# INTRODUCTION TO SMOOTH MANIFOLDS

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## 1. Chapter 1: Smooth Manifolds

### 1.1. Exercises.

**Exercise 1.1.** Show that equivalent definitions of manifolds are obtained if instead of allowing $U$ to be homeomorphic to *any* open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

*Proof.* It is clear that a "manifold" satisfying the open-ball or $\mathbb{R}^n$ definition satisfies the open-subset definition. Let $M$ be a manifold satisfying the open-subset definition. Let $x \in M$ be given and let $U, \hat{U}, \phi$ be given according to the definition. Since $\hat{U}$ is open, there exists an open ball $B$ such that $\phi(x) \in B \subset \hat{U}$. Restrict $\phi$ to $\phi^{-1}(B)$. Then $\phi^{-1}(B)$ is an open subset of $M$ containing $x$, and $\phi \mid_{\phi^{-1}(B)}$ is a homeomorphism between $\phi^{-1}(B)$ and $B$. Thus $M$ satisfies the open-ball definition.

$B(x, r) \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ by the map $(x_1 + a_1, \cdots, x_n + a_n) \mapsto (\frac{a_1}{r-a_1}, \cdots, \frac{a_n}{r-a_n})$ where $x = (x_1, \cdots, x_n)$ is the center of $B(x, r)$ and $r$ is the radius. Since the composition of two homeomorphisms gives a homeomorphism, $M$ also satisfies the $\mathbb{R}^n$ definition as well. □

**Exercise 1.6.** Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological $n$-manifold.

*Proof.* From the definition of $\pi$, it is easy to see that $\pi(B(x, r))$ is open in $\mathbb{RP}^n$ where $x \in S^n$ and $0 < r < 1$.

Let $[x], [y] \in \mathbb{RP}^n$ be given. Without loss of generality, assume $x, y \in S^n$. Let $r = \min\{|x - y|, |x + y|, 1\}/2$. Then $U_x = \pi(B(x, r)), U_y = \pi(B(y, r))$ contain $[x], [y]$, respectively. $\pi^{-1}(U_x), \pi^{-1}(U_y)$ are both open in $\mathbb{R}^{n+1} \setminus \{0\}$ which can be seen easily by writing down exactly which points belong to them, so $U_x, U_y$ are both open in $\mathbb{RP}^n$. Then $\pi^{-1}(U_x \cap U_y) = \pi^{-1}(U_x) \cap \pi^{-1}(U_y) = \emptyset$, so $U_x \cap U_y = \emptyset$. Therefore, $\mathbb{RP}^n$ is Hausdorff.

Let $\mathcal{B} = \{\pi(B(x, 1/k)) \mid x \in \mathbb{Q}^{n+1} \cap S^n, k \in \{2, 3, 4, \cdots\}\}$. Then $\mathcal{B}$ is a countable collection of open sets whose union is $\mathbb{RP}^n$. Let $U \subset \mathbb{RP}^n$ be a nonempty open set. Let $[x] \in U$. Since $\pi$ is a quotient map, $\pi^{-1}(U)$ is open. Moreover, $x \in \pi^{-1}(U)$. Without loss of generality, $x \in S^n$. Then $x \in B(x', 1/k) \subset \pi^{-1}(U)$ for some $B(x', 1/k) \in \mathcal{B}$. Then $[x] = \pi(x) \in \pi(B(x', 1/k)) \subset \pi(\pi^{-1}(U)) = U$. Therefore, $\mathcal{B}$ is a countable basis of $\mathbb{RP}^n$. □

**Exercise 1.7.** Show that $\mathbb{RP}^n$ is compact.

*Proof.* $\pi(S^n) = \mathbb{RP}^n$ and $S^n$ is compact because it is a closed, bounded subset of $\mathbb{R}^{n+1}$. (Heine-Borel) Moreover, the image of a compact set under a continuous map is compact. (See A.45(a)) Thus $\mathbb{RP}^n$ is compact. □

**Exercise 1.14.** Suppose $\mathcal{X}$ is a locally finite collection of subsets of a topological space $M$.

(a) The collection $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.
(b) $\overline{\bigcup_{X \in \mathcal{X}} X} = \bigcup_{X \in \mathcal{X}} \overline{X}$.

*Proof.*

(a) Let $p \in M$. Then there exists an open set $U$ containing $x$ such that there are only finitely many $X \in \mathcal{X}$ such that $U \cap X \neq \emptyset$. Let $X \in \mathcal{X}$.
- If $U \cap X \neq \emptyset$, then $U \cap \overline{X} \supset U \cap X \neq \emptyset$.
- If $U \cap X = \emptyset$, then $U^c$ is closed, so $\overline{X} \subset U^c$. In other words, $U \cap \overline{X} = \emptyset$.

This shows that the number of $X \in \mathcal{X}$ that intersects $U$ and the number of $\overline{X} \in \mathcal{X}$ that intersects $U$ are the same. Therefore, $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.

(b) Since the closure of a set is defined to be the intersection of all closed sets containing it, $\bigcup_{X \in \mathcal{X}} \overline{X} \subset \overline{\bigcup_{X \in \mathcal{X}} X}$. Let $x \notin \bigcup_{X \in \mathcal{X}} \overline{X}$. Then there exists a neighborhood $U$ of $x$ such that $U$ intersects only finitely many $X \in \mathcal{X}$. Let $X_1, \cdots, X_n$ denote them. By the same argument as part (a), $\overline{X_1}, \cdots, \overline{X_n}$ are the only elements in $\{\overline{X} \mid X \in \mathcal{X}\}$ that $U$ intersects. Since $x \notin \overline{X_i}$ for each $i = 1, \cdots, n$, $U^c \cup \overline{X_1} \cup \cdots \cup \overline{X_n}$ is a closed set which contains all $X \in \mathcal{X}$ but does not contain $x$. In other words, $x \notin \overline{\bigcup_{X \in \mathcal{X}} X}$.

□

**Exercise 1.18.** Let $M$ be a topological manifold. Two smooth atlases for $M$ determine the same smooth structure if and only if their union is a smooth atlas.

*Proof.* Let $\mathcal{A}, \mathcal{A}'$ be two smooth atlases.

Suppose that they determine the same smooth structure $\mathcal{B}$. Then $\mathcal{A} \cup \mathcal{A}' \subset \mathcal{B}$, so $\mathcal{A} \cup \mathcal{A}'$ must be a smooth atlas. By Proposition 1.17(a), $\mathcal{A} \cup \mathcal{A}'$ determines a unique smooth structure, but it must be $\mathcal{B}$ because $\mathcal{B}$ contains the union.

On the other hand, suppose that their union is a smooth atlas. Let $\mathcal{B}$ be the smooth structure that the union determines. Such $\mathcal{B}$ must exist by Proposition 1.17(a). By the same proposition, $\mathcal{A}, \mathcal{A}'$ must determine the unique smooth structures. However, they must be $\mathcal{B}$ because $\mathcal{B}$ contains both $\mathcal{A}$ and $\mathcal{A}'$. □

**Exercise 1.20.** Every smooth manifold has a countable basis of regular coordinate balls.

*Proof.* Let $M$ be an $n$-dimensional smooth manifold. We consider the special case that there exists a single chart $(\phi, U)$ with $U = M$. Let $x \in \hat{U}$ with rational coordinates. Then there exists $s > 0$ such that $B(x, s) \subset \hat{U}$. For each rational number $r \in (0, s)$, we consider the chart $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r)))$.

Let $\mathcal{B}$ be the collection of all such charts for each $x \in \hat{U}$ and $r$. We claim that $\mathcal{B}$ is a smooth atlas.

- Let $p \in M$. Then $\phi(p) \in \hat{U}$. Since $\hat{U}$ is open, $\phi(p) \in B(x, r) \subset \hat{U}$ for some $x$ with rational coordinates and a positive rational number $r$. Then $p \in \phi^{-1}(B(x, r))$, so the union of coordinate domains covers $M$. In other words, $\mathcal{B}$ is an atlas.
- Let $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r))), (p \mapsto \phi(p) - x', \phi^{-1}(B(x', r'))) \in \mathcal{B}$ be given. Suppose $\phi^{-1}(B(x, r)) \cap \phi^{-1}(B(x', r')) \neq \emptyset$. Let $\psi, \psi'$ denote the coordinate maps. Then $\psi' \circ \psi^{-1}$ is a composition of $\phi, \phi^{-1}$ and translation maps, so it is smooth.

Therefore, $\mathcal{B}$ is a smooth atlas.

Since $\mathcal{B}$ is a smooth atlas, there exists a smooth structure $\mathcal{A}$ on $M$ containing $\mathcal{B}$ by Proposition 1.17(a). We claim that $\mathcal{B}$, a subset of the smooth structure $\mathcal{A}$, is a countable basis of regular coordinate balls.

- $\mathcal{B}$ is a countable collection because $x \in \mathbb{Q}^n$ and $r \in \mathbb{Q}$.
- Let $(p \mapsto \phi(p)-x, \phi^{-1}(B(x,r))) \in \mathcal{B}$ be given. Then there exists a chart $(p \mapsto \phi(p)-x, \phi^{-1}(B(x,r')))$ in $\mathcal{B}$ with $r' > r$. Let $B = \phi^{-1}(B(x,r)), B' = \phi^{-1}(B(x,r'))$. Let $\psi$ denote the map $p \mapsto \phi(p) - x$. Then $\psi(B) = B(0,r)$ and $\psi(B') = B(0,r')$, respectively. Moreover, $\psi(\overline{B}) = \overline{B(0,r)}$ because $\psi$ is a homeomorphism.

Now let $M$ be an arbitrary smooth $n$-manifold. By definition, each point of $M$ is in the domain of a chart. By Proposition A.16, $M$ is covered by countably many charts $\{(U_i, \phi_i)\}$. By the previous argument, each $U_i$ has a countable basis of regular coordinate balls. Each regular coordinate ball in $U_i$ is indeed a regular coordinate ball in $M$ because $\overline{B}$ is a compact subset of $M$, which is Hausdorff, so $\overline{B}$ is closed. In other words, the closure of $B$ in $U_i$ is the same as the closure of $B$ in $M$. □

**Exercise 1.39.** Let $M$ be a topological $n$-manifold with boundary.

(a) $\operatorname{Int} M$ is an open subset of $M$ and a topological $n$-manifold without boundary.
(b) $\partial M$ is a closed subset of $M$ and a topological $(n-1)$-manifold without boundary.
(c) $M$ is a topological manifold if and only if $\partial M = \emptyset$.
(d) If $n = 0$, then $\partial M = \emptyset$ and $M$ is a 0-manifold.

*Proof.*

(a) Let $x \in \operatorname{Int} M$. Let $(\phi, U)$ be an interior chart for $x$. Then $x \in U \subset \operatorname{Int} M$ because every point in $U$ is in an interior chart $(\phi, U)$. A subspace of $M$ must be Hausdorff and second-countable by Proposition A.17(g, i), so $\operatorname{Int} M$ is a second-countable, Hausdorff space in which every point has a neighborhood homeomorphic to an open subset in $\mathbb{R}^n$. Thus $\operatorname{Int} M$ is an $n$-manifold without boundary.
(b) Since $\partial M = M \setminus \operatorname{Int} M$ and $\operatorname{Int} M$ is open in $M$, $\partial M$ is closed in $M$. Let $x \in \partial M$. Let $(\phi, U)$ be a boundary chart of $x$. If a point $y \in U$ gets mapped into $\operatorname{Int} \mathbb{H}^n$, then it is certainly an interior point. Thus $\phi(U \cap \partial M) \subset \partial \mathbb{H}^n$. Then $\pi_{n-1} \circ \phi$ is a homeomorphism that maps $U \cap \partial M$ into an open subset of $\mathbb{R}^{n-1}$ where $\pi_{n-1} : (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_{n-1})$.
(c) If $\partial M$ is empty, then $M = \operatorname{Int} M$, so (a) implies that $M$ is an $n$-dimensional manifold. If $M$ is a topological manifold, every point is an interior point. Since a point cannot be both an interior point and a boundary point, $\partial M$ is empty.
(d) If $n = 0$, then $\partial \mathbb{H}^0 = \emptyset$. Thus, the condition that $\phi(U) \cap \partial \mathbb{H}^n \neq \emptyset$ can never be satisfied, so there cannot be any boundary point.

□

**Exercise 1.41.** Let $M$ be a topological manifold with boundary.

(a) $M$ has a countable basis of precompact coordinate balls and half-balls.
(b) $M$ is locally compact.
(c) $M$ is paracompact.
(d) $M$ is locally path-connected.
(e) $M$ has countably many components, each of which is an open subset of $M$ and a connected topological manifold with boundary.
(f) The fundamental group of $M$ is countable.

*Proof.*

(a)
(b)
(c)
(d) Let $U \subset M$ be a nonempty open subset and choose $x \in U$. Then there exists a chart $(V, \phi)$ such that $x \in V$. Since $\phi(x)$ is a point in an open set $\phi(U \cap V)$, there exists $r > 0$ such that $B(\phi(x), r) \subset \phi(V)$. Then $N(x, U) = \phi^{-1}(B(\phi(x), r))$ is a path-connected neighborhood of $x$ that is contained in $U \cap V \subset U$. Therefore, $\{N(x,U) \mid \text{open } U \subset M, x \in U\}$ forms a basis of $M$ consisting of path-connected sets.

(e)
(f)

$\square$

**Exercise 1.44.** Suppose $M$ is a smooth $n$-manifold with boundary and $U$ is an open subset of $M$. Prove the following statements:

(a) $U$ is a topological $n$-manifold with boundary, and the atlas consisting of all smooth charts $(V, \phi)$ for $M$ such that $V \subset U$ defines a smooth structure on $U$. With this topology and smooth structure, $U$ is called an ***open submanifold with boundary***.
(b) If $U \subset \operatorname{Int} M$, then $U$ is actually a smooth manifold (without boundary); in this case we call it an ***open submanifold of M***.
(c) $\operatorname{Int} M$ is an open submanifold of $M$ (without boundary).

*Proof.* Let $\mathcal{T}$ denote the topology of $M$ and $\mathcal{A}$ denote the smooth structure of $M$.

(a) The subspace topology on $U$ is equivalent to $\mathcal{T}_U = \{V \in \mathcal{T} \mid V \subset U\}$ because $U$ is open. By Proposition A.17(A.18(Proof of Proposition A.17)), $U$ is Hausdorff and second-countable. For every point $p \in U$, there exists a $V \in \mathcal{T}$ with a homeomorphism $\phi : V \to \hat{V}$ where $\hat{V}$ is an open subset of $\mathbb{R}^n$ (or $\mathbb{H}^n$) Since $U \cap V$ is an open subset of $V$, $\phi$ restricted to $U \cap V$ is a homeomorphism between $U \cap V$ and $\phi(U \cap V)$, which is an open subset of $\mathbb{R}^n$ (or $\mathbb{H}^n$). Therefore, $U$ is a topological $n$-manifold with boundary.

Let $\mathcal{A}_U = \{(\phi, V) \in \mathcal{A} \mid V \subset U\}$. Then $\mathcal{A}_U$ is clearly a collection of charts on $U$ whose union covers $U$. Moreover, any two charts in $\mathcal{A}_U$ are clearly smoothly compatible. Let $(\phi, V)$ be a chart on $U$ that is smoothly compatible with every chart in $\mathcal{A}_U$. Let $(\psi, W) \in \mathcal{A}$. Then $(\psi_{W \cap U}, W \cap U)$ is a chart on $M$ and it must be smoothly compatible with every chart in $\mathcal{A}$. Therefore, $(\psi_{W \cap U}, W \cap U) \in \mathcal{A}$, so it must belong to $\mathcal{A}_U$. This implies that $(\phi, V)$ and $(\psi_{W \cap U}, W \cap U)$ are smoothly compatible. Since $V \subset W \cap U$, this implies that $(\phi, V)$ and $(\psi, W)$ are smoothly compatible.

Thus $(\phi, V)$ is smoothly compatible with every chart in $\mathcal{A}$, so $(\phi, V) \in \mathcal{A}$. This implies that $(\phi, V)$ is in $\mathcal{A}_U$, so $\mathcal{A}_U$ is indeed a maximal smooth atlas.

(b) Let $p \in U$. Then $p \in \operatorname{Int} M$, so there exists $(\phi, V) \in \mathcal{A}$ such that $p \in V$ and $\phi(V)$ is open in $\mathbb{R}^n$. Then $(\phi|_{V \cap U}, V \cap U)$ is a chart that is smoothly compatible with every chart in $\mathcal{A}$, so $(\phi|_{V \cap U}, V \cap U) \in \mathcal{A}$. Thus it must be in $\mathcal{A}_U$, so $p \in U$ is an interior point of $U$. Therefore, $U$ is a manifold without boundary.
(c) By 1.39, $\operatorname{Int} M$ is an open subset of $M$. By (b), $\operatorname{Int} M$ is an open submanifold of $M$ without boundary.

$\square$

## 1.2. **Problems.**

**Problem 1-2.** Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

*Proof.* Let $I$ denote an uncountable index set and $X = \coprod_{\alpha \in I} \mathbb{R}$. Let $(x, \alpha_0) \in X$. Define $U = \coprod_{\alpha \in I} U_\alpha$ where $U_{\alpha_0} = \mathbb{R}$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_0$. Then $U$ is an open neighborhood of $(x, \alpha_0)$ that is clearly homeomorhpic to $\mathbb{R}$. Thus $X$ is locally Euclidean.

Let $(x_1, \alpha_1) \neq (x_2, \alpha_2) \in X$. If $\alpha_1 \neq \alpha_2$, then open neighborhoods of $x_1$ and $x_2$ formed in the same way as above separate the two points. Suppose $\alpha_1 = \alpha_2$. Without loss of generality, $x_1 < x_2$. Define $U = \coprod_{\alpha \in I} U_\alpha$ where $U_{\alpha_1} = (-\infty, (x_1 + x_2)/2)$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_1$. Similarly, define $V = \coprod_{\alpha \in I} U_\alpha$ where $U_{\alpha_1} = ((x_1 + x_2)/2, \infty)$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_2$. Then such $U$ and $V$ separate the two points. Therefore, $X$ is Hausdorff.

Let $\mathcal{B}$ be a basis of $X$. For each $\alpha_0 \in I$, let $U_{\alpha_0} = \coprod_{\alpha \in I} U_\alpha$ where $U_{\alpha_0} = \mathbb{R}$ and $U_\alpha = \emptyset$ when $\alpha \neq \alpha_0$. Then for each $\alpha_0$, there must exist $B_{\alpha_0} \in \mathcal{B}$ such that $(0, \alpha_0) \in B_{\alpha_0} \subset U_{\alpha_0}$. Clearly, $B_\alpha \neq B_\beta$ if $\alpha \neq \beta$. Therefore, the cardinality of $\mathcal{B}$ is greater than or equal to that of $I$. Hence, $X$ is not second-countable. $\square$

**Problem 1-7.** Let $N$ denote the **north pole** $(0, \cdots, 0, 1) \in S^n \subset \mathbb{R}^{n+1}$, and let $S$ denote the **south pole** $(0, \cdots, 0, -1)$. Define the **stereographic projection** $\sigma : S^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1, \cdots, x^{n+1}) = \frac{(x^1, \cdots, x^n)}{1 - x^{n+1}}$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus \{S\}$.

(a) For any $x \in S^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through $N$ and $x$ intersects the linear subspace where $x^{n+1} = 0$. Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through $S$ and $x$ intersects the same subspace.

(b) Show that $\sigma$ is bijective, and

$$\sigma^{-1}(u^1, \cdots, u^n) = \frac{(2u^1, \cdots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

(c) Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(S^n \setminus \{N\}, \sigma)$ and $(S^n \setminus \{S\}, \tilde{\sigma})$ defines a smooth structure on $S^n$.

(d) Show that this smooth structure is the same as the one defined in Example 1.31.

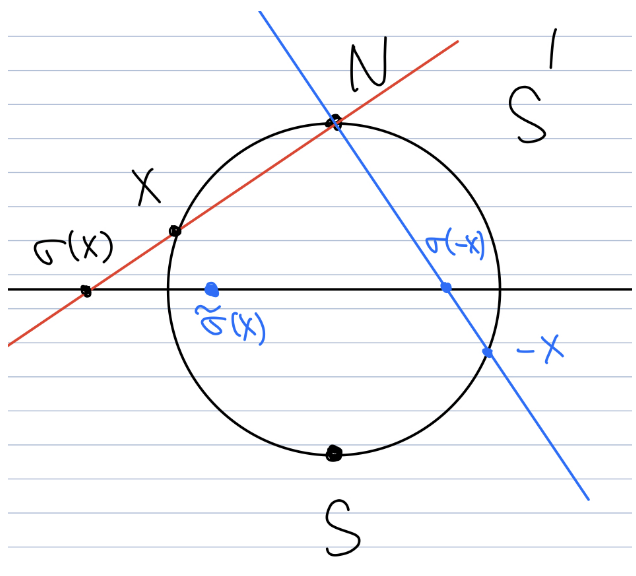


FIGURE 1. Problem 1-7

*Proof.*

(a) This is trivial from a basic trigonometry argument using the triangles $N, (0, \cdots, 0, x^{n+1}), (x^1, \cdots, x^{n+1})$ and $N, (0, \cdots, 0), \sigma(x^1, \cdots, x^{n+1})$.

(b) $\sigma \circ \sigma^{-1}$ and $\sigma^{-1} \circ \sigma$ are both the identity maps, so $\sigma$ is bijective and $\sigma^{-1}$ is its inverse.

(c) Computation shows that $\tilde{\sigma} \circ \sigma^{-1} : S^n \setminus \{N, S\} \to S^n \setminus \{N, S\}$ sends $(u^1, \cdots, u^n)$ to $(u^1, \cdots, u^n)/|u|^2$. As $|u| \neq 0$ in the domain, this map is well-defined and clearly smooth. By Proposition 1.17(a), these two charts determine a unique smooth structure.

(d) $\phi_i, \sigma, \tilde{\sigma}$ are all smooth functions of subsets of Euclidean spaces, so transition maps are always smooth. By Proposition 1.17(b), the smooth structure determined by $\sigma, \tilde{\sigma}$ is the same as the one defined in Example 1.31.

$\square$

**Problem 1-8.** By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle $S^1$ as a subset of the complex plane. An angle function on a subset $U \subset S^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$.

Show that there exists an angle function $\theta$ on an open subset $U \subset S^1$ if and only if $U \neq S^1$. For any such angle function, show that $(U, \theta)$ is a smooth coordinate chart for $S^1$ with its standard smooth structure.

*Proof.* First, we will consider the special case when $U = S^1 \backslash \{e^{it}\}$ for some $t \in \mathbb{R}$. The map $\phi : (t, t+2\pi) \to U$ defined by $\theta \mapsto e^{i\theta}$ is a bijective function. Therefore, by taking the inverse of $\phi$, which is clearly continuous, we obtain a desired angle function. The case of an arbitrary proper open subset of $U$ is the same as this special case because we simply need to restrict the domain of the map obtained above. On the other hand, suppose $U = S^1$. Suppose there exists an angle function $f$ on $U$. Define $g : S^1 \to \mathbb{R}$ by $g(z) = f(z) - f(-z)$.

- $g(1) \neq 0$ because $g(1) \neq 0 \implies f(1) = f(-1)$, which is clearly impossible.
- $g(1) > 0$ implies that $g(-1) < 0$. By the intermediate value theorem, $g(z) = 0$ for some $z \in S^1$. This is a contradiction.
- If $g(1) < 0$, $g(-1) > 0$, and we obtain a contradiction in the same manner.

Therefore, such an $f$ cannot exist. Hence, an angle function exists if and only if $U$ is an proper open subset of $S^1$.

Let $(U_i^{\pm}, \phi_i^{\pm})$ and $(U, \phi)$ be given where $\phi$ maps $U$ into $(t, t+2\pi)$ for some $t \in \mathbb{R}$. We will show that they are smoothly compatible. Let $V = U \cap U_i^{\pm}$. The map $\phi_i^{\pm} \circ \phi^{-1} : \phi(V) \to \phi_i^{\pm}(V)$ is $\phi_i^{\pm} \circ \exp$. Since it is a composition of a projection map with a smooth map, this is smooth. Therefore, $(U, \phi)$ is indeed a coordinate chart for $S^1$ with its standard smooth structure. □

**Problem 1-12(Proof of Proposition 1.45).** Suppose $M_1, \cdots, M_k$ are smooth manifolds and $N$ is a smooth manifold with boundary. Then $M_1 \times \cdots \times M_k \times N$ is a smooth manifold with boundary, and $\partial(M_1 \times \cdots \times M_k \times N) = M_1 \times \cdots \times M_k \times \partial N$.

*Proof.* By Example 1.34, $M_1 \times \cdots \times M_k$ is a smooth manifold. Thus it suffices to show that $M \times N$ is a smooth manifold with boundary if $M$ is a smooth manifold and $N$ is a smooth manifold with boundary. Let $m, n$ be the dimensions of $M, N$.

First, we show that $M \times N$ is a topological manifold with boundary and $\partial(M \times N) = M \times \partial N$. Let $(p, q) \in M \times N$. Then $p \in M$, so there exists a chart $(U, \phi)$ such that $p \in U$ and $\hat{U} = \phi(U) \subset \mathbb{R}^m$.

- Suppose $q \in \operatorname{Int} N$. Then there exists a chart $(V, \psi)$ such that $\hat{V} = \psi(V) \subset \mathbb{R}^n$. $\phi \times \psi$ is a homeomorphism between $U \times V$ and $\hat{U} \times \hat{V} \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. Thus $(U \times V, \phi \times \psi)$ is a chart for $(p, q)$.
- Suppose $q \in \operatorname{bd} N$. Then there exists a chart $(V, \psi)$ such that $\hat{V} = \psi(V) \subset \mathbb{H}^n$ and $\psi(q) \in \partial\mathbb{H}^n$. $\phi \times \psi$ is a homeomorphism between $U \times V$ and $\hat{U} \times \hat{V} \subset \mathbb{R}^m \times \mathbb{H}^n = \mathbb{H}^{m+n}$. Moreover, $(\phi \times \psi)(p, q) = (\phi(p), \psi(q)) \in \mathbb{R}^m \times \mathbb{H}^n = \mathbb{H}^{m+n}$. Thus $(U \times V, \phi \times \psi)$ is a boundary chart for $(p, q)$.

Therefore, $M \times N$ is a topological manifold with boundary and $\partial(M \times N) = M \times (\partial N)$.

Let $\mathcal{A}_M, \mathcal{A}_N$ be the smooth structures of $M, N$. Define $\mathcal{A}_{M \times N} = \{(U \times V, \phi \times \psi) \mid (U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N\}$. Then $\mathcal{A}_{M \times N}$ is an atlas because we showed earlier that each $(U \times V, \phi \times \psi)$ is a chart. Let $(U_1 \times V_1, \phi_1 \times \psi_1), (U_2 \times V_2, \phi_2 \times \psi_2) \in \mathcal{A}_{M \times N}$. Then $(\phi_2 \times \psi_2) \circ (\phi_1 \times \psi_1)^{-1} = (\phi_2 \circ \phi_1^{-1}) \times (\psi_2 \circ \psi_1^{-1})$ is a smooth map from $(\phi_1 \times \psi_1)(U_1 \times V_1)$ into $(\phi_2 \times \psi_2)(U_2 \times V_2)$. Thus every pair of charts in $\mathcal{A}_{M \times N}$ is smoothly compatible. In other words, $\mathcal{A}_{M \times N}$ is a smooth atlas.

On the other hand, $\mathcal{A}_{M \times N}$ must be maximal because the restriction of any smoothly compatible chart to $M, N$ gives a smoothly compatible chart, which must belong to $\mathcal{A}_M, \mathcal{A}_N$, respectively. Thus $M \times N$ is a smooth manifold with boundary. □

## 2. Chapter 2: Smooth Maps

### 2.1. **Exercises.**

**Exercise 2.1.** Let $M$ be a smooth manifold with or without boundary. Show that pointwise multiplication turns $C^\infty(M)$ into a commutative ring and a commutative and associative algebra over $\mathbb{R}$.

*Proof.*

- The constant map $f(p) = 0$ is clearly in $C^\infty(M)$ and it is the additive identity.
- The constant map $f(p) = 1$ is clearly in $C^\infty(M)$ and it is the multiplicative identity.

- Let $f \in C^\infty(M), g \in C^\infty(M)$. Let $p \in M$ and $(\phi, U)$ be a smooth chart for $p$. Then $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are both smooth(Exercise 2.3), real-valued maps defined on an open subset of $\mathbb{R}^n$. Thus $f+g$ is in $C^\infty(M)$ Moreover, $f+g = g+f$ because addition in $\mathbb{R}$ is commutative.
- Let $f, g, h \in C^\infty(M)$. Let $p \in M$ and $(\phi, U)$ be a smooth chart for $p$. Then $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are both smooth(Exercise 2.3), real-valued maps defined on an open subset of $\mathbb{R}^n$. Therefore, $fg$ is in $C^\infty(M)$ Moreover, $fg = gf$ and $(fg)h = f(gh)$ because multiplication in $\mathbb{R}$ is commutative and associative.
- Let $c \in \mathbb{R}, f \in C^\infty(M)$. Then $cf$ can be seen as $fg$ where $g$ is the constant function whose value is $c$. As shown above, $cf \in C^\infty(M)$.

□

**Exercise 2.2.** Let $U$ be an open submanifold of $\mathbb{R}^n$ with its standard smooth manifold structure. Show that a function $f : U \to \mathbb{R}^k$ is smooth in the sense just defined if and only if it is smooth in the sense of ordinary calculus. Do the same for an open submanifold with boundary in $\mathbb{H}^n$.

*Proof.* $f$ is smooth in the sense just defined if and only if $f \circ \mathrm{Id}^{-1}$ is smooth in the sense of ordinary calculus. Since $f \circ \mathrm{Id}^{-1} = f$, $f \circ \mathrm{Id}^{-1}$ is smooth in the sense of ordinary calculus if and only if $f$ is smooth in the sense of ordinary calculus. □

**Exercise 2.3.** Let $M$ be a smooth manifold with or without boundary, and suppose $f : M \to \mathbb{R}^k$ is a smooth function. Show that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$ is smooth for every smooth chart $(U, \phi)$ for $M$.

*Proof.* Let $\phi(x) \in \phi(U)$. Since $f$ is smooth, there exists $(V, \psi)$ such that $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}^k$ is smooth and $x \in V$. Let $W = U \cap V$. Then $f \circ \psi^{-1} : \psi(W) \to \mathbb{R}^k$ is smooth and $\psi \circ \phi^{-1} : \phi(W) \to \psi(W)$ is a diffeomorphism where $\phi(W)$ is a neighborhood of $W$. Then the restriction of $f \circ \psi^{-1}$ to $\phi(W)$ is identical to $(f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})$. Since he composition of a smooth function is smooth, $f \circ \psi^{-1}$ is smooth. □

**Exercise 2.7(Prove Proposition 2.5).** Suppose $M$ and $N$ are smooth manifolds with or without boundary, and $F : M \to N$ is a map. Then $F$ is smooth if and only if either of the following conditions is satisfied:

(a) For every $p \in M$, there exist smooth charts $(U, \phi)$ containing $p$ and $(V, \psi)$ containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in $M$ and the composite map $\psi \circ F \circ \phi^{-1}$ is smooth from $\phi(U \cap F^{-1}(V))$ to $\psi(V)$.

(b) $F$ is continuous and there exist smooth atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for $M$ and $N$, respectively, such that for each $\alpha$ and $\beta$, $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is a smooth map from $\phi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

*Proof.* Let $\mathcal{A}_M$ and $\mathcal{A}_N$ be smooth structures of $M$ and $N$. Suppose $F$ is smooth. By Proposition 2.4, $F$ is continuous. For every $p \in M$ there exist coordinate charts $(U_p, \phi_p)$ containing $p$ and $(V_p, \psi_p)$ containing $F(p)$ such that $F(U_p) \subset V_p$ and $\psi_p \circ F_p \circ \phi_p^{-1}$ is smooth from $\phi_p(U_p)$ to $\psi_p(V_p)$. Then $\{(U_p, \phi_p) \mid p \in M\} \subset \mathcal{A}_M$ and $A_n\{(V_p, \psi_p) \mid p \in M\} \subset \mathcal{A}_N$ are smooth atlases. Moreover, for every $(U_p, \phi_p)$ and $(V_q, \psi_q)$, $\psi_q \circ F \circ \phi_p^{-1}$ is a smooth map from $\phi_p(U_p \cap F^{-1}(V_q))$ to $\psi_q(V_q)$ because $\psi_q \circ F \circ \phi_p^{-1} = (\psi_q \circ \psi_p^{-1}) \circ (\psi_p \circ F \circ \phi_p^{-1})$ where $\psi_q \circ \psi_q^{-1}$ and $\psi_p \circ F \circ \phi_p^{-1}$ are smooth. Therefore, the definition implies (b).

(b) implies (a) because if $F$ is continuous, $F^{-1}(V_\beta)$ is open in $M$ for every $\beta$, so $U \cap F^{-1}(V)$ is open in $M$.

Finally, we show that (a) implies the definition. Suppose $F$ satisfies (a). Let $p \in M$. Let $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ be smooth charts satisfying the properties described in (a). Let $U' = U \cap F^{-1}(V)$ and consider $(U', \phi\mid_{U'})$. Then $(U', \phi\mid_{U'}) \in \mathcal{A}_M$ because it must be smoothly compatible with any other smooth coordinate chart in $\mathcal{A}_M$. Moreover, $F(U') \subset V$ and $\psi \circ F \circ (\phi\mid_{U'})^{-1} : \phi(U') \to \psi(V)$ is smooth. Therefore, (a) implies the definition.

Hence, (a), (b) and the definition are all equivalent. □

**Exercise 2.7(Proof of Proposition 2.6).** Let $M$ and $N$ be smooth manifolds with or without boundary, and let $F : M \to N$ be a map.

(a) If every point $p \in M$ has a neighborhood $U$ such that the restriction $F|_U$ is smooth, then $F$ is smooth.

(b) Conversely, if $F$ is smooth, then its restriction to every open subset is smooth.

*Proof.* Let $\mathcal{A}_M, \mathcal{A}_N$ be smooth structures of $M, N$, respectively.

(a) Let $p \in M$. Let $U$ be a neighborhood of $p$ such that $F|_U$ is smooth. By 1.44, $U$ is a smooth manifold with the induced smooth structure $\mathcal{A}_U = \{(V,\phi) \in \mathcal{A}_M \mid V \subset U\}$. Since $F|_U$ is smooth, there exist $(V,\phi) \in \mathcal{A}_U$ and $(W,\psi) \in \mathcal{A}_N$ such that:
  - $F|_U(V) \subset W$.
  - $\psi \circ F|_U \circ \phi^{-1} : \phi(V) \to \psi(W)$ is smooth.

  Since $V \subset U$, $F(V) \subset W, \psi \circ F \circ \phi^{-1} : \phi(V) \to \psi(W)$ is smooth, and $(V,\phi) \in \mathcal{A}$. Therefore, $F$ is smooth.

(b) Let $U \subset M$ be an open subset. By 1.44, $U$ is a smooth manifold with the induced smooth structure $\mathcal{A}_U = \{(V,\phi) \in \mathcal{A}_M \mid V \subset U\}$. Let $p \in U$. Then $p \in F$, so there exist $(V,\phi) \in \mathcal{A}_M, (W,\psi) \in \mathcal{A}_N$ such that $F(V) \subset W$ and $\psi \circ F \circ \phi^{-1} : \phi(V) \to \psi(W)$ is smooth. Then $(V \cap U, \phi|_{V\cap U})$ is a chart that is smoothly compatible with every chart in $\mathcal{A}_M$. Therefore, $(V \cap U, \phi|_{V\cap U}) \in \mathcal{A}_M$. Moreover, $\phi|_{V\cap U}(V \cap U) \subset \phi(V) \subset W$ and $\psi \circ F \circ (\phi|_{V\cap U}(V\cap))^{-1}$ is clearly smooth. Therefore, $F|_U$ is smooth.

□

**Exercise 2.9.** Suppose $F : M \to N$ is a smooth map between smooth manifolds with or without boundary. Show that the coordinate representation of $F$ with respect to *every* pair of smooth charts for $M$ and $N$ is smooth.

*Proof.* Let $(M, \mathcal{A}_M), (N, \mathcal{A}_N)$ be smooth manifolds with or without boundary. Let $F : M \to N$ be a smooth map. Let $(U,\phi) \in \mathcal{A}_M, (V,\psi) \in \mathcal{A}_N$ be given. We must show that $\hat{F} = \psi \circ F \circ \phi^{-1}$ is a smooth function from $\phi(U \cap F^{-1}(V))$ to $\psi(V)$. Let $\phi(p) \in \phi(U \cap F^{-1}(V))$. Then $p \in M$, so there exist $(U_0, \phi_0) \in \mathcal{A}_M$ and $(V_0, \psi_0) \in \mathcal{A}_N$ such that

- $p \in U_0 \subset U \cap F^{-1}(V)$;
- $\phi_0(U_0) \subset V_0$;
- $\psi_0 \circ F \circ \phi_0^{-1} : \phi_0(U_0) \to \psi(V_0)$ is smooth.

Then $\psi \circ F \circ \phi^{-1}|_{\phi(U_0)} = (\psi \circ \psi_0^{-1}) \circ (\psi_0 \circ F \circ \phi_0^{-1}) \circ (\phi_0 \circ \phi)$. Since the composition of smooth functions in Euclidean spaces is smooth, $\hat{F}$ is smooth. □

**Exercise 2.11(Proof of Proposition 2.10).** Let $M, N$ and $P$ be smooth manifolds with or without boundary.

(a) Every constant map $c : M \to N$ is smooth.
(b) The identity map of $M$ is smooth.
(c) If $U \subset M$ is an open submanifold with or without boundary, then the inclusion map $U \to M$ is smooth.

*Proof.* Let $\mathcal{A}_M, \mathcal{A}_N, \mathcal{A}_P$ be smooth structures of $M, N, P$, respectively.

(a) $F$ is clearly continuous. Moreover, for every $(U_\alpha, \phi_\alpha) \in \mathcal{A}_M, (V_\beta, \psi_\beta) \in \mathcal{A}_N$, $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is a constant map, so it is smooth. By (2.7(Prove Proposition 2.5)), $F$ is smooth.
(b) Let $p \in M$. Choose $(U,\phi) \in \mathcal{A}_M$ such that $p \in U$. Then $F(U) \subset U$ and $\phi \circ F \circ \phi^{-1} = \mathrm{Id}_U$, so it is smooth. Therefore, $F$ is smooth.
(c) By 1.44, $\mathcal{A}_U = \{(V,\phi) \mid V \subset U\}$ is a smooth structure of $U$. Let $p \in U$. Then $p \in V$ for some $(V,\phi) \in \mathcal{A}_U$. Then $(V,\phi) \in \mathcal{A}_M$, trivially. Since $F(V) \subset V$ and $\phi \circ F \circ \phi^{-1}$ is simply the identity map on $V$, $F$ is smooth.

□

**Exercise 2.16(Proof of Proposition 2.15).**

(a) Every composition of diffeomorphisms is a diffeomorphism.
(b) Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.
(c) Every diffeomorphism is a homeomorphism and an open map.
(d) The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.
(e) "Diffeomorphic" is an equivalence relation on the class of all smooth manifolds with or without boundary.

**Exercise 2.16(Proof of Proposition 2.15).** Let $(M, \mathcal{A}_M), (N, \mathcal{A}_N), (P, \mathcal{A}_P)$ be smooth manifolds with or without boundary, and let $F : M \to N, G : N \to P$ be diffeomorphisms.

(a) By Proposition 2.10(d), $G \circ F$ and $F^{-1} \circ G^{-1}$ are smooth. Then $(G \circ F) \circ (F^{-1} \circ G^{-1})$ and $(F^{-1} \circ G^{-1}) \circ (G \circ F)$ are both the identity map on the corresponding space, so $F^{-1} \circ G^{-1}$ is the smooth inverse of $G \circ F$. Therefore, $G \circ F$ is a diffeomorphism.

(b) By Example 1.34, we know that $M_1 \times \cdots \times M_k$ and $N_1 \times \cdots \times N_k$ are both smooth manifolds. Let $\mathcal{A}_{M_i}, \mathcal{A}_{N_i}, \mathcal{A}_M$ and $\mathcal{A}_N$ denote the smooth manifold structures of $M_i, N_i, M_1 \times \cdots \times M_k, N_1 \times \cdots \times N_k$, respectively. Let a smooth map $F_i : M_i \to N_i$ be given for each $i$. Let $(p_1, \cdots, p_k) \in M_1 \times \cdots M_k$ be given. Then there exist $(U_i, \phi_i) \in \mathcal{A}_{M_i}$ and $(V_i, \psi_i) \in \mathcal{A}_{N_i}$ such that $p_i \in U_i, F_i(U_i) \subset V_i, \psi_i \circ F_i \circ \phi_i^{-1} : \phi_i(U_i) \to \psi_i(V_i)$ is smooth for each $i$. This implies that $(\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots (\psi_k \circ F_k \circ \phi_k^{-1}) = (\psi_1 \times \cdots \times \psi_k) \circ (F_1 \times \cdots \times F_k) \circ (\phi_1 \times \cdots \times \phi_k)^{-1}$ is smooth.

Therefore, $F_1 \times \cdots \times F_k$ is smooth. Using the exact same argument, we can conclude that $F_1^{-1} \times \cdots \times F_k^{-1}$ is smooth. Since $(F_1 \times \cdots \times F_k)^{-1} = F_1^{-1} \times \cdots \times F_k^{-1}$, $F_1 \times \cdots \times F_k$ is a diffeomorphism.

(c) Proposition 2.4 states that every smooth map is continuous. Thus $F$ and $F^{-1}$ are both continuous. Therefore, $F$ is a homeomorphism and also an open map.

(d) Let $U \subset M$ be an open subset. By (2.7(Proof of Proposition 2.6)), $F|_U$ is smooth. Since $F$ is a homeomorphism as shown in (c), $F(U)$ is an open subset of $N$. Therefore, $F^{-1}|_{F(U)}$ is smooth by (2.7(Proof of Proposition 2.6)). Clearly, $F|_U$ and $F^{-1}|_{F(U)}$ are the inverse of each other. Therefore, $F|_U$ is a diffeomorphism.

(e) By (2.11(Proof of Proposition 2.10)), the identity map on $M$ is a diffeomorphism, so the reflexive property is satisfied. Moreover, $(F^{-1})^{-1} = F$, so the symmetric property is satisfied. By (a), the composition of two diffeomorphisms is a diffeomorphism, so the transitive property is satisfied. Therefore, "diffeomorphic" is an equivalence relation.

**Exercise 2.19(Proof of Theorem 2.18).** Suppose $M$ and $N$ are smooth manifolds with boundary and $F : M \to N$ is a diffeomorphism. Then $F(\partial M) = \partial N$, and $F$ restricts to a diffeomorphism from $\operatorname{Int} M$ to $\operatorname{Int} N$.

*Proof.* Let $\mathcal{A}_M, \mathcal{A}_N$ denote the smooth structures of $M, N$, respectively. Let $p \in \partial M$. Then there exists a chart containing $p$ that sends $p$ to $\partial \mathbb{H}^n$. By Theorem 1.46, every chart containing $p$ sends $p$ to $\partial \mathbb{H}^n$.

Since $F$ is smooth, there exist $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$ such that $F(U) \subset V$ and $\psi \circ F \circ \phi^{-1}$ is a smooth map from $\phi(U)$ to $\psi(V)$. $F^{-1}$ is a homeomorphism by (2.16(Proof of Proposition 2.15)). Then $(\phi^{-1} \circ F^{-1}, F(U))$ is a coordinate chart around $F(p)$ because we obtain a homeomorphism by restricting the composition of two injective continuous maps to its image. Moreover, we claim that $(\phi^{-1} \circ F^{-1}, F(U))$ is smoothly compatible with every chart in $\mathcal{A}_N$. Let $(\psi_1, V_1) \in \mathcal{A}_N$ be given. Then $(\phi^{-1} \circ F^{-1}) \circ \psi_1^{-1} = (\phi^{-1} \circ F^{-1} \circ \psi^{-1}) \circ (\psi \circ \psi_1^{-1})$, and the composition of two smooth maps is smooth. Therefore, $(\phi^{-1} \circ F^{-1}, F(U)) \in \mathcal{A}_N$, and this chart contains $F(p)$ and sends $F(p)$ to $\partial \mathbb{H}^n$. In other words, $F(p) \in \partial N$.

Since $F^{-1}$ is also smooth, $F^{-1}(\partial N) \subset \partial M$. $F^{-1}(\partial N) \subset \partial M \implies F(F^{-1}(\partial N)) \subset F(\partial M) \subset \partial N$. Since $F$ is a bijection, $F(F^{-1}(\partial N)) = \partial N$. Therefore, $F(\partial M) = \partial N$.

This implies that $F(\operatorname{Int} M) = \operatorname{Int} N$. By (1.44(c)) and (2.16(Proof of Proposition 2.15)(d)), $F$ is a diffeomorphism between $\operatorname{Int} M$ and $\operatorname{Int} N$. □

**Problem 2-27.** Give a counterexample to show that the conclusion of the extension lemma can be false if $A$ is not closed.

*Proof.* Let $M = \mathbb{R}, A = (0, 1), f(x) = 1/x$. Then $f$ is smooth on $A$, but $\lim_{x \to 0} f = \infty$, so $f$ cannot be extended continuously. □

2.2. **Problems.**

**Problem 2-1.** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts $(U, \phi)$ containing $x$ and $(V, \psi)$ containing $f(x)$ such that $\psi \circ f \circ \phi^{-1}$ is smooth as a map from $\phi(U \cap f^{-1}(V))$ to $\psi(V)$, but $f$ is not smooth in the sense we defined in this chapter.

*Proof.* $\phi = \psi = \mathrm{Id}$ in this solution.

If $x \geq 0$, then let $U = \mathbb{R}, V = (0, \infty)$. Then $\phi(U \cap f^{-1}(V)) = [0, \infty)$. Thus $\psi \circ f \circ \phi^{-1} : [0, \infty) \to (0, \infty)$ is the constant map that sends every number to 1. Therefore, it is smooth.

If $x < 0$, then let $U = \mathbb{R}, V = (-\infty, 1)$. Then $\phi(U \cap f^{-1}(V)) = (-\infty, 0)$. Thus $\psi \circ f \circ \phi^{-1} : (-\infty, 0) \to (-\infty, 1)$ is the constant map that sends every number to 0. Therefore, it is smooth.

It might seem that we can apply (2.7(Prove Proposition 2.5)) to show that $f$ is smooth, but (2.7(Prove Proposition 2.5)) requires that $U \cap f^{-1}(V)$ be open in $M$.

$f$ maps the interval $(-1, 1)$ to $\{0, 1\}$. Since the image of a connected set under a continuous map must be connected, $f$ cannot be continuous. By Proposition 2.4, $f$ cannot be smooth. $\square$

**Problem 2-2(Proof of Proposition 2.12).** Suppose $M_1, \cdots, M_k$ and $N$ are smooth manifolds with or without boundary, such that at most one of $M_1, \cdots, M_k$ has nonempty boundary. For each $i$, let $\pi_i : M_1 \times \cdots \times M_k \to M_i$ denote the projection onto the $M_i$ factor. A map $F : N \to M_1 \times \cdots \times M_k$ is smooth if and only if each of the component maps $F_i = \pi_i \circ F : N \to M_i$ is smooth.

*Proof.* Let $\mathcal{A}_{M_1}, \cdots, \mathcal{A}_{M_k}, \mathcal{A}_N$ be the smooth structures of $M_1, \cdots, M_k, N$. Let $d_1, \cdots, d_k$ denote the dimensions of $M_1, \cdots, M_n$, respectively. Let $d = \sum d_i$.

First, suppose that $F$ is smooth. By (2.11(Proof of Proposition 2.10)), the composition of smooth maps is smooth. Thus it suffices to show that $\pi_i : M_1 \times \cdots \times M_k \to M_i$ is smooth for each $i$. We show that $\pi_1$ is smooth and the other cases can be shown similarly.

Let $(x_1, \cdots, x_k) \in M_1 \times \cdots \times M_k$. Then for each $i$, there exist $(U_i, \phi_i) \in \mathcal{A}_{M_i}$ and $(V_i, \psi_i) \in \mathcal{A}_{M_i}$ such that $x_i \in U_i$ and $\phi_i(U_i) \subset V_i$. Then we have $(\phi_1 \times \cdots \times \phi_k)(U_1 \times \cdots \times U_k) \subset V_1 \times \cdots \times V_k$ and the composition $\phi_i \circ \pi_1 \circ (\phi_1 \times \cdots \times \phi_k)^{-1}$ is the projection of the first $d_1$ coordinates from $\mathbb{R}^n$ onto $\mathbb{R}^{d_1}$. Therefore, it is clearly smooth, so $\pi_1$ is smooth.

Suppose each $F_i = \pi_i \circ F : N \to M_i$ is smooth. Let $p \in N$. Then for each $i$, there exist $(U_i, \phi_i) \in \mathcal{A}_N$ and $(V_i, \psi_i) \in \mathcal{A}_{M_i}$ such that $p \in U_i, F_i(U_i) \subset V_i$ and $\psi_k \circ F_i \circ \phi_i^{-1}$. Let $U = U_1 \cap \cdots \cap U_k$. $U$ is a neighborhood of $p$ and the restriction of $\phi_1$ to $U$ is a homeomorphism. Then we claim that $(\phi_1, U) \in \mathcal{A}_N$ and $(\psi_1 \times \cdots \times \psi_k, V_1 \times \cdots \times V_k) \in \mathcal{A}_{M_1 \times \cdots \times M_k}$ are charts that satisfy the necessary properties.

- $F(U) \subset V_1 \times \cdots \times V_k$.
- For each $i$, $\psi_i \circ F_i \circ \phi_1^{-1} = (\psi_i \circ F_i \circ \phi_i^{-1}) \circ (\phi_i \circ \phi_1^{-1}) : \phi_1(U) \to \psi_i(V_i)$ is smooth because the composition of two smooth maps is smooth. Thus $(\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots \times (\psi_k \circ F_k \circ \phi_1^{-1}) : \phi_1(U) \to \psi_1(V_1) \times \cdots \times \psi_k(V_k)$ is smooth. Moreover, $(\psi_1 \times \cdots \times \psi_k) \circ F \circ \phi_1^{-1} = (\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots \times (\psi_k \circ F_k \circ \phi_1^{-1})$.

Therefore, $F$ is smooth. $\square$

**Problem 2-3.** For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

(a) $p_n : S^1 \to S^1$ is the $n$th power map for $n \in \mathbb{Z}$, given in complex notation by $p_n(z) = z^n$.
(b) $\alpha : S^n \to S^n$ is the antipodal map $\alpha(x) = -x$.
(c) $F : S^3 \to S^2$ is given by $F(w, z) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w})$ where we think of $S^3$ as the subset $\{(w, z) : |w|^2 + |z|^2 = 1\}$ of $\mathbb{C}^2$.

*Proof.*

(a) Example 1.31 shows the existence of a smooth structure of $S^1$ and let $\mathcal{A}$ denote it. Let $p \in S^1$. Then there exist $(U_i^\pm, \phi_i^\pm), (U_j^\pm, \phi_j^\pm) \in \mathcal{A}$ around $p, p_n(p)$, respectively. Then the composition $\phi_j^\pm \circ f \circ (\phi_i^\pm)^{-1}$ is equal to one of $\cos(n(\arccos(x))), \sin(n(\arcsin(x))), \cos(n(\arcsin(x))), \sin(n(\arccos(x)))$, all of which are clearly smooth. By Proposition 2.5(a), $p_n$ is smooth.
(b) Example 1.31 shows the existence of a smooth structure of $S^n$ and let $\mathcal{A}$ denote it. Let $p \in S^1$. Then there exists a chart $(U_i^\pm, \phi_i^\pm)$ in $\mathcal{A}$ around $p$. Then $(U_i^\mp, \phi_i^\mp)$ is a chart containing $\alpha(p)$ with $\alpha(U_i^\pm) \subset U_i^\mp$. Then $\phi_i^\mp \circ \alpha \circ \phi_i^\pm$ is the map $x \mapsto -x$, which is clearly smooth.

(c) Let $z = a + bi, w = c + di$. $z\overline{w} = ac + bd + i(bc - ad)$ and $w\overline{z} = (ac + bd) - i(bc - ad)$. Then $z\overline{w} + w\overline{z} = 2(ac + bd) = 2\operatorname{Re}(z\overline{w})$ and $i(w\overline{z} - z\overline{w}) = 2\operatorname{Im}(z\overline{w})$.

$$\begin{aligned}(2\operatorname{Re}(z\overline{w}))^2 + (2\operatorname{Im}(z\overline{w}))^2 + (|z|^2 - |w|^2)^2 &= 4|z\overline{w}|^2 + (|z|^2 - |w|^2)^2 \\ &= 4|z|^2|\overline{w}|^2 + (|z|^2 - |w|^2)^2 \\ &= (|z|^2 + |w|^2)^2 \\ &= 1.\end{aligned}$$

Therefore, $F$ indeed maps $S^3$ into $S^2$. Moreover, this map is continuous. Let $(z = a+bi, w = c+di) \in S^3$ be given. Suppose that $(U_4^+, \phi_4^+)$ and $(V_3^+, \psi_3^+)$ are charts containing $(z, w)$ and $F(z, w)$. Then $\psi_3^+ \circ F \circ \phi_4^+ : (a, b, c) \mapsto (2u, 2v)$ where $u + iv = (a + bi)(c - \sqrt{1 - a^2 - b^2 - c^2}i)$ which is a smooth map from $\phi_4^+(U_4^+) \subset \mathbb{R}^3$ into $\mathbb{R}^2$. Other cases are similar, and thus $F$ is smooth by Proposition 2.5(b).

□

**Problem 2-5.** Let $\mathbb{R}$ be the real line with its standard smooth structure, and let $\tilde{R}$ denote the same topological manifold with the smooth structure defined in Example 1.23. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is smooth in the usual sense.

(a) Show that $f$ is also smooth as a map from $\mathbb{R}$ to $\tilde{\mathbb{R}}$.
(b) Show that $f$ is smooth as a map from $\tilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever $n$ is not an integral multiple of 3.

*Proof.*

(a) The "$\psi \circ f \circ \phi^{-1}$" is simply $f^3$, which is a smooth map from $\mathbb{R}$ to $\mathbb{R}$. Thus $f : \tilde{\mathbb{R}} \to \mathbb{R}$ is smooth.
(b) Solve this!

□

**Problem 2-6.** Let $P : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Show that the map $\tilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\tilde{P}([x]) = [P(x)]$ is well defined and smooth.

*Proof.* Let $P_1, \cdots, P_{k+1}$ denote the component functions of $P$.

Suppose $[x_1 : \cdots : x_{n+1}] = [y_1 : \cdots : y_{n+1}]$. Then there exists $\lambda \neq 0$ such that $(y_1, \cdots, y_{n+1}) = (\lambda x_1, \cdots, \lambda x_{n+1})$. $P(y_1, \cdots, y_{n+1}) = P(\lambda x_1, \cdots, \lambda x_{n+1}) = \lambda^d P(x_1, \cdots, x_{n+1})$. Since $\lambda^d \neq 0$, $[P(y_1, \cdots, y_{n+1})] = [P(x_1, \cdots, x_{n+1})]$. Therefore, $\tilde{P}$ is well-defined.

Let $\tilde{p} = [p_1 : \cdots : p_{n+1}] \in \mathbb{RP}^n$ be given. Without loss of generality, assume $p_{n+1} \neq 0$. Consider the chart $(U, \psi_{n+1})$ with $U = \{[x_1 : \cdots : x_{n+1}] \mid x_{n+1} \neq 0\}$. Let $q_i = P_i(p_1, \cdots, p_{n+1})$. Without loss of generality, assume $q_{k+1} \neq 0$. Then $\tilde{P}(\tilde{p})$ is contained in $V = \{[y_1 : \cdots : y_{k+1}] \mid y_{k+1} \neq 0\}$. Since $P$ is smooth, there exists $0 < \delta < |x_{n+1}|$ such that $|(x_1, \cdots, x_{n+1}) - (p_1, \cdots, p_{n+1})| < \delta$ implies $P_{k+1}(x_1, \cdots, x_{n+1}) \neq 0$. Then $[p_1 : \cdots : p_{n+1}] \in \pi(B(p_1, \cdots, p_{n+1})) \subset U \cap F^{-1}(V)$. Therefore, $U \cap F^{-1}(V)$ is open in $\mathbb{RP}^n$.

Finally the composition map $\psi_{k+1} \cdot \tilde{P} \cdot \phi_{n+1}^{-1}$ sends $(x_1/x_{n+1}, \cdots, x_n/x_{n+1})$ to $(y_1/y_{k+1}, \cdots, y_k/y_{k+1})$ where $y_i = P_i(x_1, \cdots, x_{n+1})$. In other words, $(x_1, \cdots, x_n) \mapsto (y_1/y_{k+1}, \cdots, y_k/y_{k+1})$ where $y_i = P_i(x_1, \cdots, x_n, 1)$. Since each $P_i$ is smooth, this map must be smooth as well. By (2.7(Prove Proposition 2.5)), $\tilde{P}$ is smooth. □

**Problem 2-7.** Let $M$ be a nonempty smooth $n$-manifold with or without boundary, and suppose $n \geq 1$. Show that the vector space $C^\infty(M)$ is infinite-dimensional.

*Proof.* Let $k \in \mathbb{N}$ be given. Let $p \in M$ be chosen arbitrarily. Let $(U, \phi)$ be a smooth chart containing $p$. Then $\hat{U} = \phi(U)$ is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. In each case, we can pick $k$ distinct points $x_1, \cdots, x_k \in \hat{U}$ because $\hat{U}$ is a nonempty open subset and $n \geq 1$. Since $\hat{U}$ is open, there exist open $U_1, \cdots, U_k$ such that $x_i \in U_i \subset \hat{U}$ and $U_i \cap U_j$ whenever $i \neq j$. Moreover, $\{x_i\}$ is a closed subset. By Proposition 2.25, we obtain $k$ bump functions $f_i$ for $\{x_i\}$ supported in $U_i$. Extend each $f_i$ by setting $f_i(q) = 0$ for any $q \notin U$. Then each $f_i$ lives in $C^\infty(M)$. Clearly, $\sum c_i f_i = 0$ implies $c_i = 0$, so $\{f_1, \cdots, f_k\}$ is linearly independent. Therefore, $C^\infty(M)$ is infinite-dimensional. □

**Problem 2-14.** Suppose $A$ and $B$ are disjoint closed subsets of a smooth manifold $M$. Show that there exists $f \in C^\infty(M)$ such that $0 \le f(x) \le 1$ for all $x \in M$, $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

*Proof.* By Theorem 2.29, there exist $\alpha, \beta \in C^\infty(M)$ such that $\alpha^{-1}(0) = A$ and $\beta^{-1}(0) = B$. Then $f(x) = \alpha(x)/(\alpha(x) + \beta(x))$ is a desired map. $\square$

# 3. Chapter 3: Tangent Vectors

## 3.1. Exercises.

**Proposition 3.2.** Let $a \in \mathbb{R}^n$.

(a) For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map $D_v|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$D_v|_a f = D_v f(a) = \frac{d}{dt}\bigg|_{t=0} f(a + tv)$$

is a derivation at $a$.

(b) The map $v_a \mapsto D_v|_a$ is an isomorphism from $\mathbb{R}^n_a$ onto $T_a\mathbb{R}^n$.

*Proof.*

(a) $D_v|_a$ is linear because $D_v|_a(f + cg) = D_v(f + cg)(a) = D_v(f)(a) + cD_v g(a) = D_v|_a(f) + cD_v|_a(g)$ because directional derivatives are linear. Moreover, the product rule is satisfied because directional derivatives satisfy that. Therefore, $D_v|_a$ is a linear map that satisfies directional derivatives, so it is a derivation.

(b) Let $\phi : \mathbb{R}^n_a \to T_a\mathbb{R}^n$ be defined such that $v_a \mapsto D_v|_a$. We first claim that $\phi$ is linear.

$$\begin{aligned}
\phi(v_a + cw_a)(f) &= \phi((v + cw)_a)(f) \\
&= D_{v+cw} f(a) \\
&= D_v f(a) + cD_w f(a) \\
&= D_v|_a(f) + cD_w|_a(f) \\
&= \phi(v_a)(f) + c\phi(w_a)(f) \\
&= (\phi(v_a) + c\phi(w_a))(f).
\end{aligned}$$

Next, we claim that $\ker(\phi) = 0$. Let $v_a \in \ker(\phi) \subset \mathbb{R}^n_a$. Let $v_1, \cdots, v_n \in \mathbb{R}^n$ be chosen such that $v_a = \sum_{i=1}^n v_i e_i|_a$. For each $j$, let $x^j : \mathbb{R}^n \to \mathbb{R}$ denote the projection map of the $j$th coordinate. Then $0 = D_v|_a(x^j) = \frac{d}{dt}|_{t=0} x^j(a + tv) = v_j$ for each $j$. Therefore, $v_1 = \cdots = v_n = 0$, so $\ker(\phi) = 0$. Since $\phi$ is linear, $\phi$ must be injective.

Lastly, we claim that $\phi$ is surjective. Let $w \in T_a\mathbb{R}^n$ be given. For each $j$, let $v_j = w(x^j)$. Let $v = (v_1, \cdots, v_n)$. We claim that $\phi(v_a) = w$. Let $f \in C^\infty(\mathbb{R}^n)$. By Theorem C.15, we can write

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + \sum_{i,j=1}^n (x^i - a^i)(x^j - a^j) \int_{0,1} F(t)dt$$

where $F(t)$ is some function. Since $(x^i - a^i)$ and $(x^j - a^j)\int_{0,1} F(t)dt$ vanish at $x = a$, $w((x^i - a^i)(x^j - a^j)\int_{0,1} F(t)dt) = 0$ for any $i, j$. Therefore,

$$\begin{aligned}
w(f) &= w(f(a)) + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)(w(x^i) - w(a^i)) \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a) w(x^i) \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a) v_i \\
&= \phi(v_a)(f)
\end{aligned}$$

which proves that $\phi$ is surjective.

$\square$

**Exercise 3.5(Proof of Lemma 3.4).** Suppose $M$ is a smooth manifold with or without boundary, $p \in M, v \in T_pM$, and $f, g \in C^\infty(M)$.

(a) If $f$ is a constant function, then $vf = 0$.
(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

*Proof.* This is similar to Lemma 3.1.

(a) Let $h$ be the constant function that always takes the value 1. Then $v(h) = v(h^2) = h(p)v(h) + h(p)v(h) = 2v(h)$, so $v(h) = 0$. Since $f(p) = ch(p)$ for some $c \in \mathbb{R}$ and $v$ is linear, this implies $0 = cv(h) = v(ch) = v(f)$.
(b) $v(fg) = f(p)vg + g(p)vf = 0 + 0 = 0$.

$\square$

**Exercise 3.7(Proof of Proposition 3.6).** Let $M, N$, and $P$ be smooth manifolds with or without boundary, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.

(a) $dF_p : T_pM \to T_{F(p)}N$ is linear.
(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}P$.
(c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.
(d) If $F$ is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

*Proof.* (a) $\forall v, w \in T_pM, \forall c \in \mathbb{R}, \forall f \in C^\infty(N)$,

$$\begin{aligned} dF_p(cv + w)(f) &= (cv + w)(f \circ F) \\ &= (cv)(f \circ F) + w(f \circ F) \\ &= c(v(f \circ F)) + w(f \circ F) \\ &= c(dF_p(v)(f)) + dF_p(w)(f) \\ &= (cdF_p(v))(f) + dF_p(w)(f) \\ &= (cdF_p(v) + dF_p(w))(f). \end{aligned}$$

Therefore, $dF_p(cv + w) = cdF_p(v) + dF_p(w)$.

(b) $\forall v \in T_pM, f \in C^\infty(P)$,

$$\begin{aligned} d(G \circ F)_p(v)(f) &= v(f \circ (G \circ F)) \\ &= v((f \circ G) \circ F) \\ &= (dF_p(v))(f \circ G) \\ &= (dG_{F(p)}(dF_p(v)))(f) \\ &= ((dG_{F(p)} \circ dF_p)(v))(f) \end{aligned}$$

Therefore, $d(G \circ F)_p = dG_{F(p)} \circ dF_p$.

(c) $\forall v \in T_p(M), \forall f \in C^\infty(M)$,

$$\begin{aligned} d(\mathrm{Id}_M)_p(v)(f) &= v(f \circ \mathrm{Id}_M) \\ &= v(f). \end{aligned}$$

Therefore, $d(\mathrm{Id}_M)_p(v) = v$, so $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$.

(d) $F^{-1}$ exists and it is a smooth map since $F$ is a diffeomorphism. By combining (b) and (c), we obtain $dF_p$ and $dF^{-1}_{F(p)}$ are the inverse of each other. Therefore, $dF_p$ is an isomorphism.

$\square$

**Proposition 3.10.** If $M$ is an $n$-dimensional smooth manifold, then for each $p \in M$, the tangent space $T_pM$ is an $n$-dimensional vector space.

*Proof.* Let $\mathcal{A}$ denote the smooth structure of $M$ and let $p \in M$ be given. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Then

$$T_pM \overset{di_p}{\cong} T_pU \overset{d\phi_p}{\cong} T_{\phi(p)}\hat{U} \overset{di_{\phi(p)}}{\cong} T_{\phi(p)}\mathbb{R}^n$$

where $di_p$ is induced by the inclusion map $i : U \to M$ and $di_{\phi(p)}$ is induced by the inclusion map $: \hat{U} \to \mathbb{R}^n$. $di_p, d\phi_p, di_{\phi(p)}$ are all isomorphisms by (3.7(Proof of Proposition 3.6)(d)) and Proposition 3.9. Therefore, $\dim(T_pM) = n$. ☐

**Proposition 3.15.** Let $M$ be a smooth $n$-manifold with or without boundary, and let $p \in M$. Then $T_pM$ is an $n$-dimensional vector space, and for any smooth chart $(U, (x^i))$ containing $p$, the coordinate vectors $\partial/\partial x^1|_p, \cdots, \partial/\partial x^n|_p$ form a basis for $T_pM$.

*Proof.* By Proposition 3.12, $T_pM$ is an $n$-dimensional vector space. By Corollary 3.3, the $\partial/\partial x^i|_{\phi(p)}$ form a basis for $T_{\phi(p)}\mathbb{R}^n$. By Proposition 3.6(d), $d\phi_p : T_pM \to T_{\phi(p)}\mathbb{R}^n$ is an isomorphism. Since $d\phi_p$ is an isomorphism between vector spaces, $d\phi_p$ sends a basis to a basis. In other words, the $\partial/\partial x^i|_p = (d\phi_p)^{-1}(\partial/\partial x^i|_{\phi(p)})$ form a basis. ☐

**Remark.** *The discussion on PP.61-62 shows the connection between differentials and Jacobian matrices. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map and let $p \in \mathbb{R}^n$ be given.*

$$dF_p\left(\frac{\partial}{\partial x^i}\Big|_p\right)(f) = \frac{\partial}{\partial x^i}\Big|_p(f \circ F) \qquad \textit{(definition of d)} \tag{3.1}$$

$$= \frac{\partial(f \circ F)}{\partial x^i}(p) \qquad \textit{(Just a partial derivative of } f \circ F\textit{)} \tag{3.2}$$

$$= \sum_{j=1}^{m} \frac{\partial f}{\partial y^j}(F(p))\frac{\partial F^j}{\partial x^i}(p) \tag{3.3}$$

$$= \sum_{j=1}^{m} \frac{\partial F^j}{\partial x^i}(p)\frac{\partial f}{\partial y^j}(F(p)) \qquad \textit{(Multiplication is commutative in } \mathbb{R}\textit{)} \tag{3.4}$$

$$= \sum_{j=1}^{m} \frac{\partial F^j}{\partial x^i}(p)\frac{\partial}{\partial y^j}\Big|_{F(p)}(f). \tag{3.5}$$

*Therefore, we obtain that $dF_p(\frac{\partial}{\partial x^i}\big|_p) = \sum_{j=1}^m \frac{\partial F^j}{\partial x^i}(p)\frac{\partial}{\partial y^j}\big|_{F(p)}$. $\{\partial/\partial x^i\}$ and $\{\partial/\partial y^j\}$ form bases for $T_p\mathbb{R}^n$ and $T_{F(p)}\mathbb{R}^m$, respectively, so it makes sense to put $dF_p$ is a matrix form. Then we obtain*

$$\begin{bmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{bmatrix}$$

*which is identical to the Jacobian matrix of $F$ at $p$. Two things to note:*

- *It makes sense to discuss the Jacobian matrix of $F$ because $F$ is a map from $\mathbb{R}^m$ to $\mathbb{R}^n$.*
- *The same calculation applies if $F : U \to V$ where $U, V$ are open subsets of $\mathbb{R}^n, \mathbb{R}^m$ or where $U, V$ are open subsets of $\mathbb{H}^n, \mathbb{H}^m$.*

*We now consider a more general case when $F : M \to N$ is a smooth map between two smooth manifolds with or without boundary. Let $p \in M$ be given. Let $(U, \phi), (V, \psi)$ be smooth charts of $M, N$ that contain $p, F(p)$, respectively. Let $\hat{F} = \psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(V)$ and $\hat{p} = \phi(p)$. Then we obtain the following commutative diagram:*

$$\begin{tikzcd} U \cap F^{-1}(V) \arrow[r, "F"] \arrow[d, "\phi"] & F \arrow[d, "\psi"] \\ \phi(U \cap F^{-1}(V)) \arrow[r, "\hat{F}"] & \hat{V} \end{tikzcd}$$

*We compute*

$$
\begin{aligned}
dF_p(\frac{\partial}{\partial x^i}\Big|_p) &= dF_p(d(\phi^{-1})_{\hat{p}}(\frac{\partial}{\partial x^i}\Big|_{\hat{p}})) && \textit{(Definition of a coordinate vector)} \\
&= (dF_p \circ d(\phi^{-1})_{\hat{p}})(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}) \\
&= (d(F \circ \phi^{-1})_{\hat{p}})(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}) && (3.7(Proof of Proposition 3.6)) \\
&= d(\psi^{-1} \circ \hat{F})_{\hat{p}}(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}) && \textit{(See the diagram above)} \\
&= (d(\psi^{-1})_{\hat{F}(\hat{p})} \circ d\hat{F}_{\hat{p}})(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}) && (3.7(Proof of Proposition 3.6)) \\
&= d(\psi^{-1})_{\hat{F}(\hat{p})}(d\hat{F}_{\hat{p}}(\frac{\partial}{\partial x^i}\Big|_{\hat{p}})) \\
&= d(\psi^{-1})_{\hat{F}(\hat{p})}(\sum_{j=1}^{m} \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial y^j}\Big|_{\hat{F}(\hat{p})}) && (\textit{Discussion above}) \\
&= \sum_{j=1}^{m} \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) d(\psi^{-1})_{\hat{F}(\hat{p})}(\frac{\partial}{\partial y^j}\Big|_{\hat{F}(\hat{p})}) && (\textit{Linearity of a differential}) \\
&= \sum_{j=1}^{m} \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) d(\psi^{-1})_{\psi(F(p))}(\frac{\partial}{\partial y^j}\Big|_{\psi(F(p))}) && (\textit{Diagram above}) \\
&= \sum_{j=1}^{m} \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial y^j}\Big|_{F(p)} && (\textit{Definition of a coordinate vector}).
\end{aligned}
$$

*Therefore, even in this general case, $dF_p$ is represented in coordinate bases by the Jacobian matrix of $\hat{F}$.*

**Remark.** *The notation on P.63-64 is not easy to understand.*

*Let $M$ be an $n$-dimensional smooth manifold. Let $(U, \phi = (x^i)), (V, \psi = (\tilde{x}^i))$ be two smooth charts on $M$ and $p \in U \cap V$. The textbook denotes the transition map $\psi \circ \phi^{-1} : \phi(U \cap v) \to \psi(U \cap V)$ by*

$$\psi \circ \phi^{-1}(x) = (\tilde{x}^1(x), \cdots, \tilde{x}^n(x)).$$

*What (I think) this really means is*

$$(\psi \circ \phi^{-1})(x^1(p), \cdots, x^n(p)) = (\tilde{x}^1(p), \cdots, \tilde{x}^n(p))$$

*for each $p \in U \cap V$. The idea is that $\phi = (x^i)$ is a diffeomorphism, so the textbook decides to denote each point in $\phi(U \cap V)$ by $x$ because every point in $\phi(U \cap V)$ can be denoted by $(x^1(p), \cdots, x^n(p))$ for a unique $p \in U \cap V$.*

*Moreover, the second part of this discussion (after "By (3.9), the differential $d(\psi \circ \phi^{-1})_{\phi(p)}$ can be written") is even more confusing because:*

- *The textbook simply uses $x^i$ and $\tilde{x}^i$ to represent the coordinates of $\hat{U}$ and $\hat{V}$ instead of the coordinate functions of $\phi$ and $\psi$.*
- *$\hat{U}$ and $\hat{V}$ both live in $\mathbb{R}^n$, so it might seem unnecessary to use both $x^i$ and $\tilde{x}^i$. It is actually necessary because we want to use $\partial/\partial x^i$ to talk about the coordinate vectors induced by $\phi$ and $\partial/\partial \tilde{x}^i$ to talk about the coordinate vectors induced by $\psi$.*

*Finally, (3.12) in the textbook can be derived as following:*

$$
\begin{aligned}
v &= \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\Big|_p \\
&= \sum_{i=1}^{n} v^i (\sum_{j=1}^{n} \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial \tilde{x}^j}\Big|_p) && \text{\textit{((3.11) in the textbook)}} \\
&= \sum_{i=1}^{n} (\sum_{j=1}^{n} v^i \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial \tilde{x}^j}\Big|_p) \\
&= \sum_{j=1}^{n} [\sum_{i=1}^{n} v^i \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p})] \frac{\partial}{\partial \tilde{x}^j}\Big|_p .
\end{aligned}
$$

**Exercise 3.17.** Let $(x, y)$ denote the standard coordinates on $\mathbb{R}^2$. Verify that $(\tilde{x}, \tilde{y})$ are global smooth coordinates on $\mathbb{R}^2$, where

$$\tilde{x} = x, \quad \tilde{y} = y + x^3.$$

Let $p$ be the point $(1, 0) \in \mathbb{R}^2$ (in standard coordinates), and show that

$$\frac{\partial}{\partial x}\Big|_p \neq \frac{\partial}{\partial \tilde{x}}\Big|_p,$$

even though the coordinate function $x$ and $\tilde{x}$ are identically equal.

*Proof.* The map $(x, y) \mapsto (x, y + x^3)$ is a smooth automorphism on $\mathbb{R}^2$.

$$
\begin{aligned}
\frac{\partial}{\partial x}\Big|_p &= \frac{\partial \tilde{x}}{\partial x}(1,0) \frac{\partial}{\partial \tilde{x}}\Big|_p + \frac{\partial \tilde{y}}{\partial x}(1,0) \frac{\partial}{\partial \tilde{y}}\Big|_p && \text{((3.11) in the textbook)} \\
&= \frac{\partial}{\partial \tilde{x}}\Big|_p + 3 \frac{\partial}{\partial \tilde{y}}\Big|_p && (\frac{\partial \tilde{y}}{\partial x} = 3x^2) \\
&\neq \frac{\partial}{\partial \tilde{x}}\Big|_p .
\end{aligned}
$$

□

**Exercise 3.19.** Suppose $M$ is a smooth manifold with boundary. Show that $TM$ has a natural topology and smooth structure making it into a smooth manifold with boundary, such that if $(U, (x^i))$ is any smooth boundary chart for $M$, then rearranging the coordinates in the natural chart $(\pi^{-1}(U), (x^i, v^i))$ for $TM$ yields a boundary chart $(\pi^{-1}(U), (v^i, x^i))$.

*Proof.* The proof is similar to that of Proposition 3.18. We begin by defining the maps that will become our smooth charts. Given any smooth (possibly boundary) chart $(U, \phi)$ for $M$, note that $\pi^{-1}(U) \subset TM$ is the set of all tangent vectors to $M$ at all points of $U$. Let $(x^1, \cdots, x^n)$ denote the coordinate functions of $\phi$, and define a map $\tilde{\phi}$ that maps $\pi^{-1}(U)$ into $\mathbb{H}^{2n}$ or $\mathbb{R}^{2n}$ by

$$\tilde{\phi}(v^i \frac{\partial}{\partial x^i}\Big|_p) = (v^1, \cdots, v^n, x^1(p), \cdots, x^n(p)).$$

In case $(U, \phi)$ is a boundary chart, $\tilde{(\phi)}$ indeed maps $\pi^{-1}(U)$ into $\mathbb{H}^{2n}$ because $x^n(p) \geq 0$. Its image set is $\mathbb{R}^n \times \phi(U)$, which is an open subset of $\mathbb{R}^{2n}$ or $\mathbb{H}^{2n}$. Now suppose we are given two smooth charts $(U, \phi)$ and $(V, \psi)$ for $M$, and let $(\pi^{-1}(U), \tilde{\phi}), (\pi^{-1}(V), \tilde{\psi})$ be the corresponding charts on $TM$. The sets

$$
\begin{aligned}
\tilde{\phi}(\pi^{-1}(U) \cap \pi^{-1}(V)) &= \mathbb{R}^n \times \phi(U \cap V) && \text{and} \\
\tilde{\psi}(\pi^{-1}(U) \cap \pi^{-1}(V)) &= \mathbb{R}^n \times \psi(U \cap V)
\end{aligned}
$$

are open in $\mathbb{R}^{2n}$ or $\mathbb{H}^{2n}$, and the transition map $\tilde{\psi} \circ \tilde{\phi}$ can be written explicitly using (3.1) as

$$
\begin{aligned}
&(\tilde{\psi} \circ \tilde{\phi}^{-1})(x^1, \cdots, x^n, v^1, \cdots, v^n) \\
&= (\frac{\partial \tilde{x}^1}{\partial x^j}(x) v^j, \cdots, \frac{\partial \tilde{x}^n}{\partial x^j}(x) v^j, \tilde{x}^1(x), \cdots, \tilde{x}^n(x)).
\end{aligned}
$$

The rest of the proof is identical to that of Proposition 3.18. $\square$

**Proposition 3.21.** If $F : M \to N$ is a smooth map, then its global differential $dF : TM \to TN$ is a smooth map.

*Proof.* Let $(p_0, v_0) \in TM$ be given. It suffices to show that $F$ is smooth in an neighborhood around $(p_0, v_0)$. Let $(U, \phi), (V, \psi)$ be given such that $p_0 \in U, F(U) \subset V$ and $F(p_0) \in V$. The set $\pi^{-1}(U)$ is open in $TM$ because that is how we give $TM$ a topology in Proposition 3.18. Moreover, $\pi^{-1}(U)$ is a neighborhood of $(p_0, v_0)$ in $TM$. We will consider the charts $(\pi^{-1}, \tilde{\phi})$ and $(\pi^{-1}(V), \tilde{\psi})$ as defined in Proposition 3.18. Let $(x^1, \cdots, x^m, v^1, \cdots, v^m) \in \tilde{\phi}(\pi^{-1}(U))$. Let $x = (x^1, \cdots, x^m)$ and $p = \phi^{-1}(x)$. Then

$$dF_p(\frac{\partial}{\partial x^i}\Big|_p) = \frac{\partial F^j}{\partial x^i}(p)\frac{\partial}{\partial y^j}\Big|_{F(p)}$$

by (3.9) in the textbook. Then

$$(\tilde{\psi} \circ dF \circ \tilde{\phi}^{-1})(x^1, \cdots, x^m, v^1, \cdots, v^m) = ((\psi^1 \circ F \circ \phi^{-1})(x), \cdots, (\psi^n \circ F \circ \phi^{-1})(x), \frac{\partial F^1}{\partial x^i}(p)v^i, \cdots, \frac{\partial F^n}{\partial x^i}(p)v^i)$$

Since each component function is smooth, $\tilde{\psi} \circ dF \circ \tilde{\phi}^{-1}$ is smooth. Therefore, $dF$ is a smooth map from $TM$ to $TN$. $\square$

**Proposition 3.23.** Suppose $M$ is a smooth manifold with or without boundary and $p \in M$. Every $v \in T_pM$ is the velocity of some smooth curve in $M$.

*Proof.* First, suppose that $p \in \operatorname{Int} M$. Let $(U, \phi)$ be a smooth coordinate chart centered at $p$, and write $v = v^i \partial/\partial x^i|_p$ in terms of the coordinate basis. Without loss of generality, $\phi(p) = 0 \in \mathbb{R}^n$. Now, define $\hat{\gamma} : (-\epsilon, \epsilon) \to \hat{U}$ by $\hat{\gamma}(t) = (tv_1, \cdots, tv_n)$ for sufficiently small $\epsilon > 0$. Let $\gamma : (-\epsilon, \epsilon) \to U$ be defined by $\gamma = \phi^{-1} \circ \hat{\gamma}$. Then $\gamma$ is actually a smooth map from a 1-manifold $(-\epsilon, \epsilon)$ to an $n$-manifold $M$ (with or without boundary). We will use the formula derived in (3.1) and obtain

$$\begin{aligned}
\gamma'(0) &= d\gamma(\frac{d}{dt}\Big|_0) \\
&= \sum_{j=1}^{n} \frac{\partial \hat{\gamma}^j}{\partial t}(0)\frac{\partial}{\partial x^j}\Big|_p \\
&= \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j}\Big|_p \\
&= v.
\end{aligned}$$

If $p \in \partial M$, then we do the exact same thing as above except that the domain will be $[0, \epsilon)$ if $v_n > 0$ and $(-\epsilon, 0]$ if $v_n \leq 0$. $\square$

**Proposition 3.24.** Let $F : M \to N$ be a smooth map, and let $\gamma : J \to M$ be a smooth curve. For any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma : J \to N$ is given by

$$(F \circ \gamma)'(t_0) = dF(\gamma'(t_0)).$$

*Proof.*

$$\begin{aligned}
(F \circ \gamma)'(t_0) &= d(F \circ \gamma)(\frac{d}{dt}\Big|_{t_0}) && \text{(definition of the velocity)} \\
&= (dF \circ d\gamma)(\frac{d}{dt}\Big|_{t_0}) && \text{(Corollary 3.22(a))} \\
&= dF(d\gamma(\frac{d}{dt}\Big|_{t_0})) \\
&= dF(\gamma'(t_0)) && \text{(definition of the velocity).}
\end{aligned}$$

$\square$

**Proposition 3.25.** Suppose $F : M \to N$ is a smooth map, $p \in M$, and $v \in T_pM$. Then

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma : J \to M$ such that $0 \in J, \gamma(0) = p$, and $\gamma'(0) = v$.

*Proof.* This is a special case of (3.24) where $t_0 = 0$. $\square$

3.2. **Problems.**

**Problem 3-1.** Suppose $M$ and $N$ are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Show that $dF_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if $F$ is constant on each component of $M$.

*Proof.* Suppose $dF_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$. It suffices to show that for every $p \in M$, there exists a neighborhood of $p$ on which $F$ is constant. Let $p \in M$ and $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$ be given such that $p \in U$ and $F(U) \subset V$. Without loss of generality, we assume $\hat{U} = \phi(U)$ is an open ball in $\mathbb{R}^m$. Then for any $i, j$ and for any $q \in \hat{U}$,

$$\begin{aligned} dF_q(\frac{\partial}{\partial x^i}|_q)(\pi_j \circ \psi) = 0 &\implies (\frac{\partial}{\partial x^i}|_q)(\pi_j \circ \psi \circ F) = 0 \\ &\implies (\frac{\partial}{\partial x^i}|_{\phi(q)})(\pi_j \circ \psi \circ F \circ \phi^{-1}) = 0. \end{aligned}$$

Fix $j$. Then every partial derivative of $\pi_j \circ \psi \circ F \circ \phi^{-1}$ at every point in $\hat{U}$ is 0. The intermediate value theorem implies that $\pi_j \circ \psi \circ F \circ \phi^{-1}$ is constant on $\hat{U}$ because $\hat{U}$ is an open ball. In other words, $(\pi_j \circ \psi \circ F \circ \phi^{-1})(\hat{U}) = \{y_j\}$ for some $y_j \in \mathbb{R}$. Since this is true for every $j$ and $\pi_j$ is the projection of the $j$th coordinate, $(\psi \circ F \circ \phi^{-1})(\hat{U}) = \{y\}$ where $y = (y_1, \cdots, y_n)$. Then $(F \circ \phi^{-1})(\hat{U}) = F(U) = \psi^{-1}(y)$. Since $\psi$ is a homeomorphism, there exists exactly one point in $\psi^{-1}(U)$. In other words, $F$ is constant on $U$. Therefore, $F$ is constant on each path component.

Suppose $F$ is constant on each component of $M$. Let $p \in M$. Choose a chart $(U, \phi) \in \mathcal{A}_M$ such that $p \in U$. Then $F \circ \phi^{-1}$ is constant in a neighborhood around $\phi(p)$. For any $i$,

$$\begin{aligned} dF_p(\frac{\partial}{\partial x^i}|_p)(f) &= \frac{\partial}{\partial x^i}|_p(f \circ F) \\ &= \frac{\partial}{\partial x^i}|_{\phi(p)}(f \circ F \circ \phi^{-1}) \\ &= 0 \end{aligned}$$

because $f \circ F \circ \phi^{-1}$ is constant in a neighborhood around $\phi(p)$. By Proposition 3.15, $\partial/\partial x^i|_p$ form a basis for $T_pM$. Since $dF_p$ sends each basis element to 0, $dF_p = 0$. $\square$

**Problem 3-2(Proof of Proposition 3.14).** Let $M_1, \cdots, M_k$ be smooth manifolds, and for each $j$, let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection onto the $M_j$ factor. For any point $p = (p_1, \cdots, p_k) \in M_1 \times \cdots \times M_k$, the map

$$\alpha : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \cdots, d(\pi_k)_p(v))$$

is an isomorphism. The same is true if one of the spaces $M_i$ is a smooth manifold with boundary.

*Proof.* It suffices to show this for the case that $k = 2$ because the results extend to arbitrary $k$ by induction. Let $\mathcal{A}_{M_1}, \mathcal{A}_{M_2}, \mathcal{A}_{M_1 \times M_2}$ be the smooth structures of $M_1, M_2, M_1 \times M_2$.

We first define a lot of notations.

- Let $d_1, d_2$ denote the dimensions of $M_1, M_2$ and let $d = d_1 + d_2$ denote the dimension of $M_1 \times M_2$.
- Let $p = (p_1, p_2) \in M_1 \times M_2$ be given. Choose $(U, \phi = (x^i)) \in \mathcal{A}_{M_1}, (V, \psi = (y^i)) \in \mathcal{A}_{M_2}$ with $p_1 \in U$ and $p_2 \in V$. Let $q_1 = \phi(p_1), q_2 = \psi(p_2), q = q_1 \times q_2$.
- $(U \times V, (z^i)) \in \mathcal{A}_{M_1 \times M_2}$ and $(p_1, p_2) \in U \times V$ where $(z^i) = \phi \times \psi$. More specifically, $z^i = x^i \circ \pi_1$ for $1 \leq i \leq d_1$ and $z^i = y^i \circ \pi_2$ for $d_1 + 1 \leq i \leq d_1 + d_2$.

Note that we use $x^i, y^i, z^i, \pi_1$ to mean two different things in this solution:

- $x^i$ is either the $i$th coordinate function of $\phi$ or the $i$th projection map $\mathbb{R}^{d_1} \to \mathbb{R}$.
- $y^i$ is either the $i$th coordinate function of $\psi$ or the $i$th projection map $\mathbb{R}^{d_2} \to \mathbb{R}$.
- $z^i$ is either the $i$th coordinate function of $\phi \times \psi$ or the $i$th projection map $\mathbb{R}^{d_1+d_2} \to \mathbb{R}$.
- $\pi_1$ is either the projection map $M_1 \times M_2 \to M_1$ or the projection map $\mathbb{R}^{d_1+d_2} \to \mathbb{R}^{d_1}$.
- $\pi_2$ is either the projection map $M_1 \times M_2 \to M_2$ or the projection map $\mathbb{R}^{d_1+d_2} \to \mathbb{R}^{d_2}$.

By Proposition 3.15, $\{\partial/\partial x^1|_{p_1}, \cdots, \partial/\partial x^{d_1}|_{p_1}\}, \{\partial/\partial y^1|_{p_2}, \cdots, \partial/\partial y^{d_2}|_{p_2}\}, \{\partial/\partial z^1|_p, \cdots, \partial/\partial z^{d_1+d_2}|_p\}$ form bases for $T_{p_1}M_1, T_{p_2}M_2, T_p(M_1 \times M_2)$.

$\alpha(\partial/\partial z^1|_p) = (d(\pi_1)_p(\partial/\partial z^1|_p), d(\pi_2)_p(\partial/\partial z^1|_p))$. We claim that $d(\pi_1)_p(\partial/\partial z^1|_p) = \partial/\partial x^1|p_1$.

$$
\begin{aligned}
d(\pi_1)_p(\partial/\partial z^1|_p)(f) &= d(\pi_1)_p(d(\phi^{-1} \times \psi^{-1})_q)(\frac{\partial}{\partial z^1}|_q)(f) \\
&= (d(\pi_1)_p \circ d(\phi^{-1} \times \psi^{-1})_q)(\frac{\partial}{\partial z^1}|_q)(f) \\
&= d(\pi_1 \circ (\phi^{-1} \times \psi^{-1})_q)(\frac{\partial}{\partial z^1}|_q)(f) \\
&= \lim_{h \to 0} \frac{(f \circ \pi_1 \circ (\phi^{-1} \times \psi^{-1}))(q + e_1 h) - (f \circ \pi_1 \circ (\phi^{-1} \times \psi^{-1}))(q)}{h} \\
&= \lim_{h \to 0} \frac{(f \circ \pi_1)(\phi^{-1}(q_1 + e_1 h), p_2) - (f \circ \pi_1)(p)}{h} \\
&= \lim_{h \to 0} \frac{f(\phi^{-1}(q_1 + e_1 h)) - f(p_1)}{h} \\
&= \lim_{h \to 0} \frac{f(\phi^{-1}(q_1 + e_1 h)) - f(\phi^{-1}(q_1))}{h} \\
&= (\frac{\partial}{\partial x^1}|_{q_1})(f \circ \phi^{-1}) \\
&= d(\phi^{-1})_{q_1}(\frac{\partial}{\partial x^1}|_{q_1})(f) \\
&= (\frac{\partial}{\partial x^1}|_{p_1})(f).
\end{aligned}
$$

The same result can be shown for the other combinations of $\pi_1, \pi_2$ and $z^1, \cdots, z^{d_1+d_2}$. For any $c_1, \cdots, c_{d_1+d_2} \in \mathbb{R}$,

$$
\begin{aligned}
\alpha(\sum_{i=1}^{d_1+d_2} c_i \frac{\partial}{\partial z^i}|_p) &= \sum_{i=1}^{d_1+d_2} c_i \alpha(\frac{\partial}{\partial z^i}|_p) \\
&= \sum_{i=1}^{d_1+d_2} c_i (d(\pi_1)_p \frac{\partial}{\partial z^i}|_p, d(\pi_2)_p \frac{\partial}{\partial z^i}|_p) \\
&= \sum_{i=1}^{d_1} c_i (d(\pi_1)_p \frac{\partial}{\partial z^i}|_p, d(\pi_2)_p \frac{\partial}{\partial z^i}|_p) + \sum_{i=d_1+1}^{d_2} c_i (d(\pi_1)_p \frac{\partial}{\partial z^i}|_p, d(\pi_2)_p \frac{\partial}{\partial z^i}|_p) \\
&= \sum_{i=1}^{d_1} c_i (\frac{\partial}{\partial x^i}|_{p_1}, 0) + \sum_{i=1}^{d_2} c_{d_1+i} (0, \frac{\partial}{\partial y^i}|_{p_2}) \\
&= (c_1 \frac{\partial}{\partial x^1}|_{p_1} + \cdots + c_{d_1} \frac{\partial}{\partial x^{d_1}}|_{p_1}, c_{d_1+1} \frac{\partial}{\partial y^1}|_{p_2} + \cdots + c_{d_1+d_2} \frac{\partial}{\partial y^{d_2}}|_{p_2}).
\end{aligned}
$$

Therefore, $\alpha$ is bijective. $\square$

**Problem 3-3.** Prove that if $M$ and $N$ are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.

*Proof.* Let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the obvious projections of the corresponding coordinates. $\pi_1, \pi_2$ are clearly smooth, so Proposition 3.21 shows that $d\pi_1 : T(M \times N) \to TM$ and $d\pi_2 : T(M \times N) \to TN$ are both smooth. By (2.16(Proof of Proposition 2.15)(b)), $d\pi_1 \times d\pi_2$ is a smooth map.

By (3-2(Proof of Proposition 3.14)), $d\pi_1 \times d\pi_2$ is a bijection between $T_{(p,q)}(M \times N)$ and $T_p(M) \times T_q(N)$. Since $d\pi_1 \times d\pi_2$ sends $((p,q),\sigma)$ to $(p, d\pi_1(\sigma)) \times (q, d\pi_2(\sigma))$, we conclude that $d\pi_1 \times d\pi_2$ is bijective. $\square$

## 4. Chapter 4: Submersions, Immersions, and Embeddings

**Proposition 4.1.** Suppose $F : M \to N$ is a smooth map and $p \in M$. If $dF_p$ is surjective, then $p$ has a neighborhood $U$ such that $F|_U$ is a submersion. If $dF_p$ is injective, then $p$ has a neighborhood $U$ such that $F|_U$ is an immersion.

*Proof.* Let $(U, \phi)$ be a chart containing $p$ and $(V, \psi)$ be a chart containing $F(p)$. We may assume $F(U) \subset V$. It suffices to show that if the Jacobian of $F$ with respect to $(U, \phi)$ is full rank at $p$, then it is full rank in some neighborhood of $p$ contained in $U$. Example 1.28 in the textbook shows that the set of full rank matrices is an open subset of $M(m \times n, \mathbb{R})$. We will use the notation $J|_q$ to denote the Jacobian of $F$ with respect to $(U, \phi)$ at $q \in U$. Then $J|_p$ is an element of an open subset of $M(m \times n, \mathbb{R})$. Each entry of $J|_q$ is of the from $\frac{\partial}{\partial x^i}(\psi^j \circ F \circ \phi)(\phi(q))$ where each $(\frac{\partial}{\partial x^i}(\psi^j \circ F \circ \phi)) \circ \phi$ is a smooth function. Therefore, there exists a neighborhood of $p$ such that the Jacobian matrix of $F$ with respect to $(U, \phi)$ is full rank. $\square$

**Exercise 4.3(Verification of Example 4.2).** Verify the following claims:

(a) Suppose $M_1, \cdots, M_k$ are smooth manifolds. Then each of the projection maps $\pi_i : M_1 \times \cdots \times M_k \to M_i$ is a smooth submersion.
(b) If $\gamma : J \to M$ is a smooth curve in a smooth manifold $M$ with or without boundary, then $\gamma$ is a smooth immersion if and only if $\gamma'(t) \neq 0$ for all $t \in J$.

*Proof.*

(a) Let $d_1, \cdots, d_k$ denote the dimensions of $M_1, \cdots, M_k$, respectively. Let $M = M_1 \times \cdots \times M_k$.

(2-2(Proof of Proposition 2.12)) implies that $\pi_i$ is smooth for each $i$ by setting $F = \mathrm{Id} : M \to M$. Let $p = (p_1, \cdots, p_k) \in M$. Thus it suffices to show that the dimension of $d(\pi_i)_p(T_p(M))$ is the same as the dimension of $T_{p_i}(M_i)$.

By Proposition 3.12, $\dim(T_p(M)) = \sum d_i$. Since the $\alpha$ defined in (3-2(Proof of Proposition 3.14)) is an isomorphism,

$$\dim(d(\pi_1)_p(T_p(M)) \oplus \cdots \oplus d(\pi_k)_p(T_p(M))) = \dim(T_p(M)) = \sum d_i. \tag{4.1}$$

However, for each $i$, $d(\pi_i)_p(T_p(M)) \subset T_{p_i}M_i$. Thus $\dim(d(\pi_i)_p(T_p(M))) \leq \dim(T_{p_i}M_i) = d_i$. By (4.1), $\dim(d(\pi_i)_p(T_p(M))) = \dim(T_{p_i}M_i)$.

(b) $\gamma$ is a smooth immersion if and only if $d\gamma_t : T_tJ \to T_{\gamma(t)}M$ is injective for each $t \in J$. Since each $T_tJ$ is a 1-dimensional vector space spanned by $d/dt|_t$, $d\gamma_t$ is injective if and only if $d\gamma_t$ sends the basis element to a nonzero element. Finally, $\gamma'(t) = d\gamma(d/dt|_t)$. Therefore, $\gamma$ is a smooth immersion if and only if $\gamma'(t) \neq 0$ for all $t \in J$.

$\square$

**Exercise 4.4.** Show that a composition of smooth submersions is a smooth submersion, and a composition of smooth immersions is a smooth immersion. Give a counterexample to show that a composition of maps of constant rank need not have constant rank.

*Proof.* Let $M, N, L$ be smooth manifolds with or without boundary, and $F : M \to N, G : N \to L$ be given. If $F, G$ are submersions, $dF_p$ and $dG_{F(p)}$ are surjective for each $p$. Then $d(G \circ F)_p = dG_{F(p)} \circ dF_p$ is surjective for each $p$ by (3.7(Proof of Proposition 3.6)). Thus a composition of smooth submersions is a smooth submersion. By the exact same argument, a composition of smooth immersions is a smooth immersion.

Counterexample?

$\square$

**Proposition 4.5.** Suppose $M$ and $N$ are smooth manifolds, and $F : M \to N$ is a smooth map. If $p \in M$ is a point such that $dF_p$ is invertible, then there are connected neighborhoods $U_0$ of $p$ and $V_0$ of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.

*Proof.* Since $dF_p$ is invertible, $\dim(T_pM) = \dim(T_{F(p)}N)$. Let $n = \dim(T_pM)$. By (3.10), $n$ is the dimension of $M$ and $N$. Let $(U, \phi), (V, \psi)$ be smooth charts containing $p, F(p)$, respectively, such that $\phi(p) = \psi(F(p)) = 0 \in \mathbb{R}^n$ and $F(U) \subset V$. Let $\hat{F} = \psi \circ F \circ \phi^{-1}$. Then $\hat{F}$ is a smooth map from an open subset $\hat{U} \subset \mathbb{R}^n$ into an open subset $\hat{V} \subset \mathbb{R}^n$. Then $d\hat{F}|_0 = d\psi_{F(p)} \circ dF_p \circ d\phi_0^{-1}$. Each function on the right hand side is bijective, so $d\hat{F}|_0$ is bijective. Since the differential of a smooth map between Euclidean spaces coincides with the total derivative of the map, we may apply the ordinary inverse function theorem. Thus there exist connected open subsets $\hat{U}_0 \subset \hat{U}$ and $\hat{V}_0 \subset \hat{V}$ both containing 0 such that $\hat{F}$ is a diffeomorphism from $\hat{U}_0$ to $\hat{V}_0$. Since $\phi$ and $\psi$ are homeomorphisms, $U_0$ and $V_0$ are connected neighborhoods of $p$ and $F(p)$ respectively. Finally, since $F = \psi^{-1} \circ \hat{F} \circ \phi$, $F$ is a diffeomorphism from $U_0$ to $V_0$. □

**Proposition 4.6.**

(a) Every composition of local diffeomorphisms is a local diffeomorphism.
(b) Every finite product of local diffeomorphisms between smooth manifolds is a local diffeomorphism.
(c) Every local diffeomorphism is a local homeomorphism and an open map.
(d) The restriction of a local diffeomorphism to an open subsmanifold with or without boundary is a local diffeomorphism.
(e) Every diffeomorphism is a local diffeomorphism.
(f) Every bijective local diffeomorphism is a diffeomorphism.

*Proof.*

(a) Let $L, M, N$ be manifolds with or without boundary. Let $F : L \to M$ and $G : M \to N$ be local diffeomorphisms. Let $p \in L$. Then there exist open sets $U, V$ containing $p, F(p)$, respectively, such that $F(U), G(V)$ are open, and $F|_U, G|_V$ are diffeomorphisms. Let $W = F^{-1}(F(U) \cap V)$. Then $W$ is a neighborhood of $p$ such that $G(F(W)) = G(F(U) \cap V) = G(F(U)) \cap G(V)$, which is open in $N$. Moreover, $(G \circ F)|_W$ is clearly a diffeomorphism because a restriction of a diffeomorphism is a diffeomorphism and the composition of diffeomorphisms is a diffeomorphism.
(b) Let $M_1, \cdots, M_n, N_1, \cdots, N_n$ be $2n$ smooth manifolds and $F_i : M_i \to N_i$ be a local diffeomorphism for each $i = 1, \cdots, n$. Let $M = M_1 \times \cdots \times M_n, N = N_1 \times \cdots \times N_n$ and $F = F_1 \times \cdots \times F_n$. Let $p = (p_1, \cdots, p_n) \in M$ be given. Since each $F_i$ is a local diffeomorphism, there exists an open set $U_i$ containing $p_i$ such that $F_i(U_i)$ is open in $N_i$ and $F_i|_{U_i}$ is a diffeomorphism for each $i$.

    Then $U = U_1 \times \cdots \times U_n$ is an open subset of $M$ containing $p$ and $F(U) = F_1(U_1) \times \cdots \times F_n(U_n)$ is open in $N$. Since $F|_U = F_1|_{U_1} \times \cdots \times F_n|_{U_n}$, $F|_U$ is a diffeomorphism by (2.16(Proof of Proposition 2.15)(b)).
(c) A diffeomorphism is a homeomorphism, so a local diffeomorphism is a local homeomorphism. Let $F : M \to N$ be a local diffeomorphism and an open set $U \subset M$ be given. For every $p \in U$, there exists a neighborhood $U_p$ of $p$ such that $F(U_p)$ is open and $F|_{U_p}$ is a diffeomorphism. $U_p \cap U$ is open in $M$. Since $F|_{U_p}$ is a diffeomorphism, $F|_{U_p}(U_p \cap U) = F(U_p \cap U)$ is open in $F(U_p)$. Since $F(U_p)$ is open, $F(U_p \cap U)$ is open in $N$. Then $F(U_p \cap U) = F(U_p) \cap F(U)$ is open in $N$. Since $F(U) = \cup_{p \in U}(F(U_p) \cap F(U))$, $F(U)$ is open in $N$.
(d) Let $F : M \to N$ be a local diffeomorphism. Let $U \subset M$ be an open submanifold with or without boundary. For every $p \in U$, there exists a neighborhood $U_p$ of $p$ in $M$ such that $F(U_p)$ is open in $N$ and $F|_{U_p}$ is a diffeomorphism. Since $U_p \cap U$ is open in $M$, $F(U_p \cap U)$ is open in $N$. Moreover, $F|_{U_p \cap U}$ is a diffeomorphism. Thus $F|_U$ is a local diffeomorphism.
(e) Let $F : M \to N$ be a diffeomorphism. For every point $p \in M$, the "restriction" of $F$ to $M$ satisfies the definition.
(f) A local diffeomorphism is smooth, so a bijective local diffeomorphism is a diffeomorphism.

□

**Exercise 4.8.** Suppose $M$ and $N$ are smooth manifolds (without boundary), and $F : M \to N$ is a map.

(a) $F$ is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.

(b) If $\dim M = \dim N$ and $F$ is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.

*Proof.* Suppose that $F$ is a local diffeomorphism. Let $p \in M$. Then $p$ has a neighborhood $U$ such that $F|_U$ is a diffeomorphism. Then $d(F|_U)_p$ is an isomorphism by (3.7(Proof of Proposition 3.6)). Clearly, $dF_p = d(F|_U)_p$. Therefore, $dF_p$ is an isomorphism for each $p$. In other words, $F$ is both a smooth immersion and submersion.

Suppose that $F$ is both a smooth immersion and submersion. Then $dF_p$ is injective and surjective for each $p \in M$. Therefore, $dF_p$ is invertible for each $p \in M$. By (4.5), there exist open sets $U, V$ containing $p, F(p)$ such that $F : U \to V$ is a diffeomorphism. This is exactly the definition of a local diffeomorphism.

Since $\dim M = \dim N$, either the injectivity or surjectivity of $dF_p$ implies that $dF_p$ is an isomorphism. Then (b) follows from (a). □

**Proposition 4.13.** Let $M$ and $N$ be smooth manifolds, let $F : M \to N$ be a smooth map, and suppose $M$ is connected. Then the following are equivalent:

(a) For each $p \in M$ there exist smooth charts containing $p$ and $F(p)$ in which the coordinate representation of $F$ is linear.
(b) $F$ has constant rank.

*Proof.* Suppose (a). Let $p \in M$. Then the coordinate representation of $dF$ is linear in some neighborhood $U$ of $p$. This implies that the rank of $dF$ is constant in $U$. Since $M$ is connected, this implies that the rank of $dF$ is constant throughout $M$.

On the other hand, suppose (b). Let $p \in M$. Then the rank theorem guarantees the existence of smooth charts $(U, \phi)$ for $M$ centered at $p$ and $(V, \psi)$ for $N$ centered at $F(p)$ such that $F(U) \subset V$ in which $F$ has a coordinate representation of the form $\hat{F}(x^1, \cdots, x^r, x^{r+1}, \cdots, x^m) = (x^1, \cdots, x^r, 0, \cdots, 0)$. $\hat{F}$ is clearly linear because

$$\begin{aligned}
\hat{F}(c(x^1, \cdots, x^m) + (y^1, \cdots, y^m)) &= \hat{F}(cx^1 + y^1, \cdots, cx^m + y^m) \\
&= (cx^1 + y^1, \cdots, cx^r + y^r, 0, \cdots, 0) \\
&= c(x^1, \cdots, x^r, 0, \cdots, 0) + (y^1, \cdots, y^r, 0, \cdots, 0).
\end{aligned}$$

□

## 5. Appendix A: Review of Topology

**Exercise A.18(Proof of Proposition A.17).** Let $X$ be a topological space and let $S$ be a subspace of $X$.

(a)
(b)
(c)
(d)
(e)
(f) If $\mathcal{B}$ is a basis for the topology of $X$, then $\mathcal{B}_S = \{B \cap S \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on $S$.
(g) If $X$ is Hausdorff, then so is $S$.
(h) If $X$ is first-countable, then so is $S$.
(i) If $X$ is second-countable, then so is $S$.

*Proof.*

(a)
(b)
(c)
(d)
(e)
(f) The union of $B \cap S$ is $S$. Let $U \cap S$ be an open subset of $S$ where $U$ is open in $X$, and $x \in U \cap S$. Then there exists $B \in \mathcal{B}$ such that $x \in B \subset U$ since $\mathcal{B}$ is a basis. Therefore, $x \in B \cap S \subset U \cap S$ with $B \cap S \in \mathcal{B}_S$.

(g) Let $x \neq y \in S$. There exist two disjoint open sets $U, V$ of $X$ containing $x, y$, respectively. Then $U \cap S$ and $V \cap S$ are disjoint open sets of $X$ containing $x, y$, respectively.

(h)

(i) Let $\mathcal{B}$ be a countable basis of $X$. Then $\{B \cap S \mid B \in \mathcal{B}\}$ is a countable basis of $S$ by (f).

$\square$

**Exercise A.24(Proof of Proposition A.23).** Suppose $X_1, \cdots, X_k$ are topological spaces, and let $X_1 \times \cdots \times X_k$ be their product space.

(a) CHARACTERISTIC PROPERTY: If $B$ is a topological space, a map $F : B \to X_1 \times \cdots \times X_k$ is continuous if and only if each of its component functions $F_i = \pi_i \circ F : B \to X_i$ is continuous.

*Proof.*

(a) Suppose $F$ is continuous. Since $\pi_i$ is continuous by (c) and the composition of continuous functions is continuous, $\pi_1 \circ F$ is continuous. Suppose each component function is continuous. Let $B_1 \times \cdots \times B_k$ be a basis element of $X_1 \times \cdots \times X_k$.

$$\begin{aligned} F^{-1}(B_1 \times \cdots \times B_k) &= F^{-1}(\cap_{i=1}^k \pi_i^{-1}(B_1 \times \cdots \times B_k)) \\ &= \cap_{i=1}^k F^{-1}(\pi_i^{-1}(B_1 \times \cdots \times B_k)) \\ &= \cap_{i=1}^k (\pi_i \circ F)^{-1}(B_1 \times \cdots \times B_k). \end{aligned}$$

Since the intersection of finitely many open sets is open, $F$ is continuous.

$\square$

## 6. Appendix B: Review of Linear Algebra

**Exercise B.49.** Two norms $|\cdot|_1$ and $|\cdot|_2$ on a vector space $V$ are said to be equivalent if there are positive constants $c, C$ such that

$$c|v|_1 \leq |v|_2 \leq C|v|_1$$

for all $v \in V$. Show that equivalent norms determine the same topology.

*Proof.* Such a relation is symmetric for $c|v|_1 \leq |v|_2 \leq C|v|_1$ implies $(1/C)|v|_2 \leq |v|_1 \leq (1/c)|v|_2$. Let $\mathcal{T}_1, \mathcal{T}_2$ be the topologies induced by $|\cdot|_1, |\cdot|_2$. It suffices to show that $\forall v \in V, \forall U \in \mathcal{T}_2, (v \in U \implies \exists r > 0, B_1(v, r) \subset U)$ where $B_1(v, r)$ is the open ball centered at $v$ with the radius $r$ using the $|\cdot|_1$. Since $v \in U$ and $U$ is open, $\exists r > 0$ such that $B_2(v, r) \subset U$. Then for any $w \in V$, $|v - w|_1 \leq |v - w|_2/c$, so $B_1(v, r/c) \subset B_2(v, r)$. $\square$

## 7. Appendix C: Review of Calculus

**Exercise C.1.** Suppose that $F : U \to W$ is differentiable at $a \in U$. Show that the linear map satisfying

$$\lim_{v \to 0} \frac{|F(a + v) - F(a) - Lv|}{|v|} = 0$$

is unique.

*Proof.* Let $L, L'$ be two such linear maps.

$$\begin{aligned} \lim_{v \to 0} \frac{|Lv - L'v|}{|v|} &= \lim_{v \to 0} \frac{|(F(a + v) - F(a) - L'v) - (F(a + v) - F(a) - Lv)|}{|v|} \\ &= \lim_{v \to 0} \frac{|F(a + v) - F(a) - Lv|}{|v|} + \lim_{v \to 0} \frac{|F(a + v) - F(a) - L'v|}{|v|} \\ &= 0 + 0 = 0. \end{aligned}$$

If $L \neq L'$, $(L - L')v_0 \neq 0$ for some $v_0$. Then $\lim_{v \to 0} \frac{|Lv - L'v|}{|v|} = \lim_{h \to 0} \frac{|L(hv_0) - L'(hv_0)|}{|hv_0|} = \frac{|(L - L')v_0|}{|v_0|} \neq 0$. This is a contradiction, so $L = L'$. $\square$

# 8. Dictionary

## 8.1. Topological Manifolds.

**Definition 8.1** (Topological Manifold)**.** A *topological $n$-manifold* is a Hausdorff, second-countable topological space each point of which has a neighborhood that is homeomorphic to an open subset $\mathbb{R}^n$.

**Definition 8.2** (Coordinates)**.** Let $M$ be a topological $n$-manifold. Let $U$ be an open subset of $M$, $\hat{U}$ be an open subset of $\mathbb{R}^n$, $\phi : U \to \hat{U}$ be a homeomorphism.

- The pair $(U, \phi)$ is called a *coordinate chart* or a *chart.*
- $U$ is called a *coordinate domain* or a *coordinate neighborhood* and $\phi$ is called a *coordinate map.*
- If $\phi(U)$ is an open ball in $\mathbb{R}^n$, $U$ is called a *coordinate ball.*
- If $\phi(U)$ is an open cube in $\mathbb{R}^n$, $U$ is called a *coordinate cube.*
- The coordinate functions of $\phi$ are often denoted as $(x^1, \cdots, x^n)$. Thus a chart is sometimes denoted by $(U, (x^1, \cdots, x^n))$ or $(U, (x^i))$.

**Definition 8.3** (Atlas)**.** Let $M$ be a topological $n$-manifold. An *atlas for $M$* is a collection of charts $(U_\alpha, \phi_\alpha)$ such that $M = \bigcup_\alpha U_\alpha$.

**Definition 8.4** (Transition Map)**.** Let $M$ be a topological $n$-manifold and $(U, \phi), (V, \psi)$ be coordinate charts such that $U \cap V \neq \emptyset$. $\psi \circ \phi^{-1} : \phi(U \cap V) \mapsto \psi(U \cap V)$ is called a *transition map* from $\phi$ to $\psi$.

**Definition 8.5** (Closed Upper Half-Space)**.** $\mathbb{H}^n = \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$, and $\partial\mathbb{H}^n = \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid x^n = 0\}$.

**Definition 8.6** (Manifold With Boundary)**.** Let $M$ be a second-countable Hausdorff space and fix $n$. Suppose that for every $p \in M$, one of the following conditions is satisfied:

(1) There exists a neighborhood $U$ of $p$ and a homeomorphism $\phi : U \to \hat{U}$ where $\hat{U}$ is an open subset of $\mathbb{R}^n$. $p$ is called an *interior point* and $(U, \phi)$ is called an *interior chart.*
(2) There exists a neighborhood $U$ of $p$ and a homeomorphism $\phi : U \to \hat{U}$ where $\hat{U}$ is an open subset of $\mathbb{H}^n$ with $\phi(p) \in \partial\mathbb{H}^n$. $p$ is called a *boundary point.*

Then $M$ is called an *$n$-dimensional topological manifold with boundary.* Note that every topological manifold is a topological manifold with boundary.

**Definition 8.7** (Support)**.** If $f$ is any real-valued or vector-valued function on a topological space $M$, the *support of $f$*, denoted by $\operatorname{supp} f$, is the closure of the set of points where $f$ is nonzero:

$$\operatorname{supp} f = \overline{\{p \in M : f(p) \neq 0\}}.$$

**Definition 8.8** (Bump Function)**.** If $M$ is a topological space, $A \subset M$ is a closed subset, and $U \subset M$ is an open subset containing $A$, a continuous function $\psi : M \to \mathbb{R}$ is called a *bump function for $A$ supported in $U$* if $0 \leq \psi \leq 1$ on $M$, $\psi \equiv 1$ on $A$, and $\operatorname{supp} \psi \subset U$.

## 8.2. Smooth Manifolds.

**Definition 8.9** (Smoothly Compatible)**.** Let $M$ be a topological $n$-manifold. Two coordinate charts $(U, \phi), (V, \psi)$ are called *smoothly compatible* if $U \cap V = \emptyset$ or the transition map $\psi \circ \phi^{-1}$ is a diffeomorphism.

**Definition 8.10** (Smooth Atlas)**.** Let $M$ be a topological $n$-manifold. A *smooth atlas* is an atlas $\mathcal{A}$ such that any two charts in $\mathcal{A}$ are smoothly compatible with each other.

**Definition 8.11** (Smooth Structure)**.** If $M$ is a topological $n$-manifold, an atlas $\mathcal{A}$ that is not properly contained in any larger smooth atlas is called *maximal* or a *smooth structure on $M$*

**Definition 8.12** (Smooth Manifold)**.** A *smooth manifold* is a topological manifold equipped with a smooth structure.

**Definition 8.13.** Suppose $(M, \mathcal{A})$ is a smooth manifold.

- Any chart $(U, \phi) \in \mathcal{A}$ is called a *smooth chart.*

- Given a smooth chart $(U, \phi)$, $U$ is called a smooth coordinate domain and $\phi$ is called a *smooth coordinate map*.
- Given a smooth chart $(U, \phi)$, $U$ is called a *smooth coordinate ball* if it is a coordinate ball.

**Remark.** *One must define a smooth structure on a topological manifold before talking about a smooth chart.*

**Definition 8.14** (Smooth Maps)**.** Let $M, N$ be smooth manifolds with or without boundary and $F : M \to N$ be a map. $F$ is a *smooth map* if for every $p \in M$, there exist smooth charts $(U, \phi)$ containing $p$ and $(V, \psi)$ containing $F(p)$ such that

- $F(U) \subset V$;
- $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth.

**Definition 8.15** (Coordinate Representatin of a Smooth Map)**.** Let $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ be smooth manifolds. Let $F : M \to N$ be a smooth map and $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ be given. Then $\hat{F} = \psi \circ F \circ \phi^{-1}$ is called the coordinate representation of $F$ with respect to $(U, \phi)$ and $(V, \psi)$.

**Definition 8.16** (Diffeomorphism)**.** Let $M, N$ be smooth manifolds with or without boundary. A diffeomorphism is a smooth map $F : M \to N$ with a smooth inverse.

**Definition 8.17** (Smooth on a subset)**.** Let $M, N$ be smooth manifolds with or without boundary and $A \subset M$ be an arbitrary subset. A map $F : A \to N$ is said to be *smooth on $A$* if every $p \in A$ has an open neighborhood $W \subset M$ such that there exists a smooth map $\tilde{F} : W \to N$ with $\tilde{F}_{W \cap A} = F$.

### 8.3. **Tangent Vectors.**

**Definition 8.18** (Derivation)**.** Let $M$ be a smooth manifold with or without boundary. A derivation at $p \in M$ is a linear map $v : C^\infty(M) \to \mathbb{R}$ such that

$$v(fg) = f(p)vg + g(p)vf$$

for all $f, g \in C^\infty(M)$.

This corresponds to "arrows that are tangent to $M$ and whose basepoints are attached to $M$ at $p$" even though it may not be easy to see that from this definition.

**Definition 8.19** (Tangent Space)**.** The tangent space $T_pM$ to $M$ at $p$ is the vector space of all derivations of $C^\infty(M)$ at $p$.

| | |
|---|---|
| Derivation of $C^\infty(M)$ | Geometric tangent vector on $M$ |
| Differential of a smooth map between manifolds | Total derivative of a map between Euclidean spaces |

**Definition 8.20** (Differential)**.** $M, N$ are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. The *differential of $F$ at $p$* is the linear map $dF_p : T_pM \to T_{F(p)}N$ defined by

$$dF_p(v) := f \mapsto v(f \circ F)$$

Equivalently, $\forall v \in T_pM, \forall f \in C^\infty(N), dF_p(v)(f) = v(f \circ F)$. This corresponds to "the directional derivative of $F$ at $p$ in the direction of the arrow $v$."

**Definition 8.21** (Coordinate Vectors)**.** Let $(M, \mathcal{A})$ be a smooth manifold without boundary. Let $p \in M$ and choose a chart $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Then the *coordinate vectors at $p$*, denoted by $\frac{\partial}{\partial x^i}|_p$, are derivations $C^\infty(U) \to \mathbb{R}$ such that

$$\frac{\partial}{\partial x^i}\Big|_p := f \mapsto \frac{\partial}{\partial x^i}\Big|_{\phi(p)}(f \circ \phi^{-1}).$$

**Definition 8.22** (Tangent Bundle)**.** Let $M$ be a smooth manifold with or without boundary. The tangent bundle of $M$, denoted by $TM$, is the disjoint union $\coprod_{p \in M} T_pM$.

**Definition 8.23** (Projection Map)**.** Let $M$ be a smooth manifold with or without boundary. The projection map $\pi : TM \to M$ is the map defined by $(p, v) \mapsto p$.

**Definition 8.24** (Curve)**.** If $M$ is a manifold with or without boundary, we define a *curve in* $M$ to be a continuous map $\gamma : J \to M$ where $J \subset \mathbb{R}$ is an interval.

**Definition 8.25** (Velocity of a curve)**.** Let $\gamma : J \to M$ and $t_0 \in J$ be given. The *velocity of* $\gamma$ *at* $t_0$, denoted by $\gamma'(t_0)$ is the vector

$$\gamma'(t_0) = d\gamma(\frac{d}{dt}\Big|_{t_0}) \in T_{\gamma(t_0)}M,$$

where $d/dt|_{t_0}$ is the standard coordinate basis vector in $T_{t_0}\mathbb{R}$.

8.4. **Submersions, Immersions, and Embeddings.**

**Definition 8.26** (Rank)**.** Let $M, N$ be smooth manifolds with or without boundary and let $F : M \to N$ be a smooth map. Then the rank of $F$ at $p \in M$ is:

- The rank of the linear map $dF_p : T_pM \to T_{F(p)}N$.
- The dimension of the subspace $dF_p(T_pM)$ in the vector space $T_{F(P)}N$.

It is easy to see that the two definitions above are always equivalent.

**Definition 8.27** (Submersions and Immersions)**.** Let $M, N$ be smooth manifolds with or without boundary and let $F : M \to N$ be a smooth map.

- If $F$ has the same rank at every point $p \in M$, then $F$ is said to have *constant rank*, and the rank is denoted by $\operatorname{rank} F$.
- If the rank of $F$ at $p \in M$ is equal to $\max\{\dim M, \dim N\}$, then $F$ is said to have *full rank at* $p$.
- If $F$ has full rank everywhere, then $F$ is said to have *full rank*.
- If $F$ has constant rank and $\operatorname{rank} F = \dim N$, $F$ is called a *smooth submersion*.
- If $F$ has constant rank and $\operatorname{rank} F = \dim M$, $F$ is called a *smooth immersion*.

**Definition 8.28** (Local Diffeomorphisms)**.** Let $M, N$ be smooth manifolds with or without boundary, a map $F : M \to N$ is called a local diffeomorphism if every point $p \in M$ has a neighbhorhood $U$ such that $F(U)$ is open in $N$ and $F|_U : U \to F(U)$ is a diffeomorphism.