# THE CALCULUS OF COMPUTATION

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## 1. Chapter 1

**Exercise (1.1).**

(a) Assume that there is a falsifying interpretation $I$.
  1. $I \not\models P \wedge Q \to P \to Q$ (assumption)
  2. $I \models P \wedge Q$ (by 1 and semantics of $\to$)
  3. $I \not\models P \to Q$ (by 1 and semantics of $\to$)
  4. $I \models Q$ (by 2 and semantics of $\wedge$)
  5. $I \not\models Q$ (by 3 and semantics of $\to$)
  6. $I \models \bot$ (4 and 5 are contradictory)
  There is only one branch and it is closed. Thus $F$ is valid.

(b) By constructing a truth table which has $2^2 = 4$ rows, it is easy to see that the interpretation $I : \{P \mapsto \mathsf{false}, Q \mapsto \mathsf{true}\}$ is a falsifying interpretation.

(c) By constructing a truth table which has $2^3 = 8$ rows, it is easy to see that the interpretation $I : \{P \mapsto \mathsf{true}, Q \mapsto \mathsf{false}, R \mapsto \mathsf{false}\}$ is a falsifying interpretation.

**Exercise (1.2).**

(a) To prove that $\top \Leftrightarrow \neg\bot$, we prove that $\top \leftrightarrow \neg\bot$ is valid. Assume that there is a falsifying interpretation $I$ such that $I \not\models \top \leftrightarrow \neg\bot$. We apply the semantics of $\leftrightarrow$.
  The first branch is:
  1a. $I \models \top \wedge \neg(\neg\bot)$
  2a. $I \models \neg(\neg\bot)$ (by 1a and semantics of $\wedge$)
  3a. $I \not\models \neg\bot$ (by 2a and semantics of $\neg$)
  4a. $I \models \bot$ (by 3a and semantics of $\neg$)
  The second branch is:
  1b. $I \models \neg\top \wedge \neg\bot$
  2b. $I \models \neg\top$ (by 1b and semantics of $\wedge$)
  3b. $I \not\models \top$ (by 2b and semantics of $\neg$)
  4b. $I \models \top$ (Under any interpretation, $\top$ has value true)
  5b. $I \models \bot$ (3b and 4b are contradictory)
  Thus both branches are closed, and thus $\top \leftrightarrow \neg\bot$ is valid.

(b) We will apply a strategy similar to that of Example 1.13. To prove $\bot \Leftrightarrow \neg\top$, we prove that $F : \bot \leftrightarrow \neg\top$ is valid. Suppose $F$ is not valid; there exists an interpretation $I$ such that $I \not\models F$. There are exactly two branches.
  The first branch is:
  1a. $I \models \bot \wedge \neg(\neg\top)$ (by semantics of $\leftrightarrow$)
  2a. $I \models \bot$ (by 1a and semantics of $\wedge$)
  The second branch is:

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1b. $I \models (\neg\bot) \wedge \neg(\neg\top)$ (by semantics of $\leftrightarrow$)
2b. $I \models \neg\top$ (by 1b and semantics of $\wedge$)
3b. $I \not\models \top$ (by 2b and semantics of $\neg$)
4b. $I \models \top$ (by definition, P.7)
5b. $I \models \bot$ (by 3b and 4b)
Both of these two branches are closed; $F$ is valid.

**Exercise (1.3).**

- $\bot$ is equivalent to $\neg\top$. In other words, $\bot \Leftrightarrow \neg\top$ and that is proved in Exercise 1.2(b).

**Exercise (1.4).** We claim that $\neg F \Leftrightarrow F \barwedge F$. To prove that, we prove $\neg F \leftrightarrow F \barwedge F$ is valid.

| $F$ | $\neg F$ | $F \barwedge F$ | $\neg F \leftrightarrow F \barwedge F$ |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |

**Exercise (1.5).**

(a) By using the list of template equivalences on P.19, we can obtain the negation normal form of the original formula as following:
  - $F : \neg(P \rightarrow Q)$.
  - $F' : \neg(\neg P \vee Q)$.
  - $F'' : \neg(\neg P) \wedge \neg Q$.
  - $F''' : P \wedge \neg Q$.

  The only connectives in $F'''$ are $\neg, \wedge$, and $\vee$ and the negations appear only in literals. Thus $F'''$ is in NNF. Furthermore, $F'''$ is actually in CNF and DNF since it is the disjunction of one conjunction, and it is the conjunction of two clauses.

(b) $F' : (\neg P \vee \neg Q) \wedge R$ is the NNF of $F$ that can be obtained using the same strategy as above. $F'$ is already in CNF. $F'' : (\neg P \wedge R) \vee (\neg Q \wedge R)$ is an equivalent formula in DNF.

**Exercise (1.6).**

(a) $\bigwedge\limits_{v\in V} \bigvee\limits_{c\in C} P_v^c$.

(b) $\bigwedge\limits_{(v,w)\in E} \bigvee\limits_{c_1 \neq c_2 \in C} (\neg P_v^{c_1} \vee \neg P_w^{c_2})$.

(c) $\bigwedge\limits_{(v,w)\in E} \bigvee\limits_{c\in C} (\neg P_v^c \vee \neg P_w^c)$.

(d) No clue. The problem statement seems too ambiguous.

(e) They are already in CNF. We have $N \cdot M$ variables and $N \cdot M + K \cdot M \cdot (M-1) + K \cdot M$ clauses in the encoding above.

**Exercise (1.7).**

(a) $P_{(F)}$ contains 1 term. Each $\mathsf{En}(G)$ contains $1, 1, 2 \cdot 2 \cdot 3, 1, 1, 2 \cdot 2 \cdot 3, 3 \cdot 2 \cdot 2$ clauses. Thus the expansion would contain 1728 clauses, which is the product of the numbers of clauses.

(b)
  (i) $2^n$ clauses.

(ii) $\mathsf{Rep}(F_n) = P_{F_n}$, so it is just one clause. We have the subformula set $S_{F_n} = \{Q_1, \cdots, Q_n\} \cup \{R_1, \cdots, R_n\} \cup \{Q_i \wedge R_i \mid 1 \le i \le n\}$. We will consider how many clauses $\mathsf{En}(G)$ has for each $G \in S_{F_n}$.

- For each $Q_i$ and $R_i$, we have $\mathsf{En}(Q_i) = \mathsf{En}(R_i) = \top$. This adds $2n$ clauses to $F'$.
- $\mathsf{En}(Q_i \wedge R_i)$ contains 3 clauses as defined on P.25. This adds $3n$ clauses to $F'$.

Therefore, in total, $F'$ contains $1 + 2n + 3n = 5n + 1$ clauses. Note that $5n + 1 < 30n + 2$ where $30n + 2$ is the upper bound described on P.26.

I am not sure if a subformula can be bigger. In other words, for instance, should $(Q_1 \wedge R_1) \vee (Q_2 \wedge R_2)$ be a subformula of $F_3$? It seems unnecessary to include such cases for the purpose of CNF conversion, but it seems to be more consistent with the definition of a subformula.

(iii) When $n \le 4$, $2^n$ is smaller than $5n + 1$. For $n \ge 5$, $2^n$ is bigger than $5n + 1$.

**Exercise (1.8).**

(a) We will follow the format described in Example 1.30. Branching on $Q$ or $R$ will result in unit clauses; choose $Q$. Then $F\{Q \mapsto \top\} : (P \vee \neg R) \wedge (R)$. $P$ appear only positively, so we consider $F\{Q \mapsto \top, P \mapsto \top\} : R$. Then $R$ appears only positively, so the formula is satisfiable. In particular, $F$ is satisfied by interpretation

$$I : \{P \mapsto \mathsf{true}, Q \mapsto \mathsf{true}, R \mapsto \mathsf{true}\}.$$