INTRODUCTION TO SMOOTH MANIFOLDS

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Abstract. I am not good at smooth manifolds, so I decided to read and solve problems from Introduction to Smooth Manifolds.

Contents

1. Chapter 1

Exercise 1.1. Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to *any* open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

Proof. It is clear that a "manifold" satisfying the open-ball or $\mathbb{R}^n$ definition satisfies the open-subset definition. Let M be a manifold satisfying the open-subset definition. Let $x \in M$ be given and let $U, \hat{U}, \phi$ be given according to the definition. Since $\hat{U}$ is open, there exists an open ball B such that $\phi(x) \in B \subset \hat{U}$. Restrict ϕ to $\phi^{-1}(B)$. Then $\phi^{-1}(B)$ is an open subset of M containing x, and $\phi \mid_{\phi^{-1}(B)}$ is a homeomorphism between $\phi^{-1}(B)$ and B. Thus M satisfies the open-ball definition.

$B(x, r) \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ by the map $(x_1 + a_1, \cdots, x_n + a_n) \mapsto (\frac{a_1}{r-a_1}, \cdots, \frac{a_n}{r-a_n})$ where $x = (x_1, \cdots, x_n)$ is the center of $B(x, r)$ and r is the radius. Since the composition of two homeomorphisms gives a homeomorphism, M also satisfies the $\mathbb{R}^n$ definition as well. $\square$

Exercise 1.6. Showt that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.

Proof. Let $[x], [y] \in \mathbb{RP}^n$ be given. Without loss of generality, assume $x, y \in S^{n-1}$. Let $r = \min\{|x - y|, |x + y|\}/2$. Then $q(B(x, r)), q(B(y, r))$ contain $[x], [y]$, respectively. $q^{-1}(U_x), q^{-1}(U_y)$ are both open in $\mathbb{R}^n$, so U_x, U_y are both open in $\mathbb{RP}^n$. Let $[z] \in U_x \cap U_y$. Then $z \in q^{-1}(\{[z]\}) \subset q^{-1}(U_x \cap U_y) = q^{-1}(U_x) \cap q^{-1}(U_y) = \{0\}$, but $z \neq 0$ because $q : \mathbb{R}^n \setminus \{0\} \to \mathbb{RP}^n$. Therefore, U_x, U_y are disjoint, so $\mathbb{RP}^n$ is Hausdorff.

Show $\mathbb{RP}^n$ is second countable.

$\square$