

STELLAR CONSENSUS PROTOCOL

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1. FEDERATED BYZANTINE AGREEMENT SYSTEM

1.1. Quorums.

Definition 1.1.1 (Federated Byzantine Agreement System). Let V be a set and $Q : V \rightarrow 2^V \setminus \{\emptyset\}$ be a function such that $\forall v \in V, \forall q \in Q(v), v \in q$. Then we call the pair $\langle V, Q \rangle$ a federated Byzantine agreement system, or FBAS for short. Each q in $Q(v)$ is called a quorum slice for each $v \in V$.

Remark 1.1.2.

- For each node v , $Q(v)$ is a set of subsets of V . For instance, node v_1 may trust v_2, v_3, v_4 and may have $\{v_1, v_2, v_3, v_4\} \in Q(v_1) \subset 2^V$.
- Note that we explicitly exclude \emptyset from the co-domain. In other words, we want $Q(v) \neq \emptyset$ for all $v \in V$. If $Q(v) = \emptyset$ for some $v \in V$, it satisfies $\forall q \in Q(v), v \in q$. As we will see, each $q \in Q(v)$ is the list of nodes that v trusts. If v has no list of nodes that it trusts, v cannot really do anything. Thus we want $Q(v) \neq \emptyset$ for all $v \in V$.
- Consider the case when $V = \emptyset$. Note that $2^\emptyset = \{\emptyset\}$ and thus $2^{2^\emptyset} = \{\emptyset, \{\emptyset\}\}$. Thus V forms an FBAS where Q is the map $\emptyset \mapsto \{\{\emptyset\}\}$.

Definition 1.1.3 (Quorum). Let $\langle V, Q \rangle$ be an FBAS. $U \subset V$ is called a quorum if and only if $U \neq \emptyset$ and $\forall v \in U, \exists q \in Q(v), q \subset U$.

Theorem 1.1.4. In an FBAS $\langle V, Q \rangle$, the union of two quorums is a quorum.

Proof. Let U_1, U_2 be two quorums. Let $v \in U_1 \cup U_2$. Then $v \in U_i$ for $i = 1$ or $i = 2$. Then $q \subset U_i$ for some $q \in Q(v)$. Therefore, $q \subset U_1 \cup U_2$, so $U_1 \cup U_2$ is indeed a quorum. \square

Corollary 1.1.5. The set of quorums of a given FBAS is closed under union.

Theorem 1.1.6. In an FBAS $\langle V, Q \rangle$, V is a quorum.

Proof. For any $v \in V$, for any $q \in Q(v)$, $q \subset V$. Therefore, V is indeed a quorum. \square

Example 1.1.7. One might wonder if the intersection of quorums is always a quorum. However, this is not true in general.

Let $V = \{v_1, \dots, v_4\}$ and

- $Q(v_1) = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}\},$
- \vdots
- $Q(v_4) = \{\{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}.$

In other words, $Q(v_i) = \{U \subset V \mid |U| = 3, v_i \in U\}$.

Then $U_1 = \{v_1, v_2, v_3\}$ is a quorum, and $U_2 = \{v_2, v_3, v_4\}$ is a quorum. However, $U_1 \cap U_2 = \{v_2, v_3\}$ is not a quorum because the size of any quorum slice is 3.

Definition 1.1.8 (Quorum Intersection). Let $\langle V, Q \rangle$ be an FBAS. We say $\langle V, Q \rangle$ enjoys quorum intersection if and only if $U_1 \cap U_2 \neq \emptyset$ for any pair of quorums U_1, U_2 .

Definition 1.1.9 (Delete). Let $\langle V, Q \rangle$ be an FBAS and $B \subset V$. Then the FBAS $\langle V, Q \rangle^B$ is defined to be $\langle V \setminus B, Q^B \rangle$ where $\forall v \in V \setminus B, Q^B(v) = \{q \setminus B \mid q \in Q(v)\}$.

Remark 1.1.10. One may think that this is related to fail-stop behaviors where B is the set of nodes that stopped responding. In general, however, this is not true as we also remove nodes from quorum slices. One can think of this as the *alternate* universe where nodes from B simply did not even exist from the beginning.

Theorem 1.1.11. *Definition 1.1.9 is well-defined. In other words, if $\langle V, Q \rangle$ is an FBAS and $B \subset V$, then $\langle V, Q \rangle^B$ is an FBAS.*

Proof. If $V = B$, then we are done because of Remark 1.1.2.

Suppose otherwise. Let $v \in V \setminus B$ be given. By the definition of an FBAS, $Q(v) \neq \emptyset$. Therefore, $Q^B(v) = \{q \setminus B \mid q \in Q(v)\}$ is nonempty.

Let $q' \in Q^B(v)$ be given arbitrarily. Then $q' = q \setminus B$ for some $q \in Q(v)$. Then $v \in q$ by the definition of an FBAS, so $v \in q'$.

Thus Q^B is indeed a quorum function. Therefore, $\langle V, Q \rangle^B$ is an FBAS. \square

Theorem 1.1.12. *Let U be a quorum in FBAS $\langle V, Q \rangle$, let $B \subset V$ be a set of nodes, and let $U' = U \setminus B$. If $U' \neq \emptyset$, then U' is a quorum in $\langle V, Q \rangle^B$.*

Proof. Since $U' \neq \emptyset$, it suffices to show that $\forall v \in U', \exists q \in Q^B(v), q \subset U'$. Let $v \in U'$. Then $v \in U$. Since U is a quorum in $\langle V, Q \rangle$, we can find $q \in Q(v)$ such that $q \subset U$. Then $q' = q \setminus B \in Q^B(v)$, and $q' = q \setminus B \subset U \setminus B = U'$. Therefore, U' is a quorum in $\langle V, Q \rangle^B$. \square

Remark 1.1.13. One can think of this theorem as “A quorum in the ‘original’ universe is a quorum in the ‘alternate’ universe.”

Definition 1.1.14 (Quorum Intersection Despite B). Let $\langle V, Q \rangle$ be an FBAS and $B \subset V$ be a set of nodes. We say $\langle V, Q \rangle$ enjoys quorum intersection despite B if and only if $\langle V, Q \rangle^B$ enjoys quorum intersection.

Remark 1.1.15. Quorum intersection despite B is related to the system-level safety when nodes in B act arbitrarily. For instance, suppose $\langle V, Q \rangle$ is an FBAS, B is the set of all ill-behaved nodes, and $\langle V, Q \rangle$ enjoys quorum intersection despite B . Suppose two well-behaved nodes v_1, v_2 agree with contradictory statements a_1, a_2 in quorums q_1, q_2 , respectively.

Rewrite this!

Then $q_1 \cap q_2 \neq \emptyset$ have well-behaved nodes who agreed with both a_1 and a_2 . This is a contradiction because a well-behaved node cannot contradict itself. This example illustrates how the concept of quorum intersection despite B is related to system-level safety when nodes in B experience Byzantine failures.

Definition 1.1.16 (Quorum Availability Despite B). Let $\langle V, Q \rangle$ be an FBAS and $B \subset V$ be a set of nodes. We say $\langle V, Q \rangle$ enjoys quorum availability despite B if and only if $V \setminus B$ is a quorum in $\langle V, Q \rangle$ or $B = V$.

Theorem 1.1.17. *Let $\langle V, Q \rangle$ be an FBAS and $B \subset V$. $\langle V, Q \rangle$ enjoys quorum availability despite B if and only if $\forall v \in V \setminus B$, there exists a quorum U_v such that $v \in U_v \subset (V \setminus B)$.*

Proof. If $V = B$, we are done. Suppose otherwise.

$$\begin{aligned} \forall v \in V \setminus B, \exists \text{ a quorum } U_v, v \in U_v \subset (V \setminus B) &\implies \bigcup_{v \in V \setminus B} U_v \text{ is a quorum in } \langle V, Q \rangle \\ &\implies V \setminus B \text{ is a quorum in } \langle V, Q \rangle \end{aligned}$$

by Theorem 1.1.4. On the other hand, if $V \setminus B$ is a quorum in $\langle V, Q \rangle$, then $\forall v \in V \setminus B, \exists \text{ a quorum } U_v, v \in U_v \subset (V \setminus B)$ because we can let $U_v = V \setminus B$ for each v . \square

Remark 1.1.18. Theorem 1.1.17 shows that $\langle V, Q \rangle$ enjoys quorum availability despite B if all nodes in $V \setminus B$ can find a quorum without B . This is related to the liveness of the system. If $\langle V, Q \rangle$ enjoys quorum availability despite B , then regardless of what happens to nodes in B , nodes in $V \setminus B$ can keep going.

Definition 1.1.19 (v -blocking). Let $\langle V, Q \rangle$ be an FBAS. Let $v \in V$. A subset $B \subset V$ is called v -blocking if and only if $\forall q \in Q(v), q \cap B \neq \emptyset$.

Remark 1.1.20. Intuitively, if a subset $B \subset V$ is v -blocking, then one may think of it as “ v can’t really get by without B .” The following theorem can be interpreted as “If v can’t get by without B , v can’t get by without C for any $C \supset B$.”

Theorem 1.1.21. Let $\langle V, Q \rangle$ be an FBAS. Let $v \in V$. Then

- The union of two v -blocking sets is v -blocking.
- Any superset of a v -blocking set is v -blocking.

Proof. It suffices to only prove the second statement. If $B \subset B'$ and B is v -blocking, $q \cap B' \supset q \cap B \neq \emptyset$ for any $q \in Q(v)$. \square

Theorem 1.1.22. Let $\langle V, Q \rangle$ be an FBAS. Let $A \subsetneq V$ and U_1, U_2 be a partition of $V \setminus A$. Let $v \in U_1$. If U_2 is not v -blocking in $\langle V, Q \rangle$, then U_2 is not v -blocking in $\langle V, Q \rangle^A$.

Proof. Since U_2 is not v -blocking in $\langle V, Q \rangle$, there exists $q_v \in Q(v)$ such that $q_v \cap U_2 = \emptyset$.

$$\begin{aligned} (q_v \setminus A) \cap U_2 &= (q_v \cap U_2) \setminus (A \cap U_2) \\ &= q_v \cap U_2 = \emptyset. \end{aligned}$$

Thus U_2 is not v -blocking in $\langle V, Q \rangle^A$. \square

Theorem 1.1.23. Let $\langle V, Q \rangle$ be an FBAS. Let $B \subset V$. $\langle V, Q \rangle$ enjoys quorum availability despite B if and only if B is not v -blocking for any $v \in V \setminus B$.

Proof.

$$\begin{aligned} \forall v \in V \setminus B, \neg(B \text{ is } v\text{-blocking}) &\iff \forall v \in V \setminus B, \neg(\forall q \in Q(v), q \cap B \neq \emptyset) \\ &\iff \forall v \in V \setminus B, \exists q \in Q(v), q \cap B = \emptyset \\ &\iff \forall v \in V \setminus B, \exists q \in Q(v), q \subset V \setminus B \\ &\iff V = B \text{ or } V \setminus B \text{ is a quorum in } \langle V, Q \rangle \\ &\iff \langle V, Q \rangle \text{ enjoys quorum availability despite } B \end{aligned}$$

\square

1.2. Dispensable Sets.

Definition 1.2.1 (Dispensable Set). Let $\langle V, Q \rangle$ be an FBAS and $B \subset V$ be a set of nodes. B is called a dispensable set, or DSet, if and only if $\langle V, Q \rangle$ enjoys both quorum intersection despite B and quorum availability despite B .

We will first show some basic properties of DSets.

Theorem 1.2.2. Let $\langle V, Q \rangle$ be an FBAS. Then

- V is a DSet.
- If $\forall v \in V, Q(v) = \{V\}$, then \emptyset and V are the only DSets of $\langle V, Q \rangle$.

Proof.

- $\langle V, Q \rangle^V$ enjoys quorum intersection because there is no quorum. $\langle V, Q \rangle$ enjoys quorum availability despite V because $V = V$.
- Suppose $\forall v \in V, Q(v) = \{V\}$. As shown above, V is a DSet of $\langle V, Q \rangle$. The empty set is a DSet because
 - $\langle V, Q \rangle$ enjoys quorum intersection despite \emptyset because the only quorum is V .
 - $\langle V, Q \rangle$ enjoys quorum availability despite B because $V \setminus \emptyset = V$ is a quorum.

Let $\emptyset \subsetneq S \subsetneq V$ be given. Then $V \neq S$ and $V \setminus S$ is not a quorum for it is nonempty and does not contain any quorum slice. Therefore, $\langle V, Q \rangle$ does not enjoy quorum availability despite B , so no nonempty, proper subset of V is a DSet. □

Definition 1.2.3 (Intact and Befouled). Let $\langle V, Q \rangle$ be an FBAS and $v \in V$. v is said to be intact if and only if there exists a DSet B containing all ill-behaved nodes and $v \notin B$. v is said to be befouled if and only if v is not intact.

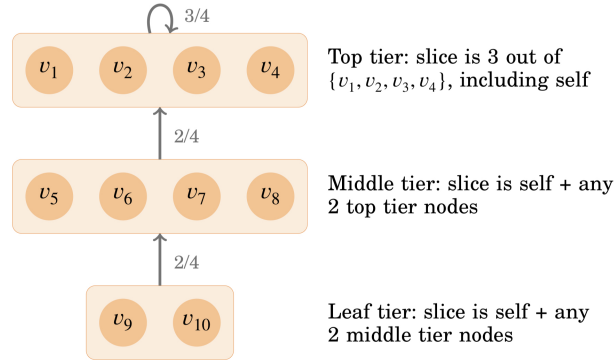


FIGURE 1. Tiered Quorum Example (P.5 of the white paper)

Example 1.2.4. We will use Figure 1 as an example.

- The smallest DSet containing v_5, v_6 in Figure 1 is $\{v_5, v_6, v_9, v_{10}\}$.
 - First, we will show that $B = \{v_5, v_6\}$ is not a DSet. By definition,

$$\begin{aligned}
 Q^B(v_9) &= \{\{v_9\}, \{v_9, v_7\}, \{v_9, v_8\}, \{v_9, v_7, v_8\}\} \\
 Q^B(v_{10}) &= \{\{v_{10}\}, \{v_{10}, v_7\}, \{v_{10}, v_8\}, \{v_{10}, v_7, v_8\}\}
 \end{aligned}$$

This implies that $U_9 = \{v_9\}$ and $U_{10} = \{v_{10}\}$ are both quorums. Then $U_9 \cap U_{10} = \emptyset$, so $\langle V, Q \rangle^B$ does not enjoy quorum intersection. Therefore, B is not a DSet. Next, we will consider $C = \{v_5, v_6, v_1\}$. Then we can use the same argument as above. $U_9 = \{v_9\} \in Q^C(v_9)$ and $U_{10} = \{v_{10}\} \in Q^C(v_{10})$, and the intersection is empty. Therefore, C is not a DSet. It is easy to see that this argument works for the case of $\{v_5, v_6, v_i\}$ for any $i = 1, 2, 3, 4$.

We will consider $D = \{v_5, v_6, v_9\}$. Similarly, $U = \{v_{10}\} \in Q^D(v_{10})$ is a quorum. Moreover, $U' = \{v_1, v_2, v_3, v_4\}$ is a quorum. Then $U \cap U' = \emptyset$, so $\langle V, Q \rangle^D$ does not enjoy quorum intersection. It is easy to see that a similar argument shows that $\{v_5, v_6, v_{10}\}$ is not a DSet.

Finally, we will show that $E = \{v_5, v_6, v_9, v_{10}\}$ is a DSet. $V \setminus E$ is a quorum in $\langle V, Q \rangle$ because every node in $V \setminus E$ has a quorum slice consisting of nodes in $V \setminus E$. If a quorum in $\langle V, Q \rangle^E$ contains v_7 or v_8 , then it must contain some of v_1, v_2, v_3, v_4 . If a quorum in $\langle V, Q \rangle^E$ contains at least one of v_1, v_2, v_3 , or v_4 , then it must contain at least three of v_1, v_2, v_3, v_4 . Therefore, any intersection of two quorums in $\langle V, Q \rangle^E$ contains at least two of v_1, v_2, v_3, v_4 by the pigeon hole principle.

Therefore, E is indeed a smallest DSet containing v_5 and v_6 .

- We showed that $B = \{v_5, v_6\}$ is not a DSet because $\langle V, Q \rangle$ does not enjoy quorum intersection despite B . What this means is that if both v_5 and v_6 either stop responding or are malicious, then it is not possible to guarantee safety for v_9 and v_{10} . For instance, consider the following situation:

- * Both v_5 and v_6 tell v_9 and v_{10} that $Q(v_5) = Q(v_6) = \{\{v_5, v_6\}\}$ convincing that $\{v_5, v_6, v_9\}$ and $\{v_5, v_6, v_{10}\}$ are both quorums.
- * Both v_5 and v_6 tell v_9 that they want to process a certain transaction. This transaction does not contradict what v_9 knows about v_5 . Moreover, everyone in the quorum $\{v_5, v_6, v_9\}$ is in favor of this transaction. Thus there is no reason for v_9 to not believe this transaction.
- * Both v_5 and v_6 tell v_{10} that they want to process a certain transaction that contradicts the transaction they told v_5 about. For the same reason, there is no reason for v_{10} to not believe this transaction.
- * Then the network processes contradicting transactions. This can let v_5 double-spend some money, for instance.
- * One can verify that this is possible by looking into the definition of accepting, confirming and such that are introduced in later chapters.

- $B = \{v_1\}$ is a DSet.

- First, we will check if $\langle V, Q \rangle$ enjoys quorum intersection despite B .

Consider $\langle V, Q \rangle^B$. Any quorum containing v_9 and/or v_{10} must contain at least two of v_5, v_6, v_7, v_8 . Any quorum containing at least one of v_5, \dots, v_8 must contain at least one of v_2, v_3, v_4 . Any quorum containing at least one of v_2, v_3, v_4 must contain at least two of v_2, v_3, v_4 . This is because $Q(v_i)^B = \{\{v_2, v_3, v_4\}, \{v_i, v_j\}, \{v_i, v_k\}\}$ where $\{i, j, k\} = \{2, 3, 4\}$.

Therefore, the intersection of any two quorums must contain at least one of v_2, v_3, v_4 by the pigeon hole principle.

Next, we need to check if $\langle V, Q \rangle$ enjoys quorum availability despite B . $V \setminus B$ is indeed a quorum in $\langle V, Q \rangle$ because each node in $V \setminus B$ has a quorum slice that does not contain v_1 .

- We showed that B is indeed a DSet. What this means is that even if v_1 stops responding or becomes malicious, the rest of the network can make progress safely. For instance, suppose that v_1 becomes malicious and tries to double-spend money. v_1 might tell v_5 that it wants to process a certain transaction. Similarly, v_1 might tell v_6 that it wants to process a contradicting transaction. However, every quorum slice of v_5 and v_6 contains at least one tier-1 node that is not v_1 . Suppose that v_5 asks v_2 what it thinks, and v_6 asks v_3 what it thinks. Then every quorum slice of v_2 and v_3 contains 3 tier-1 nodes. By the pigeon hole principle, at least one tier-1 node that is not v_1 gets asked what it thinks about the contradicting transactions from v_5 . The tier-1 node does not agree with them and v_1 's attempt to double-spend money fails.

Theorem 1.2.5. *If B_1 and B_2 are DSets in an FBAS $\langle V, Q \rangle$ enjoying quorum intersection, then $B = B_1 \cap B_2$ is a DSet, too.*

Proof. If $B_1 = V$ or $B_2 = V$, then we are done. Suppose otherwise.

First, we will show that $\langle V, Q \rangle$ enjoys quorum availability despite B . By Definition 1.1.16, it suffices to show that $V = B$ or $V \setminus B$ is a quorum in $\langle V, Q \rangle$. Since we assumed that $B_1 \neq V$ and $B_2 \neq V$, $B \neq V$. Therefore, we will show that $V \setminus B$ is a quorum in $\langle V, Q \rangle$. By basic set theory, $V \setminus B = V \setminus (B_1 \cap B_2) = (V \setminus B_1) \cup (V \setminus B_2)$. Since B_1 is a DSet, $V = B_1$ or $V \setminus B_1$ is a quorum in $\langle V, Q \rangle$. Since we assumed that $V \neq B_1$ earlier, $V \setminus B_1$ is a quorum in $\langle V, Q \rangle$. Similarly, $V \setminus B_2$ is a quorum in $\langle V, Q \rangle$. By Theorem 1.1.4, the union $(V \setminus B_1) \cup (V \setminus B_2) = V \setminus B$ is a quorum in $\langle V, Q \rangle$.

Next, we will show that $\langle V, Q \rangle$ enjoys quorum intersection despite B . Let U_a, U_b be quorums in $\langle V, Q \rangle^B$. We want to show that $U_a \cap U_b \neq \emptyset$. We will do so by proving a stronger statement, which is $(U_a \cap U_b) \setminus B_1 \neq \emptyset$. In other words, we will show that $(U_a \setminus B_1) \cap (U_b \setminus B_1) \neq \emptyset$.

Since B_1 is a DSet, $\langle V, Q \rangle$ enjoys quorum intersection despite B_1 . In other words, $\langle V, Q \rangle^{B_1}$ enjoys quorum intersection. Therefore, it suffices to show that $U_a \setminus B_1$ and $U_b \setminus B_1$ are both quorums in $\langle V, Q \rangle^{B_1}$. By Theorem 1.1.12, $U_a \setminus B_1$ and $U_b \setminus B_1$ are quorums in $(\langle V, Q \rangle^B)^{B_1}$ if $U_a \setminus B_1 \neq \emptyset$ and $U_b \setminus B_1 \neq \emptyset$. Since $(\langle V, Q \rangle^B)^{B_1} = \langle V, Q \rangle^{B_1}$, it suffices to show that $U_a \setminus B_1 \neq \emptyset$ and $U_b \setminus B_1 \neq \emptyset$.

We will first show that $U_a \setminus B_1 \neq \emptyset$. By basic set theory,

$$\begin{aligned} U_a &= U_a \setminus B \\ &= U_a \setminus (B_1 \cap B_2) \\ &= (U_a \setminus B_1) \cup (U_a \setminus B_2) \end{aligned}$$

because $U_a \cap B = \emptyset$.

This implies that $(U_a \setminus B_1) \cup (U_a \setminus B_2) \neq \emptyset$. If $U_a \setminus B_1$ is nonempty, we are done. Suppose $U_a \setminus B_2$ is nonempty. We will show that this implies that $U_a \setminus B_1 \neq \emptyset$. We will do so by first finding two quorums in $\langle V, Q \rangle^{B_2}$ whose intersection is a subset of $U_a \setminus B_1$. Since $\langle V, Q \rangle^{B_2}$ enjoys quorum intersection, the intersection of such two quorums must be nonempty, which in turn shows that $U_a \setminus B_1$ is nonempty.

- We claim that $U_a \setminus B_2$ is a quorum in $\langle V, Q \rangle^{B_2}$. U_a is a quorum in $\langle V, Q \rangle^B$. Since $U_a \setminus B_2$ is assumed to be nonempty, $U_a \setminus B_2$ is a quorum in $(\langle V, Q \rangle^B)^{B_2}$ by Theorem 1.1.12. Since $(\langle V, Q \rangle^B)^{B_2} = \langle V, Q \rangle^{B_2}$, $U_a \setminus B_2$ is a quorum in $\langle V, Q \rangle^{B_2}$.
- $V \setminus B_2 \neq \emptyset$ is a quorum in $\langle V, Q \rangle$ because $\langle V, Q \rangle$ must enjoy quorum availability despite B_2 and $B_2 \neq V$. Similarly, $V \setminus B_1 \neq \emptyset$ is a quorum in $\langle V, Q \rangle$. Because $\langle V, Q \rangle$ enjoys quorum intersection, $(V \setminus B_1) \cap (V \setminus B_2) \neq \emptyset$. In other words, $(V \setminus B_1) \setminus B_2 \neq \emptyset$. By applying Theorem 1.1.12 to the fact that $V \setminus B_1$ is a quorum in $\langle V, Q \rangle$ and $(V \setminus B_1) \setminus B_2 \neq \emptyset$, we can conclude that $(V \setminus B_1) \setminus B_2$ is a quorum in $\langle V, Q \rangle^{B_2}$.

Since $\langle V, Q \rangle^{B_2}$ enjoys quorum intersection, $(U_a \setminus B_2) \cap ((V \setminus B_1) \setminus B_2) \neq \emptyset$.

$$\begin{aligned}
\emptyset &\neq (U_a \setminus B_2) \cap ((V \setminus B_1) \setminus B_2) \\
&= (U_a \cap (V \setminus B_1)) \setminus B_2 \\
&\subset U_a \cap (V \setminus B_1) \\
&= U_a \setminus B_1.
\end{aligned}$$

Thus, $U_a \setminus B_1 \neq \emptyset$.

The same argument will show that $U_b \setminus B_1 \neq \emptyset$. □

Remark 1.2.6. Theorem 1.2.5 states that the intersection of two DSets is a DSet if the FBAS enjoys quorum intersection. One might wonder if the union of two DSets is a DSet when the FBAS enjoys quorum intersection. However, this is not true in general. Consider the FBAS $\langle V, Q \rangle$ where $V = \{v_1, v_2, v_3, v_4\}$ and $Q(v_i) = \{U \subset V \mid v_i \in U, |U| = 3\}$. This FBAS enjoys quorum intersection by the pigeon-hole principle because each quorum contains at least 3 elements. Then $B = \{v_1\}$ is a DSet because

- Quorum intersection despite B
 - Every quorum slice in $\langle V, Q \rangle^B$ contains at least 2 nodes because every quorum slice in $\langle V, Q \rangle$ contains exactly 3 nodes. This implies that any quorum in $\langle V, Q \rangle^B$ must contain at least 2 nodes. By the pigeon-hole principle, every pair of quorums in $\langle V, Q \rangle^B$ must intersect.
- Quorum availability despite B
 - $V \setminus B = \{v_2, v_3, v_4\}$ is a quorum in $\langle V, Q \rangle$ because $\{v_2, v_3, v_4\} \in Q(v_i)$ for each $i = 2, 3, 4$.

Similarly, $C = \{v_2\}$ is a DSet. However, $B \cup C = \{v_1, v_2\}$ is not a DSet because $B \cup C \neq V$ and $V \setminus (B \cup C) = \{v_3, v_4\}$ is not a quorum in $\langle V, Q \rangle$.

Theorem 1.2.7. *In an FBAS with quorum intersection, the set of befouled nodes is a DSet.*

Proof. Let $\langle V, Q \rangle$ be an FBAS with quorum intersection. Let B be the intersection of all DSets that contain all ill-behaved nodes. We will show that B is the set of befouled nodes by showing that $V \setminus B$ is the set of intact nodes.

$$\begin{aligned}
v \in V \setminus B &\iff v \notin B \\
&\iff \exists \text{ DSet } B_v \text{ that contains all ill-behaved nodes and } v \notin B_v \\
&\iff v \text{ is intact}
\end{aligned}$$

Therefore, $V \setminus B$ is precisely the set of intact nodes, and thus B is the set of befouled nodes.

By applying Theorem 1.2.5 repeatedly, we can conclude that B is a DSet. \square

Theorem 1.2.8. *Let $\langle V, Q \rangle$ be an FBAS and let $B \subset V$ be the set of befouled nodes. If B is a DSet, B is not v -blocking for any intact v .*

Proof. By Definition 1.2.3, a node $v \in V$ is intact if and only if $v \notin B$. By Theorem 1.1.23, $\langle V, Q \rangle$ enjoys quorum availability despite B if and only if B is not v -blocking for any $v \in V \setminus B$. Since B is a DSet, $\langle V, Q \rangle$ enjoys quorum availability despite B . Thus B is not v -blocking for any intact v . \square

1.3. Voting, Accepting, and Ratifying.

Definition 1.3.1 (Vote). A node v votes for a statement a if and only if v asserts

- a is valid,
- a is consistent with all statements v has accepted,
- v has never voted against a ,
- v promises never to vote for a statement that contradicts a in the future.

Definition 1.3.2 (Vote Against a). When a node v votes for a statement that contradicts a , we say v votes against a .

Definition 1.3.3 (Accept). Let $\langle V, Q \rangle$ be an FBAS, and let $v \in V$. v accepts a statement a if and only if

- It has never accepted a statement contradicting a , and
- It determines that either
 - There exists a quorum U such that $v \in U$ and each member of U either voted for a or broadcast that it has accepted a , or
 - There exists a v -blocking set B such that every member of B broadcast that it has accepted a .

When v accepts a , it must vote for the statement “an intact node accepted a .” For simplicity, we will often write “ $accept(a)$ ” to mean “an intact node accepted a .”

As you can see, the definitions of voting and accepting have a circular dependency. Note that it is possible for a node to accept a statement after voting for a contradictory statement.

Definition 1.3.4 (Ratify). A quorum U_a ratifies a statement a if and only if every member of U_a votes for a . A node v ratifies a if and only if v is a member of a quorum U_a that ratifies a .

Theorem 1.3.5. *Let $\langle V, Q \rangle$ be an FBAS. If a node $v \in V$ ratifies a statement a , then it must accept a .*

Proof. If a node v ratifies a , then it is a member of a quorum $U \subset V$ that ratifies a . Thus every member of U votes for a . This implies that v also votes for a . By Definition 1.3.1, v has never accepted a statement contradicting a . By Definition 1.3.3, v accepts a . \square

Remark 1.3.6. Theorem 1.3.5 shows that ratifying is a stronger condition than accepting.

Theorem 1.3.7. *If an FBAS enjoys quorum intersection and contains no ill-behaved node, then two contradictory statements cannot be both ratified.*

Proof. Suppose the statement is false and let a, \bar{a} denote two contradictory statements ratified in such an FBAS. Let $U_a, U_{\bar{a}}$ denote quorums ratifying such statements, respectively. By the definition of quorum intersection, $U_a \cap U_{\bar{a}} \neq \emptyset$. Let $v \in U_a \cap U_{\bar{a}}$. This implies that v voted for both a and \bar{a} . However, the definition of voting (Definition 1.3.1) explicitly prohibits that. In other words, v must be ill-behaved, which is a contradiction to our assumption. \square

Theorem 1.3.8. *Let $\langle V, Q \rangle$ be an FBAS and $B \subset V$. Suppose that B contains all the ill-behaved nodes and $\langle V, Q \rangle^B$ enjoys quorum intersection. Let $v_1 \neq v_2 \in V \setminus B$. If v_1 ratifies a statement a , then v_2 cannot ratify any statement that contradicts a .*

Proof. Suppose that the theorem is false and let U_1, U_2 be quorums of v_1, v_2 that ratify a, \bar{a} , respectively, where a and \bar{a} are contradictory. Since $v_1 \in U_1 \setminus B$, $U_1 \setminus B \neq \emptyset$. By Theorem 1.1.12, $U'_1 = U_1 \setminus B$ is a quorum in $\langle V, Q \rangle^B$. Similarly, $U'_2 = U_2 \setminus B$ is a quorum in $\langle V, Q \rangle^B$. Since $\langle V, Q \rangle^B$ enjoys quorum intersection, $U'_1 \cap U'_2 \neq \emptyset$. Choose $v \in U'_1 \cap U'_2$ arbitrarily. Then $v \in U_1 \cap U_2$. In order for U_1, U_2 to ratify a, \bar{a} , respectively, v must vote for both a and \bar{a} . However, this is against the definition of voting. v must be an ill-behaved node, so $v \in B$, which is a contradiction because $v \in U'_1 \cap U'_2 \subset U'_1 = U_1 \setminus B$ and B contains all the ill-behaved nodes. \square

Theorem 1.3.9. *Let $\langle V, Q \rangle$ be an FBAS with quorum intersection. Then two intact nodes in V cannot ratify contradictory statements.*

Proof. Let $v \neq v'$ be two intact nodes in V . Let $B \subset V$ be the set of befouled nodes. Then $v \notin B$ and $v' \notin B$. Since $\langle V, Q \rangle$ is an FBAS with quorum intersection, B is a DSet by Theorem 1.2.7. By the definition of a DSet (Definition 1.2.1), $\langle V, Q \rangle^B$ enjoys quorum intersection. By Theorem 1.3.8, v, v' cannot ratify contradictory statements. \square

Lemma 1.3.10. *Let $\langle V, Q \rangle$ be an FBAS enjoying quorum intersection and B be the set of befouled nodes. If a is accepted by an intact node in V , then a is ratified by some intact node in $\langle V, Q \rangle^B$.*

Proof. Suppose that a is accepted by one or more intact nodes in V . Since V is finite, there has to be an intact node v such that no intact nodes in V accepted a before v .

By the definition of accepting (Definition 1.3.3), v accepted a because either

- There was a quorum U of v such that every element of U either voted for a or broadcast that it has accepted a , or
- There existed a v -blocking set such that every element in it broadcast that it has accepted a .

We claim that it could not have been the second one. On the contrary, suppose that it was the second one. Since no intact nodes in V accepted a before v , such a v -blocking set must have only had befouled nodes. Therefore, such a v -blocking set must be a subset of B . Since $\langle V, Q \rangle$ enjoys quorum intersection, B is a DSet by Theorem 1.2.7. By Theorem 1.2.8, B is not v -blocking. By taking the contrapositive of Theorem 1.1.21, we conclude that no subset of B is v -blocking.

Therefore, it must have been the first case. In other words, there must have existed a quorum U of v such that, before v accepted a , every member of U either voted for a or broadcast that it has accepted a . Since no intact node accepted a before v , every intact node in U must have voted for a before v accepted a . In other words, every node in $U \setminus B$ voted for a . By Theorem 1.1.12, $U \setminus B$ is a quorum in $\langle V, Q \rangle^B$. Thus $U \setminus B$ ratified a in $\langle V, Q \rangle^B$, and thus v ratified a in $\langle V, Q \rangle^B$. Finally, v is indeed an intact node in $\langle V, Q \rangle^B$ because $\langle V, Q \rangle^B$ contains no ill-behaved nodes.

In conclusion, v is an intact node in $\langle V, Q \rangle^B$ and v ratified a in $\langle V, Q \rangle^B$. \square

Theorem 1.3.11. *Two intact nodes in an FBAS $\langle V, Q \rangle$ enjoying quorum intersection cannot accept contradictory statements.*

By Theorem 1.3.5, ratifying is a stronger condition than accepting. Therefore, Theorem 1.3.11 is a stronger version of Theorem 1.3.9.

Proof. Suppose otherwise. Let a, \bar{a} be contradictory statements accepted by two intact nodes in $\langle V, Q \rangle$. Let B denote the set of befouled nodes. By Lemma 1.3.10, a, \bar{a} are ratified by some intact nodes in $\langle V, Q \rangle^B$. By the definition of a DSet (Definition 1.2.1), $\langle V, Q \rangle$ enjoys quorum intersection despite B .

This means that $\langle V, Q \rangle^B$ enjoys quorum intersection and two contradictory statements are ratified by some intact nodes in $\langle V, Q \rangle^B$. However, this is a direct contradiction to Theorem 1.3.9. Hence, two contradictory statements cannot be accepted by two intact nodes in $\langle V, Q \rangle$. \square

1.4. Confirmation.

Definition 1.4.1 (Irrefutable). A statement a is irrefutable in an FBAS if no intact node can ever vote against it.

Definition 1.4.2 (Confirm). A quorum U_a in an FBAS confirms a statement a if and only if every element in U_a broadcasts that it has accepted a . A node confirms a if and only if it is in such a quorum.

Theorem 1.4.3. *Ratifying is stronger than confirming, and confirming is stronger than accepting.*

Proof. Let $\langle V, Q \rangle$ and $v \in V$ be given. Suppose that v ratifies a statement a . Then there exists a quorum U such that $v \in U$ and every member in U votes for a . For any $u \in U$,

- u has never accepted a statement contradicting a by the definition of voting (Definition 1.3.1), and
- U is a quorum such that $u \in U$ and every member of U voted for a .

Therefore, u accepts a . In other words, every member of U accepts a . U confirms a and thus v confirms a . Thus ratifying is stronger than confirming.

The definition of confirming (Definition 1.4.2) shows that a node must first accept a statement before confirming. Therefore, confirming is stronger than accepting. \square

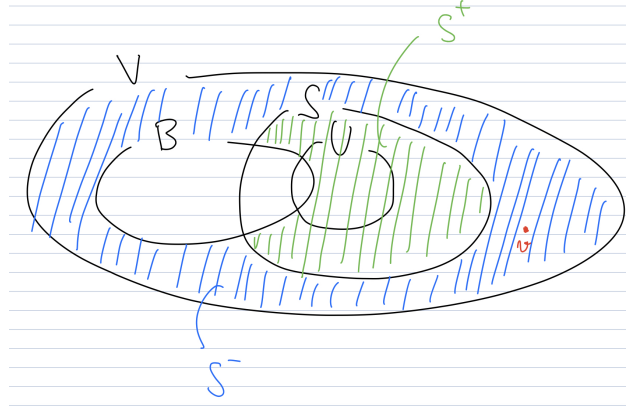


FIGURE 2. Lemma 1.4.4

Lemma 1.4.4. *Let $\langle V, Q \rangle$ be an FBAS with quorum intersection. Let B denote the set of befouled nodes. Let U be a quorum containing an intact node, and let S be a set containing U . Let S^+ be the set of intact nodes in S , and let S^- be the set of intact nodes not in S . Either $S^- = \emptyset$, or $\exists v \in S^-$ such that S^+ is v -blocking. (See Figure 2.)*

Proof.

Proofread this!

Note that $S^+ = S \setminus B$ and $S^- = (V \setminus S) \setminus B = (V \setminus B) \setminus S^+$. If $\exists v \in S^-$ such that S^+ is v -blocking, then we are done.

Suppose that $\forall v \in S^-$, S^+ is not v -blocking in $\langle V, Q \rangle$. We want to show that $S^- = \emptyset$.

For any arbitrary $v \in S^-$,

- S^+ and S^- form a partition of $V \setminus B$,
- S^+ is not v -blocking in $\langle V, Q \rangle$.

By Theorem 1.1.22, S^+ is not v -blocking in $\langle V, Q \rangle^B$ for any $v \in S^-$. Since $S^- = (V \setminus B) \setminus S^+$, S^+ is not v -blocking in $\langle V, Q \rangle^B$ for any $v \in (V \setminus B) \setminus S^+$. By Theorem 1.1.23, $\langle V, Q \rangle^B$ enjoys quorum availability despite S^+ .

By the definition of quorum availability (Definition 1.1.16), $(V \setminus B) \setminus S^+$ is a quorum in $\langle V, Q \rangle^B$, or $V \setminus B = S^+$. Suppose $(V \setminus B) \setminus S^+$ is a quorum in $\langle V, Q \rangle^B$. This leads to two contradictory claims:

- Claim 1: $\langle V, Q \rangle$ enjoys quorum intersection despite B .
 - Since $\langle V, Q \rangle$ enjoys quorum intersection, B is a DSet by Theorem 1.2.7. By the definition of a DSet (Definition 1.2.1), $\langle V, Q \rangle^B$ enjoys quorum intersection.
- Claim 2: $\langle V, Q \rangle$ does not enjoy quorum intersection despite B .
 - $U \setminus B$ is nonempty since U contains an intact node. Thus $U \setminus B$ is a quorum in $\langle V, Q \rangle^B$ by Theorem 1.1.12. We also assumed that $(V \setminus B) \setminus S^+$ is a quorum in $\langle V, Q \rangle^B$.

$$\begin{aligned}
(U \setminus B) \cap ((V \setminus B) \setminus S^+) &= (U \setminus B) \cap ((U \setminus B) \setminus S^+) \\
&= (U \setminus B) \cap S^- \\
&\subset U \cap S^- \\
&\subset S \cap S^- \\
&= \emptyset.
\end{aligned}$$

Therefore, $\langle V, Q \rangle^B$ does not enjoy quorum intersection.

Therefore, $(V \setminus B) \setminus S^+$ must not be a quorum in $\langle V, Q \rangle^B$, so $V \setminus B$ must be S^+ . In other words, S^+ contains all the intact nodes, so $S^- = \emptyset$, which is exactly what we wanted to show. \square

Theorem 1.4.5. *If an intact node in an FBAS $\langle V, Q \rangle$ with quorum intersection confirms a statement a , then every intact node will accept and confirm a once sufficient messages are delivered.*

Proof. Let B denote the set of befouled nodes. When an intact node confirms a , some quorum containing such an intact node confirms a . In other words, there exists a quorum $U \not\subset B$ such that every node in U broadcast that it accepted a . Some nodes may decide to accept a upon hearing that nodes in U broadcast that it accepted a . This may, in turn, make more nodes accept a . Thus we may experience a gradual increase in the number of nodes that accepted a over time. Since V only contains finitely many nodes, there will be a point at which the number of nodes that accepted a stops increasing. Let S be the set of nodes that accepted a at that point. We claim that S contains all intact nodes.

- U is a quorum containing an intact node.
- $U \subset S \subset V$ because every node in U accepted a in the beginning.
- Let $S^+ = S \setminus B$ be the set of intact nodes in S , and let $S^- = (V \setminus S) \setminus B$ be the set of intact nodes not in S .

By Lemma 1.4.4, S^- is empty, or S^+ is v -blocking for some $v \in S^-$. Suppose S^- is nonempty. Then v accepts a because S^+ is v -blocking and every element of S^+ broadcast that it has

accepted a . This is a contradiction because we assumed that the number of nodes that accept a stopped increasing. Therefore, S^- must be empty. If S^- is empty, then that implies that every intact node accepted and confirmed a assuming sufficient messages are delivered because

- Since S^- is empty, S^+ contains all intact nodes. $S^+ \subset S$, so S contains all intact nodes. Since every node in S accepted a , every intact node accepted a .
- Since $\langle V, Q \rangle$ enjoys quorum intersection, B is a DSet by Theorem 1.2.7. By the definition of a DSet, $V \setminus B$ is a quorum. In other words, the set of all intact nodes is a quorum.
- Since $V \setminus B$ is a quorum in which every node accepted a , every intact node confirmed a .

□

2. STELLAR CONSENSUS PROTOCOL

2.1. Nomination Protocol. Nomination is done through voting, accepting, and confirming a special type of statement in the form of *nominate x*.

Definition 2.1.1 (Nominate). A node v is said to nominate a value x if and only if it votes for the statement *nominate x*.

Definition 2.1.2 (Candidate). A node v considers a value x to be a candidate if and only if v has confirmed the statement *nominate x*. Alternatively, we say that a node v has a candidate value x .

2.2. Ballot Protocol.