

# THE CALCULUS OF COMPUTATION

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## 1. CHAPTER 1

### Exercise (1.1).

- (a) Assume that there is a falsifying interpretation  $I$ .
1.  $I \not\models P \wedge Q \rightarrow P \rightarrow Q$  (assumption)
  2.  $I \models P \wedge Q$  (by 1 and semantics of  $\rightarrow$ )
  3.  $I \not\models P \rightarrow Q$  (by 1 and semantics of  $\rightarrow$ )
  4.  $I \models Q$  (by 2 and semantics of  $\wedge$ )
  5.  $I \not\models Q$  (by 3 and semantics of  $\rightarrow$ )
  6.  $I \models \perp$  (4 and 5 are contradictory)

There is only one branch and it is closed. Thus  $F$  is valid.

- (b) By constructing a truth table which has  $2^2 = 4$  rows, it is easy to see that the interpretation  $I : \{P \mapsto \text{false}, Q \mapsto \text{true}\}$  is a falsifying interpretation.
- (c) By constructing a truth table which has  $2^3 = 8$  rows, it is easy to see that the interpretation  $I : \{P \mapsto \text{true}, Q \mapsto \text{false}, R \mapsto \text{false}\}$  is a falsifying interpretation.

### Exercise (1.2).

- (a) To prove that  $\top \Leftrightarrow \neg\perp$ , we prove that  $\top \leftrightarrow \neg\perp$  is valid. Assume that there is a falsifying interpretation  $I$  such that  $I \not\models \top \leftrightarrow \neg\perp$ . We apply the semantics of  $\leftrightarrow$ .

The first branch is:

- 1a.  $I \models \top \wedge \neg(\neg\perp)$
- 2a.  $I \models \neg(\neg\perp)$  (by 1a and semantics of  $\wedge$ )
- 3a.  $I \not\models \neg\perp$  (by 2a and semantics of  $\neg$ )
- 4a.  $I \models \perp$  (by 3a and semantics of  $\neg$ )

The second branch is:

- 1b.  $I \models \neg\top \wedge \neg\perp$
- 2b.  $I \models \neg\top$  (by 1b and semantics of  $\wedge$ )
- 3b.  $I \not\models \top$  (by 2b and semantics of  $\neg$ )
- 4b.  $I \models \top$  (Under any interpretation,  $\top$  has value true)
- 5b.  $I \models \perp$  (3b and 4b are contradictory)

Thus both branches are closed, and thus  $\top \leftrightarrow \neg\perp$  is valid.

- (b) We will apply a strategy similar to that of Example 1.13. To prove  $\perp \Leftrightarrow \neg\top$ , we prove that  $F : \perp \leftrightarrow \neg\top$  is valid. Suppose  $F$  is not valid; there exists an interpretation  $I$  such that  $I \not\models F$ . There are exactly two branches.

The first branch is:

- 1a.  $I \models \perp \wedge \neg(\neg\top)$  (by semantics of  $\leftrightarrow$ )
- 2a.  $I \models \perp$  (by 1a and semantics of  $\wedge$ )

The second branch is:

1b.  $I \models (\neg \perp) \wedge \neg(\neg \top)$  (by semantics of  $\leftrightarrow$ )

2b.  $I \models \neg \top$  (by 1b and semantics of  $\wedge$ )

3b.  $I \not\models \top$  (by 2b and semantics of  $\neg$ )

4b.  $I \models \top$  (by definition, P.7)

5b.  $I \models \perp$  (by 3b and 4b)

Both of these two branches are closed;  $F$  is valid.

**Exercise (1.3).**

- $\perp$  is equivalent to  $\neg \top$ . In other words,  $\perp \leftrightarrow \neg \top$  and that is proved in Exercise 1.2(b).

**Exercise (1.4).**

- We claim that  $\neg F \leftrightarrow F \overline{\wedge} F$ . To prove that, we prove  $\neg F \leftrightarrow F \overline{\wedge} F$  is valid.

$F$	$\neg F$	$F \overline{\wedge} F$	$\neg F \leftrightarrow F \overline{\wedge} F$
0	1	1	1
1	0	0	1

- $F_1 \vee F_2 \leftrightarrow (F_1 \overline{\wedge} F_1) \overline{\wedge} (F_2 \overline{\wedge} F_2)$ . This can be shown easily using the truth table with  $2^2 = 4$  rows.

**Exercise (1.5).**

- (a) By using the list of template equivalences on P.19, we can obtain the negation normal form of the original formula as following:

- $F : \neg(P \rightarrow Q)$ .
- $F' : \neg(\neg P \vee Q)$ .
- $F'' : \neg(\neg P) \wedge \neg Q$ .
- $F''' : P \wedge \neg Q$ .

The only connectives in  $F'''$  are  $\neg$ ,  $\wedge$ , and  $\vee$  and the negations appear only in literals. Thus  $F'''$  is in NNF. Furthermore,  $F'''$  is actually in CNF and DNF since it is the disjunction of one conjunction, and it is the conjunction of two clauses.

- (b)  $F' : (\neg P \vee \neg Q) \wedge R$  is the NNF of  $F$  that can be obtained using the same strategy as above.  $F'$  is already in CNF.  $F'' : (\neg P \wedge R) \vee (\neg Q \wedge R)$  is an equivalent formula in DNF.

- (c)  $F : (Q \wedge R \rightarrow (P \vee \neg Q)) \wedge (P \vee R)$ .

- $F_1 : (\neg(Q \wedge R) \vee (P \vee \neg Q)) \wedge (P \vee R)$ .
- $F_2 : (\neg Q \vee \neg R \vee P \vee \neg Q) \wedge (P \vee R)$ .
- $F_3 : (\neg Q \wedge (P \vee R)) \vee (\neg R \wedge (P \vee R)) \vee (P \wedge (P \vee R)) \vee (\neg Q \wedge (P \vee R))$ .
- $F_4 : (\neg Q \wedge P) \vee (\neg Q \wedge R) \vee (\neg R \wedge P) \vee (\neg R \wedge R) \vee (P \wedge P) \vee (P \wedge R) \vee (\neg Q \wedge P) \vee (\neg Q \wedge R)$ .

$F_2$  is in NNF and CNF, and  $F_4$  is in DNF.

**Exercise (1.6).**

- (a)  $\bigwedge_{v \in V} \bigvee_{c \in C} P_v^c$ .

- (b)  $\bigwedge_{(v,w) \in E} \bigvee_{c_1 \neq c_2 \in C} (\neg P_v^{c_1} \vee \neg P_w^{c_2})$ .

- (c)  $\bigwedge_{(v,w) \in E} \bigvee_{c \in C} (\neg P_v^c \vee \neg P_w^c)$ .

- (d) No clue. The problem statement seems too ambiguous.

- (e) They are already in CNF. We have  $N \cdot M$  variables and  $N \cdot M + K \cdot M \cdot (M - 1) + K \cdot M$  clauses in the encoding above.

**Exercise (1.7).**

- (a)  $P_{(F)}$  contains 1 term. Each  $\text{En}(G)$  contains  $1, 1, 2 \cdot 2 \cdot 3, 1, 1, 2 \cdot 2 \cdot 3, 3 \cdot 2 \cdot 2$  clauses. Thus the expansion would contain 1728 clauses, which is the product of the numbers of clauses.
- (b)
- (i)  $2^n$  clauses.
  - (ii)  $\text{Rep}(F_n) = P_{F_n}$ , so it is just one clause. We have the subformula set  $S_{F_n} = \{Q_1, \dots, Q_n\} \cup \{R_1, \dots, R_n\} \cup \{Q_i \wedge R_i \mid 1 \leq i \leq n\}$ . We will consider how many clauses  $\text{En}(G)$  has for each  $G \in S_{F_n}$ .
    - For each  $Q_i$  and  $R_i$ , we have  $\text{En}(Q_i) = \text{En}(R_i) = \top$ . This adds  $2n$  clauses to  $F'$ .
    - $\text{En}(Q_i \wedge R_i)$  contains 3 clauses as defined on P.25. This adds  $3n$  clauses to  $F'$ .
- Therefore, in total,  $F'$  contains  $1 + 2n + 3n = 5n + 1$  clauses. Note that  $5n + 1 < 30n + 2$  where  $30n + 2$  is the upper bound described on P.26.
- I am not sure if a subformula can be bigger. In other words, for instance, should  $(Q_1 \wedge R_1) \vee (Q_2 \wedge R_2)$  be a subformula of  $F_3$ ? It seems unnecessary to include such cases for the purpose of CNF conversion, but it seems to be more consistent with the definition of a subformula.
- (iii) When  $n \leq 4$ ,  $2^n$  is smaller than  $5n + 1$ . For  $n \geq 5$ ,  $2^n$  is bigger than  $5n + 1$ .

**Exercise (1.8).**

- (a) We will follow the format described in Example 1.30. Branching on  $Q$  or  $R$  will result in unit clauses; choose  $Q$ . Then  $F\{Q \mapsto \top\} : (P \vee \neg R) \wedge (R)$ .  $P$  appear only positively, so we consider  $F\{Q \mapsto \top, P \mapsto \top\} : R$ . Then  $R$  appears only positively, so the formula is satisfiable. In particular,  $F$  is satisfied by interpretation

$$I : \{P \mapsto \text{true}, Q \mapsto \text{true}, R \mapsto \text{true}\}.$$

- (b) Branching on  $Q$  or  $R$  will result in unit clauses. Choose  $Q$ .

$$F\{Q \mapsto \top\} : (\neg P \vee \neg R) \wedge (R).$$

$P$  appears only negatively.

$$F\{P \mapsto \perp, Q \mapsto \top\} : R.$$

$R$  appears only positively. Thus the interpretation  $I : \{P \mapsto \text{false}, Q \mapsto \text{true}, R \mapsto \text{true}\}$  satisfies  $F$ .

## 2. CHAPTER 2

**Exercise (2.1).**

- (a)  $\exists x, y. \text{day}(x) \wedge \text{day}(y) \wedge \text{length}(x) < \text{length}(y)$ .
- (b)  $\exists x. \text{place}(x) \wedge \text{home}(x) \wedge (\forall y. \text{place}(y) \wedge \text{home}(y) \rightarrow x = y)$ .
- (c)  $\forall x, y. \text{mother}(\text{me}, x) \wedge \text{mother}(x, y) \rightarrow \text{grandmother}(\text{me}, y)$ .
- (d)  $\forall x, y, z. \text{convex}(x) \wedge \text{convex}(y) \wedge \text{intersect}(x, y, z) \rightarrow \text{convex}(z)$ .

**Exercise (2.2).**

- (a) This exercise is similar to Example 2.13. To show that the given formula is invalid, we find an interpretation  $I$  such that

$$I \models (\forall x, y. p(x, y) \rightarrow p(y, x)) \wedge \exists z. \neg p(z, z).$$

Choose  $D_I = \{0, 1\}$  and  $p_I = \{(0, 1), (1, 0)\}$ . Both  $\forall x, y. p(x, y) \rightarrow p(y, x)$  and  $\exists z. \neg p(z, z)$  evaluate to **true** under  $I$  because

- $I \models \forall x, y. p(x, y) \rightarrow p(y, x)$  since  $I \triangleleft \{x \mapsto \mathbf{v}, y \mapsto \mathbf{w}\} \models p(x, y) \rightarrow p(y, x)$  for any  $\mathbf{v}, \mathbf{w} \in \{0, 1\}$ .
- $I \models \exists z. \neg p(z, z)$  since  $I \triangleleft \{z \mapsto 0\} \models \neg p(z, z)$ .

- (b) To show that the given formula is invalid, we need to find an interpretation  $I$  such that

$$I \models \exists x, y. p(x, y) \wedge p(y, x) \wedge \exists z. \neg p(z, z).$$

Choose  $D_I = \{0, 1\}$  and  $p_I = \{(0, 1), (1, 0)\}$ .

1.  $I \models \exists x, y. p(x, y) \wedge p(y, x) \wedge \exists z. \neg p(z, z)$ .
2.  $I \triangleleft \{x \mapsto 0, y \mapsto 1\} \models p(x, y) \wedge p(y, x) \wedge \exists z. \neg p(z, z)$  by 1 and the semantics of  $\exists$ .
3.  $I \triangleleft \{x \mapsto 0, y \mapsto 1\} \models p(x, y)$  by 2 and the semantics of  $\wedge$ .
4.  $I \triangleleft \{x \mapsto 0, y \mapsto 1\} \models p(y, x)$  by 2 and the semantics of  $\wedge$ .
5.  $I \triangleleft \{x \mapsto 0, y \mapsto 1\} \models \exists z. \neg p(z, z)$  by 2 and the semantics of  $\wedge$ .
6.  $I \triangleleft \{x \mapsto 0, y \mapsto 1, z \mapsto 0\} \models \neg p(z, z)$  by 5 and the semantics of  $\exists$ .

**Exercise (2.1).**

- (a) To show  $\neg(\forall x. F) \Leftrightarrow \exists x. \neg F$ , we will show the validity of  $\neg(\forall x. F) \Leftrightarrow \exists x. \neg F$ . Suppose that it is not valid. Then there exists an interpretation  $I$  such that  $I \not\models \neg(\forall x. F) \Leftrightarrow \exists x. \neg F$ . By the semantics of  $\Leftrightarrow$ , there are two branches. The semantics of  $\wedge, \neg$  appear on P.10, and that of  $\forall, \exists$  appear on P.42.

The first branch is:

- 1a.  $I \models \neg(\forall x. F) \wedge \neg(\exists x. \neg F)$ .
- 2a.  $I \models \neg(\forall x. F)$  by 1a and semantics of  $\wedge$ .
- 3a.  $I \models \neg(\exists x. \neg F)$  by 1a and semantics of  $\wedge$ .
- 4a.  $I \not\models \forall x. F$  by 2a and semantics of  $\neg$ .
- 5a.  $I \triangleleft \{x \mapsto \mathbf{v}\} \not\models F$  by 4a and semantics of  $\forall$  for some fresh  $\mathbf{v}$ .
- 6a.  $I \not\models \exists x. \neg F$  by 3a and semantics of  $\neg$ .
- 7a.  $I \triangleleft \{x \mapsto \mathbf{v}\} \not\models \neg F$  by 6a and semantics of  $\exists$  for the same  $\mathbf{v}$ .
- 8a.  $I \triangleleft \{x \mapsto \mathbf{v}\} \models F$  by 7a and semantics of  $\neg$ .
- 9a.  $I \models \perp$  by 5a and 8a.

The second branch is:

- 1b.  $I \models \neg(\neg(\forall x. F)) \wedge \exists x. \neg F$ .
- 2b.  $I \models \exists x. \neg F$  by 1b and semantics of  $\wedge$ .
- 3b.  $I \triangleleft \{x \mapsto \mathbf{v}\} \models \neg F$  by 2b and semantics of  $\exists$  for some fresh  $\mathbf{v} \in D_I$ .
- 4b.  $I \triangleleft \{x \mapsto \mathbf{v}\} \not\models F$  by 3b and semantics of  $\neg$ .
- 5b.  $I \models \neg(\neg(\forall x. F))$  by 1b and semantics of  $\wedge$ .
- 6b.  $I \not\models \neg(\forall x. F)$  by 5b and semantics of  $\neg$ .
- 7b.  $I \models \forall x. F$  by 6b and semantics of  $\neg$ .

8b.  $I \triangleleft \{x \mapsto \mathbf{v}\} \models F$  by 7b and semantics of  $\forall$  for the same  $\mathbf{v}$ .

9b.  $I \models \perp$  by 4b and 8b.

Every branch of a semantic argument proof of  $I \not\models F$  closes, so  $F$  is valid by Theorem 2.30.