INTRODUCTION TO SMOOTH MANIFOLDS

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Contents

1. Chapter 1: Smooth Manifolds

1.1. Exercises.

Exercise 1.1. Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to *any* open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

Proof. It is clear that a "manifold" satisfying the open-ball or $\mathbb{R}^n$ definition satisfies the open-subset definition. Let M be a manifold satisfying the open-subset definition. Let $x \in M$ be given and let $U, \hat{U}, \phi$ be given according to the definition. Since $\hat{U}$ is open, there exists an open ball B such that $\phi(x) \in B \subset \hat{U}$. Restrict ϕ to $\phi^{-1}(B)$. Then $\phi^{-1}(B)$ is an open subset of M containing x, and $\phi\mid_{\phi^{-1}(B)}$ is a homeomorphism between $\phi^{-1}(B)$ and B. Thus M satisfies the open-ball definition.

$B(x,r) \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ by the map $(x_1 + a_1, \cdots, x_n + a_n) \mapsto (\frac{a_1}{r-a_1}, \cdots, \frac{a_n}{r-a_n})$ where $x = (x_1, \cdots, x_n)$ is the center of $B(x,r)$ and r is the radius. Since the composition of two homeomorphisms gives a homeomorphism, M also satisfies the $\mathbb{R}^n$ definition as well. □

Exercise 1.6. Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.

Proof. From the definition of π, it is easy to see that $\pi(B(x,r))$ is open in $\mathbb{RP}^n$ where $x \in S^n$ and $0 < r < 1$.

Let $[x], [y] \in \mathbb{RP}^n$ be given. Without loss of generality, assume $x, y \in S^n$. Let $r = \min\{|x-y|, |x+y|, 1\}/2$. Then $U_x = \pi(B(x,r)), U_y = \pi(B(y,r))$ contain $[x], [y]$, respectively. $\pi^{-1}(U_x), \pi^{-1}(U_y)$ are both open in $\mathbb{R}^{n+1} \setminus \{0\}$ which can be seen easily by writing down exactly which points belong to them, so U_x, U_y are both open in $\mathbb{RP}^n$. Then $\pi^{-1}(U_x \cap U_y) = \pi^{-1}(U_x) \cap \pi^{-1}(U_y) = \emptyset$, so $U_x \cap U_y = \emptyset$. Therefore, $\mathbb{RP}^n$ is Hausdorff.

Let $\mathcal{B} = \{\pi(B(x, 1/k)) \mid x \in \mathbb{Q}^{n+1} \cap S^n, k \in \{2, 3, 4, \cdots\}\}$. Then $\mathcal{B}$ is a countable collection of open sets whose union is $\mathbb{RP}^n$. Let $U \subset \mathbb{RP}^n$ be a nonempty open set. Let $[x] \in U$. Since π is a quotient map, $\pi^{-1}(U)$ is open. Moreover, $x \in \pi^{-1}(U)$. Without loss of generality, $x \in S^n$. Then $x \in B(x', 1/k) \subset \pi^{-1}(U)$ for some $B(x', 1/k) \in \mathcal{B}$. Then $[x] = \pi(x) \in \pi(B(x', 1/k)) \subset \pi(\pi^{-1}(U)) = U$. Therefore, $\mathcal{B}$ is a countable basis of $\mathbb{RP}^n$. □

Exercise 1.7. Show that $\mathbb{RP}^n$ is compact.

Proof. $\pi(S^n) = \mathbb{RP}^n$ and S^n is compact because it is a closed, bounded subset of $\mathbb{R}^{n+1}$. (Heine-Borel) Moreover, the image of a compact set under a continuous map is compact. (See A.45(a)) Thus $\mathbb{RP}^n$ is compact. □

Exercise 1.14. Suppose $\mathcal{X}$ is a locally finite collection of subsets of a topological space M.

(a) The collection $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.
(b) $\overline{\bigcup_{X \in \mathcal{X}} X} = \bigcup_{X \in \mathcal{X}} \overline{X}$.

Proof.

(a) Let $p \in M$. Then there exists an open set U containing x such that there are only finitely many $X \in \mathcal{X}$ such that $U \cap X \neq \emptyset$. Let $X \in \mathcal{X}$.
- If $U \cap X \neq \emptyset$, then $U \cap \overline{X} \supset U \cap X \neq \emptyset$.
- If $U \cap X = \emptyset$, then U^c is closed, so $\overline{X} \subset U^c$. In other words, $U \cap \overline{X} = \emptyset$.

This shows that the number of $X \in \mathcal{X}$ that intersects U and the number of $\overline{X} \in \mathcal{X}$ that intersects U are the same. Therefore, $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.

(b) Since the closure of a set is defined to be the intersection of all closed sets containing it, $\bigcup_{X \in \mathcal{X}} \overline{X} \subset \overline{\bigcup_{X \in \mathcal{X}} X}$. Let $x \notin \bigcup_{X \in \mathcal{X}} \overline{X}$. Then there exists a neighborhood U of x such that U intersects only finitely many $X \in \mathcal{X}$. Let $X_1, \cdots, X_n$ denote them. By the same argument as part (a), $\overline{X_1}, \cdots, \overline{X_n}$ are the only elements in $\{\overline{X} \mid X \in \mathcal{X}\}$ that U intersects. Since $x \notin \overline{X_i}$ for each $i = 1, \cdots, n$, $U^c \cup \overline{X_1} \cup \cdots \cup \overline{X_n}$ is a closed set which contains all $X \in \mathcal{X}$ but does not contain x. In other words, $x \notin \overline{\bigcup_{X \in \mathcal{X}} X}$.

□

Exercise 1.18. Let M be a topological manifold. Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Proof. Let $\mathcal{A}, \mathcal{A}'$ be two smooth atlases.

Suppose that they determine the same smooth structure $\mathcal{B}$. Then $\mathcal{A} \cup \mathcal{A}' \subset \mathcal{B}$, so $\mathcal{A} \cup \mathcal{A}'$ must be a smooth atlas. By Proposition 1.17(a), $\mathcal{A} \cup \mathcal{A}'$ determines a unique smooth structure, but it must be $\mathcal{B}$ because $\mathcal{B}$ contains the union.

On the other hand, suppose that their union is a smooth atlas. Let $\mathcal{B}$ be the smooth structure that the union determines. Such $\mathcal{B}$ must exist by Proposition 1.17(a). By the same proposition, $\mathcal{A}, \mathcal{A}'$ must determine the unique smooth structures. However, they must be $\mathcal{B}$ because $\mathcal{B}$ contains both $\mathcal{A}$ and $\mathcal{A}'$. □

Exercise 1.20. Every smooth manifold has a countable basis of regular coordinate balls.

Proof. Let M be an n-dimensional smooth manifold. We consider the special case that there exists a single chart (ϕ, U) with $U = M$. Let $x \in \hat{U}$ with rational coordinates. Then there exists $s > 0$ such that $B(x, s) \subset \hat{U}$. For each rational number $r \in (0, s)$, we consider the chart $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r)))$.

Let $\mathcal{B}$ be the collection of all such charts for each $x \in \hat{U}$ and r. We claim that $\mathcal{B}$ is a smooth atlas.

- Let $p \in M$. Then $\phi(p) \in \hat{U}$. Since $\hat{U}$ is open, $\phi(p) \in B(x, r) \subset \hat{U}$ for some x with rational coordinates and a positive rational number r. Then $p \in \phi^{-1}(B(x, r))$, so the union of coordinate domains covers M. In other words, $\mathcal{B}$ is an atlas.
- Let $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r))), (p \mapsto \phi(p) - x', \phi^{-1}(B(x', r'))) \in \mathcal{B}$ be given. Suppose $\phi^{-1}(B(x, r)) \cap \phi^{-1}(B(x', r')) \neq \emptyset$. Let ψ, ψ' denote the coordinate maps. Then $\psi' \circ \psi^{-1}$ is a composition of ϕ, ϕ^{-1} and translation maps, so it is smooth.

Therefore, $\mathcal{B}$ is a smooth atlas.

Since $\mathcal{B}$ is a smooth atlas, there exists a smooth structure $\mathcal{A}$ on M containing $\mathcal{B}$ by Proposition 1.17(a). We claim that $\mathcal{B}$, a subset of the smooth structure $\mathcal{A}$, is a countable basis of regular coordinate balls.

- $\mathcal{B}$ is a countable collection because $x \in \mathbb{Q}^n$ and $r \in \mathbb{Q}$.
- Let $(p \mapsto \phi(p)-x, \phi^{-1}(B(x,r))) \in \mathcal{B}$ be given. Then there exists a chart $(p \mapsto \phi(p)-x, \phi^{-1}(B(x,r')))$ in $\mathcal{B}$ with $r' > r$. Let $B = \phi^{-1}(B(x,r)), B' = \phi^{-1}(B(x,r'))$. Let ψ denote the map $p \mapsto \phi(p) - x$. Then $\psi(B) = B(0,r)$ and $\psi(B') = B(0,r')$, respectively. Moreover, $\psi(\overline{B}) = \overline{B(0,r)}$ because ψ is a homeomorphism.

Now let M be an arbitrary smooth n-manifold. By definition, each point of M is in the domain of a chart. By Proposition A.16, M is covered by countably many charts $\{(U_i, \phi_i)\}$. By the previous argument, each U_i has a countable basis of regular coordinate balls. Each regular coordinate ball in U_i is indeed a regular coordinate ball in M because $\overline{B}$ is a compact subset of M, which is Hausdorff, so $\overline{B}$ is closed. In other words, the closure of B in U_i is the same as the closure of B in M. □

Exercise 1.39. Let M be a topological n-manifold with boundary.

(a) $\operatorname{Int} M$ is an open subset of M and a topological n-manifold without boundary.
(b) ∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.
(c) M is a topological manifold if and only if $\partial M = \emptyset$.
(d) If $n = 0$, then $\partial M = \emptyset$ and M is a 0-manifold.

Proof.

(a) Let $x \in \operatorname{Int} M$. Let (ϕ, U) be an interior chart for x. Then $x \in U \subset \operatorname{Int} M$ because every point in U is in an interior chart (ϕ, U). A subspace of M must be Hausdorff and second-countable by Proposition A.17(g, i), so $\operatorname{Int} M$ is a second-countable, Hausdorff space in which every point has a neighborhood homeomorphic to an open subset in $\mathbb{R}^n$. Thus $\operatorname{Int} M$ is an n-manifold without boundary.

(b) Since $\partial M = M \setminus \operatorname{Int} M$ and $\operatorname{Int} M$ is open in M, ∂M is closed in M. Let $x \in \partial M$. Let (ϕ, U) be a boundary chart of x. If a point $y \in U$ gets mapped into $\operatorname{Int} \mathbb{H}^n$, then it is certainly an interior point. Thus $\phi(U \cap \partial M) \subset \partial \mathbb{H}^n$. Then $\pi_{n-1} \circ \phi$ is a homeomorphism that maps $U \cap \partial M$ into an open subset of $\mathbb{R}^{n-1}$ where $\pi_{n-1} : (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_{n-1})$.

(c) If ∂M is empty, then $M = \operatorname{Int} M$, so (a) implies that M is an n-dimensional manifold. If M is a topological manifold, every point is an interior point. Since a point cannot be both an interior point and a boundary point, ∂M is empty.

(d) If $n = 0$, then $\partial \mathbb{H}^0 = \emptyset$. Thus, the condition that $\phi(U) \cap \partial \mathbb{H}^n \neq \emptyset$ can never be satisfied, so there cannot be any boundary point.

□

Exercise 1.41. Let M be a topological manifold with boundary.

(a) M has a countable basis of precompact coordinate balls and half-balls.
(b) M is locally compact.
(c) M is paracompact.
(d) M is locally path-connected.
(e) M has countably many components, each of which is an open subset of M and a connected topological manifold with boundary.
(f) The fundamental group of M is countable.

Proof.

(a)
(b)
(c)
(d) Let $U \subset M$ be a nonempty open subset and choose $x \in U$. Then there exists a chart (V, ϕ) such that $x \in V$. Since $\phi(x)$ is a point in an open set $\phi(U \cap V)$, there exists $r > 0$ such that $B(\phi(x), r) \subset \phi(V)$. Then $N(x, U) = \phi^{-1}(B(\phi(x), r))$ is a path-connected neighborhood of x that is contained in $U \cap V \subset U$. Therefore, $\{N(x, U) \mid \text{open } U \subset M, x \in U\}$ forms a basis of M consisting of path-connected sets.

(e)
(f)

□

Exercise 1.44. Suppose M is a smooth n-manifold with boundary and U is an open subset of M. Prove the following statements:

(a) U is a topological n-manifold with boundary, and the atlas consisting of all smooth charts (V, ϕ) for M such that $V \subset U$ defines a smooth structure on U. With this topology and smooth structure, U is called an ***open submanifold with boundary***.
(b) If $U \subset \operatorname{Int} M$, then U is actually a smooth manifold (without boundary); in this case we call it an ***open submanifold of M***.
(c) $\operatorname{Int} M$ is an open submanifold of M (without boundary).

Proof. Let $\mathcal{T}$ denote the topology of M and $\mathcal{A}$ denote the smooth structure of M.

(a) The subspace topology on U is equivalent to $\mathcal{T}_U = \{V \in \mathcal{T} \mid V \subset U\}$ because U is open. By Proposition A.17(A.18(Proof of Proposition A.17)), U is Hausdorff and second-countable. For every point $p \in U$, there exists a $V \in \mathcal{T}$ with a homeomorphism $\phi : V \to \hat{V}$ where $\hat{V}$ is an open subset of $\mathbb{R}^n$ (or $\mathbb{H}^n$) Since $U \cap V$ is an open subset of V, ϕ restricted to $U \cap V$ is a homeomorphism between $U \cap V$ and $\phi(U \cap V)$, which is an open subset of $\mathbb{R}^n$ (or $\mathbb{H}^n$). Therefore, U is a topological n-manifold with boundary.
Let $\mathcal{A}_U = \{(\phi, V) \in \mathcal{A} \mid V \subset U\}$. Then $\mathcal{A}_U$ is clearly a collection of charts on U whose union covers U. Moreover, any two charts in $\mathcal{A}_U$ are clearly smoothly compatible. Let (ϕ, V) be a chart on U that is smoothly compatible with every chart in $\mathcal{A}_U$. Let $(\psi, W) \in \mathcal{A}$. Then $(\psi_{W\cap U}, W \cap U)$ is a chart on M and it must be smoothly compatible with every chart in $\mathcal{A}$. Therefore, $(\psi_{W\cap U}, W \cap U) \in \mathcal{A}$, so it must belong to $\mathcal{A}_U$. This implies that (ϕ, V) and $(\psi_{W\cap U}, W \cap U)$ are smoothly compatible. Since $V \subset W \cap U$, this implies that (ϕ, V) and (ψ, W) are smoothly compatible.
Thus (ϕ, V) is smoothly compatible with every chart in $\mathcal{A}$, so $(\phi, V) \in \mathcal{A}$. This implies that (ϕ, V) is in $\mathcal{A}_U$, so $\mathcal{A}_U$ is indeed a maximal smooth atlas.
(b) Let $p \in U$. Then $p \in \operatorname{Int} M$, so there exists $(\phi, V) \in \mathcal{A}$ such that $p \in V$ and $\phi(V)$ is open in $\mathbb{R}^n$. Then $(\phi|_{V\cap U}, V \cap U)$ is a chart that is smoothly compatible with every chart in $\mathcal{A}$, so $(\phi|_{V\cap U}, V \cap U) \in \mathcal{A}$. Thus it must be in $\mathcal{A}_U$, so $p \in U$ is an interior point of U. Therefore, U is a manifold without boundary.
(c) By 1.39, $\operatorname{Int} M$ is an open subset of M. By (b), $\operatorname{Int} M$ is an open submanifold of M without boundary.

□

1.2. **Problems.**

Problem 1-2. Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

Proof. Let I denote an uncountable index set and $X = \coprod_{\alpha \in I} \mathbb{R}$. Let $(x, \alpha_0) \in X$. Define $U = \coprod_{\alpha \in I} U_\alpha$ where $U_{\alpha_0} = \mathbb{R}$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_0$. Then U is an open neighborhood of (x, α_0) that is clearly homeomorhpic to $\mathbb{R}$. Thus X is locally Euclidean.

Let $(x_1, \alpha_1) \neq (x_2, \alpha_2) \in X$. If $\alpha_1 \neq \alpha_2$, then open neighborhoods of x_1 and x_2 formed in the same way as above separate the two points. Suppose $\alpha_1 = \alpha_2$. Without loss of generality, $x_1 < x_2$. Define $U = \coprod_{\alpha \in I} U_\alpha$ where $U_{\alpha_1} = (-\infty, (x_1 + x_2)/2)$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_1$. Similarly, define $V = \coprod_{\alpha \in I} U_\alpha$ where $U_{\alpha_1} = ((x_1 + x_2)/2, \infty)$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_2$. Then such U and V separate the two points. Therefore, X is Hausdorff.

Let $\mathcal{B}$ be a basis of X. For each $\alpha_0 \in I$, let $U_{\alpha_0} = \coprod_{\alpha \in I} U_\alpha$ where $U_{\alpha_0} = \mathbb{R}$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_0$. Then for each α_0, there must exist $B_{\alpha_0} \in \mathcal{B}$ such that $(0, \alpha_0) \in B_{\alpha_0} \subset U_{\alpha_0}$. Clearly, $B_\alpha \neq B_\beta$ if $\alpha \neq \beta$. Therefore, the cardinality of $\mathcal{B}$ is greater than or equal to that of I. Hence, X is not second-countable. □

Problem 1-7. Let N denote the **north pole** $(0, \cdots, 0, 1) \in S^n \subset \mathbb{R}^{n+1}$, and let S denote the **south pole** $(0, \cdots, 0, -1)$. Define the **stereographic projection** $\sigma : S^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1, \cdots, x^{n+1}) = \frac{(x^1, \cdots, x^n)}{1 - x^{n+1}}$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus \{S\}$.

(a) For any $x \in S^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$. Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace.

(b) Show that σ is bijective, and

$$\sigma^{-1}(u^1, \cdots, u^n) = \frac{(2u^1, \cdots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

(c) Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(S^n \setminus \{N\}, \sigma)$ and $(S^n \setminus \{S\}, \tilde{\sigma})$ defines a smooth structure on S^n.

(d) Show that this smooth structure is the same as the one defined in Example 1.31.

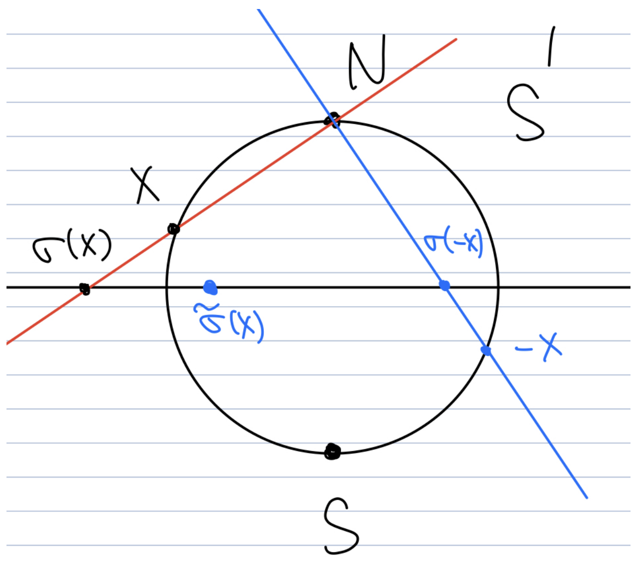

FIGURE 1. Problem 1-7

Proof.

(a) This is trivial from a basic trigonometry argument using the triangles $N, (0, \cdots, 0, x^{n+1}), (x^1, \cdots, x^{n+1})$ and $N, (0, \cdots, 0), \sigma(x^1, \cdots, x^{n+1})$.

(b) $\sigma \circ \sigma^{-1}$ and $\sigma^{-1} \circ \sigma$ are both the identity maps, so σ is bijective and σ^{-1} is its inverse.

(c) Computation shows that $\tilde{\sigma} \circ \sigma^{-1} : S^n \setminus \{N, S\} \to S^n \setminus \{N, S\}$ sends $(u^1, \cdots, u^n)$ to $(u^1, \cdots, u^n)/|u|^2$. As $|u| \neq 0$ in the domain, this map is well-defined and clearly smooth. By Proposition 1.17(a), these two charts determine a unique smooth structure.

(d) $\phi_i, \sigma, \tilde{\sigma}$ are all smooth functions of subsets of Euclidean spaces, so transition maps are always smooth. By Proposition 1.17(b), the smooth structure determined by $\sigma, \tilde{\sigma}$ is the same as the one defined in Example 1.31.

□

Problem 1-8. By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle S^1 as a subset of the complex plane. An angle function on a subset $U \subset S^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$.

Show that there exists an angle function θ on an open subset $U \subset S^1$ if and only if $U \neq S^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for S^1 with its standard smooth structure.

Proof. First, we will consider the special case when $U = S^1 \backslash \{e^{it}\}$ for some $t \in \mathbb{R}$. The map $\phi : (t, t+2\pi) \to U$ defined by $\theta \mapsto e^{i\theta}$ is a bijective function. Therefore, by taking the inverse of ϕ, which is clearly continuous, we obtain a desired angle function. The case of an arbitrary proper open subset of U is the same as this special case because we simply need to restrict the domain of the map obtained above. On the other hand, suppose $U = S^1$. Suppose there exists an angle function f on U. Define $g : S^1 \to \mathbb{R}$ by $g(z) = f(z) - f(-z)$.

- $g(1) \neq 0$ because $g(1) \neq 0 \implies f(1) = f(-1)$, which is clearly impossible.
- $g(1) > 0$ implies that $g(-1) < 0$. By the intermediate value theorem, $g(z) = 0$ for some $z \in S^1$. This is a contradiction.
- If $g(1) < 0$, $g(-1) > 0$, and we obtain a contradiction in the same manner.

Therefore, such an f cannot exist. Hence, an angle function exists if and only if U is an proper open subset of S^1.

Let $(U_i^{\pm}, \phi_i^{\pm})$ and (U, ϕ) be given where ϕ maps U into $(t, t+2\pi)$ for some $t \in \mathbb{R}$. We will show that they are smoothly compatible. Let $V = U \cap U_i^{\pm}$. The map $\phi_i^{\pm} \circ \phi^{-1} : \phi(V) \to \phi_i^{\pm}(V)$ is $\phi_i^{\pm} \circ \exp$. Since it is a composition of a projection map with a smooth map, this is smooth. Therefore, (U, ϕ) is indeed a coordinate chart for S^1 with its standard smooth structure. □

Problem 1-12(Proof of Proposition 1.45). Suppose $M_1, \cdots, M_k$ are smooth manifolds and N is a smooth manifold with boundary. Then $M_1 \times \cdots \times M_k \times N$ is a smooth manifold with boundary, and $\partial(M_1 \times \cdots \times M_k \times N) = M_1 \times \cdots \times M_k \times \partial N$.

Proof. By Example 1.34, $M_1 \times \cdots \times M_k$ is a smooth manifold. Thus it suffices to show that $M \times N$ is a smooth manifold with boundary if M is a smooth manifold and N is a smooth manifold with boundary. Let m, n be the dimensions of M, N.

First, we show that $M \times N$ is a topological manifold with boundary and $\partial(M \times N) = M \times \partial N$. Let $(p, q) \in M \times N$. Then $p \in M$, so there exists a chart (U, ϕ) such that $p \in U$ and $\hat{U} = \phi(U) \subset \mathbb{R}^m$.

- Suppose $q \in \operatorname{Int} N$. Then there exists a chart (V, ψ) such that $\hat{V} = \psi(V) \subset \mathbb{R}^n$. $\phi \times \psi$ is a homeomorphism between $U \times V$ and $\hat{U} \times \hat{V} \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. Thus $(U \times V, \phi \times \psi)$ is a chart for (p, q).
- Suppose $q \in \operatorname{bd} N$. Then there exists a chart (V, ψ) such that $\hat{V} = \psi(V) \subset \mathbb{H}^n$ and $\psi(q) \in \partial\mathbb{H}^n$. $\phi \times \psi$ is a homeomorphism between $U \times V$ and $\hat{U} \times \hat{V} \subset \mathbb{R}^m \times \mathbb{H}^n = \mathbb{H}^{m+n}$. Moreover, $(\phi \times \psi)(p, q) = (\phi(p), \psi(q)) \in \mathbb{R}^m \times \mathbb{H}^n = \mathbb{H}^{m+n}$. Thus $(U \times V, \phi \times \psi)$ is a boundary chart for (p, q).

Therefore, $M \times N$ is a topological manifold with boundary and $\partial(M \times N) = M \times (\partial N)$.

Let $\mathcal{A}_M, \mathcal{A}_N$ be the smooth structures of M, N. Define $\mathcal{A}_{M \times N} = \{(U \times V, \phi \times \psi) \mid (U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N\}$. Then $\mathcal{A}_{M \times N}$ is an atlas because we showed earlier that each $(U \times V, \phi \times \psi)$ is a chart. Let $(U_1 \times V_1, \phi_1 \times \psi_1), (U_2 \times V_2, \phi_2 \times \psi_2) \in \mathcal{A}_{M \times N}$. Then $(\phi_2 \times \psi_2) \circ (\phi_1 \times \psi_1)^{-1} = (\phi_2 \circ \phi_1^{-1}) \times (\psi_2 \circ \psi_1^{-1})$ is a smooth map from $(\phi_1 \times \psi_1)(U_1 \times V_1)$ into $(\phi_2 \times \psi_2)(U_2 \times V_2)$. Thus every pair of charts in $\mathcal{A}_{M \times N}$ is smoothly compatible. In other words, $\mathcal{A}_{M \times N}$ is a smooth atlas.

On the other hand, $\mathcal{A}_{M \times N}$ must be maximal because the restriction of any smoothly compatible chart to M, N gives a smoothly compatible chart, which must belong to $\mathcal{A}_M, \mathcal{A}_N$, respectively. Thus $M \times N$ is a smooth manifold with boundary. □

2. Chapter 2: Smooth Maps

2.1. Exercises.

Exercise 2.1. Let M be a smooth manifold with or without boundary. Show that pointwise multiplication turns $C^\infty(M)$ into a commutative ring and a commutative and associative algebra over $\mathbb{R}$.

Proof.

- The constant map $f(p) = 0$ is clearly in $C^\infty(M)$ and it is the additive identity.
- The constant map $f(p) = 1$ is clearly in $C^\infty(M)$ and it is the multiplicative identity.

- Let $f \in C^\infty(M), g \in C^\infty(M)$. Let $p \in M$ and (ϕ, U) be a smooth chart for p. Then $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are both smooth(Exercise 2.3), real-valued maps defined on an open subset of $\mathbb{R}^n$. Thus $f + g$ is in $C^\infty(M)$ Moreover, $f + g = g + f$ because addition in $\mathbb{R}$ is commutative.
- Let $f, g, h \in C^\infty(M)$. Let $p \in M$ and (ϕ, U) be a smooth chart for p. Then $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are both smooth(Exercise 2.3), real-valued maps defined on an open subset of $\mathbb{R}^n$. Therefore, fg is in $C^\infty(M)$ Moreover, $fg = gf$ and $(fg)h = f(gh)$ because multiplication in $\mathbb{R}$ is commutative and associative.
- Let $c \in \mathbb{R}, f \in C^\infty(M)$. Then cf can be seen as fg where g is the constant function whose value is c. As shown above, $cf \in C^\infty(M)$.

□

Exercise 2.2. Let U be an open submanifold of $\mathbb{R}^n$ with its standard smooth manifold structure. Show that a function $f : U \to \mathbb{R}^k$ is smooth in the sense just defined if and only if it is smooth in the sense of ordinary calculus. Do the same for an open submanifold with boundary in $\mathbb{H}^n$.

Proof. f is smooth in the sense just defined if and only if $f \circ \mathrm{Id}^{-1}$ is smooth in the sense of ordinary calculus. Since $f \circ \mathrm{Id}^{-1} = f$, $f \circ \mathrm{Id}^{-1}$ is smooth in the sense of ordinary calculus if and only if f is smooth in the sense of ordinary calculus. □

Exercise 2.3. Let M be a smooth manifold with or without boundary, and suppose $f : M \to \mathbb{R}^k$ is a smooth function. Show that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$ is smooth for every smooth chart (U, ϕ) for M.

Proof. Let $\phi(x) \in \phi(U)$. Since f is smooth, there exists (V, ψ) such that $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}^k$ is smooth and $x \in V$. Let $W = U \cap V$. Then $f \circ \psi^{-1} : \psi(W) \to \mathbb{R}^k$ is smooth and $\psi \circ \phi^{-1} : \phi(W) \to \psi(W)$ is a diffeomorphism where $\phi(W)$ is a neighborhood of W. Then the restriction of $f \circ \psi^{-1}$ to $\phi(W)$ is identical to $(f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})$. Since he composition of a smooth function is smooth, $f \circ \psi^{-1}$ is smooth. □

Exercise 2.7(Prove Proposition 2.5). Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a map. Then F is smooth if and only if either of the following conditions is satisfied:

(a) For every $p \in M$, there exist smooth charts (U, ϕ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \phi^{-1}$ is smooth from $\phi(U \cap F^{-1}(V))$ to $\psi(V)$.

(b) F is continuous and there exist smooth atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is a smooth map from $\phi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

Proof. Let $\mathcal{A}_M$ and $\mathcal{A}_N$ be smooth structures of M and N. Suppose F is smooth. By Proposition 2.4, F is continuous. For every $p \in M$ there exist coordinate charts (U_p, ϕ_p) containing p and (V_p, ψ_p) containing $F(p)$ such that $F(U_p) \subset V_p$ and $\psi_p \circ F_p \circ \phi_p^{-1}$ is smooth from $\phi_p(U_p)$ to $\psi_p(V_p)$. Then $\{(U_p, \phi_p) \mid p \in M\} \subset \mathcal{A}_M$ and $A_n\{(V_p, \psi_p) \mid p \in M\} \subset \mathcal{A}_N$ are smooth atlases. Moreover, for every (U_p, ϕ_p) and (V_q, ψ_q), $\psi_q \circ F \circ \phi_p^{-1}$ is a smooth map from $\phi_p(U_p \cap F^{-1}(V_q))$ to $\psi_q(V_q)$ because $\psi_q \circ F \circ \phi_p^{-1} = (\psi_q \circ \psi_p^{-1}) \circ (\psi_p \circ F \circ \phi_p^{-1})$ where $\psi_q \circ \psi_q^{-1}$ and $\psi_p \circ F \circ \phi_p^{-1}$ are smooth. Therefore, the definition implies (b).

(b) implies (a) because if F is continuous, $F^{-1}(V_\beta)$ is open in M for every β, so $U \cap F^{-1}(V)$ is open in M.

Finally, we show that (a) implies the definition. Suppose F satisfies (a). Let $p \in M$. Let $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ be smooth charts satisfying the properties described in (a). Let $U' = U \cap F^{-1}(V)$ and consider $(U', \phi \mid_{U'})$. Then $(U', \phi \mid_{U'}) \in \mathcal{A}_M$ because it must be smoothly compatible with any other smooth coordinate chart in $\mathcal{A}_M$. Moreover, $F(U') \subset V$ and $\psi \circ F \circ (\phi \mid_{U'})^{-1} : \phi(U') \to \psi(V)$ is smooth. Therefore, (a) implies the definition.

Hence, (a), (b) and the definition are all equivalent. □

Exercise 2.7(Proof of Proposition 2.6). Let M and N be smooth manifolds with or without boundary, and let $F : M \to N$ be a map.

(a) If every point $p \in M$ has a neighborhood U such that the restriction $F|_U$ is smooth, then F is smooth.

(b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

Proof. Let $\mathcal{A}_M, \mathcal{A}_N$ be smooth structures of M, N, respectively.

(a) Let $p \in M$. Let U be a neighborhood of p such that $F|_U$ is smooth. By 1.44, U is a smooth manifold with the induced smooth structure $\mathcal{A}_U = \{(V,\phi) \in \mathcal{A}_M \mid V \subset U\}$. Since $F|_U$ is smooth, there exist $(V,\phi) \in \mathcal{A}_U$ and $(W,\psi) \in \mathcal{A}_N$ such that:
  - $F|_U(V) \subset W$.
  - $\psi \circ F|_U \circ \phi^{-1} : \phi(V) \to \psi(W)$ is smooth.

  Since $V \subset U$, $F(V) \subset W, \psi \circ F \circ \phi^{-1} : \phi(V) \to \psi(W)$ is smooth, and $(V,\phi) \in \mathcal{A}$. Therefore, F is smooth.

(b) Let $U \subset M$ be an open subset. By 1.44, U is a smooth manifold with the induced smooth structure $\mathcal{A}_U = \{(V,\phi) \in \mathcal{A}_M \mid V \subset U\}$. Let $p \in U$. Then $p \in F$, so there exist $(V,\phi) \in \mathcal{A}_M, (W,\psi) \in \mathcal{A}_N$ such that $F(V) \subset W$ and $\psi \circ F \circ \phi^{-1} : \phi(V) \to \psi(W)$ is smooth. Then $(V \cap U, \phi|_{V \cap U})$ is a chart that is smoothly compatible with every chart in $\mathcal{A}_M$. Therefore, $(V \cap U, \phi|_{V\cap U}) \in \mathcal{A}_M$. Moreover, $\phi|_{V\cap U}(V \cap U) \subset \phi(V) \subset W$ and $\psi \circ F \circ (\phi|_{V\cap U}(V\cap))^{-1}$ is clearly smooth. Therefore, $F|_U$ is smooth.

□

Exercise 2.9. Suppose $F : M \to N$ is a smooth map between smooth manifolds with or without boundary. Show that the coordinate representation of F with respect to *every* pair of smooth charts for M and N is smooth.

Proof. Let $(M, \mathcal{A}_M), (N, \mathcal{A}_N)$ be smooth manifolds with or without boundary. Let $F : M \to N$ be a smooth map. Let $(U,\phi) \in \mathcal{A}_M, (V,\psi) \in \mathcal{A}_N$ be given. We must show that $\hat{F} = \psi \circ F \circ \phi^{-1}$ is a smooth function from $\phi(U \cap F^{-1}(V))$ to $\psi(V)$. Let $\phi(p) \in \phi(U \cap F^{-1}(V))$. Then $p \in M$, so there exist $(U_0, \phi_0) \in \mathcal{A}_M$ and $(V_0, \psi_0) \in \mathcal{A}_N$ such that

- $p \in U_0 \subset U \cap F^{-1}(V)$;
- $\phi_0(U_0) \subset V_0$;
- $\psi_0 \circ F \circ \phi_0^{-1} : \phi_0(U_0) \to \psi(V_0)$ is smooth.

Then $\psi \circ F \circ \phi^{-1}|_{\phi(U_0)} = (\psi \circ \psi_0^{-1}) \circ (\psi_0 \circ F \circ \phi_0^{-1}) \circ (\phi_0 \circ \phi)$. Since the composition of smooth functions in Euclidean spaces is smooth, $\hat{F}$ is smooth. □

Exercise 2.11(Proof of Proposition 2.10). Let M, N and P be smooth manifolds with or without boundary.

(a) Every constant map $c : M \to N$ is smooth.
(b) The identity map of M is smooth.
(c) If $U \subset M$ is an open submanifold with or without boundary, then the inclusion map $U \to M$ is smooth.

Proof. Let $\mathcal{A}_M, \mathcal{A}_N, \mathcal{A}_P$ be smooth structures of M, N, P, respectively.

(a) F is clearly continuous. Moreover, for every $(U_\alpha, \phi_\alpha) \in \mathcal{A}_M, (V_\beta, \psi_\beta) \in \mathcal{A}_N$, $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is a constant map, so it is smooth. By (2.7(Prove Proposition 2.5)), F is smooth.
(b) Let $p \in M$. Choose $(U,\phi) \in \mathcal{A}_M$ such that $p \in U$. Then $F(U) \subset U$ and $\phi \circ F \circ \phi^{-1} = \mathrm{Id}_U$, so it is smooth. Therefore, F is smooth.
(c) By 1.44, $\mathcal{A}_U = \{(V,\phi) \mid V \subset U\}$ is a smooth structure of U. Let $p \in U$. Then $p \in V$ for some $(V,\phi) \in \mathcal{A}_U$. Then $(V,\phi) \in \mathcal{A}_M$, trivially. Since $F(V) \subset V$ and $\phi \circ F \circ \phi^{-1}$ is simply the identity map on V, F is smooth.

□

Exercise 2.16(Proof of Proposition 2.15).

(a) Every composition of diffeomorphisms is a diffeomorphism.
(b) Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.
(c) Every diffeomorphism is a homeomorphism and an open map.
(d) The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.
(e) "Diffeomorphic" is an equivalence relation on the class of all smooth manifolds with or without boundary.

Exercise 2.16(Proof of Proposition 2.15). Let $(M, \mathcal{A}_M), (N, \mathcal{A}_N), (P, \mathcal{A}_P)$ be smooth manifolds with or without boundary, and let $F : M \to N, G : N \to P$ be diffeomorphisms.

(a) By Proposition 2.10(d), $G \circ F$ and $F^{-1} \circ G^{-1}$ are smooth. Then $(G \circ F) \circ (F^{-1} \circ G^{-1})$ and $(F^{-1} \circ G^{-1}) \circ (G \circ F)$ are both the identity map on the corresponding space, so $F^{-1} \circ G^{-1}$ is the smooth inverse of $G \circ F$. Therefore, $G \circ F$ is a diffeomorphism.

(b) By Example 1.34, we know that $M_1 \times \cdots \times M_k$ and $N_1 \times \cdots \times N_k$ are both smooth manifolds. Let $\mathcal{A}_{M_i}, \mathcal{A}_{N_i}, \mathcal{A}_M$ and $\mathcal{A}_N$ denote the smooth manifold structures of $M_i, N_i, M_1 \times \cdots \times M_k, N_1 \times \cdots \times N_k$, respectively. Let a smooth map $F_i : M_i \to N_i$ be given for each i. Let $(p_1, \cdots, p_k) \in M_1 \times \cdots M_k$ be given. Then there exist $(U_i, \phi_i) \in \mathcal{A}_{M_i}$ and $(V_i, \psi_i) \in \mathcal{A}_{N_i}$ such that $p_i \in U_i, F_i(U_i) \subset V_i, \psi_i \circ F_i \circ \phi_i^{-1} : \phi_i(U_i) \to \psi_i(V_i)$ is smooth for each i. This implies that $(\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots (\psi_k \circ F_k \circ \phi_k^{-1}) = (\psi_1 \times \cdots \times \psi_k) \circ (F_1 \times \cdots \times F_k) \circ (\phi_1 \times \cdots \times \phi_k)^{-1}$ is smooth.

Therefore, $F_1 \times \cdots \times F_k$ is smooth. Using the exact same argument, we can conclude that $F_1^{-1} \times \cdots \times F_k^{-1}$ is smooth. Since $(F_1 \times \cdots \times F_k)^{-1} = F_1^{-1} \times \cdots \times F_k^{-1}$, $F_1 \times \cdots \times F_k$ is a diffeomorphism.

(c) Proposition 2.4 states that every smooth map is continuous. Thus F and F^{-1} are both continuous. Therefore, F is a homeomorphism and also an open map.

(d) Let $U \subset M$ be an open subset. By (2.7(Proof of Proposition 2.6)), $F|_U$ is smooth. Since F is a homeomorphism as shown in (c), $F(U)$ is an open subset of N. Therefore, $F^{-1}|_{F(U)}$ is smooth by (2.7(Proof of Proposition 2.6)). Clearly, $F|_U$ and $F^{-1}|_{F(U)}$ are the inverse of each other. Therefore, $F|_U$ is a diffeomorphism.

(e) By (2.11(Proof of Proposition 2.10)), the identity map on M is a diffeomorphism, so the reflexive property is satisfied. Moreover, $(F^{-1})^{-1} = F$, so the symmetric property is satisfied. By (a), the composition of two diffeomorphisms is a diffeomorphism, so the transitive property is satisfied. Therefore, "diffeomorphic" is an equivalence relation.

Exercise 2.19(Proof of Theorem 2.18). Suppose M and N are smooth manifolds with boundary and $F : M \to N$ is a diffeomorphism. Then $F(\partial M) = \partial N$, and F restricts to a diffeomorphism from $\operatorname{Int} M$ to $\operatorname{Int} N$.

Proof. Let $\mathcal{A}_M, \mathcal{A}_N$ denote the smooth structures of M, N, respectively. Let $p \in \partial M$. Then there exists a chart containing p that sends p to $\partial\mathbb{H}^n$. By Theorem 1.46, every chart containing p sends p to $\partial\mathbb{H}^n$.

Since F is smooth, there exist $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$ such that $F(U) \subset V$ and $\psi \circ F \circ \phi^{-1}$ is a smooth map from $\phi(U)$ to $\psi(V)$. F^{-1} is a homeomorphism by (2.16(Proof of Proposition 2.15)). Then $(\phi^{-1} \circ F^{-1}, F(U))$ is a coordinate chart around $F(p)$ because we obtain a homeomorphism by restricting the composition of two injective continuous maps to its image. Moreover, we claim that $(\phi^{-1} \circ F^{-1}, F(U))$ is smoothly compatible with every chart in $\mathcal{A}_N$. Let $(\psi_1, V_1) \in \mathcal{A}_N$ be given. Then $(\phi^{-1} \circ F^{-1}) \circ \psi_1^{-1} = (\phi^{-1} \circ F^{-1} \circ \psi^{-1}) \circ (\psi \circ \psi_1^{-1})$, and the composition of two smooth maps is smooth. Therefore, $(\phi^{-1} \circ F^{-1}, F(U)) \in \mathcal{A}_N$, and this chart contains $F(p)$ and sends $F(p)$ to $\partial\mathbb{H}^n$. In other words, $F(p) \in \partial N$.

Since F^{-1} is also smooth, $F^{-1}(\partial N) \subset \partial M$. $F^{-1}(\partial N) \subset \partial M \implies F(F^{-1}(\partial N)) \subset F(\partial M) \subset \partial N$. Since F is a bijection, $F(F^{-1}(\partial N)) = \partial N$. Therefore, $F(\partial M) = \partial N$.

This implies that $F(\operatorname{Int} M) = \operatorname{Int} N$. By (1.44(c)) and (2.16(Proof of Proposition 2.15)(d)), F is a diffeomorphism between $\operatorname{Int} M$ and $\operatorname{Int} N$. ☐

Problem 2-27. Give a counterexample to show that the conclusion of the extension lemma can be false if A is not closed.

Proof. Let $M = \mathbb{R}, A = (0, 1), f(x) = 1/x$. Then f is smooth on A, but $\lim_{x\to 0} f = \infty$, so f cannot be extended continuously. ☐

2.2. **Problems.**

Problem 2-1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, ϕ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \phi^{-1}$ is smooth as a map from $\phi(U \cap f^{-1}(V))$ to $\psi(V)$, but f is not smooth in the sense we defined in this chapter.

Proof. $\phi = \psi = \mathrm{Id}$ in this solution.

If $x \geq 0$, then let $U = \mathbb{R}, V = (0, \infty)$. Then $\phi(U \cap f^{-1}(V)) = [0, \infty)$. Thus $\psi \circ f \circ \phi^{-1} : [0, \infty) \to (0, \infty)$ is the constant map that sends every number to 1. Therefore, it is smooth.

If $x < 0$, then let $U = \mathbb{R}, V = (-\infty, 1)$. Then $\phi(U \cap f^{-1}(V)) = (-\infty, 0)$. Thus $\psi \circ f \circ \phi^{-1} : (-\infty, 0) \to (-\infty, 1)$ is the constant map that sends every number to 0. Therefore, it is smooth.

It might seem that we can apply (2.7(Prove Proposition 2.5)) to show that f is smooth, but (2.7(Prove Proposition 2.5)) requires that $U \cap f^{-1}(V)$ be open in M.

f maps the interval $(-1, 1)$ to $\{0, 1\}$. Since the image of a connected set under a continuous map must be connected, f cannot be continuous. By Proposition 2.4, f cannot be smooth. □

Problem 2-2(Proof of Proposition 2.12). Suppose $M_1, \cdots, M_k$ and N are smooth manifolds with or without boundary, such that at most one of $M_1, \cdots, M_k$ has nonempty boundary. For each i, let $\pi_i : M_1 \times \cdots \times M_k \to M_i$ denote the projection onto the M_i factor. A map $F : N \to M_1 \times \cdots \times M_k$ is smooth if and only if each of the component maps $F_i = \pi_i \circ F : N \to M_i$ is smooth.

Proof. Let $\mathcal{A}_{M_1}, \cdots, \mathcal{A}_{M_k}, \mathcal{A}_N$ be the smooth structures of $M_1, \cdots, M_k, N$. Let $d_1, \cdots, d_k$ denote the dimensions of $M_1, \cdots, M_n$, respectively. Let $d = \sum d_i$.

First, suppose that F is smooth. By (2.11(Proof of Proposition 2.10)), the composition of smooth maps is smooth. Thus it suffices to show that $\pi_i : M_1 \times \cdots \times M_k \to M_i$ is smooth for each i. We show that π_1 is smooth and the other cases can be shown similarly.

Let $(x_1, \cdots, x_k) \in M_1 \times \cdots \times M_k$. Then for each i, there exist $(U_i, \phi_i) \in \mathcal{A}_{M_i}$ and $(V_i, \psi_i) \in \mathcal{A}_{M_i}$ such that $x_i \in U_i$ and $\phi_i(U_i) \subset V_i$. Then we have $(\phi_1 \times \cdots \times \phi_k)(U_1 \times \cdots \times U_k) \subset V_1 \times \cdots \times V_k$ and the composition $\phi_i \circ \pi_1 \circ (\phi_1 \times \cdots \times \phi_k)^{-1}$ is the projection of the first d_1 coordinates from $\mathbb{R}^n$ onto $\mathbb{R}^{d_1}$. Therefore, it is clearly smooth, so π_1 is smooth.

Suppose each $F_i = \pi_i \circ F : N \to M_i$ is smooth. Let $p \in N$. Then for each i, there exist $(U_i, \phi_i) \in \mathcal{A}_N$ and $(V_i, \psi_i) \in \mathcal{A}_{M_i}$ such that $p \in U_i, F_i(U_i) \subset V_i$ and $\psi_k \circ F_i \circ \phi_i^{-1}$. Let $U = U_1 \cap \cdots \cap U_k$. U is a neighborhood of p and the restriction of ϕ_1 to U is a homeomorphism. Then we claim that $(\phi_1, U) \in \mathcal{A}_N$ and $(\psi_1 \times \cdots \times \psi_k, V_1 \times \cdots \times V_k) \in \mathcal{A}_{M_1 \times \cdots \times M_k}$ are charts that satisfy the necessary properties.

- $F(U) \subset V_1 \times \cdots \times V_k$.
- For each i, $\psi_i \circ F_i \circ \phi_1^{-1} = (\psi_i \circ F_i \circ \phi_i^{-1}) \circ (\phi_i \circ \phi_1^{-1}) : \phi_1(U) \to \psi_i(V_i)$ is smooth because the composition of two smooth maps is smooth. Thus $(\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots \times (\psi_k \circ F_k \circ \phi_1^{-1}) : \phi_1(U) \to \psi_1(V_1) \times \cdots \times \psi_k(V_k)$ is smooth. Moreover, $(\psi_1 \times \cdots \times \psi_k) \circ F \circ \phi_1^{-1} = (\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots \times (\psi_k \circ F_k \circ \phi_1^{-1})$.

Therefore, F is smooth. □

Problem 2-3. For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

(a) $p_n : S^1 \to S^1$ is the nth power map for $n \in \mathbb{Z}$, given in complex notation by $p_n(z) = z^n$.
(b) $\alpha : S^n \to S^n$ is the antipodal map $\alpha(x) = -x$.
(c) $F : S^3 \to S^2$ is given by $F(w, z) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w})$ where we think of S^3 as the subset $\{(w, z) : |w|^2 + |z|^2 = 1\}$ of $\mathbb{C}^2$.

Proof.

(a) Example 1.31 shows the existence of a smooth structure of S^1 and let $\mathcal{A}$ denote it. Let $p \in S^1$. Then there exist $(U_i^{\pm}, \phi_i^{\pm}), (U_j^{\pm}, \phi_j^{\pm}) \in \mathcal{A}$ around $p, p_n(p)$, respectively. Then the composition $\phi_j^{\pm} \circ f \circ (\phi_i^{\pm})^{-1}$ is equal to one of $\cos(n(\arccos(x))), \sin(n(\arcsin(x))), \cos(n(\arcsin(x))), \sin(n(\arccos(x)))$, all of which are clearly smooth. By Proposition 2.5(a), p_n is smooth.
(b) Example 1.31 shows the existence of a smooth structure of S^n and let $\mathcal{A}$ denote it. Let $p \in S^1$. Then there exists a chart $(U_i^{\pm}, \phi_i^{\pm})$ in $\mathcal{A}$ around p. Then $(U_i^{\mp}, \phi_i^{\mp})$ is a chart containing $\alpha(p)$ with $\alpha(U_i^{\pm}) \subset U_i^{\mp}$. Then $\phi_i^{\mp} \circ \alpha \circ \phi_i^{\pm}$ is the map $x \mapsto -x$, which is clearly smooth.

(c) Let $z = a + bi, w = c + di$. $z\overline{w} = ac + bd + i(bc - ad)$ and $w\overline{z} = (ac + bd) - i(bc - ad)$. Then $z\overline{w} + w\overline{z} = 2(ac + bd) = 2\operatorname{Re}(z\overline{w})$ and $i(w\overline{z} - z\overline{w}) = 2\operatorname{Im}(z\overline{w})$.

$$\begin{aligned}(2\operatorname{Re}(z\overline{w}))^2 + (2\operatorname{Im}(z\overline{w}))^2 + (|z|^2 - |w|^2)^2 &= 4|z\overline{w}|^2 + (|z|^2 - |w|^2)^2 \\ &= 4|z|^2|\overline{w}|^2 + (|z|^2 - |w|^2)^2 \\ &= (|z|^2 + |w|^2)^2 \\ &= 1.\end{aligned}$$

Therefore, F indeed maps S^3 into S^2. Moreover, this map is continuous. Let $(z = a+bi, w = c+di) \in S^3$ be given. Suppose that (U_4^+, ϕ_4^+) and (V_3^+, ψ_3^+) are charts containing (z, w) and $F(z, w)$. Then $\psi_3^+ \circ F \circ \phi_4^+ : (a, b, c) \mapsto (2u, 2v)$ where $u + iv = (a + bi)(c - \sqrt{1 - a^2 - b^2 - c^2}i)$ which is a smooth map from $\phi_4^+(U_4^+) \subset \mathbb{R}^3$ into $\mathbb{R}^2$. Other cases are similar, and thus F is smooth by Proposition 2.5(b).

□

Problem 2-5. Let $\mathbb{R}$ be the real line with its standard smooth structure, and let $\tilde{R}$ denote the same topological manifold with the smooth structure defined in Example 1.23. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is smooth in the usual sense.

(a) Show that f is also smooth as a map from $\mathbb{R}$ to $\tilde{\mathbb{R}}$.
(b) Show that f is smooth as a map from $\tilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3.

Proof.

(a) The "$\psi \circ f \circ \phi^{-1}$" is simply f^3, which is a smooth map from $\mathbb{R}$ to $\mathbb{R}$. Thus $f : \tilde{\mathbb{R}} \to \mathbb{R}$ is smooth.
(b) Solve this!

□

Problem 2-6. Let $P : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Show that the map $\tilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\tilde{P}([x]) = [P(x)]$ is well defined and smooth.

Proof. Let $P_1, \cdots, P_{k+1}$ denote the component functions of P.

Suppose $[x_1 : \cdots : x_{n+1}] = [y_1 : \cdots : y_{n+1}]$. Then there exists $\lambda \neq 0$ such that $(y_1, \cdots, y_{n+1}) = (\lambda x_1, \cdots, \lambda x_{n+1})$. $P(y_1, \cdots, y_{n+1}) = P(\lambda x_1, \cdots, \lambda x_{n+1}) = \lambda^d P(x_1, \cdots, x_{n+1})$. Since $\lambda^d \neq 0$, $[P(y_1, \cdots, y_{n+1})] = [P(x_1, \cdots, x_{n+1})]$. Therefore, $\tilde{P}$ is well-defined.

Let $\tilde{p} = [p_1 : \cdots : p_{n+1}] \in \mathbb{RP}^n$ be given. Without loss of generality, assume $p_{n+1} \neq 0$. Consider the chart (U, ψ_{n+1}) with $U = \{[x_1 : \cdots : x_{n+1}] \mid x_{n+1} \neq 0\}$. Let $q_i = P_i(p_1, \cdots, p_{n+1})$. Without loss of generality, assume $q_{k+1} \neq 0$. Then $\tilde{P}(\tilde{p})$ is contained in $V = \{[y_1 : \cdots : y_{k+1}] \mid y_{k+1} \neq 0\}$. Since P is smooth, there exists $0 < \delta < |x_{n+1}|$ such that $|(x_1, \cdots, x_{n+1}) - (p_1, \cdots, p_{n+1})| < \delta$ implies $P_{k+1}(x_1, \cdots, x_{n+1}) \neq 0$. Then $[p_1 : \cdots : p_{n+1}] \in \pi(B(p_1, \cdots, p_{n+1})) \subset U \cap F^{-1}(V)$. Therefore, $U \cap F^{-1}(V)$ is open in $\mathbb{RP}^n$.

Finally the composition map $\psi_{k+1} \cdot \tilde{P} \cdot \phi_{n+1}^{-1}$ sends $(x_1/x_{n+1}, \cdots, x_n/x_{n+1})$ to $(y_1/y_{k+1}, \cdots, y_k/y_{k+1})$ where $y_i = P_i(x_1, \cdots, x_{n+1})$. In other words, $(x_1, \cdots, x_n) \mapsto (y_1/y_{k+1}, \cdots, y_k/y_{k+1})$ where $y_i = P_i(x_1, \cdots, x_n, 1)$. Since each P_i is smooth, this map must be smooth as well. By (2.7(Prove Proposition 2.5)), $\tilde{P}$ is smooth. □

Problem 2-7. Let M be a nonempty smooth n-manifold with or without boundary, and suppose $n \geq 1$. Show that the vector space $C^\infty(M)$ is infinite-dimensional.

Proof. Let $k \in \mathbb{N}$ be given. Let $p \in M$ be chosen arbitrarily. Let (U, ϕ) be a smooth chart containing p. Then $\hat{U} = \phi(U)$ is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. In each case, we can pick k distinct points $x_1, \cdots, x_k \in \hat{U}$ because $\hat{U}$ is a nonempty open subset and $n \geq 1$. Since $\hat{U}$ is open, there exist open $U_1, \cdots, U_k$ such that $x_i \in U_i \subset \hat{U}$ and $U_i \cap U_j$ whenever $i \neq j$. Moreover, $\{x_i\}$ is a closed subset. By Proposition 2.25, we obtain k bump functions f_i for $\{x_i\}$ supported in U_i. Extend each f_i by setting $f_i(q) = 0$ for any $q \notin U$. Then each f_i lives in $C^\infty(M)$. Clearly, $\sum c_i f_i = 0$ implies $c_i = 0$, so $\{f_1, \cdots, f_k\}$ is linearly independent. Therefore, $C^\infty(M)$ is infinite-dimensional. □

Problem 2-14. Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

Proof. By Theorem 2.29, there exist $\alpha, \beta \in C^\infty(M)$ such that $\alpha^{-1}(0) = A$ and $\beta^{-1}(0) = B$. Then $f(x) = \alpha(x)/(\alpha(x) + \beta(x))$ is a desired map. ☐

3. Chapter 3: Tangent Vectors

3.1. **Exercises.**

Proposition 3.2. Let $a \in \mathbb{R}^n$.

(a) For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map $D_v|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$D_v|_a f = D_v f(a) = \frac{d}{dt}\Big|_{t=0} f(a + tv)$$

is a derivation at a.

(b) The map $v_a \mapsto D_v|_a$ is an isomorphism from $\mathbb{R}^n_a$ onto $T_a\mathbb{R}^n$.

Proof.

(a) $D_v|_a$ is linear because $D_v|_a(f + cg) = D_v(f + cg)(a) = D_v(f)(a) + cD_v g(a) = D_v|_a(f) + cD_v|_a(g)$ because directional derivatives are linear. Moreover, the product rule is satisfied because directional derivatives satisfy that. Therefore, $D_v|_a$ is a linear map that satisfies directional derivatives, so it is a derivation.

(b) Let $\phi : \mathbb{R}^n_a \to T_a\mathbb{R}^n$ be defined such that $v_a \mapsto D_v|_a$. We first claim that ϕ is linear.

$$\begin{aligned}
\phi(v_a + cw_a)(f) &= \phi((v + cw)_a)(f) \\
&= D_{v+cw} f(a) \\
&= D_v f(a) + cD_w f(a) \\
&= D_v|_a(f) + cD_w|_a(f) \\
&= \phi(v_a)(f) + c\phi(w_a)(f) \\
&= (\phi(v_a) + c\phi(w_a))(f).
\end{aligned}$$

Next, we claim that $\ker(\phi) = 0$. Let $v_a \in \ker(\phi) \subset \mathbb{R}^n_a$. Let $v_1, \cdots, v_n \in \mathbb{R}^n$ be chosen such that $v_a = \sum_{i=1}^n v_i e_i|_a$. For each j, let $x^j : \mathbb{R}^n \to \mathbb{R}$ denote the projection map of the jth coordinate. Then $0 = D_v|_a(x^j) = \frac{d}{dt}|_{t=0} x^j(a + tv) = v_j$ for each j. Therefore, $v_1 = \cdots = v_n = 0$, so $\ker(\phi) = 0$. Since ϕ is linear, ϕ must be injective.

Lastly, we claim that ϕ is surjective. Let $w \in T_a\mathbb{R}^n$ be given. For each j, let $v_j = w(x^j)$. Let $v = (v_1, \cdots, v_n)$. We claim that $\phi(v_a) = w$. Let $f \in C^\infty(\mathbb{R}^n)$. By Theorem C.15, we can write

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + \sum_{i,j=1}^n (x^i - a^i)(x^j - a^j) \int_{0,1} F(t)dt$$

where $F(t)$ is some function. Since $(x^i - a^i)$ and $(x^j - a^j)\int_{0,1} F(t)dt$ vanish at $x = a$, $w((x^i - a^i)(x^j - a^j)\int_{0,1} F(t)dt) = 0$ for any i, j. Therefore,

$$\begin{aligned}
w(f) &= w(f(a)) + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)(w(x^i) - w(a^i)) \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a) w(x^i) \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a) v_i \\
&= \phi(v_a)(f)
\end{aligned}$$

which proves that ϕ is surjective.

□

Exercise 3.5(Proof of Lemma 3.4). Suppose M is a smooth manifold with or without boundary, $p \in M, v \in T_pM$, and $f, g \in C^\infty(M)$.

(a) If f is a constant function, then $vf = 0$.
(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

Proof. This is similar to Lemma 3.1.

(a) Let h be the constant function that always takes the value 1. Then $v(h) = v(h^2) = h(p)v(h) + h(p)v(h) = 2v(h)$, so $v(h) = 0$. Since $f(p) = ch(p)$ for some $c \in \mathbb{R}$ and v is linear, this implies $0 = cv(h) = v(ch) = v(f)$.
(b) $v(fg) = f(p)vg + g(p)vf = 0 + 0 = 0$.

□

Exercise 3.7(Proof of Proposition 3.6). Let M, N, and P be smooth manifolds with or without boundary, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.

(a) $dF_p : T_pM \to T_{F(p)}N$ is linear.
(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}P$.
(c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.
(d) If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Proof. (a) $\forall v, w \in T_pM, \forall c \in \mathbb{R}, \forall f \in C^\infty(N)$,

$$\begin{aligned}
dF_p(cv + w)(f) &= (cv + w)(f \circ F) \\
&= (cv)(f \circ F) + w(f \circ F) \\
&= c(v(f \circ F)) + w(f \circ F) \\
&= c(dF_p(v)(f)) + dF_p(w)(f) \\
&= (cdF_p(v))(f) + dF_p(w)(f) \\
&= (cdF_p(v) + dF_p(w))(f).
\end{aligned}$$

Therefore, $dF_p(cv + w) = cdF_p(v) + dF_p(w)$.

(b) $\forall v \in T_pM, f \in C^\infty(P)$,

$$\begin{aligned}
d(G \circ F)_p(v)(f) &= v(f \circ (G \circ F)) \\
&= v((f \circ G) \circ F) \\
&= (dF_p(v))(f \circ G) \\
&= (dG_{F(p)}(dF_p(v)))(f) \\
&= ((dG_{F(p)} \circ dF_p)(v))(f)
\end{aligned}$$

Therefore, $d(G \circ F)_p = dG_{F(p)} \circ dF_p$.

(c) $\forall v \in T_p(M), \forall f \in C^\infty(M)$,

$$\begin{aligned}
d(\mathrm{Id}_M)_p(v)(f) &= v(f \circ \mathrm{Id}_M) \\
&= v(f).
\end{aligned}$$

Therefore, $d(\mathrm{Id}_M)_p(v) = v$, so $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$.

(d) F^{-1} exists and it is a smooth map since F is a diffeomorphism. By combining (b) and (c), we obtain dF_p and $dF^{-1}_{F(p)}$ are the inverse of each other. Therefore, dF_p is an isomorphism.

□

Proposition 3.10. If M is an n-dimensional smooth manifold, then for each $p \in M$, the tangent space T_pM is an n-dimensional vector space.

Proof. Let $\mathcal{A}$ denote the smooth structure of M and let $p \in M$ be given. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Then

$$T_pM \overset{di_p}{\cong} T_pU \overset{d\phi_p}{\cong} T_{\phi(p)}\hat{U} \overset{di_{\phi(p)}}{\cong} T_{\phi(p)}\mathbb{R}^n$$

where di_p is induced by the inclusion map $i : U \to M$ and $di_{\phi(p)}$ is induced by the inclusion map $: \hat{U} \to \mathbb{R}^n$. $di_p, d\phi_p, di_{\phi(p)}$ are all isomorphisms by (3.7(Proof of Proposition 3.6)(d)) and Proposition 3.9. Therefore, $\dim(T_pM) = n$. $\square$

Proposition 3.15. Let M be a smooth n-manifold with or without boundary, and let $p \in M$. Then T_pM is an n-dimensional vector space, and for any smooth chart $(U, (x^i))$ containing p, the coordinate vectors $\partial/\partial x^1|_p, \cdots, \partial/\partial x^n|_p$ form a basis for T_pM.

Proof. By Proposition 3.12, T_pM is an n-dimensional vector space. By Corollary 3.3, the $\partial/\partial x^i|_{\phi(p)}$ form a basis for $T_{\phi(p)}\mathbb{R}^n$. By Proposition 3.6(d), $d\phi_p : T_pM \to T_{\phi(p)}\mathbb{R}^n$ is an isomorphism. Since $d\phi_p$ is an isomorphism between vector spaces, $d\phi_p$ sends a basis to a basis. In other words, the $\partial/\partial x^i|_p = (d\phi_p)^{-1}(\partial/\partial x^i|_{\phi(p)})$ form a basis. $\square$

Remark. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map and let $p \in \mathbb{R}^n$ be given.*

$$\begin{aligned}
dF_p(\frac{\partial}{\partial x^i}\Big|_p)(f) &= \frac{\partial}{\partial x^i}\Big|_p(f \circ F) && \textit{(definition of d)} \\
&= \frac{\partial(f \circ F)}{\partial x^i}(p) && \textit{(Just a partial derivative of } f \circ F\textit{)} \\
&= \sum_{j=1}^m \frac{\partial f}{\partial y^j}(F(p))\frac{\partial F^j}{\partial x^i}(p) \\
&= \sum_{j=1}^m \frac{\partial F^j}{\partial x^i}(p)\frac{\partial f}{\partial y^j}(F(p)) && \textit{(Multiplication is commutative in } \mathbb{R}\textit{)} \\
&= \sum_{j=1}^m \frac{\partial F^j}{\partial x^i}(p)\frac{\partial}{\partial y^j}\Big|_{F(p)}(f).
\end{aligned}$$

Therefore, we obtain that $dF_p(\frac{\partial}{\partial x^i}\big|_p) = \sum_{j=1}^m \frac{\partial F^j}{\partial x^i}(p)\frac{\partial}{\partial y^j}\big|_{F(p)}$. $\{\partial/\partial x^i\}$ and $\{\partial/\partial y^j\}$ form bases for $T_p\mathbb{R}^n$ and $T_{F(p)}\mathbb{R}^m$, respectively, so it makes sense to put dF_p is a matrix form. Then we obtain

$$\begin{bmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{bmatrix}$$

which is identical to the Jacobian matrix of F at p. (Note that it makes sense to discussion the Jacobian matrix of F because F is a map from $\mathbb{R}^m$ to $\mathbb{R}^n$.)

3.2. **Problems.**

Problem 3-1. Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Show that $dF_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

Proof. Suppose $dF_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$. It suffices to show that for every $p \in M$, there exists a neighborhood of p on which F is constant. Let $p \in M$ and $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$ be given such that $p \in U$ and $F(U) \subset V$. Without loss of generality, we assume $\hat{U} = \phi(U)$ is an open ball in $\mathbb{R}^m$. Then for any i, j and for any $q \in \hat{U}$,

$$\begin{aligned}
dF_q(\frac{\partial}{\partial x^i}|_q)(\pi_j \circ \psi) = 0 &\implies (\frac{\partial}{\partial x^i}|_q)(\pi_j \circ \psi \circ F) = 0 \\
&\implies (\frac{\partial}{\partial x^i}|_{\phi(q)})(\pi_j \circ \psi \circ F \circ \phi^{-1}) = 0.
\end{aligned}$$

Fix j. Then every partial derivative of $\pi_j \circ \psi \circ F \circ \phi^{-1}$ at every point in $\hat{U}$ is 0. The intermediate value theorem implies that $\pi_j \circ \psi \circ F \circ \phi^{-1}$ is constant on $\hat{U}$ because $\hat{U}$ is an open ball. In other words, $(\pi_j \circ \psi \circ F \circ \phi^{-1})(\hat{U}) = \{y_j\}$ for some $y_j \in \mathbb{R}$. Since this is true for every j and π_j is the projection of the jth coordinate, $(\psi \circ F \circ \phi^{-1})(\hat{U}) = \{y\}$ where $y = (y_1, \cdots, y_n)$. Then $(F \circ \phi^{-1})(\hat{U}) = F(U) = \psi^{-1}(y)$. Since ψ is a homeomorphism, there exists exactly one point in $\psi^{-1}(U)$. In other words, F is constant on U. Therefore, F is constant on each path component.

Suppose F is constant on each component of M. Let $p \in M$. Choose a chart $(U, \phi) \in \mathcal{A}_M$ such that $p \in U$. Then $F \circ \phi^{-1}$ is constant in a neighborhood around $\phi(p)$. For any i,

$$\begin{aligned} dF_p(\frac{\partial}{\partial x^i}|_p)(f) &= \frac{\partial}{\partial x^i}|_p(f \circ F) \\ &= \frac{\partial}{\partial x^i}|_{\phi(p)}(f \circ F \circ \phi^{-1}) \\ &= 0 \end{aligned}$$

because $f \circ F \circ \phi^{-1}$ is constant in a neighborhood around $\phi(p)$. By Proposition 3.15, $\partial/\partial x^i|_p$ form a basis for T_pM. Since dF_p sends each basis element to 0, $dF_p = 0$. □

Problem 3-2(Proof of Proposition 3.14). Let $M_1, \cdots, M_k$ be smooth manifolds, and for each j, let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection onto the M_j factor. For any point $p = (p_1, \cdots, p_k) \in M_1 \times \cdots \times M_k$, the map

$$\alpha : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \cdots, d(\pi_k)_p(v))$$

is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.

Proof. It suffices to show this for the case that $k = 2$ because the results extend to arbitrary k by induction. Let $\mathcal{A}_{M_1}, \mathcal{A}_{M_2}, \mathcal{A}_{M_1 \times M_2}$ be the smooth structures of $M_1, M_2, M_1 \times M_2$.

We first define a lot of notations.

- Let d_1, d_2 denote the dimensions of M_1, M_2 and let $d = d_1 + d_2$ denote the dimension of $M_1 \times M_2$.
- Let $p = (p_1, p_2) \in M_1 \times M_2$ be given. Choose $(U, \phi = (x^i)) \in \mathcal{A}_{M_1}, (V, \psi = (y^i)) \in \mathcal{A}_{M_2}$ with $p_1 \in U$ and $p_2 \in V$. Let $q_1 = \phi(p_1), q_2 = \psi(p_2), q = q_1 \times q_2$.
- $(U \times V, (z^i)) \in \mathcal{A}_{M_1 \times M_2}$ and $(p_1, p_2) \in U \times V$ where $(z^i) = \phi \times \psi$. More specifically, $z^i = x^i \circ \pi_1$ for $1 \le i \le d_1$ and $z^i = y^i \circ \pi_2$ for $d_1 + 1 \le i \le d_1 + d_2$.

Note that we use x^i, y^i, z^i, π_1 to mean two different things in this solution:

- x^i is either the ith coordinate function of ϕ or the ith projection map $\mathbb{R}^{d_1} \to \mathbb{R}$.
- y^i is either the ith coordinate function of ψ or the ith projection map $\mathbb{R}^{d_2} \to \mathbb{R}$.
- z^i is either the ith coordinate function of $\phi \times \psi$ or the ith projection map $\mathbb{R}^{d_1+d_2} \to \mathbb{R}$.
- π_1 is either the projection map $M_1 \times M_2 \to M_1$ or the projection map $\mathbb{R}^{d_1+d_2} \to \mathbb{R}^{d_1}$.
- π_2 is either the projection map $M_1 \times M_2 \to M_2$ or the projection map $\mathbb{R}^{d_1+d_2} \to \mathbb{R}^{d_2}$.

By Proposition 3.15, $\{\partial/\partial x^1|_{p_1}, \cdots, \partial/\partial x^{d_1}|_{p_1}\}, \{\partial/\partial y^1|_{p_2}, \cdots, \partial/\partial y^{d_2}|_{p_2}\}, \{\partial/\partial z^1|_p, \cdots, \partial/\partial z^{d_1+d_2}|_p\}$ form bases for $T_{p_1}M_1, T_{p_2}M_2, T_p(M_1 \times M_2)$.

$\alpha(\partial/\partial z^1|_p) = (d(\pi_1)_p(\partial/\partial z^1|_p), d(\pi_2)_p(\partial/\partial z^1|_p))$. We claim that $d(\pi_1)_p(\partial/\partial z^1|_p) = \partial/\partial x^1|p_1$.

$$
\begin{aligned}
d(\pi_1)_p(\partial/\partial z^1|_p)(f) &= d(\pi_1)_p(d(\phi^{-1}\times\psi^{-1})_q)(\frac{\partial}{\partial z^1}|_q)(f)\\
&= (d(\pi_1)_p\circ d(\phi^{-1}\times\psi^{-1})_q)(\frac{\partial}{\partial z^1}|_q)(f)\\
&= d(\pi_1\circ(\phi^{-1}\times\psi^{-1})_q)(\frac{\partial}{\partial z^1}|_q)(f)\\
&= \lim_{h\to 0}\frac{(f\circ\pi_1\circ(\phi^{-1}\times\psi^{-1}))(q+e_1h)-(f\circ\pi_1\circ(\phi^{-1}\times\psi^{-1}))(q)}{h}\\
&= \lim_{h\to 0}\frac{(f\circ\pi_1)(\phi^{-1}(q_1+e_1h),p_2)-(f\circ\pi_1)(p)}{h}\\
&= \lim_{h\to 0}\frac{f(\phi^{-1}(q_1+e_1h))-f(p_1)}{h}\\
&= \lim_{h\to 0}\frac{f(\phi^{-1}(q_1+e_1h))-f(\phi^{-1}(q_1))}{h}\\
&= (\frac{\partial}{\partial x^1}|_{q_1})(f\circ\phi^{-1})\\
&= d(\phi^{-1})_{q_1}(\frac{\partial}{\partial x^1}|_{q_1})(f)\\
&= (\frac{\partial}{\partial x^1}|_{p_1})(f).
\end{aligned}
$$

The same result can be shown for the other combinations of π_1, π_2 and $z^1, \cdots, z^{d_1+d_2}$. For any $c_1, \cdots, c_{d_1+d_2} \in \mathbb{R}$,

$$
\begin{aligned}
\alpha(\sum_{i=1}^{d_1+d_2} c_i\frac{\partial}{\partial z^i}|_p) &= \sum_{i=1}^{d_1+d_2} c_i\alpha(\frac{\partial}{\partial z^i}|_p)\\
&= \sum_{i=1}^{d_1+d_2} c_i(d(\pi_1)_p\frac{\partial}{\partial z^i}|_p, d(\pi_2)_p\frac{\partial}{\partial z^i}|_p)\\
&= \sum_{i=1}^{d_1} c_i(d(\pi_1)_p\frac{\partial}{\partial z^i}|_p, d(\pi_2)_p\frac{\partial}{\partial z^i}|_p) + \sum_{i=d_1+1}^{d_2} c_i(d(\pi_1)_p\frac{\partial}{\partial z^i}|_p, d(\pi_2)_p\frac{\partial}{\partial z^i}|_p)\\
&= \sum_{i=1}^{d_1} c_i(\frac{\partial}{\partial x^i}|_{p_1}, 0) + \sum_{i=1}^{d_2} c_{d_1+i}(0, \frac{\partial}{\partial y^i}|_{p_2})\\
&= (c_1\frac{\partial}{\partial x^1}|_{p_1} + \cdots + c_{d_1}\frac{\partial}{\partial x^{d_1}}|_{p_1}, c_{d_1+1}\frac{\partial}{\partial y^1}|_{p_2} + \cdots + c_{d_1+d_2}\frac{\partial}{\partial y^{d_2}}|_{p_2}).
\end{aligned}
$$

Therefore, α is bijective. □

4. Chapter 4: Submersions, Immersions, and Embeddings

Exercise 4.3(Verification of Example 4.2). Verify the following claims:

(a) Suppose $M_1, \cdots, M_k$ are smooth manifolds. Then each of the projection maps $\pi_i : M_1 \times \cdots \times M_k \to M_i$ is a smooth submersion.

Proof.

(a) Let $d_1, \cdots, d_k$ denote the dimensions of $M_1, \cdots, M_k$, respectively. Let $M = M_1 \times \cdots \times M_k$.
(2-2(Proof of Proposition 2.12)) implies that π_i is smooth for each i by setting $F = \mathrm{Id} : M \to M$. Let $p = (p_1, \cdots, p_k) \in M$. Thus it suffices to show that the dimension of $d(\pi_i)_p(T_p(M))$ is the same as the dimension of $T_{p_i}(M_i)$.

By Proposition 3.12, $\dim(T_p(M)) = \sum d_i$. Since the α defined in (3-2(Proof of Proposition 3.14)) is an isomorphism,

$$\dim(d(\pi_1)_p(T_p(M)) \oplus \cdots \oplus d(\pi_k)_p(T_p(M))) = \dim(T_p(M)) = \sum d_i. \tag{4.1}$$

However, for each i, $d(\pi_i)_p(T_p(M)) \subset T_{p_i}M_i$. Thus $\dim(d(\pi_i)_p(T_p(M))) \leq \dim(T_{p_i}M_i) = d_i$. By (4.1), $\dim(d(\pi_i)_p(T_p(M))) = \dim(T_{p_i}M_i)$.

□

5. Appendix A: Review of Topology

Exercise A.18(Proof of Proposition A.17). Let X be a topological space and let S be a subspace of X.

(a)
(b)
(c)
(d)
(e)
(f) If $\mathcal{B}$ is a basis for the topology of X, then $\mathcal{B}_S = \{B \cap S \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on S.
(g) If X is Hausdorff, then so is S.
(h) If X is first-countable, then so is S.
(i) If X is second-countable, then so is S.

Proof.

(a)
(b)
(c)
(d)
(e)
(f) The union of $B \cap S$ is S. Let $U \cap S$ be an open subset of S where U is open in X, and $x \in U \cap S$. Then there exists $B \in \mathcal{B}$ such that $x \in B \subset U$ since $\mathcal{B}$ is a basis. Therefore, $x \in B \cap S \subset U \cap S$ with $B \cap S \in \mathcal{B}_S$.
(g) Let $x \neq y \in S$. There exist two disjoint open sets U, V of X containing x, y, respectively. Then $U \cap S$ and $V \cap S$ are disjoint open sets of X containing x, y, respectively.
(h)
(i) Let $\mathcal{B}$ be a countable basis of X. Then $\{B \cap S \mid B \in \mathcal{B}\}$ is a countable basis of S by (f).

□

Exercise A.24(Proof of Proposition A.23). Suppose $X_1, \cdots, X_k$ are topological spaces, and let $X_1 \times \cdots \times X_k$ be their product space.

(a) CHARACTERISTIC PROPERTY: If B is a topological space, a map $F : B \to X_1 \times \cdots \times X_k$ is continuous if and only if each of its component functions $F_i = \pi_i \circ F : B \to X_i$ is continuous.

Proof.

(a) Suppose F is continuous. Since π_i is continuous by (c) and the composition of continuous functions is continuous, $\pi_1 \circ F$ is continuous. Suppose each component function is continuous. Let $B_1 \times \cdots \times B_k$ be a basis element of $X_1 \times \cdots \times X_k$.

$$\begin{aligned} F^{-1}(B_1 \times \cdots \times B_k) &= F^{-1}(\cap_{i=1}^k \pi_i^{-1}(B_1 \times \cdots \times B_k)) \\ &= \cap_{i=1}^k F^{-1}(\pi_i^{-1}(B_1 \times \cdots \times B_k)) \\ &= \cap_{i=1}^k (\pi_i \circ F)^{-1}(B_1 \times \cdots \times B_k). \end{aligned}$$

Since the intersection of finitely many open sets is open, F is continuous.

□

6. Appendix B: Review of Linear Algebra

Exercise B.49. Two norms $|\cdot|_1$ and $|\cdot|_2$ on a vector space V are said to be equivalent if there are positive constants c, C such that

$$c|v|_1 \le |v|_2 \le C|v|_1$$

for all $v \in V$. Show that equivalent norms determine the same topology.

Proof. Such a relation is symmetric for $c|v|_1 \le |v|_2 \le C|v|_1$ implies $(1/C)|v|_2 \le |v|_1 \le (1/c)|v|_2$. Let $\mathcal{T}_1, \mathcal{T}_2$ be the topologies induced by $|\cdot|_1, |\cdot|_2$. It suffices to show that $\forall v \in V, \forall U \in \mathcal{T}_2, (v \in U \implies \exists r > 0, B_1(v,r) \subset U)$ where $B_1(v,r)$ is the open ball centered at v with the radius r using the $|\cdot|_1$. Since $v \in U$ and U is open, $\exists r > 0$ such that $B_2(v,r) \subset U$. Then for any $w \in V$, $|v-w|_1 \le |v-w|_2/c$, so $B_1(v, r/c) \subset B_2(v,r)$. $\square$

7. Appendix C: Review of Calculus

Exercise C.1. Suppose that $F : U \to W$ is differentiable at $a \in U$. Show that the linear map satisfying

$$\lim_{v \to 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} = 0$$

is unique.

Proof. Let L, L' be two such linear maps.

$$\begin{aligned}
\lim_{v\to 0} \frac{|Lv - L'v|}{|v|} &= \lim_{v \to 0} \frac{|(F(a+v) - F(a) - L'v) - (F(a+v) - F(a) - Lv)|}{|v|} \\
&= \lim_{v\to 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} + \lim_{v\to 0} \frac{|F(a+v) - F(a) - L'v|}{|v|} \\
&= 0 + 0 = 0.
\end{aligned}$$

If $L \ne L'$, $(L - L')v_0 \ne 0$ for some v_0. Then $\lim_{v\to 0} \frac{|Lv - L'v|}{|v|} = \lim_{h \to 0} \frac{|L(hv_0) - L'(hv_0)|}{|hv_0|} = \frac{|(L-L')v_0|}{|v_0|} \ne 0$. This is a contradiction, so $L = L'$. $\square$

8. Dictionary

8.1. Topological Manifolds.

Definition 8.1 (Topological Manifold)**.** A *topological n-manifold* is a Hausdorff, second-countable topological space each point of which has a neighborhood that is homeomorphic to an open subset $\mathbb{R}^n$.

Definition 8.2 (Coordinates)**.** Let M be a topological n-manifold. Let U be an open subset of M, $\hat{U}$ be an open subset of $\mathbb{R}^n$, $\phi : U \to \hat{U}$ be a homeomorphism.

- The pair (U, ϕ) is called a *coordinate chart* or a *chart*.
- U is called a *coordinate domain* or a *coordinate neighborhood* and ϕ is called a *coordinate map*.
- If $\phi(U)$ is an open ball in $\mathbb{R}^n$, U is called a *coordinate ball*.
- If $\phi(U)$ is an open cube in $\mathbb{R}^n$, U is called a *coordinate cube*.
- The coordinate functions of ϕ are often denoted as $(x^1, \cdots, x^n)$. Thus a chart is sometimes denoted by $(U, (x^1, \cdots, x^n))$ or $(U, (x^i))$.

Definition 8.3 (Atlas)**.** Let M be a topological n-manifold. An *atlas for M* is a collection of charts (U_α, ϕ_α) such that $M = \bigcup_\alpha U_\alpha$.

Definition 8.4 (Transition Map)**.** Let M be a topological n-manifold and $(U, \phi), (V, \psi)$ be coordinate charts such that $U \cap V \ne \emptyset$. $\psi \circ \phi^{-1} : \phi(U \cap V) \mapsto \psi(U \cap V)$ is called a *transition map* from ϕ to ψ.

Definition 8.5 (Closed Upper Half-Space)**.** $\mathbb{H}^n = \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid x^n \ge 0\}$, and $\partial\mathbb{H}^n = \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid x^n = 0\}$.

Definition 8.6 (Manifold With Boundary)**.** Let M be a second-countable Hausdorff space and fix n. Suppose that for every $p \in M$, one of the following conditions is satisfied:

(1) There exists a neighborhood U of p and a homeomorphism $\phi : U \to \hat{U}$ where $\hat{U}$ is an open subset of $\mathbb{R}^n$. p is called an *interior point* and (U, ϕ) is called an *interior chart.*
(2) There exists a neighborhood U of p and a homeomorphism $\phi : U \to \hat{U}$ where $\hat{U}$ is an open subset of $\mathbb{H}^n$ with $\phi(p) \in \partial\mathbb{H}^n$. p is called a *boundary point.*

Then M is called an *n-dimensional topological manifold with boundary.* Note that every topological manifold is a topological manifold with boundary.

Definition 8.7 (Support)**.** If f is any real-valued or vector-valued function on a topological space M, the *support of f*, denoted by $\operatorname{supp} f$, is the closure of the set of points where f is nonzero:

$$\operatorname{supp} f = \overline{\{p \in M : f(p) \neq 0\}}.$$

Definition 8.8 (Bump Function)**.** If M is a topological space, $A \subset M$ is a closed subset, and $U \subset M$ is an open subset containing A, a continuous function $\psi : M \to \mathbb{R}$ is called a *bump function for A supported in U* if $0 \leq \psi \leq 1$ on M, $\psi \equiv 1$ on A, and $\operatorname{supp} \psi \subset U$.

8.2. **Smooth Manifolds.**

Definition 8.9 (Smoothly Compatible)**.** Let M be a topological n-manifold. Two coordinate charts $(U, \phi), (V, \psi)$ are called *smoothly compatible* if $U \cap V = \emptyset$ or the transition map $\psi \circ \phi^{-1}$ is a diffeomorphism.

Definition 8.10 (Smooth Atlas)**.** Let M be a topological n-manifold. A *smooth atlas* is an atlas $\mathcal{A}$ such that any two charts in $\mathcal{A}$ are smoothly compatible with each other.

Definition 8.11 (Smooth Structure)**.** If M is a topological n-manifold, an atlas $\mathcal{A}$ that is not properly contained in any larger smooth atlas is called *maximal* or a *smooth structure on M*

Definition 8.12 (Smooth Manifold)**.** A *smooth manifold* is a topological manifold equipped with a smooth structure.

Definition 8.13. Suppose $(M, \mathcal{A})$ is a smooth manifold.

- Any chart $(U, \phi) \in \mathcal{A}$ is called a *smooth chart.*
- Given a smooth chart (U, ϕ), U is called a smooth coordinate domain and ϕ is called a *smooth coordinate map.*
- Given a smooth chart (U, ϕ), U is called a *smooth coordinate ball* if it is a coordinate ball.

Remark. *One must define a smooth structure on a topological manifold before talking about a smooth chart.*

Definition 8.14 (Smooth Maps)**.** Let M, N be smooth manifolds with or without boundary and $F : M \to N$ be a map. F is a *smooth map* if for every $p \in M$, there exist smooth charts (U, ϕ) containing p and (V, ψ) containing $F(p)$ such that

- $F(U) \subset V$;
- $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth.

Definition 8.15 (Coordinate Representatin of a Smooth Map)**.** Let $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ be smooth manifolds. Let $F : M \to N$ be a smooth map and $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ be given. Then $\hat{F} = \psi \circ F \circ \phi^{-1}$ is called the coordinate representation of F with respect to (U, ϕ) and (V, ψ).

Definition 8.16 (Diffeomorphism)**.** Let M, N be smooth manifolds with or without boundary. A diffeomorphism is a smooth map $F : M \to N$ with a smooth inverse.

Definition 8.17 (Smooth on a subset)**.** Let M, N be smooth manifolds with or without boundary and $A \subset M$ be an arbitrary subset. A map $F : A \to N$ is said to be *smooth on A* if every $p \in A$ has an open neighborhood $W \subset M$ such that there exists a smooth map $\tilde{F} : W \to N$ with $\tilde{F}_{W \cap A} = F$.

8.3. **Tangent Vectors.**

Definition 8.18 (Derivation)**.** Let M be a smooth manifold with or without boundary. A derivation at $p \in M$ is a linear map $v : C^\infty(M) \to \mathbb{R}$ such that

$$v(fg) = f(p)vg + g(p)vf$$

for all $f, g \in C^\infty(M)$.

This corresponds to "arrows that are tangent to M and whose basepoints are attached to M at p" even though it may not be easy to see that from this definition.

Definition 8.19 (Tangent Space)**.** The tangent space T_pM to M at p is the vector space of all derivations of $C^\infty(M)$ at p.

Derivation of $C^\infty(M)$	Geometric tangent vector on M
Differential of a smooth map between manifolds	Total derivative of a map between Euclidean spaces

Definition 8.20 (Differential)**.** M, N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. The *differential of F at p* is the linear map $dF_p : T_pM \to T_{F(p)}N$ defined by

$$dF_p(v) := f \mapsto v(f \circ F)$$

Equivalently, $\forall v \in T_pM, \forall f \in C^\infty(N), dF_p(v)(f) = v(f \circ F)$. This corresponds to "the directional derivative of F at p in the direction of the arrow v."

Definition 8.21 (Coordinate Vectors)**.** Let $(M, \mathcal{A})$ be a smooth manifold without boundary. Let $p \in M$ and choose a chart $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Then the *coordinate vectors at p*, denoted by $\frac{\partial}{\partial x^i}|_p$, are derivations $C^\infty(U) \to \mathbb{R}$ such that

$$\frac{\partial}{\partial x^i}\Big|_p := f \mapsto \frac{\partial}{\partial x^i}\Big|_{\phi(p)}(f \circ \phi^{-1}).$$

Definition 8.22 (Tangent Bundle)**.** Let M be a smooth manifold with or without boundary. The tangent bundle of M, denoted by TM, is the disjoint union $\coprod_{p \in M} T_pM$.

Definition 8.23 (Projection Map)**.** Let M be a smooth manifold with or without boundary. The projection map $\pi : TM \to M$ is the map defined by $(p, v) \mapsto p$.

8.4. **Submersions, Immersions, and Embeddings.**

Definition 8.24 (Rank)**.** Let M, N be smooth manifolds with or without boundary and let $F : M \to N$ be a smooth map. Then the rank of F at $p \in M$ is:

- The rank of the linear map $dF_p : T_pM \to T_{F(p)}N$.
- The dimension of the subspace $dF_p(T_pM)$ in the vector space $T_{F(P)}N$.

It is easy to see that the two definitions above are always equivalent.

Definition 8.25 (Submersions and Immersions)**.** Let M, N be smooth manifolds with or without boundary and let $F : M \to N$ be a smooth map.

- If F has the same rank at every point $p \in M$, then F is said to have *constant rank*, and the rank is denoted by $\operatorname{rank} F$.
- If the rank of F at $p \in M$ is equal to $\max\{\dim M, \dim N\}$, then F is said to have *full rank at p*.
- If F has full rank everywhere, then F is said to have *full rank*.
- If F has constant rank and $\operatorname{rank} F = \dim N$, F is called a *smooth submersion.*
- If F has constant rank and $\operatorname{rank} F = \dim M$, F is called a *smooth immersion.*

Definition 8.26 (Curve)**.** If M is a manifold with or without boundary, we define a *curve in M* to be a continuous map $\gamma : J \to M$ where $J \subset \mathbb{R}$ is an interval.