

# STELLAR CONSENSUS PROTOCOL

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## 1. FEDERATED BYZANTINE AGREEMENT SYSTEM

### 1.1. Quorums.

**Definition 1.1.1 (Federated Byzantine Agreement System).** Let  $V$  be a set and  $Q : V \rightarrow 2^V \setminus \{\emptyset\}$  be a function such that  $\forall v \in V, \forall q \in Q(v), v \in q$ . Then we call the pair  $\langle V, Q \rangle$  a federated Byzantine agreement system, or FBAS for short. Each  $q$  in  $Q(v)$  is called a quorum slice for each  $v \in V$ .

*Remark 1.1.2.*

- For each node  $v$ ,  $Q(v)$  is a set of subsets of  $V$ . For instance, node  $v_1$  may trust  $v_2, v_3, v_4$  and may have  $\{v_1, v_2, v_3, v_4\} \in Q(v_1) \subset 2^V$ .
- Note that we explicitly exclude  $\emptyset$  from the co-domain. In other words, we want  $Q(v) \neq \emptyset$  for all  $v \in V$ . If  $Q(v) = \emptyset$  for some  $v \in V$ , it satisfies  $\forall q \in Q(v), v \in q$ . As we will see, each  $q \in Q(v)$  is the list of nodes that  $v$  trusts. If  $v$  has no list of nodes that it trusts,  $v$  cannot really do anything. Thus we want  $Q(v) \neq \emptyset$  for all  $v \in V$ .
- Consider the case when  $V = \emptyset$ . Note that  $2^\emptyset = \{\emptyset\}$  and thus  $2^{2^\emptyset} = \{\emptyset, \{\emptyset\}\}$ . Thus  $V$  forms an FBAS where  $Q$  is the map  $\emptyset \mapsto \{\{\emptyset\}\}$ .

**Definition 1.1.3 (Quorum).** Let  $\langle V, Q \rangle$  be an FBAS.  $U \subset V$  is called a quorum if and only if  $U \neq \emptyset$  and  $\forall v \in U, \exists q \in Q(v), q \subset U$ .

**Theorem 1.1.4.** In an FBAS  $\langle V, Q \rangle$ , the union of two quorums is a quorum.

*Proof.* Let  $U_1, U_2$  be two quorums. Let  $v \in U_1 \cup U_2$ . Then  $v \in U_i$  for  $i = 1$  or  $i = 2$ . Then  $q \subset U_i$  for some  $q \in Q(v)$ . Therefore,  $q \subset U_1 \cup U_2$ , so  $U_1 \cup U_2$  is indeed a quorum.  $\square$

**Corollary 1.1.5.** The set of quorums of a given FBAS is closed under union.

**Theorem 1.1.6.** In an FBAS  $\langle V, Q \rangle$ ,  $V$  is a quorum.

*Proof.* For any  $v \in V$ , for any  $q \in Q(v)$ ,  $q \subset V$ . Therefore,  $V$  is indeed a quorum.  $\square$

**Example 1.1.7.** One might wonder if the intersection of quorums is always a quorum. However, this is not true in general.

Let  $V = \{v_1, \dots, v_4\}$  and

- $Q(v_1) = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}\},$
- $\vdots$
- $Q(v_4) = \{\{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}.$

In other words,  $Q(v_i) = \{U \subset V \mid |U| = 3, v_i \in U\}$ .

Then  $U_1 = \{v_1, v_2, v_3\}$  is a quorum, and  $U_2 = \{v_2, v_3, v_4\}$  is a quorum. However,  $U_1 \cap U_2 = \{v_2, v_3\}$  is not a quorum because the size of any quorum slice is 3.

**Definition 1.1.8 (Quorum Intersection).** Let  $\langle V, Q \rangle$  be an FBAS. We say  $\langle V, Q \rangle$  enjoys quorum intersection if and only if  $U_1 \cap U_2 \neq \emptyset$  for any pair of quorums  $U_1, U_2$ .

**Definition 1.1.9 (Delete).** Let  $\langle V, Q \rangle$  be an FBAS and  $B \subset V$ . Then the FBAS  $\langle V, Q \rangle^B$  is defined to be  $\langle V \setminus B, Q^B \rangle$  where  $\forall v \in V \setminus B, Q^B(v) = \{q \setminus B \mid q \in Q(v)\}$ .

*Remark 1.1.10.* One may think that this is related to fail-stop behaviors where  $B$  is the set of nodes that stopped responding. In general, however, this is not true as we also remove nodes from quorum slices. One can think of this as the *alternate* universe where nodes from  $B$  simply did not even exist from the beginning.

**Theorem 1.1.11.** *Definition 1.1.9 is well-defined. In other words, if  $\langle V, Q \rangle$  is an FBAS and  $B \subset V$ , then  $\langle V, Q \rangle^B$  is an FBAS.*

*Proof.* If  $V = B$ , then we are done because of Remark 1.1.2.

Suppose otherwise. Let  $v \in V \setminus B$  be given. By the definition of an FBAS,  $Q(v) \neq \emptyset$ . Therefore,  $Q^B(v) = \{q \setminus B \mid q \in Q(v)\}$  is nonempty.

Let  $q' \in Q^B(v)$  be given arbitrarily. Then  $q' = q \setminus B$  for some  $q \in Q(v)$ . Then  $v \in q$  by the definition of an FBAS, so  $v \in q'$ .

Thus  $Q^B$  is indeed a quorum function. Therefore,  $\langle V, Q \rangle^B$  is an FBAS.  $\square$

**Theorem 1.1.12.** *Let  $U$  be a quorum in FBAS  $\langle V, Q \rangle$ , let  $B \subset V$  be a set of nodes, and let  $U' = U \setminus B$ . If  $U' \neq \emptyset$ , then  $U'$  is a quorum in  $\langle V, Q \rangle^B$ .*

*Proof.* Since  $U' \neq \emptyset$ , it suffices to show that  $\forall v \in U', \exists q \in Q^B(v), q \subset U'$ . Let  $v \in U'$ . Then  $v \in U$ . Since  $U$  is a quorum in  $\langle V, Q \rangle$ , we can find  $q \in Q(v)$  such that  $q \subset U$ . Then  $q' = q \setminus B \in Q^B(v)$ , and  $q' = q \setminus B \subset U \setminus B = U'$ . Therefore,  $U'$  is a quorum in  $\langle V, Q \rangle^B$ .  $\square$

*Remark 1.1.13.* One can think of this theorem as “A quorum in the ‘original’ universe is a quorum in the ‘alternate’ universe.”

**Definition 1.1.14 (Quorum Intersection Despite  $B$ ).** Let  $\langle V, Q \rangle$  be an FBAS and  $B \subset V$  be a set of nodes. We say  $\langle V, Q \rangle$  enjoys quorum intersection despite  $B$  if and only if  $\langle V, Q \rangle^B$  enjoys quorum intersection.

*Remark 1.1.15.* Quorum intersection despite  $B$  is related to the system-level safety when nodes in  $B$  act arbitrarily. For instance, suppose  $\langle V, Q \rangle$  is an FBAS,  $B$  is the set of all ill-behaved nodes, and  $\langle V, Q \rangle$  enjoys quorum intersection despite  $B$ . Suppose two well-behaved nodes  $v_1, v_2$  agree with contradictory statements  $a_1, a_2$  in quorums  $q_1, q_2$ , respectively.

Rewrite this!

Then  $q_1 \cap q_2 \neq \emptyset$  have well-behaved nodes who agreed with both  $a_1$  and  $a_2$ . This is a contradiction because a well-behaved node cannot contradict itself. This example illustrates how the concept of quorum intersection despite  $B$  is related to system-level safety when nodes in  $B$  experience Byzantine failures.

**Definition 1.1.16 (Quorum Availability Despite  $B$ ).** Let  $\langle V, Q \rangle$  be an FBAS and  $B \subset V$  be a set of nodes. We say  $\langle V, Q \rangle$  enjoys quorum availability despite  $B$  if and only if  $V \setminus B$  is a quorum in  $\langle V, Q \rangle$  or  $B = V$ .

**Theorem 1.1.17.** *Let  $\langle V, Q \rangle$  be an FBAS and  $B \subset V$ .  $\langle V, Q \rangle$  enjoys quorum availability despite  $B$  if and only if  $\forall v \in V \setminus B$ , there exists a quorum  $U_v$  such that  $v \in U_v \subset (V \setminus B)$ .*

*Proof.* If  $V = B$ , we are done. Suppose otherwise.

$$\begin{aligned} \forall v \in V \setminus B, \exists \text{ a quorum } U_v, v \in U_v \subset (V \setminus B) &\implies \bigcup_{v \in V \setminus B} U_v \text{ is a quorum in } \langle V, Q \rangle \\ &\implies V \setminus B \text{ is a quorum in } \langle V, Q \rangle \end{aligned}$$

by Theorem 1.1.4. On the other hand, if  $V \setminus B$  is a quorum in  $\langle V, Q \rangle$ , then  $\forall v \in V \setminus B, \exists \text{ a quorum } U_v, v \in U_v \subset (V \setminus B)$  because we can let  $U_v = V \setminus B$  for each  $v$ .  $\square$

*Remark 1.1.18.* Theorem 1.1.17 shows that  $\langle V, Q \rangle$  enjoys quorum availability despite  $B$  if all nodes in  $V \setminus B$  can find a quorum without  $B$ . This is related to the liveness of the system. If  $\langle V, Q \rangle$  enjoys quorum availability despite  $B$ , then regardless of what happens to nodes in  $B$ , nodes in  $V \setminus B$  can keep going.

**Definition 1.1.19 ( $v$ -blocking).** Let  $\langle V, Q \rangle$  be an FBAS. Let  $v \in V$ . A subset  $B \subset V$  is called  $v$ -blocking if and only if  $\forall q \in Q(v), q \cap B \neq \emptyset$ .

*Remark 1.1.20.* Intuitively, if a subset  $B \subset V$  is  $v$ -blocking, then one may think of it as “ $v$  can't really get by without  $B$ .” The following theorem can be interpreted as “If  $v$  can't get by without  $B$ ,  $v$  can't get by without  $C$  for any  $C \supset B$ .”

**Theorem 1.1.21.** Let  $\langle V, Q \rangle$  be an FBAS. Let  $v \in V$ . Then

- The union of two  $v$ -blocking sets is  $v$ -blocking.
- Any superset of a  $v$ -blocking set is  $v$ -blocking.

*Proof.* It suffices to only prove the second statement. If  $B \subset B'$  and  $B$  is  $v$ -blocking,  $q \cap B' \supset q \cap B \neq \emptyset$  for any  $q \in Q(v)$ .  $\square$

**Theorem 1.1.22.** Let  $\langle V, Q \rangle$  be an FBAS. Let  $A \subsetneq V$  and  $U_1, U_2$  be a partition of  $V \setminus A$ . Let  $v \in U_1$ . If  $U_2$  is not  $v$ -blocking in  $\langle V, Q \rangle$ , then  $U_2$  is not  $v$ -blocking in  $\langle V, Q \rangle^A$ .

*Proof.* Since  $U_2$  is not  $v$ -blocking in  $\langle V, Q \rangle$ , there exists  $q_v \in Q(v)$  such that  $q_v \cap U_2 = \emptyset$ .

$$\begin{aligned} (q_v \setminus A) \cap U_2 &= (q_v \cap U_2) \setminus (A \cap U_2) \\ &= q_v \cap U_2 = \emptyset. \end{aligned}$$

Thus  $U_2$  is not  $v$ -blocking in  $\langle V, Q \rangle^A$ .  $\square$

**Theorem 1.1.23.** Let  $\langle V, Q \rangle$  be an FBAS. Let  $B \subset V$ .  $\langle V, Q \rangle$  enjoys quorum availability despite  $B$  if and only if  $B$  is not  $v$ -blocking for any  $v \in V \setminus B$ .

*Proof.*

$$\begin{aligned} \forall v \in V \setminus B, \neg(B \text{ is } v\text{-blocking}) &\iff \forall v \in V \setminus B, \neg(\forall q \in Q(v), q \cap B \neq \emptyset) \\ &\iff \forall v \in V \setminus B, \exists q \in Q(v), q \cap B = \emptyset \\ &\iff \forall v \in V \setminus B, \exists q \in Q(v), q \subset V \setminus B \\ &\iff V = B \text{ or } V \setminus B \text{ is a quorum in } \langle V, Q \rangle \\ &\iff \langle V, Q \rangle \text{ enjoys quorum availability despite } B \end{aligned}$$

$\square$

## 1.2. Dispensable Sets.

**Definition 1.2.1 (Dispensable Set).** Let  $\langle V, Q \rangle$  be an FBAS and  $B \subset V$  be a set of nodes.  $B$  is called a dispensable set, or DSet, if and only if  $\langle V, Q \rangle$  enjoys both quorum intersection despite  $B$  and quorum availability despite  $B$ .

We will first show some basic properties of DSets.

**Theorem 1.2.2.** Let  $\langle V, Q \rangle$  be an FBAS. Then

- $V$  is a DSet.
- If  $\forall v \in V, Q(v) = \{V\}$ , then  $\emptyset$  and  $V$  are the only DSets of  $\langle V, Q \rangle$ .

*Proof.*

- $\langle V, Q \rangle^V$  enjoys quorum intersection because there is no quorum.  $\langle V, Q \rangle$  enjoys quorum availability despite  $V$  because  $V = V$ .
- Suppose  $\forall v \in V, Q(v) = \{V\}$ . As shown above,  $V$  is a DSet of  $\langle V, Q \rangle$ . The empty set is a DSet because
  - $\langle V, Q \rangle$  enjoys quorum intersection despite  $\emptyset$  because the only quorum is  $V$ .
  - $\langle V, Q \rangle$  enjoys quorum availability despite  $B$  because  $V \setminus \emptyset = V$  is a quorum.

Let  $\emptyset \subsetneq S \subsetneq V$  be given. Then  $V \neq S$  and  $V \setminus S$  is not a quorum for it is nonempty and does not contain any quorum slice. Therefore,  $\langle V, Q \rangle$  does not enjoy quorum availability despite  $B$ , so no nonempty, proper subset of  $V$  is a DSet. □

**Definition 1.2.3 (Intact and Befouled).** Let  $\langle V, Q \rangle$  be an FBAS and  $v \in V$ .  $v$  is said to be intact if and only if there exists a DSet  $B$  containing all ill-behaved nodes and  $v \notin B$ .  $v$  is said to be befouled if and only if  $v$  is not intact.

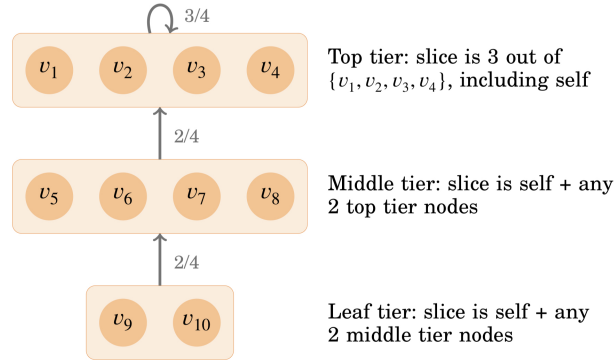


FIGURE 1. Tiered Quorum Example (P.5 of the white paper)

**Example 1.2.4.** We will use Figure 1 as an example.

- The smallest DSet containing  $v_5, v_6$  in Figure 1 is  $\{v_5, v_6, v_9, v_{10}\}$ .
  - First, we will show that  $B = \{v_5, v_6\}$  is not a DSet. By definition,

$$\begin{aligned}
 Q^B(v_9) &= \{\{v_9\}, \{v_9, v_7\}, \{v_9, v_8\}, \{v_9, v_7, v_8\}\} \\
 Q^B(v_{10}) &= \{\{v_{10}\}, \{v_{10}, v_7\}, \{v_{10}, v_8\}, \{v_{10}, v_7, v_8\}\}
 \end{aligned}$$

This implies that  $U_9 = \{v_9\}$  and  $U_{10} = \{v_{10}\}$  are both quorums. Then  $U_9 \cap U_{10} = \emptyset$ , so  $\langle V, Q \rangle^B$  does not enjoy quorum intersection. Therefore,  $B$  is not a DSet. Next, we will consider  $C = \{v_5, v_6, v_1\}$ . Then we can use the same argument as above.  $U_9 = \{v_9\} \in Q^C(v_9)$  and  $U_{10} = \{v_{10}\} \in Q^C(v_{10})$ , and the intersection is empty. Therefore,  $C$  is not a DSet. It is easy to see that this argument works for the case of  $\{v_5, v_6, v_i\}$  for any  $i = 1, 2, 3, 4$ .

We will consider  $D = \{v_5, v_6, v_9\}$ . Similarly,  $U = \{v_{10}\} \in Q^D(v_{10})$  is a quorum. Moreover,  $U' = \{v_1, v_2, v_3, v_4\}$  is a quorum. Then  $U \cap U' = \emptyset$ , so  $\langle V, Q \rangle^D$  does not enjoy quorum intersection. It is easy to see that a similar argument shows that  $\{v_5, v_6, v_{10}\}$  is not a DSet.

Finally, we will show that  $E = \{v_5, v_6, v_9, v_{10}\}$  is a DSet.  $V \setminus E$  is a quorum in  $\langle V, Q \rangle$  because every node in  $V \setminus E$  has a quorum slice consisting of nodes in  $V \setminus E$ . If a quorum in  $\langle V, Q \rangle^E$  contains  $v_7$  or  $v_8$ , then it must contain some of  $v_1, v_2, v_3, v_4$ . If a quorum in  $\langle V, Q \rangle^E$  contains at least one of  $v_1, v_2, v_3$ , or  $v_4$ , then it must contain at least three of  $v_1, v_2, v_3, v_4$ . Therefore, any intersection of two quorums in  $\langle V, Q \rangle^E$  contains at least two of  $v_1, v_2, v_3, v_4$  by the pigeon hole principle.

Therefore,  $E$  is indeed a smallest DSet containing  $v_5$  and  $v_6$ .

- We showed that  $B = \{v_5, v_6\}$  is not a DSet because  $\langle V, Q \rangle$  does not enjoy quorum intersection despite  $B$ . What this means is that if both  $v_5$  and  $v_6$  either stop responding or are malicious, then it is not possible to guarantee safety for  $v_9$  and  $v_{10}$ . For instance, consider the following situation:

- \* Both  $v_5$  and  $v_6$  tell  $v_9$  and  $v_{10}$  that  $Q(v_5) = Q(v_6) = \{\{v_5, v_6\}\}$  convincing that  $\{v_5, v_6, v_9\}$  and  $\{v_5, v_6, v_{10}\}$  are both quorums.
- \* Both  $v_5$  and  $v_6$  tell  $v_9$  that they want to process a certain transaction. This transaction does not contradict what  $v_9$  knows about  $v_5$ . Moreover, everyone in the quorum  $\{v_5, v_6, v_9\}$  is in favor of this transaction. Thus there is no reason for  $v_9$  to not believe this transaction.
- \* Both  $v_5$  and  $v_6$  tell  $v_{10}$  that they want to process a certain transaction that contradicts the transaction they told  $v_5$  about. For the same reason, there is no reason for  $v_{10}$  to not believe this transaction.
- \* Then the network processes contradicting transactions. This can let  $v_5$  double-spend some money, for instance.
- \* One can verify that this is possible by looking into the definition of accepting, confirming and such that are introduced in later chapters.

- $B = \{v_1\}$  is a DSet.

- First, we will check if  $\langle V, Q \rangle$  enjoys quorum intersection despite  $B$ .

Consider  $\langle V, Q \rangle^B$ . Any quorum containing  $v_9$  and/or  $v_{10}$  must contain at least two of  $v_5, v_6, v_7, v_8$ . Any quorum containing at least one of  $v_5, \dots, v_8$  must contain at least one of  $v_2, v_3, v_4$ . Any quorum containing at least one of  $v_2, v_3, v_4$  must contain at least two of  $v_2, v_3, v_4$ . This is because  $Q(v_i)^B = \{\{v_2, v_3, v_4\}, \{v_i, v_j\}, \{v_i, v_k\}\}$  where  $\{i, j, k\} = \{2, 3, 4\}$ .

Therefore, the intersection of any two quorums must contain at least one of  $v_2, v_3, v_4$  by the pigeon hole principle.

Next, we need to check if  $\langle V, Q \rangle$  enjoys quorum availability despite  $B$ .  $V \setminus B$  is indeed a quorum in  $\langle V, Q \rangle$  because each node in  $V \setminus B$  has a quorum slice that does not contain  $v_1$ .

- We showed that  $B$  is indeed a DSet. What this means is that even if  $v_1$  stops responding or becomes malicious, the rest of the network can make progress safely. For instance, suppose that  $v_1$  becomes malicious and tries to double-spend money.  $v_1$  might tell  $v_5$  that it wants to process a certain transaction. Similarly,  $v_1$  might tell  $v_6$  that it wants to process a contradicting transaction. However, every quorum slice of  $v_5$  and  $v_6$  contains at least one tier-1 node that is not  $v_1$ . Suppose that  $v_5$  asks  $v_2$  what it thinks, and  $v_6$  asks  $v_3$  what it thinks. Then every quorum slice of  $v_2$  and  $v_3$  contains 3 tier-1 nodes. By the pigeon hole principle, at least one tier-1 node that is not  $v_1$  gets asked what it thinks about the contradicting transactions from  $v_5$ . The tier-1 node does not agree with them and  $v_1$ 's attempt to double-spend money fails.

**Theorem 1.2.5.** *If  $B_1$  and  $B_2$  are DSets in an FBAS  $\langle V, Q \rangle$  enjoying quorum intersection, then  $B = B_1 \cap B_2$  is a DSet, too.*

*Proof.* If  $B_1 = V$  or  $B_2 = V$ , then we are done. Suppose otherwise.

First, we will show that  $\langle V, Q \rangle$  enjoys quorum availability despite  $B$ . By Definition 1.1.16, it suffices to show that  $V = B$  or  $V \setminus B$  is a quorum in  $\langle V, Q \rangle$ . Since we assumed that  $B_1 \neq V$  and  $B_2 \neq V$ ,  $B \neq V$ . Therefore, we will show that  $V \setminus B$  is a quorum in  $\langle V, Q \rangle$ . By basic set theory,  $V \setminus B = V \setminus (B_1 \cap B_2) = (V \setminus B_1) \cup (V \setminus B_2)$ . Since  $B_1$  is a DSet,  $V = B_1$  or  $V \setminus B_1$  is a quorum in  $\langle V, Q \rangle$ . Since we assumed that  $V \neq B_1$  earlier,  $V \setminus B_1$  is a quorum in  $\langle V, Q \rangle$ . Similarly,  $V \setminus B_2$  is a quorum in  $\langle V, Q \rangle$ . By Theorem 1.1.4, the union  $(V \setminus B_1) \cup (V \setminus B_2) = V \setminus B$  is a quorum in  $\langle V, Q \rangle$ .

Next, we will show that  $\langle V, Q \rangle$  enjoys quorum intersection despite  $B$ . Let  $U_a, U_b$  be quorums in  $\langle V, Q \rangle^B$ . We want to show that  $U_a \cap U_b \neq \emptyset$ . We will do so by proving a stronger statement, which is  $(U_a \cap U_b) \setminus B_1 \neq \emptyset$ . In other words, we will show that  $(U_a \setminus B_1) \cap (U_b \setminus B_1) \neq \emptyset$ .

Since  $B_1$  is a DSet,  $\langle V, Q \rangle$  enjoys quorum intersection despite  $B_1$ . In other words,  $\langle V, Q \rangle^{B_1}$  enjoys quorum intersection. Therefore, it suffices to show that  $U_a \setminus B_1$  and  $U_b \setminus B_1$  are both quorums in  $\langle V, Q \rangle^{B_1}$ . By Theorem 1.1.12,  $U_a \setminus B_1$  and  $U_b \setminus B_1$  are quorums in  $(\langle V, Q \rangle^B)^{B_1}$  if  $U_a \setminus B_1 \neq \emptyset$  and  $U_b \setminus B_1 \neq \emptyset$ . Since  $(\langle V, Q \rangle^B)^{B_1} = \langle V, Q \rangle^{B_1}$ , it suffices to show that  $U_a \setminus B_1 \neq \emptyset$  and  $U_b \setminus B_1 \neq \emptyset$ .

We will first show that  $U_a \setminus B_1 \neq \emptyset$ . By basic set theory,

$$\begin{aligned} U_a &= U_a \setminus B \\ &= U_a \setminus (B_1 \cap B_2) \\ &= (U_a \setminus B_1) \cup (U_a \setminus B_2) \end{aligned}$$

because  $U_a \cap B = \emptyset$ .

This implies that  $(U_a \setminus B_1) \cup (U_a \setminus B_2) \neq \emptyset$ . If  $U_a \setminus B_1$  is nonempty, we are done. Suppose  $U_a \setminus B_2$  is nonempty. We will show that this implies that  $U_a \setminus B_1 \neq \emptyset$ . We will do so by first finding two quorums in  $\langle V, Q \rangle^{B_2}$  whose intersection is a subset of  $U_a \setminus B_1$ . Since  $\langle V, Q \rangle^{B_2}$  enjoys quorum intersection, the intersection of such two quorums must be nonempty, which in turn shows that  $U_a \setminus B_1$  is nonempty.



- We claim that  $U_a \setminus B_2$  is a quorum in  $\langle V, Q \rangle^{B_2}$ .  $U_a$  is a quorum in  $\langle V, Q \rangle^B$ . Since  $U_a \setminus B_2$  is assumed to be nonempty,  $U_a \setminus B_2$  is a quorum in  $(\langle V, Q \rangle^B)^{B_2}$  by Theorem 1.1.12. Since  $(\langle V, Q \rangle^B)^{B_2} = \langle V, Q \rangle^{B_2}$ ,  $U_a \setminus B_2$  is a quorum in  $\langle V, Q \rangle^{B_2}$ .
- $V \setminus B_2 \neq \emptyset$  is a quorum in  $\langle V, Q \rangle$  because  $\langle V, Q \rangle$  must enjoy quorum availability despite  $B_2$  and  $B_2 \neq V$ . Similarly,  $V \setminus B_1 \neq \emptyset$  is a quorum in  $\langle V, Q \rangle$ . Because  $\langle V, Q \rangle$  enjoys quorum intersection,  $(V \setminus B_1) \cap (V \setminus B_2) \neq \emptyset$ . In other words,  $(V \setminus B_1) \setminus B_2 \neq \emptyset$ . By applying Theorem 1.1.12 to the fact that  $V \setminus B_1$  is a quorum in  $\langle V, Q \rangle$  and  $(V \setminus B_1) \setminus B_2 \neq \emptyset$ , we can conclude that  $(V \setminus B_1) \setminus B_2$  is a quorum in  $\langle V, Q \rangle^{B_2}$ .

Since  $\langle V, Q \rangle^{B_2}$  enjoys quorum intersection,  $(U_a \setminus B_2) \cap ((V \setminus B_1) \setminus B_2) \neq \emptyset$ .

$$\begin{aligned}
\emptyset &\neq (U_a \setminus B_2) \cap ((V \setminus B_1) \setminus B_2) \\
&= (U_a \cap (V \setminus B_1)) \setminus B_2 \\
&\subset U_a \cap (V \setminus B_1) \\
&= U_a \setminus B_1.
\end{aligned}$$

Thus,  $U_a \setminus B_1 \neq \emptyset$ .

The same argument will show that  $U_b \setminus B_1 \neq \emptyset$ . □

*Remark 1.2.6.* Theorem 1.2.5 states that the intersection of two DSets is a DSet if the FBAS enjoys quorum intersection. One might wonder if the union of two DSets is a DSet when the FBAS enjoys quorum intersection. However, this is not true in general. Consider the FBAS  $\langle V, Q \rangle$  where  $V = \{v_1, v_2, v_3, v_4\}$  and  $Q(v_i) = \{U \subset V \mid v_i \in U, |U| = 3\}$ . This FBAS enjoys quorum intersection by the pigeon-hole principle because each quorum contains at least 3 elements. Then  $B = \{v_1\}$  is a DSet because

- Quorum intersection despite  $B$ 
  - Every quorum slice in  $\langle V, Q \rangle^B$  contains at least 2 nodes because every quorum slice in  $\langle V, Q \rangle$  contains exactly 3 nodes. This implies that any quorum in  $\langle V, Q \rangle^B$  must contain at least 2 nodes. By the pigeon-hole principle, every pair of quorums in  $\langle V, Q \rangle^B$  must intersect.
- Quorum availability despite  $B$ 
  - $V \setminus B = \{v_2, v_3, v_4\}$  is a quorum in  $\langle V, Q \rangle$  because  $\{v_2, v_3, v_4\} \in Q(v_i)$  for each  $i = 2, 3, 4$ .

Similarly,  $C = \{v_2\}$  is a DSet. However,  $B \cup C = \{v_1, v_2\}$  is not a DSet because  $B \cup C \neq V$  and  $V \setminus (B \cup C) = \{v_3, v_4\}$  is not a quorum in  $\langle V, Q \rangle$ .

**Theorem 1.2.7.** *In an FBAS with quorum intersection, the set of befouled nodes is a DSet.*

*Proof.* Let  $\langle V, Q \rangle$  be an FBAS with quorum intersection. Let  $B$  be the intersection of all DSets that contain all ill-behaved nodes. We will show that  $B$  is the set of befouled nodes by showing that  $V \setminus B$  is the set of intact nodes.

$$\begin{aligned}
v \in V \setminus B &\iff v \notin B \\
&\iff \exists \text{ DSet } B_v \text{ that contains all ill-behaved nodes and } v \notin B_v \\
&\iff v \text{ is intact}
\end{aligned}$$

Therefore,  $V \setminus B$  is precisely the set of intact nodes, and thus  $B$  is the set of befouled nodes.

By applying Theorem 1.2.5 repeatedly, we can conclude that  $B$  is a DSet.  $\square$

**Theorem 1.2.8.** *Let  $\langle V, Q \rangle$  be an FBAS and let  $B \subset V$  be the set of befouled nodes. If  $B$  is a DSet,  $B$  is not  $v$ -blocking for any intact  $v$ .*

*Proof.* By Definition 1.2.3, a node  $v \in V$  is intact if and only if  $v \notin B$ . By Theorem 1.1.23,  $\langle V, Q \rangle$  enjoys quorum availability despite  $B$  if and only if  $B$  is not  $v$ -blocking for any  $v \in V \setminus B$ . Since  $B$  is a DSet,  $\langle V, Q \rangle$  enjoys quorum availability despite  $B$ . Thus  $B$  is not  $v$ -blocking for any intact  $v$ .  $\square$

### 1.3. Voting, Accepting, and Ratifying.

**Definition 1.3.1 (Vote).** A node  $v$  votes for a statement  $a$  if and only if  $v$  asserts

- $a$  is valid,
- $a$  is consistent with all statements  $v$  has accepted,
- $v$  has never voted against  $a$ ,
- $v$  promises never to vote for a statement that contradicts  $a$  in the future.

**Definition 1.3.2 (Vote Against  $a$ ).** When a node  $v$  votes for a statement that contradicts  $a$ , we say  $v$  votes against  $a$ .

**Definition 1.3.3 (Accept).** Let  $\langle V, Q \rangle$  be an FBAS, and let  $v \in V$ .  $v$  accepts a statement  $a$  if and only if

- It has never accepted a statement contradicting  $a$ , and
- It determines that either
  - There exists a quorum  $U$  such that  $v \in U$  and each member of  $U$  either voted for  $a$  or broadcast that it has accepted  $a$ , or
  - There exists a  $v$ -blocking set  $B$  such that every member of  $B$  broadcast that it has accepted  $a$ .

When  $v$  accepts  $a$ , it must vote for the statement “an intact node accepted  $a$ .” For simplicity, we will often write “ $accept(a)$ ” to mean “an intact node accepted  $a$ .”

As you can see, the definitions of voting and accepting have a circular dependency. Note that it is possible for a node to accept a statement after voting for a contradictory statement.

**Definition 1.3.4 (Ratify).** A quorum  $U_a$  ratifies a statement  $a$  if and only if every member of  $U_a$  votes for  $a$ . A node  $v$  ratifies  $a$  if and only if  $v$  is a member of a quorum  $U_a$  that ratifies  $a$ .

**Theorem 1.3.5.** *Let  $\langle V, Q \rangle$  be an FBAS. If a node  $v \in V$  ratifies a statement  $a$ , then it must accept  $a$ .*

*Proof.* If a node  $v$  ratifies  $a$ , then it is a member of a quorum  $U \subset V$  that ratifies  $a$ . Thus every member of  $U$  votes for  $a$ . This implies that  $v$  also votes for  $a$ . By Definition 1.3.1,  $v$  has never accepted a statement contradicting  $a$ . By Definition 1.3.3,  $v$  accepts  $a$ .  $\square$

*Remark 1.3.6.* Theorem 1.3.5 shows that ratifying is a stronger condition than accepting.

**Theorem 1.3.7.** *If an FBAS enjoys quorum intersection and contains no ill-behaved node, then two contradictory statements cannot be both ratified.*

*Proof.* Suppose the statement is false and let  $a, \bar{a}$  denote two contradictory statements ratified in such an FBAS. Let  $U_a, U_{\bar{a}}$  denote quorums ratifying such statements, respectively. By the definition of quorum intersection,  $U_a \cap U_{\bar{a}} \neq \emptyset$ . Let  $v \in U_a \cap U_{\bar{a}}$ . This implies that  $v$  voted for both  $a$  and  $\bar{a}$ . However, the definition of voting (Definition 1.3.1) explicitly prohibits that. In other words,  $v$  must be ill-behaved, which is a contradiction to our assumption.  $\square$

**Theorem 1.3.8.** *Let  $\langle V, Q \rangle$  be an FBAS and  $B \subset V$ . Suppose that  $B$  contains all the ill-behaved nodes and  $\langle V, Q \rangle^B$  enjoys quorum intersection. Let  $v_1 \neq v_2 \in V \setminus B$ . If  $v_1$  ratifies a statement  $a$ , then  $v_2$  cannot ratify any statement that contradicts  $a$ .*

*Proof.* Suppose that the theorem is false and let  $U_1, U_2$  be quorums of  $v_1, v_2$  that ratify  $a, \bar{a}$ , respectively, where  $a$  and  $\bar{a}$  are contradictory. Since  $v_1 \in U_1 \setminus B$ ,  $U_1 \setminus B \neq \emptyset$ . By Theorem 1.1.12,  $U'_1 = U_1 \setminus B$  is a quorum in  $\langle V, Q \rangle^B$ . Similarly,  $U'_2 = U_2 \setminus B$  is a quorum in  $\langle V, Q \rangle^B$ . Since  $\langle V, Q \rangle^B$  enjoys quorum intersection,  $U'_1 \cap U'_2 \neq \emptyset$ . Choose  $v \in U'_1 \cap U'_2$  arbitrarily. Then  $v \in U_1 \cap U_2$ . In order for  $U_1, U_2$  to ratify  $a, \bar{a}$ , respectively,  $v$  must vote for both  $a$  and  $\bar{a}$ . However, this is against the definition of voting.  $v$  must be an ill-behaved node, so  $v \in B$ , which is a contradiction because  $v \in U'_1 \cap U'_2 \subset U'_1 = U_1 \setminus B$  and  $B$  contains all the ill-behaved nodes.  $\square$

**Theorem 1.3.9.** *Let  $\langle V, Q \rangle$  be an FBAS with quorum intersection. Then two intact nodes in  $V$  cannot ratify contradictory statements.*

*Proof.* Let  $v \neq v'$  be two intact nodes in  $V$ . Let  $B \subset V$  be the set of befouled nodes. Then  $v \notin B$  and  $v' \notin B$ . Since  $\langle V, Q \rangle$  is an FBAS with quorum intersection,  $B$  is a DSet by Theorem 1.2.7. By the definition of a DSet (Definition 1.2.1),  $\langle V, Q \rangle^B$  enjoys quorum intersection. By Theorem 1.3.8,  $v, v'$  cannot ratify contradictory statements.  $\square$

**Lemma 1.3.10.** *Let  $\langle V, Q \rangle$  be an FBAS enjoying quorum intersection and  $B$  be the set of befouled nodes. If  $a$  is accepted by an intact node in  $V$ , then  $a$  is ratified by some intact node in  $\langle V, Q \rangle^B$ .*

*Proof.* Suppose that  $a$  is accepted by one or more intact nodes in  $V$ . Since  $V$  is finite, there has to be an intact node  $v$  such that no intact nodes in  $V$  accepted  $a$  before  $v$ .

By the definition of accepting (Definition 1.3.3),  $v$  accepted  $a$  because either

- There was a quorum  $U$  of  $v$  such that every element of  $U$  either voted for  $a$  or broadcast that it has accepted  $a$ , or
- There existed a  $v$ -blocking set such that every element in it broadcast that it has accepted  $a$ .

We claim that it could not have been the second one. On the contrary, suppose that it was the second one. Since no intact nodes in  $V$  accepted  $a$  before  $v$ , such a  $v$ -blocking set must have only had befouled nodes. Therefore, such a  $v$ -blocking set must be a subset of  $B$ . Since  $\langle V, Q \rangle$  enjoys quorum intersection,  $B$  is a DSet by Theorem 1.2.7. By Theorem 1.2.8,  $B$  is not  $v$ -blocking. By taking the contrapositive of Theorem 1.1.21, we conclude that no subset of  $B$  is  $v$ -blocking.

Therefore, it must have been the first case. In other words, there must have existed a quorum  $U$  of  $v$  such that, before  $v$  accepted  $a$ , every member of  $U$  either voted for  $a$  or broadcast that it has accepted  $a$ . Since no intact node accepted  $a$  before  $v$ , every intact node in  $U$  must have voted for  $a$  before  $v$  accepted  $a$ . In other words, every node in  $U \setminus B$  voted for  $a$ . By Theorem 1.1.12,  $U \setminus B$  is a quorum in  $\langle V, Q \rangle^B$ . Thus  $U \setminus B$  ratified  $a$  in  $\langle V, Q \rangle^B$ , and thus  $v$  ratified  $a$  in  $\langle V, Q \rangle^B$ . Finally,  $v$  is indeed an intact node in  $\langle V, Q \rangle^B$  because  $\langle V, Q \rangle^B$  contains no ill-behaved nodes.

In conclusion,  $v$  is an intact node in  $\langle V, Q \rangle^B$  and  $v$  ratified  $a$  in  $\langle V, Q \rangle^B$ .  $\square$

**Theorem 1.3.11.** *Two intact nodes in an FBAS  $\langle V, Q \rangle$  enjoying quorum intersection cannot accept contradictory statements.*

By Theorem 1.3.5, ratifying is a stronger condition than accepting. Therefore, Theorem 1.3.11 is a stronger version of Theorem 1.3.9.

*Proof.* Suppose otherwise. Let  $a, \bar{a}$  be contradictory statements accepted by two intact nodes in  $\langle V, Q \rangle$ . Let  $B$  denote the set of befouled nodes. By Lemma 1.3.10,  $a, \bar{a}$  are ratified by some intact nodes in  $\langle V, Q \rangle^B$ . By the definition of a DSet (Definition 1.2.1),  $\langle V, Q \rangle$  enjoys quorum intersection despite  $B$ .

This means that  $\langle V, Q \rangle^B$  enjoys quorum intersection and two contradictory statements are ratified by some intact nodes in  $\langle V, Q \rangle^B$ . However, this is a direct contradiction to Theorem 1.3.9. Hence, two contradictory statements cannot be accepted by two intact nodes in  $\langle V, Q \rangle$ .  $\square$

#### 1.4. Confirmation.

**Definition 1.4.1 (Irrefutable).** A statement  $a$  is irrefutable in an FBAS if no intact node can ever vote against it.

**Definition 1.4.2 (Confirm).** A quorum  $U_a$  in an FBAS confirms a statement  $a$  if and only if every element in  $U_a$  broadcasts that it has accepted  $a$ . A node confirms  $a$  if and only if it is in such a quorum.

**Theorem 1.4.3.** *Ratifying is stronger than confirming, and confirming is stronger than accepting.*

*Proof.* Let  $\langle V, Q \rangle$  and  $v \in V$  be given. Suppose that  $v$  ratifies a statement  $a$ . Then there exists a quorum  $U$  such that  $v \in U$  and every member in  $U$  votes for  $a$ . For any  $u \in U$ ,

- $u$  has never accepted a statement contradicting  $a$  by the definition of voting (Definition 1.3.1), and
- $U$  is a quorum such that  $u \in U$  and every member of  $U$  voted for  $a$ .

Therefore,  $u$  accepts  $a$ . In other words, every member of  $U$  accepts  $a$ .  $U$  confirms  $a$  and thus  $v$  confirms  $a$ . Thus ratifying is stronger than confirming.

The definition of confirming (Definition 1.4.2) shows that a node must first accept a statement before confirming. Therefore, confirming is stronger than accepting.  $\square$

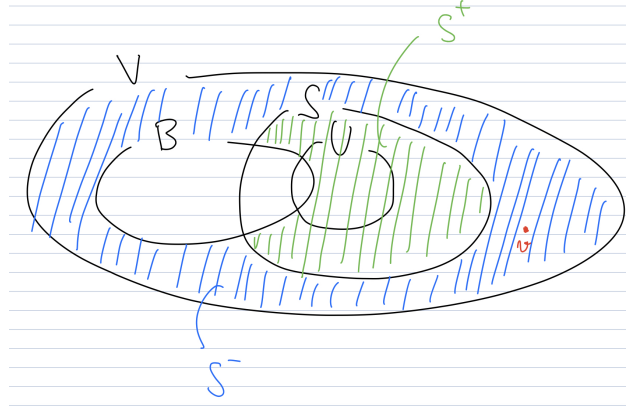


FIGURE 2. Lemma 1.4.4

**Lemma 1.4.4.** *Let  $\langle V, Q \rangle$  be an FBAS with quorum intersection. Let  $B$  denote the set of befouled nodes. Let  $U$  be a quorum containing an intact node, and let  $S$  be a set containing  $U$ . Let  $S^+$  be the set of intact nodes in  $S$ , and let  $S^-$  be the set of intact nodes not in  $S$ . Either  $S^- = \emptyset$ , or  $\exists v \in S^-$  such that  $S^+$  is  $v$ -blocking. (See Figure 2.)*

*Proof.*

Proofread this one more time!

Note that  $S^+ = S \setminus B$  and  $S^- = (V \setminus S) \setminus B = (V \setminus B) \setminus S^+$ . If  $\exists v \in S^-$  such that  $S^+$  is  $v$ -blocking, then we are done.

Suppose that  $\forall v \in S^-$ ,  $S^+$  is not  $v$ -blocking in  $\langle V, Q \rangle$ . We want to show that  $S^- = \emptyset$ .

For any arbitrary  $v \in S^-$ ,

- $S^+$  and  $S^-$  form a partition of  $V \setminus B$ ,
- $S^+$  is not  $v$ -blocking in  $\langle V, Q \rangle$ .

By Theorem 1.1.22,  $S^+$  is not  $v$ -blocking in  $\langle V, Q \rangle^B$  for any  $v \in S^-$ . Since  $S^- = (V \setminus B) \setminus S^+$ ,  $S^+$  is not  $v$ -blocking in  $\langle V, Q \rangle^B$  for any  $v \in (V \setminus B) \setminus S^+$ . By applying Theorem 1.1.23 to the FBAS  $\langle V, Q \rangle^B$  and subset  $S^+$ ,  $\langle V, Q \rangle^B$  enjoys quorum availability despite  $S^+$ .

By the definition of quorum availability (Definition 1.1.16),  $(V \setminus B) \setminus S^+$  is a quorum in  $\langle V, Q \rangle^B$ , or  $V \setminus B = S^+$ . Suppose  $(V \setminus B) \setminus S^+$  is a quorum in  $\langle V, Q \rangle^B$ . This leads to two contradictory claims:

- Claim 1:  $\langle V, Q \rangle$  enjoys quorum intersection despite  $B$ .
  - Since  $\langle V, Q \rangle$  enjoys quorum intersection,  $B$  is a DSet by Theorem 1.2.7. By the definition of a DSet (Definition 1.2.1),  $\langle V, Q \rangle^B$  enjoys quorum intersection.
- Claim 2:  $\langle V, Q \rangle$  does not enjoy quorum intersection despite  $B$ .
  - $U \setminus B$  is nonempty since  $U$  contains an intact node. Thus  $U \setminus B$  is a quorum in  $\langle V, Q \rangle^B$  by Theorem 1.1.12. We also assumed that  $(V \setminus B) \setminus S^+$  is a quorum in  $\langle V, Q \rangle^B$ .

$$\begin{aligned}
 (U \setminus B) \cap ((V \setminus B) \setminus S^+) &\subset (U \setminus B) \cap S^- \\
 &\subset (S \setminus B) \cap S^- \\
 &\subset S^+ \cap S^- \\
 &= \emptyset.
 \end{aligned}$$

Therefore,  $\langle V, Q \rangle^B$  does not enjoy quorum intersection.

Therefore,  $(V \setminus B) \setminus S^+$  must not be a quorum in  $\langle V, Q \rangle^B$ , so  $V \setminus B$  must be  $S^+$ . In other words,  $S^+$  contains all the intact nodes, so  $S^- = \emptyset$ , which is exactly what we wanted to show.  $\square$

**Theorem 1.4.5.** *If an intact node in an FBAS  $\langle V, Q \rangle$  with quorum intersection confirms a statement  $a$ , then every intact node will accept and confirm  $a$  once sufficient messages are delivered.*

*Proof.* Let  $B$  denote the set of befouled nodes. When an intact node confirms  $a$ , some quorum containing such an intact node confirms  $a$ . In other words, there exists a quorum  $U \not\subset B$  such that every node in  $U$  broadcast that it accepted  $a$ . Some nodes may decide to accept  $a$  upon hearing that nodes in  $U$  broadcast that it accepted  $a$ . This may, in turn, make more nodes accept  $a$ . Thus we may experience a gradual increase in the number of nodes that accept  $a$  over time. Since  $V$  only contains finitely many nodes, there will be a point at which the number of nodes that accept  $a$  stops increasing. Let  $S$  be the set of nodes that accept  $a$  at that point. We claim that  $S$  contains all intact nodes.

- $U$  is a quorum containing an intact node.
- $U \subset S \subset V$  because every node in  $U$  accepted  $a$  in the beginning.
- Let  $S^+ = S \setminus B$  be the set of intact nodes in  $S$ , and let  $S^- = (V \setminus S) \setminus B$  be the set of intact nodes not in  $S$ .

By Lemma 1.4.4,  $S^-$  is empty, or  $S^+$  is  $v$ -blocking for some  $v \in S^-$ . Suppose  $S^-$  is nonempty. Then  $v$  accepts  $a$  because  $S^+$  is  $v$ -blocking and every element of  $S^+$  broadcast that it has accepted  $a$ . This is a contradiction because we assumed that the number of nodes that

accept  $a$  stopped increasing. Therefore,  $S^-$  must be empty. If  $S^-$  is empty, then that implies that every intact node accepted and confirmed  $a$  assuming sufficient messages are delivered because

- Since  $S^-$  is empty,  $S^+$  contains all intact nodes in  $V$ .  $S^+ \subset S$ , so  $S$  contains all intact nodes. Since every node in  $S$  accepted  $a$ , every intact node accepted  $a$ .
- Since  $\langle V, Q \rangle$  enjoys quorum intersection,  $B$  is a DSet by Theorem 1.2.7. By the definition of a DSet,  $V \setminus B$  is a quorum. In other words, the set of all intact nodes is a quorum.
- Since  $V \setminus B$  is a quorum in which every node accepted  $a$ , every intact node confirmed  $a$ .

□



## 2. STELLAR CONSENSUS PROTOCOL

**2.1. Nomination Protocol.** Nomination is done through voting, accepting, and confirming a special type of statement in the form of *nominate x*.

**Definition 2.1.1 (Nominate).** A node  $v$  is said to nominate a value  $x$  if and only if it votes for the statement *nominate x*.

**Definition 2.1.2 (Candidate).** A node  $v$  considers a value  $x$  to be a candidate if and only if  $v$  has confirmed the statement *nominate x*. Alternatively, we say that a node  $v$  has a candidate value  $x$ .

## 2.2. Ballot Protocol.