

# INTRODUCTION TO SMOOTH MANIFOLDS

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## CONTENTS

1. Chapter 1: Smooth Manifolds	1
1.1. Exercises	1
1.2. Problems	4
2. Chapter 2: Smooth Maps	6
2.1. Exercises	6
2.2. Problems	9
3. Chapter 3: Tangent Vectors	12
3.1. Exercises	12
3.2. Problems	18
4. Chapter 4: Submersions, Immersions, and Embeddings	20
5. Appendix A: Review of Topology	20
6. Appendix B: Review of Linear Algebra	21
7. Appendix C: Review of Calculus	22
8. Dictionary	22
8.1. Topological Manifolds	22
8.2. Smooth Manifolds	23
8.3. Tangent Vectors	23
8.4. Submersions, Immersions, and Embeddings	24

## 1. CHAPTER 1: SMOOTH MANIFOLDS

### 1.1. Exercises.

**Exercise 1.1.** Show that equivalent definitions of manifolds are obtained if instead of allowing  $U$  to be homeomorphic to *any* open subset of  $\mathbb{R}^n$ , we require it to be homeomorphic to an open ball in  $\mathbb{R}^n$ , or to  $\mathbb{R}^n$  itself.

*Proof.* It is clear that a “manifold” satisfying the open-ball or  $\mathbb{R}^n$  definition satisfies the open-subset definition. Let  $M$  be a manifold satisfying the open-subset definition. Let  $x \in M$  be given and let  $U, \hat{U}, \phi$  be given according to the definition. Since  $\hat{U}$  is open, there exists an open ball  $B$  such that  $\phi(x) \in B \subset \hat{U}$ . Restrict  $\phi$  to  $\phi^{-1}(B)$ . Then  $\phi^{-1}(B)$  is an open subset of  $M$  containing  $x$ , and  $\phi|_{\phi^{-1}(B)}$  is a homeomorphism between  $\phi^{-1}(B)$  and  $B$ . Thus  $M$  satisfies the open-ball definition.

$B(x, r) \subset \mathbb{R}^n$  is homeomorphic to  $\mathbb{R}^n$  by the map  $(x_1 + a_1, \dots, x_n + a_n) \mapsto (\frac{a_1}{r-a_1}, \dots, \frac{a_n}{r-a_n})$  where  $x = (x_1, \dots, x_n)$  is the center of  $B(x, r)$  and  $r$  is the radius. Since the composition of two homeomorphisms gives a homeomorphism,  $M$  also satisfies the  $\mathbb{R}^n$  definition as well.  $\square$

**Exercise 1.6.** Show that  $\mathbb{RP}^n$  is Hausdorff and second-countable, and is therefore a topological  $n$ -manifold.

*Proof.* From the definition of  $\pi$ , it is easy to see that  $\pi(B(x, r))$  is open in  $\mathbb{RP}^n$  where  $x \in S^n$  and  $0 < r < 1$ .

Let  $[x], [y] \in \mathbb{RP}^n$  be given. Without loss of generality, assume  $x, y \in S^n$ . Let  $r = \min\{|x - y|, |x + y|, 1\}/2$ . Then  $U_x = \pi(B(x, r)), U_y = \pi(B(y, r))$  contain  $[x], [y]$ , respectively.  $\pi^{-1}(U_x), \pi^{-1}(U_y)$  are both open in  $\mathbb{R}^{n+1} \setminus \{0\}$  which can be seen easily by writing down exactly which points belong to them, so  $U_x, U_y$  are both open in  $\mathbb{RP}^n$ . Then  $\pi^{-1}(U_x \cap U_y) = \pi^{-1}(U_x) \cap \pi^{-1}(U_y) = \emptyset$ , so  $U_x \cap U_y = \emptyset$ . Therefore,  $\mathbb{RP}^n$  is Hausdorff.

Let  $\mathcal{B} = \{\pi(B(x, 1/k)) \mid x \in \mathbb{Q}^{n+1} \cap S^n, k \in \{2, 3, 4, \dots\}\}$ . Then  $\mathcal{B}$  is a countable collection of open sets whose union is  $\mathbb{RP}^n$ . Let  $U \subset \mathbb{RP}^n$  be a nonempty open set. Let  $[x] \in U$ . Since  $\pi$  is a quotient map,  $\pi^{-1}(U)$  is open. Moreover,  $x \in \pi^{-1}(U)$ . Without loss of generality,  $x \in S^n$ . Then  $x \in B(x', 1/k) \subset \pi^{-1}(U)$  for some  $B(x', 1/k) \in \mathcal{B}$ . Then  $[x] = \pi(x) \in \pi(B(x', 1/k)) \subset \pi(\pi^{-1}(U)) = U$ . Therefore,  $\mathcal{B}$  is a countable basis of  $\mathbb{RP}^n$ .  $\square$

**Exercise 1.7.** Show that  $\mathbb{RP}^n$  is compact.

*Proof.*  $\pi(S^n) = \mathbb{RP}^n$  and  $S^n$  is compact because it is a closed, bounded subset of  $\mathbb{R}^{n+1}$ . (Heine-Borel) Moreover, the image of a compact set under a continuous map is compact. (See A.45(a)) Thus  $\mathbb{RP}^n$  is compact.  $\square$

**Exercise 1.14.** Suppose  $\mathcal{X}$  is a locally finite collection of subsets of a topological space  $M$ .

- (a) The collection  $\{\overline{X} : X \in \mathcal{X}\}$  is also locally finite.
- (b)  $\overline{\bigcup_{X \in \mathcal{X}} X} = \bigcup_{X \in \mathcal{X}} \overline{X}$ .

*Proof.*

- (a) Let  $p \in M$ . Then there exists an open set  $U$  containing  $x$  such that there are only finitely many  $X \in \mathcal{X}$  such that  $U \cap X \neq \emptyset$ . Let  $X \in \mathcal{X}$ .
  - If  $U \cap X \neq \emptyset$ , then  $U \cap \overline{X} \supset U \cap X \neq \emptyset$ .
  - If  $U \cap X = \emptyset$ , then  $U^c$  is closed, so  $\overline{X} \subset U^c$ . In other words,  $U \cap \overline{X} = \emptyset$ .
This shows that the number of  $X \in \mathcal{X}$  that intersects  $U$  and the number of  $\overline{X} \in \mathcal{X}$  that intersects  $U$  are the same. Therefore,  $\{\overline{X} : X \in \mathcal{X}\}$  is also locally finite.
- (b) Since the closure of a set is defined to be the intersection of all closed sets containing it,  $\bigcup_{X \in \mathcal{X}} \overline{X} \subset \overline{\bigcup_{X \in \mathcal{X}} X}$ . Let  $x \notin \bigcup_{X \in \mathcal{X}} \overline{X}$ . Then there exists a neighborhood  $U$  of  $x$  such that  $U$  intersects only finitely many  $X \in \mathcal{X}$ . Let  $X_1, \dots, X_n$  denote them. By the same argument as part (a),  $\overline{X_1}, \dots, \overline{X_n}$  are the only elements in  $\{\overline{X} \mid X \in \mathcal{X}\}$  that  $U$  intersects. Since  $x \notin \overline{X_i}$  for each  $i = 1, \dots, n$ ,  $U^c \cup \overline{X_1} \cup \dots \cup \overline{X_n}$  is a closed set which contains all  $X \in \mathcal{X}$  but does not contain  $x$ . In other words,  $x \notin \overline{\bigcup_{X \in \mathcal{X}} X}$ .  $\square$

**Exercise 1.18.** Let  $M$  be a topological manifold. Two smooth atlases for  $M$  determine the same smooth structure if and only if their union is a smooth atlas.

*Proof.* Let  $\mathcal{A}, \mathcal{A}'$  be two smooth atlases.

Suppose that they determine the same smooth structure  $\mathcal{B}$ . Then  $\mathcal{A} \cup \mathcal{A}' \subset \mathcal{B}$ , so  $\mathcal{A} \cup \mathcal{A}'$  must be a smooth atlas. By Proposition 1.17(a),  $\mathcal{A} \cup \mathcal{A}'$  determines a unique smooth structure, but it must be  $\mathcal{B}$  because  $\mathcal{B}$  contains the union.

On the other hand, suppose that their union is a smooth atlas. Let  $\mathcal{B}$  be the smooth structure that the union determines. Such  $\mathcal{B}$  must exist by Proposition 1.17(a). By the same proposition,  $\mathcal{A}, \mathcal{A}'$  must determine the unique smooth structures. However, they must be  $\mathcal{B}$  because  $\mathcal{B}$  contains both  $\mathcal{A}$  and  $\mathcal{A}'$ .  $\square$

**Exercise 1.20.** Every smooth manifold has a countable basis of regular coordinate balls.

*Proof.* Let  $M$  be an  $n$ -dimensional smooth manifold. We consider the special case that there exists a single chart  $(\phi, U)$  with  $U = M$ . Let  $x \in \hat{U}$  with rational coordinates. Then there exists  $s > 0$  such that  $B(x, s) \subset \hat{U}$ . For each rational number  $r \in (0, s)$ , we consider the chart  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r)))$ .

Let  $\mathcal{B}$  be the collection of all such charts for each  $x \in \hat{U}$  and  $r$ . We claim that  $\mathcal{B}$  is a smooth atlas.

- Let  $p \in M$ . Then  $\phi(p) \in \hat{U}$ . Since  $\hat{U}$  is open,  $\phi(p) \in B(x, r) \subset \hat{U}$  for some  $x$  with rational coordinates and a positive rational number  $r$ . Then  $p \in \phi^{-1}(B(x, r))$ , so the union of coordinate domains covers  $M$ . In other words,  $\mathcal{B}$  is an atlas.
- Let  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r))), (p \mapsto \phi(p) - x', \phi^{-1}(B(x', r')))) \in \mathcal{B}$  be given. Suppose  $\phi^{-1}(B(x, r)) \cap \phi^{-1}(B(x', r')) \neq \emptyset$ . Let  $\psi, \psi'$  denote the coordinate maps. Then  $\psi' \circ \psi^{-1}$  is a composition of  $\phi, \phi^{-1}$  and translation maps, so it is smooth.

Therefore,  $\mathcal{B}$  is a smooth atlas.

Since  $\mathcal{B}$  is a smooth atlas, there exists a smooth structure  $\mathcal{A}$  on  $M$  containing  $\mathcal{B}$  by Proposition 1.17(a). We claim that  $\mathcal{B}$ , a subset of the smooth structure  $\mathcal{A}$ , is a countable basis of regular coordinate balls.

- $\mathcal{B}$  is a countable collection because  $x \in \mathbb{Q}^n$  and  $r \in \mathbb{Q}$ .
- Let  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r))) \in \mathcal{B}$  be given. Then there exists a chart  $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r')))$  in  $\mathcal{B}$  with  $r' > r$ . Let  $B = \phi^{-1}(B(x, r))$ ,  $B' = \phi^{-1}(B(x, r'))$ . Let  $\psi$  denote the map  $p \mapsto \phi(p) - x$ . Then  $\psi(B) = B(0, r)$  and  $\psi(B') = B(0, r')$ , respectively. Moreover,  $\psi(\overline{B}) = \overline{B(0, r)}$  because  $\psi$  is a homeomorphism.

Now let  $M$  be an arbitrary smooth  $n$ -manifold. By definition, each point of  $M$  is in the domain of a chart. By Proposition A.16,  $M$  is covered by countably many charts  $\{(U_i, \phi_i)\}$ . By the previous argument, each  $U_i$  has a countable basis of regular coordinate balls. Each regular coordinate ball in  $U_i$  is indeed a regular coordinate ball in  $M$  because  $\overline{B}$  is a compact subset of  $M$ , which is Hausdorff, so  $\overline{B}$  is closed. In other words, the closure of  $B$  in  $U_i$  is the same as the closure of  $B$  in  $M$ .  $\square$

**Exercise 1.39.** Let  $M$  be a topological  $n$ -manifold with boundary.

- Int  $M$  is an open subset of  $M$  and a topological  $n$ -manifold without boundary.
- $\partial M$  is a closed subset of  $M$  and a topological  $(n - 1)$ -manifold without boundary.
- $M$  is a topological manifold if and only if  $\partial M = \emptyset$ .
- If  $n = 0$ , then  $\partial M = \emptyset$  and  $M$  is a 0-manifold.

*Proof.*

- Let  $x \in \text{Int } M$ . Let  $(\phi, U)$  be an interior chart for  $x$ . Then  $x \in U \subset \text{Int } M$  because every point in  $U$  is in an interior chart  $(\phi, U)$ . A subspace of  $M$  must be Hausdorff and second-countable by Proposition A.17(g, i), so  $\text{Int } M$  is a second-countable, Hausdorff space in which every point has a neighborhood homeomorphic to an open subset in  $\mathbb{R}^n$ . Thus  $\text{Int } M$  is an  $n$ -manifold without boundary.
- Since  $\partial M = M \setminus \text{Int } M$  and  $\text{Int } M$  is open in  $M$ ,  $\partial M$  is closed in  $M$ . Let  $x \in \partial M$ . Let  $(\phi, U)$  be a boundary chart of  $x$ . If a point  $y \in U$  gets mapped into  $\text{Int } \mathbb{H}^n$ , then it is certainly an interior point. Thus  $\phi(U \cap \partial M) \subset \partial \mathbb{H}^n$ . Then  $\pi_{n-1} \circ \phi$  is a homeomorphism that maps  $U \cap \partial M$  into an open subset of  $\mathbb{R}^{n-1}$  where  $\pi_{n-1} : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$ .
- If  $\partial M$  is empty, then  $M = \text{Int } M$ , so (a) implies that  $M$  is an  $n$ -dimensional manifold. If  $M$  is a topological manifold, every point is an interior point. Since a point cannot be both an interior point and a boundary point,  $\partial M$  is empty.
- If  $n = 0$ , then  $\partial \mathbb{H}^0 = \emptyset$ . Thus, the condition that  $\phi(U) \cap \partial \mathbb{H}^n \neq \emptyset$  can never be satisfied, so there cannot be any boundary point.  $\square$

**Exercise 1.41.** Let  $M$  be a topological manifold with boundary.

- $M$  has a countable basis of precompact coordinate balls and half-balls.
- $M$  is locally compact.
- $M$  is paracompact.
- $M$  is locally path-connected.
- $M$  has countably many components, each of which is an open subset of  $M$  and a connected topological manifold with boundary.
- The fundamental group of  $M$  is countable.

*Proof.*

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- 
- 
- Let  $U \subset M$  be a nonempty open subset and choose  $x \in U$ . Then there exists a chart  $(V, \phi)$  such that  $x \in V$ . Since  $\phi(x)$  is a point in an open set  $\phi(U \cap V)$ , there exists  $r > 0$  such that  $B(\phi(x), r) \subset \phi(V)$ . Then  $N(x, U) = \phi^{-1}(B(\phi(x), r))$  is a path-connected neighborhood of  $x$  that is contained in  $U \cap V \subset U$ . Therefore,  $\{N(x, U) \mid \text{open } U \subset M, x \in U\}$  forms a basis of  $M$  consisting of path-connected sets.

- (e)
- (f)

□

**Exercise 1.44.** Suppose  $M$  is a smooth  $n$ -manifold with boundary and  $U$  is an open subset of  $M$ . Prove the following statements:

- (a)  $U$  is a topological  $n$ -manifold with boundary, and the atlas consisting of all smooth charts  $(V, \phi)$  for  $M$  such that  $V \subset U$  defines a smooth structure on  $U$ . With this topology and smooth structure,  $U$  is called an **open submanifold with boundary**.
- (b) If  $U \subset \text{Int } M$ , then  $U$  is actually a smooth manifold (without boundary); in this case we call it an **open submanifold of  $M$** .
- (c)  $\text{Int } M$  is an open submanifold of  $M$  (without boundary).

*Proof.* Let  $\mathcal{T}$  denote the topology of  $M$  and  $\mathcal{A}$  denote the smooth structure of  $M$ .

- (a) The subspace topology on  $U$  is equivalent to  $\mathcal{T}_U = \{V \in \mathcal{T} \mid V \subset U\}$  because  $U$  is open. By Proposition A.17(A.18(Proof of Proposition A.17)),  $U$  is Hausdorff and second-countable. For every point  $p \in U$ , there exists a  $V \in \mathcal{T}$  with a homeomorphism  $\phi : V \rightarrow \hat{V}$  where  $\hat{V}$  is an open subset of  $\mathbb{R}^n$  (or  $\mathbb{H}^n$ ). Since  $U \cap V$  is an open subset of  $V$ ,  $\phi$  restricted to  $U \cap V$  is a homeomorphism between  $U \cap V$  and  $\phi(U \cap V)$ , which is an open subset of  $\mathbb{R}^n$  (or  $\mathbb{H}^n$ ). Therefore,  $U$  is a topological  $n$ -manifold with boundary.

Let  $\mathcal{A}_U = \{(\phi, V) \in \mathcal{A} \mid V \subset U\}$ . Then  $\mathcal{A}_U$  is clearly a collection of charts on  $U$  whose union covers  $U$ . Moreover, any two charts in  $\mathcal{A}_U$  are clearly smoothly compatible. Let  $(\phi, V)$  be a chart on  $U$  that is smoothly compatible with every chart in  $\mathcal{A}_U$ . Let  $(\psi, W) \in \mathcal{A}$ . Then  $(\psi_{W \cap U}, W \cap U)$  is a chart on  $M$  and it must be smoothly compatible with every chart in  $\mathcal{A}$ . Therefore,  $(\psi_{W \cap U}, W \cap U) \in \mathcal{A}$ , so it must belong to  $\mathcal{A}_U$ . This implies that  $(\phi, V)$  and  $(\psi_{W \cap U}, W \cap U)$  are smoothly compatible. Since  $V \subset W \cap U$ , this implies that  $(\phi, V)$  and  $(\psi, W)$  are smoothly compatible.

Thus  $(\phi, V)$  is smoothly compatible with every chart in  $\mathcal{A}$ , so  $(\phi, V) \in \mathcal{A}$ . This implies that  $(\phi, V)$  is in  $\mathcal{A}_U$ , so  $\mathcal{A}_U$  is indeed a maximal smooth atlas.

- (b) Let  $p \in U$ . Then  $p \in \text{Int } M$ , so there exists  $(\phi, V) \in \mathcal{A}$  such that  $p \in V$  and  $\phi(V)$  is open in  $\mathbb{R}^n$ . Then  $(\phi|_{V \cap U}, V \cap U)$  is a chart that is smoothly compatible with every chart in  $\mathcal{A}$ , so  $(\phi|_{V \cap U}, V \cap U) \in \mathcal{A}$ . Thus it must be in  $\mathcal{A}_U$ , so  $p \in U$  is an interior point of  $U$ . Therefore,  $U$  is a manifold without boundary.
- (c) By 1.39,  $\text{Int } M$  is an open subset of  $M$ . By (b),  $\text{Int } M$  is an open submanifold of  $M$  without boundary.

□

## 1.2. Problems.

**Problem 1-2.** Show that a disjoint union of uncountably many copies of  $\mathbb{R}$  is locally Euclidean and Hausdorff, but not second-countable.

*Proof.* Let  $I$  denote an uncountable index set and  $X = \coprod_{\alpha \in I} \mathbb{R}$ . Let  $(x, \alpha_0) \in X$ . Define  $U = \coprod_{\alpha \in I} U_\alpha$  where  $U_{\alpha_0} = \mathbb{R}$  and  $U_\alpha = \emptyset$  when  $\alpha \neq \alpha_0$ . Then  $U$  is an open neighborhood of  $(x, \alpha_0)$  that is clearly homeomorphic to  $\mathbb{R}$ . Thus  $X$  is locally Euclidean.

Let  $(x_1, \alpha_1) \neq (x_2, \alpha_2) \in X$ . If  $\alpha_1 \neq \alpha_2$ , then open neighborhoods of  $x_1$  and  $x_2$  formed in the same way as above separate the two points. Suppose  $\alpha_1 = \alpha_2$ . Without loss of generality,  $x_1 < x_2$ . Define  $U = \coprod_{\alpha \in I} U_\alpha$  where  $U_{\alpha_1} = (-\infty, (x_1 + x_2)/2)$  and  $U_\alpha = \emptyset$  when  $\alpha \neq \alpha_1$ . Similarly, define  $V = \coprod_{\alpha \in I} U_\alpha$  where  $U_{\alpha_1} = ((x_1 + x_2)/2, \infty)$  and  $U_\alpha = \emptyset$  when  $\alpha \neq \alpha_1$ . Then such  $U$  and  $V$  separate the two points. Therefore,  $X$  is Hausdorff.

Let  $\mathcal{B}$  be a basis of  $X$ . For each  $\alpha_0 \in I$ , let  $U_{\alpha_0} = \coprod_{\alpha \in I} U_\alpha$  where  $U_{\alpha_0} = \mathbb{R}$  and  $U_\alpha = \emptyset$  when  $\alpha \neq \alpha_0$ . Then for each  $\alpha_0$ , there must exist  $B_{\alpha_0} \in \mathcal{B}$  such that  $(0, \alpha_0) \in B_{\alpha_0} \subset U_{\alpha_0}$ . Clearly,  $B_\alpha \neq B_\beta$  if  $\alpha \neq \beta$ . Therefore, the cardinality of  $\mathcal{B}$  is greater than or equal to that of  $I$ . Hence,  $X$  is not second-countable. □

**Problem 1-7.** Let  $N$  denote the **north pole**  $(0, \dots, 0, 1) \in S^n \subset \mathbb{R}^{n+1}$ , and let  $S$  denote the **south pole**  $(0, \dots, 0, -1)$ . Define the **stereographic projection**  $\sigma : S^n \setminus \{N\} \rightarrow \mathbb{R}^n$  by

$$\sigma(x^1, \dots, x^{n+1}) = \frac{(x^1, \dots, x^n)}{1 - x^{n+1}}$$

Let  $\tilde{\sigma}(x) = -\sigma(-x)$  for  $x \in S^n \setminus \{S\}$ .

- (a) For any  $x \in S^n \setminus \{N\}$ , show that  $\sigma(x) = u$ , where  $(u, 0)$  is the point where the line through  $N$  and  $x$  intersects the linear subspace where  $x^{n+1} = 0$ . Similarly, show that  $\tilde{\sigma}(x)$  is the point where the line through  $S$  and  $x$  intersects the same subspace.
- (b) Show that  $\sigma$  is bijective, and

$$\sigma^{-1}(u^1, \dots, u^n) = \frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

- (c) Compute the transition map  $\tilde{\sigma} \circ \sigma^{-1}$  and verify that the atlas consisting of the two charts  $(S^n \setminus \{N\}, \sigma)$  and  $(S^n \setminus \{S\}, \tilde{\sigma})$  defines a smooth structure on  $S^n$ .
- (d) Show that this smooth structure is the same as the one defined in Example 1.31.



FIGURE 1. Problem 1-7

*Proof.*

- (a) This is trivial from a basic trigonometry argument using the triangles  $N, (0, \dots, 0, x^{n+1}), (x^1, \dots, x^{n+1})$  and  $N, (0, \dots, 0), \sigma(x^1, \dots, x^{n+1})$ .
- (b)  $\sigma \circ \sigma^{-1}$  and  $\sigma^{-1} \circ \sigma$  are both the identity maps, so  $\sigma$  is bijective and  $\sigma^{-1}$  is its inverse.
- (c) Computation shows that  $\tilde{\sigma} \circ \sigma^{-1} : S^n \setminus \{N, S\} \rightarrow S^n \setminus \{N, S\}$  sends  $(u^1, \dots, u^n)$  to  $(u^1, \dots, u^n)/|u|^2$ . As  $|u| \neq 0$  in the domain, this map is well-defined and clearly smooth. By Proposition 1.17(a), these two charts determine a unique smooth structure.
- (d)  $\phi_i, \sigma, \tilde{\sigma}$  are all smooth functions of subsets of Euclidean spaces, so transition maps are always smooth. By Proposition 1.17(b), the smooth structure determined by  $\sigma, \tilde{\sigma}$  is the same as the one defined in Example 1.31.

□

**Problem 1-8.** By identifying  $\mathbb{R}^2$  with  $\mathbb{C}$ , we can think of the unit circle  $S^1$  as a subset of the complex plane. An angle function on a subset  $U \subset S^1$  is a continuous function  $\theta : U \rightarrow \mathbb{R}$  such that  $e^{i\theta(z)} = z$  for all  $z \in U$ .

Show that there exists an angle function  $\theta$  on an open subset  $U \subset S^1$  if and only if  $U \neq S^1$ . For any such angle function, show that  $(U, \theta)$  is a smooth coordinate chart for  $S^1$  with its standard smooth structure.

*Proof.* First, we will consider the special case when  $U = S^1 \setminus \{e^{it}\}$  for some  $t \in \mathbb{R}$ . The map  $\phi : (t, t+2\pi) \rightarrow U$  defined by  $\theta \mapsto e^{i\theta}$  is a bijective function. Therefore, by taking the inverse of  $\phi$ , which is clearly continuous, we obtain a desired angle function. The case of an arbitrary proper open subset of  $U$  is the same as this special case because we simply need to restrict the domain of the map obtained above. On the other hand, suppose  $U = S^1$ . Suppose there exists an angle function  $f$  on  $U$ . Define  $g : S^1 \rightarrow \mathbb{R}$  by  $g(z) = f(z) - f(-z)$ .

- $g(1) \neq 0$  because  $g(1) \neq 0 \implies f(1) = f(-1)$ , which is clearly impossible.
- $g(1) > 0$  implies that  $g(-1) < 0$ . By the intermediate value theorem,  $g(z) = 0$  for some  $z \in S^1$ . This is a contradiction.
- If  $g(1) < 0$ ,  $g(-1) > 0$ , and we obtain a contradiction in the same manner.

Therefore, such an  $f$  cannot exist. Hence, an angle function exists if and only if  $U$  is a proper open subset of  $S^1$ .

Let  $(U_i^\pm, \phi_i^\pm)$  and  $(U, \phi)$  be given where  $\phi$  maps  $U$  into  $(t, t+2\pi)$  for some  $t \in \mathbb{R}$ . We will show that they are smoothly compatible. Let  $V = U \cap U_i^\pm$ . The map  $\phi_i^\pm \circ \phi^{-1} : \phi(V) \rightarrow \phi_i^\pm(V)$  is  $\phi_i^\pm \circ \exp$ . Since it is a composition of a projection map with a smooth map, this is smooth. Therefore,  $(U, \phi)$  is indeed a coordinate chart for  $S^1$  with its standard smooth structure.  $\square$

**Problem 1-12(Proof of Proposition 1.45).** Suppose  $M_1, \dots, M_k$  are smooth manifolds and  $N$  is a smooth manifold with boundary. Then  $M_1 \times \dots \times M_k \times N$  is a smooth manifold with boundary, and  $\partial(M_1 \times \dots \times M_k \times N) = M_1 \times \dots \times M_k \times \partial N$ .

*Proof.* By Example 1.34,  $M_1 \times \dots \times M_k$  is a smooth manifold. Thus it suffices to show that  $M \times N$  is a smooth manifold with boundary if  $M$  is a smooth manifold and  $N$  is a smooth manifold with boundary. Let  $m, n$  be the dimensions of  $M, N$ .

First, we show that  $M \times N$  is a topological manifold with boundary and  $\partial(M \times N) = M \times \partial N$ . Let  $(p, q) \in M \times N$ . Then  $p \in M$ , so there exists a chart  $(U, \phi)$  such that  $p \in U$  and  $\hat{U} = \phi(U) \subset \mathbb{R}^m$ .

- Suppose  $q \in \text{Int } N$ . Then there exists a chart  $(V, \psi)$  such that  $\hat{V} = \psi(V) \subset \mathbb{R}^n$ .  $\phi \times \psi$  is a homeomorphism between  $U \times V$  and  $\hat{U} \times \hat{V} \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ . Thus  $(U \times V, \phi \times \psi)$  is a chart for  $(p, q)$ .
- Suppose  $q \in \text{bd } N$ . Then there exists a chart  $(V, \psi)$  such that  $\hat{V} = \psi(V) \subset \mathbb{H}^n$  and  $\psi(q) \in \partial \mathbb{H}^n$ .  $\phi \times \psi$  is a homeomorphism between  $U \times V$  and  $\hat{U} \times \hat{V} \subset \mathbb{R}^m \times \mathbb{H}^n = \mathbb{H}^{m+n}$ . Moreover,  $(\phi \times \psi)(p, q) = (\phi(p), \psi(q)) \in \mathbb{R}^m \times \mathbb{H}^n = \mathbb{H}^{m+n}$ . Thus  $(U \times V, \phi \times \psi)$  is a boundary chart for  $(p, q)$ .

Therefore,  $M \times N$  is a topological manifold with boundary and  $\partial(M \times N) = M \times (\partial N)$ .

Let  $\mathcal{A}_M, \mathcal{A}_N$  be the smooth structures of  $M, N$ . Define  $\mathcal{A}_{M \times N} = \{(U \times V, \phi \times \psi) \mid (U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N\}$ . Then  $\mathcal{A}_{M \times N}$  is an atlas because we showed earlier that each  $(U \times V, \phi \times \psi)$  is a chart. Let  $(U_1 \times V_1, \phi_1 \times \psi_1), (U_2 \times V_2, \phi_2 \times \psi_2) \in \mathcal{A}_{M \times N}$ . Then  $(\phi_2 \times \psi_2) \circ (\phi_1 \times \psi_1)^{-1} = (\phi_2 \circ \phi_1^{-1}) \times (\psi_2 \circ \psi_1^{-1})$  is a smooth map from  $(\phi_1 \times \psi_1)(U_1 \times V_1)$  into  $(\phi_2 \times \psi_2)(U_2 \times V_2)$ . Thus every pair of charts in  $\mathcal{A}_{M \times N}$  is smoothly compatible. In other words,  $\mathcal{A}_{M \times N}$  is a smooth atlas.

On the other hand,  $\mathcal{A}_{M \times N}$  must be maximal because the restriction of any smoothly compatible chart to  $M, N$  gives a smoothly compatible chart, which must belong to  $\mathcal{A}_M, \mathcal{A}_N$ , respectively. Thus  $M \times N$  is a smooth manifold with boundary.  $\square$

## 2. CHAPTER 2: SMOOTH MAPS

### 2.1. Exercises.

**Exercise 2.1.** Let  $M$  be a smooth manifold with or without boundary. Show that pointwise multiplication turns  $C^\infty(M)$  into a commutative ring and a commutative and associative algebra over  $\mathbb{R}$ .

*Proof.*

- The constant map  $f(p) = 0$  is clearly in  $C^\infty(M)$  and it is the additive identity.
- The constant map  $f(p) = 1$  is clearly in  $C^\infty(M)$  and it is the multiplicative identity.

- Let  $f \in C^\infty(M), g \in C^\infty(M)$ . Let  $p \in M$  and  $(\phi, U)$  be a smooth chart for  $p$ . Then  $f \circ \phi^{-1}$  and  $g \circ \phi^{-1}$  are both smooth (Exercise 2.3), real-valued maps defined on an open subset of  $\mathbb{R}^n$ . Thus  $f + g$  is in  $C^\infty(M)$ . Moreover,  $f + g = g + f$  because addition in  $\mathbb{R}$  is commutative.
- Let  $f, g, h \in C^\infty(M)$ . Let  $p \in M$  and  $(\phi, U)$  be a smooth chart for  $p$ . Then  $f \circ \phi^{-1}$  and  $g \circ \phi^{-1}$  are both smooth (Exercise 2.3), real-valued maps defined on an open subset of  $\mathbb{R}^n$ . Therefore,  $fg$  is in  $C^\infty(M)$ . Moreover,  $fg = gf$  and  $(fg)h = f(gh)$  because multiplication in  $\mathbb{R}$  is commutative and associative.
- Let  $c \in \mathbb{R}, f \in C^\infty(M)$ . Then  $cf$  can be seen as  $fg$  where  $g$  is the constant function whose value is  $c$ . As shown above,  $cf \in C^\infty(M)$ .

□

**Exercise 2.2.** Let  $U$  be an open submanifold of  $\mathbb{R}^n$  with its standard smooth manifold structure. Show that a function  $f : U \rightarrow \mathbb{R}^k$  is smooth in the sense just defined if and only if it is smooth in the sense of ordinary calculus. Do the same for an open submanifold with boundary in  $\mathbb{H}^n$ .

*Proof.*  $f$  is smooth in the sense just defined if and only if  $f \circ \text{Id}^{-1}$  is smooth in the sense of ordinary calculus. Since  $f \circ \text{Id}^{-1} = f$ ,  $f \circ \text{Id}^{-1}$  is smooth in the sense of ordinary calculus if and only if  $f$  is smooth in the sense of ordinary calculus. □

**Exercise 2.3.** Let  $M$  be a smooth manifold with or without boundary, and suppose  $f : M \rightarrow \mathbb{R}^k$  is a smooth function. Show that  $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}^k$  is smooth for every smooth chart  $(U, \phi)$  for  $M$ .

*Proof.* Let  $\phi(x) \in \phi(U)$ . Since  $f$  is smooth, there exists  $(V, \psi)$  such that  $f \circ \psi^{-1} : \psi(V) \rightarrow \mathbb{R}^k$  is smooth and  $x \in V$ . Let  $W = U \cap V$ . Then  $f \circ \psi^{-1} : \psi(W) \rightarrow \mathbb{R}^k$  is smooth and  $\psi \circ \phi^{-1} : \phi(W) \rightarrow \psi(W)$  is a diffeomorphism where  $\phi(W)$  is a neighborhood of  $\phi(x)$ . Then the restriction of  $f \circ \psi^{-1}$  to  $\phi(W)$  is identical to  $(f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})$ . Since the composition of a smooth function is smooth,  $f \circ \phi^{-1}$  is smooth. □

**Exercise 2.7(Prove Proposition 2.5).** Suppose  $M$  and  $N$  are smooth manifolds with or without boundary, and  $F : M \rightarrow N$  is a map. Then  $F$  is smooth if and only if either of the following conditions is satisfied:

- For every  $p \in M$ , there exist smooth charts  $(U, \phi)$  containing  $p$  and  $(V, \psi)$  containing  $F(p)$  such that  $U \cap F^{-1}(V)$  is open in  $M$  and the composite map  $\psi \circ F \circ \phi^{-1}$  is smooth from  $\phi(U \cap F^{-1}(V))$  to  $\psi(V)$ .
- $F$  is continuous and there exist smooth atlases  $\{(U_\alpha, \phi_\alpha)\}$  and  $\{(V_\beta, \psi_\beta)\}$  for  $M$  and  $N$ , respectively, such that for each  $\alpha$  and  $\beta$ ,  $\psi_\beta \circ F \circ \phi_\alpha^{-1}$  is a smooth map from  $\phi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$  to  $\psi_\beta(V_\beta)$ .

*Proof.* Let  $\mathcal{A}_M$  and  $\mathcal{A}_N$  be smooth structures of  $M$  and  $N$ . Suppose  $F$  is smooth. By Proposition 2.4,  $F$  is continuous. For every  $p \in M$  there exist coordinate charts  $(U_p, \phi_p)$  containing  $p$  and  $(V_p, \psi_p)$  containing  $F(p)$  such that  $F(U_p) \subset V_p$  and  $\psi_p \circ F \circ \phi_p^{-1}$  is smooth from  $\phi_p(U_p)$  to  $\psi_p(V_p)$ . Then  $\{(U_p, \phi_p) \mid p \in M\} \subset \mathcal{A}_M$  and  $\mathcal{A}_N \setminus \{(V_p, \psi_p) \mid p \in M\} \subset \mathcal{A}_N$  are smooth atlases. Moreover, for every  $(U_p, \phi_p)$  and  $(V_q, \psi_q)$ ,  $\psi_q \circ F \circ \phi_p^{-1}$  is a smooth map from  $\phi_p(U_p \cap F^{-1}(V_q))$  to  $\psi_q(V_q)$  because  $\psi_q \circ F \circ \phi_p^{-1} = (\psi_q \circ \psi_p^{-1}) \circ (\psi_p \circ F \circ \phi_p^{-1})$  where  $\psi_q \circ \psi_p^{-1}$  and  $\psi_p \circ F \circ \phi_p^{-1}$  are smooth. Therefore, the definition implies (b).

(b) implies (a) because if  $F$  is continuous,  $F^{-1}(V_\beta)$  is open in  $M$  for every  $\beta$ , so  $U \cap F^{-1}(V)$  is open in  $M$ .

Finally, we show that (a) implies the definition. Suppose  $F$  satisfies (a). Let  $p \in M$ . Let  $(U, \phi) \in \mathcal{A}_M$  and  $(V, \psi) \in \mathcal{A}_N$  be smooth charts satisfying the properties described in (a). Let  $U' = U \cap F^{-1}(V)$  and consider  $(U', \phi|_{U'})$ . Then  $(U', \phi|_{U'}) \in \mathcal{A}_M$  because it must be smoothly compatible with any other smooth coordinate chart in  $\mathcal{A}_M$ . Moreover,  $F(U') \subset V$  and  $\psi \circ F \circ (\phi|_{U'})^{-1} : \phi(U') \rightarrow \psi(V)$  is smooth. Therefore, (a) implies the definition.

Hence, (a), (b) and the definition are all equivalent. □

**Exercise 2.7(Proof of Proposition 2.6).** Let  $M$  and  $N$  be smooth manifolds with or without boundary, and let  $F : M \rightarrow N$  be a map.

- If every point  $p \in M$  has a neighborhood  $U$  such that the restriction  $F|_U$  is smooth, then  $F$  is smooth.
- Conversely, if  $F$  is smooth, then its restriction to every open subset is smooth.

*Proof.* Let  $\mathcal{A}_M, \mathcal{A}_N$  be smooth structures of  $M, N$ , respectively.

- (a) Let  $p \in M$ . Let  $U$  be a neighborhood of  $p$  such that  $F|_U$  is smooth. By 1.44,  $U$  is a smooth manifold with the induced smooth structure  $\mathcal{A}_U = \{(V, \phi) \in \mathcal{A}_M \mid V \subset U\}$ . Since  $F|_U$  is smooth, there exist  $(V, \phi) \in \mathcal{A}_U$  and  $(W, \psi) \in \mathcal{A}_N$  such that:

- $F|_U(V) \subset W$ .
- $\psi \circ F|_U \circ \phi^{-1} : \phi(V) \rightarrow \psi(W)$  is smooth.

Since  $V \subset U$ ,  $F(V) \subset W$ ,  $\psi \circ F \circ \phi^{-1} : \phi(V) \rightarrow \psi(W)$  is smooth, and  $(V, \phi) \in \mathcal{A}$ . Therefore,  $F$  is smooth.

- (b) Let  $U \subset M$  be an open subset. By 1.44,  $U$  is a smooth manifold with the induced smooth structure  $\mathcal{A}_U = \{(V, \phi) \in \mathcal{A}_M \mid V \subset U\}$ . Let  $p \in U$ . Then  $p \in F$ , so there exist  $(V, \phi) \in \mathcal{A}_M, (W, \psi) \in \mathcal{A}_N$  such that  $F(V) \subset W$  and  $\psi \circ F \circ \phi^{-1} : \phi(V) \rightarrow \psi(W)$  is smooth. Then  $(V \cap U, \phi|_{V \cap U})$  is a chart that is smoothly compatible with every chart in  $\mathcal{A}_M$ . Therefore,  $(V \cap U, \phi|_{V \cap U}) \in \mathcal{A}_M$ . Moreover,  $\phi|_{V \cap U}(V \cap U) \subset \phi(V) \subset W$  and  $\psi \circ F \circ (\phi|_{V \cap U}(V \cap U))^{-1}$  is clearly smooth. Therefore,  $F|_U$  is smooth.  $\square$

**Exercise 2.9.** Suppose  $F : M \rightarrow N$  is a smooth map between smooth manifolds with or without boundary. Show that the coordinate representation of  $F$  with respect to *every* pair of smooth charts for  $M$  and  $N$  is smooth.

*Proof.* Let  $(M, \mathcal{A}_M), (N, \mathcal{A}_N)$  be smooth manifolds with or without boundary. Let  $F : M \rightarrow N$  be a smooth map. Let  $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$  be given. We must show that  $\hat{F} = \psi \circ F \circ \phi^{-1}$  is a smooth function from  $\phi(U \cap F^{-1}(V))$  to  $\psi(V)$ . Let  $\phi(p) \in \phi(U \cap F^{-1}(V))$ . Then  $p \in M$ , so there exist  $(U_0, \phi_0) \in \mathcal{A}_M$  and  $(V_0, \psi_0) \in \mathcal{A}_N$  such that

- $p \in U_0 \subset U \cap F^{-1}(V)$ ;
- $\phi_0(U_0) \subset V_0$ ;
- $\psi_0 \circ F \circ \phi_0^{-1} : \phi_0(U_0) \rightarrow \psi(V_0)$  is smooth.

Then  $\psi \circ F \circ \phi^{-1}|_{\phi(U_0)} = (\psi \circ \psi_0^{-1}) \circ (\psi_0 \circ F \circ \phi_0^{-1}) \circ (\phi_0 \circ \phi)$ . Since the composition of smooth functions in Euclidean spaces is smooth,  $\hat{F}$  is smooth.  $\square$

**Exercise 2.11(Proof of Proposition 2.10).** Let  $M, N$  and  $P$  be smooth manifolds with or without boundary.

- (a) Every constant map  $c : M \rightarrow N$  is smooth.  
(b) The identity map of  $M$  is smooth.  
(c) If  $U \subset M$  is an open submanifold with or without boundary, then the inclusion map  $U \rightarrow M$  is smooth.

*Proof.* Let  $\mathcal{A}_M, \mathcal{A}_N, \mathcal{A}_P$  be smooth structures of  $M, N, P$ , respectively.

- (a)  $F$  is clearly continuous. Moreover, for every  $(U_\alpha, \phi_\alpha) \in \mathcal{A}_M, (V_\beta, \psi_\beta) \in \mathcal{A}_N$ ,  $\psi_\beta \circ F \circ \phi_\alpha^{-1}$  is a constant map, so it is smooth. By (2.7(Prove Proposition 2.5)),  $F$  is smooth.  
(b) Let  $p \in M$ . Choose  $(U, \phi) \in \mathcal{A}_M$  such that  $p \in U$ . Then  $F(U) \subset U$  and  $\phi \circ F \circ \phi^{-1} = \text{Id}_U$ , so it is smooth. Therefore,  $F$  is smooth.  
(c) By 1.44,  $\mathcal{A}_U = \{(V, \phi) \mid V \subset U\}$  is a smooth structure of  $U$ . Let  $p \in U$ . Then  $p \in V$  for some  $(V, \phi) \in \mathcal{A}_U$ . Then  $(V, \phi) \in \mathcal{A}_M$ , trivially. Since  $F(V) \subset V$  and  $\phi \circ F \circ \phi^{-1}$  is simply the identity map on  $V$ ,  $F$  is smooth.  $\square$

**Exercise 2.16(Proof of Proposition 2.15).**

- (a) Every composition of diffeomorphisms is a diffeomorphism.  
(b) Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.  
(c) Every diffeomorphism is a homeomorphism and an open map.  
(d) The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.  
(e) “Diffeomorphic” is an equivalence relation on the class of all smooth manifolds with or without boundary.



**Exercise 2.16(Proof of Proposition 2.15).** Let  $(M, \mathcal{A}_M), (N, \mathcal{A}_N), (P, \mathcal{A}_P)$  be smooth manifolds with or without boundary, and let  $F : M \rightarrow N, G : N \rightarrow P$  be diffeomorphisms.

- (a) By Proposition 2.10(d),  $G \circ F$  and  $F^{-1} \circ G^{-1}$  are smooth. Then  $(G \circ F) \circ (F^{-1} \circ G^{-1})$  and  $(F^{-1} \circ G^{-1}) \circ (G \circ F)$  are both the identity map on the corresponding space, so  $F^{-1} \circ G^{-1}$  is the smooth inverse of  $G \circ F$ . Therefore,  $G \circ F$  is a diffeomorphism.
- (b) By Example 1.34, we know that  $M_1 \times \cdots \times M_k$  and  $N_1 \times \cdots \times N_k$  are both smooth manifolds. Let  $\mathcal{A}_{M_i}, \mathcal{A}_{N_i}, \mathcal{A}_M$  and  $\mathcal{A}_N$  denote the smooth manifold structures of  $M_i, N_i, M_1 \times \cdots \times M_k, N_1 \times \cdots \times N_k$ , respectively. Let a smooth map  $F_i : M_i \rightarrow N_i$  be given for each  $i$ . Let  $(p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$  be given. Then there exist  $(U_i, \phi_i) \in \mathcal{A}_{M_i}$  and  $(V_i, \psi_i) \in \mathcal{A}_{N_i}$  such that  $p_i \in U_i, F_i(U_i) \subset V_i, \psi_i \circ F_i \circ \phi_i^{-1} : \phi_i(U_i) \rightarrow \psi_i(V_i)$  is smooth for each  $i$ . This implies that  $(\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \cdots (\psi_k \circ F_k \circ \phi_k^{-1}) = (\psi_1 \times \cdots \times \psi_k) \circ (F_1 \times \cdots \times F_k) \circ (\phi_1 \times \cdots \times \phi_k)^{-1}$  is smooth. Therefore,  $F_1 \times \cdots \times F_k$  is smooth. Using the exact same argument, we can conclude that  $F_1^{-1} \times \cdots \times F_k^{-1}$  is smooth. Since  $(F_1 \times \cdots \times F_k)^{-1} = F_1^{-1} \times \cdots \times F_k^{-1}$ ,  $F_1 \times \cdots \times F_k$  is a diffeomorphism.
- (c) Proposition 2.4 states that every smooth map is continuous. Thus  $F$  and  $F^{-1}$  are both continuous. Therefore,  $F$  is a homeomorphism and also an open map.
- (d) Let  $U \subset M$  be an open subset. By (2.7(Proof of Proposition 2.6)),  $F|_U$  is smooth. Since  $F$  is a homeomorphism as shown in (c),  $F(U)$  is an open subset of  $N$ . Therefore,  $F^{-1}|_{F(U)}$  is smooth by (2.7(Proof of Proposition 2.6)). Clearly,  $F|_U$  and  $F^{-1}|_{F(U)}$  are the inverse of each other. Therefore,  $F|_U$  is a diffeomorphism.
- (e) By (2.11(Proof of Proposition 2.10)), the identity map on  $M$  is a diffeomorphism, so the reflexive property is satisfied. Moreover,  $(F^{-1})^{-1} = F$ , so the symmetric property is satisfied. By (a), the composition of two diffeomorphisms is a diffeomorphism, so the transitive property is satisfied. Therefore, “diffeomorphic” is an equivalence relation.

**Exercise 2.19(Proof of Theorem 2.18).** Suppose  $M$  and  $N$  are smooth manifolds with boundary and  $F : M \rightarrow N$  is a diffeomorphism. Then  $F(\partial M) = \partial N$ , and  $F$  restricts to a diffeomorphism from  $\text{Int } M$  to  $\text{Int } N$ .

*Proof.* Let  $\mathcal{A}_M, \mathcal{A}_N$  denote the smooth structures of  $M, N$ , respectively. Let  $p \in \partial M$ . Then there exists a chart containing  $p$  that sends  $p$  to  $\partial \mathbb{H}^n$ . By Theorem 1.46, every chart containing  $p$  sends  $p$  to  $\partial \mathbb{H}^n$ .

Since  $F$  is smooth, there exist  $(U, \phi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N$  such that  $F(U) \subset V$  and  $\psi \circ F \circ \phi^{-1}$  is a smooth map from  $\phi(U)$  to  $\psi(V)$ .  $F^{-1}$  is a homeomorphism by (2.16(Proof of Proposition 2.15)). Then  $(\phi^{-1} \circ F^{-1}, F(U))$  is a coordinate chart around  $F(p)$  because we obtain a homeomorphism by restricting the composition of two injective continuous maps to its image. Moreover, we claim that  $(\phi^{-1} \circ F^{-1}, F(U))$  is smoothly compatible with every chart in  $\mathcal{A}_N$ . Let  $(\psi_1, V_1) \in \mathcal{A}_N$  be given. Then  $(\phi^{-1} \circ F^{-1}) \circ \psi_1^{-1} = (\phi^{-1} \circ F^{-1} \circ \psi_1^{-1}) \circ (\psi \circ \psi_1^{-1})$ , and the composition of two smooth maps is smooth. Therefore,  $(\phi^{-1} \circ F^{-1}, F(U)) \in \mathcal{A}_N$ , and this chart contains  $F(p)$  and sends  $F(p)$  to  $\partial \mathbb{H}^n$ . In other words,  $F(p) \in \partial N$ .

Since  $F^{-1}$  is also smooth,  $F^{-1}(\partial N) \subset \partial M$ .  $F^{-1}(\partial N) \subset \partial M \implies F(F^{-1}(\partial N)) \subset F(\partial M) \subset \partial N$ . Since  $F$  is a bijection,  $F(F^{-1}(\partial N)) = \partial N$ . Therefore,  $F(\partial M) = \partial N$ .

This implies that  $F(\text{Int } M) = \text{Int } N$ . By (1.44(c)) and (2.16(Proof of Proposition 2.15)(d)),  $F$  is a diffeomorphism between  $\text{Int } M$  and  $\text{Int } N$ .  $\square$

**Problem 2-27.** Give a counterexample to show that the conclusion of the extension lemma can be false if  $A$  is not closed.

*Proof.* Let  $M = \mathbb{R}, A = (0, 1), f(x) = 1/x$ . Then  $f$  is smooth on  $A$ , but  $\lim_{x \rightarrow 0} f = \infty$ , so  $f$  cannot be extended continuously.  $\square$

## 2.2. Problems.

**Problem 2-1.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that for every  $x \in \mathbb{R}$ , there are smooth coordinate charts  $(U, \phi)$  containing  $x$  and  $(V, \psi)$  containing  $f(x)$  such that  $\psi \circ f \circ \phi^{-1}$  is smooth as a map from  $\phi(U \cap f^{-1}(V))$  to  $\psi(V)$ , but  $f$  is not smooth in the sense we defined in this chapter.

*Proof.*  $\phi = \psi = \text{Id}$  in this solution.

If  $x \geq 0$ , then let  $U = \mathbb{R}, V = (0, \infty)$ . Then  $\phi(U \cap f^{-1}(V)) = [0, \infty)$ . Thus  $\psi \circ f \circ \phi^{-1} : [0, \infty) \rightarrow (0, \infty)$  is the constant map that sends every number to 1. Therefore, it is smooth.

If  $x < 0$ , then let  $U = \mathbb{R}, V = (-\infty, 1)$ . Then  $\phi(U \cap f^{-1}(V)) = (-\infty, 0)$ . Thus  $\psi \circ f \circ \phi^{-1} : (-\infty, 0) \rightarrow (-\infty, 1)$  is the constant map that sends every number to 0. Therefore, it is smooth.

It might seem that we can apply (2.7(Prove Proposition 2.5)) to show that  $f$  is smooth, but (2.7(Prove Proposition 2.5)) requires that  $U \cap f^{-1}(V)$  be open in  $M$ .

$f$  maps the interval  $(-1, 1)$  to  $\{0, 1\}$ . Since the image of a connected set under a continuous map must be connected,  $f$  cannot be continuous. By Proposition 2.4,  $f$  cannot be smooth.  $\square$

**Problem 2-2(Proof of Proposition 2.12).** Suppose  $M_1, \dots, M_k$  and  $N$  are smooth manifolds with or without boundary, such that at most one of  $M_1, \dots, M_k$  has nonempty boundary. For each  $i$ , let  $\pi_i : M_1 \times \dots \times M_k \rightarrow M_i$  denote the projection onto the  $M_i$  factor. A map  $F : N \rightarrow M_1 \times \dots \times M_k$  is smooth if and only if each of the component maps  $F_i = \pi_i \circ F : N \rightarrow M_i$  is smooth.

*Proof.* Let  $\mathcal{A}_{M_1}, \dots, \mathcal{A}_{M_k}, \mathcal{A}_N$  be the smooth structures of  $M_1, \dots, M_k, N$ . Let  $d_1, \dots, d_k$  denote the dimensions of  $M_1, \dots, M_k$ , respectively. Let  $d = \sum d_i$ .

First, suppose that  $F$  is smooth. By (2.11(Proof of Proposition 2.10)), the composition of smooth maps is smooth. Thus it suffices to show that  $\pi_i : M_1 \times \dots \times M_k \rightarrow M_i$  is smooth for each  $i$ . We show that  $\pi_1$  is smooth and the other cases can be shown similarly.

Let  $(x_1, \dots, x_k) \in M_1 \times \dots \times M_k$ . Then for each  $i$ , there exist  $(U_i, \phi_i) \in \mathcal{A}_{M_i}$  and  $(V_i, \psi_i) \in \mathcal{A}_{M_i}$  such that  $x_i \in U_i$  and  $\phi_i(U_i) \subset V_i$ . Then we have  $(\phi_1 \times \dots \times \phi_k)(U_1 \times \dots \times U_k) \subset V_1 \times \dots \times V_k$  and the composition  $\phi_i \circ \pi_1 \circ (\phi_1 \times \dots \times \phi_k)^{-1}$  is the projection of the first  $d_1$  coordinates from  $\mathbb{R}^n$  onto  $\mathbb{R}^{d_1}$ . Therefore, it is clearly smooth, so  $\pi_1$  is smooth.

Suppose each  $F_i = \pi_i \circ F : N \rightarrow M_i$  is smooth. Let  $p \in N$ . Then for each  $i$ , there exist  $(U_i, \phi_i) \in \mathcal{A}_N$  and  $(V_i, \psi_i) \in \mathcal{A}_{M_i}$  such that  $p \in U_i, F_i(U_i) \subset V_i$  and  $\psi_i \circ F_i \circ \phi_i^{-1}$ . Let  $U = U_1 \cap \dots \cap U_k$ .  $U$  is a neighborhood of  $p$  and the restriction of  $\phi_1$  to  $U$  is a homeomorphism. Then we claim that  $(\phi_1, U) \in \mathcal{A}_N$  and  $(\psi_1 \times \dots \times \psi_k, V_1 \times \dots \times V_k) \in \mathcal{A}_{M_1 \times \dots \times M_k}$  are charts that satisfy the necessary properties.

- $F(U) \subset V_1 \times \dots \times V_k$ .
- For each  $i$ ,  $\psi_i \circ F_i \circ \phi_i^{-1} = (\psi_i \circ F_i \circ \phi_i^{-1}) \circ (\phi_i \circ \phi_1^{-1}) : \phi_1(U) \rightarrow \psi_i(V_i)$  is smooth because the composition of two smooth maps is smooth. Thus  $(\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \dots \times (\psi_k \circ F_k \circ \phi_k^{-1}) : \phi_1(U) \rightarrow \psi_1(V_1) \times \dots \times \psi_k(V_k)$  is smooth. Moreover,  $(\psi_1 \times \dots \times \psi_k) \circ F \circ \phi_1^{-1} = (\psi_1 \circ F_1 \circ \phi_1^{-1}) \times \dots \times (\psi_k \circ F_k \circ \phi_k^{-1})$ .

Therefore,  $F$  is smooth.  $\square$

**Problem 2-3.** For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

- $p_n : S^1 \rightarrow S^1$  is the  $n$ th power map for  $n \in \mathbb{Z}$ , given in complex notation by  $p_n(z) = z^n$ .
- $\alpha : S^n \rightarrow S^n$  is the antipodal map  $\alpha(x) = -x$ .
- $F : S^3 \rightarrow S^2$  is given by  $F(w, z) = (z\bar{w} + w\bar{z}, iw\bar{z} - iz\bar{w}, z\bar{z} - w\bar{w})$  where we think of  $S^3$  as the subset  $\{(w, z) : |w|^2 + |z|^2 = 1\}$  of  $\mathbb{C}^2$ .

*Proof.*

- Example 1.31 shows the existence of a smooth structure of  $S^1$  and let  $\mathcal{A}$  denote it. Let  $p \in S^1$ . Then there exist  $(U_i^\pm, \phi_i^\pm), (U_j^\pm, \phi_j^\pm) \in \mathcal{A}$  around  $p, p_n(p)$ , respectively. Then the composition  $\phi_j^\pm \circ f \circ (\phi_i^\pm)^{-1}$  is equal to one of  $\cos(n(\arccos(x))), \sin(n(\arcsin(x))), \cos(n(\arcsin(x))), \sin(n(\arccos(x)))$ , all of which are clearly smooth. By Proposition 2.5(a),  $p_n$  is smooth.
- Example 1.31 shows the existence of a smooth structure of  $S^n$  and let  $\mathcal{A}$  denote it. Let  $p \in S^1$ . Then there exists a chart  $(U_i^\pm, \phi_i^\pm)$  in  $\mathcal{A}$  around  $p$ . Then  $(U_i^\mp, \phi_i^\mp)$  is a chart containing  $\alpha(p)$  with  $\alpha(U_i^\pm) \subset U_i^\mp$ . Then  $\phi_i^\mp \circ \alpha \circ \phi_i^\pm$  is the map  $x \mapsto -x$ , which is clearly smooth.

- (c) Let  $z = a + bi, w = c + di$ .  $z\bar{w} = ac + bd + i(bc - ad)$  and  $w\bar{z} = (ac + bd) - i(bc - ad)$ . Then  $z\bar{w} + w\bar{z} = 2(ac + bd) = 2\operatorname{Re}(z\bar{w})$  and  $i(w\bar{z} - z\bar{w}) = 2\operatorname{Im}(z\bar{w})$ .

$$\begin{aligned} (2\operatorname{Re}(z\bar{w}))^2 + (2\operatorname{Im}(z\bar{w}))^2 + (|z|^2 - |w|^2)^2 &= 4|z\bar{w}|^2 + (|z|^2 - |w|^2)^2 \\ &= 4|z|^2|\bar{w}|^2 + (|z|^2 - |w|^2)^2 \\ &= (|z|^2 + |w|^2)^2 \\ &= 1. \end{aligned}$$

Therefore,  $F$  indeed maps  $S^3$  into  $S^2$ . Moreover, this map is continuous. Let  $(z = a + bi, w = c + di) \in S^3$  be given. Suppose that  $(U_4^+, \phi_4^+)$  and  $(V_3^+, \psi_3^+)$  are charts containing  $(z, w)$  and  $F(z, w)$ . Then  $\psi_3^+ \circ F \circ \phi_4^+ : (a, b, c) \mapsto (2u, 2v)$  where  $u + iv = (a + bi)(c - \sqrt{1 - a^2 - b^2 - c^2}i)$  which is a smooth map from  $\phi_4^+(U_4^+) \subset \mathbb{R}^3$  into  $\mathbb{R}^2$ . Other cases are similar, and thus  $F$  is smooth by Proposition 2.5(b). □

**Problem 2-5.** Let  $\mathbb{R}$  be the real line with its standard smooth structure, and let  $\tilde{R}$  denote the same topological manifold with the smooth structure defined in Example 1.23. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function that is smooth in the usual sense.

- (a) Show that  $f$  is also smooth as a map from  $\mathbb{R}$  to  $\tilde{\mathbb{R}}$ .  
(b) Show that  $f$  is smooth as a map from  $\tilde{\mathbb{R}}$  to  $\mathbb{R}$  if and only if  $f^{(n)}(0) = 0$  whenever  $n$  is not an integral multiple of 3.

*Proof.*

- (a) The “ $\psi \circ f \circ \phi^{-1}$ ” is simply  $f^3$ , which is a smooth map from  $\mathbb{R}$  to  $\mathbb{R}$ . Thus  $f : \tilde{\mathbb{R}} \rightarrow \mathbb{R}$  is smooth.

- (b) Solve this! □

**Problem 2-6.** Let  $P : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}^{k+1} \setminus \{0\}$  be a smooth function, and suppose that for some  $d \in \mathbb{Z}$ ,  $P(\lambda x) = \lambda^d P(x)$  for all  $\lambda \in \mathbb{R} \setminus \{0\}$  and  $x \in \mathbb{R}^{n+1} \setminus \{0\}$ . Show that the map  $\tilde{P} : \mathbb{RP}^n \rightarrow \mathbb{RP}^k$  defined by  $\tilde{P}([x]) = [P(x)]$  is well defined and smooth.

*Proof.* Let  $P_1, \dots, P_{k+1}$  denote the component functions of  $P$ .

Suppose  $[x_1 : \dots : x_{n+1}] = [y_1 : \dots : y_{n+1}]$ . Then there exists  $\lambda \neq 0$  such that  $(y_1, \dots, y_{n+1}) = (\lambda x_1, \dots, \lambda x_{n+1})$ .  $P(y_1, \dots, y_{n+1}) = P(\lambda x_1, \dots, \lambda x_{n+1}) = \lambda^d P(x_1, \dots, x_{n+1})$ . Since  $\lambda^d \neq 0$ ,  $[P(y_1, \dots, y_{n+1})] = [P(x_1, \dots, x_{n+1})]$ . Therefore,  $\tilde{P}$  is well-defined.

Let  $\tilde{p} = [p_1 : \dots : p_{n+1}] \in \mathbb{RP}^n$  be given. Without loss of generality, assume  $p_{n+1} \neq 0$ . Consider the chart  $(U, \psi_{n+1})$  with  $U = \{[x_1 : \dots : x_{n+1}] \mid x_{n+1} \neq 0\}$ . Let  $q_i = P_i(p_1, \dots, p_{n+1})$ . Without loss of generality, assume  $q_{k+1} \neq 0$ . Then  $\tilde{P}(\tilde{p})$  is contained in  $V = \{[y_1 : \dots : y_{k+1}] \mid y_{k+1} \neq 0\}$ . Since  $P$  is smooth, there exists  $0 < \delta < |x_{n+1}|$  such that  $|(x_1, \dots, x_{n+1}) - (p_1, \dots, p_{n+1})| < \delta$  implies  $P_{k+1}(x_1, \dots, x_{n+1}) \neq 0$ . Then  $[p_1 : \dots : p_{n+1}] \in \pi(B(p_1, \dots, p_{n+1})) \subset U \cap F^{-1}(V)$ . Therefore,  $U \cap F^{-1}(V)$  is open in  $\mathbb{RP}^n$ .

Finally the composition map  $\psi_{k+1} \circ \tilde{P} \circ \phi_{n+1}^{-1}$  sends  $(x_1/x_{n+1}, \dots, x_n/x_{n+1})$  to  $(y_1/y_{k+1}, \dots, y_k/y_{k+1})$  where  $y_i = P_i(x_1, \dots, x_{n+1})$ . In other words,  $(x_1, \dots, x_n) \mapsto (y_1/y_{k+1}, \dots, y_k/y_{k+1})$  where  $y_i = P_i(x_1, \dots, x_n, 1)$ . Since each  $P_i$  is smooth, this map must be smooth as well. By (2.7(Prove Proposition 2.5)),  $\tilde{P}$  is smooth. □

**Problem 2-7.** Let  $M$  be a nonempty smooth  $n$ -manifold with or without boundary, and suppose  $n \geq 1$ . Show that the vector space  $C^\infty(M)$  is infinite-dimensional.

*Proof.* Let  $k \in \mathbb{N}$  be given. Let  $p \in M$  be chosen arbitrarily. Let  $(U, \phi)$  be a smooth chart containing  $p$ . Then  $\hat{U} = \phi(U)$  is an open subset of  $\mathbb{R}^n$  or  $\mathbb{H}^n$ . In each case, we can pick  $k$  distinct points  $x_1, \dots, x_k \in \hat{U}$  because  $\hat{U}$  is a nonempty open subset and  $n \geq 1$ . Since  $\hat{U}$  is open, there exist open  $U_1, \dots, U_k$  such that  $x_i \in U_i \subset \hat{U}$  and  $U_i \cap U_j = \emptyset$  whenever  $i \neq j$ . Moreover,  $\{x_i\}$  is a closed subset. By Proposition 2.25, we obtain  $k$  bump functions  $f_i$  for  $\{x_i\}$  supported in  $U_i$ . Extend each  $f_i$  by setting  $f_i(q) = 0$  for any  $q \notin U$ . Then each  $f_i$  lives in  $C^\infty(M)$ . Clearly,  $\sum c_i f_i = 0$  implies  $c_i = 0$ , so  $\{f_1, \dots, f_k\}$  is linearly independent. Therefore,  $C^\infty(M)$  is infinite-dimensional. □

**Problem 2-14.** Suppose  $A$  and  $B$  are disjoint closed subsets of a smooth manifold  $M$ . Show that there exists  $f \in C^\infty(M)$  such that  $0 \leq f(x) \leq 1$  for all  $x \in M$ ,  $f^{-1}(0) = A$  and  $f^{-1}(1) = B$ .

*Proof.* By Theorem 2.29, there exist  $\alpha, \beta \in C^\infty(M)$  such that  $\alpha^{-1}(0) = A$  and  $\beta^{-1}(0) = B$ . Then  $f(x) = \alpha(x)/(\alpha(x) + \beta(x))$  is a desired map.  $\square$

### 3. CHAPTER 3: TANGENT VECTORS

#### 3.1. Exercises.

**Proposition 3.2.** Let  $a \in \mathbb{R}^n$ .

- (a) For each geometric tangent vector  $v_a \in \mathbb{R}_a^n$ , the map  $D_v|_a : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$  defined by

$$D_v|_a f = D_v f(a) = \left. \frac{d}{dt} \right|_{t=0} f(a + tv)$$

is a derivation at  $a$ .

- (b) The map  $v_a \mapsto D_v|_a$  is an isomorphism from  $\mathbb{R}_a^n$  onto  $T_a \mathbb{R}^n$ .

*Proof.*

- (a)  $D_v|_a$  is linear because  $D_v|_a(f + cg) = D_v(f + cg)(a) = D_v f(a) + cD_v g(a) = D_v|_a(f) + cD_v|_a(g)$  because directional derivatives are linear. Moreover, the product rule is satisfied because directional derivatives satisfy that. Therefore,  $D_v|_a$  is a linear map that satisfies directional derivatives, so it is a derivation.
- (b) Let  $\phi : \mathbb{R}_a^n \rightarrow T_a \mathbb{R}^n$  be defined such that  $v_a \mapsto D_v|_a$ . We first claim that  $\phi$  is linear.

$$\begin{aligned} \phi(v_a + cw_a)(f) &= \phi((v + cw)_a)(f) \\ &= D_{v+cw} f(a) \\ &= D_v f(a) + cD_w f(a) \\ &= D_v|_a(f) + cD_w|_a(f) \\ &= \phi(v_a)(f) + c\phi(w_a)(f) \\ &= (\phi(v_a) + c\phi(w_a))(f). \end{aligned}$$

Next, we claim that  $\ker(\phi) = 0$ . Let  $v_a \in \ker(\phi) \subset \mathbb{R}_a^n$ . Let  $v_1, \dots, v_n \in \mathbb{R}^n$  be chosen such that  $v_a = \sum_{i=1}^n v_i e_i|_a$ . For each  $j$ , let  $x^j : \mathbb{R}^n \rightarrow \mathbb{R}$  denote the projection map of the  $j$ th coordinate. Then  $0 = D_v|_a(x^j) = \left. \frac{d}{dt} \right|_{t=0} x^j(a + tv) = v_j$  for each  $j$ . Therefore,  $v_1 = \dots = v_n = 0$ , so  $\ker(\phi) = 0$ . Since  $\phi$  is linear,  $\phi$  must be injective.

Lastly, we claim that  $\phi$  is surjective. Let  $w \in T_a \mathbb{R}^n$  be given. For each  $j$ , let  $v_j = w(x^j)$ . Let  $v = (v_1, \dots, v_n)$ . We claim that  $\phi(v_a) = w$ . Let  $f \in C^\infty(\mathbb{R}^n)$ . By Theorem C.15, we can write

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + \sum_{i,j=1}^n (x^i - a^i)(x^j - a^j) \int_{0,1} F(t) dt$$

where  $F(t)$  is some function. Since  $(x^i - a^i)$  and  $(x^j - a^j) \int_{0,1} F(t) dt$  vanish at  $x = a$ ,  $w((x^i - a^i)(x^j - a^j) \int_{0,1} F(t) dt) = 0$  for any  $i, j$ . Therefore,

$$\begin{aligned} w(f) &= w(f(a)) + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)(w(x^i) - w(a^i)) \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)w(x^i) \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)v_i \\ &= \phi(v_a)(f) \end{aligned}$$

which proves that  $\phi$  is surjective.

□

**Exercise 3.5(Proof of Lemma 3.4).** Suppose  $M$  is a smooth manifold with or without boundary,  $p \in M$ ,  $v \in T_p M$ , and  $f, g \in C^\infty(M)$ .

- (a) If  $f$  is a constant function, then  $vf = 0$ .
- (b) If  $f(p) = g(p) = 0$ , then  $v(fg) = 0$ .

*Proof.* This is similar to Lemma 3.1.

- (a) Let  $h$  be the constant function that always takes the value 1. Then  $v(h) = v(h^2) = h(p)v(h) + h(p)v(h) = 2v(h)$ , so  $v(h) = 0$ . Since  $f(p) = ch(p)$  for some  $c \in \mathbb{R}$  and  $v$  is linear, this implies  $0 = cv(h) = v(ch) = v(f)$ .
- (b)  $v(fg) = f(p)vg + g(p)vf = 0 + 0 = 0$ .

□

**Exercise 3.7(Proof of Proposition 3.6).** Let  $M, N$ , and  $P$  be smooth manifolds with or without boundary, let  $F : M \rightarrow N$  and  $G : N \rightarrow P$  be smooth maps, and let  $p \in M$ .

- (a)  $dF_p : T_p M \rightarrow T_{F(p)} N$  is linear.
- (b)  $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_p M \rightarrow T_{G \circ F(p)} P$ .
- (c)  $d(\text{Id}_M)_p = \text{Id}_{T_p M} : T_p M \rightarrow T_p M$ .
- (d) If  $F$  is a diffeomorphism, then  $dF_p : T_p M \rightarrow T_{F(p)} N$  is an isomorphism, and  $(dF_p)^{-1} = d(F^{-1})_{F(p)}$ .

*Proof.* (a)  $\forall v, w \in T_p M, \forall c \in \mathbb{R}, \forall f \in C^\infty(N)$ ,

$$\begin{aligned} dF_p(cv + w)(f) &= (cv + w)(f \circ F) \\ &= (cv)(f \circ F) + w(f \circ F) \\ &= c(v(f \circ F)) + w(f \circ F) \\ &= c(dF_p(v)(f)) + dF_p(w)(f) \\ &= (cdF_p(v))(f) + dF_p(w)(f) \\ &= (cdF_p(v) + dF_p(w))(f). \end{aligned}$$

Therefore,  $dF_p(cv + w) = cdF_p(v) + dF_p(w)$ .

- (b)  $\forall v \in T_p M, f \in C^\infty(P)$ ,

$$\begin{aligned} d(G \circ F)_p(v)(f) &= v(f \circ (G \circ F)) \\ &= v((f \circ G) \circ F) \\ &= (dF_p(v))(f \circ G) \\ &= (dG_{F(p)}(dF_p(v)))(f) \\ &= ((dG_{F(p)} \circ dF_p)(v))(f) \end{aligned}$$

Therefore,  $d(G \circ F)_p = dG_{F(p)} \circ dF_p$ .

- (c)  $\forall v \in T_p(M), \forall f \in C^\infty(M)$ ,

$$\begin{aligned} d(\text{Id}_M)_p(v)(f) &= v(f \circ \text{Id}_M) \\ &= v(f). \end{aligned}$$

Therefore,  $d(\text{Id}_M)_p(v) = v$ , so  $d(\text{Id}_M)_p = \text{Id}_{T_p M}$ .

- (d)  $F^{-1}$  exists and it is a smooth map since  $F$  is a diffeomorphism. By combining (b) and (c), we obtain  $dF_p$  and  $dF_{F(p)}^{-1}$  are the inverse of each other. Therefore,  $dF_p$  is an isomorphism.

□

**Proposition 3.10.** If  $M$  is an  $n$ -dimensional smooth manifold, then for each  $p \in M$ , the tangent space  $T_p M$  is an  $n$ -dimensional vector space.

*Proof.* Let  $\mathcal{A}$  denote the smooth structure of  $M$  and let  $p \in M$  be given. Choose  $(U, \phi) \in \mathcal{A}$  such that  $p \in U$ . Then

$$T_p M \xrightarrow{di_p} T_p U \xrightarrow{d\phi_p} T_{\phi(p)} \hat{U} \xrightarrow{di_{\phi(p)}} T_{\phi(p)} \mathbb{R}^n$$

where  $di_p$  is induced by the inclusion map  $i : U \rightarrow M$  and  $di_{\phi(p)}$  is induced by the inclusion map  $\hat{U} \rightarrow \mathbb{R}^n$ .  $di_p, d\phi_p, di_{\phi(p)}$  are all isomorphisms by (3.7(Proof of Proposition 3.6)(d)) and Proposition 3.9. Therefore,  $\dim(T_p M) = n$ .  $\square$

**Proposition 3.15.** Let  $M$  be a smooth  $n$ -manifold with or without boundary, and let  $p \in M$ . Then  $T_p M$  is an  $n$ -dimensional vector space, and for any smooth chart  $(U, (x^i))$  containing  $p$ , the coordinate vectors  $\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p$  form a basis for  $T_p M$ .

*Proof.* By Proposition 3.12,  $T_p M$  is an  $n$ -dimensional vector space. By Corollary 3.3, the  $\partial/\partial x^i|_{\phi(p)}$  form a basis for  $T_{\phi(p)} \mathbb{R}^n$ . By Proposition 3.6(d),  $d\phi_p : T_p M \rightarrow T_{\phi(p)} \mathbb{R}^n$  is an isomorphism. Since  $d\phi_p$  is an isomorphism between vector spaces,  $d\phi_p$  sends a basis to a basis. In other words, the  $\partial/\partial x^i|_p = (d\phi_p)^{-1}(\partial/\partial x^i|_{\phi(p)})$  form a basis.  $\square$

**Remark.** The discussion on PP.61-62 shows the connection between differentials and Jacobian matrices. Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a smooth map and let  $p \in \mathbb{R}^n$  be given.

$$(3.1) \quad dF_p\left(\frac{\partial}{\partial x^i}\Big|_p\right)(f) = \frac{\partial}{\partial x^i}\Big|_p (f \circ F) \quad (\text{definition of } d)$$

$$(3.2) \quad = \frac{\partial(f \circ F)}{\partial x^i}(p) \quad (\text{Just a partial derivative of } f \circ F)$$

$$(3.3) \quad = \sum_{j=1}^m \frac{\partial f}{\partial y^j}(F(p)) \frac{\partial F^j}{\partial x^i}(p)$$

$$(3.4) \quad = \sum_{j=1}^m \frac{\partial F^j}{\partial x^i}(p) \frac{\partial f}{\partial y^j}(F(p)) \quad (\text{Multiplication is commutative in } \mathbb{R})$$

$$(3.5) \quad = \sum_{j=1}^m \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}\Big|_{F(p)}(f).$$

Therefore, we obtain that  $dF_p(\frac{\partial}{\partial x^i}|_p) = \sum_{j=1}^m \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}|_{F(p)}$ .  $\{\partial/\partial x^i\}$  and  $\{\partial/\partial y^j\}$  form bases for  $T_p \mathbb{R}^n$  and  $T_{F(p)} \mathbb{R}^m$ , respectively, so it makes sense to put  $dF_p$  is a matrix form. Then we obtain

$$\begin{bmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{bmatrix}$$

which is identical to the Jacobian matrix of  $F$  at  $p$ . Two things to note:

- It makes sense to discuss the Jacobian matrix of  $F$  because  $F$  is a map from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ .
- The same calculation applies if  $F : U \rightarrow V$  where  $U, V$  are open subsets of  $\mathbb{R}^n, \mathbb{R}^m$  or where  $U, V$  are open subsets of  $\mathbb{H}^n, \mathbb{H}^m$ .

We now consider a more general case when  $F : M \rightarrow N$  is a smooth map between two smooth manifolds with or without boundary. Let  $p \in M$  be given. Let  $(U, \phi), (V, \psi)$  be smooth charts of  $M, N$  that contain  $p, F(p)$ , respectively. Let  $\hat{F} = \psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \rightarrow \psi(V)$  and  $\hat{p} = \phi(p)$ . Then we obtain the following commutative diagram:

$$\begin{array}{ccc} U \cap F^{-1}(V) & \xrightarrow{F} & F \\ \downarrow \phi & & \downarrow \psi \\ \phi(U \cap F^{-1}(V)) & \xrightarrow{\hat{F}} & \hat{V} \end{array}$$

We compute

$$\begin{aligned}
dF_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) &= dF_p(d(\phi^{-1})_{\hat{p}}\left(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}\right)) && \text{(Definition of a coordinate vector)} \\
&= (dF_p \circ d(\phi^{-1})_{\hat{p}})\left(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}\right) \\
&= (d(F \circ \phi^{-1})_{\hat{p}})\left(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}\right) && \text{(3.7(Proof of Proposition 3.6))} \\
&= d(\psi^{-1} \circ \hat{F})_{\hat{p}}\left(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}\right) && \text{(See the diagram above)} \\
&= (d(\psi^{-1})_{\hat{F}(\hat{p})} \circ d\hat{F}_{\hat{p}})\left(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}\right) && \text{(3.7(Proof of Proposition 3.6))} \\
&= d(\psi^{-1})_{\hat{F}(\hat{p})}(d\hat{F}_{\hat{p}}\left(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}\right)) \\
&= d(\psi^{-1})_{\hat{F}(\hat{p})}\left(\sum_{j=1}^m \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial y^j}\Big|_{\hat{F}(\hat{p})}\right) && \text{(Discussion above)} \\
&= \sum_{j=1}^m \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) d(\psi^{-1})_{\hat{F}(\hat{p})}\left(\frac{\partial}{\partial y^j}\Big|_{\hat{F}(\hat{p})}\right) && \text{(Linearity of a differential)} \\
&= \sum_{j=1}^m \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) d(\psi^{-1})_{\psi(F(p))}\left(\frac{\partial}{\partial y^j}\Big|_{\psi(F(p))}\right) && \text{(Diagram above)} \\
&= \sum_{j=1}^m \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial y^j}\Big|_{F(p)} && \text{(Definition of a coordinate vector).}
\end{aligned}$$

Therefore, even in this general case,  $dF_p$  is represented in coordinate bases by the Jacobian matrix of  $\hat{F}$ .

**Remark.** The notation on P.63-64 is not easy to understand.

Let  $M$  be an  $n$ -dimensional smooth manifold. Let  $(U, \phi = (x^i)), (V, \psi = (\tilde{x}^i))$  be two smooth charts on  $M$  and  $p \in U \cap V$ . The textbook denotes the transition map  $\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$  by

$$\psi \circ \phi^{-1}(x) = (\tilde{x}^1(x), \dots, \tilde{x}^n(x)).$$

What (I think) this really means is

$$(\psi \circ \phi^{-1})(x^1(p), \dots, x^n(p)) = (\tilde{x}^1(p), \dots, \tilde{x}^n(p))$$

for each  $p \in U \cap V$ . The idea is that  $\phi = (x^i)$  is a diffeomorphism, so the textbook decides to denote each point in  $\phi(U \cap V)$  by  $x$  because every point in  $\phi(U \cap V)$  can be denoted by  $(x^1(p), \dots, x^n(p))$  for a unique  $p \in U \cap V$ .

Moreover, the second part of this discussion (after “By (3.9), the differential  $d(\psi \circ \phi^{-1})_{\phi(p)}$  can be written”) is even more confusing because:

- The textbook simply uses  $x^i$  and  $\tilde{x}^i$  to represent the coordinates of  $\hat{U}$  and  $\hat{V}$  instead of the coordinate functions of  $\phi$  and  $\psi$ .
- $\hat{U}$  and  $\hat{V}$  both live in  $\mathbb{R}^n$ , so it might seem unnecessary to use both  $x^i$  and  $\tilde{x}^i$ . It is actually necessary because we want to use  $\partial/\partial x^i$  to talk about the coordinate vectors induced by  $\phi$  and  $\partial/\partial \tilde{x}^i$  to talk about the coordinate vectors induced by  $\psi$ .

Finally, (3.12) in the textbook can be derived as following:

$$\begin{aligned}
v &= \sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \Big|_p \\
&= \sum_{i=1}^n v^i \left( \sum_{j=1}^n \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial \tilde{x}^j} \Big|_p \right) && ((3.11) \text{ in the textbook}) \\
&= \sum_{i=1}^n \left( \sum_{j=1}^n v^i \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p}) \right) \frac{\partial}{\partial \tilde{x}^j} \Big|_p \\
&= \sum_{j=1}^n \left[ \sum_{i=1}^n v^i \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p}) \right] \frac{\partial}{\partial \tilde{x}^j} \Big|_p.
\end{aligned}$$

**Exercise 3.17.** Let  $(x, y)$  denote the standard coordinates on  $\mathbb{R}^2$ . Verify that  $(\tilde{x}, \tilde{y})$  are global smooth coordinates on  $\mathbb{R}^2$ , where

$$\tilde{x} = x, \quad \tilde{y} = y + x^3.$$

Let  $p$  be the point  $(1, 0) \in \mathbb{R}^2$  (in standard coordinates), and show that

$$\frac{\partial}{\partial x} \Big|_p \neq \frac{\partial}{\partial \tilde{x}} \Big|_p,$$

even though the coordinate function  $x$  and  $\tilde{x}$  are identically equal.

*Proof.* The map  $(x, y) \mapsto (x, y + x^3)$  is a smooth automorphism on  $\mathbb{R}^2$ .

$$\begin{aligned}
\frac{\partial}{\partial x} \Big|_p &= \frac{\partial \tilde{x}}{\partial x}(1, 0) \frac{\partial}{\partial \tilde{x}} \Big|_p + \frac{\partial \tilde{y}}{\partial x}(1, 0) \frac{\partial}{\partial \tilde{y}} \Big|_p && ((3.11) \text{ in the textbook}) \\
&= \frac{\partial}{\partial \tilde{x}} \Big|_p + 3 \frac{\partial}{\partial \tilde{y}} \Big|_p && \left( \frac{\partial \tilde{y}}{\partial x} = 3x^2 \right) \\
&\neq \frac{\partial}{\partial \tilde{x}} \Big|_p.
\end{aligned}$$

□

**Exercise 3.19.** Suppose  $M$  is a smooth manifold with boundary. Show that  $TM$  has a natural topology and smooth structure making it into a smooth manifold with boundary, such that if  $(U, (x^i))$  is any smooth boundary chart for  $M$ , then rearranging the coordinates in the natural chart  $(\pi^{-1}(U), (x^i, v^i))$  for  $TM$  yields a boundary chart  $(\pi^{-1}(U), (v^i, x^i))$ .

*Proof.* The proof is similar to that of Proposition 3.18. We begin by defining the maps that will become our smooth charts. Given any smooth (possibly boundary) chart  $(U, \phi)$  for  $M$ , note that  $\pi^{-1}(U) \subset TM$  is the set of all tangent vectors to  $M$  at all points of  $U$ . Let  $(x^1, \dots, x^n)$  denote the coordinate functions of  $\phi$ , and define a map  $\tilde{\phi}$  that maps  $\pi^{-1}(U)$  into  $\mathbb{H}^{2n}$  or  $\mathbb{R}^{2n}$  by

$$\tilde{\phi} \left( v^i \frac{\partial}{\partial x^i} \Big|_p \right) = (v^1, \dots, v^n, x^1(p), \dots, x^n(p)).$$

In case  $(U, \phi)$  is a boundary chart,  $\tilde{\phi}$  indeed maps  $\pi^{-1}(U)$  into  $\mathbb{H}^{2n}$  because  $x^n(p) \geq 0$ . Its image set is  $\mathbb{R}^n \times \phi(U)$ , which is an open subset of  $\mathbb{R}^{2n}$  or  $\mathbb{H}^{2n}$ . Now suppose we are given two smooth charts  $(U, \phi)$  and  $(V, \psi)$  for  $M$ , and let  $(\pi^{-1}(U), \tilde{\phi}), (\pi^{-1}(V), \tilde{\psi})$  be the corresponding charts on  $TM$ . The sets

$$\begin{aligned}
\tilde{\phi}(\pi^{-1}(U) \cap \pi^{-1}(V)) &= \mathbb{R}^n \times \phi(U \cap V) && \text{and} \\
\tilde{\psi}(\pi^{-1}(U) \cap \pi^{-1}(V)) &= \mathbb{R}^n \times \psi(U \cap V)
\end{aligned}$$

are open in  $\mathbb{R}^{2n}$  or  $\mathbb{H}^{2n}$ , and the transition map  $\tilde{\psi} \circ \tilde{\phi}^{-1}$  can be written explicitly using (3.1) as

$$\begin{aligned}
&(\tilde{\psi} \circ \tilde{\phi}^{-1})(x^1, \dots, x^n, v^1, \dots, v^n) \\
&= \left( \frac{\partial \tilde{x}^1}{\partial x^j}(x) v^j, \dots, \frac{\partial \tilde{x}^n}{\partial x^j}(x) v^j, \tilde{x}^1(x), \dots, \tilde{x}^n(x) \right).
\end{aligned}$$



The rest of the proof is identical to that of Proposition 3.18.  $\square$

**Proposition 3.21.** If  $F : M \rightarrow N$  is a smooth map, then its global differential  $dF : TM \rightarrow TN$  is a smooth map.

*Proof.* Let  $(p_0, v_0) \in TM$  be given. It suffices to show that  $F$  is smooth in a neighborhood around  $(p_0, v_0)$ . Let  $(U, \phi), (V, \psi)$  be given such that  $p_0 \in U, F(U) \subset V$  and  $F(p_0) \in V$ . The set  $\pi^{-1}(U)$  is open in  $TM$  because that is how we give  $TM$  a topology in Proposition 3.18. Moreover,  $\pi^{-1}(U)$  is a neighborhood of  $(p_0, v_0)$  in  $TM$ . We will consider the charts  $(\pi^{-1}, \tilde{\phi})$  and  $(\pi^{-1}(V), \tilde{\psi})$  as defined in Proposition 3.18. Let  $(x^1, \dots, x^m, v^1, \dots, v^m) \in \tilde{\phi}(\pi^{-1}(U))$ . Let  $x = (x^1, \dots, x^m)$  and  $p = \phi^{-1}(x)$ . Then

$$dF_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}\Big|_{F(p)}$$

by (3.9) in the textbook. Then

$$(\tilde{\psi} \circ dF \circ \tilde{\phi}^{-1})(x^1, \dots, x^m, v^1, \dots, v^m) = ((\psi^1 \circ F \circ \phi^{-1})(x), \dots, (\psi^n \circ F \circ \phi^{-1})(x), \frac{\partial F^1}{\partial x^i}(p)v^i, \dots, \frac{\partial F^n}{\partial x^i}(p)v^i)$$

Since each component function is smooth,  $\tilde{\psi} \circ dF \circ \tilde{\phi}^{-1}$  is smooth. Therefore,  $dF$  is a smooth map from  $TM$  to  $TN$ .  $\square$

**Proposition 3.23.** Suppose  $M$  is a smooth manifold with or without boundary and  $p \in M$ . Every  $v \in T_p M$  is the velocity of some smooth curve in  $M$ .

*Proof.* First, suppose that  $p \in \text{Int } M$ . Let  $(U, \phi)$  be a smooth coordinate chart centered at  $p$ , and write  $v = v^i \partial/\partial x^i|_p$  in terms of the coordinate basis. Without loss of generality,  $\phi(p) = 0 \in \mathbb{R}^n$ . Now, define  $\hat{\gamma} : (-\epsilon, \epsilon) \rightarrow \hat{U}$  by  $\hat{\gamma}(t) = (tv_1, \dots, tv_n)$  for sufficiently small  $\epsilon > 0$ . Let  $\gamma : (-\epsilon, \epsilon) \rightarrow U$  be defined by  $\gamma = \phi^{-1} \circ \hat{\gamma}$ . Then  $\gamma$  is actually a smooth map from a 1-manifold  $(-\epsilon, \epsilon)$  to an  $n$ -manifold  $M$  (with or without boundary). We will use the formula derived in (3.1) and obtain

$$\begin{aligned} \gamma'(0) &= d\gamma\left(\frac{d}{dt}\Big|_0\right) \\ &= \sum_{j=1}^n \frac{\partial \hat{\gamma}^j}{\partial t}(0) \frac{\partial}{\partial x^j}\Big|_p \\ &= \sum_{j=1}^n v^j \frac{\partial}{\partial x^j}\Big|_p \\ &= v. \end{aligned}$$

If  $p \in \partial M$ , then we do the exact same thing as above except that the domain will be  $[0, \epsilon)$  if  $v_n > 0$  and  $(-\epsilon, 0]$  if  $v_n \leq 0$ .  $\square$

**Proposition 3.24.** Let  $F : M \rightarrow N$  be a smooth map, and let  $\gamma : J \rightarrow M$  be a smooth curve. For any  $t_0 \in J$ , the velocity at  $t = t_0$  of the composite curve  $F \circ \gamma : J \rightarrow N$  is given by

$$(F \circ \gamma)'(t_0) = dF(\gamma'(t_0)).$$

*Proof.*

$$\begin{aligned} (F \circ \gamma)'(t_0) &= d(F \circ \gamma)\left(\frac{d}{dt}\Big|_{t_0}\right) && \text{(definition of the velocity)} \\ &= (dF \circ d\gamma)\left(\frac{d}{dt}\Big|_{t_0}\right) && \text{(Corollary 3.22(a))} \\ &= dF\left(d\gamma\left(\frac{d}{dt}\Big|_{t_0}\right)\right) \\ &= dF(\gamma'(t_0)) && \text{(definition of the velocity).} \end{aligned}$$

$\square$

**Proposition 3.25.** Suppose  $F : M \rightarrow N$  is a smooth map,  $p \in M$ , and  $v \in T_p M$ . Then

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve  $\gamma : J \rightarrow M$  such that  $0 \in J$ ,  $\gamma(0) = p$ , and  $\gamma'(0) = v$ .

*Proof.* This is a special case of (3.24) where  $t_0 = 0$ . □

### 3.2. Problems.

**Problem 3-1.** Suppose  $M$  and  $N$  are smooth manifolds with or without boundary, and  $F : M \rightarrow N$  is a smooth map. Show that  $dF_p : T_p M \rightarrow T_{F(p)} N$  is the zero map for each  $p \in M$  if and only if  $F$  is constant on each component of  $M$ .

*Proof.* Suppose  $dF_p : T_p M \rightarrow T_{F(p)} N$  is the zero map for each  $p \in M$ . It suffices to show that for every  $p \in M$ , there exists a neighborhood of  $p$  on which  $F$  is constant. Let  $p \in M$  and  $(U, \phi) \in \mathcal{A}_M$ ,  $(V, \psi) \in \mathcal{A}_N$  be given such that  $p \in U$  and  $F(U) \subset V$ . Without loss of generality, we assume  $\hat{U} = \phi(U)$  is an open ball in  $\mathbb{R}^m$ . Then for any  $i, j$  and for any  $q \in \hat{U}$ ,

$$\begin{aligned} dF_q\left(\frac{\partial}{\partial x^i}\Big|_q\right)(\pi_j \circ \psi) &= 0 \implies \left(\frac{\partial}{\partial x^i}\Big|_q\right)(\pi_j \circ \psi \circ F) = 0 \\ &\implies \left(\frac{\partial}{\partial x^i}\Big|_{\phi(q)}\right)(\pi_j \circ \psi \circ F \circ \phi^{-1}) = 0. \end{aligned}$$

Fix  $j$ . Then every partial derivative of  $\pi_j \circ \psi \circ F \circ \phi^{-1}$  at every point in  $\hat{U}$  is 0. The intermediate value theorem implies that  $\pi_j \circ \psi \circ F \circ \phi^{-1}$  is constant on  $\hat{U}$  because  $\hat{U}$  is an open ball. In other words,  $(\pi_j \circ \psi \circ F \circ \phi^{-1})(\hat{U}) = \{y_j\}$  for some  $y_j \in \mathbb{R}$ . Since this is true for every  $j$  and  $\pi_j$  is the projection of the  $j$ th coordinate,  $(\psi \circ F \circ \phi^{-1})(\hat{U}) = \{y\}$  where  $y = (y_1, \dots, y_n)$ . Then  $(F \circ \phi^{-1})(\hat{U}) = F(U) = \psi^{-1}(y)$ . Since  $\psi$  is a homeomorphism, there exists exactly one point in  $\psi^{-1}(U)$ . In other words,  $F$  is constant on  $U$ . Therefore,  $F$  is constant on each path component.

Suppose  $F$  is constant on each component of  $M$ . Let  $p \in M$ . Choose a chart  $(U, \phi) \in \mathcal{A}_M$  such that  $p \in U$ . Then  $F \circ \phi^{-1}$  is constant in a neighborhood around  $\phi(p)$ . For any  $i$ ,

$$\begin{aligned} dF_p\left(\frac{\partial}{\partial x^i}\Big|_p\right)(f) &= \frac{\partial}{\partial x^i}\Big|_p(f \circ F) \\ &= \frac{\partial}{\partial x^i}\Big|_{\phi(p)}(f \circ F \circ \phi^{-1}) \\ &= 0 \end{aligned}$$

because  $f \circ F \circ \phi^{-1}$  is constant in a neighborhood around  $\phi(p)$ . By Proposition 3.15,  $\partial/\partial x^i|_p$  form a basis for  $T_p M$ . Since  $dF_p$  sends each basis element to 0,  $dF_p = 0$ . □

**Problem 3-2(Proof of Proposition 3.14).** Let  $M_1, \dots, M_k$  be smooth manifolds, and for each  $j$ , let  $\pi_j : M_1 \times \dots \times M_k \rightarrow M_j$  be the projection onto the  $M_j$  factor. For any point  $p = (p_1, \dots, p_k) \in M_1 \times \dots \times M_k$ , the map

$$\alpha : T_p(M_1 \times \dots \times M_k) \rightarrow T_{p_1} M_1 \oplus \dots \oplus T_{p_k} M_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \dots, d(\pi_k)_p(v))$$

is an isomorphism. The same is true if one of the spaces  $M_i$  is a smooth manifold with boundary.

*Proof.* It suffices to show this for the case that  $k = 2$  because the results extend to arbitrary  $k$  by induction. Let  $\mathcal{A}_{M_1}, \mathcal{A}_{M_2}, \mathcal{A}_{M_1 \times M_2}$  be the smooth structures of  $M_1, M_2, M_1 \times M_2$ .

We first define a lot of notations.

- Let  $d_1, d_2$  denote the dimensions of  $M_1, M_2$  and let  $d = d_1 + d_2$  denote the dimension of  $M_1 \times M_2$ .
- Let  $p = (p_1, p_2) \in M_1 \times M_2$  be given. Choose  $(U, \phi = (x^i)) \in \mathcal{A}_{M_1}$ ,  $(V, \psi = (y^i)) \in \mathcal{A}_{M_2}$  with  $p_1 \in U$  and  $p_2 \in V$ . Let  $q_1 = \phi(p_1)$ ,  $q_2 = \psi(p_2)$ ,  $q = q_1 \times q_2$ .
- $(U \times V, (z^i)) \in \mathcal{A}_{M_1 \times M_2}$  and  $(p_1, p_2) \in U \times V$  where  $(z^i) = \phi \times \psi$ . More specifically,  $z^i = x^i \circ \pi_1$  for  $1 \leq i \leq d_1$  and  $z^i = y^i \circ \pi_2$  for  $d_1 + 1 \leq i \leq d_1 + d_2$ .

Note that we use  $x^i, y^i, z^i, \pi_1$  to mean two different things in this solution:

- $x^i$  is either the  $i$ th coordinate function of  $\phi$  or the  $i$ th projection map  $\mathbb{R}^{d_1} \rightarrow \mathbb{R}$ .
- $y^i$  is either the  $i$ th coordinate function of  $\psi$  or the  $i$ th projection map  $\mathbb{R}^{d_2} \rightarrow \mathbb{R}$ .
- $z^i$  is either the  $i$ th coordinate function of  $\phi \times \psi$  or the  $i$ th projection map  $\mathbb{R}^{d_1+d_2} \rightarrow \mathbb{R}$ .
- $\pi_1$  is either the projection map  $M_1 \times M_2 \rightarrow M_1$  or the projection map  $\mathbb{R}^{d_1+d_2} \rightarrow \mathbb{R}^{d_1}$ .
- $\pi_2$  is either the projection map  $M_1 \times M_2 \rightarrow M_2$  or the projection map  $\mathbb{R}^{d_1+d_2} \rightarrow \mathbb{R}^{d_2}$ .

By Proposition 3.15,  $\{\partial/\partial x^1|_{p_1}, \dots, \partial/\partial x^{d_1}|_{p_1}\}, \{\partial/\partial y^1|_{p_2}, \dots, \partial/\partial y^{d_2}|_{p_2}\}, \{\partial/\partial z^1|_p, \dots, \partial/\partial z^{d_1+d_2}|_p\}$  form bases for  $T_{p_1}M_1, T_{p_2}M_2, T_p(M_1 \times M_2)$ .

$\alpha(\partial/\partial z^1|_p) = (d(\pi_1)_p(\partial/\partial z^1|_p), d(\pi_2)_p(\partial/\partial z^1|_p))$ . We claim that  $d(\pi_1)_p(\partial/\partial z^1|_p) = \partial/\partial x^1|_{p_1}$ .

$$\begin{aligned}
d(\pi_1)_p(\partial/\partial z^1|_p)(f) &= d(\pi_1)_p(d(\phi^{-1} \times \psi^{-1})_q)(\frac{\partial}{\partial z^1}|_q)(f) \\
&= (d(\pi_1)_p \circ d(\phi^{-1} \times \psi^{-1})_q)(\frac{\partial}{\partial z^1}|_q)(f) \\
&= d(\pi_1 \circ (\phi^{-1} \times \psi^{-1}))_q(\frac{\partial}{\partial z^1}|_q)(f) \\
&= \lim_{h \rightarrow 0} \frac{(f \circ \pi_1 \circ (\phi^{-1} \times \psi^{-1}))(q + e_1 h) - (f \circ \pi_1 \circ (\phi^{-1} \times \psi^{-1}))(q)}{h} \\
&= \lim_{h \rightarrow 0} \frac{(f \circ \pi_1)(\phi^{-1}(q_1 + e_1 h), p_2) - (f \circ \pi_1)(p)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(\phi^{-1}(q_1 + e_1 h)) - f(p_1)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(\phi^{-1}(q_1 + e_1 h)) - f(\phi^{-1}(q_1))}{h} \\
&= (\frac{\partial}{\partial x^1}|_{q_1})(f \circ \phi^{-1}) \\
&= d(\phi^{-1})_{q_1}(\frac{\partial}{\partial x^1}|_{q_1})(f) \\
&= (\frac{\partial}{\partial x^1}|_{p_1})(f).
\end{aligned}$$

The same result can be shown for the other combinations of  $\pi_1, \pi_2$  and  $z^1, \dots, z^{d_1+d_2}$ . For any  $c_1, \dots, c_{d_1+d_2} \in \mathbb{R}$ ,

$$\begin{aligned}
\alpha(\sum_{i=1}^{d_1+d_2} c_i \frac{\partial}{\partial z^i}|_p) &= \sum_{i=1}^{d_1+d_2} c_i \alpha(\frac{\partial}{\partial z^i}|_p) \\
&= \sum_{i=1}^{d_1+d_2} c_i (d(\pi_1)_p \frac{\partial}{\partial z^i}|_p, d(\pi_2)_p \frac{\partial}{\partial z^i}|_p) \\
&= \sum_{i=1}^{d_1} c_i (d(\pi_1)_p \frac{\partial}{\partial z^i}|_p, d(\pi_2)_p \frac{\partial}{\partial z^i}|_p) + \sum_{i=d_1+1}^{d_2} c_i (d(\pi_1)_p \frac{\partial}{\partial z^i}|_p, d(\pi_2)_p \frac{\partial}{\partial z^i}|_p) \\
&= \sum_{i=1}^{d_1} c_i (\frac{\partial}{\partial x^i}|_{p_1}, 0) + \sum_{i=1}^{d_2} c_{d_1+i} (0, \frac{\partial}{\partial y^i}|_{p_2}) \\
&= (c_1 \frac{\partial}{\partial x^1}|_{p_1} + \dots + c_{d_1} \frac{\partial}{\partial x^{d_1}}|_{p_1}, c_{d_1+1} \frac{\partial}{\partial y^1}|_{p_2} + \dots + c_{d_1+d_2} \frac{\partial}{\partial y^{d_2}}|_{p_2}).
\end{aligned}$$

Therefore,  $\alpha$  is bijective. □

**Problem 3-3.** Prove that if  $M$  and  $N$  are smooth manifolds, then  $T(M \times N)$  is diffeomorphic to  $TM \times TN$ .

*Proof.* Let  $\pi_1 : M \times N \rightarrow M$  and  $\pi_2 : M \times N \rightarrow N$  be the obvious projections of the corresponding coordinates.  $\pi_1, \pi_2$  are clearly smooth, so Proposition 3.21 shows that  $d\pi_1 : T(M \times N) \rightarrow TM$  and  $d\pi_2 : T(M \times N) \rightarrow TN$  are both smooth. By (2.16(Proof of Proposition 2.15)(b)),  $d\pi_1 \times d\pi_2$  is a smooth map.

By (3-2(Proof of Proposition 3.14)),  $d\pi_1 \times d\pi_2$  is a bijection between  $T_{(p,q)}(M \times N)$  and  $T_p(M) \times T_q(N)$ . Since  $d\pi_1 \times d\pi_2$  sends  $((p, q), \sigma)$  to  $(p, d\pi_1(\sigma)) \times (q, d\pi_2(\sigma))$ , we conclude that  $d\pi_1 \times d\pi_2$  is bijective.  $\square$

#### 4. CHAPTER 4: SUBMERSIONS, IMMERSIONS, AND EMBEDDINGS

**Proposition 4.1.** Suppose  $F : M \rightarrow N$  is a smooth map and  $p \in M$ . If  $dF_p$  is surjective, then  $p$  has a neighborhood  $U$  such that  $F|_U$  is a submersion. If  $dF_p$  is injective, then  $p$  has a neighborhood  $U$  such that  $F|_U$  is an immersion.

*Proof.* Let  $(U, \phi)$  be a chart containing  $p$  and  $(V, \psi)$  be a chart containing  $F(p)$ . We may assume  $F(U) \subset V$ . It suffices to show that if the Jacobian of  $F$  with respect to  $(U, \phi)$  is full rank at  $p$ , then it is full rank in some neighborhood of  $p$  contained in  $U$ . Example 1.28 in the textbook shows that the set of full rank matrices is an open subset of  $M(m \times n, \mathbb{R})$ . We will use the notation  $J|_q$  to denote the Jacobian of  $F$  with respect to  $(U, \phi)$  at  $q \in U$ . Then  $J|_p$  is an element of an open subset of  $M(m \times n, \mathbb{R})$ . Each entry of  $J|_q$  is of the form  $\frac{\partial}{\partial x^i}(\psi^j \circ F \circ \phi)(\phi(q))$  where each  $(\frac{\partial}{\partial x^i}(\psi^j \circ F \circ \phi)) \circ \phi$  is a smooth function. Therefore, there exists a neighborhood of  $p$  such that the Jacobian matrix of  $F$  with respect to  $(U, \phi)$  is full rank.  $\square$

**Exercise 4.3(Verification of Example 4.2).** Verify the following claims:

- (a) Suppose  $M_1, \dots, M_k$  are smooth manifolds. Then each of the projection maps  $\pi_i : M_1 \times \dots \times M_k \rightarrow M_i$  is a smooth submersion.

*Proof.*

- (a) Let  $d_1, \dots, d_k$  denote the dimensions of  $M_1, \dots, M_k$ , respectively. Let  $M = M_1 \times \dots \times M_k$ . (2-2(Proof of Proposition 2.12)) implies that  $\pi_i$  is smooth for each  $i$  by setting  $F = \text{Id} : M \rightarrow M$ . Let  $p = (p_1, \dots, p_k) \in M$ . Thus it suffices to show that the dimension of  $d(\pi_i)_p(T_p(M))$  is the same as the dimension of  $T_{p_i}(M_i)$ .

By Proposition 3.12,  $\dim(T_p(M)) = \sum d_i$ . Since the  $\alpha$  defined in (3-2(Proof of Proposition 3.14)) is an isomorphism,

$$(4.1) \quad \dim(d(\pi_1)_p(T_p(M)) \oplus \dots \oplus d(\pi_k)_p(T_p(M))) = \dim(T_p(M)) = \sum d_i.$$

However, for each  $i$ ,  $d(\pi_i)_p(T_p(M)) \subset T_{p_i}M_i$ . Thus  $\dim(d(\pi_i)_p(T_p(M))) \leq \dim(T_{p_i}M_i) = d_i$ . By (4.1),  $\dim(d(\pi_i)_p(T_p(M))) = \dim(T_{p_i}M_i)$ .  $\square$

**Exercise 4.4.** Show that a composition of smooth submersions is a smooth submersion, and a composition of smooth immersions is a smooth immersion. Give a counterexample to show that a composition of maps of constant rank need not have constant rank.

*Proof.* Let  $M, N, L$  be smooth manifolds with or without boundary, and  $F : M \rightarrow N, G : N \rightarrow L$  be given. If  $F, G$  are submersions,  $dF_p$  and  $dG_{F(p)}$  are surjective for each  $p$ . Then  $d(G \circ F)_p = dG_{F(p)} \circ dF_p$  is surjective for each  $p$  by (3.7(Proof of Proposition 3.6)). Thus a composition of smooth submersions is a smooth submersion. By the exact same argument, a composition of smooth immersions is a smooth immersion.

Counterexample?

$\square$

#### 5. APPENDIX A: REVIEW OF TOPOLOGY

**Exercise A.18(Proof of Proposition A.17).** Let  $X$  be a topological space and let  $S$  be a subspace of  $X$ .

- (a)  
(b)  
(c)  
(d)  
(e)

- (f) If  $\mathcal{B}$  is a basis for the topology of  $X$ , then  $\mathcal{B}_S = \{B \cap S \mid B \in \mathcal{B}\}$  is a basis for the subspace topology on  $S$ .
- (g) If  $X$  is Hausdorff, then so is  $S$ .
- (h) If  $X$  is first-countable, then so is  $S$ .
- (i) If  $X$  is second-countable, then so is  $S$ .

*Proof.*

- (a)
- (b)
- (c)
- (d)
- (e)
- (f) The union of  $B \cap S$  is  $S$ . Let  $U \cap S$  be an open subset of  $S$  where  $U$  is open in  $X$ , and  $x \in U \cap S$ . Then there exists  $B \in \mathcal{B}$  such that  $x \in B \subset U$  since  $\mathcal{B}$  is a basis. Therefore,  $x \in B \cap S \subset U \cap S$  with  $B \cap S \in \mathcal{B}_S$ .
- (g) Let  $x \neq y \in S$ . There exist two disjoint open sets  $U, V$  of  $X$  containing  $x, y$ , respectively. Then  $U \cap S$  and  $V \cap S$  are disjoint open sets of  $X$  containing  $x, y$ , respectively.
- (h)
- (i) Let  $\mathcal{B}$  be a countable basis of  $X$ . Then  $\{B \cap S \mid B \in \mathcal{B}\}$  is a countable basis of  $S$  by (f).

□

**Exercise A.24(Proof of Proposition A.23).** Suppose  $X_1, \dots, X_k$  are topological spaces, and let  $X_1 \times \dots \times X_k$  be their product space.

- (a) CHARACTERISTIC PROPERTY: If  $B$  is a topological space, a map  $F : B \rightarrow X_1 \times \dots \times X_k$  is continuous if and only if each of its component functions  $F_i = \pi_i \circ F : B \rightarrow X_i$  is continuous.

*Proof.*

- (a) Suppose  $F$  is continuous. Since  $\pi_i$  is continuous by (c) and the composition of continuous functions is continuous,  $\pi_i \circ F$  is continuous. Suppose each component function is continuous. Let  $B_1 \times \dots \times B_k$  be a basis element of  $X_1 \times \dots \times X_k$ .

$$\begin{aligned} F^{-1}(B_1 \times \dots \times B_k) &= F^{-1}(\cap_{i=1}^k \pi_i^{-1}(B_1 \times \dots \times B_k)) \\ &= \cap_{i=1}^k F^{-1}(\pi_i^{-1}(B_1 \times \dots \times B_k)) \\ &= \cap_{i=1}^k (\pi_i \circ F)^{-1}(B_1 \times \dots \times B_k). \end{aligned}$$

Since the intersection of finitely many open sets is open,  $F$  is continuous.

□

## 6. APPENDIX B: REVIEW OF LINEAR ALGEBRA

**Exercise B.49.** Two norms  $|\cdot|_1$  and  $|\cdot|_2$  on a vector space  $V$  are said to be equivalent if there are positive constants  $c, C$  such that

$$c|v|_1 \leq |v|_2 \leq C|v|_1$$

for all  $v \in V$ . Show that equivalent norms determine the same topology.

*Proof.* Such a relation is symmetric for  $c|v|_1 \leq |v|_2 \leq C|v|_1$  implies  $(1/C)|v|_2 \leq |v|_1 \leq (1/c)|v|_2$ . Let  $\mathcal{T}_1, \mathcal{T}_2$  be the topologies induced by  $|\cdot|_1, |\cdot|_2$ . It suffices to show that  $\forall v \in V, \forall U \in \mathcal{T}_2, (v \in U \implies \exists r > 0, B_1(v, r) \subset U)$  where  $B_1(v, r)$  is the open ball centered at  $v$  with the radius  $r$  using the  $|\cdot|_1$ . Since  $v \in U$  and  $U$  is open,  $\exists r > 0$  such that  $B_2(v, r) \subset U$ . Then for any  $w \in V$ ,  $|v - w|_1 \leq |v - w|_2/c$ , so  $B_1(v, r/c) \subset B_2(v, r)$ . □

## 7. APPENDIX C: REVIEW OF CALCULUS

**Exercise C.1.** Suppose that  $F : U \rightarrow W$  is differentiable at  $a \in U$ . Show that the linear map satisfying

$$\lim_{v \rightarrow 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} = 0$$

is unique.

*Proof.* Let  $L, L'$  be two such linear maps.

$$\begin{aligned} \lim_{v \rightarrow 0} \frac{|Lv - L'v|}{|v|} &= \lim_{v \rightarrow 0} \frac{|(F(a+v) - F(a) - L'v) - (F(a+v) - F(a) - Lv)|}{|v|} \\ &= \lim_{v \rightarrow 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} + \lim_{v \rightarrow 0} \frac{|F(a+v) - F(a) - L'v|}{|v|} \\ &= 0 + 0 = 0. \end{aligned}$$

If  $L \neq L'$ ,  $(L - L')v_0 \neq 0$  for some  $v_0$ . Then  $\lim_{v \rightarrow 0} \frac{|Lv - L'v|}{|v|} = \lim_{h \rightarrow 0} \frac{|L(hv_0) - L'(hv_0)|}{|hv_0|} = \frac{|(L - L')v_0|}{|v_0|} \neq 0$ . This is a contradiction, so  $L = L'$ .  $\square$

## 8. DICTIONARY

### 8.1. Topological Manifolds.

**Definition 8.1** (Topological Manifold). A *topological  $n$ -manifold* is a Hausdorff, second-countable topological space each point of which has a neighborhood that is homeomorphic to an open subset  $\mathbb{R}^n$ .

**Definition 8.2** (Coordinates). Let  $M$  be a topological  $n$ -manifold. Let  $U$  be an open subset of  $M$ ,  $\hat{U}$  be an open subset of  $\mathbb{R}^n$ ,  $\phi : U \rightarrow \hat{U}$  be a homeomorphism.

- The pair  $(U, \phi)$  is called a *coordinate chart* or a *chart*.
- $U$  is called a *coordinate domain* or a *coordinate neighborhood* and  $\phi$  is called a *coordinate map*.
- If  $\phi(U)$  is an open ball in  $\mathbb{R}^n$ ,  $U$  is called a *coordinate ball*.
- If  $\phi(U)$  is an open cube in  $\mathbb{R}^n$ ,  $U$  is called a *coordinate cube*.
- The coordinate functions of  $\phi$  are often denoted as  $(x^1, \dots, x^n)$ . Thus a chart is sometimes denoted by  $(U, (x^1, \dots, x^n))$  or  $(U, (x^i))$ .

**Definition 8.3** (Atlas). Let  $M$  be a topological  $n$ -manifold. An *atlas for  $M$*  is a collection of charts  $(U_\alpha, \phi_\alpha)$  such that  $M = \bigcup_\alpha U_\alpha$ .

**Definition 8.4** (Transition Map). Let  $M$  be a topological  $n$ -manifold and  $(U, \phi), (V, \psi)$  be coordinate charts such that  $U \cap V \neq \emptyset$ .  $\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$  is called a *transition map* from  $\phi$  to  $\psi$ .

**Definition 8.5** (Closed Upper Half-Space).  $\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$ , and  $\partial\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n = 0\}$ .

**Definition 8.6** (Manifold With Boundary). Let  $M$  be a second-countable Hausdorff space and fix  $n$ . Suppose that for every  $p \in M$ , one of the following conditions is satisfied:

- (1) There exists a neighborhood  $U$  of  $p$  and a homeomorphism  $\phi : U \rightarrow \hat{U}$  where  $\hat{U}$  is an open subset of  $\mathbb{R}^n$ .  $p$  is called an *interior point* and  $(U, \phi)$  is called an *interior chart*.
- (2) There exists a neighborhood  $U$  of  $p$  and a homeomorphism  $\phi : U \rightarrow \hat{U}$  where  $\hat{U}$  is an open subset of  $\mathbb{H}^n$  with  $\phi(p) \in \partial\mathbb{H}^n$ .  $p$  is called a *boundary point*.

Then  $M$  is called an  *$n$ -dimensional topological manifold with boundary*. Note that every topological manifold is a topological manifold with boundary.

**Definition 8.7** (Support). If  $f$  is any real-valued or vector-valued function on a topological space  $M$ , the *support of  $f$* , denoted by  $\text{supp } f$ , is the closure of the set of points where  $f$  is nonzero:

$$\text{supp } f = \overline{\{p \in M : f(p) \neq 0\}}.$$

**Definition 8.8** (Bump Function). If  $M$  is a topological space,  $A \subset M$  is a closed subset, and  $U \subset M$  is an open subset containing  $A$ , a continuous function  $\psi : M \rightarrow \mathbb{R}$  is called a *bump function for  $A$  supported in  $U$*  if  $0 \leq \psi \leq 1$  on  $M$ ,  $\psi \equiv 1$  on  $A$ , and  $\text{supp } \psi \subset U$ .

## 8.2. Smooth Manifolds.

**Definition 8.9** (Smoothly Compatible). Let  $M$  be a topological  $n$ -manifold. Two coordinate charts  $(U, \phi), (V, \psi)$  are called *smoothly compatible* if  $U \cap V = \emptyset$  or the transition map  $\psi \circ \phi^{-1}$  is a diffeomorphism.

**Definition 8.10** (Smooth Atlas). Let  $M$  be a topological  $n$ -manifold. A *smooth atlas* is an atlas  $\mathcal{A}$  such that any two charts in  $\mathcal{A}$  are smoothly compatible with each other.

**Definition 8.11** (Smooth Structure). If  $M$  is a topological  $n$ -manifold, an atlas  $\mathcal{A}$  that is not properly contained in any larger smooth atlas is called *maximal* or a *smooth structure on  $M$* .

**Definition 8.12** (Smooth Manifold). A *smooth manifold* is a topological manifold equipped with a smooth structure.

**Definition 8.13.** Suppose  $(M, \mathcal{A})$  is a smooth manifold.

- Any chart  $(U, \phi) \in \mathcal{A}$  is called a *smooth chart*.
- Given a smooth chart  $(U, \phi)$ ,  $U$  is called a smooth coordinate domain and  $\phi$  is called a *smooth coordinate map*.
- Given a smooth chart  $(U, \phi)$ ,  $U$  is called a *smooth coordinate ball* if it is a coordinate ball.

**Remark.** One must define a smooth structure on a topological manifold before talking about a smooth chart.

**Definition 8.14** (Smooth Maps). Let  $M, N$  be smooth manifolds with or without boundary and  $F : M \rightarrow N$  be a map.  $F$  is a *smooth map* if for every  $p \in M$ , there exist smooth charts  $(U, \phi)$  containing  $p$  and  $(V, \psi)$  containing  $F(p)$  such that

- $F(U) \subset V$ ;
- $\psi \circ F \circ \phi^{-1} : \phi(U) \rightarrow \psi(V)$  is smooth.

**Definition 8.15** (Coordinate Representatin of a Smooth Map). Let  $(M, \mathcal{A}_M)$  and  $(N, \mathcal{A}_N)$  be smooth manifolds. Let  $F : M \rightarrow N$  be a smooth map and  $(U, \phi) \in \mathcal{A}_M$  and  $(V, \psi) \in \mathcal{A}_N$  be given. Then  $\hat{F} = \psi \circ F \circ \phi^{-1}$  is called the coordinate representation of  $F$  with respect to  $(U, \phi)$  and  $(V, \psi)$ .

**Definition 8.16** (Diffeomorphism). Let  $M, N$  be smooth manifolds with or without boundary. A diffeomorphism is a smooth map  $F : M \rightarrow N$  with a smooth inverse.

**Definition 8.17** (Smooth on a subset). Let  $M, N$  be smooth manifolds with or without boundary and  $A \subset M$  be an arbitrary subset. A map  $F : A \rightarrow N$  is said to be *smooth on  $A$*  if every  $p \in A$  has an open neighborhood  $W \subset M$  such that there exists a smooth map  $\tilde{F} : W \rightarrow N$  with  $\tilde{F}_{W \cap A} = F$ .

## 8.3. Tangent Vectors.

**Definition 8.18** (Derivation). Let  $M$  be a smooth manifold with or without boundary. A derivation at  $p \in M$  is a linear map  $v : C^\infty(M) \rightarrow \mathbb{R}$  such that

$$v(fg) = f(p)vg + g(p)vf$$

for all  $f, g \in C^\infty(M)$ .

This corresponds to “arrows that are tangent to  $M$  and whose basepoints are attached to  $M$  at  $p$ ” even though it may not be easy to see that from this definition.

**Definition 8.19** (Tangent Space). The tangent space  $T_p M$  to  $M$  at  $p$  is the vector space of all derivations of  $C^\infty(M)$  at  $p$ .

Derivation of $C^\infty(M)$	Geometric tangent vector on $M$
Differential of a smooth map between manifolds	Total derivative of a map between Euclidean spaces

**Definition 8.20** (Differential).  $M, N$  are smooth manifolds with or without boundary, and  $F : M \rightarrow N$  is a smooth map. The *differential of  $F$  at  $p$*  is the linear map  $dF_p : T_p M \rightarrow T_{F(p)} N$  defined by

$$dF_p(v) := f \mapsto v(f \circ F)$$

Equivalently,  $\forall v \in T_p M, \forall f \in C^\infty(N), dF_p(v)(f) = v(f \circ F)$ . This corresponds to “the directional derivative of  $F$  at  $p$  in the direction of the arrow  $v$ .”

**Definition 8.21** (Coordinate Vectors). Let  $(M, \mathcal{A})$  be a smooth manifold without boundary. Let  $p \in M$  and choose a chart  $(U, \phi) \in \mathcal{A}$  such that  $p \in U$ . Then the *coordinate vectors at  $p$* , denoted by  $\frac{\partial}{\partial x^i} \Big|_p$ , are derivations  $C^\infty(U) \rightarrow \mathbb{R}$  such that

$$\frac{\partial}{\partial x^i} \Big|_p := f \mapsto \frac{\partial}{\partial x^i} \Big|_{\phi(p)} (f \circ \phi^{-1}).$$

**Definition 8.22** (Tangent Bundle). Let  $M$  be a smooth manifold with or without boundary. The tangent bundle of  $M$ , denoted by  $TM$ , is the disjoint union  $\coprod_{p \in M} T_p M$ .

**Definition 8.23** (Projection Map). Let  $M$  be a smooth manifold with or without boundary. The projection map  $\pi : TM \rightarrow M$  is the map defined by  $(p, v) \mapsto p$ .

**Definition 8.24** (Curve). If  $M$  is a manifold with or without boundary, we define a *curve in  $M$*  to be a continuous map  $\gamma : J \rightarrow M$  where  $J \subset \mathbb{R}$  is an interval.

#### 8.4. Submersions, Immersions, and Embeddings.

**Definition 8.25** (Rank). Let  $M, N$  be smooth manifolds with or without boundary and let  $F : M \rightarrow N$  be a smooth map. Then the rank of  $F$  at  $p \in M$  is:

- The rank of the linear map  $dF_p : T_p M \rightarrow T_{F(p)} N$ .
- The dimension of the subspace  $dF_p(T_p M)$  in the vector space  $T_{F(p)} N$ .

It is easy to see that the two definitions above are always equivalent.

**Definition 8.26** (Submersions and Immersions). Let  $M, N$  be smooth manifolds with or without boundary and let  $F : M \rightarrow N$  be a smooth map.

- If  $F$  has the same rank at every point  $p \in M$ , then  $F$  is said to have *constant rank*, and the rank is denoted by  $\text{rank } F$ .
- If the rank of  $F$  at  $p \in M$  is equal to  $\max\{\dim M, \dim N\}$ , then  $F$  is said to have *full rank at  $p$* .
- If  $F$  has full rank everywhere, then  $F$  is said to have *full rank*.
- If  $F$  has constant rank and  $\text{rank } F = \dim N$ ,  $F$  is called a *smooth submersion*.
- If  $F$  has constant rank and  $\text{rank } F = \dim M$ ,  $F$  is called a *smooth immersion*.