

# INTRODUCTION TO SMOOTH MANIFOLDS

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## 1. CHAPTER 1: SMOOTH MANIFOLDS

**Exercise 1.1.** Show that equivalent definitions of manifolds are obtained if instead of allowing  $U$  to be homeomorphic to *any* open subset of  $\mathbb{R}^n$ , we require it to be homeomorphic to an open ball in  $\mathbb{R}^n$ , or to  $\mathbb{R}^n$  itself.

*Proof.* It is clear that a “manifold” satisfying the open-ball or  $\mathbb{R}^n$  definition satisfies the open-subset definition. Let  $M$  be a manifold satisfying the open-subset definition. Let  $x \in M$  be given and let  $U, \hat{U}, \phi$  be given according to the definition. Since  $\hat{U}$  is open, there exists an open ball  $B$  such that  $\phi(x) \in B \subset \hat{U}$ . Restrict  $\phi$  to  $\phi^{-1}(B)$ . Then  $\phi^{-1}(B)$  is an open subset of  $M$  containing  $x$ , and  $\phi|_{\phi^{-1}(B)}$  is a homeomorphism between  $\phi^{-1}(B)$  and  $B$ . Thus  $M$  satisfies the open-ball definition.

$B(x, r) \subset \mathbb{R}^n$  is homeomorphic to  $\mathbb{R}^n$  by the map  $(x_1 + a_1, \dots, x_n + a_n) \mapsto (\frac{a_1}{r-a_1}, \dots, \frac{a_n}{r-a_n})$  where  $x = (x_1, \dots, x_n)$  is the center of  $B(x, r)$  and  $r$  is the radius. Since the composition of two homeomorphisms gives a homeomorphism,  $M$  also satisfies the  $\mathbb{R}^n$  definition as well.  $\square$

**Exercise 1.6.** Show that  $\mathbb{RP}^n$  is Hausdorff and second-countable, and is therefore a topological  $n$ -manifold.

*Proof.* From the definition of  $\pi$ , it is easy to see that  $\pi(B(x, r))$  is open in  $\mathbb{RP}^n$  where  $x \in S^n$  and  $0 < r < 1$ .

Let  $[x], [y] \in \mathbb{RP}^n$  be given. Without loss of generality, assume  $x, y \in S^n$ . Let  $r = \min\{|x - y|, |x + y|, 1\}/2$ . Then  $U_x = \pi(B(x, r)), U_y = \pi(B(y, r))$  contain  $[x], [y]$ , respectively.  $\pi^{-1}(U_x), \pi^{-1}(U_y)$  are both open in  $\mathbb{R}^{n+1} \setminus \{0\}$  which can be seen easily by writing down exactly which points belong to them, so  $U_x, U_y$  are both open in  $\mathbb{RP}^n$ . Then  $\pi^{-1}(U_x \cap U_y) = \pi^{-1}(U_x) \cap \pi^{-1}(U_y) = \emptyset$ , so  $U_x \cap U_y = \emptyset$ . Therefore,  $\mathbb{RP}^n$  is Hausdorff.

Let  $\mathcal{B} = \{\pi(B(x, 1/k)) \mid x \in \mathbb{Q}^{n+1} \cap S^n, k \in \{2, 3, 4, \dots\}\}$ . Then  $\mathcal{B}$  is a countable collection of open sets whose union is  $\mathbb{RP}^n$ . Let  $U \subset \mathbb{RP}^n$  be a nonempty open set. Let  $[x] \in U$ . Since  $\pi$  is a quotient map,  $\pi^{-1}(U)$  is open. Moreover,  $x \in \pi^{-1}(U)$ . Without loss of generality,  $x \in S^n$ . Then  $x \in B(x', 1/k) \subset \pi^{-1}(U)$  for some  $B(x', 1/k) \in \mathcal{B}$ . Then  $[x] = \pi(x) \in \pi(B(x', 1/k)) \subset \pi(\pi^{-1}(U)) = U$ . Therefore,  $\mathcal{B}$  is a countable basis of  $\mathbb{RP}^n$ .  $\square$

**Exercise 1.7.** Show that  $\mathbb{RP}^n$  is compact.

*Proof.*  $\pi(S^n) = \mathbb{RP}^n$  and  $S^n$  is compact because it is a closed, bounded subset of  $\mathbb{R}^{n+1}$ . (Heine-Borel) Moreover, the image of a compact set under a continuous map is compact. (See A.45(a)) Thus  $\mathbb{RP}^n$  is compact.  $\square$

**Exercise 1.14.** Suppose  $\mathcal{X}$  is a locally finite collection of subsets of a topological space  $M$ .

- (a) The collection  $\{\overline{X} : X \in \mathcal{X}\}$  is also locally finite.
- (b)  $\overline{\bigcup_{X \in \mathcal{X}} X} = \bigcup_{X \in \mathcal{X}} \overline{X}$ .

*Proof.*

- (a) Let  $p \in M$ . Then there exists an open set  $U$  containing  $x$  such that there are only finitely many  $X \in \mathcal{X}$  such that  $U \cap X \neq \emptyset$ . Let  $X \in \mathcal{X}$ .
  - If  $U \cap X \neq \emptyset$ , then  $U \cap \overline{X} \supset U \cap X \neq \emptyset$ .
  - If  $U \cap X = \emptyset$ , then  $U^c$  is closed, so  $\overline{X} \subset U^c$ . In other words,  $U \cap \overline{X} = \emptyset$ .

This shows that the number of  $X \in \mathcal{X}$  that intersects  $U$  and the number of  $\overline{X} \in \mathcal{X}$  that intersects  $U$  are the same. Therefore,  $\{\overline{X} : X \in \mathcal{X}\}$  is also locally finite.

- (b) Since the closure of a set is defined to be the intersection of all closed sets containing it,  $\bigcup_{X \in \mathcal{X}} \overline{X} \subset \overline{\bigcup_{X \in \mathcal{X}} X}$ . Let  $x \notin \bigcup_{X \in \mathcal{X}} \overline{X}$ . Then there exists a neighborhood  $U$  of  $x$  such that  $U$  intersects only finitely many  $X \in \mathcal{X}$ . Let  $X_1, \dots, X_n$  denote them. By the same argument as part (a),  $\overline{X_1}, \dots, \overline{X_n}$  are the only elements in  $\{\overline{X} : X \in \mathcal{X}\}$  that  $U$  intersects. Since  $x \notin \overline{X_i}$  for each  $i = 1, \dots, n$ ,  $U^c \cup \overline{X_1} \cup \dots \cup \overline{X_n}$  is a closed set which contains all  $X \in \mathcal{X}$  but does not contain  $x$ . In other words,  $x \notin \overline{\bigcup_{X \in \mathcal{X}} X}$ .

□

**Exercise 1.18.** Let  $M$  be a topological manifold. Two smooth atlases for  $M$  determine the same smooth structure if and only if their union is a smooth atlas.

*Proof.* Let  $\mathcal{A}, \mathcal{A}'$  be two smooth atlases.

Suppose that they determine the same smooth structure  $\mathcal{B}$ . Then  $\mathcal{A} \cup \mathcal{A}' \subset \mathcal{B}$ , so  $\mathcal{A} \cup \mathcal{A}'$  must be a smooth atlas. By Proposition 1.17(a),  $\mathcal{A} \cup \mathcal{A}'$  determines a unique smooth structure, but it must be  $\mathcal{B}$  because  $\mathcal{B}$  contains the union.

On the other hand, suppose that their union is a smooth atlas. Let  $\mathcal{B}$  be the smooth structure that the union determines. Such  $\mathcal{B}$  must exist by Proposition 1.17(a). By the same proposition,  $\mathcal{A}, \mathcal{A}'$  must determine the unique smooth structures. However, they must be  $\mathcal{B}$  because  $\mathcal{B}$  contains both  $\mathcal{A}$  and  $\mathcal{A}'$ . □

**Exercise 1.20.** Every smooth manifold has a countable basis of regular coordinate balls.

*Proof.* Let  $M$  be an  $n$ -dimensional smooth manifold. We consider the special case that there exists a single chart  $(\phi, U)$  with  $U = M$ . Let  $x \in \hat{U}$  with rational coordinates. Then there exists  $s > 0$  such that  $B(x, s) \subset \hat{U}$ . For each rational number  $r \in (0, s)$ , we consider the chart  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r)))$ .

Let  $\mathcal{B}$  be the collection of all such charts for each  $x \in \hat{U}$  and  $r$ . We claim that  $\mathcal{B}$  is a smooth atlas.

- Let  $p \in M$ . Then  $\phi(p) \in \hat{U}$ . Since  $\hat{U}$  is open,  $\phi(p) \in B(x, r) \subset \hat{U}$  for some  $x$  with rational coordinates and a positive rational number  $r$ . Then  $p \in \phi^{-1}(B(x, r))$ , so the union of coordinate domains covers  $M$ . In other words,  $\mathcal{B}$  is an atlas.
- Let  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r))), (p \mapsto \phi(p) - x', \phi^{-1}(B(x', r')))) \in \mathcal{B}$  be given. Suppose  $\phi^{-1}(B(x, r)) \cap \phi^{-1}(B(x', r')) \neq \emptyset$ . Let  $\psi, \psi'$  denote the coordinate maps. Then  $\psi' \circ \psi^{-1}$  is a composition of  $\phi, \phi^{-1}$  and translation maps, so it is smooth.

Therefore,  $\mathcal{B}$  is a smooth atlas.

Since  $\mathcal{B}$  is a smooth atlas, there exists a smooth structure  $\mathcal{A}$  on  $M$  containing  $\mathcal{B}$  by Proposition 1.17(a). We claim that  $\mathcal{B}$ , a subset of the smooth structure  $\mathcal{A}$ , is a countable basis of regular coordinate balls.

- $\mathcal{B}$  is a countable collection because  $x \in \mathbb{Q}^n$  and  $r \in \mathbb{Q}$ .
- Let  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r))) \in \mathcal{B}$  be given. Then there exists a chart  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r')))$  in  $\mathcal{B}$  with  $r' > r$ . Let  $B = \phi^{-1}(B(x, r))$ ,  $B' = \phi^{-1}(B(x, r'))$ . Let  $\psi$  denote the map  $p \mapsto \phi(p) - x$ . Then  $\psi(B) = B(0, r)$  and  $\psi(B') = B(0, r')$ , respectively. Moreover,  $\psi(\overline{B}) = \overline{B(0, r)}$  because  $\psi$  is a homeomorphism.

Work on the case when there is no chart that covers the entire manifold.

□

**Exercise 1.39.** Let  $M$  be a topological  $n$ -manifold with boundary.

- (a)  $\text{Int } M$  is an open subset of  $M$  and a topological  $n$ -manifold without boundary.
- (b)  $\partial M$  is a closed subset of  $M$  and a topological  $(n-1)$ -manifold without boundary.
- (c)  $M$  is a topological manifold if and only if  $\partial M = \emptyset$ .
- (d) If  $n = 0$ , then  $\partial M = \emptyset$  and  $M$  is a 0-manifold.

*Proof.*

- (a) Let  $x \in \text{Int } M$ . Let  $(\phi, U)$  be an interior chart for  $x$ . Then  $x \in U \subset \text{Int } M$  because every point in  $U$  is in an interior chart  $(\phi, U)$ . A subspace of  $M$  must be Hausdorff and second-countable by Proposition A.17(g, i), so  $\text{Int } M$  is a second-countable, Hausdorff space in which every point has a neighborhood homeomorphic to an open subset in  $\mathbb{R}^n$ . Thus  $\text{Int } M$  is an  $n$ -manifold without boundary.
- (b) Since  $\partial M = M \setminus \text{Int } M$  and  $\text{Int } M$  is open in  $M$ ,  $\partial M$  is closed in  $M$ . Let  $x \in \partial M$ . Let  $(\phi, U)$  be a boundary chart of  $x$ . If a point  $y \in U$  gets mapped into  $\text{Int } \mathbb{H}^n$ , then it is certainly an interior point. Thus  $\phi(U \cap \partial M) \subset \partial \mathbb{H}^n$ . Then  $\pi_{n-1} \circ \phi$  is a homeomorphism that maps  $U \cap \partial M$  into an open subset of  $\mathbb{R}^{n-1}$  where  $\pi_{n-1} : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$ .
- (c) If  $\partial M$  is empty, then  $M = \text{Int } M$ , so (a) implies that  $M$  is an  $n$ -dimensional manifold. If  $M$  is a topological manifold, every point is an interior point. Since a point cannot be both an interior point and a boundary point,  $\partial M$  is empty.
- (d) If  $n = 0$ , then  $\partial \mathbb{H}^0 = \emptyset$ . Thus, the condition that  $\phi(U) \cap \partial \mathbb{H}^n \neq \emptyset$  can never be satisfied, so there cannot be any boundary point.

□

**Exercise 1.44.** Suppose  $M$  is a smooth  $n$ -manifold with boundary and  $U$  is an open subset of  $M$ . Prove the following statements:

- (a)  $U$  is a topological  $n$ -manifold with boundary, and the atlas consisting of all smooth charts  $(V, \phi)$  for  $M$  such that  $V \subset U$  defines a smooth structure on  $U$ . With this topology and smooth structure,  $U$  is called an **open submanifold with boundary**.
- (b) If  $U \subset \text{Int } M$ , then  $U$  is actually a smooth manifold (without boundary); in this case we call it an **open submanifold of  $M$** .
- (c)  $\text{Int } M$  is an open submanifold of  $M$  (without boundary).

*Proof.* Let  $\mathcal{T}$  denote the topology of  $M$  and  $\mathcal{A}$  denote the smooth structure of  $M$ .

- (a) The subspace topology on  $U$  is equivalent to  $\mathcal{T}_U = \{V \in \mathcal{T} \mid V \subset U\}$  because  $U$  is open. By Proposition A.17(A.18(Proof of Proposition A.17)),  $U$  is Hausdorff and second-countable. For every point  $p \in U$ , there exists a  $V \in \mathcal{T}$  with a homeomorphism  $\phi : V \rightarrow \hat{V}$  where  $\hat{V}$  is an open subset of  $\mathbb{R}^n$  (or  $\mathbb{H}^n$ ). Since  $U \cap V$  is an open subset of  $V$ ,  $\phi$  restricted to  $U \cap V$  is a homeomorphism between  $U \cap V$  and  $\phi(U \cap V)$ , which is an open subset of  $\mathbb{R}^n$  (or  $\mathbb{H}^n$ ). Therefore,  $U$  is a topological  $n$ -manifold with boundary.

Let  $\mathcal{A}_U = \{(\phi, V) \in \mathcal{A} \mid V \subset U\}$ . Then  $\mathcal{A}_U$  is clearly a collection of charts on  $U$  whose union covers  $U$ . Moreover, any two charts in  $\mathcal{A}_U$  are clearly smoothly compatible. Let  $(\phi, V)$  be a chart on  $U$  that is smoothly compatible with every chart in  $\mathcal{A}_U$ . Let  $(\psi, W) \in \mathcal{A}$ . Then  $(\psi_{W \cap U}, W \cap U)$  is a chart on  $M$  and it must be smoothly compatible with every chart in  $\mathcal{A}$ . Therefore,  $(\psi_{W \cap U}, W \cap U) \in \mathcal{A}$ , so it must belong to  $\mathcal{A}_U$ . This implies that  $(\phi, V)$  and  $(\psi_{W \cap U}, W \cap U)$  are smoothly compatible. Since  $V \subset W \cap U$ , this implies that  $(\phi, V)$  and  $(\psi, W)$  are smoothly compatible.

Thus  $(\phi, V)$  is smoothly compatible with every chart in  $\mathcal{A}$ , so  $(\phi, V) \in \mathcal{A}$ . This implies that  $(\phi, V)$  is in  $\mathcal{A}_U$ , so  $\mathcal{A}_U$  is indeed a maximal smooth atlas.

- (b) Let  $p \in U$ . Then  $p \in \text{Int } M$ , so there exists  $(\phi, V) \in \mathcal{A}$  such that  $p \in V$  and  $\phi(V)$  is open in  $\mathbb{R}^n$ . Then  $(\phi|_{V \cap U}, V \cap U)$  is a chart that is smoothly compatible with every chart in  $\mathcal{A}$ , so  $(\phi|_{V \cap U}, V \cap U) \in \mathcal{A}$ . Thus it must be in  $\mathcal{A}_U$ , so  $p \in U$  is an interior point of  $U$ . Therefore,  $U$  is a manifold without boundary.
- (c) By 1.39,  $\text{Int } M$  is an open subset of  $M$ . By (b),  $\text{Int } M$  is an open submanifold of  $M$  without boundary.

□

## 2. CHAPTER 2: SMOOTH MAPS

**Exercise 2.1.** Let  $M$  be a smooth manifold with or without boundary. Show that pointwise multiplication turns  $C^\infty(M)$  into a commutative ring and a commutative and associative algebra over  $\mathbb{R}$ .

*Proof.*

- The constant map  $f(p) = 0$  is clearly in  $C^\infty(M)$  and it is the additive identity.
- The constant map  $f(p) = 1$  is clearly in  $C^\infty(M)$  and it is the multiplicative identity.
- Let  $f, g \in C^\infty(M)$ . Let  $p \in M$  and  $(\phi, U)$  be a smooth chart for  $p$ . Then  $f \circ \phi^{-1}$  and  $g \circ \phi^{-1}$  are both smooth (Exercise 2.3), real-valued maps defined on an open subset of  $\mathbb{R}^n$ . Thus  $f + g$  is in  $C^\infty(M)$ . Moreover,  $f + g = g + f$  because addition in  $\mathbb{R}$  is commutative.
- Let  $f, g, h \in C^\infty(M)$ . Let  $p \in M$  and  $(\phi, U)$  be a smooth chart for  $p$ . Then  $f \circ \phi^{-1}$  and  $g \circ \phi^{-1}$  are both smooth (Exercise 2.3), real-valued maps defined on an open subset of  $\mathbb{R}^n$ . Therefore,  $fg$  is in  $C^\infty(M)$ . Moreover,  $fg = gf$  and  $(fg)h = f(gh)$  because multiplication in  $\mathbb{R}$  is commutative and associative.
- Let  $c \in \mathbb{R}$ ,  $f \in C^\infty(M)$ . Then  $cf$  can be seen as  $fg$  where  $g$  is the constant function whose value is  $c$ . As shown above,  $cf \in C^\infty(M)$ .

□

**Exercise 2.2.** Let  $U$  be an open submanifold of  $\mathbb{R}^n$  with its standard smooth manifold structure. Show that a function  $f : U \rightarrow \mathbb{R}^k$  is smooth in the sense just defined if and only if it is smooth in the sense of ordinary calculus. Do the same for an open submanifold with boundary in  $\mathbb{H}^n$ .

*Proof.*  $f$  is smooth in the sense just defined if and only if  $f \circ \text{Id}^{-1}$  is smooth in the sense of ordinary calculus. Since  $f \circ \text{Id}^{-1} = f$ ,  $f \circ \text{Id}^{-1}$  is smooth in the sense of ordinary calculus if and only if  $f$  is smooth in the sense of ordinary calculus. □

**Exercise 2.3.** Let  $M$  be a smooth manifold with or without boundary, and suppose  $f : M \rightarrow \mathbb{R}^k$  is a smooth function. Show that  $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}^k$  is smooth for every smooth chart  $(U, \phi)$  for  $M$ .

*Proof.* Let  $\phi(x) \in \phi(U)$ . Since  $f$  is smooth, there exists  $(V, \psi)$  such that  $f \circ \psi^{-1} : \psi(V) \rightarrow \mathbb{R}^k$  is smooth and  $x \in V$ . Let  $W = U \cap V$ . Then  $f \circ \psi^{-1} : \psi(W) \rightarrow \mathbb{R}^k$  is smooth and  $\psi \circ \phi^{-1} : \phi(W) \rightarrow \psi(W)$  is a diffeomorphism where  $\phi(W)$  is a neighborhood of  $W$ . Then the restriction of  $f \circ \psi^{-1}$  to  $\phi(W)$  is identical to  $(f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})$ . Since the composition of a smooth function is smooth,  $f \circ \psi^{-1}$  is smooth. □

**Exercise 2.7 (Prove Proposition 2.5).** Suppose  $M$  and  $N$  are smooth manifolds with or without boundary, and  $F : M \rightarrow N$  is a map. Then  $F$  is smooth if and only if either of the following conditions is satisfied:

- (a) For every  $p \in M$ , there exist smooth charts  $(U, \phi)$  containing  $p$  and  $(V, \psi)$  containing  $F(p)$  such that  $U \cap F^{-1}(V)$  is open in  $M$  and the composite map  $\psi \circ F \circ \phi^{-1}$  is smooth from  $\phi(U \cap F^{-1}(V))$  to  $\psi(V)$ .
- (b)  $F$  is continuous and there exist smooth atlases  $\{(U_\alpha, \phi_\alpha)\}$  and  $\{(V_\beta, \psi_\beta)\}$  for  $M$  and  $N$ , respectively, such that for each  $\alpha$  and  $\beta$ ,  $\psi_\beta \circ F \circ \phi_\alpha^{-1}$  is a smooth map from  $\phi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$  to  $\psi_\beta(V_\beta)$ .

*Proof.* Let  $\mathcal{A}_M$  and  $\mathcal{A}_N$  be smooth structures of  $M$  and  $N$ . Suppose  $F$  is smooth. By Proposition 2.4,  $F$  is continuous. For every  $p \in M$  there exist coordinate charts  $(U_p, \phi_p)$  containing  $p$  and  $(V_p, \psi_p)$  containing  $F(p)$  such that  $F(U_p) \subset V_p$  and  $\psi_p \circ F \circ \phi_p^{-1}$  is smooth from  $\phi_p(U_p)$  to  $\psi_p(V_p)$ . Then  $\{(U_p, \phi_p) \mid p \in M\} \subset \mathcal{A}_M$  and  $\{(V_p, \psi_p) \mid p \in M\} \subset \mathcal{A}_N$  are smooth atlases. Moreover, for every  $(U_p, \phi_p)$  and  $(V_q, \psi_q)$ ,  $\psi_q \circ F \circ \phi_p^{-1}$  is a smooth map from  $\phi_p(U_p \cap F^{-1}(V_q))$  to  $\psi_q(V_q)$  because  $\psi_q \circ F \circ \phi_p^{-1} = (\psi_q \circ \psi_p^{-1}) \circ (\psi_p \circ F \circ \phi_p^{-1})$  where  $\psi_q \circ \psi_p^{-1}$  and  $\psi_p \circ F \circ \phi_p^{-1}$  are smooth. Therefore, the definition implies (b).

(b) implies (a) because if  $F$  is continuous,  $F^{-1}(V_\beta)$  is open in  $M$  for every  $\beta$ , so  $U \cap F^{-1}(V)$  is open in  $M$ .

Finally, we show that (a) implies the definition. Suppose  $F$  satisfies (a). Let  $p \in M$ . Let  $(U, \phi) \in \mathcal{A}_M$  and  $(V, \psi) \in \mathcal{A}_N$  be smooth charts satisfying the properties described in (a). Let  $U' = U \cap F^{-1}(V)$  and consider  $(U', \phi|_{U'})$ . Then  $(U', \phi|_{U'}) \in \mathcal{A}_M$  because it must be smoothly compatible with any other smooth coordinate chart in  $\mathcal{A}_M$ . Moreover,  $F(U') \subset V$  and  $\psi \circ F \circ (\phi|_{U'})^{-1} : \phi(U') \rightarrow \psi(V)$  is smooth. Therefore, (a) implies the definition.

Hence, (a), (b) and the definition are all equivalent. □

**Exercise 2.7(Proof of Proposition 2.6).** Let  $M$  and  $N$  be smooth manifolds with or without boundary, and let  $F : M \rightarrow N$  be a map.

- (a) If every point  $p \in M$  has a neighborhood  $U$  such that the restriction  $F|_U$  is smooth, then  $F$  is smooth.
- (b) Conversely, if  $F$  is smooth, then its restriction to every open subset is smooth.

*Proof.* Let  $\mathcal{A}_M, \mathcal{A}_N$  be smooth structures of  $M, N$ , respectively.

- (a) Let  $p \in M$ . Let  $U$  be a neighborhood of  $p$  such that  $F|_U$  is smooth. By 1.44,  $U$  is a smooth manifold with the induced smooth structure  $\mathcal{A}_U = \{(V, \phi) \in \mathcal{A}_M \mid V \subset U\}$ . Since  $F|_U$  is smooth, there exist  $(V, \phi) \in \mathcal{A}_U$  and  $(W, \psi) \in \mathcal{A}_N$  such that:

- $F|_U(V) \subset W$ .
- $\psi \circ F|_U \circ \phi^{-1} : \phi(V) \rightarrow \psi(W)$  is smooth.

Since  $V \subset U$ ,  $F(V) \subset W$ ,  $\psi \circ F \circ \phi^{-1} : \phi(V) \rightarrow \psi(W)$  is smooth, and  $(V, \phi) \in \mathcal{A}$ . Therefore,  $F$  is smooth.

- (b) Let  $U \subset M$  be an open subset. By 1.44,  $U$  is a smooth manifold with the induced smooth structure  $\mathcal{A}_U = \{(V, \phi) \in \mathcal{A}_M \mid V \subset U\}$ . Let  $p \in U$ . Then  $p \in F$ , so there exist  $(V, \phi) \in \mathcal{A}_M, (W, \psi) \in \mathcal{A}_N$  such that  $F(V) \subset W$  and  $\psi \circ F \circ \phi^{-1} : \phi(V) \rightarrow \psi(W)$  is smooth. Then  $(V \cap U, \phi|_{V \cap U})$  is a chart that is smoothly compatible with every chart in  $\mathcal{A}_M$ . Therefore,  $(V \cap U, \phi|_{V \cap U}) \in \mathcal{A}_M$ . Moreover,  $\phi|_{V \cap U}(V \cap U) \subset \phi(V) \subset W$  and  $\psi \circ F \circ (\phi|_{V \cap U}(V \cap U))^{-1}$  is clearly smooth. Therefore,  $F|_U$  is smooth. □

**Exercise 2.9.** Suppose  $F : M \rightarrow N$  is a smooth map between smooth manifolds with or without boundary. Show that the coordinate representation of  $F$  with respect to *every* pair of smooth charts for  $M$  and  $N$  is smooth.

*Proof.* Let  $(M, \mathcal{A}_M), (N, \mathcal{A}_N)$  be smooth manifolds with or without boundary. Let  $F : M \rightarrow N$  be a smooth map. Let  $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$  be given. We must show that  $\hat{F} = \psi \circ F \circ \phi^{-1}$  is a smooth function from  $\phi(U \cap F^{-1}(V))$  to  $\psi(V)$ . Let  $\phi(p) \in \phi(U \cap F^{-1}(V))$ . Then  $p \in M$ , so there exist  $(U_0, \phi_0) \in \mathcal{A}_M$  and  $(V_0, \psi_0) \in \mathcal{A}_N$  such that

- $p \in U_0 \subset U \cap F^{-1}(V)$ ;
- $\phi_0(U_0) \subset \phi_0(U)$ ;
- $\psi_0 \circ F \circ \phi_0^{-1} : \phi_0(U_0) \rightarrow \psi(V_0)$  is smooth.

Then  $\psi \circ F \circ \phi^{-1}|_{\phi(U_0)} = (\psi \circ \psi_0^{-1}) \circ (\psi_0 \circ F \circ \phi_0^{-1}) \circ (\phi_0 \circ \phi)^{-1}$ . Since the composition of smooth functions in Euclidean spaces is smooth,  $\hat{F}$  is smooth. □

**Exercise 2.11(Proof of Proposition 2.10).** Let  $M, N$  and  $P$  be smooth manifolds with or without boundary.

- (a) Every constant map  $c : M \rightarrow N$  is smooth.
- (b) The identity map of  $M$  is smooth.
- (c) If  $U \subset M$  is an open submanifold with or without boundary, then the inclusion map  $U \rightarrow M$  is smooth.

*Proof.* Let  $\mathcal{A}_M, \mathcal{A}_N, \mathcal{A}_P$  be smooth structures of  $M, N, P$ , respectively.

- (a)  $F$  is clearly continuous. Moreover, for every  $(U_\alpha, \phi_\alpha) \in \mathcal{A}_M, (V_\beta, \psi_\beta) \in \mathcal{A}_N$ ,  $\psi_\beta \circ F \circ \phi_\alpha^{-1}$  is a constant map, so it is smooth. By 2.7(Proof of Proposition 2.6),  $F$  is smooth.
- (b) Let  $p \in M$ . Choose  $(U, \phi) \in \mathcal{A}_M$  such that  $p \in U$ . Then  $F(U) \subset U$  and  $\phi \circ F \circ \phi^{-1} = \text{Id}_U$ , so it is smooth. Therefore,  $F$  is smooth.
- (c) By 1.44,  $\mathcal{A}_U = \{(V, \phi) \mid V \subset U\}$  is a smooth structure of  $U$ . Let  $p \in U$ . Then  $p \in V$  for some  $(V, \phi) \in \mathcal{A}_U$ . Then  $(V, \phi) \in \mathcal{A}_M$ , trivially. Since  $F(V) \subset V$  and  $\phi \circ F \circ \phi^{-1}$  is simply the identity map on  $V$ ,  $F$  is smooth. □

**Exercise 2.16(Proof of Proposition 2.15).**

- (a) Every composition of diffeomorphisms is a diffeomorphism.

- (b) Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.
- (c) Every diffeomorphism is a homeomorphism and an open map.
- (d) The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.
- (e) “Diffeomorphic” is an equivalence relation on the class of all smooth manifolds with or without boundary.

**Exercise 2.16(Proof of Proposition 2.15).** Let  $(M, \mathcal{A}_M), (N, \mathcal{A}_N), (P, \mathcal{A}_P)$  be smooth manifolds with or without boundary, and let  $F : M \rightarrow N, G : N \rightarrow P$  be diffeomorphisms.

- (a) By Proposition 2.10(d),  $G \circ F$  and  $F^{-1} \circ G^{-1}$  are smooth. Then  $(G \circ F) \circ (F^{-1} \circ G^{-1})$  and  $(F^{-1} \circ G^{-1}) \circ (G \circ F)$  are both the identity map on the corresponding space, so  $F^{-1} \circ G^{-1}$  is the smooth inverse of  $G \circ F$ . Therefore,  $G \circ F$  is a diffeomorphism.

Read Example 1.34(Smooth Product Manifolds)!

- (b)
- (c) Proposition 2.4 states that every smooth map is continuous. Thus  $F$  and  $F^{-1}$  are both continuous. Therefore,  $F$  is a homeomorphism and also an open map.
- (d) Let  $U \subset M$  be an open subset. By (2.7(Proof of Proposition 2.6)),  $F|_U$  is smooth. Since  $F$  is a homeomorphism as shown in (c),  $F(U)$  is an open subset of  $N$ . Therefore,  $F^{-1}|_{F(U)}$  is smooth by (2.7(Proof of Proposition 2.6)). Clearly,  $F|_U$  and  $F^{-1}|_{F(U)}$  are the inverse of each other. Therefore,  $F|_U$  is a diffeomorphism.
- (e) By (2.11(Proof of Proposition 2.10)), the identity map on  $M$  is a diffeomorphism, so the reflexive property is satisfied. Moreover,  $(F^{-1})^{-1} = F$ , so the symmetric property is satisfied. By (a), the composition of two diffeomorphisms is a diffeomorphism, so the transitive property is satisfied. Therefore, “diffeomorphic” is an equivalence relation.

### 3. CHAPTER 3: TANGENT VECTORS

**Exercise 3.5(Proof of Lemma 3.4).** Suppose  $M$  is a smooth manifold with or without boundary,  $p \in M, v \in T_p M$ , and  $f, g \in C^\infty(M)$ .

- (a) If  $f$  is a constant function, then  $vf = 0$ .
- (b) If  $f(p) = g(p) = 0$ , then  $v(fg) = 0$ .

*Proof.*

- (a) Let  $h$  be the constant function that always takes the value 1. Then  $f(p) = ch(p)$  for some  $c \in \mathbb{R}$ . Then  $v(fh) = f(p)vf + f(p)vf$ , so  $c^2v(h) = c^2v(h) + c^2v(h)$ . Therefore,  $c^2v(h) = 0$ , so  $cv(h) = 0$ . Since  $v$  is linear, this implies  $0 = v(ch) = v(f)$ , so  $v(f) = 0$ .
- (b)  $v(fg) = f(p)vg + g(p)vf = 0 + 0 = 0$ .

□

**Exercise 3.7(Proof of Proposition 3.6).** Let  $M, N$ , and  $P$  be smooth manifolds with or without boundary, let  $F : M \rightarrow N$  and  $G : N \rightarrow P$  be smooth maps, and let  $p \in M$ .

- (a)  $dF_p : T_p M \rightarrow T_{F(p)} N$  is linear.
- (b)  $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_p M \rightarrow T_{G \circ F(p)} P$ .
- (c)  $d(\text{Id}_M)_p = \text{Id}_{T_p M} : T_p M \rightarrow T_p M$ .
- (d) If  $F$  is a diffeomorphism, then  $dF_p : T_p M \rightarrow T_{F(p)} N$  is an isomorphism, and  $(dF_p)^{-1} = d(F^{-1})_{F(p)}$ .

*Proof.* (a)  $\forall v, w \in T_p M, \forall c \in \mathbb{R}, \forall f \in C^\infty(N)$ ,

$$\begin{aligned}
 dF_p(cv + w)(f) &= (cv + w)(f \circ F) \\
 &= (cv)(f \circ F) + w(f \circ F) \\
 &= c(v(f \circ F)) + w(f \circ F) \\
 &= c(dF_p(v)(f)) + dF_p(w)(f) \\
 &= (cdF_p(v))(f) + dF_p(w)(f) \\
 &= (cdF_p(v) + dF_p(w))(f).
 \end{aligned}$$

- Therefore,  $dF_p(cv + w) = cdF_p(v) + dF_p(w)$ .  
 (b)  $\forall v \in T_p M, f \in C^\infty(P)$ ,

$$\begin{aligned}
 d(G \circ F)_p(v)(f) &= v(f \circ (G \circ F)) \\
 &= v((f \circ G) \circ F) \\
 &= (dF_p(v))(f \circ G) \\
 &= (dG_{F(p)}(dF_p(v)))(f) \\
 &= ((dG_{F(p)} \circ dF_p)(v))(f)
 \end{aligned}$$

- Therefore,  $d(G \circ F)_p = dG_{F(p)} \circ dF_p$ .  
 (c)  $\forall v \in T_p(M), \forall f \in C^\infty(M)$ ,

$$\begin{aligned}
 d(\text{Id}_M)_p(v)(f) &= v(f \circ \text{Id}_M) \\
 &= v(f).
 \end{aligned}$$

- Therefore,  $d(\text{Id}_M)_p(v) = v$ , so  $d(\text{Id}_M)_p = \text{Id}_{T_p M}$ .  
 (d)  $F^{-1}$  exists and it is a smooth map since  $F$  is a diffeomorphism. By combining (b) and (c), we obtain  $dF_p$  and  $dF_{F(p)}^{-1}$  are the inverse of each other. Therefore,  $dF_p$  is an isomorphism.  $\square$

#### 4. APPENDIX A: REVIEW OF TOPOLOGY

**Exercise A.18(Proof of Proposition A.17).** Let  $X$  be a topological space and let  $S$  be a subspace of  $X$ .

- (a)
- (b)
- (c)
- (d)
- (e)
- (f) If  $\mathcal{B}$  is a basis for the topology of  $X$ , then  $\mathcal{B}_S = \{B \cap S \mid B \in \mathcal{B}\}$  is a basis for the subspace topology on  $S$ .
- (g) If  $X$  is Hausdorff, then so is  $S$ .
- (h) If  $X$  is first-countable, then so is  $S$ .
- (i) If  $X$  is second-countable, then so is  $S$ .

*Proof.*

- (a)
- (b)
- (c)
- (d)
- (e)
- (f) The union of  $B \cap S$  is  $S$ . Let  $U \cap S$  be an open subset of  $S$  where  $U$  is open in  $X$ , and  $x \in U \cap S$ . Then there exists  $B \in \mathcal{B}$  such that  $x \in B \subset U$  since  $\mathcal{B}$  is a basis. Therefore,  $x \in B \cap S \subset U \cap S$  with  $B \cap S \in \mathcal{B}_S$ .
- (g) Let  $x \neq y \in S$ . There exist two disjoint open sets  $U, V$  of  $X$  containing  $x, y$ , respectively. Then  $U \cap S$  and  $V \cap S$  are disjoint open sets of  $X$  containing  $x, y$ , respectively.
- (h)
- (i) Let  $\mathcal{B}$  be a countable basis of  $X$ . Then  $\{B \cap S \mid B \in \mathcal{B}\}$  is a countable basis of  $S$  by (f).  $\square$

**Exercise A.24(Proof of Proposition A.23).** Suppose  $X_1, \dots, X_k$  are topological spaces, and let  $X_1 \times \dots \times X_k$  be their product space.

- (a) **CHARACTERISTIC PROPERTY:** If  $B$  is a topological space, a map  $F : B \rightarrow X_1 \times \dots \times X_k$  is continuous if and only if each of its component functions  $F_i = \pi_i \circ F : B \rightarrow X_i$  is continuous.

*Proof.*

- (a) Suppose  $F$  is continuous. Since  $\pi_i$  is continuous by (c) and the composition of continuous functions is continuous,  $\pi_1 \circ F$  is continuous. Suppose each component function is continuous. Let  $B_1 \times \cdots \times B_k$  be a basis element of  $X_1 \times \cdots \times X_k$ .

$$\begin{aligned} F^{-1}(B_1 \times \cdots \times B_k) &= F^{-1}(\cap_{i=1}^k \pi_i^{-1}(B_1 \times \cdots \times B_k)) \\ &= \cap_{i=1}^k F^{-1}(\pi_i^{-1}(B_1 \times \cdots \times B_k)) \\ &= \cap_{i=1}^k (\pi_i \circ F)^{-1}(B_1 \times \cdots \times B_k). \end{aligned}$$

Since the intersection of finitely many open sets is open,  $F$  is continuous. □

## 5. APPENDIX C: REVIEW OF CALCULUS

**Exercise C.1.** Suppose that  $F : U \rightarrow W$  is differentiable at  $a \in U$ . Show that the linear map satisfying

$$\lim_{v \rightarrow 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} = 0$$

is unique.

*Proof.* Let  $L, L'$  be two such linear maps.

$$\begin{aligned} \lim_{v \rightarrow 0} \frac{|Lv - L'v|}{|v|} &= \lim_{v \rightarrow 0} \frac{|(F(a+v) - F(a) - L'v) - (F(a+v) - F(a) - Lv)|}{|v|} \\ &= \lim_{v \rightarrow 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} + \lim_{v \rightarrow 0} \frac{|F(a+v) - F(a) - L'v|}{|v|} \\ &= 0 + 0 = 0. \end{aligned}$$

If  $L \neq L'$ ,  $(L - L')v_0 \neq 0$  for some  $v_0$ . Then  $\lim_{v \rightarrow 0} \frac{|Lv - L'v|}{|v|} = \lim_{h \rightarrow 0} \frac{|L(hv_0) - L'(hv_0)|}{|hv_0|} = \frac{|(L - L')v_0|}{|v_0|} \neq 0$ . This is a contradiction, so  $L = L'$ . □

## 6. DICTIONARY

### 6.1. Topological Manifolds.

**Definition 6.1** (Topological Manifold). A *topological  $n$ -manifold* is a Hausdorff, second-countable topological space each point of which has a neighborhood that is homeomorphic to an open subset  $\mathbb{R}^n$ .

**Definition 6.2** (Coordinates). Let  $M$  be a topological  $n$ -manifold. Let  $U$  be an open subset of  $M$ ,  $\hat{U}$  be an open subset of  $\mathbb{R}^n$ ,  $\phi : U \rightarrow \hat{U}$  be a homeomorphism.

- The pair  $(U, \phi)$  is called a *coordinate chart* or a *chart*.
- $U$  is called a *coordinate domain* or a *coordinate neighborhood* and  $\phi$  is called a *coordinate map*.
- If  $\phi(U)$  is an open ball in  $\mathbb{R}^n$ ,  $U$  is called a *coordinate ball*.
- If  $\phi(U)$  is an open cube in  $\mathbb{R}^n$ ,  $U$  is called a *coordinate cube*.
- The coordinate functions of  $\phi$  are often denoted as  $(x^1, \dots, x^n)$ . Thus a chart is sometimes denoted by  $(U, (x^1, \dots, x^n))$  or  $(U, (x^i))$ .

**Definition 6.3** (Atlas). Let  $M$  be a topological  $n$ -manifold. An *atlas* for  $M$  is a collection of charts  $(U_\alpha, \phi_\alpha)$  such that  $M = \bigcup_\alpha U_\alpha$ .

**Definition 6.4** (Transition Map). Let  $M$  be a topological  $n$ -manifold and  $(U, \phi), (V, \psi)$  be coordinate charts such that  $U \cap V \neq \emptyset$ .  $\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$  is called a *transition map* from  $\phi$  to  $\psi$ .

**Definition 6.5** (Closed Upper Half-Space).  $\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$ , and  $\partial\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n = 0\}$ .

**Definition 6.6** (Manifold With Boundary). Let  $M$  be a second-countable Hausdorff space and fix  $n$ . Suppose that for every  $p \in M$ , one of the following conditions is satisfied:



- (1) There exists a neighborhood  $U$  of  $p$  and a homeomorphism  $\phi : U \rightarrow \hat{U}$  where  $\hat{U}$  is an open subset of  $\mathbb{R}^n$ .  $p$  is called an *interior point* and  $(U, \phi)$  is called an *interior chart*.
- (2) There exists a neighborhood  $U$  of  $p$  and a homeomorphism  $\phi : U \rightarrow \hat{U}$  where  $\hat{U}$  is an open subset of  $\mathbb{H}^n$  with  $\hat{U} \cap \partial\mathbb{H}^n \neq \emptyset$ .  $p$  is called a *boundary point* and  $(U, \phi)$  is called a *boundary chart*.

Then  $M$  is called an  *$n$ -dimensional topological manifold with boundary*. Note that every topological manifold is a topological manifold with boundary.

## 6.2. Smooth Manifolds.

**Definition 6.7** (Smoothly Compatible). Let  $M$  be a topological  $n$ -manifold. Two coordinate charts  $(U, \phi), (V, \psi)$  are called *smoothly compatible* if  $U \cap V = \emptyset$  or the transition map  $\psi \circ \phi^{-1}$  is a diffeomorphism.

**Definition 6.8** (Smooth Atlas). Let  $M$  be a topological  $n$ -manifold. A *smooth atlas* is an atlas  $\mathcal{A}$  such that any two charts in  $\mathcal{A}$  are smoothly compatible with each other.

**Definition 6.9** (Smooth Structure). If  $M$  is a topological  $n$ -manifold, an atlas  $\mathcal{A}$  that is not properly contained in any larger smooth atlas is called *maximal* or a *smooth structure on  $M$* .

**Definition 6.10** (Smooth Manifold). A *smooth manifold* is a topological manifold equipped with a smooth structure.

**Definition 6.11.** Suppose  $(M, \mathcal{A})$  is a smooth manifold.

- Any chart  $(U, \phi) \in \mathcal{A}$  is called a *smooth chart*.
- Given a smooth chart  $(U, \phi)$ ,  $U$  is called a smooth coordinate domain and  $\phi$  is called a *smooth coordinate map*.
- Given a smooth chart  $(U, \phi)$ ,  $U$  is called a *smooth coordinate ball* if it is a coordinate ball.

*Remark 6.12.* One must define a smooth structure on a topological manifold before talking about a smooth chart.

**Definition 6.13** (Smooth Maps). Let  $M, N$  be smooth manifolds with or without boundary and  $F : M \rightarrow N$  be a map.  $F$  is a *smooth map* if for every  $p \in M$ , there exist smooth charts  $(U, \phi)$  containing  $p$  and  $(V, \psi)$  containing  $F(p)$  such that

- $F(U) \subset V$ ;
- $\psi \circ F \circ \phi^{-1} : \phi(U) \rightarrow \psi(V)$  is smooth.

**Definition 6.14** (Coordinate Representatin of a Smooth Map). Let  $(M, \mathcal{A}_M)$  and  $(N, \mathcal{A}_N)$  be smooth manifolds. Let  $F : M \rightarrow N$  be a smooth map and  $(U, \phi) \in \mathcal{A}_M$  and  $(V, \psi) \in \mathcal{A}_N$  be given. Then  $\hat{F} = \psi \circ F \circ \phi^{-1}$  is called the coordinate representation of  $F$  with respect to  $(U, \phi)$  and  $(V, \psi)$ .

**Definition 6.15** (Diffeomorphism). Let  $M, N$  be smooth manifolds with or without boundary. A diffeomorphism is a smooth map  $F : M \rightarrow N$  with a smooth inverse.

## 6.3. Tangent Vectors.

**Definition 6.16** (Derivation). Let  $M$  be a smooth manifold with or without boundary. A derivation at  $p \in M$  is a linear map  $v : C^\infty(M) \rightarrow \mathbb{R}$  such that

$$v(fg) = f(p)v g + g(p)v f$$

for all  $f, g \in C^\infty(M)$ .

**Definition 6.17** (Tangent Space). The tangent space  $T_p M$  to  $M$  at  $p$  is the vector space of all derivations of  $C^\infty(M)$  at  $p$ .

**Definition 6.18** (Differential).  $M, N$  are smooth manifolds with or without boundary, and  $F : M \rightarrow N$  is a smooth map. The *differential of  $F$  at  $p$*  is the linear map  $dF_p : T_p M \rightarrow T_{F(p)} N$  defined by

$$dF_p(v) := f \mapsto v(f \circ F)$$

Equivalently,  $\forall v \in T_p M, \forall f \in C^\infty(N), dF_p(v)(f) = v(f \circ F)$ .

**Definition 6.19** (Coordinate Vectors). Let  $(M, \mathcal{A})$  be a smooth manifold without boundary. Let  $p \in M$  and choose a chart  $(U, \phi) \in \mathcal{A}$  such that  $p \in U$ . Then the *coordinate vectors at  $p$* , denoted by  $\frac{\partial}{\partial x^i} \Big|_p$ , are derivations  $C^\infty(U) \rightarrow \mathbb{R}$  such that

$$\frac{\partial}{\partial x^i} \Big|_p := f \mapsto \frac{\partial}{\partial x^i} \Big|_{\phi(p)} (f \circ \phi^{-1}).$$