . Then  $cf$  can be seen as  $fg$  where  $g$  is the constant function whose value is  $c$ . As shown above,  $cf \in C^\infty(M)$ .  $\square$

**Exercise 2.2.** Let  $U$  be an open submanifold of  $\mathbb{R}^n$  with its standard smooth manifold structure. Show that a function  $f : U \rightarrow \mathbb{R}^k$  is smooth in the sense just defined if and only if it is smooth in the sense of ordinary calculus. Do the same for an open submanifold with boundary in  $\mathbb{H}^n$ .

*Proof.*  $f$  is smooth in the sense just defined if and only if  $f \circ \text{Id}^{-1}$  is smooth in the sense of ordinary calculus. Since  $f \circ \text{Id}^{-1} = f$ ,  $f \circ \text{Id}^{-1}$  is smooth in the sense of ordinary calculus if and only if  $f$  is smooth in the sense of ordinary calculus.  $\square$

**Exercise 2.3.** Let  $M$  be a smooth manifold with or without boundary, and suppose  $f : M \rightarrow \mathbb{R}^k$  is a smooth function. Show that  $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}^k$  is smooth for every smooth chart  $(U, \phi)$  for  $M$ .

*Proof.* Let  $\phi(x) \in \phi(U)$ . Since  $f$  is smooth, there exists  $(V, \psi)$  such that  $f \circ \psi^{-1} : \psi(V) \rightarrow \mathbb{R}^k$  is smooth and  $x \in V$ . Let  $W = U \cap V$ . Then  $f \circ \psi^{-1} : \psi(W) \rightarrow \mathbb{R}^k$  is smooth and  $\psi \circ \phi^{-1} : \phi(W) \rightarrow \psi(W)$  is a

diffeomorphism where  $\phi(W)$  is a neighborhood of  $W$ . Then the restriction of  $f \circ \psi^{-1}$  to  $\phi(W)$  is identical to  $(f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})$ . Since the composition of a smooth function is smooth,  $f \circ \psi^{-1}$  is smooth.  $\square$

**Exercise 2.7(Prove Proposition 2.5).** Suppose  $M$  and  $N$  are smooth manifolds with or without boundary, and  $F : M \rightarrow N$  is a map. Then  $F$  is smooth if and only if either of the following conditions is satisfied:

- (a) For every  $p \in M$ , there exist smooth charts  $(U, \phi)$  containing  $p$  and  $(V, \psi)$  containing  $F(p)$  such that  $U \cap F^{-1}(V)$  is open in  $M$  and the composite map  $\psi \circ F \circ \phi^{-1}$  is smooth from  $\phi(U \cap F^{-1}(V))$  to  $\psi(V)$ .
- (b)  $F$  is continuous and there exist smooth atlases  $\{(U_\alpha, \phi_\alpha)\}$  and  $\{(V_\beta, \psi_\beta)\}$  for  $M$  and  $N$ , respectively, such that for each  $\alpha$  and  $\beta$ ,  $\psi_\beta \circ F \circ \phi_\alpha^{-1}$  is a smooth map from  $\phi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$  to  $\psi_\beta(V_\beta)$ .

*Proof.* Let  $\mathcal{A}_M$  and  $\mathcal{A}_N$  be smooth structures of  $M$  and  $N$ . Suppose  $F$  is smooth. By Proposition 2.4,  $F$  is continuous. For every  $p \in M$  there exist coordinate charts  $(U_p, \phi_p)$  containing  $p$  and  $(V_p, \psi_p)$  containing  $F(p)$  such that  $F(U_p) \subset V_p$  and  $\psi_p \circ F \circ \phi_p^{-1}$  is smooth from  $\phi_p(U_p)$  to  $\psi_p(V_p)$ . Then  $\{(U_p, \phi_p) \mid p \in M\} \subset \mathcal{A}_M$  and  $\mathcal{A}_N \setminus \{(V_p, \psi_p) \mid p \in M\} \subset \mathcal{A}_N$  are smooth atlases. Moreover, for every  $(U_p, \phi_p)$  and  $(V_q, \psi_q)$ ,  $\psi_q \circ F \circ \phi_p^{-1}$  is a smooth map from  $\phi_p(U_p \cap F^{-1}(V_q))$  to  $\psi_q(V_q)$  because  $\psi_q \circ F \circ \phi_p^{-1} = (\psi_q \circ \psi_p^{-1}) \circ (\psi_p \circ F \circ \phi_p^{-1})$  where  $\psi_q \circ \psi_p^{-1}$  and  $\psi_p \circ F \circ \phi_p^{-1}$  are smooth. Therefore, the definition implies (b).

(b) implies (a) because if  $F$  is continuous,  $F^{-1}(V_\beta)$  is open in  $M$  for every  $\beta$ , so  $U \cap F^{-1}(V)$  is open in  $M$ .

Finally, we show that (a) implies the definition. Suppose  $F$  satisfies (a). Let  $p \in M$ . Let  $(U, \phi) \in \mathcal{A}_M$  and  $(V, \psi) \in \mathcal{A}_N$  be smooth charts satisfying the properties described in (a). Let  $U' = U \cap F^{-1}(V)$  and consider  $(U', \phi|_{U'})$ . Then  $(U', \phi|_{U'}) \in \mathcal{A}_M$  because it must be smoothly compatible with any other smooth coordinate chart in  $\mathcal{A}_M$ . Moreover,  $F(U') \subset V$  and  $\psi \circ F \circ (\phi|_{U'})^{-1} : \phi(U') \rightarrow \psi(V)$  is smooth. Therefore, (a) implies the definition.

Hence, (a), (b) and the definition are all equivalent.  $\square$

**Exercise 2.7(Proof of Proposition 2.6).** Let  $M$  and  $N$  be smooth manifolds with or without boundary, and let  $F : M \rightarrow N$  be a map.

- (a) If every point  $p \in M$  has a neighborhood  $U$  such that the restriction  $F|_U$  is smooth, then  $F$  is smooth.
- (b) Conversely, if  $F$  is smooth, then its restriction to every open subset is smooth.

*Proof.* Let  $\mathcal{A}_M, \mathcal{A}_N$  be smooth structures of  $M, N$ , respectively.

- (a) Let  $p \in M$ . Let  $U$  be a neighborhood of  $p$  such that  $F|_U$  is smooth. By 1.44,  $U$  is a smooth manifold with the induced smooth structure  $\mathcal{A}_U = \{(V, \phi) \in \mathcal{A}_M \mid V \subset U\}$ . Since  $F|_U$  is smooth, there exist  $(V, \phi) \in \mathcal{A}_U$  and  $(W, \psi) \in \mathcal{A}_N$  such that:
  - $F|_U(V) \subset W$ .
  - $\psi \circ F|_U \circ \phi^{-1} : \phi(V) \rightarrow \psi(W)$  is smooth.

Since  $V \subset U$ ,  $F(V) \subset W$ ,  $\psi \circ F \circ \phi^{-1} : \phi(V) \rightarrow \psi(W)$  is smooth, and  $(V, \phi) \in \mathcal{A}$ . Therefore,  $F$  is smooth.

- (b) Let  $U \subset M$  be an open subset. By 1.44,  $U$  is a smooth manifold with the induced smooth structure  $\mathcal{A}_U = \{(V, \phi) \in \mathcal{A}_M \mid V \subset U\}$ . Let  $p \in U$ . Then  $p \in F$ , so there exist  $(V, \phi) \in \mathcal{A}_M, (W, \psi) \in \mathcal{A}_N$  such that  $F(V) \subset W$  and  $\psi \circ F \circ \phi^{-1} : \phi(V) \rightarrow \psi(W)$  is smooth. Then  $(V \cap U, \phi|_{V \cap U})$  is a chart that is smoothly compatible with every chart in  $\mathcal{A}_M$ . Therefore,  $(V \cap U, \phi|_{V \cap U}) \in \mathcal{A}_M$ . Moreover,  $\phi|_{V \cap U}(V \cap U) \subset \phi(V) \subset W$  and  $\psi \circ F \circ (\phi|_{V \cap U})^{-1}$  is clearly smooth. Therefore,  $F|_U$  is smooth.  $\square$

**Exercise 2.9.** Suppose  $F : M \rightarrow N$  is a smooth map between smooth manifolds with or without boundary. Show that the coordinate representation of  $F$  with respect to *every* pair of smooth charts for  $M$  and  $N$  is smooth.

*Proof.* Let  $(M, \mathcal{A}_M), (N, \mathcal{A}_N)$  be smooth manifolds with or without boundary. Let  $F : M \rightarrow N$  be a smooth map. Let  $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$  be given. We must show that  $\hat{F} = \psi \circ F \circ \phi^{-1}$  is a smooth function from  $\phi(U \cap F^{-1}(V))$  to  $\psi(V)$ . Let  $\phi(p) \in \phi(U \cap F^{-1}(V))$ . Then  $p \in M$ , so there exist  $(U_0, \phi_0) \in \mathcal{A}_M$  and  $(V_0, \psi_0) \in \mathcal{A}_N$  such that

- $p \in U_0 \subset U \cap F^{-1}(V)$ ;
- $\phi_0(U_0) \subset V_0$ ;
- $\psi_0 \circ F \circ \phi_0^{-1} : \phi_0(U_0) \rightarrow \psi(V_0)$  is smooth.

Then  $\psi \circ F \circ \phi^{-1}|_{\phi(U_0)} = (\psi \circ \psi_0^{-1}) \circ (\psi_0 \circ F \circ \phi_0^{-1}) \circ (\phi_0 \circ \phi)$ . Since the composition of smooth functions in Euclidean spaces is smooth,  $\hat{F}$  is smooth.  $\square$

**Exercise 2.11(Proof of Proposition 2.10).** Let  $M, N$  and  $P$  be smooth manifolds with or without boundary.

- Every constant map  $c : M \rightarrow N$  is smooth.
- The identity map of  $M$  is smooth.
- If  $U \subset M$  is an open submanifold with or without boundary, then the inclusion map  $U \rightarrow M$  is smooth.

*Proof.* Let  $\mathcal{A}_M, \mathcal{A}_N, \mathcal{A}_P$  be smooth structures of  $M, N, P$ , respectively.

- $F$  is clearly continuous. Moreover, for every  $(U_\alpha, \phi_\alpha) \in \mathcal{A}_M, (V_\beta, \psi_\beta) \in \mathcal{A}_N, \psi_\beta \circ F \circ \phi_\alpha^{-1}$  is a constant map, so it is smooth. By (2.7(Prove Proposition 2.5)),  $F$  is smooth.
- Let  $p \in M$ . Choose  $(U, \phi) \in \mathcal{A}_M$  such that  $p \in U$ . Then  $F(U) \subset U$  and  $\phi \circ F \circ \phi^{-1} = \text{Id}_U$ , so it is smooth. Therefore,  $F$  is smooth.
- By 1.44,  $\mathcal{A}_U = \{(V, \phi) \mid V \subset U\}$  is a smooth structure of  $U$ . Let  $p \in U$ . Then  $p \in V$  for some  $(V, \phi) \in \mathcal{A}_U$ . Then  $(V, \phi) \in \mathcal{A}_M$ , trivially. Since  $F(V) \subset V$  and  $\phi \circ F \circ \phi^{-1}$  is simply the identity map on  $V$ ,  $F$  is smooth.

$\square$

**Exercise 2.16(Proof of Proposition 2.15).**

- Every composition of diffeomorphisms is a diffeomorphism.
- Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.
- Every diffeomorphism is a homeomorphism and an open map.
- The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.
- “Diffeomorphic” is an equivalence relation on the class of all smooth manifolds with or without boundary.

**Exercise 2.16(Proof of Proposition 2.15).** Let  $(M, \mathcal{A}_M), (N, \mathcal{A}_N), (P, \mathcal{A}_P)$  be smooth manifolds with or without boundary, and let  $F : M \rightarrow N, G : N \rightarrow P$  be diffeomorphisms.

- By Proposition 2.10(d),  $G \circ F$  and  $F^{-1} \circ G^{-1}$  are smooth. Then  $(G \circ F) \circ (F^{-1} \circ G^{-1})$  and  $(F^{-1} \circ G^{-1}) \circ (G \circ F)$  are both the identity map on the corresponding space, so  $F^{-1} \circ G^{-1}$  is the smooth inverse of  $G \circ F$ . Therefore,  $G \circ F$  is a diffeomorphism.
- By Example 1.34, we know that  $M_1 \times \cdots \times M_k$  and  $N_1 \times \cdots \times N_k$  are both smooth manifolds. Let  $\mathcal{A}_{M_i}, \mathcal{A}_{N_i}, \mathcal{A}_M$  and  $\mathcal{A}_N$  denote the smooth manifold structures of  $M_i, N_i, M_1 \times \cdots \times M_k, N_1 \times \cdots \times N_k$ , respectively. Let a smooth map  $F_i : M_i \rightarrow N_i$  be given for each  $i$ . Let  $(p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$  be given. Then there exist  $(U_i, \phi_i) \in \mathcal{A}_{M_i}$  and  $(V_i, \psi_i) \in \mathcal{A}_{N_i}$  such that  $p_i \in U_i, F_i(U_i) \subset V_i, \psi_i \circ F_i \circ \phi_i^{-1} : \phi_i(U_i) \rightarrow \psi_i(V_i)$  is smooth for each  $i$ . This implies that  $(\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots (\psi_k \circ F_k \circ \phi_k^{-1}) = (\psi_1 \times \cdots \times \psi_k) \circ (F_1 \times \cdots \times F_k) \circ (\phi_1 \times \cdots \times \phi_k)^{-1}$  is smooth.

Therefore,  $F_1 \times \cdots \times F_k$  is smooth. Using the exact same argument, we can conclude that  $F_1^{-1} \times \cdots \times F_k^{-1}$  is smooth. Since  $(F_1 \times \cdots \times F_k)^{-1} = F_1^{-1} \times \cdots \times F_k^{-1}$ ,  $F_1 \times \cdots \times F_k$  is a diffeomorphism.

- Proposition 2.4 states that every smooth map is continuous. Thus  $F$  and  $F^{-1}$  are both continuous. Therefore,  $F$  is a homeomorphism and also an open map.
- Let  $U \subset M$  be an open subset. By (2.7(Proof of Proposition 2.6)),  $F|_U$  is smooth. Since  $F$  is a homeomorphism as shown in (c),  $F(U)$  is an open subset of  $N$ . Therefore,  $F^{-1}|_{F(U)}$  is smooth by (2.7(Proof of Proposition 2.6)). Clearly,  $F|_U$  and  $F^{-1}|_{F(U)}$  are the inverse of each other. Therefore,  $F|_U$  is a diffeomorphism.

- (e) By (2.11(Proof of Proposition 2.10)), the identity map on  $M$  is a diffeomorphism, so the reflexive property is satisfied. Moreover,  $(F^{-1})^{-1} = F$ , so the symmetric property is satisfied. By (a), the composition of two diffeomorphisms is a diffeomorphism, so the transitive property is satisfied. Therefore, “diffeomorphic” is an equivalence relation.

**Exercise 2.19(Proof of Theorem 2.18).** Suppose  $M$  and  $N$  are smooth manifolds with boundary and  $F : M \rightarrow N$  is a diffeomorphism. Then  $F(\partial M) = \partial N$ , and  $F$  restricts to a diffeomorphism from  $\text{Int } M$  to  $\text{Int } N$ .

*Proof.* Let  $\mathcal{A}_M, \mathcal{A}_N$  denote the smooth structures of  $M, N$ , respectively. Let  $p \in \partial M$ . Then there exists a chart containing  $p$  that sends  $p$  to  $\partial \mathbb{H}^n$ . By Theorem 1.46, every chart containing  $p$  sends  $p$  to  $\partial \mathbb{H}^n$ .

Since  $F$  is smooth, there exist  $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$  such that  $F(U) \subset V$  and  $\psi \circ F \circ \phi^{-1}$  is a smooth map from  $\phi(U)$  to  $\psi(V)$ .  $F^{-1}$  is a homeomorphism by (2.16(Proof of Proposition 2.15)). Then  $(\phi^{-1} \circ F^{-1}, F(U))$  is a coordinate chart around  $F(p)$  because we obtain a homeomorphism by restricting the composition of two injective continuous maps to its image. Moreover, we claim that  $(\phi^{-1} \circ F^{-1}, F(U))$  is smoothly compatible with every chart in  $\mathcal{A}_N$ . Let  $(\psi_1, V_1) \in \mathcal{A}_N$  be given. Then  $(\phi^{-1} \circ F^{-1}) \circ \psi_1^{-1} = (\phi^{-1} \circ F^{-1} \circ \psi_1^{-1}) \circ (\psi \circ \psi_1^{-1})$ , and the composition of two smooth maps is smooth. Therefore,  $(\phi^{-1} \circ F^{-1}, F(U)) \in \mathcal{A}_N$ , and this chart contains  $F(p)$  and sends  $F(p)$  to  $\partial \mathbb{H}^n$ . In other words,  $F(p) \in \partial N$ .

Since  $F^{-1}$  is also smooth,  $F^{-1}(\partial N) \subset \partial M$ .  $F^{-1}(\partial N) \subset \partial M \implies F(F^{-1}(\partial N)) \subset F(\partial M) \subset \partial N$ . Since  $F$  is a bijection,  $F(F^{-1}(\partial N)) = \partial N$ . Therefore,  $F(\partial M) = \partial N$ .

This implies that  $F(\text{Int } M) = \text{Int } N$ . By (1.44(c)) and (2.16(Proof of Proposition 2.15)(d)),  $F$  is a diffeomorphism between  $\text{Int } M$  and  $\text{Int } N$ .  $\square$

## 2.2. Problems.

**Problem 2-1.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that for every  $x \in \mathbb{R}$ , there are smooth coordinate charts  $(U, \phi)$  containing  $x$  and  $(V, \psi)$  containing  $f(x)$  such that  $\psi \circ f \circ \phi^{-1}$  is smooth as a map from  $\phi(U \cap f^{-1}(V))$  to  $\psi(V)$ , but  $f$  is not smooth in the sense we defined in this chapter.

*Proof.*  $\phi = \psi = \text{Id}$  in this solution.

If  $x \geq 0$ , then let  $U = \mathbb{R}, V = (0, \infty)$ . Then  $\phi(U \cap f^{-1}(V)) = [0, \infty)$ . Thus  $\psi \circ f \circ \phi^{-1} : [0, \infty) \rightarrow (0, \infty)$  is the constant map that sends every number to 1. Therefore, it is smooth.

If  $x < 0$ , then let  $U = \mathbb{R}, V = (-\infty, 1)$ . Then  $\phi(U \cap f^{-1}(V)) = (-\infty, 0)$ . Thus  $\psi \circ f \circ \phi^{-1} : (-\infty, 0) \rightarrow (-\infty, 1)$  is the constant map that sends every number to 0. Therefore, it is smooth.

It might seem that we can apply (2.7(Prove Proposition 2.5)) to show that  $f$  is smooth, but (2.7(Prove Proposition 2.5)) requires that  $U \cap f^{-1}(V)$  be open in  $M$ .

$f$  maps the interval  $(-1, 1)$  to  $\{0, 1\}$ . Since the image of a connected set under a continuous map must be connected,  $f$  cannot be continuous. By Proposition 2.4,  $f$  cannot be smooth.  $\square$

**Problem 2-2(Proof of Proposition 2.12).** Suppose  $M_1, \dots, M_k$  and  $N$  are smooth manifolds with or without boundary, such that at most one of  $M_1, \dots, M_k$  has nonempty boundary. For each  $i$ , let  $\pi_i : M_1 \times \dots \times M_k \rightarrow M_i$  denote the projection onto the  $M_i$  factor. A map  $F : N \rightarrow M_1 \times \dots \times M_k$  is smooth if and only if each of the component maps  $F_i = \pi_i \circ F : N \rightarrow M_i$  is smooth.

*Proof.* Let  $\mathcal{A}_{M_1}, \dots, \mathcal{A}_{M_k}, \mathcal{A}_N$  be the smooth structures of  $M_1, \dots, M_k, N$ . Let  $d_1, \dots, d_k$  denote the dimensions of  $M_1, \dots, M_k$ , respectively. Let  $d = \sum d_i$ .

First, suppose that  $F$  is smooth. By (2.11(Proof of Proposition 2.10)), the composition of smooth maps is smooth. Thus it suffices to show that  $\pi_i : M_1 \times \dots \times M_k \rightarrow M_i$  is smooth for each  $i$ . We show that  $\pi_1$  is smooth and the other cases can be shown similarly.

Let  $(x_1, \dots, x_k) \in M_1 \times \dots \times M_k$ . Then for each  $i$ , there exist  $(U_i, \phi_i) \in \mathcal{A}_{M_i}$  and  $(V_i, \psi_i) \in \mathcal{A}_{M_i}$  such that  $x_i \in U_i$  and  $\phi_i(U_i) \subset V_i$ . Then we have  $(\phi_1 \times \dots \times \phi_k)(U_1 \times \dots \times U_k) \subset V_1 \times \dots \times V_k$  and the composition  $\phi_i \circ \pi_1 \circ (\phi_1 \times \dots \times \phi_k)^{-1}$  is the projection of the first  $d_1$  coordinates from  $\mathbb{R}^n$  onto  $\mathbb{R}^{d_1}$ . Therefore, it is clearly smooth, so  $\pi_1$  is smooth.



Suppose each  $F_i = \pi_i \circ F : N \rightarrow M_i$  is smooth. Let  $p \in N$ . Then for each  $i$ , there exist  $(U_i, \phi_i) \in \mathcal{A}_N$  and  $(V_i, \psi_i) \in \mathcal{A}_{M_i}$  such that  $p \in U_i$ ,  $F_i(U_i) \subset V_i$  and  $\psi_i \circ F_i \circ \phi_i^{-1}$ . Let  $U = U_1 \cap \cdots \cap U_k$ .  $U$  is a neighborhood of  $p$  and the restriction of  $\phi_1$  to  $U$  is a homeomorphism. Then we claim that  $(\phi_1, U) \in \mathcal{A}_N$  and  $(\psi_1 \times \cdots \times \psi_k, V_1 \times \cdots \times V_k) \in \mathcal{A}_{M_1 \times \cdots \times M_k}$  are charts that satisfy the necessary properties.

- $F(U) \subset V_1 \times \cdots \times V_k$ .
- For each  $i$ ,  $\psi_i \circ F_i \circ \phi_i^{-1} = (\psi_i \circ F_i \circ \phi_i^{-1}) \circ (\phi_i \circ \phi_1^{-1}) : \phi_1(U) \rightarrow \psi_i(V_i)$  is smooth because the composition of two smooth maps is smooth. Thus  $(\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots \times (\psi_k \circ F_k \circ \phi_k^{-1}) : \phi_1(U) \rightarrow \psi_1(V_1) \times \cdots \times \psi_k(V_k)$  is smooth. Moreover,  $(\psi_1 \times \cdots \times \psi_k) \circ F \circ \phi_1^{-1} = (\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots \times (\psi_k \circ F_k \circ \phi_k^{-1})$ .

Therefore,  $F$  is smooth.  $\square$

**Problem 2-3.** For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

- $p_n : S^1 \rightarrow S^1$  is the  $n$ th power map for  $n \in \mathbb{Z}$ , given in complex notation by  $p_n(z) = z^n$ .
- $\alpha : S^n \rightarrow S^n$  is the antipodal map  $\alpha(x) = -x$ .
- $F : S^3 \rightarrow S^2$  is given by  $F(w, z) = (z\bar{w} + w\bar{z}, iw\bar{z} - iz\bar{w}, z\bar{z} - w\bar{w})$  where we think of  $S^3$  as the subset  $\{(w, z) : |w|^2 + |z|^2 = 1\}$  of  $\mathbb{C}^2$ .

*Proof.*

- Example 1.31 shows the existence of a smooth structure of  $S^1$  and let  $\mathcal{A}$  denote it. Let  $p \in S^1$ . Then there exist  $(U_i^\pm, \phi_i^\pm), (U_j^\pm, \phi_j^\pm) \in \mathcal{A}$  around  $p, p_n(p)$ , respectively. Then the composition  $\phi_j^\pm \circ f \circ (\phi_i^\pm)^{-1}$  is equal to one of  $\cos(n(\arccos(x))), \sin(n(\arcsin(x))), \cos(n(\arcsin(x))), \sin(n(\arccos(x)))$ , all of which are clearly smooth. By Proposition 2.5(a),  $p_n$  is smooth.
- Example 1.31 shows the existence of a smooth structure of  $S^n$  and let  $\mathcal{A}$  denote it. Let  $p \in S^1$ . Then there exists a chart  $(U_i^\pm, \phi_i^\pm)$  in  $\mathcal{A}$  around  $p$ . Then  $(U_i^\mp, \phi_i^\mp)$  is a chart containing  $\alpha(p)$  with  $\alpha(U_i^\pm) \subset U_i^\mp$ . Then  $\phi_i^\mp \circ \alpha \circ \phi_i^\pm$  is the map  $x \mapsto -x$ , which is clearly smooth.
- Let  $z = a + bi, w = c + di$ .  $z\bar{w} = ac + bd + i(bc - ad)$  and  $w\bar{z} = (ac + bd) - i(bc - ad)$ . Then  $z\bar{w} + w\bar{z} = 2(ac + bd) = 2\operatorname{Re}(z\bar{w})$  and  $i(w\bar{z} - z\bar{w}) = 2\operatorname{Im}(z\bar{w})$ .

$$\begin{aligned} (2\operatorname{Re}(z\bar{w}))^2 + (2\operatorname{Im}(z\bar{w}))^2 + (|z|^2 - |w|^2)^2 &= 4|z\bar{w}|^2 + (|z|^2 - |w|^2)^2 \\ &= 4|z|^2|\bar{w}|^2 + (|z|^2 - |w|^2)^2 \\ &= (|z|^2 + |w|^2)^2 \\ &= 1. \end{aligned}$$

Therefore,  $F$  indeed maps  $S^3$  into  $S^2$ . Moreover, this map is continuous.

Show that  $F$  is smooth.

$\square$

### 3. CHAPTER 3: TANGENT VECTORS

#### 3.1. Exercises.

**Exercise 3.5(Proof of Lemma 3.4).** Suppose  $M$  is a smooth manifold with or without boundary,  $p \in M, v \in T_p M$ , and  $f, g \in C^\infty(M)$ .

- If  $f$  is a constant function, then  $vf = 0$ .
- If  $f(p) = g(p) = 0$ , then  $v(fg) = 0$ .

*Proof.*

- Let  $h$  be the constant function that always takes the value 1. Then  $f(p) = ch(p)$  for some  $c \in \mathbb{R}$ . Then  $v(fh) = f(p)vf + f(p)vf$ , so  $c^2v(h) = c^2v(h) + c^2v(h)$ . Therefore,  $c^2v(h) = 0$ , so  $cv(h) = 0$ . Since  $v$  is linear, this implies  $0 = v(ch) = v(f)$ , so  $vf = 0$ .
- $v(fg) = f(p)vg + g(p)vf = 0 + 0 = 0$ .

$\square$

**Exercise 3.7(Proof of Proposition 3.6).** Let  $M, N$ , and  $P$  be smooth manifolds with or without boundary, let  $F : M \rightarrow N$  and  $G : N \rightarrow P$  be smooth maps, and let  $p \in M$ .

- (a)  $dF_p : T_p M \rightarrow T_{F(p)} N$  is linear.
- (b)  $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_p M \rightarrow T_{G \circ F(p)} P$ .
- (c)  $d(\text{Id}_M)_p = \text{Id}_{T_p M} : T_p M \rightarrow T_p M$ .
- (d) If  $F$  is a diffeomorphism, then  $dF_p : T_p M \rightarrow T_{F(p)} N$  is an isomorphism, and  $(dF_p)^{-1} = d(F^{-1})_{F(p)}$ .

*Proof.* (a)  $\forall v, w \in T_p M, \forall c \in \mathbb{R}, \forall f \in C^\infty(N)$ ,

$$\begin{aligned} dF_p(cv + w)(f) &= (cv + w)(f \circ F) \\ &= (cv)(f \circ F) + w(f \circ F) \\ &= c(v(f \circ F)) + w(f \circ F) \\ &= c(dF_p(v)(f)) + dF_p(w)(f) \\ &= (cdF_p(v))(f) + dF_p(w)(f) \\ &= (cdF_p(v) + dF_p(w))(f). \end{aligned}$$

Therefore,  $dF_p(cv + w) = cdF_p(v) + dF_p(w)$ .

- (b)  $\forall v \in T_p M, f \in C^\infty(P)$ ,

$$\begin{aligned} d(G \circ F)_p(v)(f) &= v(f \circ (G \circ F)) \\ &= v((f \circ G) \circ F) \\ &= (dF_p(v))(f \circ G) \\ &= (dG_{F(p)}(dF_p(v)))(f) \\ &= ((dG_{F(p)} \circ dF_p)(v))(f) \end{aligned}$$

Therefore,  $d(G \circ F)_p = dG_{F(p)} \circ dF_p$ .

- (c)  $\forall v \in T_p(M), \forall f \in C^\infty(M)$ ,

$$\begin{aligned} d(\text{Id}_M)_p(v)(f) &= v(f \circ \text{Id}_M) \\ &= v(f). \end{aligned}$$

Therefore,  $d(\text{Id}_M)_p(v) = v$ , so  $d(\text{Id}_M)_p = \text{Id}_{T_p M}$ .

- (d)  $F^{-1}$  exists and it is a smooth map since  $F$  is a diffeomorphism. By combining (b) and (c), we obtain  $dF_p$  and  $dF_{F(p)}^{-1}$  are the inverse of each other. Therefore,  $dF_p$  is an isomorphism.

□

### 3.2. Problems.

**Problem 3-1.** Suppose  $M$  and  $N$  are smooth manifolds with or without boundary, and  $F : M \rightarrow N$  is a smooth map. Show that  $dF_p : T_p M \rightarrow T_{F(p)} N$  is the zero map for each  $p \in M$  if and only if  $F$  is constant on each component of  $M$ .

*Proof.* Suppose  $dF_p : T_p M \rightarrow T_{F(p)} N$  is the zero map for each  $p \in M$ . It suffices to show that for every  $p \in M$ , there exists a neighborhood of  $p$  on which  $F$  is constant. Let  $p \in M$  and  $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$  be given such that  $p \in U$  and  $F(U) \subset V$ . Without loss of generality, we assume  $\hat{U} = \phi(U)$  is an open ball in  $\mathbb{R}^m$ . Then for any  $i, j$  and for any  $q \in \hat{U}$ ,

$$\begin{aligned} dF_q\left(\frac{\partial}{\partial x^i}\bigg|_q\right)(\pi_j \circ \psi) &= 0 \implies \left(\frac{\partial}{\partial x^i}\bigg|_q\right)(\pi_j \circ \psi \circ F) = 0 \\ &\implies \left(\frac{\partial}{\partial x^i}\bigg|_{\phi(q)}\right)(\pi_j \circ \psi \circ F \circ \phi^{-1}) = 0. \end{aligned}$$

Fix  $j$ . Then every partial derivative of  $\pi_j \circ \psi \circ F \circ \phi^{-1}$  at every point in  $\hat{U}$  is 0. The intermediate value theorem implies that  $\pi_j \circ \psi \circ F \circ \phi^{-1}$  is constant on  $\hat{U}$  because  $\hat{U}$  is an open ball. In other words,  $(\pi_j \circ \psi \circ F \circ \phi^{-1})(\hat{U}) = \{y_j\}$  for some  $y_j \in \mathbb{R}$ . Since this is true for every  $j$  and  $\pi_j$  is the projection of

the  $j$ th coordinate,  $(\psi \circ F \circ \phi^{-1})(\hat{U}) = \{y\}$  where  $y = (y_1, \dots, y_n)$ . Then  $(F \circ \phi^{-1})(\hat{U}) = F(U) = \psi^{-1}(y)$ . Since  $\psi$  is a homeomorphism, there exists exactly one point in  $\psi^{-1}(U)$ . In other words,  $F$  is constant on  $U$ . Therefore,  $F$  is constant on each path component.

Suppose  $F$  is constant on each component of  $M$ . Let  $p \in M$ . Choose a chart  $(U, \phi) \in \mathcal{A}_M$  such that  $p \in U$ . Then  $F \circ \phi^{-1}$  is constant in a neighborhood around  $\phi(p)$ . For any  $i$ ,

$$\begin{aligned} dF_p\left(\frac{\partial}{\partial x^i}\Big|_p\right)(f) &= \frac{\partial}{\partial x^i}\Big|_p(f \circ F) \\ &= \frac{\partial}{\partial x^i}\Big|_{\phi(p)}(f \circ F \circ \phi^{-1}) \\ &= 0 \end{aligned}$$

because  $f \circ F \circ \phi^{-1}$  is constant in a neighborhood around  $\phi(p)$ . By Proposition 3.15,  $\partial/\partial x^i|_p$  form a basis for  $T_p M$ . Since  $dF_p$  sends each basis element to 0,  $dF_p = 0$ .  $\square$

**Problem 3-2(Proof of Proposition 3.14).** Let  $M_1, \dots, M_k$  be smooth manifolds, and for each  $j$ , let  $\pi_j : M_1 \times \dots \times M_k \rightarrow M_j$  be the projection onto the  $M_j$  factor. For any point  $p = (p_1, \dots, p_k) \in M_1 \times \dots \times M_k$ , the map

$$\alpha : T_p(M_1 \times \dots \times M_k) \rightarrow T_{p_1} M_1 \oplus \dots \oplus T_{p_k} M_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \dots, d(\pi_k)_p(v))$$

is an isomorphism. The same is true if one of the spaces  $M_i$  is a smooth manifold with boundary.

*Proof.* It suffices to show this for the case that  $k = 2$  because the results extend to arbitrary  $k$  by induction. Let  $\mathcal{A}_{M_1}, \mathcal{A}_{M_2}, \mathcal{A}_{M_1 \times M_2}$  be the smooth structures of  $M_1, M_2, M_1 \times M_2$ .

We first define a lot of notations.

- Let  $d_1, d_2$  denote the dimensions of  $M_1, M_2$  and let  $d = d_1 + d_2$  denote the dimension of  $M_1 \times M_2$ .
- Let  $p = (p_1, p_2) \in M_1 \times M_2$  be given. Choose  $(U, \phi = (x^i)) \in \mathcal{A}_{M_1}, (V, \psi = (y^i)) \in \mathcal{A}_{M_2}$  with  $p_1 \in U$  and  $p_2 \in V$ . Let  $q_1 = \phi(p_1), q_2 = \psi(p_2), q = q_1 \times q_2$ .
- $(U \times V, (z^i)) \in \mathcal{A}_{M_1 \times M_2}$  and  $(p_1, p_2) \in U \times V$  where  $(z^i) = \phi \times \psi$ . More specifically,  $z^i = x^i \circ \pi_1$  for  $1 \leq i \leq d_1$  and  $z^i = y^i \circ \pi_2$  for  $d_1 + 1 \leq i \leq d_1 + d_2$ .

Note that we use  $x^i, y^i, z^i, \pi_1$  to mean two different things in this solution:

- $x^i$  is either the  $i$ th coordinate function of  $\phi$  or the  $i$ th projection map  $\mathbb{R}^{d_1} \rightarrow \mathbb{R}$ .
- $y^i$  is either the  $i$ th coordinate function of  $\psi$  or the  $i$ th projection map  $\mathbb{R}^{d_2} \rightarrow \mathbb{R}$ .
- $z^i$  is either the  $i$ th coordinate function of  $\phi \times \psi$  or the  $i$ th projection map  $\mathbb{R}^{d_1+d_2} \rightarrow \mathbb{R}$ .
- $\pi_1$  is either the projection map  $M_1 \times M_2 \rightarrow M_1$  or the projection map  $\mathbb{R}^{d_1+d_2} \rightarrow \mathbb{R}^{d_1}$ .
- $\pi_2$  is either the projection map  $M_1 \times M_2 \rightarrow M_2$  or the projection map  $\mathbb{R}^{d_1+d_2} \rightarrow \mathbb{R}^{d_2}$ .

By Proposition 3.15,  $\{\partial/\partial x^1|_{p_1}, \dots, \partial/\partial x^{d_1}|_{p_1}\}, \{\partial/\partial y^1|_{p_2}, \dots, \partial/\partial y^{d_2}|_{p_2}\}, \{\partial/\partial z^1|_p, \dots, \partial/\partial z^{d_1+d_2}|_p\}$  form bases for  $T_{p_1} M_1, T_{p_2} M_2, T_p(M_1 \times M_2)$ .

$\alpha(\partial/\partial z^1|_p) = (d(\pi_1)_p(\partial/\partial z^1|_p), d(\pi_2)_p(\partial/\partial z^1|_p))$ . We claim that  $d(\pi_1)_p(\partial/\partial z^1|_p) = \partial/\partial x^1|_{p_1}$ .

$$\begin{aligned}
d(\pi_1)_p(\partial/\partial z^1|_p)(f) &= d(\pi_1)_p(d(\phi^{-1} \times \psi^{-1})_q)(\frac{\partial}{\partial z^1}|_q)(f) \\
&= (d(\pi_1)_p \circ d(\phi^{-1} \times \psi^{-1})_q)(\frac{\partial}{\partial z^1}|_q)(f) \\
&= d(\pi_1 \circ (\phi^{-1} \times \psi^{-1})_q)(\frac{\partial}{\partial z^1}|_q)(f) \\
&= \lim_{h \rightarrow 0} \frac{(f \circ \pi_1 \circ (\phi^{-1} \times \psi^{-1}))(q + e_1 h) - (f \circ \pi_1 \circ (\phi^{-1} \times \psi^{-1}))(q)}{h} \\
&= \lim_{h \rightarrow 0} \frac{(f \circ \pi_1)(\phi^{-1}(q_1 + e_1 h), p_2) - (f \circ \pi_1)(p)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(\phi^{-1}(q_1 + e_1 h)) - f(p_1)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(\phi^{-1}(q_1 + e_1 h)) - f(\phi^{-1}(q_1))}{h} \\
&= (\frac{\partial}{\partial x^1}|_{q_1})(f \circ \phi^{-1}) \\
&= d(\phi^{-1})_{q_1}(\frac{\partial}{\partial x^1}|_{q_1})(f) \\
&= (\frac{\partial}{\partial x^1}|_{p_1})(f).
\end{aligned}$$

The same result can be shown for the other combinations of  $\pi_1, \pi_2$  and  $z^1, \dots, z^{d_1+d_2}$ . For any  $c_1, \dots, c_{d_1+d_2} \in \mathbb{R}$ ,

$$\begin{aligned}
\alpha(\sum_{i=1}^{d_1+d_2} c_i \frac{\partial}{\partial z^i}|_p) &= \sum_{i=1}^{d_1+d_2} c_i \alpha(\frac{\partial}{\partial z^i}|_p) \\
&= \sum_{i=1}^{d_1+d_2} c_i (d(\pi_1)_p \frac{\partial}{\partial z^i}|_p, d(\pi_2)_p \frac{\partial}{\partial z^i}|_p) \\
&= \sum_{i=1}^{d_1} c_i (d(\pi_1)_p \frac{\partial}{\partial z^i}|_p, d(\pi_2)_p \frac{\partial}{\partial z^i}|_p) + \sum_{i=d_1+1}^{d_2} c_i (d(\pi_1)_p \frac{\partial}{\partial z^i}|_p, d(\pi_2)_p \frac{\partial}{\partial z^i}|_p) \\
&= \sum_{i=1}^{d_1} c_i (\frac{\partial}{\partial x^i}|_{p_1}, 0) + \sum_{i=1}^{d_2} c_{d_1+i} (0, \frac{\partial}{\partial y^i}|_{p_2}) \\
&= (c_1 \frac{\partial}{\partial x^1}|_{p_1} + \dots + c_{d_1} \frac{\partial}{\partial x^{d_1}}|_{p_1}, c_{d_1+1} \frac{\partial}{\partial y^1}|_{p_2} + \dots + c_{d_1+d_2} \frac{\partial}{\partial y^{d_2}}|_{p_2}).
\end{aligned}$$

Therefore,  $\alpha$  is bijective. □

#### 4. CHAPTER 4: SUBMERSIONS, IMMERSIONS, AND EMBEDDINGS

**Exercise 4.3(Verification of Example 4.2).** Verify the following claims:

- (a) Suppose  $M_1, \dots, M_k$  are smooth manifolds. Then each of the projection maps  $\pi_i : M_1 \times \dots \times M_k \rightarrow M_i$  is a smooth submersion.

*Proof.*

- (a) Let  $d_1, \dots, d_k$  denote the dimensions of  $M_1, \dots, M_k$ , respectively. Let  $M = M_1 \times \dots \times M_k$ . (2-2(Proof of Proposition 2.12)) implies that  $\pi_i$  is smooth for each  $i$  by setting  $F = \text{Id} : M \rightarrow M$ . Let  $p = (p_1, \dots, p_k) \in M$ . Thus it suffices to show that the dimension of  $d(\pi_i)_p(T_p(M))$  is the same as the dimension of  $T_{p_i}(M_i)$ .

By Proposition 3.12,  $\dim(T_p(M)) = \sum d_i$ . Since the  $\alpha$  defined in (3-2(Proof of Proposition 3.14)) is an isomorphism,

$$(4.1) \quad \dim(d(\pi_1)_p(T_p(M)) \oplus \cdots \oplus d(\pi_k)_p(T_p(M))) = \dim(T_p(M)) = \sum d_i.$$

However, for each  $i$ ,  $d(\pi_i)_p(T_p(M)) \subset T_{p_i}M_i$ . Thus  $\dim(d(\pi_i)_p(T_p(M))) \leq \dim(T_{p_i}M_i) = d_i$ . By (4.1),  $\dim(d(\pi_i)_p(T_p(M))) = \dim(T_{p_i}M_i)$ . □

## 5. APPENDIX A: REVIEW OF TOPOLOGY

**Exercise A.18(Proof of Proposition A.17).** Let  $X$  be a topological space and let  $S$  be a subspace of  $X$ .

- (a)
- (b)
- (c)
- (d)
- (e)
- (f) If  $\mathcal{B}$  is a basis for the topology of  $X$ , then  $\mathcal{B}_S = \{B \cap S \mid B \in \mathcal{B}\}$  is a basis for the subspace topology on  $S$ .
- (g) If  $X$  is Hausdorff, then so is  $S$ .
- (h) If  $X$  is first-countable, then so is  $S$ .
- (i) If  $X$  is second-countable, then so is  $S$ .

*Proof.*

- (a)
- (b)
- (c)
- (d)
- (e)
- (f) The union of  $B \cap S$  is  $S$ . Let  $U \cap S$  be an open subset of  $S$  where  $U$  is open in  $X$ , and  $x \in U \cap S$ . Then there exists  $B \in \mathcal{B}$  such that  $x \in B \subset U$  since  $\mathcal{B}$  is a basis. Therefore,  $x \in B \cap S \subset U \cap S$  with  $B \cap S \in \mathcal{B}_S$ .
- (g) Let  $x \neq y \in S$ . There exist two disjoint open sets  $U, V$  of  $X$  containing  $x, y$ , respectively. Then  $U \cap S$  and  $V \cap S$  are disjoint open sets of  $X$  containing  $x, y$ , respectively.
- (h)
- (i) Let  $\mathcal{B}$  be a countable basis of  $X$ . Then  $\{B \cap S \mid B \in \mathcal{B}\}$  is a countable basis of  $S$  by (f). □

**Exercise A.24(Proof of Proposition A.23).** Suppose  $X_1, \dots, X_k$  are topological spaces, and let  $X_1 \times \cdots \times X_k$  be their product space.

- (a) CHARACTERISTIC PROPERTY: If  $B$  is a topological space, a map  $F : B \rightarrow X_1 \times \cdots \times X_k$  is continuous if and only if each of its component functions  $F_i = \pi_i \circ F : B \rightarrow X_i$  is continuous.

*Proof.*

- (a) Suppose  $F$  is continuous. Since  $\pi_i$  is continuous by (c) and the composition of continuous functions is continuous,  $\pi_i \circ F$  is continuous. Suppose each component function is continuous. Let  $B_1 \times \cdots \times B_k$  be a basis element of  $X_1 \times \cdots \times X_k$ .

$$\begin{aligned} F^{-1}(B_1 \times \cdots \times B_k) &= F^{-1}(\cap_{i=1}^k \pi_i^{-1}(B_1 \times \cdots \times B_k)) \\ &= \cap_{i=1}^k F^{-1}(\pi_i^{-1}(B_1 \times \cdots \times B_k)) \\ &= \cap_{i=1}^k (\pi_i \circ F)^{-1}(B_1 \times \cdots \times B_k). \end{aligned}$$

Since the intersection of finitely many open sets is open,  $F$  is continuous. □

## 6. APPENDIX C: REVIEW OF CALCULUS

**Exercise C.1.** Suppose that  $F : U \rightarrow W$  is differentiable at  $a \in U$ . Show that the linear map satisfying

$$\lim_{v \rightarrow 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} = 0$$

is unique.

*Proof.* Let  $L, L'$  be two such linear maps.

$$\begin{aligned} \lim_{v \rightarrow 0} \frac{|Lv - L'v|}{|v|} &= \lim_{v \rightarrow 0} \frac{|(F(a+v) - F(a) - L'v) - (F(a+v) - F(a) - Lv)|}{|v|} \\ &= \lim_{v \rightarrow 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} + \lim_{v \rightarrow 0} \frac{|F(a+v) - F(a) - L'v|}{|v|} \\ &= 0 + 0 = 0. \end{aligned}$$

If  $L \neq L'$ ,  $(L - L')v_0 \neq 0$  for some  $v_0$ . Then  $\lim_{v \rightarrow 0} \frac{|Lv - L'v|}{|v|} = \lim_{h \rightarrow 0} \frac{|L(hv_0) - L'(hv_0)|}{|hv_0|} = \frac{|(L - L')v_0|}{|v_0|} \neq 0$ . This is a contradiction, so  $L = L'$ .  $\square$

## 7. DICTIONARY

### 7.1. Topological Manifolds.

**Definition 7.1** (Topological Manifold). A *topological  $n$ -manifold* is a Hausdorff, second-countable topological space each point of which has a neighborhood that is homeomorphic to an open subset  $\mathbb{R}^n$ .

**Definition 7.2** (Coordinates). Let  $M$  be a topological  $n$ -manifold. Let  $U$  be an open subset of  $M$ ,  $\hat{U}$  be an open subset of  $\mathbb{R}^n$ ,  $\phi : U \rightarrow \hat{U}$  be a homeomorphism.

- The pair  $(U, \phi)$  is called a *coordinate chart* or a *chart*.
- $U$  is called a *coordinate domain* or a *coordinate neighborhood* and  $\phi$  is called a *coordinate map*.
- If  $\phi(U)$  is an open ball in  $\mathbb{R}^n$ ,  $U$  is called a *coordinate ball*.
- If  $\phi(U)$  is an open cube in  $\mathbb{R}^n$ ,  $U$  is called a *coordinate cube*.
- The coordinate functions of  $\phi$  are often denoted as  $(x^1, \dots, x^n)$ . Thus a chart is sometimes denoted by  $(U, (x^1, \dots, x^n))$  or  $(U, (x^i))$ .

**Definition 7.3** (Atlas). Let  $M$  be a topological  $n$ -manifold. An *atlas* for  $M$  is a collection of charts  $(U_\alpha, \phi_\alpha)$  such that  $M = \bigcup_\alpha U_\alpha$ .

**Definition 7.4** (Transition Map). Let  $M$  be a topological  $n$ -manifold and  $(U, \phi), (V, \psi)$  be coordinate charts such that  $U \cap V \neq \emptyset$ .  $\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$  is called a *transition map* from  $\phi$  to  $\psi$ .

**Definition 7.5** (Closed Upper Half-Space).  $\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$ , and  $\partial\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n = 0\}$ .

**Definition 7.6** (Manifold With Boundary). Let  $M$  be a second-countable Hausdorff space and fix  $n$ . Suppose that for every  $p \in M$ , one of the following conditions is satisfied:

- (1) There exists a neighborhood  $U$  of  $p$  and a homeomorphism  $\phi : U \rightarrow \hat{U}$  where  $\hat{U}$  is an open subset of  $\mathbb{R}^n$ .  $p$  is called an *interior point* and  $(U, \phi)$  is called an *interior chart*.
- (2) There exists a neighborhood  $U$  of  $p$  and a homeomorphism  $\phi : U \rightarrow \hat{U}$  where  $\hat{U}$  is an open subset of  $\mathbb{H}^n$  with  $\phi(p) \in \partial\mathbb{H}^n$ .  $p$  is called a *boundary point*.

Then  $M$  is called an  *$n$ -dimensional topological manifold with boundary*. Note that every topological manifold is a topological manifold with boundary.

## 7.2. Smooth Manifolds.

**Definition 7.7** (Smoothly Compatible). Let  $M$  be a topological  $n$ -manifold. Two coordinate charts  $(U, \phi), (V, \psi)$  are called *smoothly compatible* if  $U \cap V = \emptyset$  or the transition map  $\psi \circ \phi^{-1}$  is a diffeomorphism.

**Definition 7.8** (Smooth Atlas). Let  $M$  be a topological  $n$ -manifold. A *smooth atlas* is an atlas  $\mathcal{A}$  such that any two charts in  $\mathcal{A}$  are smoothly compatible with each other.

**Definition 7.9** (Smooth Structure). If  $M$  is a topological  $n$ -manifold, an atlas  $\mathcal{A}$  that is not properly contained in any larger smooth atlas is called *maximal* or a *smooth structure on  $M$* .

**Definition 7.10** (Smooth Manifold). A *smooth manifold* is a topological manifold equipped with a smooth structure.

**Definition 7.11.** Suppose  $(M, \mathcal{A})$  is a smooth manifold.

- Any chart  $(U, \phi) \in \mathcal{A}$  is called a *smooth chart*.
- Given a smooth chart  $(U, \phi)$ ,  $U$  is called a smooth coordinate domain and  $\phi$  is called a *smooth coordinate map*.
- Given a smooth chart  $(U, \phi)$ ,  $U$  is called a *smooth coordinate ball* if it is a coordinate ball.

*Remark 7.12.* One must define a smooth structure on a topological manifold before talking about a smooth chart.

**Definition 7.13** (Smooth Maps). Let  $M, N$  be smooth manifolds with or without boundary and  $F : M \rightarrow N$  be a map.  $F$  is a *smooth map* if for every  $p \in M$ , there exist smooth charts  $(U, \phi)$  containing  $p$  and  $(V, \psi)$  containing  $F(p)$  such that

- $F(U) \subset V$ ;
- $\psi \circ F \circ \phi^{-1} : \phi(U) \rightarrow \psi(V)$  is smooth.

**Definition 7.14** (Coordinate Representatin of a Smooth Map). Let  $(M, \mathcal{A}_M)$  and  $(N, \mathcal{A}_N)$  be smooth manifolds. Let  $F : M \rightarrow N$  be a smooth map and  $(U, \phi) \in \mathcal{A}_M$  and  $(V, \psi) \in \mathcal{A}_N$  be given. Then  $\hat{F} = \psi \circ F \circ \phi^{-1}$  is called the coordinate representation of  $F$  with respect to  $(U, \phi)$  and  $(V, \psi)$ .

**Definition 7.15** (Diffeomorphism). Let  $M, N$  be smooth manifolds with or without boundary. A diffeomorphism is a smooth map  $F : M \rightarrow N$  with a smooth inverse.

## 7.3. Tangent Vectors.

**Definition 7.16** (Derivation). Let  $M$  be a smooth manifold with or without boundary. A derivation at  $p \in M$  is a linear map  $v : C^\infty(M) \rightarrow \mathbb{R}$  such that

$$v(fg) = f(p)vg + g(p)vf$$

for all  $f, g \in C^\infty(M)$ .

This corresponds to “arrows that are tangent to  $M$  and whose basepoints are attached to  $M$  at  $p$ ” even though it may not be easy to see that from this definition.

**Definition 7.17** (Tangent Space). The tangent space  $T_p M$  to  $M$  at  $p$  is the vector space of all derivations of  $C^\infty(M)$  at  $p$ .

**Definition 7.18** (Differential).  $M, N$  are smooth manifolds with or without boundary, and  $F : M \rightarrow N$  is a smooth map. The *differential of  $F$  at  $p$*  is the linear map  $dF_p : T_p M \rightarrow T_{F(p)} N$  defined by

$$dF_p(v) := f \mapsto v(f \circ F)$$

Equivalently,  $\forall v \in T_p M, \forall f \in C^\infty(N), dF_p(v)(f) = v(f \circ F)$ . This corresponds to “the directional derivative of  $F$  at  $p$  in the direction of the arrow  $v$ .”

**Definition 7.19** (Coordinate Vectors). Let  $(M, \mathcal{A})$  be a smooth manifold without boundary. Let  $p \in M$  and choose a chart  $(U, \phi) \in \mathcal{A}$  such that  $p \in U$ . Then the *coordinate vectors at  $p$* , denoted by  $\frac{\partial}{\partial x^i}|_p$ , are derivations  $C^\infty(U) \rightarrow \mathbb{R}$  such that

$$\frac{\partial}{\partial x^i}\Big|_p := f \mapsto \frac{\partial}{\partial x^i}\Big|_{\phi(p)} (f \circ \phi^{-1}).$$

**Definition 7.20** (Tangent Bundle). Let  $M$  be a smooth manifold with or without boundary. The tangent bundle of  $M$ , denoted by  $TM$ , is the disjoint union  $\coprod_{p \in M} T_p M$ .

**Definition 7.21** (Projection Map). Let  $M$  be a smooth manifold with or without boundary. The projection map  $\pi : TM \rightarrow M$  is the map defined by  $(p, v) \mapsto p$ .

#### 7.4. Submersions, Immersions, and Embeddings.

**Definition 7.22** (Rank). Let  $M, N$  be smooth manifolds with or without boundary and let  $F : M \rightarrow N$  be a smooth map. Then the rank of  $F$  at  $p \in M$  is:

- The rank of the linear map  $dF_p : T_p M \rightarrow T_{F(p)} N$ .
- The dimension of the subspace  $dF_p(T_p M)$  in the vector space  $T_{F(p)} N$ .

It is easy to see that the two definitions above are always equivalent.

**Definition 7.23** (Submersions and Immersions). Let  $M, N$  be smooth manifolds with or without boundary and let  $F : M \rightarrow N$  be a smooth map.

- If  $F$  has the same rank at every point  $p \in M$ , then  $F$  is said to have *constant rank*, and the rank is denoted by  $\text{rank } F$ .
- If the rank of  $F$  at  $p \in M$  is equal to  $\max\{\dim M, \dim N\}$ , then  $F$  is said to have *full rank at  $p$* .
- If  $F$  has full rank everywhere, then  $F$  is said to have *full rank*.
- If  $F$  has constant rank and  $\text{rank } F = \dim N$ ,  $F$  is called a *smooth submersion*.
- If  $F$  has constant rank and  $\text{rank } F = \dim M$ ,  $F$  is called a *smooth immersion*.