

# STELLAR CONSENSUS PROTOCOL

HIDENORI SHINOHARA

ABSTRACT. This is my personal notes on the Stellar consensus protocol. This roughly follows the structure of the white paper on <https://www.stellar.org/>.

## CONTENTS

1. Basic Properties of Quorums	1
2. Dispensable Sets	3
3. Voting, Accepting, and Ratifying	4
4. Confirmation	6

## 1. BASIC PROPERTIES OF QUORUMS

**Definition 1.1.** Let  $V$  be a set and  $Q : V \rightarrow 2^V \setminus \{\emptyset\}$  be a function such that  $\forall v \in V, \forall q \in Q(v), v \in q$ . Then we call the pair  $\langle V, Q \rangle$  a federated Byzantine agreement system, or FBAS for short.

**Definition 1.2.** Let  $\langle V, Q \rangle$  be an FBAS.  $U \subset V$  is called a quorum if and only if  $\forall v \in U, \exists q \in Q(v), q \subset U$ .

**Theorem 1.3.** In an FBAS  $\langle V, Q \rangle$ , the union of two quorums is a quorum.

*Proof.* Let  $U_1, U_2$  be two quorums. Let  $v \in U_1 \cup U_2$ . Then  $v \in U_i$  for  $i = 1$  or  $i = 2$ . Then  $q \subset U_i$  for some  $q \in Q(v)$ . Therefore,  $q \subset U_1 \cup U_2$ , so  $U_1 \cup U_2$  is indeed a quorum.  $\square$

**Theorem 1.4.** In an FBAS  $\langle V, Q \rangle$ ,  $V$  is a quorum.

*Proof.* For any  $v \in V$ , for any  $q \in Q(v)$ ,  $q \subset V$ . Therefore,  $V$  is indeed a quorum.  $\square$

**Example 1.5.** One might wonder if the intersection of quorums is always a quorum. However, this is not true in general.

Let  $V = \{v_1, \dots, v_4\}$  and

- $Q(v_1) = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}\},$
- $\vdots$
- $Q(v_4) = \{\{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}.$

In other words,  $Q(v_i) = \{U \mid U \in 2^V, v_i \in U\}$ .

Then  $U_1 = \{v_1, v_2, v_3\}$  is a quorum, and  $U_2 = \{v_2, v_3, v_4\}$  is a quorum. However,  $U_1 \cap U_2 = \{v_2, v_3\}$  is not a quorum because the size of any quorum slice is 3.

**Definition 1.6.** Let  $\langle V, Q \rangle$  be an FBAS. We say  $\langle V, Q \rangle$  enjoys quorum intersection if and only if for any pair of quorums  $U_1, U_2$ ,  $U_1 \cap U_2 \neq \emptyset$ .

**Definition 1.7.** Let  $\langle V, Q \rangle$  be an FBAS and  $B \subset V$ . Then the FBAS  $\langle V, Q \rangle^B$  is defined to be  $\langle V \setminus B, Q^B \rangle$  where  $\forall v \in V, Q^B(v) = \{q \setminus B \mid q \in Q(v)\}$ .

**Theorem 1.8.** *Definition 1.7 is well-defined. In other words, if  $\langle V, Q \rangle$  is an FBAS and  $B \subset V$ , then  $\langle V, Q \rangle^B$  is an FBAS.*

*Proof.* Let  $v \in V \setminus B, q' \in Q^B(v)$  be given. Then  $q' = q \setminus B$  for some  $q \in Q(v)$ . By the definition of an FBAS,  $v \in q$ . Since  $v \notin B$ ,  $v \in q \setminus B = q'$ . Therefore,  $\langle V, Q \rangle^B$  is an FBAS.  $\square$

**Theorem 1.9.** *Let  $U$  be a quorum in FBAS  $\langle V, Q \rangle$ , let  $B \subset V$  be a set of nodes, and let  $U' = U \setminus B$ . If  $U' \neq \emptyset$ , then  $U'$  is a quorum in  $\langle V, Q \rangle^B$ .*

*Proof.* Since  $U' \neq \emptyset$ , it suffices to show that  $\forall v \in U', \exists q \in Q^B(v), q \subset U'$ . Let  $v \in U'$ . Then  $v \in U$ . Since  $U$  is a quorum in  $\langle V, Q \rangle$ , we can find  $q \in Q(v)$  such that  $q \subset U$ . Then  $q' = q \setminus B \in Q^B(v)$ , and  $q' = q \setminus B \subset U \setminus B = U'$ . Therefore,  $U'$  is a quorum in  $\langle V, Q \rangle^B$ .  $\square$

**Definition 1.10.** Let  $\langle V, Q \rangle$  be an FBAS and  $B \subset V$  be a set of nodes. We say  $\langle V, Q \rangle$  enjoys quorum intersection despite  $B$  if and only if  $\langle V, Q \rangle^B$  enjoys quorum intersection.

**Definition 1.11.** Let  $\langle V, Q \rangle$  be an FBAS and  $B \subset V$  be a set of nodes. We say  $\langle V, Q \rangle$  enjoys quorum availability despite  $B$  if and only if  $V \setminus B$  is a quorum in  $\langle V, Q \rangle$  or  $B = V$ .

**Definition 1.12.** Let  $\langle V, Q \rangle$  be an FBAS. Let  $v \in V$ . A subset  $B \subset V$  is called  $v$ -blocking if and only if  $\forall q \in Q(v), q \cap B \neq \emptyset$ .

**Theorem 1.13.** *Let  $\langle V, Q \rangle$  be an FBAS. Let  $v \in V$ . Then*

- *The union of two  $v$ -blocking sets is  $v$ -blocking.*
- *Any superset of a  $v$ -blocking set is  $v$ -blocking.*

*Proof.* If  $B, B'$  are  $v$ -blocking, then  $q \cap (B \cup B') = (q \cap B) \cup (q \cap B') \neq \emptyset$  for any  $q \in Q(v)$  because the union of two nonempty sets is nonempty. If  $B \subset B'$  and  $B$  is  $v$ -blocking,  $q \cap B' \supset q \cap B \neq \emptyset$  for any  $q \in Q(v)$ .  $\square$

**Theorem 1.14.** *Let  $\langle V, Q \rangle$  be an FBAS. Let  $A \subsetneq V$  and  $U_1, U_2$  be a partition of  $V \setminus A$ . Let  $v \in U_1$ . If  $U_2$  is not  $v$ -blocking in  $\langle V, Q \rangle$ , then  $U_2$  is not  $v$ -blocking in  $\langle V, Q \rangle^A$ .*

*Proof.* Since  $U_2$  is not  $v$ -blocking in  $\langle V, Q \rangle$ , there exists  $q_v \in Q(v)$  such that  $q_v \cap U_2 \neq \emptyset$ .

$$\begin{aligned} (q_v \setminus A) \cap U_2 &= (q_v \cap U_2) \setminus (A \cap U_2) \\ &= q_v \cap U_2 \neq \emptyset. \end{aligned}$$

Thus  $U_2$  is not  $v$ -blocking in  $\langle V, Q \rangle^A$ .  $\square$

**Theorem 1.15.** *Let  $\langle V, Q \rangle$  be an FBAS. Let  $B \subset V$ .  $\langle V, Q \rangle$  enjoys quorum availability despite  $B$  if and only if  $B$  is not  $v$ -blocking for any  $v \in V \setminus B$ .*

*Proof.*

$$\begin{aligned} \forall v \in V \setminus B, \neg(B \text{ is } v\text{-blocking}) &\iff \forall v \in V \setminus B, \neg(\forall q \in Q(v), q \cap B \neq \emptyset) \\ &\iff \forall v \in V \setminus B, \exists q \in Q(v), q \cap B = \emptyset \\ &\iff \forall v \in V \setminus B, \exists q \in Q(v), q \subset V \setminus B \\ &\iff V = B \text{ or } V \setminus B \text{ is a quorum in } \langle V, Q \rangle \\ &\iff \langle V, Q \rangle \text{ enjoys quorum availability despite } B \end{aligned}$$

□

## 2. DISPENSABLE SETS

**Definition 2.1.** Let  $\langle V, Q \rangle$  be an FBAS and  $B \subset V$  be a set of nodes.  $B$  is called a dispensable set, or DSet, if and only if  $\langle V, Q \rangle$  enjoys quorum intersection and availability despite  $B$ .

**Definition 2.2.** Let  $\langle V, Q \rangle$  be an FBAS and  $v \in V$ .  $v$  is said to be intact if and only if there exists a DSet  $B$  containing all ill-behaved nodes and  $v \notin B$ .  $v$  is said to be befouled if and only if  $v$  is not intact.

**Theorem 2.3.** If  $B_1$  and  $B_2$  are DSets in an FBAS  $\langle V, Q \rangle$  enjoying quorum intersection, then  $B = B_1 \cap B_2$  is a DSet, too.

*Proof.* If  $B_1 = V$  or  $B_2 = V$ , then we are done. Suppose otherwise.

For any  $v \in V$ ,

$$\begin{aligned}
 v \in V \setminus B &\iff v \in V \wedge v \notin B \\
 &\iff v \in V \wedge (v \notin B_1 \vee v \notin B_2) \\
 &\iff (v \in V \wedge v \notin B_1) \vee (v \in V \wedge v \notin B_2) \\
 &\iff (v \in (V \setminus B_1)) \vee (v \in (V \setminus B_2)) \\
 &\iff v \in ((V \setminus B_1) \cup (V \setminus B_2)).
 \end{aligned}$$

Thus,  $V \setminus B = (V \setminus B_1) \cup (V \setminus B_2)$ . By the definition of a DSet,  $V \setminus B_1$  and  $V \setminus B_2$  are both quorums in  $\langle V, Q \rangle$ . By Theorem 1.3,  $V \setminus B$  is a quorum in  $\langle V, Q \rangle$ .

We must now show quorum intersection despite  $B$ . Let  $U_a, U_b$  be quorums in  $\langle V, Q \rangle^B$ .

- $U_a \setminus B_1$  is a quorum in  $(\langle V, Q \rangle^B)^{B_1} = \langle V, Q \rangle^{B_1}$  by Theorem 1.7.
- Similarly,  $U_b \setminus B_1$  is a quorum in  $\langle V, Q \rangle^{B_1}$ , and  $U_a \setminus B_2$  and  $U_b \setminus B_2$  are both quorums in  $\langle V, Q \rangle^{B_2}$ .

$$\begin{aligned}
 (U_a \setminus B_1) \cup (U_a \setminus B_2) &= U_a \setminus (B_1 \cap B_2) \\
 &= U_a \setminus B \\
 &= U_a
 \end{aligned}$$

because  $U_a$  is a quorum in  $\langle V, Q \rangle^B$ . In other words,  $(U_a \setminus B_1) \cup (U_a \setminus B_2) \neq \emptyset$ . Similarly,  $(U_b \setminus B_1) \cup (U_b \setminus B_2) \neq \emptyset$ .

Without loss of generality, assume that  $U_a \setminus B_1 \neq \emptyset$ .

- $V \setminus B_1$  is a quorum in  $\langle V, Q \rangle$  because  $B_1$  is a DSet. Similarly,  $V \setminus B_2$  is a quorum in  $\langle V, Q \rangle$ . Because  $\langle V, Q \rangle$  enjoys quorum intersection,  $(V \setminus B_1) \cap (V \setminus B_2) \neq \emptyset$ . In other words,  $(V \setminus B_2) \setminus B_1$  is a quorum. By Theorem 1.7,  $(V \setminus B_2) \setminus B_1$  is a quorum in  $\langle V, Q \rangle^{B_1}$ .
- $U_a \setminus B_1$  is a quorum in  $(\langle V, Q \rangle^B)^{B_1} = \langle V, Q \rangle^{B_1}$  for the same reason.

Because  $B_1$  is a DSet in  $\langle V, Q \rangle$ ,  $\langle V, Q \rangle^{B_1}$  enjoys quorum intersection. Therefore,  $(U_a \setminus B_1) \cap ((V \setminus B_2) \setminus B_1) \neq \emptyset$ .

$$\begin{aligned} (U_a \setminus B_1) \cap ((V \setminus B_2) \setminus B_1) &= (U_a \cap (V \setminus B_2)) \setminus B_1 \\ &\subset U_a \cap (V \setminus B_2) \\ &= (U_a \cap V) \setminus B_2 \\ &= U_a \setminus B_2. \end{aligned}$$

Thus,  $U_a \setminus B_2 \neq \emptyset$ . Using the same argument, we can show that  $U_b \setminus B_1 \neq \emptyset$  and  $U_b \setminus B_2 \neq \emptyset$ . Since  $U_a \setminus B_1$  and  $U_b \setminus B_1$  are quorums in  $\langle V, Q \rangle^{B_1}$  and  $B_1$  is a DSet,  $(U_a \setminus B_1) \cap (U_b \setminus B_1) \neq \emptyset$  by the definition of a DSet. This implies  $(U_a \cap U_b) \setminus B_1 \neq \emptyset$ . Therefore,  $U_a \cap U_b \neq \emptyset$ .  $\square$

**Theorem 2.4.** *In an FBAS with quorum intersection, the set of befouled nodes is a DSet.*

*Proof.* Let  $\langle V, Q \rangle$  be an FBAS with quorum intersection. Let  $B$  be the intersection of all DSets that contain all ill-behaved nodes. By Theorem 2.3,  $B$  is a DSet.

- Case 1:  $v \in B$ . Then there exists no DSet  $B_v$  such that  $B_v$  contains all ill-behaved nodes and  $v \notin B_v$ . Therefore,  $v$  is not an intact node. In other words,  $v$  is a befouled node.
- Case 2:  $v \notin B$ . Then there exists a DSet  $B_v$  that contains all ill-behaved nodes and  $v \notin B_v$ . In other words,  $v$  is intact and thus  $v$  is not a befouled node.

Therefore,  $B$  is precisely the set of befouled nodes and it is a DSet.  $\square$

**Theorem 2.5.** *Let  $\langle V, Q \rangle$  be an FBAS and let  $B \subset V$  be the set of befouled nodes. If  $B$  is a DSet,  $B$  is not  $v$ -blocking for any intact  $v$ .*

*Proof.* By Definition 2.2, a node  $v \in V$  is intact if and only if  $v \notin B$ . By Theorem 1.15,  $\langle V, Q \rangle$  enjoys quorum availability despite  $B$  if and only if  $B$  is not  $v$ -blocking for any  $v \in V \setminus B$ . Since  $B$  is a DSet,  $\langle V, Q \rangle$  enjoys quorum availability despite  $B$ . Thus  $B$  is not  $v$ -blocking for any intact  $v$ .  $\square$

### 3. VOTING, ACCEPTING, AND RATIFYING

**Definition 3.1.** A node  $v$  votes for a statement  $a$  if and only if  $v$  asserts

- $a$  is valid,
- $a$  is consistent with all statements  $v$  has accepted,
- $v$  has never voted against  $a$ ,
- $v$  promises never to vote against  $a$  in the future.

**Definition 3.2.** Let  $\langle V, Q \rangle$  be an FBAS, and let  $v \in V$ .  $v$  accepts a statement  $a$  if and only if

- It has never accepted a statement contradicting  $a$ .
- It determines that either
  - There exists a quorum such that  $v \in U$  and each member of  $U$  either voted for  $a$  or broadcasted that it has accepted  $a$ , or
  - There exists a  $v$ -blocking set  $B$  such that every member of  $B$  broadcast that it has accepted  $a$ .

Note that it is possible for a node to accept a statement it did not vote for. Furthermore, it is possible for a node to accept a statement after voting for a contradictory statement.

**Definition 3.3.** A quorum  $U_a$  ratifies a statement  $a$  if and only if every member of  $U_a$  votes for  $a$ . A node  $v$  ratifies  $a$  if and only if  $v$  is a member of a quorum  $U_a$  that ratifies  $a$ .

**Theorem 3.4.** Let  $\langle V, Q \rangle$  be an FBAS. If a node  $v \in V$  ratifies a statement  $a$ , then it must accept  $a$ .

*Proof.* If a node  $v$  ratifies  $a$ , then it is a member of a quorum  $U \subset V$  that ratifies  $a$ . Thus every member of  $U$  votes for  $a$ . This implies that  $v$  also votes for  $a$ . By Definition 3.1,  $v$  has never accepted a statement contradicting  $a$ . By Definition 3.2,  $v$  accepts  $a$ .  $\square$

**Theorem 3.5.** If an FBAS enjoys quorum intersection and contains no ill-behaved node, then two contradictory statements cannot be both ratified.

*Proof.* Suppose the statement is false and let  $a, \bar{a}$  denote two contradictory statements ratified in such an FBAS. Let  $U_a, U_{\bar{a}}$  denote quorums ratifying such statements, respectively. By the definition of quorum intersection,  $U_a \cap U_{\bar{a}} \neq \emptyset$ . Let  $v \in U_a \cap U_{\bar{a}}$ . This implies that  $v$  voted for both  $a$  and  $\bar{a}$ . However, this goes against the definition of voting. In other words,  $v$  must be ill-behaved, which is a contradiction to our assumption.  $\square$

**Theorem 3.6.** Let  $\langle V, Q \rangle$  be an FBAS. Let  $B \subsetneq V$  be a subset containing all the ill-behaved nodes and suppose that  $\langle V, Q \rangle^B$  enjoys quorum intersection. Let  $v_1 \neq v_2 \in V \setminus B$ . If  $v_1$  ratifies a statement  $a$ , then  $v_2$  cannot ratify any statement that contradicts  $a$ .

*Proof.* Suppose that the theorem is false and let  $U_1, U_2$  be quorums of  $v_1, v_2$  that ratify  $a, \bar{a}$ , respectively where  $a$  and  $\bar{a}$  are contradictory. Since  $v_1 \in U_1 \setminus B$ ,  $U_1 \setminus B \neq \emptyset$ . By Theorem 1.9,  $U'_1 = U_1 \setminus B$  is a quorum in  $\langle V, Q \rangle^B$ . Similarly,  $U'_2 = U_2 \setminus B$  is a quorum in  $\langle V, Q \rangle^B$ . Since  $\langle V, Q \rangle^B$  enjoys quorum intersection,  $U'_1 \cap U'_2 \neq \emptyset$ . Let  $v \in U'_1 \cap U'_2$ . Then  $v \in U_1 \cap U_2$ . In order for  $U_1, U_2$  to ratify  $a, \bar{a}$ , respectively,  $v$  must vote for both  $a$  and  $\bar{a}$ . However, this is against the definition of voting.  $v$  must be an ill-behaved node, so  $v \in B$ , which is a contradiction because  $v \in U_1 \setminus B$ .  $\square$

**Theorem 3.7.** Let  $\langle V, Q \rangle$  be an FBAS with quorum intersection. Then two intact nodes in  $V$  cannot ratify contradictory statements.

*Proof.* Let  $v \neq v'$  be two intact nodes in  $V$ . Let  $B \subset V$  be the set of befouled nodes. Then  $v \notin B$  and  $v' \notin B$ . Since  $\langle V, Q \rangle$  is an FBAS with quorum intersection,  $B$  is a DSet by Theorem 2.4. By the definition of a DSet (Definition 2.1),  $\langle V, Q \rangle^B$  enjoys quorum intersection. By Theorem 3.6,  $v, v'$  cannot ratify contradictory statements.  $\square$

**Lemma 3.8.** Let  $\langle V, Q \rangle$  be an FBAS enjoying quorum intersection and  $B$  be the set of befouled nodes. If  $a$  is accepted by an intact node in  $V$ , then  $a$  is ratified by some intact node in  $\langle V, Q \rangle^B$ .

*Proof.* Since  $V$  is finite, there has to be an intact node  $v$  such that no intact nodes in  $V$  accepted  $a$  before  $v$ .

Since  $\langle V, Q \rangle$  enjoys quorum intersection,  $B$  is a DSet by Theorem 2.4. By Theorem 2.5,  $B$  is not  $v$ -blocking. Therefore, by Definition 3.2, there must exist a quorum  $U$  of  $v$  such that, before  $v$  accepted  $a$ , every member of  $U$  either voted for  $a$  or broadcast that it has

accepted  $a$ . Because of the way we picked  $v$ , every intact node in  $U$  must have voted for  $a$  before  $v$  accepted  $a$ . In other words, every node in  $U \setminus B$  voted for  $a$ . By Theorem 1.9,  $v$  ratified  $a$  in  $\langle V, Q \rangle^B$ . Finally,  $v$  is indeed an intact node in  $\langle V, Q \rangle^B$  because  $\langle V, Q \rangle^B$  contains no ill-behaved nodes.  $\square$

**Theorem 3.9.** *Two intact nodes in an FBAS  $\langle V, Q \rangle$  enjoying quorum intersection cannot accept contradictory statements.*

Note that Theorem 3.9 is a stronger version of Theorem 3.7 by Theorem 3.4.

*Proof.* Suppose otherwise. Let  $a, \bar{a}$  be contradictory statements accepted by two intact nodes in  $\langle V, Q \rangle$ . By Lemma 3.8,  $a, \bar{a}$  are ratified by some intact nodes in  $\langle V, Q \rangle^B$ . By Definition 2.1,  $\langle V, Q \rangle$  enjoys quorum intersection despite  $B$ . In other words,  $\langle V, Q \rangle^B$  enjoys quorum intersection. By Theorem 3.7,  $a, \bar{a}$  cannot be ratified by  $v, v'$  in  $\langle V, Q \rangle^B$ , which is a contradiction.  $\square$

#### 4. CONFIRMATION

**Definition 4.1.** A statement  $a$  is irrefutable in an FBAS if no intact node can ever vote against it.

**Definition 4.2.** A quorum  $U_a$  in an FBAS confirms a statement  $a$  if and only if every element in  $U_a$  broadcasts that it has accepted  $a$ . A node confirms  $a$  if and only if it is in such a quorum.

Let  $\langle V, Q \rangle$  be an FBAS with quorum intersection. Let  $B$  denote the set of befouled nodes. Let  $U$  be a quorum containing an intact node. Let  $S$  be a set such that  $U \subset S \subset V$ . Let  $S^+ = S \setminus B$  be the set of intact nodes in  $S$ , and let  $S^- = (V \setminus S) \setminus B$  be the set of intact nodes not in  $S$ . Either  $S^- = \emptyset$ , or  $\exists v \in S^-$  such that  $S^+$  is  $v$ -blocking.

*Proof.* If  $\exists v \in S^-$  such that  $S^+$  is  $v$ -blocking, then we are done. Suppose that  $\forall v \in S^-$ ,  $S^+$  is not  $v$ -blocking in  $\langle V, Q \rangle$ . By Theorem 1.14,  $S^+$  is not  $v$ -blocking in  $\langle V, Q \rangle^B$  for any  $v \in S^- = (V \setminus B) \setminus S^+$ . By Theorem 1.15,  $\langle V, Q \rangle^B$  enjoys quorum availability despite  $S^+$ . By Definition 1.11,  $(V \setminus B) \setminus S^+$  is a quorum in  $\langle V, Q \rangle^B$ , or  $V \setminus B = S^+$ . If  $V \setminus B = S^+$ , then  $S^- = \emptyset$ , and we are done. Suppose  $(V \setminus B) \setminus S^+$  is a quorum in  $\langle V, Q \rangle^B$ .

- $U \setminus B$  is a quorum in  $\langle V, Q \rangle^B$  by Theorem 1.9.
- Since  $B$  is a DSet by Theorem 2.4,  $\langle V, Q \rangle^B$  enjoys quorum intersection by Definition 2.1.
- However,

$$\begin{aligned} (U \setminus B) \cap ((V \setminus B) \setminus S^+) &= (U \setminus B) \cap S^- \\ &\subset S \cap S^- \\ &= \emptyset. \end{aligned}$$

This is a contradiction.  $\square$

**Theorem 4.3.** *If an intact node in an FBAS  $\langle V, Q \rangle$  with quorum intersection confirms a statement  $a$ , then every intact node will accept and confirm  $a$  once sufficient messages are delivered.*

*Proof.* Let  $B$  denote the set of befouled nodes. Then there exists a quorum  $U \not\subset B$  such that every node in  $U$  broadcast that it accepted  $a$ . After every node in  $U$  broadcast it accepted  $a$ , there may be a node  $v$  that accept  $a$  since  $U$  is  $v$ -blocking. After all such nodes broadcast that they accepted  $a$ , there may be other nodes that accept  $a$  as well. Since  $V$  is a finite set, there is a point in time where the number of nodes that accept  $a$  does not increase. Let  $S$  be the set of all nodes that accepted  $a$  and broadcast it.

- $U$  is a quorum containing an intact node.
- $U \subset S \subset V$ .
- Let  $S^+ = S \setminus B$  be the set of intact nodes in  $S$ , and let  $S^- = (V \setminus S) \setminus B$  be the set of intact nodes not in  $S$ .

By Lemma 4,  $S^-$  is empty, or  $S^+$  is  $v$ -blocking for some  $v \in S^-$ . However, the latter is impossible because it would imply that  $v$  would accept  $a$ . Therefore,  $S^-$  is empty, and thus every intact node accepted  $a$ .  $\square$