THE CALCULUS OF COMPUTATION

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1. Chapter 1

Exercise (1.1).

(a) Assume that there is a falsifying interpretation I.

1. $I \models P \wedge Q \rightarrow P \rightarrow Q$ (assumption)
2. $I \models P \wedge Q$ (by 1 and semantics of $\rightarrow$)
3. $I \not\models P \rightarrow Q$ (by 1 and semantics of $\rightarrow$)
4. $I \models Q$ (by 2 and semantics of $\wedge$)
5. $I \not\models Q$ (by 3 and semantics of $\rightarrow$)
6. $I \models \perp$ (4 and 5 are contradictory)

There is only one branch and it is closed. Thus F is valid.

Exercise (1.2).

(a) To prove that $\top \Leftrightarrow \neg\perp$, we prove that $\top \leftrightarrow \neg\perp$ is valid. Assume that there is a falsifying interpretation I such that $I \not\models \top \leftrightarrow \neg\perp$. We apply the semantics of $\leftrightarrow$.

The first branch is:

1a. $I \models \top \wedge \neg(\neg\perp)$
2a. $I \models \neg(\neg\perp)$ (by 1a and semantics of $\wedge$)
3a. $I \not\models \neg\perp$ (by 2a and semantics of $\neg$)
4a. $I \models \perp$ (by 3a and semantics of $\neg$)

The second branch is:

1b. $I \models \neg\top \wedge \neg\perp$
2b. $I \models \neg\top$ (by 1b and semantics of $\wedge$)
3b. $I \not\models \top$ (by 2b and semantics of $\neg$)
4b. $I \models \top$ (Under any interpretation, $\top$ has value true)
5b. $I \models \perp$ (3b and 4b are contradictory)

Thus both branches are closed, and thus $\top \leftrightarrow \neg\perp$ is valid.

(b) We will apply a strategy similar to that of Example 1.13. To prove $\perp \Leftrightarrow \neg\top$, we prove that $F : \perp \leftrightarrow \neg\top$ is valid. Suppose F is not valid; there exists an interpretation I such that $I \not\models F$. There are exactly two branches.

The first branch is:

1a. $I \models \perp \wedge \neg(\neg\top)$ (by semantics of $\leftrightarrow$)
2a. $I \models \perp$ (by 1a and semantics of $\wedge$)

The second branch is:

1b. $I \models (\neg\perp) \wedge \neg(\neg\top)$ (by semantics of $\leftrightarrow$)
2b. $I \models \neg\top$ (by 1b and semantics of $\wedge$)
3b. $I \not\models \top$ (by 2b and semantics of $\neg$)
4b. $I \models \top$ (by definition, P.7)

5b. $I \models \bot$ (by 3b and 4b)
Both of these two branches are closed; F is valid.

Exercise (1.3).

- $\bot$ is equivalent to $\neg\top$. In other words, $\bot \Leftrightarrow \neg\top$ and that is proved in Exercise 1.2(b).

Exercise (1.5).

(a) By using the list of template equivalences on P.19, we can obtain the negation normal form of the original formula as following:
- $F : \neg(P \to Q)$.
- $F' : \neg(\neg P \vee Q)$.
- $F'' : \neg(\neg P) \wedge \neg Q$.
- $F''' : P \wedge \neg Q$.

The only connectives in F''' are $\neg$, $\wedge$, and $\vee$ and the negations appear only in literals. Thus F''' is in NNF. Furthermore, F''' is actually in CNF and DNF since it is the disjunction of one conjunction, and it is the conjunction of two clauses.

Exercise (1.6).

(a) $\bigwedge\limits_{v \in V} \bigvee\limits_{c \in C} P_v^c$.
(b) $\bigwedge\limits_{(v,w) \in E} \bigvee\limits_{c_1 \neq c_2 \in C} (\neg P_v^{c_1} \vee \neg P_w^{c_2})$.
(c) $\bigwedge\limits_{(v,w) \in E} \bigvee\limits_{c \in C} (\neg P_v^c \vee \neg P_w^c)$.
(d) No clue. The problem statement seems too ambiguous.
(e) They are already in CNF. We have $N \cdot M$ variables and $N \cdot M + K \cdot M \cdot (M-1) + K \cdot M$ clauses in the encoding above.

Exercise (1.7).

(a) $P_{(F)}$ contains 1 term. Each $\mathsf{En}(G)$ contains $1, 1, 2 \cdot 2 \cdot 3, 1, 1, 2 \cdot 2 \cdot 3, 3 \cdot 2 \cdot 2$ clauses. Thus the expansion would contain 1728 clauses, which is the product of the numbers of clauses.
(b)
(i) 2^n clauses.
(ii) $\mathsf{Rep}(F_n) = P_{F_n}$, so it is just one clause. We have the subformula set $S_{F_n} = \{Q_1, \cdots, Q_n\} \cup \{R_1, \cdots, R_n\} \cup \{Q_i \wedge R_i \mid 1 \leq i \leq n\}$. We will consider how many clauses $\mathsf{En}(G)$ has for each $G \in S_{F_n}$.
- For each Q_i and R_i, we have $\mathsf{En}(Q_i) = \mathsf{En}(R_i) = \top$. This adds $2n$ clauses to F'.
- $\mathsf{En}(Q_i \wedge R_i)$ contains 3 clauses as defined on P.25. This adds $3n$ clauses to F'.

Therefore, in total, F' contains $1 + 2n + 3n = 5n + 1$ clauses. Note that $5n + 1 < 30n + 2$ where $30n + 2$ is the upper bound described on P.26.
I am not sure if a subformula can be bigger. In other words, for instance, should $(Q_1 \wedge R_1) \vee (Q_2 \wedge R_2)$ be a subformula of F_3? It seems unnecessary to include such cases for the purpose of CNF conversion, but it seems to be more consistent with the definition of a subformula.

(iii) When $n \leq 4$, 2^n is smaller than $5n+1$. For $n \geq 5$, 2^n is bigger than $5n+1$.

Exercise (1.8).

(a) We will follow the format described in Example 1.30. Branching on Q or R will result in unit clauses; choose Q. Then $F\{Q \mapsto \top\} : (P \vee \neg R) \wedge (R)$. P appear only positively, so we consider $F\{Q \mapsto \top, P \mapsto \top\} : R$. Then R appears only positively, so the formula is satisfiable. In particular, F is satisfied by interpretation

$$I : \{P \mapsto \mathsf{true}, Q \mapsto \mathsf{true}, R \mapsto \mathsf{true}\}.$$