THE CALCULUS OF COMPUTATION

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1. Chapter 1

Exercise (1.1).

(a) Assume that there is a falsifying interpretation I.
  1. $I \not\models P \wedge Q \to P \to Q$ (assumption)
  2. $I \models P \wedge Q$ (by 1 and semantics of $\to$)
  3. $I \not\models P \to Q$ (by 1 and semantics of $\to$)
  4. $I \models Q$ (by 2 and semantics of $\wedge$)
  5. $I \not\models Q$ (by 3 and semantics of $\to$)
  6. $I \models \bot$ (4 and 5 are contradictory)

  There is only one branch and it is closed. P.13 states that it is a proof of the validity of F if every branch is closed. Thus F is valid.

(b) By the relative precedence of the logical connectives mentioned on P.5, the given formula is the same as

$$F : (P \to Q) \vee (P \wedge \neg Q).$$

  Suppose that there is a falsifying interpretation I. We will use the semantics mentioned on P.10 and P.11.
  1. $I \not\models (P \to Q) \vee (P \wedge \neg Q)$ (assumption)
  2. $I \not\models P \to Q$ (by 1 and semantics of $\vee$)
  3. $I \not\models P \wedge \neg Q$ (by 1 and semantics of $\vee$)
  4. $I \models P$ (by 2 and semantics of $\to$)
  5. $I \not\models Q$ (by 2 and semantics of $\to$)

  By 3 and the semantics of $\wedge$, we obtain two branches.

  The first branch is:

  1a. $I \not\models P$ (by 3 and semantics of $\wedge$)
  2a. $I \models \bot$ (by 4 and 1a)

  The second branch is:

  1b. $I \not\models \neg Q$ (by 3 and semantics of $\wedge$)
  2b. $I \models Q$ (by 1b and $\neg$)
  3b. $I \models \bot$ (by 5 and 2b)

  Every branch is closed, so this is a proof of validity of F.

(c) We claim that the interpretation $I : \{P \mapsto \mathsf{true}, Q \mapsto \mathsf{false}, R \mapsto \mathsf{false}\}$ is a falsifying interpretation. We will use the inductive definition on P.7 and P.8.

$$
\begin{aligned}
& I \not\models (P \to Q \to R) \to P \to R \\
\text{iff} \quad & I \models P \to Q \to R \text{ and } I \not\models P \to R \\
\text{iff} \quad & I \models P \to Q \to R \text{ and } I \models P \text{ and } I \not\models R \\
\text{iff} \quad & I \models P \to Q \to R \text{ and } I[P] = \mathsf{true} \text{ and } I[R] = \mathsf{false} \\
\text{iff} \quad & \text{if } I \models P \text{, then } I \models Q \to R \\
\text{iff} \quad & \text{if } I[P] = \mathsf{true}\text{, then } I \models Q \to R \\
\text{iff} \quad & I \models Q \to R \\
\text{iff} \quad & \text{if } I \models Q \text{, then } I \models R \\
\text{iff} \quad & \text{if } I[Q] = \mathsf{true}\text{, then } I \models R \\
\text{iff} \quad & I \models R \\
\text{iff} \quad & I[R] = \mathsf{true}.
\end{aligned}
$$

Therefore, I is indeed a falsifying interpretation of the given formula.

Exercise (1.2).

(a) To prove that $\top \Leftrightarrow \neg\bot$, we prove that $\top \leftrightarrow \neg\bot$ is valid. Assume that there is a falsifying interpretation I such that $I \not\models \top \leftrightarrow \neg\bot$. We apply the semantics of $\leftrightarrow$.

The first branch is:

1a. $I \models \top \wedge \neg(\neg\bot)$
2a. $I \models \neg(\neg\bot)$ (by 1a and semantics of $\wedge$)
3a. $I \not\models \neg\bot$ (by 2a and semantics of $\neg$)
4a. $I \models \bot$ (by 3a and semantics of $\neg$)

The second branch is:

1b. $I \models \neg\top \wedge \neg\bot$
2b. $I \models \neg\top$ (by 1b and semantics of $\wedge$)
3b. $I \not\models \top$ (by 2b and semantics of $\neg$)
4b. $I \models \top$ (Under any interpretation, $\top$ has value true)
5b. $I \models \bot$ (3b and 4b are contradictory)

Thus both branches are closed, and thus $\top \leftrightarrow \neg\bot$ is valid.

(b) We will apply a strategy similar to that of Example 1.13. To prove $\bot \Leftrightarrow \neg\top$, we prove that $F : \bot \leftrightarrow \neg\top$ is valid. Suppose F is not valid; there exists an interpretation I such that $I \not\models F$. There are exactly two branches.

The first branch is:

1a. $I \models \bot \wedge \neg(\neg\top)$ (by semantics of $\leftrightarrow$)
2a. $I \models \bot$ (by 1a and semantics of $\wedge$)

The second branch is:

1b. $I \models (\neg\bot) \wedge \neg(\neg\top)$ (by semantics of $\leftrightarrow$)
2b. $I \models \neg\top$ (by 1b and semantics of $\wedge$)
3b. $I \not\models \top$ (by 2b and semantics of $\neg$)
4b. $I \models \top$ (by definition, P.7)
5b. $I \models \bot$ (by 3b and 4b)

Both of these two branches are closed; F is valid.

Exercise (1.3).

- $\bot$ is equivalent to $\neg\top$. In other words, $\bot \Leftrightarrow \neg\top$ and that is proved in Exercise 1.2(b).

Exercise (1.4).

- We claim that $\neg F \Leftrightarrow F \barwedge F$. To prove that, we prove $\neg F \leftrightarrow F \barwedge F$ is valid.

F	$\neg F$	$F \barwedge F$	$\neg F \leftrightarrow F \barwedge F$
0	1	1	1
1	0	0	1

- $F_1 \vee F_2 \Leftrightarrow (F_1 \barwedge F_1) \barwedge (F_2 \barwedge F_2)$. This can be shown easily using the truth table with $2^2 = 4$ rows.

Exercise (1.5).

(a) By using the list of template equivalences on P.19, we can obtain the negation normal form of the original formula as following:
  - $F : \neg(P \to Q)$.
  - $F' : \neg(\neg P \vee Q)$.
  - $F'' : \neg(\neg P) \wedge \neg Q$.
  - $F''' : P \wedge \neg Q$.

  The only connectives in F''' are $\neg$, $\wedge$, and $\vee$ and the negations appear only in literals. Thus F''' is in NNF. Furthermore, F''' is actually in CNF and DNF since it is the disjunction of one conjunction, and it is the conjunction of two clauses.

(b) $F' : (\neg P \vee \neg Q) \wedge R$ is the NNF of F that can be obtained using the same strategy as above. F' is already in CNF. $F'' : (\neg P \wedge R) \vee (\neg Q \wedge R)$ is an equivalent formula in DNF.

(c) $F : (Q \wedge R \to (P \vee \neg Q)) \wedge (P \vee R)$.
  - $F_1 : (\neg(Q \wedge R) \vee (P \vee \neg Q)) \wedge (P \vee R)$.
  - $F_2 : (\neg Q \vee \neg R \vee P \vee \neg Q) \wedge (P \vee R)$.
  - $F_3 : (\neg Q \wedge (P \vee R)) \vee (\neg R \wedge (P \vee R)) \vee (P \wedge (P \vee R)) \vee (\neg Q \wedge (P \vee R))$.
  - $F_4 : (\neg Q \wedge P) \vee (\neg Q \wedge R) \vee (\neg R \wedge P) \vee (\neg R \wedge R) \vee (P \wedge P) \vee (P \wedge R) \vee (\neg Q \wedge P) \vee (\neg Q \wedge R)$.

  F_2 is in NNF and CNF, and F_4 is in DNF.

Exercise (1.6).

(a) $\bigwedge_{v \in V} \bigvee_{c \in C} P_v^c$.

(b) $\bigwedge_{(v,w) \in E} \bigvee_{c_1 \neq c_2 \in C} (\neg P_v^{c_1} \vee \neg P_w^{c_2})$.

(c) $\bigwedge_{(v,w) \in E} \bigvee_{c \in C} (\neg P_v^c \vee \neg P_w^c)$.

(d) No clue. The problem statement seems too ambiguous.

(e) They are already in CNF. We have $N \cdot M$ variables and $N \cdot M + K \cdot M \cdot (M-1) + K \cdot M$ clauses in the encoding above.

Exercise (1.7).

(a) $P_{(F)}$ contains 1 term. Each $\mathsf{En}(G)$ contains $1, 1, 2 \cdot 2 \cdot 3, 1, 1, 2 \cdot 2 \cdot 3, 3 \cdot 2 \cdot 2$ clauses. Thus the expansion would contain 1728 clauses, which is the product of the numbers of clauses.

(b)
 (i) 2^n clauses.
 (ii) $\mathsf{Rep}(F_n) = P_{F_n}$, so it is just one clause. We have the subformula set $S_{F_n} = \{Q_1, \cdots, Q_n\} \cup \{R_1, \cdots, R_n\} \cup \{Q_i \wedge R_i \mid 1 \leq i \leq n\}$. We will consider how many clauses $\mathsf{En}(G)$ has for each $G \in S_{F_n}$.
  - For each Q_i and R_i, we have $\mathsf{En}(Q_i) = \mathsf{En}(R_i) = \top$. This adds $2n$ clauses to F'.
  - $\mathsf{En}(Q_i \wedge R_i)$ contains 3 clauses as defined on P.25. This adds $3n$ clauses to F'.

 Therefore, in total, F' contains $1 + 2n + 3n = 5n + 1$ clauses. Note that $5n+1 < 30n + 2$ where $30n + 2$ is the upper bound described on P.26.
 I am not sure if a subformula can be bigger. In other words, for instance, should $(Q_1 \wedge R_1) \vee (Q_2 \wedge R_2)$ be a subformula of F_3? It seems unnecessary to include such cases for the purpose of CNF conversion, but it seems to be more consistent with the definition of a subformula.
 This solution seems wrong, and I should update this.
 (iii) When $n \leq 4$, 2^n is smaller than $5n + 1$. For $n \geq 5$, 2^n is bigger than $5n + 1$.

Exercise (1.8).

(a) We will follow the format described in Example 1.30. Branching on Q or R will result in unit clauses; choose Q. Then $F\{Q \mapsto \top\} : (P \vee \neg R) \wedge (R)$. P appear only positively, so we consider $F\{Q \mapsto \top, P \mapsto \top\} : R$. Then R appears only positively, so the formula is satisfiable. In particular, F is satisfied by interpretation

$$I : \{P \mapsto \mathsf{true}, Q \mapsto \mathsf{true}, R \mapsto \mathsf{true}\}.$$

(b) Branching on Q or R will result in unit clauses. Choose Q.

$$F\{Q \mapsto \top\} : (\neg P \vee \neg R) \wedge (R).$$

P appears only negatively.

$$F\{P \mapsto \bot, Q \mapsto \top\} : R.$$

R appears only positively. Thus the interpretation $I : \{P \mapsto \mathsf{false}, Q \mapsto \mathsf{true}, R \mapsto \mathsf{true}\}$ satisfies F.

2. Chapter 2

Exercise (2.1).

(a) $\exists x, y.\, \mathrm{day}(x) \wedge \mathrm{day}(y) \wedge \mathrm{length}(x) < \mathrm{length}(y)$.
(b) $\exists x.\, \mathrm{place}(x) \wedge \mathrm{home}(x) \wedge (\forall y.\, \mathrm{place}(y) \wedge \mathrm{home}(y) \to x = y)$.
(c) $\forall x, y.\, \mathrm{mother}(\mathrm{me}, x) \wedge \mathrm{mother}(x, y) \to \mathrm{grandmother}(\mathrm{me}, y)$.
(d) $\forall x, y, z.\, \mathrm{convex}(x) \wedge \mathrm{convex}(y) \wedge \mathrm{intersect}(x, y, z) \to \mathrm{convex}(z)$.

Exercise (2.2).

(a) This exercise is similar to Example 2.13. To show that the given formula is invalid, we find an interpretation I such that

$$I \models \neg((\forall x, y.\, p(x,y) \rightarrow p(y,x)) \rightarrow \forall z. p(z,z)).$$

We will first find the NNF as following:

- $\neg((\forall x, y.\, p(x,y) \rightarrow p(y,x)) \rightarrow \forall z. p(z,z))$.
- $\neg(\neg(\forall x, y. p(x,y) \rightarrow p(y,x)) \vee \forall z. p(z,z))$.
- $(\forall x, y. p(x,y) \rightarrow p(y,x)) \wedge \neg\forall z. p(z,z)$.
- $(\forall x, y. \neg p(x,y) \vee p(y,x)) \wedge \exists z. \neg p(z,z)$.

Using the inductive steps described on P.40 and P.41,

$$\begin{aligned}
&I \models \neg((\forall x, y.\, p(x,y) \rightarrow p(y,x)) \rightarrow \forall z.\, p(z,z)) \\
&\quad \text{iff } I \models (\forall x, y.\, \neg p(x,y) \vee p(y,x)) \wedge \exists z.\, \neg p(z,z) \\
&\quad \text{iff } I \models \forall x, y.\, \neg p(x,y) \vee p(y,x) \text{ and } I \models \exists z.\, \neg p(z,z) \\
&\quad \text{iff } I \triangleleft \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models \neg p(x,y) \vee p(y,x) \text{ and } I \triangleleft \{z \mapsto \mathsf{u}\} \models \neg p(z,z) \\
&\quad \text{iff } \big[I \triangleleft \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \not\models p(x,y) \text{ or } I \triangleleft \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models p(y,x)\big] \\
&\qquad\quad \text{and } I \triangleleft \{z \mapsto \mathsf{u}\} \models \neg p(z,z).
\end{aligned}$$

where each line with v, w should be followed by "for all v, w in D_I and for some u in D_I." Choose $D_I = \{0, 1\}$ and $p_I = \{(0,1), (1,0)\}$, then it is easy to see that the last line is true. In other words, I is indeed a falsifying interpretation, and thus the given formula is invalid.

(b) To show that the given formula is invalid, we need to find an interpretation I such that

$$I \models \neg\forall x, y.\, p(x,y) \rightarrow p(y,x) \rightarrow \forall z.\, p(z,z).$$

We will find the NNF as following:

- $\neg\forall x, y.\, p(x,y) \rightarrow p(y,x) \rightarrow \forall z.\, p(z,z)$.
- $\neg\forall x, y.\, p(x,y) \rightarrow (p(y,x) \rightarrow \forall z.\, p(z,z))$.
- $\neg\forall x, y.\, p(x,y) \rightarrow (\neg p(y,x) \vee \forall z.\, p(z,z))$.
- $\neg\forall x, y.\, \neg p(x,y) \vee (\neg p(y,x) \vee \forall z.\, p(z,z))$.
- $\exists x, y.\, p(x,y) \wedge \neg(\neg p(y,x) \vee \forall z.\, p(z,z))$.
- $\exists x, y.\, p(x,y) \wedge p(y,x) \wedge \neg\forall z.\, p(z,z)$.
- $\exists x, y.\, p(x,y) \wedge p(y,x) \wedge \exists z.\, \neg p(z,z)$.

Using the inductive steps described on P.40 and P.41,

$$
\begin{aligned}
& I \models \neg\forall x, y.\, p(x, y) \to p(y, x) \to \forall z.\, p(z, z) \\
\text{iff} \quad & I \models \exists x, y.\, p(x, y) \wedge p(y, x) \wedge \exists z.\, \neg p(z, z) \\
\text{iff} \quad & I \lhd \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models p(x, y) \wedge p(y, x) \wedge \exists z.\, \neg p(z, z) \\
& \text{for some } \mathsf{v},\, \mathsf{w} \text{ in } D_I \\
\text{iff} \quad & I \lhd \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models p(x, y) \text{ and} \\
& I \lhd \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models p(y, x) \text{ and} \\
& I \lhd \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models \exists z.\, \neg p(z, z) \\
& \text{for some } \mathsf{v},\, \mathsf{w} \text{ in } D_I \\
\text{iff} \quad & I \lhd \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models p(x, y) \text{ and} \\
& I \lhd \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models p(y, x) \text{ and} \\
& I \lhd \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}, z \mapsto \mathsf{u}\} \not\models p(z, z) \\
& \text{for some } \mathsf{v},\, \mathsf{w},\, \mathsf{u} \text{ in } D_I
\end{aligned}
$$

Choose $D_I = \{0, 1\}$ and $p_I = \{(0, 1), (1, 0)\}$, then the last line is clearly true for we can set $\mathsf{v} = 0, \mathsf{w} = 1, \mathsf{u} = 0$. Therefore, I is a falsifying interpretation, and thus the original formula is invalid.

(c) To show that the given formula is invalid, we need to find an interpretation I such that

$$I \models \neg((\exists x.\, p(x)) \to (\forall y.\, p(y))).$$

It suffices to show that the negation normal form is satisfied by I.

- $\neg((\exists x.\, p(x)) \to (\forall y.\, p(y)))$.
- $\neg(\neg(\exists x.\, p(x)) \vee (\forall y.\, p(y)))$.
- $(\exists x.\, p(x)) \wedge \neg(\forall y.\, p(y))$.
- $(\exists x.\, p(x)) \wedge (\exists y.\, \neg p(y))$.

Then we have

$$
\begin{aligned}
& I \models \neg((\exists x.\, p(x)) \to (\forall y.\, p(y))) \\
& \text{iff } I \models (\exists x. p(x)) \wedge (\exists y. \neg p(y)) && \text{(NNF)} \\
& \text{iff } I \models (\exists x. p(x)) \text{ and } (\exists y. \neg p(y)) && \text{(P.40)} \\
& \text{iff } I \models (\exists x. p(x)) \text{ and } (\exists y. \neg p(y)) && \text{(P.40)} \\
& \text{iff } I \lhd \{x \mapsto \mathsf{v}\} \models p(x) \text{ and } I \lhd \{y \mapsto \mathsf{w}\} \models \neg p(y) \text{ for some } \mathsf{v},\, \mathsf{w} && \text{(P.41)}
\end{aligned}
$$

Let $D_I = \{0, 1\}$ and $p_I = \{0\}$. Then the last statement is true since we can set $\mathsf{v} = 0$ and $\mathsf{w} = 1$. In other words, such I is a falsifying interpretation.

(d) To prove the validity of $F : (\forall x.\, p(x)) \to (\exists y.\, p(y))$, we will assume that it is not. Then there must exist a falsifying interpretation.

1. $I \not\models (\forall x.\, p(x)) \to (\exists y.\, p(y))$.
2. $I \models \forall x.\, p(x)$ by 1 and semantics of $\to$ on P.10.
3. $I \not\models \exists y.\, p(y)$ by 1 and semantics of $\to$ on P.10.
4. $I \lhd \{x \mapsto \mathsf{v}\} \models p(x)$ by 2 and semantics of $\forall$ on P.42 for any $\mathsf{v} \in D_I$.

5. $I \lhd \{y \mapsto \mathsf{v}\} \not\models p(y)$ by 3 and semantics of $\exists$ on P.42 for the same v as above.
6. $I \models \bot$, contradiction as shown on P.43.

Since every branch of a semantic argument proof of $I \not\models F$ closes, F is valid by Theorem 2.30

(e)

The following solution is wrong. The formula is actually valid because If $p(x, y)$ is false for some x, y, then those x, y can be chosen. If p is always true, we are done.

In order to show that $\exists x, y.\, (p(x, y) \to (p(y, x) \to \forall z.\, p(z, z)))$ is invalid, we need to find an interpretation I such that

$$I \not\models \exists x, y.\, (p(x, y) \to (p(y, x) \to \forall z.\, p(z, z))).$$

We claim that $D_I = \{0, 1\}$ and $p_I = \{(0, 1), (1, 0)\}$ define such an interpretation. By the semantics of $\exists$ on P.42, the above expression is equivalent to $I \lhd \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models p(x, y) \to (p(y, x) \to \forall z.\, p(z, z))$ for any v, w in D_I. This is true if and only if $J \models p(x, y)$ and $J \not\models p(y, x) \to \forall z.\, p(z, z)$ where J is the interpretation $I \lhd \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\}$. $J \not\models p(y, x) \to \forall z.\, p(z, z)$ is true if and only if $J \models p(y, x)$ and $J \not\models \forall z.\, p(z, z)$. Finally, $J \not\models \forall z.\, p(z, z)$ if and only if $J \lhd \{z \mapsto \mathsf{u}\} \not\models p(z, z)$ for a fresh u in D_I. It is easy to see that each expression evaluates to true for v, w and u. Therefore, I is a falsifying interpretation.

Exercise (2.3).

(a) This is similar to Example 2.21. To show $\neg(\forall x.\, F) \Leftrightarrow \exists x.\, \neg F$, we will show the validity of $\neg(\forall x.\, F) \leftrightarrow \exists x.\, \neg F$. Suppose that it is not valid. Then there exists an interpretation I such that $I \not\models \neg(\forall x.\, F) \leftrightarrow \exists x.\, \neg F$. By the semantiocs of $\leftrightarrow$, there are two branches. The semantics of $\wedge, \neg$ appear on P.10, and that of $\forall, \exists$ appear on P.42.

The first branch is:

1a. $I \models \neg(\forall x.\, F) \wedge \neg(\exists x.\, \neg F)$.
2a. $I \models \neg(\forall x.\, F)$ by 1a and semantics of $\wedge$.
3a. $I \models \neg(\exists x.\, \neg F)$ by 1a and semantics of $\wedge$.
4a. $I \not\models \forall x.\, F$ by 2a and semantics of $\neg$.
5a. $I \lhd \{x \mapsto \mathsf{v}\} \not\models F$ by 4a and semantics of $\forall$ for some fresh v.
6a. $I \not\models \exists x.\, \neg F$ by 3a and semantics of $\neg$.
7a. $I \lhd \{x \mapsto \mathsf{v}\} \not\models \neg F$ by 6a and semantics of $\exists$ for the same v.
8a. $I \lhd \{x \mapsto \mathsf{v}\} \models F$ by 7a and semantics of $\neg$.
9a. $I \models \bot$ by 5a and 8a.

The second branch is:

1b. $I \models \neg(\neg(\forall x.\, F)) \wedge \exists x.\, \neg F$.
2b. $I \models \exists x.\, \neg F$ by 1b and semantics of $\wedge$.
3b. $I \lhd \{x \mapsto \mathsf{v}\} \models \neg F$ by 2b and semantics of $\exists$ for some fresh $\mathsf{v} \in D_I$.
4b. $I \lhd \{x \mapsto \mathsf{v}\} \not\models F$ by 3b and semantics of $\neg$.
5b. $I \models \neg(\neg(\forall x.\, F))$ by 1b and semantics of $\wedge$.
6b. $I \not\models \neg(\forall x.\, F)$ by 5b and semantics of $\neg$.
7b. $I \models \forall x.\, F$ by 6b and semantics of $\neg$.
8b. $I \lhd \{x \mapsto \mathsf{v}\} \models F$ by 7b and semantics of $\forall$ for the same v.

9b. $I \models \bot$ by 4b and 8b.

Every branch of a semantic argument proof of $I \not\models F$ closes, so F is valid by Theorem 2.30.

(b) To show $\neg(\exists x.\, F) \Leftrightarrow \forall x.\, \neg F$, it suffices to show the validity of $\neg(\exists x.\, F) \leftrightarrow \forall x.\, \neg F$. Suppose otherwise. Then there exists an interpretation I such that $I \not\models \neg(\exists x.\, F) \leftrightarrow (\forall x.\, \neg F)$. We will use the semantics of $\wedge, \neg, \leftrightarrow$ on P.10 and 11, and that of $\forall, \exists$ on P.42.

The first branch is:

1a. $I \models \neg(\exists x.\, F) \wedge \neg(\forall x.\, \neg F)$.
2a. $I \models \neg(\exists x.\, F)$ by 1a and $\wedge$.
3a. $I \models \neg(\forall x.\, \neg F)$ by 1a and $\wedge$.
4a. $I \not\models \forall x.\, \neg F$ by 3a and $\neg$.
5a. $I \lhd \{x \mapsto \mathsf{v}\} \not\models \neg F$ by 4a and $\forall$ for a fresh $\mathsf{v} \in D_I$.
6a. $I \lhd \{x \mapsto \mathsf{v}\} \models F$ by 5a and $\neg$.
7a. $I \not\models \exists x.\, F$ by 2a and $\neg$.
8a. $I \lhd \{x \mapsto \mathsf{v}\} \not\models F$ by 7a and $\exists$ for the same v.
9a. $I \models \bot$ by 6a and 8a.

The second branch is:

1b. $I \models \neg(\neg(\exists x. F)) \wedge \forall x. \neg F$.
2b. $I \models \neg(\neg(\exists x. F))$ by 1b and $\wedge$.
3b. $I \models \forall x. \neg F$ by 1b and $\wedge$.
4b. $I \models \exists x. F$ by applying the semantics of $\neg$ twice to 2b.
5b. $I \lhd \{x \mapsto \mathsf{v}\} \models F$ by 4b and $\exists$ for a fresh v.
6b. $I \lhd \{x \mapsto \mathsf{v}\} \models \neg F$ by 3b and $\forall$ for the same v.
7b. $I \lhd \{x \mapsto \mathsf{v}\} \not\models F$ by 6b and $\neg$.
8b. $I \models \bot$ by 5b and 7b.

Exercise (2.4).

(a)
- $(\forall x.\, \exists y.\, p(x, y)) \to \forall x.\, p(x, x)$.
- $\neg(\forall x.\, \exists y.\, p(x, y)) \vee \forall x.\, p(x, x)$.
- $(\exists x.\, \neg(\exists y.\, p(x, y))) \vee \forall x.\, p(x, x)$.
- $(\exists x.\, \forall y.\, \neg p(x, y)) \vee \forall x.\, p(x, x)$.
- $(\exists x.\, \forall y.\, \neg p(x, y)) \vee \forall w.\, p(w, w)$.
- $\neg p(x, y)) \vee p(w, w)$.
- $\exists x.\, \forall y.\, \forall w.\, \neg p(x, y)) \vee p(w, w)$.

(b)
- $\exists z.\, (\forall x.\, \exists y.\, p(x, y)) \to \forall x.\, p(x, z)$.
- $\exists z.\, \neg(\forall x.\, \exists y.\, p(x, y)) \vee \forall x.\, p(x, z)$.
- $\exists z.\, (\exists x.\, \forall y.\, \neg p(x, y)) \vee \forall x.\, p(x, z)$.
- $\exists z.\, (\exists x.\, \forall y.\, \neg p(x, y)) \vee \forall w.\, p(w, z)$.
- $\neg p(x, y) \vee p(w, z)$.
- $\exists z.\, \exists x.\, \forall y.\, \forall w.\, \neg p(x, y) \vee p(w, z)$.

3. Chapter 3

Exercise (3.1).

(a) First, as mentioned on P.42, it technically does not make sense to discuss validity of an open formula. The convention is to take the universal closure of the formula.

Thus we will prove that the formula $F : \forall x, y.\, f(x,y) = f(y,x) \rightarrow f(a,y) = f(y,a)$ is invalid. We will do so by finding a falsifying T_{E}-interpretation. In other words, we need to show that

- We have an interpretation I which satisfies all the axioms of T_{E}, and
- $I \not\models F$.

Let an interpretation $I : \{D_I, \alpha_I\}$ be defined such that

- $D_I = \{\circ, \bullet\}$ and the equaltiy among $\circ$ and $\bullet$ is defined in the most trivial way. (e.g., $\circ = \circ$ and $\circ \neq \bullet$.)
- $\alpha_I[a] = \circ$.
- $\alpha_I[f]$ is the left projection map. In other words,
  - $\alpha_I[f](\circ, \circ) = \circ$,
  - $\alpha_I[f](\circ, \bullet) = \circ$,
  - $\alpha_I[f](\bullet, \circ) = \bullet$,
  - $\alpha_I[f](\bullet, \bullet) = \bullet$.

Then I is a T_{E}-interpetation for it satisfies all the axioms of T_{E} on P.71. For instance, I satisfies reflexivity since

$$
\begin{aligned}
& I \models \forall x.\, x = x \\
\text{iff} \quad & I \triangleleft \{x \mapsto \mathsf{v}\} \models x = x \text{ for any } \mathsf{v} \in D_I \text{ (P.41)}
\end{aligned}
$$

which we can easily see as true by examining the two cases where v is $\circ$ and $\bullet$. It is easy to show that I satisfies the other four axioms (symmetry, transitivity, function congruence, predicate congruence) in a similar way.

Now that we have established that I is indeed a T_{E}-interpretation, we claim that it is a falsifying interpretation of the given formula. In other words, we want to show that $I \not\models F$.

$$
\begin{aligned}
& I \not\models \forall x, y.\, f(x,y) = f(y,x) \rightarrow f(a,y) = f(y,a) \\
\text{iff} \quad & I \models \neg(\forall x, y.\, f(x,y) = f(y,x) \rightarrow f(a,y) = f(y,a)) \quad \text{(P.40)} \\
\text{iff} \quad & I \models \exists x, y.\, f(x,y) = f(y,x) \wedge \neg(f(a,y) = f(y,a)) \quad \text{(NNF)} \\
\text{iff} \quad & \text{for some } \mathsf{v}, \mathsf{w},\ I_{\mathsf{vw}} \models f(x,y) = f(y,x) \wedge \neg(f(a,y) = f(y,a)) \quad \text{(P.41)} \\
\text{iff} \quad & \text{for some } \mathsf{v}, \mathsf{w},\ I_{\mathsf{vw}} \models f(x,y) = f(y,x) \text{ and } I_{\mathsf{vw}} \not\models f(a,y) = f(y,a) \quad \text{(P.40)}
\end{aligned}
$$

where I_{vw} is used as a shorthand for $I \triangleleft \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\}$. The last line is true because when we set $\mathsf{v} = \mathsf{w} = \bullet$, we have

$$
\begin{aligned}
& I_{\bullet\bullet} \models f(x,y) = f(y,x) \\
\text{iff} \quad & \alpha_{I_{\bullet\bullet}}[f](\alpha_{I_{\bullet\bullet}}[x], \alpha_{I_{\bullet\bullet}}[y]) = \alpha_{I_{\bullet\bullet}}[f](\alpha_{I_{\bullet\bullet}}[y], \alpha_{I_{\bullet\bullet}}[x]) \\
\text{iff} \quad & \alpha_I[f](\bullet, \bullet) = \alpha_I[f](\bullet, \bullet) \\
\text{iff} \quad & \bullet = \bullet,
\end{aligned}
$$

and

$$\begin{array}{ll} & I_{\bullet\bullet} \not\models f(a,y) = f(y,a) \\ \text{iff} & \alpha_{I_{\bullet\bullet}}[f](\alpha_{I_{\bullet\bullet}}[a], \alpha_{I_{\bullet\bullet}}[y]) = \alpha_{I_{\bullet\bullet}}[f](\alpha_{I_{\bullet\bullet}}[y], \alpha_{I_{\bullet\bullet}}[a]) \text{ is } \mathsf{false} \\ \text{iff} & \alpha_I[f](\circ, \bullet) = \alpha_I[f](\bullet, \circ) \text{ is } \mathsf{false} \\ \text{iff} & \circ = \bullet \text{ is } \mathsf{false}. \end{array}$$

(b) Suppose F is invalid. Then there must exist a falsifying T_{E}-interpretation I.

1. $I \not\models \forall x, y.\, f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \rightarrow g(f(x)) = x$.
2. $I_{\mathsf{vw}} \not\models f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \rightarrow g(f(x)) = x$ for fresh v, w where $I_{\mathsf{vw}} : I \lhd \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\}$ (by 1 and $\forall$ on P.62)
3. $I_{\mathsf{vw}} \models f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x$ (by 2 and $\rightarrow$ on P.10)
4. $I_{\mathsf{vw}} \not\models g(f(x)) = x$ (by 2 and $\rightarrow$ on P.10)
5. $I_{\mathsf{vw}} \models f(g(x)) = g(f(x))$ (by 3 and $\wedge$ on P.10)
6. $I_{\mathsf{vw}} \models f(g(f(y))) = x$ (by 3 and $\wedge$ on P.10)
7. $I_{\mathsf{vw}} \models f(y) = x$ (by 3 and $\wedge$ on P.10)
8. $I_{\mathsf{vw}} \models g(f(y)) = g(x)$ (by 7 and function congruence)
9. $I_{\mathsf{vw}} \models f(g(f(y))) = f(g(x))$ (by 8 and function congruence)
10. $I_{\mathsf{vw}} \models x = f(g(f(y)))$ (by 6 and symmetry)
11. $I_{\mathsf{vw}} \models x = f(g(x))$ (by 9, 10 and transitivity)
12. $I_{\mathsf{vw}} \models x = g(f(x))$ (by 5, 11 and transitivity)
13. $I_{\mathsf{vw}} \models g(f(x)) = x$ (by 12 and symmetry)
14. $I_{\mathsf{vw}} \models \perp$ (by 4 and 13)

Since the only one branch closes, the given formula is valid. Note that we are able to apply (function congruence), (symmetry), and (transitivity) because I is a T_{E}-interpretation.