

# STELLAR CONSENSUS PROTOCOL

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ABSTRACT. This is my personal notes on the Stellar consensus protocol. This roughly follows the structure of the white paper on <https://www.stellar.org/>.

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### 1. BASIC PROPERTIES OF QUORUMS

**Definition 1.1.** Let  $V$  be a set and  $Q : V \rightarrow 2^V \setminus \{\emptyset\}$  be a function such that  $\forall v \in V, \forall q \in Q(v), v \in q$ . Then we call the pair  $\langle V, Q \rangle$  a federated Byzantine agreement system, or FBAS for short.

**Definition 1.2.** Let  $\langle V, Q \rangle$  be an FBAS.  $U \subset V$  is called a quorum if and only if  $\forall v \in U, \exists q \in Q(v), q \subset U$ .

**Theorem 1.3.** In an FBAS  $\langle V, Q \rangle$ , the union of two quorums is a quorum.

*Proof.* Let  $U_1, U_2$  be two quorums. Let  $v \in U_1 \cup U_2$ . Then  $v \in U_i$  for  $i = 1$  or  $i = 2$ . Then  $q \subset U_i$  for some  $q \in Q(v)$ . Therefore,  $q \subset U_1 \cup U_2$ , so  $U_1 \cup U_2$  is indeed a quorum.  $\square$

**Theorem 1.4.** In an FBAS  $\langle V, Q \rangle$ ,  $V$  is a quorum.

*Proof.* For any  $v \in V$ , for any  $q \in Q(v)$ ,  $q \subset V$ . Therefore,  $V$  is indeed a quorum.  $\square$

**Example 1.5.** One might wonder if the intersection of quorums is always a quorum. However, this is not true in general.

Let  $V = \{v_1, \dots, v_4\}$  and

- $Q(v_1) = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}\},$
- $\vdots$
- $Q(v_4) = \{\{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}.$

In other words,  $Q(v_i) = \{U \mid U \in 2^V, v_i \in U\}$ .

Then  $U_1 = \{v_1, v_2, v_3\}$  is a quorum, and  $U_2 = \{v_2, v_3, v_4\}$  is a quorum. However,  $U_1 \cap U_2 = \{v_2, v_3\}$  is not a quorum because the size of any quorum slice is 3.

**Definition 1.6.** Let  $\langle V, Q \rangle$  be an FBAS. We say  $\langle V, Q \rangle$  enjoys quorum intersection if and only if for any pair of quorums  $U_1, U_2$ ,  $U_1 \cap U_2 \neq \emptyset$ .

**Definition 1.7.** Let  $\langle V, Q \rangle$  be an FBAS and  $B \subset V$ . Then the FBAS  $\langle V, Q \rangle^B$  is defined to be  $\langle V \setminus B, Q^B \rangle$  where  $\forall v \in V, Q^B(v) = \{q \setminus B \mid q \in Q(v)\}$ .

**Theorem 1.8.** *Definition 1.7 is well-defined. In other words, if  $\langle V, Q \rangle$  is an FBAS and  $B \subset V$ , then  $\langle V, Q \rangle^B$  is an FBAS.*

*Proof.* Let  $v \in V \setminus B, q' \in Q^B(v)$  be given. Then  $q' = q \setminus B$  for some  $q \in Q(v)$ . By the definition of an FBAS,  $v \in q$ . Since  $v \notin B$ ,  $v \in q \setminus B = q'$ . Therefore,  $\langle V, Q \rangle^B$  is an FBAS.  $\square$

**Theorem 1.9.** *Let  $U$  be a quorum in FBAS  $\langle V, Q \rangle$ , let  $B \subset V$  be a set of nodes, and let  $U' = U \setminus B$ . If  $U' \neq \emptyset$ , then  $U'$  is a quorum in  $\langle V, Q \rangle^B$ .*

*Proof.* Since  $U' \neq \emptyset$ , it suffices to show that  $\forall v \in U', \exists q \in Q^B(v), q \subset U'$ . Let  $v \in U'$ . Then  $v \in U$ . Since  $U$  is a quorum in  $\langle V, Q \rangle$ , we can find  $q \in Q(v)$  such that  $q \subset U$ . Then  $q' = q \setminus B \in Q^B(v)$ , and  $q' = q \setminus B \subset U \setminus B = U'$ . Therefore,  $U'$  is a quorum in  $\langle V, Q \rangle^B$ .  $\square$

**Definition 1.10.** Let  $\langle V, Q \rangle$  be an FBAS and  $B \subset V$  be a set of nodes. We say  $\langle V, Q \rangle$  enjoys quorum intersection despite  $B$  if and only if  $\langle V, Q \rangle^B$  enjoys quorum intersection.

**Definition 1.11.** Let  $\langle V, Q \rangle$  be an FBAS and  $B \subset V$  be a set of nodes. We say  $\langle V, Q \rangle$  enjoys quorum availability despite  $B$  if and only if  $V \setminus B$  is a quorum in  $\langle V, Q \rangle$  or  $B = V$ .

## 2. DISPENSABLE SETS

**Definition 2.1.** Let  $\langle V, Q \rangle$  be an FBAS and  $B \subset V$  be a set of nodes.  $B$  is called a dispensable set, or DSet, if and only if  $\langle V, Q \rangle$  enjoys quorum intersection and availability despite  $B$ .

**Definition 2.2.** Let  $\langle V, Q \rangle$  be an FBAS and  $v \in V$ .  $v$  is said to be intact if and only if there exists a DSet  $B$  containing all ill-behaved nodes and  $v \notin B$ .  $v$  is said to be befouled if and only if  $v$  is not intact.

**Theorem 2.3.** *If  $B_1$  and  $B_2$  are DSets in an FBAS  $\langle V, Q \rangle$  enjoying quorum intersection, then  $B = B_1 \cap B_2$  is a DSet, too.*

*Proof.* If  $B_1 = V$  or  $B_2 = V$ , then we are done. Suppose otherwise.

For any  $v \in V$ ,

$$\begin{aligned} v \in V \setminus B &\iff v \in V \wedge v \notin B \\ &\iff v \in V \wedge (v \notin B_1 \vee v \notin B_2) \\ &\iff (v \in V \wedge v \notin B_1) \vee (v \in V \wedge v \notin B_2) \\ &\iff (v \in (V \setminus B_1)) \vee (v \in (V \setminus B_2)) \\ &\iff v \in ((V \setminus B_1) \cup (V \setminus B_2)). \end{aligned}$$

Thus,  $V \setminus B = (V \setminus B_1) \cup (V \setminus B_2)$ . By the definition of a DSet,  $V \setminus B_1$  and  $V \setminus B_2$  are both quorums in  $\langle V, Q \rangle$ . By Theorem 1.3,  $V \setminus B$  is a quorum in  $\langle V, Q \rangle$ .

We must now show quorum intersection despite  $B$ . Let  $U_a, U_b$  be quorums in  $\langle V, Q \rangle^B$ .

- $U_a \setminus B_1$  is a quorum in  $(\langle V, Q \rangle^B)^{B_1} = \langle V, Q \rangle^{B_1}$  by Theorem 1.7.

- Similarly,  $U_b \setminus B_1$  is a quorum in  $\langle V, Q \rangle^{B_1}$ , and  $U_a \setminus B_2$  and  $U_b \setminus B_2$  are both quorums in  $\langle V, Q \rangle^{B_2}$ .

$$\begin{aligned}(U_a \setminus B_1) \cup (U_a \setminus B_2) &= U_a \setminus (B_1 \cap B_2) \\ &= U_a \setminus B \\ &= U_a\end{aligned}$$

because  $U_a$  is a quorum in  $\langle V, Q \rangle^B$ . In other words,  $(U_a \setminus B_1) \cup (U_a \setminus B_2) \neq \emptyset$ . Similarly,  $(U_b \setminus B_1) \cup (U_b \setminus B_2) \neq \emptyset$ .

Without loss of generality, assume that  $U_a \setminus B_1 \neq \emptyset$ .

- $V \setminus B_1$  is a quorum in  $\langle V, Q \rangle$  because  $B_1$  is a DSet. Similarly,  $V \setminus B_2$  is a quorum in  $\langle V, Q \rangle$ . Because  $\langle V, Q \rangle$  enjoys quorum intersection,  $(V \setminus B_1) \cap (V \setminus B_2) \neq \emptyset$ . In other words,  $(V \setminus B_2) \setminus B_1$  is a quorum. By Theorem 1.7,  $(V \setminus B_2) \setminus B_1$  is a quorum in  $\langle V, Q \rangle^{B_1}$ .
- $U_a \setminus B_1$  is a quorum in  $(\langle V, Q \rangle^B)^{B_1} = \langle V, Q \rangle^{B_1}$  for the same reason.

Because  $B_1$  is a DSet in  $\langle V, Q \rangle$ ,  $\langle V, Q \rangle^{B_1}$  enjoys quorum intersection. Therefore,  $(U_a \setminus B_1) \cap ((V \setminus B_2) \setminus B_1) \neq \emptyset$ .

$$\begin{aligned}(U_a \setminus B_1) \cap ((V \setminus B_2) \setminus B_1) &= (U_a \cap (V \setminus B_2)) \setminus B_1 \\ &\subset U_a \cap (V \setminus B_2) \\ &= (U_a \cap V) \setminus B_2 \\ &= U_a \setminus B_2.\end{aligned}$$

Thus,  $U_a \setminus B_2 \neq \emptyset$ . Using the same argument, we can show that  $U_b \setminus B_1 \neq \emptyset$  and  $U_b \setminus B_2 \neq \emptyset$ . Since  $U_a \setminus B_1$  and  $U_b \setminus B_1$  are quorums in  $\langle V, Q \rangle^{B_1}$  and  $B_1$  is a DSet,  $(U_a \setminus B_1) \cap (U_b \setminus B_1) \neq \emptyset$  by the definition of a DSet. This implies  $(U_a \cap U_b) \setminus B_1 \neq \emptyset$ . Therefore,  $U_a \cap U_b \neq \emptyset$ .  $\square$

**Theorem 2.4.** *In an FBAS with quorum intersection, the set of befouled nodes is a DSet.*

*Proof.* Let  $\langle V, Q \rangle$  be an FBAS with quorum intersection. Let  $B$  be the intersection of all DSets that contain all ill-behaved nodes. By Theorem 2.3,  $B$  is a DSet.

- Case 1:  $v \in B$ . Then there exists no DSet  $B_v$  such that  $B_v$  contains all ill-behaved nodes and  $v \notin B_v$ . Therefore,  $v$  is not an intact node. In other words,  $v$  is a befouled node.
- Case 2:  $v \notin B$ . Then there exists a DSet  $B_v$  that contains all ill-behaved nodes and  $v \notin B_v$ . In other words,  $v$  is intact and thus  $v$  is not a befouled node.

Therefore,  $B$  is precisely the set of befouled nodes and it is a DSet.  $\square$

### 3. VOTING

**Definition 3.1.** A node  $v$  votes for a statement  $a$  if and only if  $v$  asserts

- $a$  is valid,
- $a$  is consistent with all statements  $v$  has accepted,
- $v$  has never voted against  $a$ ,
- $v$  promises never to vote against  $a$  in the future.

**Definition 3.2.** A quorum  $U_a$  ratifies a statement  $a$  if and only if every member of  $U_a$  votes for  $a$ . A node  $v$  ratifies  $a$  if and only if  $v$  is a member of a quorum  $U_a$  that ratifies  $a$ .

**Theorem 3.3.** *If an FBAS enjoys quorum intersection and contains no ill-behaved node, then two contradictory statements cannot be both ratified.*

*Proof.* Suppose the statement is false and let  $a, \bar{a}$  denote two contradictory statements ratified in such an FBAS. Let  $U_a, U_{\bar{a}}$  denote quorums ratifying such statements, respectively. By the definition of quorum intersection,  $U_a \cap U_{\bar{a}} \neq \emptyset$ . Let  $v \in U_a \cap U_{\bar{a}}$ . This implies that  $v$  voted for both  $a$  and  $\bar{a}$ . However, this goes against the definition of voting. In other words,  $v$  must be ill-behaved, which is a contradiction to our assumption.  $\square$

**Theorem 3.4.** *Let  $\langle V, Q \rangle$  be an FBAS. Let  $B \subsetneq V$  be a subset containing all the ill-behaved nodes and suppose that  $\langle V, Q \rangle^B$  enjoys quorum intersection. Let  $v_1 \neq v_2 \in V \setminus B$ . If  $v_1$  ratifies a statement  $a$ , then  $v_2$  cannot ratify any statement that contradicts  $a$ .*

*Proof.* Suppose that the theorem is false and let  $U_1, U_2$  be quorums of  $v_1, v_2$  that ratify  $a, \bar{a}$ , respectively where  $a$  and  $\bar{a}$  are contradictory. Since  $v_1 \in U_1 \setminus B$ ,  $U_1 \setminus B \neq \emptyset$ . By Theorem 1.9,  $U'_1 = U_1 \setminus B$  is a quorum in  $\langle V, Q \rangle^B$ . Similarly,  $U'_2 = U_2 \setminus B$  is a quorum in  $\langle V, Q \rangle^B$ . Since  $\langle V, Q \rangle^B$  enjoys quorum intersection,  $U'_1 \cap U'_2 \neq \emptyset$ . Let  $v \in U'_1 \cap U'_2$ . Then  $v \in U_1 \cap U_2$ . In order for  $U_1, U_2$  to ratify  $a, \bar{a}$ , respectively,  $v$  must vote for both  $a$  and  $\bar{a}$ . However, this is against the definition of voting.  $v$  must be an ill-behaved node, so  $v \in B$ , which is a contradiction because  $v \in U_1 \setminus B$ .  $\square$

**Theorem 3.5.** *Let  $\langle V, Q \rangle$  be an FBAS with quorum intersection. Then two intact nodes in  $V$  cannot ratify contradictory statements.*

*Proof.* Let  $v \neq v'$  be two intact nodes in  $V$ . Let  $B \subset V$  be the set of befouled nodes. Then  $v \notin B$  and  $v' \notin B$ . Since  $\langle V, Q \rangle$  is an FBAS with quorum intersection,  $B$  is a DSet by Theorem 2.4. By the definition of a DSet(2.1),  $\langle V, Q \rangle^B$  enjoys quorum intersection. By Theorem 3.4,  $v, v'$  cannot ratify contradictory statements.  $\square$

**Definition 3.6.** Let  $\langle V, Q \rangle$  be an FBAS. Let  $v \in V$ . A subset  $B \subset V$  is called  $v$ -blocking if and only if  $\forall q \in Q(v), q \cap B \neq \emptyset$ .

**Theorem 3.7.** *Let  $\langle V, Q \rangle$  be an FBAS. Let  $B \subset V$ .  $\langle V, Q \rangle$  enjoys quorum availability despite  $B$  if and only if  $B$  is not  $v$ -blocking for any  $v \in V \setminus B$ .*

*Proof.*

$$\begin{aligned}
\forall v \in V \setminus B, \neg(B \text{ is } v\text{-blocking}) &\iff \forall v \in V \setminus B, \neg(\forall q \in Q(v), q \cap B \neq \emptyset) \\
&\iff \forall v \in V \setminus B, \exists q \in Q(v), q \cap B = \emptyset \\
&\iff \forall v \in V \setminus B, \exists q \in Q(v), q \subset V \setminus B \\
&\iff V = B \text{ or } V \setminus B \text{ is a quorum in } \langle V, Q \rangle \\
&\iff \langle V, Q \rangle \text{ enjoys quorum availability despite } B
\end{aligned}$$

$\square$