THE CALCULUS OF COMPUTATION

HIDENORI SHINOHARA

Contents

1. Chapter 1

Exercise (1.1).

(a) Assume that there is a falsifying interpretation I.

1. $I \not\models P \wedge Q \to P \to Q$ (assumption)
2. $I \models P \wedge Q$ (by 1 and semantics of $\to$)
3. $I \not\models P \to Q$ (by 1 and semantics of $\to$)
4. $I \models Q$ (by 2 and semantics of $\wedge$)
5. $I \not\models Q$ (by 3 and semantics of $\to$)
6. $I \models \perp$ (4 and 5 are contradictory)

There is only one branch and it is closed. P.13 states that it is a proof of the validity of F if every branch is closed. Thus F is valid.

(b) By the relative precedence of the logical connectives mentioned on P.5, the given formula is the same as

$$F : (P \to Q) \vee (P \wedge \neg Q).$$

Suppose that there is a falsifying interpretation I. We will use the semantics mentioned on P.10 and P.11.

1. $I \not\models (P \to Q) \vee (P \wedge \neg Q)$ (assumption)
2. $I \not\models P \to Q$ (by 1 and semantics of $\vee$)
3. $I \not\models P \wedge \neg Q$ (by 1 and semantics of $\vee$)
4. $I \models P$ (by 2 and semantics of $\to$)
5. $I \not\models Q$ (by 2 and semantics of $\to$)

By 3 and the semantics of $\wedge$, we obtain two branches.

The first branch is:

1a. $I \not\models P$ (by 3 and semantics of $\wedge$)
2a. $I \models \perp$ (by 4 and 1a)

The second branch is:

1b. $I \not\models \neg Q$ (by 3 and semantics of $\wedge$)
2b. $I \models Q$ (by 1b and $\neg$)

3b. $I \models \bot$ (by 5 and 2b)

Every branch is closed, so this is a proof of validity of F.

(c) We claim that the interpretation $I : \{P \mapsto \mathsf{true}, Q \mapsto \mathsf{false}, R \mapsto \mathsf{false}\}$ is a falsifying interpretation. We will use the inductive definition on P.7 and P.8.

$$
\begin{aligned}
& I \not\models (P \to Q \to R) \to P \to R \\
\text{iff} \quad & I \models P \to Q \to R \text{ and } I \not\models P \to R \\
\text{iff} \quad & I \models P \to Q \to R \text{ and } I \models P \text{ and } I \not\models R \\
\text{iff} \quad & I \models P \to Q \to R \text{ and } I[P] = \mathsf{true} \text{ and } I[R] = \mathsf{false} \\
\text{iff} \quad & \text{if } I \models P, \text{ then } I \models Q \to R \\
\text{iff} \quad & \text{if } I[P] = \mathsf{true}, \text{ then } I \models Q \to R \\
\text{iff} \quad & I \models Q \to R \\
\text{iff} \quad & \text{if } I \models Q, \text{ then } I \models R \\
\text{iff} \quad & \text{if } I[Q] = \mathsf{true}, \text{ then } I \models R \\
\text{iff} \quad & I \models R \\
\text{iff} \quad & I[R] = \mathsf{true}.
\end{aligned}
$$

Therefore, I is indeed a falsifying interpretation of the given formula.

(d) We claim that the interpretation $I : \{P \mapsto \mathsf{true}, Q \mapsto \mathsf{true}, R \mapsto \mathsf{false}\}$ is a falsifying interpretation. We will use the inductive definition on P.7 and P.8.

$$
\begin{aligned}
& I \not\models (P \to Q \vee R) \to P \to R \\
\text{iff} \quad & I \models P \to Q \vee R \text{ and } I \not\models P \to R \\
\text{iff} \quad & I \models P \to Q \vee R \text{ and } I \models P \text{ and } I \not\models R \\
\text{iff} \quad & I \models P \to Q \vee R \\
\text{iff} \quad & \text{if } I \models P \text{ then } I \models Q \vee R \\
\text{iff} \quad & \text{if } I \models P \text{ then } (I \models Q \text{ or } I \models R)
\end{aligned}
$$

where the last statement is true since $I \models Q$. Therefore, I is indeed a falsifying interpretation of the given formula.

(e) Assume that there is a falsifying interpretation I.

1. $I \not\models \neg(P \wedge Q) \to R \to \neg R \to Q$.
2. $I \models P \wedge Q$.
3. $I \models R \to \neg R \to Q$.
4. $I \models R$.
5. $I \not\models \neg R \to Q$.
6. $I \models R$.

4 and 6 lead to a contradiction.

(f) By the relative precedence of the logical connectives on P.5, we have $(P \wedge Q) \vee \neg P \vee (\neg Q \to \neg P)$. We claim that $I : \{P \mapsto \mathsf{true}, Q \mapsto \mathsf{false}\}$ is a falsifying interpretation.

We will use the inductive definition on P.7 and P.8.

$$\begin{array}{ll} & I \not\models (P \land Q) \lor \neg P \lor (\neg Q \to \neg P) \\ \text{iff} & I \not\models P \land Q \text{ and } I \not\models \neg P \text{ and } I \not\models \neg Q \to \neg P \\ \text{iff} & I \not\models P \land Q \text{ and } I \models P \text{ and } I \models \neg Q \text{ and } I \not\models \neg P \\ \text{iff} & I \not\models P \land Q \end{array}$$

We know that $I \not\models P \land Q$ by $I \not\models Q$ and the semantics of $\land$. (See Example 1.4) Therefore, I is indeed a falsifying interpretation, and thus the given formula is invalid.

Exercise (1.2).

(a) To prove that $\top \Leftrightarrow \neg\bot$, we prove that $\top \leftrightarrow \neg\bot$ is valid. Assume that there is a falsifying interpretation I such that $I \not\models \top \leftrightarrow \neg\bot$. We apply the semantics of $\leftrightarrow$.

The first branch is:

1a. $I \models \top \land \neg(\neg\bot)$
2a. $I \models \neg(\neg\bot)$ (by 1a and semantics of $\land$)
3a. $I \not\models \neg\bot$ (by 2a and semantics of $\neg$)
4a. $I \models \bot$ (by 3a and semantics of $\neg$)

The second branch is:

1b. $I \models \neg\top \land \neg\bot$
2b. $I \models \neg\top$ (by 1b and semantics of $\land$)
3b. $I \not\models \top$ (by 2b and semantics of $\neg$)
4b. $I \models \top$ (Under any interpretation, $\top$ has value `true`)
5b. $I \models \bot$ (3b and 4b are contradictory)

Thus both branches are closed, and thus $\top \leftrightarrow \neg\bot$ is valid.

(b) We will apply a strategy similar to that of Example 1.13. To prove $\bot \Leftrightarrow \neg\top$, we prove that $F : \bot \leftrightarrow \neg\top$ is valid. Suppose F is not valid; there exists an interpretation I such that $I \not\models F$. There are exactly two branches.

The first branch is:

1a. $I \models \bot \land \neg(\neg\top)$ (by semantics of $\leftrightarrow$)
2a. $I \models \bot$ (by 1a and semantics of $\land$)

The second branch is:

1b. $I \models (\neg\bot) \land \neg(\neg\top)$ (by semantics of $\leftrightarrow$)
2b. $I \models \neg\top$ (by 1b and semantics of $\land$)
3b. $I \not\models \top$ (by 2b and semantics of $\neg$)
4b. $I \models \top$ (by definition, P.7)
5b. $I \models \bot$ (by 3b and 4b)

Both of these two branches are closed; F is valid.

(q)

(r)

(s) Let $F : (F_1 \to F_2) \leftrightarrow (\neg F_2 \to \neg F_1)$. We will use a truth table to show F is valid.

F_1	F_2	$F_1 \to F_2$	$\neg F_2 \to \neg F_1$	F
0	0	1	1	1
0	1	1	1	1
1	0	0	0	1
1	1	1	1	1

(x) Let $F : ((F_1 \to F_2) \wedge (F_1 \to F_3)) \leftrightarrow (F_1 \to F_2 \wedge F_3)$. We will use a truth table to show F is valid.

F_1	F_2	F_3	$F_1 \to F_2$	$F_1 \to F_3$	$(F_1 \to F_2) \wedge (F_1 \to F_3)$	$F_2 \wedge F_3$	$F_1 \to F_2 \wedge F_3$	F
0	0	0	1	1	1	0	1	1
0	0	1	1	1	1	0	1	1
0	1	0	1	1	1	0	1	1
0	1	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0	1
1	0	1	0	1	0	0	0	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

Exercise (1.3).

- $\perp$ is equivalent to $\neg\top$. In other words, $\perp \Leftrightarrow \neg\top$ and that is proved in Exercise 1.2(b).

Exercise (1.4).

- We claim that $\neg F \Leftrightarrow F \overline{\wedge} F$. To prove that, we prove $\neg F \leftrightarrow F \overline{\wedge} F$ is valid.

F	$\neg F$	$F \overline{\wedge} F$	$\neg F \leftrightarrow F \overline{\wedge} F$
0	1	1	1
1	0	0	1

- $F_1 \vee F_2 \Leftrightarrow (F_1 \overline{\wedge} F_1) \overline{\wedge} (F_2 \overline{\wedge} F_2)$.

F_1	F_2	$F_1 \overline{\wedge} F_1$	$F_2 \overline{\wedge} F_2$	$(F_1 \overline{\wedge} F_1) \overline{\wedge} (F_2 \overline{\wedge} F_2)$
0	0	1	1	0
1	0	0	1	1
0	1	1	0	1
1	1	0	0	1

Exercise (1.5).

(a) By using the list of template equivalences on P.19, we can obtain the negation normal form of the original formula as following:
- $F : \neg(P \to Q)$.
- $F' : \neg(\neg P \vee Q)$.
- $F'' : \neg(\neg P) \wedge \neg Q$.
- $F''' : P \wedge \neg Q$.

The only connectives in F''' are $\neg$, $\wedge$, and $\vee$ and the negations appear only in literals. Thus F''' is in NNF. Furthermore, F''' is actually in CNF and DNF since it is the disjunction of one conjunction, and it is the conjunction of two clauses.

(b) $F' : (\neg P \vee \neg Q) \wedge R$ is the NNF of F that can be obtained using the same strategy as above. F' is already in CNF. $F'' : (\neg P \wedge R) \vee (\neg Q \wedge R)$ is an equivalent formula in DNF.

(c) $F : (Q \wedge R \to (P \vee \neg Q)) \wedge (P \vee R)$.
- $F_1 : (\neg(Q \wedge R) \vee (P \vee \neg Q)) \wedge (P \vee R)$.
- $F_2 : (\neg Q \vee \neg R \vee P \vee \neg Q) \wedge (P \vee R)$.
- $F_3 : (\neg Q \wedge (P \vee R)) \vee (\neg R \wedge (P \vee R)) \vee (P \wedge (P \vee R)) \vee (\neg Q \wedge (P \vee R))$.

- $F_4 : (\neg Q \wedge P) \vee (\neg Q \wedge R) \vee (\neg R \wedge P) \vee (\neg R \wedge R) \vee (P \wedge P) \vee (P \wedge R) \vee (\neg Q \wedge P) \vee (\neg Q \wedge R)$.

F_2 is in NNF and CNF, and F_4 is in DNF.

Exercise (1.6).

(a) $\bigwedge_{v \in V} \bigvee_{c \in C} P_v^c$.

(b) $\bigwedge_{(v,w) \in E} \bigvee_{c_1 \neq c_2 \in C} (\neg P_v^{c_1} \vee \neg P_w^{c_2})$.

(c) $\bigwedge_{(v,w) \in E} \bigvee_{c \in C} (\neg P_v^c \vee \neg P_w^c)$.

(d) No clue. The problem statement seems too ambiguous.

(e) They are already in CNF. We have $N \cdot M$ variables and $N \cdot M + K \cdot M \cdot (M-1) + K \cdot M$ clauses in the encoding above.

Exercise (1.7).

(a) $P_{(F)}$ contains 1 term. Each $\mathsf{En}(G)$ contains $1, 1, 2 \cdot 2 \cdot 3, 1, 1, 2 \cdot 2 \cdot 3, 3 \cdot 2 \cdot 2$ clauses. Thus the expansion would contain 1728 clauses, which is the product of the numbers of clauses.

(b)

(i) 2^n clauses.

(ii) $\mathsf{Rep}(F_n) = P_{F_n}$, so it is just one clause. We have the subformula set $S_{F_n} = \{Q_1, \cdots, Q_n\} \cup \{R_1, \cdots, R_n\} \cup \{Q_i \wedge R_i \mid 1 \leq i \leq n\} \cup \{(\bigvee_{i=1}^{k-1} Q_i \wedge R_i) \vee (Q_k \wedge R_k) \mid k = 2, \cdots, n\}$. We will consider how many clauses $\mathsf{En}(G)$ has for each $G \in S_{F_n}$.

- For each Q_i and R_i, we have $\mathsf{En}(Q_i) = \mathsf{En}(R_i) = \top$. This adds $2n$ clauses to F'.
- $\mathsf{En}(Q_i \wedge R_i)$ contains 3 clauses as defined on P.25. This adds $3n$ clauses to F'.
- Each $\mathsf{En}((\bigvee_{i=1}^{k-1} Q_i \wedge R_i) \vee (Q_k \wedge R_k))$ contains 3 clauses as defined on P.25. This adds $3(n-1)$ clauses to F'.

Therefore, in total, F' contains $1 + 2n + 3n + 3(n-1) = 8n - 2$ clauses. Note that $8n + 2 < 30n + 2$ where $30n + 2$ is the upper bound described on P.26.

(iii) When $n \leq 5$, 2^n is smaller than $8n - 2$. For $n \geq 6$, 2^n is bigger than $8n - 2$.

Exercise (1.8).

(a) We will follow the format described in Example 1.30. Branching on Q or R will result in unit clauses; choose Q. Then $F\{Q \mapsto \top\} : (P \vee \neg R) \wedge (R)$. P appear only positively, so we consider $F\{Q \mapsto \top, P \mapsto \top\} : R$. Then R appears only positively, so the formula is satisfiable. In particular, F is satisfied by interpretation

$$I : \{P \mapsto \mathsf{true}, Q \mapsto \mathsf{true}, R \mapsto \mathsf{true}\}.$$

(b) Branching on Q or R will result in unit clauses. Choose Q.

$$F\{Q \mapsto \top\} : (\neg P \vee \neg R) \wedge (R).$$

P appears only negatively.

$$F\{P \mapsto \bot, Q \mapsto \top\} : R.$$

R appears only positively. Thus the interpretation $I : \{P \mapsto \mathsf{false}, Q \mapsto \mathsf{true}, R \mapsto \mathsf{true}\}$ satisfies F.

Exercise (2.1).

(a) $\exists x, y.\, \text{day}(x) \wedge \text{day}(y) \wedge \text{length}(x) < \text{length}(y)$.
(b) $\exists x.\, \text{place}(x) \wedge \text{home}(x) \wedge (\forall y.\, \text{place}(y) \wedge \text{home}(y) \rightarrow x = y)$.
(c) $\forall x, y.\, \text{mother}(\text{me}, x) \wedge \text{mother}(x, y) \rightarrow \text{grandmother}(\text{me}, y)$.
(d) $\forall x, y, z.\, \text{convex}(x) \wedge \text{convex}(y) \wedge \text{intersect}(x, y, z) \rightarrow \text{convex}(z)$.

Exercise (2.2).

(a) This exercise is similar to Example 2.13. To show that the given formula is invalid, we find an interpretation I such that

$$I \models \neg((\forall x, y.\, p(x, y) \rightarrow p(y, x)) \rightarrow \forall z.p(z, z)).$$

We will first find the NNF as following:
- $\neg((\forall x, y.\, p(x, y) \rightarrow p(y, x)) \rightarrow \forall z.p(z, z))$.
- $\neg(\neg(\forall x, y.p(x, y) \rightarrow p(y, x)) \vee \forall z.p(z, z))$.
- $(\forall x, y.p(x, y) \rightarrow p(y, x)) \wedge \neg\forall z.p(z, z)$.
- $(\forall x, y.\neg p(x, y) \vee p(y, x)) \wedge \exists z.\neg p(z, z)$.

Using the inductive steps described on P.40 and P.41,

$$\begin{aligned}
&I \models \neg((\forall x, y.\, p(x, y) \rightarrow p(y, x)) \rightarrow \forall z.\, p(z, z)) \\
&\quad \text{iff } I \models (\forall x, y.\, \neg p(x, y) \vee p(y, x)) \wedge \exists z.\, \neg p(z, z) \\
&\quad \text{iff } I \models \forall x, y.\, \neg p(x, y) \vee p(y, x) \text{ and } I \models \exists z.\, \neg p(z, z) \\
&\quad \text{iff } I \lhd \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models \neg p(x, y) \vee p(y, x) \text{ and } I \lhd \{z \mapsto \mathsf{u}\} \models \neg p(z, z) \\
&\quad \text{iff } \big[I \lhd \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \not\models p(x, y) \text{ or } I \lhd \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models p(y, x)\big] \\
&\qquad\quad \text{and } I \lhd \{z \mapsto \mathsf{u}\} \models \neg p(z, z).
\end{aligned}$$

where each line with v, w should be followed by "for all v, w in D_I and for some u in D_I." Choose $D_I = \{0, 1\}$ and $p_I = \{(0, 1), (1, 0)\}$, then it is easy to see that the last line is true. In other words, I is indeed a falsifying interpretation, and thus the given formula is invalid.

(b) To show that the given formula is invalid, we need to find an interpretation I such that

$$I \models \neg\forall x, y.\, p(x, y) \rightarrow p(y, x) \rightarrow \forall z.\, p(z, z).$$

We will find the NNF as following:
- $\neg\forall x, y.\, p(x, y) \rightarrow p(y, x) \rightarrow \forall z.\, p(z, z)$.
- $\neg\forall x, y.\, p(x, y) \rightarrow (p(y, x) \rightarrow \forall z.\, p(z, z))$.
- $\neg\forall x, y.\, p(x, y) \rightarrow (\neg p(y, x) \vee \forall z.\, p(z, z))$.
- $\neg\forall x, y.\, \neg p(x, y) \vee (\neg p(y, x) \vee \forall z.\, p(z, z))$.
- $\exists x, y.\, p(x, y) \wedge \neg(\neg p(y, x) \vee \forall z.\, p(z, z))$.
- $\exists x, y.\, p(x, y) \wedge p(y, x) \wedge \neg\forall z.\, p(z, z)$.
- $\exists x, y.\, p(x, y) \wedge p(y, x) \wedge \exists z.\, \neg p(z, z)$.

Using the inductive steps described on P.40 and P.41,

$$\begin{aligned}
& I \models \neg\forall x, y.\, p(x,y) \to p(y,x) \to \forall z.\, p(z,z) \\
\text{iff} \quad & I \models \exists x, y.\, p(x,y) \wedge p(y,x) \wedge \exists z.\, \neg p(z,z) \\
\text{iff} \quad & I \triangleleft \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models p(x,y) \wedge p(y,x) \wedge \exists z.\, \neg p(z,z) \\
& \text{for some } \mathsf{v}, \mathsf{w} \text{ in } D_I \\
\text{iff} \quad & I \triangleleft \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models p(x,y) \text{ and} \\
& I \triangleleft \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models p(y,x) \text{ and} \\
& I \triangleleft \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models \exists z.\, \neg p(z,z) \\
& \text{for some } \mathsf{v}, \mathsf{w} \text{ in } D_I \\
\text{iff} \quad & I \triangleleft \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models p(x,y) \text{ and} \\
& I \triangleleft \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models p(y,x) \text{ and} \\
& I \triangleleft \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}, z \mapsto \mathsf{u}\} \not\models p(z,z) \\
& \text{for some } \mathsf{v}, \mathsf{w}, \mathsf{u} \text{ in } D_I
\end{aligned}$$

Choose $D_I = \{0,1\}$ and $p_I = \{(0,1),(1,0)\}$, then the last line is clearly true for we can set $\mathsf{v} = 0, \mathsf{w} = 1, \mathsf{u} = 0$. Therefore, I is a falsifying interpretation, and thus the original formula is invalid.

(c) To show that the given formula is invalid, we need to find an interpretation I such that

$$I \models \neg((\exists x.\, p(x)) \to (\forall y.\, p(y))).$$

It suffices to show that the negation normal form is satisfied by I.

- $\neg((\exists x.\, p(x)) \to (\forall y.\, p(y)))$.
- $\neg(\neg(\exists x.\, p(x)) \vee (\forall y.\, p(y)))$.
- $(\exists x.\, p(x)) \wedge \neg(\forall y.\, p(y))$.
- $(\exists x.\, p(x)) \wedge (\exists y.\, \neg p(y))$.

Then we have

$$\begin{aligned}
& I \models \neg((\exists x.\, p(x)) \to (\forall y.\, p(y))) && \\
& \text{iff } I \models (\exists x. p(x)) \wedge (\exists y. \neg p(y)) && \text{(NNF)} \\
& \text{iff } I \models (\exists x. p(x)) \text{ and } (\exists y. \neg p(y)) && \text{(P.40)} \\
& \text{iff } I \models (\exists x. p(x)) \text{ and } (\exists y. \neg p(y)) && \text{(P.40)} \\
& \text{iff } I \triangleleft \{x \mapsto \mathsf{v}\} \models p(x) \text{ and } I \triangleleft \{y \mapsto \mathsf{w}\} \models \neg p(y) \text{ for some } \mathsf{v}, \mathsf{w} && \text{(P.41)}
\end{aligned}$$

Let $D_I = \{0,1\}$ and $p_I = \{0\}$. Then the last statement is true since we can set $\mathsf{v} = 0$ and $\mathsf{w} = 1$. In other words, such I is a falsifying interpretation.

(d) To prove the validity of $F : (\forall x.\, p(x)) \to (\exists y.\, p(y))$, we will assume that it is not. Then there must exist a falsifying interpretation.

1. $I \not\models (\forall x.\, p(x)) \to (\exists y.\, p(y))$.
2. $I \models \forall x.\, p(x)$ by 1 and semantics of $\to$ on P.10.
3. $I \not\models \exists y.\, p(y)$ by 1 and semantics of $\to$ on P.10.
4. $I \triangleleft \{x \mapsto \mathsf{v}\} \models p(x)$ by 2 and semantics of $\forall$ on P.42 for any $\mathsf{v} \in D_I$.

5. $I \triangleleft \{y \mapsto \mathsf{v}\} \not\models p(y)$ by 3 and semantics of $\exists$ on P.42 for the same v as above.
6. $I \models \bot$, contradiction as shown on P.43.

Since every branch of a semantic argument proof of $I \not\models F$ closes, F is valid by Theorem 2.30

(e)

The following solution is wrong. The formula is actually valid because If $p(x, y)$ is false for some x, y, then those x, y can be chosen. If p is always true, we are done.

In order to show that $\exists x, y.\, (p(x, y) \rightarrow (p(y, x) \rightarrow \forall z.\, p(z, z)))$ is invalid, we need to find an interpretation I such that

$$I \not\models \exists x, y.\, (p(x, y) \rightarrow (p(y, x) \rightarrow \forall z.\, p(z, z))).$$

We claim that $D_I = \{0, 1\}$ and $p_I = \{(0, 1), (1, 0)\}$ define such an interpretation. By the semantics of $\exists$ on P.42, the above expression is equivalent to $I \triangleleft \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\} \models p(x, y) \rightarrow (p(y, x) \rightarrow \forall z.\, p(z, z))$ for any v, w in D_I. This is true if and only if $J \models p(x, y)$ and $J \not\models p(y, x) \rightarrow \forall z.\, p(z, z)$ where J is the interpretation $I \triangleleft \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\}$. $J \not\models p(y, x) \rightarrow \forall z.\, p(z, z)$ is true if and only if $J \models p(y, x)$ and $J \not\models \forall z.\, p(z, z)$. Finally, $J \not\models \forall z.\, p(z, z)$ if and only if $J \triangleleft \{z \mapsto \mathsf{u}\} \not\models p(z, z)$ for a fresh u in D_I. It is easy to see that each expression evaluates to true for v, w and u. Therefore, I is a falsifying interpretation.

Exercise (2.3).

(a) This is similar to Example 2.21. To show $\neg(\forall x.\, F) \Leftrightarrow \exists x.\, \neg F$, we will show the validity of $\neg(\forall x.\, F) \leftrightarrow \exists x.\, \neg F$. Suppose that it is not valid. Then there exists an interpretation I such that $I \not\models \neg(\forall x.\, F) \leftrightarrow \exists x.\, \neg F$. By the semantiocs of $\leftrightarrow$, there are two branches. The semantics of $\wedge, \neg$ appear on P.10, and that of $\forall, \exists$ appear on P.42.

The first branch is:

1a. $I \models \neg(\forall x.\, F) \wedge \neg(\exists x.\, \neg F)$.
2a. $I \models \neg(\forall x.\, F)$ by 1a and semantics of $\wedge$.
3a. $I \models \neg(\exists x.\, \neg F)$ by 1a and semantics of $\wedge$.
4a. $I \not\models \forall x.\, F$ by 2a and semantics of $\neg$.
5a. $I \triangleleft \{x \mapsto \mathsf{v}\} \not\models F$ by 4a and semantics of $\forall$ for some fresh v.
6a. $I \not\models \exists x.\, \neg F$ by 3a and semantics of $\neg$.
7a. $I \triangleleft \{x \mapsto \mathsf{v}\} \not\models \neg F$ by 6a and semantics of $\exists$ for the same v.
8a. $I \triangleleft \{x \mapsto \mathsf{v}\} \models F$ by 7a and semantics of $\neg$.
9a. $I \models \bot$ by 5a and 8a.

The second branch is:

1b. $I \models \neg(\neg(\forall x.\, F)) \wedge \exists x.\, \neg F$.
2b. $I \models \exists x.\, \neg F$ by 1b and semantics of $\wedge$.
3b. $I \triangleleft \{x \mapsto \mathsf{v}\} \models \neg F$ by 2b and semantics of $\exists$ for some fresh $\mathsf{v} \in D_I$.
4b. $I \triangleleft \{x \mapsto \mathsf{v}\} \not\models F$ by 3b and semantics of $\neg$.
5b. $I \models \neg(\neg(\forall x.\, F))$ by 1b and semantics of $\wedge$.
6b. $I \not\models \neg(\forall x.\, F)$ by 5b and semantics of $\neg$.
7b. $I \models \forall x.\, F$ by 6b and semantics of $\neg$.
8b. $I \triangleleft \{x \mapsto \mathsf{v}\} \models F$ by 7b and semantics of $\forall$ for the same v.

9b. $I \models \bot$ by 4b and 8b.

Every branch of a semantic argument proof of $I \not\models F$ closes, so F is valid by Theorem 2.30.

(b) To show $\neg(\exists x.\, F) \Leftrightarrow \forall x.\, \neg F$, it suffices to show the validity of $\neg(\exists x.\, F) \leftrightarrow \forall x.\, \neg F$. Suppose otherwise. Then there exists an interpretation I such that $I \not\models \neg(\exists x.\, F) \leftrightarrow (\forall x.\, \neg F)$. We will use the semantics of $\wedge, \neg, \leftrightarrow$ on P.10 and 11, and that of $\forall, \exists$ on P.42.

The first branch is:

1a. $I \models \neg(\exists x.\, F) \wedge \neg(\forall x.\, \neg F)$.
2a. $I \models \neg(\exists x.\, F)$ by 1a and $\wedge$.
3a. $I \models \neg(\forall x.\, \neg F)$ by 1a and $\wedge$.
4a. $I \not\models \forall x.\, \neg F$ by 3a and $\neg$.
5a. $I \triangleleft \{x \mapsto \mathsf{v}\} \not\models \neg F$ by 4a and $\forall$ for a fresh $\mathsf{v} \in D_I$.
6a. $I \triangleleft \{x \mapsto \mathsf{v}\} \models F$ by 5a and $\neg$.
7a. $I \not\models \exists x.\, F$ by 2a and $\neg$.
8a. $I \triangleleft \{x \mapsto \mathsf{v}\} \not\models F$ by 7a and $\exists$ for the same v.
9a. $I \models \bot$ by 6a and 8a.

The second branch is:

1b. $I \models \neg(\neg(\exists x.F)) \wedge \forall x.\neg F$.
2b. $I \models \neg(\neg(\exists x.F))$ by 1b and $\wedge$.
3b. $I \models \forall x.\neg F$ by 1b and $\wedge$.
4b. $I \models \exists x.F$ by applying the semantics of $\neg$ twice to 2b.
5b. $I \triangleleft \{x \mapsto \mathsf{v}\} \models F$ by 4b and $\exists$ for a fresh v.
6b. $I \triangleleft \{x \mapsto \mathsf{v}\} \models \neg F$ by 3b and $\forall$ for the same v.
7b. $I \triangleleft \{x \mapsto \mathsf{v}\} \not\models F$ by 6b and $\neg$.
8b. $I \models \bot$ by 5b and 7b.

Exercise (2.4).

(a)
- $(\forall x.\, \exists y.\, p(x,y)) \rightarrow \forall x.\, p(x,x)$.
- $\neg(\forall x.\, \exists y.\, p(x,y)) \vee \forall x.\, p(x,x)$.
- $(\exists x.\, \neg(\exists y.\, p(x,y))) \vee \forall x.\, p(x,x)$.
- $(\exists x.\, \forall y.\, \neg p(x,y)) \vee \forall x.\, p(x,x)$.
- $(\exists x.\, \forall y.\, \neg p(x,y)) \vee \forall w.\, p(w,w)$.
- $\neg p(x,y)) \vee p(w,w)$.
- $\exists x.\, \forall y.\, \forall w.\, \neg p(x,y)) \vee p(w,w)$.

(b)
- $\exists z.\, (\forall x.\, \exists y.\, p(x,y)) \rightarrow \forall x.\, p(x,z)$.
- $\exists z.\, \neg(\forall x.\, \exists y.\, p(x,y)) \vee \forall x.\, p(x,z)$.
- $\exists z.\, (\exists x.\, \forall y.\, \neg p(x,y)) \vee \forall x.\, p(x,z)$.
- $\exists z.\, (\exists x.\, \forall y.\, \neg p(x,y)) \vee \forall w.\, p(w,z)$.
- $\neg p(x,y) \vee p(w,z)$.
- $\exists z.\, \exists x.\, \forall y.\, \forall w.\, \neg p(x,y) \vee p(w,z)$.

3. Chapter 3

Exercise (3.1).

(a) First, as mentioned on P.42, it technically does not make sense to discuss validity of an open formula. The convention is to take the universal closure of the formula.

Thus we will prove that the formula $F : \forall x, y.\, f(x,y) = f(y,x) \rightarrow f(a,y) = f(y,a)$ is invalid. We will do so by finding a falsifying T_{E}-interpretation. In other words, we need to show that

- We have an interpretation I which satisfies all the axioms of T_{E}, and
- $I \not\models F$.

Let an interpretation $I : \{D_I, \alpha_I\}$ be defined such that

- $D_I = \{\circ, \bullet\}$ and the equaltiy among $\circ$ and $\bullet$ is defined in the most trivial way. (e.g., $\circ = \circ$ and $\circ \neq \bullet$.)
- $\alpha_I[a] = \circ$.
- $\alpha_I[f]$ is the left projection map. In other words,
  - $\alpha_I[f](\circ, \circ) = \circ$,
  - $\alpha_I[f](\circ, \bullet) = \circ$,
  - $\alpha_I[f](\bullet, \circ) = \bullet$,
  - $\alpha_I[f](\bullet, \bullet) = \bullet$.

Then I is a T_{E}-interpretation for it satisfies all the axioms of T_{E} on P.71. For instance, I satisfies reflexivity since

$$
\begin{array}{ll}
 & I \models \forall x.\, x = x \\
\text{iff} & I \lhd \{x \mapsto \mathsf{v}\} \models x = x \text{ for any } \mathsf{v} \in D_I \text{ (P.41)}
\end{array}
$$

which we can easily see as `true` by examining the two cases where v is $\circ$ and $\bullet$. It is easy to show that I satisfies the other four axioms (symmetry, transitivity, function congruence, predicate congruence) in a similar way.

Now that we have established that I is indeed a T_{E}-interpretation, we claim that it is a falsifying interpretation of the given formula. In other words, we want to show that $I \not\models F$.

$$
\begin{array}{ll}
 & I \not\models \forall x, y.\, f(x,y) = f(y,x) \rightarrow f(a,y) = f(y,a) \\
\text{iff} & I \models \neg(\forall x, y.\, f(x,y) = f(y,x) \rightarrow f(a,y) = f(y,a)) \quad \text{(P.40)} \\
\text{iff} & I \models \exists x, y.\, f(x,y) = f(y,x) \wedge \neg(f(a,y) = f(y,a)) \quad \text{(NNF)} \\
\text{iff} & \text{for some } \mathsf{v}, \mathsf{w},\ I_{\mathsf{vw}} \models f(x,y) = f(y,x) \wedge \neg(f(a,y) = f(y,a)) \quad \text{(P.41)} \\
\text{iff} & \text{for some } \mathsf{v}, \mathsf{w},\ I_{\mathsf{vw}} \models f(x,y) = f(y,x) \text{ and } I_{\mathsf{vw}} \not\models f(a,y) = f(y,a) \quad \text{(P.40)}
\end{array}
$$

where I_{vw} is used as a shorthand for $I \lhd \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\}$. The last line is true because when we set $\mathsf{v} = \mathsf{w} = \bullet$, we have

$$
\begin{array}{ll}
 & I_{\bullet\bullet} \models f(x,y) = f(y,x) \\
\text{iff} & \alpha_{I_{\bullet\bullet}}[f](\alpha_{I_{\bullet\bullet}}[x], \alpha_{I_{\bullet\bullet}}[y]) = \alpha_{I_{\bullet\bullet}}[f](\alpha_{I_{\bullet\bullet}}[y], \alpha_{I_{\bullet\bullet}}[x]) \\
\text{iff} & \alpha_I[f](\bullet, \bullet) = \alpha_I[f](\bullet, \bullet) \\
\text{iff} & \bullet = \bullet,
\end{array}
$$

and

$$
\begin{aligned}
& I_{\bullet\bullet} \not\models f(a,y) = f(y,a) \\
\text{iff}\quad & \alpha_{I_{\bullet\bullet}}[f](\alpha_{I_{\bullet\bullet}}[a], \alpha_{I_{\bullet\bullet}}[y]) = \alpha_{I_{\bullet\bullet}}[f](\alpha_{I_{\bullet\bullet}}[y], \alpha_{I_{\bullet\bullet}}[a]) \text{ is } \mathsf{false} \\
\text{iff}\quad & \alpha_I[f](\circ, \bullet) = \alpha_I[f](\bullet, \circ) \text{ is } \mathsf{false} \\
\text{iff}\quad & \circ = \bullet \text{ is } \mathsf{false}.
\end{aligned}
$$

(b) Suppose F is invalid. Then there must exist a falsifying T_{E}-interpretation I.

1. $I \not\models \forall x, y.\, f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \rightarrow g(f(x)) = x$.
2. $I_{\mathsf{vw}} \not\models f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \rightarrow g(f(x)) = x$ for fresh v, w where $I_{\mathsf{vw}} : I \lhd \{x \mapsto \mathsf{v}, y \mapsto \mathsf{w}\}$ (by 1 and $\forall$ on P.62)
3. $I_{\mathsf{vw}} \models f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x$ (by 2 and $\rightarrow$ on P.10)
4. $I_{\mathsf{vw}} \not\models g(f(x)) = x$ (by 2 and $\rightarrow$ on P.10)
5. $I_{\mathsf{vw}} \models f(g(x)) = g(f(x))$ (by 3 and $\wedge$ on P.10)
6. $I_{\mathsf{vw}} \models f(g(f(y))) = x$ (by 3 and $\wedge$ on P.10)
7. $I_{\mathsf{vw}} \models f(y) = x$ (by 3 and $\wedge$ on P.10)
8. $I_{\mathsf{vw}} \models g(f(y)) = g(x)$ (by 7 and (function congruence))
9. $I_{\mathsf{vw}} \models f(g(f(y))) = f(g(x))$ (by 8 and (function congruence))
10. $I_{\mathsf{vw}} \models x = f(g(f(y)))$ (by 6 and (symmetry))
11. $I_{\mathsf{vw}} \models x = f(g(x))$ (by 9, 10 and (transitivity))
12. $I_{\mathsf{vw}} \models x = g(f(x))$ (by 5, 11 and (transitivity))
13. $I_{\mathsf{vw}} \models g(f(x)) = x$ (by 12 and (symmetry))
14. $I_{\mathsf{vw}} \models \bot$ (by 4 and 13)

Since the only one branch closes, the given formula is valid. Note that we are able to apply (function congruence), (symmetry), and (transitivity) because I is a T_{E}-interpretation.

(c) Suppose F is invalid. Then there must exist a falsifying T_{E}-interpretation I.

1. $I \not\models f(f(f(a))) = f(f(a)) \wedge f(f(f(f(a)))) = a \rightarrow f(a) = a$.
2. $I \models f(f(f(a))) = f(f(a)) \wedge f(f(f(f(a)))) = a$ (by 1 and $\rightarrow$)
3. $I \not\models f(a) = a$ (by 1 and $\rightarrow$)
4. $I \models f(f(f(a))) = f(f(a))$ (by 2 and $\wedge$)
5. $I \models f(f(f(f(a)))) = a$ (by 2 and $\wedge$)
6. $I \models f(f(a)) = f(f(f(a)))$ (by 4 and (symmetry))
7. $I \models f(f(f(a))) = f(f(f(f(a))))$ (by 6 and (function congruence))
8. $I \models f(f(f(a))) = a$ (by 5, 7, and (transitivity))
9. $I \models f(f(f(f(a)))) = f(a)$ (by 8 and (function congruence))
10. $I \models f(a) = f(f(f(f(a))))$ (by 9 and (symmetry))
11. $I \models f(a) = a$ (by 5, 10 and (transitivity))
12. $I \models \bot$ (by 3 and 11)

Thus the given formula is valid.

(d) We claim that F is invalid by identifying a falsifying T_{E}-interpretation. Let $I : (\alpha_I, D_I)$ be defined such that

- $D_I = \{0, 1\}$.
- $\alpha_I[f](0) = 1, \alpha_I[f](1) = 0$.
- $a \mapsto 0$.

Then I is a T_{E}-interpretation as it satisfies all the axioms of T_{E}. For instance, it satisfies function congruence since $I \models \forall x, y.\, x = y \implies f(x) = f(y)$.

$$\begin{aligned}
& I \not\models f(f(f(a))) = f(a) \land f(f(a)) = a \to f(a) = a \\
\text{iff} \quad & I \models f(f(f(a))) = f(a) \land f(f(a)) = a \land \neg(f(a) = a) \quad \text{(NNF)} \\
\text{iff} \quad & I \models f(f(f(a))) = f(a) \text{ and } I \models f(f(a)) = a \text{ and } I \models \neg(f(a) = a) \quad (\text{by } \land)
\end{aligned}$$

where each of the three expressions in the last line is true. For instance, $\alpha_I[f](\alpha_I[f](\alpha_I[a])) = \alpha_I[a]$ if and only if $0 = 0$. The other two expressions can be evaluated in the same way. Therefore, I is indeed a falsifying T_{E}-interpretation of F, and thus F is invalid.

Exercise (3.2).

(a) We claim that any $T_{\mathbb{Z}}$-interpretation is a counterexample. Let I be a $T_{\mathbb{Z}}$-interpretation. Given a, b, $\mathsf{c} \in D_I$, let $I_{\mathsf{abc}} : I \lhd \{x \mapsto \mathsf{a}, y \mapsto \mathsf{b}, z \mapsto \mathsf{c}\}$ for convenience.

$$\begin{aligned}
& I \not\models \forall x, y, z.\, x \le y \land z = x + 1 \to z \le y \\
\text{iff} \quad & I \models \exists x, y, z.\, x \le y \land z = x + 1 \land \neg(z \le y) \\
\text{iff} \quad & \text{For some } \mathsf{a}, \mathsf{b}, \mathsf{c},\ I_{\mathsf{abc}} \models x \le y \land z = x + 1 \land \neg(z \le y) \\
\text{iff} \quad & \text{For some } \mathsf{a}, \mathsf{b}, \mathsf{c},\ I_{\mathsf{abc}} \models x \le y \text{ and } I_{\mathsf{abc}} \models z = x + 1 \text{ and } I_{\mathsf{abc}} \models \neg(z \le y).
\end{aligned}$$

The last line is equivalent to asking the existence of a, b, c such that $\mathsf{a} \le \mathsf{b}$, $\mathsf{c} = \mathsf{a} + 1$, and $\mathsf{c} > \mathsf{b}$. It is true because, for instance, the tuple $(\mathsf{a}, \mathsf{b}, \mathsf{c}) = (0, 0, 1)$ satisfies all the three expressions simultaneously.

(b) We claim that the given formula is valid. Suppose otherwise. Let I be a falsifying $T_{\mathbb{Z}}$-interpretation.

1. $I \not\models \forall x, y, z.\, x \le y \land z = x - 1 \to z \le y$.
2. $I \models \exists x, y, z.\, x \le y \land z = x - 1 \land \neg(z \le y)$.
3. $I_{\mathsf{abc}} \models x \le y \land z = x - 1 \land \neg(z \le y)$ where $I_{\mathsf{abc}} : I \lhd \{x \mapsto \mathsf{a}, y \mapsto \mathsf{b}, z \mapsto \mathsf{c}\}$ for some a, b, $\mathsf{c} \in D_I$.
4. $I_{\mathsf{abc}} \models x \le y$, $I_{\mathsf{abc}} \models z = x - 1$ and $I_{\mathsf{abc}} \models \neg(z \le y)$ where $I_{\mathsf{abc}} : I \lhd \{x \mapsto \mathsf{a}, y \mapsto \mathsf{b}, z \mapsto \mathsf{c}\}$ for some a, b, $\mathsf{c} \in D_I$.

The last line is equivalent to the existence of a, b, c such that $\mathsf{a} \le \mathsf{b}$, $\mathsf{c} = \mathsf{a} - 1$, and $\mathsf{c} > \mathsf{b}$. Since $\mathsf{c} = \mathsf{a} - 1 < \mathsf{a} \le \mathsf{b} < \mathsf{c}$ would imply that $\mathsf{c} < \mathsf{c}$, there is no such tuple. Thus the original formula must be valid.

(c) We claim that the given formula is valid. Suppose otherwise. Let I be a falsifying $T_{\mathbb{Z}}$-interpretation.

1. $I \not\models \forall x.\, 3x = 2 \to x \le 0$.
2. $I \models \exists x.\, 3x = 2 \land \neg(x \le 0)$.
3. $I_{\mathsf{a}} \models 3x = 2 \land \neg(x \le 0)$ for some $\mathsf{a} \in D_I$.
4. $I_{\mathsf{a}} \models 3x = 2$ and $I_{\mathsf{a}} \models \neg(x \le 0)$ for some $\mathsf{a} \in D_I$.

The last line is equivalent to the existence of a such that $3\mathsf{a} = 2$ and $\mathsf{a} > 0$. Since no integer satisfies $3\mathsf{a} = 2$, this is a contradiction. Therefore, the original formula is valid.

Exercise (3.3).

(a) We claim that any $T_{\mathbb{Q}}$-interpretation is a falsifying interpretation. Let I be a $T_{\mathbb{Q}}$-interpretation.

$$\begin{array}{ll} & I \not\models \forall x.\, 3x = 2 \rightarrow x \leq 0 \\ \text{iff} & I \models \exists x.\, 3x = 2 \land \neg(x \leq 0) \\ \text{iff} & I \lhd \{x \mapsto \mathsf{a}\} \models 3x = 2 \land \neg(x \leq 0) \text{ for some } \mathsf{a} \in \mathbb{Q} \\ \text{iff} & \text{There exists a rational number } a \text{ such that } 3a = 2 \text{ and } a \geq 0 \end{array}$$

where the last line is clearly true since 3/2 is such a rational number. Thus the given formula is invalid. In fact, it is unsatisfiable because no interpretation satisfies the formula.

Exercise (3.4).

(a) Let $I = (\alpha_I, D_I)$ be an interpretation with
- $D_I = \{\bullet\}$,
- $\alpha_I[\mathsf{atom}](\bullet) = \mathsf{true}$,
- $\alpha_I[\mathsf{car}](\bullet) = \bullet$,
- $\alpha_I[\mathsf{cdr}](\bullet) = \bullet$,
- α_I maps $\mathsf{cons}, \mathsf{car}, \mathsf{cdr}$ and atom so as to satisfy the axioms of T_{cons}.

It is easy to see that this is indeed possible, and also it is clear that I is indeed a T_{cons}-interpretation. Note that the behavior of car and cdr on atoms is undefined as mentioned on P.85, so it is okay to declare it as we did above.

$$\begin{array}{ll} & I \not\models \forall x, y, z.\ \mathsf{car}(x) = y \land \mathsf{cdr}(x) = z \rightarrow x = \mathsf{cons}(y, z) \\ \text{iff} & I \models \exists x, y, z.\ \mathsf{car}(x) = y \land \mathsf{cdr}(x) = z \land \neg(x = \mathsf{cons}(y, z)) \\ \text{iff} & J \models \mathsf{car}(x) = y \land \mathsf{cdr}(x) = z \land \neg(x = \mathsf{cons}(y, z)) \\ \text{iff} & J \models \mathsf{car}(x) = y \text{ and } J \models \mathsf{cdr}(x) = z \text{ and } J \not\models x = \mathsf{cons}(y, z) \\ \text{iff} & \alpha_J[\mathsf{car}](\bullet) = \bullet \text{ and } \alpha_J[\mathsf{cdr}](\bullet) = \bullet \text{ and } J \not\models x = \mathsf{cons}(y, z) \\ \text{iff} & J \not\models x = \mathsf{cons}(y, z) \end{array}$$

where $J : I \lhd \{x \mapsto \bullet, y \mapsto \bullet, z \mapsto \bullet\}$.

From the definition of I and the axiom (atom), we know that $J \models \mathsf{atom}(x)$ and $J \not\models \mathsf{atom}(\mathsf{cons}(y, z))$. Thus $J \not\models \mathsf{atom}(x) \leftrightarrow \mathsf{atom}(\mathsf{cons}(y, z))$. By taking the contrapositive of (predicate congruence) as mentioned on P.89, we obtain that $J \not\models x = \mathsf{cons}(y, z)$.

Therefore, the last line is true, so I is indeed a falsifying T_{cons}-interpretation.

(b) We claim that F is valid. Suppose otherwise. Let I be a falsifying T_{cons}-interpretation.
1. $I \not\models \forall *.\, \neg\,\mathsf{atom}(x) \land \mathsf{car}(x) = y \land \mathsf{cdr}(x) = z \rightarrow x = \mathsf{cons}(y, z)$.
2. $I \models \exists *.\, \neg\,\mathsf{atom}(x) \land \mathsf{car}(x) = y \land \mathsf{cdr}(x) = z \land \neg(x = \mathsf{cons}(y, z))$.
3. $I_{\mathsf{abc}} \models \neg\,\mathsf{atom}(x) \land \mathsf{car}(x) = y \land \mathsf{cdr}(x) = z \land \neg(x = \mathsf{cons}(y, z))$ for fresh a, b, c.
4. $I_{\mathsf{abc}} \models \neg\,\mathsf{atom}(x)$.
5. $I_{\mathsf{abc}} \models \mathsf{car}(x) = y$.
6. $I_{\mathsf{abc}} \models \mathsf{cdr}(x) = z$.
7. $I_{\mathsf{abc}} \models \neg(x = \mathsf{cons}(y, z))$.
8. $I_{\mathsf{abc}} \models \mathsf{cons}(\mathsf{car}(x), \mathsf{cdr}(x)) = \mathsf{cons}(y, z)$. (by 5, 6 and (function congruence)
9. $I_{\mathsf{abc}} \models \mathsf{cons}(\mathsf{car}(x), \mathsf{cdr}(x)) = x$. (by 4 and (construction)

10. $I_{\mathsf{abc}} \models x = \mathsf{cons}(y, z)$. (by 8, 9, (symmetry), and (transitivity))
11. $I_{\mathsf{abc}} \models \bot$. (by 7 and 10)

Every branch of a semantic argument proof closes, so F is valid.

Exercise (3.4).

(a) Let $I = (\alpha_I, D_I)$ be a T_{A}-interpretation with $D_I = \{\bullet\}$. It is easy to see such an interpretation exists. We claim that I is a falsifying interpretation.

$$\begin{array}{ll} & I \not\models \forall i, j.\, a\,\langle i \lhd e\rangle\,[j] = e \rightarrow i = j \\ \text{iff} & I \models \exists i, j.\, a\,\langle i \lhd e\rangle\,[j] = e \wedge \neg(i = j) \\ \text{iff} & \text{There exist } \mathsf{v}, \mathsf{w} \text{ such that } I_{\mathsf{vw}} \models a\,\langle i \lhd e\rangle\,[j] = e \wedge \neg(i = j) \\ \text{iff} & \text{There exist } \mathsf{v}, \mathsf{w} \text{ such that } I_{\mathsf{vw}} \models a\,\langle i \lhd e\rangle\,[j] = e \text{ and } I_{\mathsf{vw}} \models \neg(i = j) \end{array}$$

The last statement is true when we set $\mathsf{v} = 0$ and $\mathsf{w} = 1$, for instance. Note that since the cardinality of D_I is 0, any read operation returns $\bullet$, and every constant is mapped to $\bullet$.

(b) Let $I = (\alpha_I, D_I)$ be a T_{A}-interpretation such that
- $D_I = \{0, 1\}$.
- $\alpha_I[a][k] = k$ for each $k = 0, 1$.
- $\alpha_I[e] = 1$.

It is easy to see that such an interpretation exists. We claim that it is a falsifying interpretation.

$$\begin{array}{ll} & I \not\models \forall i, j.\, a\,\langle i \lhd e\rangle\,[j] = e \rightarrow a[j] = e \\ \text{iff} & I \models \exists i, j.\, a\,\langle i \lhd e\rangle\,[j] = e \wedge \neg(a[j] = e) \\ \text{iff} & \text{For some } \mathsf{v}, \mathsf{w}, I_{\mathsf{vw}} \models a\,\langle i \lhd e\rangle\,[j] = e \text{ and } I_{\mathsf{vw}} \models \neg(a[j] = e) \end{array}$$

If $\mathsf{v} = \mathsf{w} = 0$, then $\alpha_{I_{\mathsf{vw}}}[a]\,\langle 0 \lhd 1\rangle\,[0] = 1$ and $\alpha_{I_{\mathsf{vw}}}[a][0] \neq 1$. Thereore, the last line is true, so I is indeed a falsifying interpretation.

(c) Suppose that the formula is invalid. Let I be a falsifying T_{A}-interpretation.
1. $I \not\models \forall i, j.\, a\,\langle i \lhd e\rangle\,[j] = e \rightarrow i = j \vee a[j] = e$.
2. $I \models \exists i, j.\, a\,\langle i \lhd e\rangle\,[j] = e \wedge i \neq j \wedge a[j] \neq e$.
3. $I_{\mathsf{vw}} \models a\,\langle i \lhd e\rangle\,[j] = e \wedge i \neq j \wedge a[j] \neq e$ for fresh v and w.
4. $I_{\mathsf{vw}} \models a\,\langle i \lhd e\rangle\,[j] = e$.
5. $I_{\mathsf{vw}} \models i \neq j$.
6. $I_{\mathsf{vw}} \models a[j] \neq e$.
7. $I_{\mathsf{vw}} \models a\,\langle i \lhd e\rangle\,[j] = a[j]$. (by 5 and (read-over-write 2))
8. $I_{\mathsf{vw}} \models a[j] = e$. (by 4, 7, (symmetry) and (transitivity))
9. $I_{\mathsf{vw}} \models \bot$. (by 6 and 8)

Since every branch closes, the given formula must be valid.

(d) Suppose that the given formula is invalid. Let I be a falsifying T_{A}-interpretation.
1. $I \not\models \forall i, j, k.\, a\,\langle i \lhd e\rangle\,\langle j \lhd f\rangle\,[k] = g \wedge j \neq k \wedge i = j \rightarrow a[k] = g$.
2. $I \models \exists i, j, k.\, a\,\langle i \lhd e\rangle\,\langle j \lhd f\rangle\,[k] = g \wedge j \neq k \wedge i = j \wedge a[k] \neq g$.
3. $I_{\mathsf{uvw}} \models a\,\langle i \lhd e\rangle\,\langle j \lhd f\rangle\,[k] = g \wedge j \neq k \wedge i = j \wedge a[k] \neq g$ for fresh u, v, and w.

4. $I_{\mathsf{uvw}} \models a\,\langle i \lhd e\rangle\,\langle j \lhd f\rangle\,[k] = g$.
5. $I_{\mathsf{uvw}} \models j \neq k$.
6. $I_{\mathsf{uvw}} \models i = j$.
7. $I_{\mathsf{uvw}} \models a[k] \neq g$.
8. $I_{\mathsf{uvw}} \models a\,\langle i \lhd e\rangle\,\langle j \lhd f\rangle\,[k] = a\,\langle i \lhd e\rangle\,[k]$. (by 5 and (read-over-write 2))
9. $I_{\mathsf{uvw}} \models i \neq k$. (by 5, 6 and the contrapositive of (transitivity))
10. $I_{\mathsf{uvw}} \models a\,\langle i \lhd e\rangle\,[k] = a[k]$. (by 9 and (read-over-write 2))
11. $I_{\mathsf{uvw}} \models a[k] = g$. (by 4, 8, 10, (symmtry) and (transitivity))
12. $I_{\mathsf{uvw}} \models \bot$. (by 7 and 11)

Therefore, the original formula must be valid.

4. Chapter 4

Exercise (4.1).

(a)

Proofread!

This proof is rather complicated since we use stepwise induction inside another stepwise induction proof. Let $G_u[v]$ be defined to be the formula $\mathit{flat}(u) \land \mathit{flat}(v) \to \mathit{flat}(\mathit{concat}(u, v))$. Let $F[u]$ be defined to be the formula $\forall v.\, G_u[v]$. We prove by stepwise induction on u that $F[u]$.

- For the base case, we consider arbitrary $\mathsf{atom}\, u$ and prove $F[u]$. In other words, we want to show that $\forall v.\, G_u[v]$.
  - For the base case, we consider arbitrary $\mathsf{atom}\, v$ and prove $G_u[v]$. In other words, we want to prove $\mathit{flat}(u) \land \mathit{flat}(v) \to \mathit{flat}(\mathit{concat}(u, v))$. Since $\mathsf{atom}\, u$ and $\mathsf{atom}\, v$, $\mathit{flat}(\mathit{concat}(u, v)) = \mathit{flat}(\mathsf{cons}(u, v)) \leftrightarrow \mathsf{atom}(u) \land \mathit{flat}(v) = \mathsf{true}$ by (concat atom), (flat atom), (flat list).
  - Assume as the inductive hypothesis that for arbitrary list y, $G_u[y]$. Let x be given. We want to show that $G_u[\mathsf{cons}(x, y)]$. In other words, we want to show that $\mathit{flat}(u) \land \mathit{flat}(\mathsf{cons}(x, y)) \to \mathit{flat}(\mathit{concat}(u, \mathsf{cons}(x, y)))$.

$$\begin{aligned}
&\mathit{flat}(u) \land \mathit{flat}(\mathsf{cons}(x, y)) \\
&\quad \leftrightarrow \mathsf{atom}(u) \land \mathit{flat}(\mathsf{cons}(x, y)) && (\mathsf{atom}(u)) \\
&\quad \leftrightarrow \mathit{flat}(\mathsf{cons}(u, \mathsf{cons}(x, y))) && (\text{flat list}) \\
&\quad \leftrightarrow \mathit{flat}(\mathit{concat}(u, \mathsf{cons}(x, y))) && (\text{concat atom})
\end{aligned}$$

  Therefore, the inductive step has been shown.

  By stepwise induction, we have shown that $\forall v.\, G_u[v]$.
- Assume as the inductive hypothesis that $F[y]$ for arbitrary y. In other words, we are assuming that $\forall v.\, G_y[v]$, which is equivalent to $\forall v.\, \mathit{flat}(y) \land \mathit{flat}(v) \to \mathit{flat}(\mathit{concat}(y, v))$. Let x be given. We want to show that $F[\mathsf{cons}(x, y)]$. In other words, $\forall v.\, G_{\mathsf{cons}(x,y)}[v]$, which is equivalent to $\forall v.\, \mathit{flat}(\mathsf{cons}(x, y)) \land \mathit{flat}(v) \to$

$\mathit{flat}(\mathit{concat}(\mathsf{cons}(x,y),v))$. We will first examine the hypothesis:

$$\begin{aligned}
&\mathit{flat}(\mathsf{cons}(x,y)) \wedge \mathit{flat}(v) \\
&\leftrightarrow \mathsf{atom}\, x \wedge \mathit{flat}(y) \wedge \mathit{flat}(v) && \text{(flat list)}
\end{aligned}$$

Therefore, if $\neg\, \mathsf{atom}\, x$, we are done. Suppose $\mathsf{atom}\, x$.

$$\begin{aligned}
&\mathsf{atom}\, x \wedge \mathit{flat}(y) \wedge \mathit{flat}(v) \\
&\leftrightarrow \mathit{flat}(y) \wedge \mathit{flat}(v) \\
&\rightarrow \mathit{flat}(\mathit{concat}(y,v)) && \text{(inductive hypothesis)} \\
&\leftrightarrow \mathit{flat}(\mathsf{cons}(x, \mathit{concat}(y,v))) && \text{(flat list)} \\
&= \mathit{flat}(\mathit{concat}(\mathsf{cons}(x,y),v)) && \text{(concat list)}
\end{aligned}$$

Therefore, we have shown that $\forall u.\, F[u]$.

(b) Let $F[u] : \mathit{flat}(u) \rightarrow \mathit{flat}(\mathit{rvs}(u))$. We will show that $F[u]$ is true for any u by stepwise induction on u. For the base case, we consider arbitrary $\mathsf{atom}\, u$ and prove $F[u]$. This is trivial by (reverse atom) and (flat atom). Assume as the inductive hypothesis that for arbtrary list v, $F[v]$ is true. We want to prove that, for arbitrary list u, $F[\mathsf{cons}(u,v)]$. Suppose that $\mathsf{atom}\, u$

$$\begin{aligned}
\mathit{flat}(\mathsf{cons}(u,v)) &\leftrightarrow \mathsf{atom}(u) \wedge \mathit{flat}(v) && \text{(flat list)} \\
&\rightarrow \mathsf{atom}(u) \wedge \mathit{flat}(\mathit{rvs}(v)) && \text{(inductive hypothesis)} \\
&\leftrightarrow \mathit{flat}(\mathsf{cons}(u, \mathit{rvs}(v))) && \text{(flat list)} \\
&= \mathit{flat}(\mathit{concat}(u, \mathit{rvs}(v))) && \text{(concat atom)} \\
&= \mathit{flat}(\mathit{concat}(\mathit{rvs}(u), \mathit{rvs}(v))) && \text{(reverse atom)} \\
&= \mathit{flat}(\mathit{rvs}(\mathit{concat}(u,v))) && \text{(reverse list)}
\end{aligned}$$

Thus $\mathit{flat}(\mathsf{cons}(u,v)) \rightarrow \mathit{flat}(\mathit{rvs}(\mathit{concat}(u,v)))$. Suppose that $\neg\, \mathsf{atom}\, u$. Then

$$\mathit{flat}(\mathsf{cons}(u,v)) \leftrightarrow \mathsf{atom}(u) \wedge \mathit{flat}(v)$$

by (flat list), but $\mathsf{atom}(u)$ is false, so we are done. Thus we have shown that $F[u]$ is true for any arbitrary u.

Exercise (4.2).

(a) We will apply the stepwise induction principle for lists. Let $F[u] = u \preceq_{\mathsf{C}} u$. It suffices to show that

- $\forall\, \mathsf{atom}\, u.\, F[u]$, and
- $\forall u, v.\, F[v] \rightarrow F[\mathsf{cons}(u,v)]$.

The base case is a direct consequence of ($\preceq_{\mathsf{C}}$ (1)). $F[\mathsf{cons}(u,v)]$ is equivalent to $\mathsf{cons}(u,v) \preceq_{\mathsf{C}} \mathsf{cons}(u,v)$, which is equivalent to $(u = u \wedge v \preceq_{\mathsf{C}} v) \vee \mathsf{cons}(u,v) \leq v$. Since we have $v \preceq_{\mathsf{C}} v$ by the inductive hypothesis, we are done with the inductive step. THerefore, $F[u]$ is true for any u by the stepwise induction principle.