INTRODUCTION TO SMOOTH MANIFOLDS

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Contents

1. Dictionary

1.1. Topological Manifolds.

Definition 1.1 (Topological Manifold)**.** A *topological n-manifold* is a Hausdorff, second-countable topological space each point of which has a neighborhood that is homeomorphic to an open subset $\mathbb{R}^n$.

Definition 1.2 (Coordinates)**.** Let M be a topological n-manifold. Let U be an open subset of M, $\hat{U}$ be an open subset of $\mathbb{R}^n$, $\phi : U \to \hat{U}$ be a homeomorphism.

- The pair (U, ϕ) is called a *coordinate chart* or a *chart.*
- U is called a *coordinate domain* or a *coordinate neighborhood* and ϕ is called a *coordinate map.*
- If $\phi(U)$ is an open ball in $\mathbb{R}^n$, U is called a *coordinate ball.*
- If $\phi(U)$ is an open cube in $\mathbb{R}^n$, U is called a *coordinate cube.*

Definition 1.3 (Atlas)**.** Let M be a topological n-manifold. An *atlas for M* is a collection of charts (U_α, ϕ_α) such that $M = \bigcup_\alpha U_\alpha$.

Definition 1.4 (Transition Map)**.** Let M be a topological n-manifold and $(U, \phi), (V, \psi)$ be coordinate charts such that $U \cap V \neq \emptyset$. $\psi \circ \phi^{-1} : \phi(U \cap V) \mapsto \psi(U \cap V)$ is called a *transition map* from ϕ to ψ.

1.2. Smooth Manifolds.

Definition 1.5 (Smoothly Compatible)**.** Let M be a topological n-manifold. Two coordinate charts $(U, \phi), (V, \psi)$ are called *smoothly compatible* if $U \cap V = \emptyset$ or the transition map $\psi \circ \phi^{-1}$ is a diffeomorphism.

Definition 1.6 (Smooth Atlas)**.** Let M be a topological n-manifold. A *smooth atlas* is an atlas $\mathcal{A}$ such that any two charts in $\mathcal{A}$ are smoothly compatible with each other.

Definition 1.7 (Smooth Structure)**.** If M is a topological n-manifold, an atlas $\mathcal{A}$ that is not properly contained in any larger smooth atlas is called *maximal* or a *smooth structure on M*

Definition 1.8 (Smooth Manifold)**.** A *smooth manifold* is a topological manifold equipped with a smooth structure.

Definition 1.9. Suppose $(M, \mathcal{A})$ is a smooth manifold.

- Any chart $(U, \phi) \in \mathcal{A}$ is called a *smooth chart.*
- Given a smooth chart (U, ϕ), U is called a smooth coordinate domain and ϕ is called a *smooth coordinate map.*

- Given a smooth chart (U, ϕ), U is called a *smooth coordinate ball* if it is a coordinate ball.

Remark 1.10. One must define a smooth structure on a topological manifold before talking about a smooth chart.

2. Chapter 1: Smooth Manifolds

Exercise 1.1. Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to *any* open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

Proof. It is clear that a "manifold" satisfying the open-ball or $\mathbb{R}^n$ definition satisfies the open-subset definition. Let M be a manifold satisfying the open-subset definition. Let $x \in M$ be given and let $U, \hat{U}, \phi$ be given according to the definition. Since $\hat{U}$ is open, there exists an open ball B such that $\phi(x) \in B \subset \hat{U}$. Restrict ϕ to $\phi^{-1}(B)$. Then $\phi^{-1}(B)$ is an open subset of M containing x, and $\phi\mid_{\phi^{-1}(B)}$ is a homeomorphism between $\phi^{-1}(B)$ and B. Thus M satisfies the open-ball definition.

$B(x, r) \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ by the map $(x_1 + a_1, \cdots, x_n + a_n) \mapsto (\frac{a_1}{r-a_1}, \cdots, \frac{a_n}{r-a_n})$ where $x = (x_1, \cdots, x_n)$ is the center of $B(x, r)$ and r is the radius. Since the composition of two homeomorphisms gives a homeomorphism, M also satisfies the $\mathbb{R}^n$ definition as well. $\square$

Exercise 1.6. Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.

Proof. From the definition of π, it is easy to see that $\pi(B(x, r))$ is open in $\mathbb{RP}^n$ where $x \in S^n$ and $0 < r < 1$.

Let $[x], [y] \in \mathbb{RP}^n$ be given. Without loss of generality, assume $x, y \in S^n$. Let $r = \min\{|x - y|, |x + y|, 1\}/2$. Then $U_x = \pi(B(x, r)), U_y = \pi(B(y, r))$ contain $[x], [y]$, respectively. $\pi^{-1}(U_x), \pi^{-1}(U_y)$ are both open in $\mathbb{R}^{n+1} \setminus \{0\}$ which can be seen easily by writing down exactly which points belong to them, so U_x, U_y are both open in $\mathbb{RP}^n$. Then $\pi^{-1}(U_x \cap U_y) = \pi^{-1}(U_x) \cap \pi^{-1}(U_y) = \emptyset$, so $U_x \cap U_y = \emptyset$. Therefore, $\mathbb{RP}^n$ is Hausdorff.

Let $\mathcal{B} = \{\pi(B(x, 1/k)) \mid x \in \mathbb{Q}^{n+1} \cap S^n, k \in \{2, 3, 4, \cdots\}\}$. Then $\mathcal{B}$ is a countable collection of open sets whose union is $\mathbb{RP}^n$. Let $U \subset \mathbb{RP}^n$ be a nonempty open set. Let $[x] \in U$. Since π is a quotient map, $\pi^{-1}(U)$ is open. Moreover, $x \in \pi^{-1}(U)$. Without loss of generality, $x \in S^n$. Then $x \in B(x', 1/k) \subset \pi^{-1}(U)$ for some $B(x', 1/k) \in \mathcal{B}$. Then $[x] = \pi(x) \in \pi(B(x', 1/k)) \subset \pi(\pi^{-1}(U)) = U$. Therefore, $\mathcal{B}$ is a countable basis of $\mathbb{RP}^n$. $\square$

Exercise 1.7. Show that $\mathbb{RP}^n$ is compact.

Proof. $\pi(S^n) = \mathbb{RP}^n$ and S^n is compact because it is a closed, bounded subset of $\mathbb{R}^{n+1}$. (Heine-Borel) Moreover, the image of a compact set under a continuous map is compact. (See A.45(a)) Thus $\mathbb{RP}^n$ is compact. $\square$

Exercise 1.14. Suppose $\mathcal{X}$ is a locally finite collection of subsets of a topological space M.

(a) The collection $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.
(b) $\overline{\bigcup_{X\in\mathcal{X}} X} = \bigcup_{X\in\mathcal{X}} \overline{X}$.

Proof.

(a) Let $p \in M$. Then there exists an open set U containing x such that there are only finitely many $X \in \mathcal{X}$ such that $U \cap X \neq \emptyset$. Let $X \in \mathcal{X}$.
  - If $U \cap X \neq \emptyset$, then $U \cap \overline{X} \supset U \cap X \neq \emptyset$.
  - If $U \cap X = \emptyset$, then U^c is closed, so $\overline{X} \subset U^c$. In other words, $U \cap \overline{X} = \emptyset$.

  This shows that the number of $X \in \mathcal{X}$ that intersects U and the number of $\overline{X} \in \mathcal{X}$ that intersects U are the same. Therefore, $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.

(b) Since the closure of a set is defined to be the intersection of all closed sets containing it, $\bigcup_{X\in\mathcal{X}} \overline{X} \subset \overline{\bigcup_{X\in\mathcal{X}} X}$. Let $x \notin \bigcup_{X\in\mathcal{X}} \overline{X}$. Then there exists a neighborhood U of x such that U intersects only finitely many $X \in \mathcal{X}$. Let $X_1, \cdots, X_n$ denote them. By the same argument as part (a), $\overline{X_1}, \cdots, \overline{X_n}$ are the only elements in $\{\overline{X} \mid X \in \mathcal{X}\}$ that U intersects. Since $x \notin \overline{X_i}$ for each $i = 1, \cdots, n$, $U^c \cup \overline{X_1} \cup \cdots \cup \overline{X_n}$ is a closed set which contains all $X \in \mathcal{X}$ but does not contain x. In other words, $x \notin \overline{\bigcup_{X\in\mathcal{X}} X}$.

$\square$

Exercise 1.18. Let M be a topological manifold. Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Proof. Let $\mathcal{A}, \mathcal{A}'$ be two smooth atlases.

Suppose that they determine the same smooth structure $\mathcal{B}$. Then $\mathcal{A} \cup \mathcal{A}' \subset \mathcal{B}$, so $\mathcal{A} \cup \mathcal{A}'$ must be a smooth atlas. By Proposition 1.17(a), $\mathcal{A} \cup \mathcal{A}'$ determines a unique smooth structure, but it must be $\mathcal{B}$ because $\mathcal{B}$ contains the union.

On the other hand, suppose that their union is a smooth atlas. Let $\mathcal{B}$ be the smooth structure that the union determines. Such $\mathcal{B}$ must exist by Proposition 1.17(a). By the same proposition, $\mathcal{A}, \mathcal{A}'$ must determine the unique smooth structures. However, they must be $\mathcal{B}$ because $\mathcal{B}$ contains both $\mathcal{A}$ and $\mathcal{A}'$. $\square$

Exercise 1.20. Every smooth manifold has a countable basis of regular coordinate balls.

Proof. Let M be an n-dimensional smooth manifold. We consider the special case that there exists a single chart (ϕ, U) with $U = M$. Let $x \in \hat{U}$ with rational coordinates. Then there exists $s > 0$ such that $B(x, s) \subset \hat{U}$. For each rational number $r \in (0, s)$, we consider the chart $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r)))$.

Let $\mathcal{B}$ be the collection of all such charts for each $x \in \hat{U}$ and r. We claim that $\mathcal{B}$ is a smooth atlas.

- Let $p \in M$. Then $\phi(p) \in \hat{U}$. Since $\hat{U}$ is open, $\phi(p) \in B(x, r) \subset \hat{U}$ for some x with rational coordinates and a positive rational number r. Then $p \in \phi^{-1}(B(x, r))$, so the union of coordinate domains covers M. In other words, $\mathcal{B}$ is an atlas.
- Let $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r))), (p \mapsto \phi(p) - x', \phi^{-1}(B(x', r'))) \in \mathcal{B}$ be given. Suppose $\phi^{-1}(B(x, r)) \cap \phi^{-1}(B(x', r')) \neq \emptyset$. Let ψ, ψ' denote the coordinate maps. Then $\psi' \circ \psi^{-1}$ is a composition of ϕ, ϕ^{-1} and translation maps, so it is smooth.

Therefore, $\mathcal{B}$ is a smooth atlas.

Since $\mathcal{B}$ is a smooth atlas, there exists a smooth structure $\mathcal{A}$ on M containing $\mathcal{B}$ by Proposition 1.17(a). We claim that $\mathcal{B}$, a subset of the smooth structure $\mathcal{A}$, is a countable basis of regular coordinate balls.

- $\mathcal{B}$ is a countable collection because $x \in \mathbb{Q}^n$ and $r \in \mathbb{Q}$.
- Let $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r))) \in \mathcal{B}$ be given. Then there exists a chart $(p \mapsto \phi(p) - x, \phi^{-1}(B(x, r')))$ in $\mathcal{B}$ with $r' > r$. Let $B = \phi^{-1}(B(x, r)), B' = \phi^{-1}(B(x, r'))$. Let ψ denote the map $p \mapsto \phi(p) - x$. Then $\psi(B) = B(0, r)$ and $\psi(B') = B(0, r')$, respectively.

Work on the case when there is no chart that covers the entire manifold.

$\square$

Exercise 1.39. Let M be a topological n-manifold with boundary.

(a) $\operatorname{Int} M$ is an open subset of M and a topological n-manifold without boundary.
(b) ∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.
(c) M is a topological manifold if and only if $\partial M = \emptyset$.
(d) If $n = 0$, then $\partial M = \emptyset$ and M is a 0-manifold.

Proof.

(a) Let $x \in \operatorname{Int} M$. Let (ϕ, U) be an interior chart for x. Then $x \in U \subset \operatorname{Int} M$ because every point in U is in an interior chart (ϕ, U). A subspace of M must be Hausdorff and second-countable by Proposition A.17(g, i), so $\operatorname{Int} M$ is a second-countable, Hausdorff space in which every point has a neighborhood homeomorphic to an open subset in $\mathbb{R}^n$. Thus $\operatorname{Int} M$ is an n-manifold without boundary.

(b) Since $\partial M = M \setminus \operatorname{Int} M$ and $\operatorname{Int} M$ is open in M, ∂M is closed in M. Let $x \in \partial M$. Let (ϕ, U) be a boundary chart of x. If a point $y \in U$ gets mapped into $\operatorname{Int} \mathbb{H}^n$, then it is certainly an interior point. Thus $\phi(U \cap \partial M) \subset \partial \mathbb{H}^n$. Then $\pi_{n-1} \circ \phi$ is a homeomorphism that maps $U \cap \partial M$ into an open subset of $\mathbb{R}^{n-1}$ where $\pi_{n-1} : (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_{n-1})$.

(c) If ∂M is empty, then $M = \operatorname{Int} M$, so (a) implies that M is an n-dimensional manifold. If M is a topological manifold, every point is an interior point. Since a point cannot be both an interior point and a boundary point, ∂M is empty.

(d) If $n = 0$, then $\partial \mathbb{H}^0 = \emptyset$. Thus, the condition that $\phi(U) \cap \partial \mathbb{H}^n \neq \emptyset$ can never be satisfied, so there cannot be any boundary point.

$\square$

3. Chapter 2: Smooth Maps

Exercise 2.1. Let M be a smooth manifold with or without boundary. Show that pointwise multiplication turns $C^\infty(M)$ into a commutative ring and a commutative and associative algebra over $\mathbb{R}$.

Proof.

- The constant map $f(p) = 0$ is clearly in $C^\infty(M)$ and it is the additive identity.
- The constant map $f(p) = 1$ is clearly in $C^\infty(M)$ and it is the multiplicative identity.
- Let $f \in C^\infty(M), g \in C^\infty(M)$. Let $p \in M$ and (ϕ, U) be a smooth chart for p. Then $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are both smooth(Exercise 2.3), real-valued maps defined on an open subset of $\mathbb{R}^n$. Thus $f + g$ is in $C^\infty(M)$ Moreover, $f + g = g + f$ because addition in $\mathbb{R}$ is commutative.
- Let $f, g, h \in C^\infty(M)$. Let $p \in M$ and (ϕ, U) be a smooth chart for p. Then $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are both smooth(Exercise 2.3), real-valued maps defined on an open subset of $\mathbb{R}^n$. Therefore, fg is in $C^\infty(M)$ Moreover, $fg = gf$ and $(fg)h = f(gh)$ because multiplication in $\mathbb{R}$ is commutative and associative.
- Let $c \in \mathbb{R}, f \in C^\infty(M)$. Then cf can be seen as fg where g is the constant function whose value is c. As shown above, $cf \in C^\infty(M)$.

□

Exercise 2.2. Let U be an open submanifold of $\mathbb{R}^n$ with its standard smooth manifold structure. Show that a function $f : U \to \mathbb{R}^k$ is smooth in the sense just defined if and only if it is smooth in the sense of ordinary calculus. Do the same for an open submanifold with boundary in $\mathbb{H}^n$.

Proof. f is smooth in the sense just defined if and only if $f \circ \mathrm{Id}^{-1}$ is smooth in the sense of ordinary calculus. Since $f \circ \mathrm{Id}^{-1} = f$, $f \circ \mathrm{Id}^{-1}$ is smooth in the sense of ordinary calculus if and only if f is smooth in the sense of ordinary calculus. □

Exercise 2.3. Let M be a smooth manifold with or without boundary, and suppose $f : M \to \mathbb{R}^k$ is a smooth function. Show that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$ is smooth for every smooth chart (U, ϕ) for M.

Proof. Let $\phi(x) \in \phi(U)$. Since f is smooth, there exists (V, ψ) such that $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}^k$ is smooth and $x \in V$. Let $W = U \cap V$. Then $f \circ \psi^{-1} : \psi(W) \to \mathbb{R}^k$ is smooth and $\psi \circ \phi^{-1} : \phi(W) \to \psi(W)$ is a diffeomorphism where $\phi(W)$ is a neighborhood of W. Then the restriction of $f \circ \psi^{-1}$ to $\phi(W)$ is identical to $(f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})$. Since he composition of a smooth function is smooth, $f \circ \psi^{-1}$ is smooth. □

4. Appendix C: Review of Calculus

Exercise C.1. Suppose that $F : U \to W$ is differentiable at $a \in U$. Show that the linear map satisfying

$$\lim_{v \to 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} = 0$$

is unique.

Proof. Let L, L' be two such linear maps.

$$\begin{aligned}
\lim_{v \to 0} \frac{|Lv - L'v|}{|v|} &= \lim_{v \to 0} \frac{|(F(a+v) - F(a) - L'v) - (F(a+v) - F(a) - Lv)|}{|v|} \\
&= \lim_{v \to 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} + \lim_{v \to 0} \frac{|F(a+v) - F(a) - L'v|}{|v|} \\
&= 0 + 0 = 0.
\end{aligned}$$

If $L \neq L'$, $(L - L')v_0 \neq 0$ for some v_0. Then $\lim_{v \to 0} \frac{|Lv - L'v|}{|v|} = \lim_{h \to 0} \frac{|L(hv_0) - L'(hv_0)|}{|hv_0|} = \frac{|(L-L')v_0|}{|v_0|} \neq 0$. This is a contradiction, so $L = L'$. □