階数 3 の等質開凸錐の線形非同値類について

On linearly non-isomorphism classes
of homogeneous cones of rank 3

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Motivation

Theorem 1 (Ishi 2006).

Any homogeneous cone can be described in a matrix form as

$$\left\{ x = \begin{pmatrix} x_1 I_{n_1} & {}^t X_{21} & \cdots & {}^t X_{r1} \\ X_{21} & x_2 I_{n_2} & & {}^t X_{r2} \\ \vdots & & \ddots & \vdots \\ X_{r1} & X_{r2} & \cdots & x_r I_{n_r} \end{pmatrix}; \quad \begin{array}{l} x \gg 0, \\ x_1, \ldots, x_r \in \mathbb{R}, \\ X_{kj} \in \mathcal{V}_{kj} \ (j < k) \end{array} \right\}.$$

Here, $\mathcal{V}_{kj}$ are vector subspaces of $\mathrm{Mat}(n_k, n_j; \mathbb{R})$ satisfying

$$X_{kj} X_{ji} \in \mathcal{V}_{ki}, \quad X_{ki} \, {}^t X_{ji} \in \mathcal{V}_{kj}, \quad X_{kj} \, {}^t X_{kj} \in \mathbb{R} I_{n_k}$$

for $X_{kj} \in \mathcal{V}_{kj}$.

We want to know more concrete structures of $\mathcal{V}_{kj}$'s.

Examples

- $\Omega = \mathrm{Herm}(3, \mathbb{C})^*$ is linearly isomorphic to

$$\left\{ X = \begin{pmatrix} a & 0 & y_1 & -y_2 & z_1 \\ 0 & a & y_2 & y_1 & z_2 \\ y_1 & y_2 & b & 0 & x_1 \\ -y_2 & y_1 & 0 & b & x_2 \\ z_1 & z_2 & x_1 & x_2 & c \end{pmatrix} ;\ X \text{ is positive definite} \right\}.$$

- $\Omega = \mathrm{Herm}(3, \mathbb{H})^*$ is linearly isomorphic to

$$\left\{ X = \begin{pmatrix} aI_4 & R_y & \boldsymbol{z} \\ {}^t R_y & bI_4 & \boldsymbol{x} \\ {}^t\boldsymbol{z} & {}^t\boldsymbol{x} & c \end{pmatrix} ;\ X \text{ is positive definite} \right\}.$$

Here, we identify $\mathbb{H}$ with $\mathbb{R}^4$ via the standard basis and R_α $(\alpha \in \mathbb{H})$ is a right multiplication op. of quaternion.

One parameter family

- Vinberg (1963, p. 397) says

> "If $n_{12} = n_{23} = 2$, $n_{13} = 4$, then we obtain a one-parameter family of non-isomorphic T-algebras. It corresponds to a *one-parameter family of nonisomorphic convex homogeneous cones of dimension* 11."

- The following question arises naturally.

> How do we realize such cones with parameters in a matrix form?

  - This question is answered by Yamasaki–Nomura 2015.

One parameter family

Theorem 2 (Yamasaki–Nomura 2015).

The one-parameter family of non-isomorphic homogeneous cones can be realized in a matrix form as

$$\Omega_\lambda := \left\{ \begin{pmatrix} aI_4 & X_\lambda(\boldsymbol{y}) & \boldsymbol{z} \\ {}^t X_\lambda(\boldsymbol{y}) & bI_2 & \boldsymbol{x} \\ {}^t\boldsymbol{z} & {}^t\boldsymbol{x} & c \end{pmatrix} ; \begin{array}{l} a, b, c \in \mathbb{R} \\ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2 \\ \boldsymbol{z} \in \mathbb{R}^4 \end{array} \right\} \cap \mathcal{S}_7^*,$$

where $X_\lambda(\boldsymbol{y})$ $(\lambda \in [0, 1])$ is defined as

$$X_\lambda(\boldsymbol{y}) := \begin{pmatrix} y_1 & \lambda y_2 \\ y_2 & -\lambda y_1 \\ 0 & \lambda' y_2 \\ 0 & \lambda' y_1 \end{pmatrix} \quad \begin{cases} \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \\ \lambda' := \sqrt{1 - \lambda^2}. \end{cases}$$

Here, $\mathcal{S}_7^* = \{X \in \mathrm{Sym}(7, \mathbb{R});\ X \text{ is positive definite}\}$.

What we consider

In this talk, we shall deal with homogeneous cones of rank 3.

Problem. Let Ω be a homogeneous cone of rank 3.

1. Describe Ω in a matrix form similar to Yamasaki–Nomura.
2. Determine a parameter set Λ of inequivalent classes, e.g. $\Lambda \simeq [0, 1]$ in the case of Yamasaki–Nomura.

Today's topic.

1. Give a method to describe Ω in the matrix form.
2. Give an explicit formula of the parameter sets Λ for lower rank cases.

Notations

- V : a finite-dimensional real vector space,
- Ω: a homogeneous cone of rank 3 in V.
  - One has the normal decomposition of V:

$$V = \begin{pmatrix} \mathbb{R}c_1 & V_{21} & V_{31} \\ V_{21} & \mathbb{R}c_2 & V_{32} \\ V_{31} & V_{32} & \mathbb{R}c_3 \end{pmatrix}, \quad n_{kj} = \dim V_{kj}.$$

  - Then, we call Ω a homogeneous cone of type (n_{32}, n_{21}, n_{31}).

- $N = V_{32} \oplus V_{21} \oplus V_{31}$: the corresponding N-algebra.
- $r, s, n \in \mathbb{N}$: $\quad r \leftrightarrow n_{32}, \quad s \leftrightarrow n_{21}, \quad n \leftrightarrow n_{31}$.

$$\begin{aligned} \mathcal{S}_m^* &:= \{x \in \mathrm{Sym}(m, \mathbb{R});\ x \text{ is positive definite}\}, \\ \overline{\mathcal{S}_m^*} &:= \{x \in \mathrm{Sym}(m, \mathbb{R});\ x \text{ is positive semi-definite}\}, \\ \mathrm{Alt}_m &:= \{X \in \mathrm{Mat}(m, \mathbb{R});\ {}^tX = -X\}. \end{aligned}$$

Isomorphic classes

Homogeneous cones of rank 3

$\updownarrow$ (Vinberg 1963)

N-algebras of rank 3

$\updownarrow$ (Definition of N-algebras)

Bilinear products $\bullet$ s.t. $\|\boldsymbol{x}_{32} \bullet \boldsymbol{x}_{21}\|_{31} = \|\boldsymbol{x}_{32}\|_{32} \times \|\boldsymbol{x}_{21}\|_{21}$

$\updownarrow$ (Fix an ONB of N-algebras, Kaneyuki–Tsuji 1974)

Structure constants of the product $\bullet$ correspond to

$$\left(I_{n_{32}\times n_{21}} + \mathrm{Alt}_{n_{32}} \otimes \mathrm{Alt}_{n_{21}}\right) \cap \overline{\mathcal{S}^*_{n_{32}\times n_{21}}}$$

Analysis of the bilinear product $\bullet$

- $\{\boldsymbol{e}_{kj}^a\}_a$: ONB of V_{kj}
- β_c^{ab}: the structure constants of $\bullet$: $\boldsymbol{e}_{32}^a \bullet \boldsymbol{e}_{21}^b = \sum_{c=1}^{n_{31}} \beta_c^{ab} \boldsymbol{e}_{31}^c$.

The norm condition yields that, if $a \neq a'$ or $b \neq b'$ in (ii),

$$\text{(i)} \ \sum_{c=1}^{n_{31}} \left(\beta_c^{ab}\right)^2 = 1 \quad \text{(ii)} \ \sum_{c=1}^{n_{31}} \left(\beta_c^{ab}\beta_c^{a'b'} + \beta_c^{ab'}\beta_c^{a'b}\right) = 0.$$

- (ij): double indices, put in the lexicographic order
  ($1 \leq i \leq n_{32}$ and $1 \leq j \leq n_{21}$)

The above equations show that $B := (\beta_c^{ab})_{(ab)\times c}$ satisfies

$$B\,{}^tB \in \left(I_{n_{32}\times n_{21}} + \mathrm{Alt}_{n_{32}} \otimes \mathrm{Alt}_{n_{21}}\right) \cap \overline{\mathcal{S}^*_{n_{32}\times n_{21}}}.$$

Analysis of the bilinear product $\bullet$

Define. For $r, s, n \in \mathbb{N}$, we set

$$\mathcal{A}^*(r,s) := \left(I_{r\times s} + \mathrm{Alt}_r \otimes \mathrm{Alt}_s\right) \cap \overline{\mathcal{S}^*_{r\times s}},$$

$$\mathcal{B}(r,s;n) := \left\{B \in \mathrm{Mat}(rs, n;\, \mathbb{R});\ B\,{}^tB \in \mathcal{A}^*(r,s)\right\}.$$

- Ω: homogeneous cone of type (r, s, n)

$\Rightarrow$ $N = V_{32} \oplus V_{21} \oplus V_{31}$: the corresponding N-algebra of rank 3

$\Rightarrow$ $\bullet\colon V_{32} \times V_{21} \to V_{31}$: the product of the N-algebra

$\Rightarrow$ $B = (\beta^{ab}_c)_{(ab)\times c}$: matrix of structure constants w.r.t. $\bullet$

$\Rightarrow$ $B\,{}^tB \in \mathcal{A}^*(r,s) \qquad \Leftrightarrow \qquad B \in \mathcal{B}(r,s;\,n)$

Analysis of the bilinear product $\bullet$

- Conversely, take $B = \begin{pmatrix} B_1 \\ \vdots \\ B_r \end{pmatrix} \in \mathcal{B}(r, s;\, n)$, $B_i \in \mathrm{Mat}(s, n;\, \mathbb{R})$.
- Put
$$X_B(\boldsymbol{y}) := {}^tB \cdot (I_r \otimes \boldsymbol{y}) = \left({}^tB_1\boldsymbol{y}, \dots, {}^tB_r\boldsymbol{y}\right)$$
and define a bilinear map $\bullet\colon \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ by
$$\boldsymbol{x} \bullet \boldsymbol{y} := X_B(\boldsymbol{y})\boldsymbol{x} \quad (\boldsymbol{x} \in \mathbb{R}^r,\ \boldsymbol{y} \in \mathbb{R}^s).$$
  - Namely, $X_B(\boldsymbol{y})$ is a right multiplication operator.
- Then, one has $\|\boldsymbol{x} \bullet \boldsymbol{y}\|_{31} = \|\boldsymbol{x}\|_{32} \times \|\boldsymbol{y}\|_{21}$.

$\Rightarrow \exists\ \Omega\colon$ a homogeneous cone of type (r, s, n).

Matrix realizations

Theorem 3.

Assume that $\mathcal{B}(r,s;\,n) \neq \emptyset$. Associated with $B \in \mathcal{B}(r,s;\,n)$, there exists a homogeneous cone of type (r,s,n) which is isomorphic to

$$\Omega_B := \left\{ X = \begin{pmatrix} aI_n & X_B(\boldsymbol{y}) & \boldsymbol{z} \\ {}^tX_B(\boldsymbol{y}) & bI_r & \boldsymbol{x} \\ {}^t\boldsymbol{z} & {}^t\boldsymbol{x} & c \end{pmatrix} ;\ \begin{array}{l} X \gg 0, \\ a,b,c \in \mathbb{R}, \\ \boldsymbol{x} \in \mathbb{R}^r,\ \boldsymbol{y} \in \mathbb{R}^s \\ \boldsymbol{z} \in \mathbb{R}^n \end{array} \right\},$$

where $X_B(\boldsymbol{y}) = {}^tB \cdot (I_r \otimes \boldsymbol{y}) \in \mathrm{Mat}(n,r;\,\mathbb{R})$.

When are two $B, B' \in \mathcal{B}(r,s;\,n)$ mutually linearly isomorphic?

Linearly isomorphic classes

Let $N = V_{32} \oplus V_{21} \oplus V_{31}$ be an N-algebra of rank 3 with $n_{kj} \geq 1$.

Lemma (Vinberg 1963). Two N-algebras N, N' are isomorphic if and only if one has $n_{kj} = n'_{kj}$ and there exist an algebra isomorphism $k \colon N \to N'$ s.t.

$$k|_{V_{kj}} \colon V_{kj} \longrightarrow V'_{kj} \text{ are norm preserving maps.}$$

Namely, it is sufficient to consider only orthogonal transformations.

Lemma. Let $\bullet$ be the product satisfying the norm condition, and let B, B' be matrices of structure constants of $\bullet$ with respect to ONBs $\{\boldsymbol{e}^a_{kj}\}$ and $\{\boldsymbol{f}^a_{kj}\}$. Then, $\exists k_{kj} \in O(n_{kj})$ s.t.

$$B' = {}^t(k_{32} \otimes k_{21}) B \, k_{31}.$$

Linearly isomorphic classes

Therefore, one has the following

Theorem 4.

Assume that $r, s, n \geq 1$. Two homogeneous cones Ω_B and $\Omega_{B'}$ of type (r, s, n) are linearly isomorphic if and only if

$$\exists k \in O(r),\ k' \in O(s) \text{ and } k'' \in O(n) \text{ s.t. } B' = {}^t(k \otimes k')B\,k''.$$

The parameter set Λ can be described symbolically as

$$\Lambda \simeq (O(r) \times O(s)) \backslash \mathcal{B}(r, s;\, n)/O(n).$$

Linearly isomorphic classes

Theorem 5.

Assume that $r, s, n \geq 1$ and $\mathcal{B}(r, s;\, n) \neq \emptyset$. Then, mutually linearly inequivalent homogeneous cones of type (r, s, n) are realized as

$$\Omega_B := \left\{ X = \begin{pmatrix} aI_n & X_B(\boldsymbol{y}) & \boldsymbol{z} \\ {}^t X_B(\boldsymbol{y}) & bI_r & \boldsymbol{x} \\ {}^t\boldsymbol{z} & {}^t\boldsymbol{x} & c \end{pmatrix} ;\ \begin{array}{l} X \gg 0, \\ a, b, c \in \mathbb{R}, \\ \boldsymbol{x} \in \mathbb{R}^r,\ \boldsymbol{y} \in \mathbb{R}^s \\ \boldsymbol{z} \in \mathbb{R}^n \end{array} \right\},$$

with a parameter $B \in \Lambda \simeq (O(r) \times O(s)) \backslash \mathcal{B}(r, s;\, n)/O(n)$.

Problem. For given r, s, n, determine the parameter set $\Lambda \simeq (O(r) \times O(s)) \backslash \mathcal{B}(r, s;\, n)/O(n)$ concretely.

Parameter sets Λ

Recall that $B \in \mathcal{B}(r,s;\,n)$ satisfies $B\,{}^tB \in \mathcal{A}^*(r,s)$.

1. Consider the orbit decomposition w.r.t the following action:

$$\rho(k_1,k_2)X := {}^t(k_1 \otimes k_2)X(k_1 \otimes k_2),$$

   where $k_1 \in O(r)$, $k_2 \in O(s)$ and $X \in \mathcal{A}^*(r,s)$.

   - Denote the set of representatives by $\mathcal{A}^*(r,s)/(O(r) \times O(s))$.

2. Take X in $\mathcal{A}^*(r,s)/(O(r) \times O(s))$. Then, X is decomposed as

$$X = (Lk)\,{}^t(Lk) \quad \begin{cases} L \text{ is lower triangular,} \\ k \text{ is orthogonal.} \end{cases}$$

   The representative of $(O(r) \times O(s))\backslash\mathcal{B}(r,s;\,n)/O(n)$ can be taken as lower triangular (Diagonal entries may 0 in general).

Parameter sets Λ

- Write $B = \begin{pmatrix} B_{11} & O & \\ B_{21} & B_{22} & \\ \vdots & \vdots & \ddots \\ B_{r1} & B_{r2} & \cdots \end{pmatrix}$, where $B_{kj} \in \mathrm{Mat}(s, \mathbb{R})$.
- Since $B\,{}^tB \in \mathcal{A}^*(r,s)$, there exist $X_{kj} \in \mathrm{Alt}_s$ such that

$$\begin{pmatrix} B_{11}\,{}^tB_{11} & B_{11}\,{}^tB_{21} & \cdots \\ B_{21}\,{}^tB_{11} & \ddots & \\ \vdots & & \ddots \end{pmatrix} = \begin{pmatrix} I_s & -X_{21} & \cdots \\ X_{21} & I_s & \\ \vdots & & \ddots \end{pmatrix} \in \mathcal{A}^*(r,s).$$

- This implies that

$$B_{11}\,{}^tB_{11} = I_s, \quad \text{and} \quad B_{k1}\,{}^tB_{11} = X_{k1} \quad (k = 2, \ldots, r).$$

- Since B_{11} is lower triangular, we have

$$B_{11} = I_s \quad \text{and} \quad B_{k1} \in \mathrm{Alt}_s \quad (k = 2, \ldots, r).$$

Parameter sets Λ

- B is described as $\begin{pmatrix} I_s & O & \cdots \\ B_{21} & B_{22} & \\ \vdots & & \ddots \end{pmatrix}$.
- If we write $X_B(\boldsymbol{y}) = (\widetilde{\boldsymbol{y}}_1, \ldots, \widetilde{\boldsymbol{y}}_r)$, then we have $\widetilde{\boldsymbol{y}}_1 = \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{0} \end{pmatrix}$.

Namely, $X_B(\boldsymbol{y})$ is of the form

$$
{}^tX_B(\boldsymbol{y}) = \begin{pmatrix} y_1 & \cdots & y_s & 0 & \cdots & 0 \\ * & \cdots & * & \cdots & \cdots & \cdots \\ * & \cdots & * & \cdots & \cdots & \cdots \end{pmatrix}.
$$

We cannot derive more information in a general scheme, and so in what follows we shall consider lower dimensional cases.

Settings

Associated with $B \in \mathcal{B}(r,s;\,n)$, we have a homogeneous cone of type (r,s,n) isomorphic to

$$\Omega_B := \left\{ X = \begin{pmatrix} aI_n & X_B(\boldsymbol{y}) & \boldsymbol{z} \\ {}^t X_B(\boldsymbol{y}) & bI_r & \boldsymbol{x} \\ {}^t\boldsymbol{z} & {}^t\boldsymbol{x} & c \end{pmatrix};\ \begin{array}{l} X \gg 0, \\ a,b,c \in \mathbb{R}, \\ \boldsymbol{x} \in \mathbb{R}^r,\ \boldsymbol{y} \in \mathbb{R}^s \\ \boldsymbol{z} \in \mathbb{R}^n \end{array} \right\},$$

For given triplet (r,s,n), we consider the following.

1. Determine the parameter set Λ;

$$\Lambda \simeq (O(r) \times O(s)) \backslash \mathcal{B}(r,s;\,n)/O(n).$$

2. Describe $X_B(\boldsymbol{y})$ explicitly for $B \in \Lambda$.

$(r, s, n) = (2, 2, 4)$

The case of Yamasaki–Nomura

- Set $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The space $\mathcal{A}^*(2,2)$ is given as

$$\begin{aligned} \mathcal{A}^*(2,2) &= \left\{ X_\lambda = \begin{pmatrix} I_2 & -\lambda J \\ \lambda J & I_2 \end{pmatrix} ;\ X_\lambda \text{ is positive semi-definite} \right\} \\ &= \left\{ X_\lambda = \begin{pmatrix} I_2 & -\lambda J \\ \lambda J & I_2 \end{pmatrix} ;\ \lambda \in [-1, 1] \right\}. \end{aligned}$$

- Since $g_{X_\lambda}(t) = \left((t-1)^2 - \lambda^2\right)^2$, the eigenvalues are $1 \pm \lambda$.
- The action of $O(2)$ on Alt_2 is equivalent to $\{\pm 1\}$ on λ, so that

$$\mathcal{A}^*(2,2)/(O(2) \times O(2)) = \left\{ X_\lambda = \begin{pmatrix} I_2 & -\lambda J \\ \lambda J & I_2 \end{pmatrix} ;\ \lambda \in [0, 1] \right\}.$$

$(r, s, n) = (2, 2, 4)$

The case of Yamasaki–Nomura

- Set $B = \begin{pmatrix} I_2 & O \\ aJ & L \end{pmatrix}$, $L \in \mathrm{Mat}(2, \mathbb{R})$: lower triangular.
- Take $X_\lambda \in \mathcal{A}^*(2,2)/(O(2) \times O(2)) \quad (\lambda \in [0,1])$.

$$B\,{}^tB = X_\lambda \iff \begin{pmatrix} I_2 & -aJ \\ aJ & a^2 I_2 + L\,{}^tL \end{pmatrix} = \begin{pmatrix} I_2 & -\lambda J \\ \lambda J & I_2 \end{pmatrix}.$$

- Therefore, one has $a = \lambda$ and since L is lower triangular,

$$L\,{}^tL = (1-\lambda^2) I_2 \implies L = \begin{pmatrix} \sqrt{1-\lambda^2} & 0 \\ 0 & \sqrt{1-\lambda^2} \end{pmatrix}.$$

- We have $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\lambda & \lambda' & 0 \\ \lambda & 0 & 0 & \lambda' \end{pmatrix}$ where $\lambda' := \sqrt{1-\lambda^2}$.

$(r, s, n) = (2, 2, 4)$

The case of Yamasaki–Nomura

- We have $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\lambda & \lambda' & 0 \\ \lambda & 0 & 0 & \lambda' \end{pmatrix}$.
- Then, $X_B(\boldsymbol{y}) = {}^tB \cdot (I_2 \otimes \boldsymbol{y})$ is calculated as

$$X_B(\boldsymbol{y}) = \begin{pmatrix} 1 & 0 & 0 & \lambda \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & \lambda' & 0 \\ 0 & 0 & 0 & \lambda' \end{pmatrix} \begin{pmatrix} y_1 & 0 \\ y_2 & 0 \\ 0 & y_1 \\ 0 & y_2 \end{pmatrix} = \begin{pmatrix} y_1 & \lambda y_2 \\ y_2 & -\lambda y_1 \\ 0 & \lambda' y_1 \\ 0 & \lambda' y_2 \end{pmatrix}.$$

- In Yamasaki–Nomura, $X_\lambda(\boldsymbol{y})$ is given as

$$X_\lambda(\boldsymbol{y}) = \begin{pmatrix} y_1 & \lambda y_2 \\ y_2 & -\lambda y_1 \\ 0 & \lambda' y_2 \\ 0 & \lambda' y_1 \end{pmatrix} \because \text{they take } B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\lambda & 0 & \lambda' \\ \lambda & 0 & \lambda' & 0 \end{pmatrix}.$$

$(r, s, n) = (2, 2k, 4k)$

For $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$, we set $\boldsymbol{\lambda}' := \left(\sqrt{1-\lambda_1^2}, \ldots, \sqrt{1-\lambda_k^2}\right)$ and

$$d_{\boldsymbol{\lambda}} := \mathrm{diag}(\lambda_1, \ldots, \lambda_k), \quad J_{\boldsymbol{\lambda}} = \begin{pmatrix} O & -d_{\boldsymbol{\lambda}} \\ d_{\boldsymbol{\lambda}} & O \end{pmatrix}.$$

Proposition 6.

In this case, we have, for $\boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{pmatrix} \in \mathbb{R}^{2k}$,

$$\Lambda = \left\{\boldsymbol{\lambda} \in \mathbb{R}^k;\ 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq 1\right\},$$

$$X_B(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{y}_1 & d_{\boldsymbol{\lambda}}\boldsymbol{y}_2 \\ \boldsymbol{y}_2 & -d_{\boldsymbol{\lambda}}\boldsymbol{y}_1 \\ \mathbf{0} & d_{\boldsymbol{\lambda}'}\boldsymbol{y}_1 \\ \mathbf{0} & d_{\boldsymbol{\lambda}'}\boldsymbol{y}_2 \end{pmatrix} \in \mathrm{Mat}(4k, 2; \mathbb{R}).$$

$(r, s, n) = (2, 2k, 4k)$

- The space $(I_{4k} + \mathrm{Alt}_2 \otimes \mathrm{Alt}_{2k})/(O(2) \times O(2k))$ is given as
$$\left\{ X = \begin{pmatrix} I_{2k} & -J_{\boldsymbol{\lambda}} \\ J_{\boldsymbol{\lambda}} & I_{2k} \end{pmatrix};\ 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \right\}.$$
  - Thus, we have $\Lambda = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^k;\ 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq 1 \right\}$.
- Let $B = \begin{pmatrix} I_{2k} & O \\ J_{\boldsymbol{\lambda}} & L \end{pmatrix}$ $\quad L \in \mathrm{Mat}(2k, \mathbb{R})$: lower triangular.
- Take $X_{\boldsymbol{\lambda}} \in \mathcal{A}^*(2, 2k)/(O(2) \times O(2k)) \quad (\boldsymbol{\lambda} \in \Lambda)$.
$$B\,{}^tB = X_{\boldsymbol{\lambda}} \quad \Longleftrightarrow \quad \begin{pmatrix} I_{2k} & -J_{\boldsymbol{\lambda}} \\ J_{\boldsymbol{\lambda}} & L\,{}^tL - J_{\boldsymbol{\lambda}}^2 \end{pmatrix} = \begin{pmatrix} I_{2k} & -J_{\boldsymbol{\lambda}} \\ J_{\boldsymbol{\lambda}} & I_{2k} \end{pmatrix}.$$
- Therefore, one has $L = \begin{pmatrix} d_{\boldsymbol{\lambda}'} & O \\ O & d_{\boldsymbol{\lambda}'} \end{pmatrix}$.

$(r, s, n) = (2, 2k, 4k)$

This shows that $B = \begin{pmatrix} I_k & O & O & O \\ O & I_k & O & O \\ O & -d_{\boldsymbol{\lambda}} & d_{\boldsymbol{\lambda}'} & O \\ d_{\boldsymbol{\lambda}} & O & O & d_{\boldsymbol{\lambda}'} \end{pmatrix}$, and hence we have

$$X_B(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{y}_1 & d_{\boldsymbol{\lambda}}\boldsymbol{y}_2 \\ \boldsymbol{y}_2 & -d_{\boldsymbol{\lambda}}\boldsymbol{y}_1 \\ \mathbf{0} & d_{\boldsymbol{\lambda}'}\boldsymbol{y}_1 \\ \mathbf{0} & d_{\boldsymbol{\lambda}'}\boldsymbol{y}_2 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{pmatrix} \in \mathbb{R}^{2k}.$$

We can give an explicit description for any triplet $(2, s, n)$.

$(r, s, n) = (3, 3, n)$

Proposition 7.

In this case, we have

$$\Lambda = \left\{ \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3);\ \begin{array}{l} 0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le 1, \\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\lambda_1\lambda_2\lambda_3 \le 1 \end{array} \right\}$$

and ${}^tX_{\boldsymbol{\lambda}}(\boldsymbol{y})$ is of the form

$$\begin{pmatrix} y_1 & y_2 & y_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 y_3 & -\lambda_1 y_2 & y_1 & \lambda_1' y_2 & \lambda_1' y_3 & 0 & 0 & 0 \\ -\lambda_2 y_3 & 0 & \lambda_2 y_1 & \lambda_3 y_2 & -\gamma_{\boldsymbol{\lambda}} y_1 & 0 & \delta_{\boldsymbol{\lambda}} y_1 & \lambda_3' y_2 & \lambda_2' y_3 \end{pmatrix}.$$

Here, if $\lambda_1 = 1$ then $\gamma_{\boldsymbol{\lambda}} = \delta_{\boldsymbol{\lambda}} = 0$, and otherwise one has

$$\gamma_{\boldsymbol{\lambda}} = \frac{\lambda_3 - \lambda_1\lambda_2}{\sqrt{1-\lambda_1^2}}, \qquad \delta_{\boldsymbol{\lambda}} = \frac{\sqrt{1-\lambda_1^2-\lambda_2^2-\lambda_3^2+2\lambda_1\lambda_2\lambda_3}}{\sqrt{1-\lambda_1^2}}.$$

$(r, s, n) = (3, 3, n)$

- Put $G = O(3) \times O(3)$ and identify $\mathrm{Alt}_3 \otimes \mathrm{Alt}_3$ with $\mathrm{Mat}(3, \mathbb{R})$.
- We consider an action of G on $\mathrm{Mat}(3, \mathbb{R})$ defined by

$$\kappa(k_1, k_2)M := {}^t k_1 M k_2 \quad \begin{cases} (k_1, k_2) \in G, \\ M \in \mathrm{Mat}(3, \mathbb{R}). \end{cases}$$

Put $D := \{\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3;\ 0 \le \lambda_1 \le \lambda_2 \le \lambda_3\}$.
The G-orbit of $\mathrm{Mat}(3, \mathbb{R})$ w.r.t. κ is given as

$$\mathrm{Mat}(3, \mathbb{R}) = \bigsqcup_{\boldsymbol{\lambda} \in D} \kappa(G) \begin{pmatrix} 0 & 0 & \lambda_3 \\ 0 & \lambda_2 & 0 \\ \lambda_1 & 0 & 0 \end{pmatrix} \quad \text{(disjoint union).}$$

- The action κ is equivalent to the action of G on $\mathcal{A}(3, 3)$.

cf. $\quad {}^t k \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} k \longleftrightarrow {}^t k \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad k \in O(3).$

$(r, s, n) = (3, 3, n)$

Let us consider the element $X_{\boldsymbol{\lambda}} \in \mathcal{A}^*(3,3)/(O(3) \times O(3))$ defined by

$$X_{\boldsymbol{\lambda}} = \begin{pmatrix} I_3 & -\lambda_1 \mathcal{X} & \lambda_2 \mathcal{Y} \\ \lambda_1 \mathcal{X} & I_3 & -\lambda_3 \mathcal{Z} \\ -\lambda_2 \mathcal{Y} & \lambda_3 \mathcal{Z} & I_3 \end{pmatrix}.$$

Here, $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are alternative matrices defined by

$$\mathcal{X} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{Y} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{Z} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $X_{\boldsymbol{\lambda}}$ corresponds to $\begin{pmatrix} 0 & 0 & \lambda_3 \\ 0 & \lambda_2 & 0 \\ \lambda_1 & 0 & 0 \end{pmatrix}$.

$(r, s, n) = (3, 3, n)$

- The characteristic polynomial of $X_{\boldsymbol{\lambda}}$ is calculated as

$$((t-1)^2 - \lambda_1^2)^2 \times ((t-1)^2 - \lambda_2^2)^2 \times ((t-1)^2 - \lambda_3^2)^2$$
$$\times \Big((t-1)^3 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(t-1) - 2\lambda_1\lambda_2\lambda_3\Big).$$

- Set $f_{\boldsymbol{\lambda}}(t) := (t-1)^3 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(t-1) - 2\lambda_1\lambda_2\lambda_3$.
- We note that

$$f_{\boldsymbol{\lambda}}(t) = \det\left(tI_3 - \begin{pmatrix} 1 & \lambda_1 & \lambda_2 \\ \lambda_1 & 1 & \lambda_3 \\ \lambda_2 & \lambda_3 & 1 \end{pmatrix}\right).$$

- The eigenvalues are $1 \pm \lambda_i$ $(i = 1, 2, 3)$ and roots of $f_{\boldsymbol{\lambda}}(t)$.
- Since $X_{\boldsymbol{\lambda}}$ is positive semi-definite, one has

$$\lambda_i \in [-1, 1] \quad (i = 1, 2, 3).$$

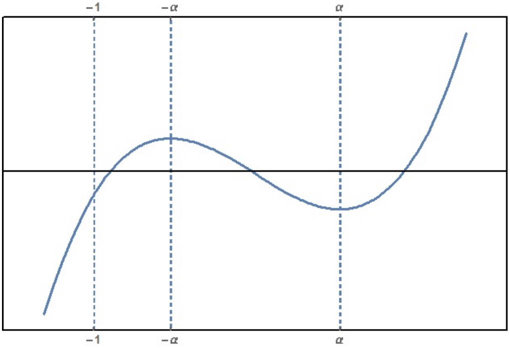
$(r, s, n) = (3, 3, n)$

We investigate roots of $f_{\boldsymbol{\lambda}}(t)$.

- Set $g_{\boldsymbol{\lambda}}(x) := f_{\boldsymbol{\lambda}}(x+1) = x^3 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)x - 2\lambda_1\lambda_2\lambda_3$.
- Obviously, we have

$$X_{\boldsymbol{\lambda}} \text{ is positive semi-definite} \iff (\text{all roots of } g_{\boldsymbol{\lambda}}(x)) \geq -1.$$

- Let α be the positive root of $g'_{\boldsymbol{\lambda}}(x)$.



(i) $g_{\boldsymbol{\lambda}}(-1) \leq 0$,

(ii) $g_{\boldsymbol{\lambda}}(\alpha) \leq 0$,

(iii) $g_{\boldsymbol{\lambda}}(-\alpha) \geq 0$

(iv) $\alpha \leq 1$.

$(r,s,n) = (3,3,n)$

$g_{\boldsymbol{\lambda}}(x) = x^3 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)x - 2\lambda_1\lambda_2\lambda_3.$

- The condition (i) $g_{\boldsymbol{\lambda}}(-1) \le 0$ implies that
$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\lambda_1\lambda_2\lambda_3 \le 1.$$
- The conditions (ii) $g_{\boldsymbol{\lambda}}(\alpha) \le 0$ and (iii) $g_{\boldsymbol{\lambda}}(-\alpha) \ge 0$:
$$\alpha = \frac{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}{\sqrt{3}}, \quad g(\varepsilon\alpha) = -2\varepsilon(\alpha^3 + \lambda_1\lambda_2\lambda_3),$$
where $\varepsilon = \pm 1$. Namely, the condition (iv) $\alpha \le 1$ always holds.
- By the inequality of arithmetic and geometric means, we have
$$\alpha^3 = \left(\frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{3}\right)^{\frac{3}{2}} \ge |\lambda_1\lambda_2\lambda_3|,$$
which implies two conditions (ii) and (iii) are always satisfied.

$(r, s, n) = (3, 3, n)$

Therefore, we obtain

$$\Lambda := \left\{ \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3);\ \begin{array}{l} 0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le 1, \\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\lambda_1\lambda_2\lambda_3 \le 1 \end{array} \right\}.$$

A simple calculation yields that ${}^tX_{\boldsymbol{\lambda}}(\boldsymbol{y})$ is of the form

$$\begin{pmatrix} y_1 & y_2 & y_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 y_3 & -\lambda_1 y_2 & y_1 & \lambda_1' y_2 & \lambda_1' y_3 & 0 & 0 & 0 \\ -\lambda_2 y_3 & 0 & \lambda_2 y_1 & \lambda_3 y_2 & -\gamma_{\boldsymbol{\lambda}} y_1 & 0 & \delta_{\boldsymbol{\lambda}} y_1 & \lambda_3' y_2 & \lambda_2' y_3 \end{pmatrix}.$$

Here, if $\lambda_1 = 1$ then $\gamma_{\boldsymbol{\lambda}} = \delta_{\boldsymbol{\lambda}} = 0$, and otherwise one has

$$\gamma_{\boldsymbol{\lambda}} = \frac{\lambda_3 - \lambda_1\lambda_2}{\sqrt{1-\lambda_1^2}}, \qquad \delta_{\boldsymbol{\lambda}} = \frac{\sqrt{1-\lambda_1^2-\lambda_2^2-\lambda_3^2+2\lambda_1\lambda_2\lambda_3}}{\sqrt{1-\lambda_1^2}}.$$

$(r, s, n) = (3, 3, n)$

Proposition 8.

Put $\Lambda' = \{\boldsymbol{\lambda};\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\lambda_1\lambda_2\lambda_3 = 1\}$.

One has $$\operatorname{rank} X_{\boldsymbol{\lambda}} = \begin{cases} 4 & (\boldsymbol{\lambda} = (1,1,1)) \\ 8 & (\boldsymbol{\lambda} \in \Lambda' \setminus \{(1,1,1)\}) \\ 9 & (\boldsymbol{\lambda} \in \Lambda \setminus \Lambda') \end{cases}$$

Note that ${}^t X_{\boldsymbol{\lambda}}(\boldsymbol{y})$ is of the form

$$\begin{pmatrix} y_1 & y_2 & y_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 y_3 & -\lambda_1 y_2 & y_1 & \lambda_1' y_2 & \lambda_1' y_3 & 0 & 0 & 0 \\ -\lambda_2 y_3 & 0 & \lambda_2 y_1 & \lambda_3 y_2 & -\gamma_{\boldsymbol{\lambda}} y_1 & 0 & \delta_{\boldsymbol{\lambda}} y_1 & \lambda_3' y_2 & \lambda_2' y_3 \end{pmatrix}.$$

$(r, s, n) = (3, 4, n)$

In this case, we do not have a complete answer yet.

Set $G = O(3) \times O(4)$ and identify $\mathrm{Alt}_3 \otimes \mathrm{Alt}_4$ with $\mathrm{Mat}(6, 3; \mathbb{R})$. We consider the action τ of G on $\mathrm{Mat}(6, 3; \mathbb{R})$ defined by

$$\tau(k_1, k_2)M := \rho_4(k_1)Mk_2 \quad \begin{cases} (k_1, k_2) \in G, \\ M \in \mathrm{Mat}(6, 3; \mathbb{R}), \end{cases}$$

where ρ_4 is the action of $O(4)$ on Alt_4 :

$$\rho_4(k)X := {}^t kXk \quad (k \in O(4),\ X \in \mathrm{Alt}_4).$$

Since the Dynkin diagram of $\mathfrak{so}(4, \mathbb{R}) \simeq \mathrm{Alt}_4$ is ◯ ◯, $\mathfrak{so}(4, \mathbb{R})$ can be decomposed into two 3-dimensional ideals.

$(r, s, n) = (3, 4, n)$

$\mathbb{H}$: the ring of quaternions, $\{1, i, j, k\}$: its standard basis.
We identify $\mathbb{H}$ with $\mathbb{R}^4$ through the standard basis.

$$L_\alpha w := \alpha w, \quad R_\alpha w := w\alpha \quad (\alpha, w \in \mathbb{H}).$$

Lemma (cf. Helgason 1978).
The following set form a basis of $\mathrm{so}(4, \mathbb{R}) \simeq \mathrm{Alt}_4$:

$$\{L_i,\ L_j,\ L_k,\ {}^tR_i,\ {}^tR_j,\ {}^tR_k\}.$$

Moreover, the following two subspaces are ideals of $\mathrm{so}(4, \mathbb{R})$:

$$\mathrm{Span}(L_i, L_j, L_j), \quad \mathrm{Span}({}^tR_i, {}^tR_j, {}^tR_k).$$

These are both isomorphic to $\mathfrak{so}(3, \mathbb{R})$.

$(r, s, n) = (3, 4, n)$

- We take this basis of Alt_4.
- The action of $O(4)$ on Alt_4 is described by $k_1, k_2 \in O(3)$ as

$$\begin{pmatrix} {}^t k_1 & O \\ O & {}^t k_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} \in \mathbb{R}^6 \simeq \mathrm{Alt}_4.$$

- The representative of the action of G on $\mathrm{Mat}(6, 3; \mathbb{R})$ is

$$\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \\ b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix}.$$

$(r, s, n) = (3, 4, n)$

We consider for the case

$$M = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} \quad \begin{cases} D_1 = \operatorname{diag}(a_1, a_2, a_3), \\ D_2 = \operatorname{diag}(b_1, b_2, b_3). \end{cases}$$

Namely, we consider the following element in $I_{12} + \mathrm{Alt}_3 \otimes \mathrm{Alt}_4$:

$$X_{\boldsymbol{\lambda}} = \begin{pmatrix} I_4 & a_3\,{}^tL_k + b_3 R_k & a_2 L_j + b_2\,{}^tR_j \\ a_3 L_k + b_3\,{}^tR_k & I_4 & a_1\,{}^tL_i + b_1 R_i \\ a_2\,{}^tL_j + b_2 R_j & a_1 L_i + b_1\,{}^tR_i & I_4 \end{pmatrix}.$$

In this case, we can calculate its characteristic polynomial $g_{X_{\boldsymbol{\lambda}}}(t)$ explicitly.

$(r, s, n) = (3, 4, n)$

- Put $P[\alpha, \beta, \gamma](x) := x^3 - (\alpha^2 + \beta^2 + \gamma^2)x - 2\alpha\beta\gamma$.
- $g_{X_{\boldsymbol{\lambda}}}(t)$ is factorized into the following four polynomials:

$$P[a_1 + b_1, a_2 - b_2, a_3 - b_3](t-1),$$
$$P[a_1 - b_1, a_2 + b_2, a_3 - b_3](t-1),$$
$$P[a_1 - b_1, a_2 - b_2, a_3 + b_3](t-1),$$
$$P[a_1 + b_1, a_2 + b_2, a_3 + b_3](t-1).$$

- If we set $b_1 = b_2 = b_3 = 0$, then one has

$$g_{X_{\boldsymbol{\lambda}}}(t) = \left((t-1)^3 - (a_1^2 + a_2^2 + a_3^2)(t-1) - 2a_1a_2a_3\right)^4.$$

- If D_2 is not diagonal, then the characteristic polynomial of $X_{\boldsymbol{\lambda}}$ can not be factored in general.

$(r, s, n) = (3, 4, n)$

Set $\Lambda := \left\{ (a_1, a_2, a_3);\ \begin{array}{l} 0 \leq a_3 \leq a_2 \leq a_1 \leq 1, \\ a_1^2 + a_2^2 + a_3^2 - 2a_1a_2a_3 \leq 1 \end{array} \right\}$.

Proposition 9. For $\boldsymbol{\lambda} \in \Lambda$, one has

$$X_{\boldsymbol{\lambda}}(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{y} & a_3\,{}^tL_k\boldsymbol{y} & a_2 L_j \boldsymbol{y} \\ \mathbf{0} & a_3'\boldsymbol{y} & \gamma_{\boldsymbol{\lambda}}\,{}^tL_i\boldsymbol{y} \\ \mathbf{0} & \mathbf{0} & \delta_{\boldsymbol{\lambda}}\boldsymbol{y} \end{pmatrix} \quad (\boldsymbol{y} \in \mathbb{R}^4),$$

where $\gamma_{\boldsymbol{\lambda}}$ and $\delta_{\boldsymbol{\lambda}}$ are constants given as

$$\gamma_{\boldsymbol{\lambda}} = \frac{a_1 - a_2a_3}{\sqrt{1-a_3^2}}, \quad \delta_{\boldsymbol{\lambda}} = \frac{\sqrt{1-(a_1^2+a_2^2+a_3^2)+2a_1a_2a_3}}{\sqrt{1-a_3^2}}.$$

$(r, s, n) = (3, 4, n)$

- One can calculate similarly for general $a_1, a_2, a_3, b_1, b_2, b_3$.

$$X_{\boldsymbol{\lambda}}(\boldsymbol{y}) = \begin{pmatrix} y_1 & \beta_3 y_4 & -\beta_2 y_3 \\ y_2 & \alpha_3 y_3 & \alpha_2 y_4 \\ y_3 & -\alpha_3 y_2 & \beta_2 y_1 \\ y_4 & -\beta_3 y_1 & -\alpha_2 y_2 \\ 0 & \beta_3' y_1 & \gamma_{\beta_1 \alpha_2 \beta_3} y_2 \\ 0 & \alpha_3' y_2 & -\gamma_{\beta_1 \beta_2 \alpha_3} y_1 \\ 0 & \alpha_3' y_3 & \gamma_{\alpha_1 \alpha_2 \alpha_3} y_4 \\ 0 & \beta_3' y_4 & -\gamma_{\alpha_1 \alpha_2 \beta_3} y_3 \\ 0 & 0 & \delta_{\beta_1 \beta_2 \alpha_3} y_1 \\ 0 & 0 & \delta_{\beta_1 \alpha_2 \beta_3} y_2 \\ 0 & 0 & \delta_{\alpha_1 \beta_2 \beta_3} y_3 \\ 0 & 0 & \delta_{\alpha_1 \alpha_2 \alpha_3} y_4 \end{pmatrix}$$

$$\alpha_i = a_i + b_i$$
$$\beta_i = a_i - b_i$$

$$(\alpha_1, \alpha_2, \alpha_3) \in \Lambda,$$
$$(\alpha_1, \beta_2, \beta_3) \in \Lambda,$$
$$(\beta_1, \alpha_2, \beta_3) \in \Lambda,$$
$$(\beta_1, \beta_2, \alpha_3) \in \Lambda$$

$$\gamma_{abc} := \frac{a - bc}{\sqrt{1 - c^2}}, \quad \delta_{abc} := \frac{\sqrt{1 - (a^2 + b^2 + c^2) + 2abc}}{\sqrt{1 - c^2}}$$

Future works

- Complete calculation for the case $(3, 4, n)$.
- Explore a geometric interpretation of Λ.
- For general (r, s, n).
- Consider higher-rank homogeneous cones.
- Determine the best possibility n for the triplet $(16, 16, n)$.
- Change coefficient fields (rings). What happens?