The Enemy of My Enemy Is My Friend: A New Condition for Stable Networks *

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Abstract

This paper examines if an ancient principle, "the enemy of my enemy is my friend," is a good predictor of group formation. I model coalition formation as a static network formation game with complementarities between a pair of adjacent nodes. I demonstrate that the ancient proverb is indeed a sufficient condition for the existence of a stable network that is also efficient.

Keywords: Network, Matching with Complementarities, Core

JEL Codes: C62, C68, C71, C78, D44, D47, D50

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The king who is situated anywhere immediately on the circumference of the conqueror's territory is termed the enemy. The king who is likewise situated close to the enemy, but separated from the conqueror only by the enemy, is termed the friend [of the conqueror].—Kautilya, Arthasastra

1 Introduction

It is not hard to find in a daily life an example in which two people get along with each other *and* have a common enemy. This state of a relation[1] is equivalent to an ancient principle, "the enemy of my enemy is my friend." It is empirically known that in addition to the state in which everyone is friends with each other, this state of a relation is most commonly observed in social networks such as individual human relations on massive online game experiments (Szell et al. (2010)), international relations (Maoz et al. (2007)), inter-gang violence (Nakamura et al. (2019)), trust/distrust networks among the users of a product review website (Facchetti et al. (2011)), friend/foes networks of a technological news site (Facchetti et al. (2011)), and elections of the Wikipedia administrators (Facchetti et al. (2011)).

The intention of this paper is to theoretically examine if the ancient proverb is a good predictor of stable networks. Note that the paper does not intend to provide any *normative* arguments for such relations. For example, historically, the West cooperated with Hitler, Mussolini, and Franco when its enemy of the 1930s was Stalin (Saperstein (2004)). Therefore, such a condition for stability does not justify any normative arguments for peace. Meanwhile,

[1] In this paper, I will use "relation" instead of "relationship" since the settings of interests in this paper include large groups of people or entities such as international relations and companies. See `https://www.clarkandmiller.com/relation-vs-relationship-difference/`

Maoz et al. (2007) empirically find that while there are many exceptions, enemies of enemies are three times more likely to become allies than by random chance. In this paper, my intention is to theoretically provide a *positive* argument for the prediction of social networks at an equilibrium.

I model the formation of social networks as a static network formation game similar to the canonical one from Jackson and Wolinsky (1996). The main difference from Jackson and Wolinsky (1996) is the addition of pairwise complementarities/substitutabilities between adjacency nodes to an agent's preferences. That is, when agent A thinks about forming relations with agent B and C, agent A cares about, following the terminology from Jackson and Wolinsky (1996), the *intrinsic* relation between agent B and C even if agent B and C do not have a formal relation. For instance, it was only until the South-North Korean relation improved without a formal relation in 2000 that England made a formal relation with North Korea.

In my model, if agent A is thinking to become an ally/friend of agent B and C, agent A welcomes a positive intrinsic relation between B and C while depreciates a negative intrinsic relation between B and C. I assume such complementarities can be broken down to pairwise addition. For instance, suppose that agent D is considering forming relations with A, B, and C. In addition to the benefits from forming a relation individually with these agents, agent D cares about pairwise (intrinsic) relations between A and B, A and C, and B and C, but *not* a group-wise relation among A, B, and C.

I call such preferences with additively separable pairwise complementarities as *binary quadratic programs* (BQP) preferences. The restriction to BQP preferences allows us to explicitly express parameters on the benefits/damages from indirect relations with those to whom their adjacent nodes are linked, and consequently allows us to find how to group agents in such a way to prevent a blocking coalition.

A BQP preference is a natural way to incorporate pairwise complementarities as it is often used in the auction literature. Ausubel et al. (1997), for example, uses a form of BQP preferences in spectrum auction settings with pairwise complementarities, while Bertsimas et al. (1999), Candogan et al. (2015), and Candogan et al. (2018) deal with combinatorial auction problems. However, the issue of complementarities through indirect connections is not well studied in the network formation context. Therefore, I believe that this paper provides a new class of preferences that are understudied but particularly relevant to the network formation literature.

Meanwhile, I admit the limitation of BQP preferences. One major limitation is that BQP preferences do not capture the influence of group-wise complementarities. For instance, one agent may check if an agent belongs to a group such as the Association of Southeast Asian Nations (ASEAN) while considering forming a relation with the agent. This study does not incorporate the effects of such groups.

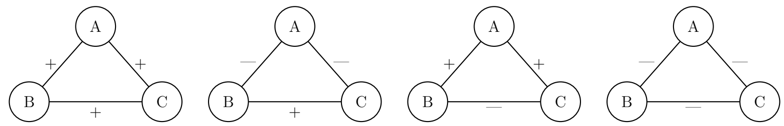
With the BQP preferences, there are two main conditions for the existence of a stable network. One is that all agents agree on which pair of agents have a good (bad) relation, which I call the *sign-consistency* condition. Second, a graph of agents features the principle that the enemy of my enemy is my friend (and the friend of my friend is my friend). Mathematically, this principle is translated into the condition that the graph can be partitioned into a pair of subgraphs in which each of the subgraphs consists of positive edges but the two subgraphs are connected by negative edges. I call this condition the *sign-balance* condition. I prove that these conditions are sufficient conditions for the existence of a stable network that is also efficient.

Note that the concept of stability in networks is similar to Ostrovsky (2008). The concept is not strategic, and I do not analyze the dynamics of network formation nor "what-if" scenarios considered by agents who consider temporarily forming a coalition in the hopes of influencing the entire network in a way beneficial to them. Rather, the concept is "closer in spirit to general equilibrium models, where agents perceive conditions surrounding them as given, and optimize given those conditions" (Ostrovsky (2008), p.899). To solve the model with complementarities, I exploit linear programming techniques from the auction and matching literature that involve complementarities.

While it tends to be hard to give intuitive mathematical reasoning for linear programming proof approaches, the intuition behind my main result is chewable and relatable based on experiences. Consider the four possible triads among agent A, B, and C, as depicted in Figure 1. Plus means a positive intrinsic relation, while minus indicates a negative intrinsic relation. The case with all positives (sign-balanced) or all negatives (sign-unbalanced) is easy to solve; everyone forms a relation with each other in the former, while no one forms a relation with each other in the latter. The second one from the left is the sign-balanced case with a common enemy. It is easy to imagine the stability of a formal relation between B and C; B and C will not form a relation with A and do not incur any damage from indirect relations. Now, consider the third triad with two positives and one negative case that is sign-unbalanced. I believe that based on our experiences, it is not hard to predict the instability of this relationship; if you have two friends who do not get along with each other, you know that things can get awkward.

Meanwhile, the third case can be stable depending on the magnitude of the benefits from

Figure 1: Examples of a cycle in balanced and unbalanced graphs



The two graphs at the left are balanced, while the other graphs are unbalanced

direct formal relations and damages from indirect intrinsic relations. If the benefits from direct formal relations are larger than the damages from indirect intrinsic relations, then the network of interests may be stable. However, it is hard to precisely pin down such conditions. Consider a case in which an agent who has 99 friends is trying to decide if she wants to form a relation with another agent who has negative intrinsic relations with all the 99 friends. Even if each of these negative intrinsic values is smaller than the positive direct value of forming a friendship with the 100th agent, the sum of all the negative values can dominate the positive direct value.

The contribution of this paper is two-fold. One is that it theoretically proves that the ancient principle is indeed a good predictor of social network formation with complementarities. Note that the principle is only a sufficient condition for existence; there can be a stable network that does not feature the principle. The other is that it contributes to the network formation and matching literature[2] that provides an understanding of how restrictions on preferences over complementarities/substitutabilities ensure the existence of a stable network/matching[3].

[2] Note that the strategic complementarity literature usually focuses on continuous divisible activities while the matching literature usually studies discrete indivisible activities.

[3] In addition to the aforementioned network and auction literature, for the matching literature, see, for example, Kelso and Crawford (1982), Roth (1984), Hatfield and Milgrom (2005), Bikhchandani and Mamer (1997), Gul et al. (1999), Gul et al. (2000), Milgrom (2000), Bikhchandani and Ostroy (2002), Bikhchandani et al. (2002), Demange (2004), Ausubel et al. (2006), Sun and Yang (2006), Sun and Yang (2009), Ostrovsky (2008), Echenique and Oviedo (2006), Pycia (2012), Hatfield et al. (2013), Kojima et al. (2013), Azevedo et al. (2013), Sun and Yang (2014), Azevedo and Hatfield (2015), Che et al. (2019).

1.1 Related Literature

In particular, the closest existing condition to mine is the gross substitutes and complements (GSC) condition suggested by Sun and Yang (2006)[4]. The GSC condition allows for preference structures more general than the preferences in this paper and is satisfied if goods can be divided into two groups, and within groups, goods are gross substitutes, and across groups, goods are gross complements[5]. However, in general, the condition would imply that a friend of my friend has to be my enemy, which seems less plausible than mine in settings that involve humans (including countries and institutions). Whether in school, the workplace, a community, a country, or the globe, we tend to see that one joins a group of people with whom he or she finds comfortable, and across these groups, people fight.

Another set of papers close to mine are Candogan et al. (2015), Candogan et al. (2018), Nguyen and Vohra (2018), and Baldwin and Klemperer (2019). Nguyen and Vohra (2018) study two-sided matching markets with couples in the presence of capacity constraints, while allowing for general preferences in the context of nontransferable utility. In the presence of complementarities coupled with capacity constraints, their setting inevitably encounters potential emptiness of the core set, and they overcome it by (possibly) minimally perturbing the capacity constraints. While their findings and algorithm are extremely powerful, the direct application of their results to network formation is not immediate due to the lack of capacity constraints in network formation problems.

Baldwin and Klemperer (2019) provide a novel and powerful characterization of classes of valuations that result in Walrasian equilibria. In the sphere of their demand types, they

[4] The same-side substitutability and cross-side complementarity from Ostrovsky (2008) is similar.

[5] Note that while the GSC condition may sound *qualitatively* opposite to our condition, precisely speaking, it is not mathematically opposite in the class of preferences with additively separable pairwise complementarities.

provide necessary and sufficient conditions for such an equilibrium to exist. One may think that an appropriate basis change might allow their results to be applied to my network formation setting. However, this does not need to be the case. As demonstrated by an example in Appendix that is same as Example 3.2 of the Supplemental Appendix in Candogan et al. (2015), the results of Baldwin and Klemperer (2019) do not allow for establishing the existence of a Walrasian equilibrium for sign-consistent tree (graph) valuations. Note that the sign-consistent tree valuation class in Candogan et al. (2015) is a subset of my sign-consistent sign-balanced valuation class. Thus, there may not be such a basis change for the results of Baldwin and Klemperer (2019) to be applicable to my model. Note that the results of Candogan et al. (2015) do not contradict the necessary and sufficient condition of Baldwin and Klemperer (2019) "since for sign-consistent tree valuations, it is implicitly the case that each item has a single copy, and while the equilibrium need not exist in the sense of [Definition 4.2 from Baldwin and Klemperer (2019)], it always exists when we restrict attention to this single-copy setting" (p.34, the Online Appendix of Candogan et al. (2015)). Similarly, in my case, an agent cannot form multiple relations with another agent, and implicitly, I assume no one agent is identical to another.

Candogan et al. (2015) provide necessary and sufficient conditions on agents' valuations to *guarantee* the existence of a Walrasian equilibrium in one auctioneer and many bidders with multiple items[6]. They also exploit the BQP preferences and employ similar proof strategies for existence results. Furthermore, their conditions, the sign-consistency and tree

[6]To avoid readers' confusion, while Candogan et al. (2015) state "we establish that the sign-consistency and tree graph assumptions are necessary and sufficient for our existence results," by showing examples where violating one of the assumptions can lead to non-existence of a Walrasian equilibrium, these assumptions are, strictly speaking, not technically necessary since there can be many instances without the assumptions to have a Walrasian economy. So what they really mean is that violating one of their conditions *can* lead to a lack of a Walrasian equilibrium.

structure on agents' valuation graph for a bundle of commodities, are similar to my conditions. I employ the same sign-consistency assumption, while I expand their tree valuation graph restriction to a sign-balance valuation graph that is a superset of theirs. For instance, the tree condition requires that if there are three goods A, B, and C, and non-zero complementarity/substitutability value attached to {A, B} pair and {A, C} pair, then there cannot be a non-zero complementarity/substitutability value attached to {B, C} pair to ensure the absence of a cycle in the entire graph. On the other hand, my sign-balance condition allows for a cycle (and a tree of course), while it restricts the structure of each cycle.

Aside from the difference in the domain of valuation, the major difference is that Candogan et al. (2015) focus on welfare-maximizing Walrasian equilibria of one-seller-many-buyer economies, while I focus on the core of network formation games. In the presence of complementarities, as implied in Shapley and Scarf (1974), a Walrasian equilibrium may not have the core property. In network formation settings, the literature has paid greater attention to the possibility of a blocking coalition formation and thus the core/stability property. Therefore, in settings concerning social networks, I believe that the core is the most suitable property to analyze.

Candogan et al. (2018) provide powerful results that within BQP preferences and a more generalized version of these preferences, there always exists a certain pricing scheme to clear the market of a one-seller-many-buyer economy, as long as the pricing scheme is as complex as preference structures. Note that this paper's network formation settings are outside of the scope of their one-seller-many-buyer settings. Thus, their results are not directly applicable to my settings. Furthermore, my main result is applicable to both transferable utility and non-transferable utility games, and therefore, applicable to settings in which there is no market price (e.g., friendship, international relations, etc.).

2 Environment

I follow notations from Jackson and Wolinsky (1996) with some modifications to facilitate the exposition of my proof. Importantly, unlike Jackson and Wolinsky (1996), when I say graphs, I mean value graphs[7]. In other words, a graph itself does not specify which agents actually form a relation with which.

Let $\mathcal{N} = \{1, \ldots, N\}$ be the finite set of agents. The network relations among these agents from the perspective of agent i are represented by the (complete) non-directed value graph $g^i = (\mathcal{N}, E^i)$, where node $j \in \mathcal{N}$ indicates agent j, while edge $(i, j) \in E^i$ captures the value of potential relations from the perspective of i, $w^i_{ij} \in \mathbb{R}$. One can regard this value as the net value of forming a relation between agent i and j after subtracting i's cost of maintaining the relation. I assume that $w^i_{ii} = w^i_{jj} = 0$ for any i and j. Notice that the value on each pair of agents can be different among different agents, to account for heterogeneity in valuation and costs of maintaining such relations. For existence, I will later impose a restriction on these values.

Furthermore, I say, if $w^i_{ij} = 0$, there is no value connection or zero complementarity between i and j, if $w^i_{ij} > 0$, there is a positive value connection or positive complementarity, and if $w^i_{ij} < 0$, there is a negative value connection or negative complementarity, from the perspective of agent i[8].

Let $N(g^i) = \{i | \exists j \text{ s.t. } w^i_{ij} \neq 0\}$. *A path in g^i connecting i_1 and i_n in value* is a set of

[7]See, e.g., Candogan et al. (2018)

[8]Substitutabilities do not correspond to negative complementarities in BQP preferences. See Koizumi (2019).

distinct nodes $\{i_1, i_2, \ldots, i_n\} \subset N(g)$ such that $w_{i_1 i_2}, w_{i_2 i_3}, \ldots, w_{i_{n-1} i_n} \neq 0$. A *cycle* is a path in which no node except the first, which is also the last, appears more than once.

A matching function $\mu : \mathcal{N} \to 2^{\mathcal{N}}$ determines which agents form a bilateral relation with a specific agent. Following the convention, I assume that if $\mu(i) \ni j$, then $\mu(j) \ni i$. Given a set of agents to which agent i is matched, $\mu(i)$, a payoff function of agent i, u_i, lies in the class of BQP preferences, first *with transfer*:

$$u_i(\mu(i)) = \sum_{j \in \mu(i)} (w^i_{ij} - p_{ij}) + \sum_{j,k \in \mu(i) : j \neq k \neq i} w^i_{jk}, \tag{1}$$

where $p_{ij} \in \mathbb{R}$ is a transfer from agent i to j. Utility is normalized to be zero for those agents who form no relation with any other agent under a matching. As mentioned in the introduction, I assume that by forming a relation with agent j and k, agent i receives indirect benefits if j and k have a good intrinsic relation ($w^i_{jk} > 0$) and indirect drawbacks if j and k have a bad intrinsic relation ($w^i_{jk} < 0$), regardless of a formal relation between j and k. Because of this indirect value, it is possible for agent i to form a relation with j even if $w^i_{ij} < 0$ as long as the indirect values through j outweigh the direct value. The BQP-preference payoff function *without transfer* is the one without p:

$$u_i(\mu(i)) = \sum_{j \in \mu(i)} w^i_{ij} + \sum_{j,k \in \mu(i) : j \neq k \neq i} w^i_{jk}, \tag{2}$$

A *network* of $\mathcal{N}$ is the outcome of a matching that specifies which pair of agents form a relation for every pair of agents in $\mathcal{N}$.

Finally, the total value of a matching function μ for any proper subset of agents $S \in \mathcal{N}$,

$V(S)$, is defined as:

$$v(S) = \sum_{i \in S} u_i(\mu(i)). \tag{3}$$

That is, the total value is equivalent to the standard utilitarian welfare function.

2.1 Stability

Since an agent's payoff can be indirectly influenced by its ally's relation with another agent, the standard pairwise stability concept is not suitable to my setting. Subsequently, I employ a groupwise stability notion. To do so, I adopt the standard blocking coalition definition.

Definition 2.1. A matching function μ is *blocked* by a coalition, $S \subseteq \mathcal{N}$, with a new matching function μ' if there exists at least one agent $i \in S$ that obtains a higher payoff under the new matching μ' than the original matching μ,

$$u_i\left(\mu'(i)\right) > u_i(\mu(i)) \quad \text{for some } i \in S, \tag{4}$$

while all the other agents in the coalition receive at least as good payoffs as before

$$u_j(\mu'(j)) \geq u_j(\mu(j)) \quad \forall j \in S \tag{5}$$

A *stable* matching function is a matching function that is not blocked by any coalition with another matching function. Accordingly, a network of $\mathcal{N}$ with matching μ is stable if there is no blocking coalition with another matching μ'.

Additionally, I shall define the core of this network game following the standard definition, first with transfer, as below:

Definition 2.2. Denote by $(\mathcal{N}, v)$ a cooperative game with transferable utility (TU). Then, for any subset $T \subseteq \mathcal{N}$, the core is a set of *imputations* $\pi \in \mathbb{R}^N$ satisfying

(i) Efficiency: $\sum_{i \in N} \pi_i = v(\mathcal{N})$

(ii) Coalitional Rationality: $\sum_{i \in T} \pi_i \geq v(T)$

Notice that any point in the core induces a stable matching function by definition. The definition of the core without transfer is the following.

Definition 2.3. Denote by $(\mathcal{N}, V)$ a cooperative game with non-transferable utility (NTU), where $V \subset \mathbb{R}^n$ satisfies the following four conditions:

1. If $T \neq \emptyset$, then $V(T)$ is non-empty and closed; and $V(\emptyset) = \emptyset$
2. For every $i \in N$, there is a V_i such that for all $x \in \mathbb{R}^n, x \in V(i)$ if and only if $x_i \leq V_i$
3. If $x \in V(T)$ and $y \in \mathbb{R}^n$ with $y_i \leq x_i$ for all $i \in T$, then $y \in V(T)$
4. The set $\left\{x \in V(\mathcal{N}) : x_i \geq V_i\right\}$ is compact.

The core of the NTU game $(\mathcal{N}, V)$ is

$$V(N) \setminus \bigcup_{S \subseteq \mathcal{N}} \mathrm{int} V(S), \tag{6}$$

where $\mathrm{int}A$ indicates the set of all interior points of A.

In words, the core of an NTU game consists of all payoff vectors that are feasible for the grand coalition $\mathcal{N}$ and that cannot be improved upon by any coalition, including $\mathcal{N}$ itself. Suppose $x \in V(\mathcal{N})$. Coalition S can improve upon x if there is $y \in V(S)$ with $y_i > x_i$ for all $i \in S$.

3 Existence

I first introduce the so-called sign-consistency assumption introduced by Candogan et al. (2015). The idea is that roughly speaking, all agents agree on which pair of agents are good (bad) matches.

Assumption 3.1. *(Sign Consistency). For some* $(i, j) \in E^k$ *and* $k \in \mathcal{N}$*, if* $w_{ij}^k > 0$*, then* $w_{ij}^l \geq 0$ *for all* $l \in \mathcal{N}$*, and similarly, if* $w_{ij}^k < 0$*, then* $w_{ij}^l \leq 0$ *for all* $l \in \mathcal{N}$*.*

Next, I introduce the so-called sign-balance assumption[9]. Colloquially, the condition requires that the enemy of my enemy is my friend. The important property of a sign-balance (value) graph is so-called *clusterability* (Cartwright and Harary (1956)); one can regroup the nodes of the graph into two subgroups within which $w_{ij}^k > 0$ or $w_{ij}^k = 0$ and across which $w_{ij}^k < 0$. This seems qualitatively the opposite of the GSC condition, although the two conditions are mathematically not opposite due to the numerical restrictions of the GSC condition under BQP preferences[10]. Figure 1 shows examples of a sign-balance graph. The two graphs at the left are sign balanced, while the other two are not.

Assumption 3.2. *(Sign Balance). Let* $g^i = (\mathcal{N}, E^i)$ *be the value graph of agent* i*. For any* i*,* g^i *is a sign-balance value graph–i.e., any cycle in the graph contains an even number of negative edges.*

My proof for the main result exploits the primal-dual relation between welfare-maximizing solutions and the core. In particular, I first show that the following quadratic program (QP1) has an integer-value solution:

[9] It is sometimes called structural balance condition.

[10] See Murota (2003), Murota and Shioura (2004), Iwamasa (2018), and Koizumi (2019).

$$
\begin{aligned}
\text{maximize} & \sum_{k\in\mathcal{N}} \left(\sum_{i\neq k} w_{ik}^k x_i^k + \sum_{i\neq j\neq k} w_{ij}^k x_i^k x_j^k \right) \\
\text{subject to } & x_i^k = x_k^i \quad \forall i, k \in \mathcal{N}, \\
& 0 \le x_i^k \le 1 \quad \forall i, k \in \mathcal{N},
\end{aligned}
$$

where $x_i^k = 1$ if agent i is matched to agent k, $0 < x_i^k < 1$ if *a fraction* of agent i is matched to agent k, and $x_i^k = 0$ if agent i does not form a relation with agent k. For convenience, define $x_k^k = 0$ for all $k \in \mathcal{N}$. The constraint, $\sum_{i\in\mathcal{N}} x_i^k \le N$, ensures that agent k forms relations with no more than N agents including itself.

I prove the existence of an integral solution by extending the version of the proof for the tree-valuation graph from Candogan et al. (2015) written in one of Vohra's blog posts (2014)[11]. His proof uses induction, in particular showing that an extreme point in the polyhedron of the welfare-maximizing problem formulated in the linear programming manner is integral for every natural number of the cardinality of the maximal connected components of the valuation graph after deleting negative edges. Exploiting the tree structure, he divides the graph into one connected component and the complement of the component, which allows him to formulate the original problem as the convex combination of the two parts of the graph, both of which have integral solutions.

The proof for the network formation setting turns out to be much simpler, since the setting does not involve typical constraints of one good to one agent from buyer-seller or optimal

[11] https://theoryclass.wordpress.com/2014/02/10/combinatorial-auctions-and-binary-quadratic-valuations-postscript/

assignment problems. Using the sign-consistency assumption, we can categorize each edge from each agent's value graph into either non-negative or strictly negative. Combined with the sign-balance assumptions, the absence of such constraints makes the induction problem into two simple cases, ignoring the degenerate case in which all the edges are negative. One is that the sign of all the edges is all non-negative. In this case, the solution is easy since matching everyone to each other (i.e., the complete network) is the solution, and thus there exists a solution to (QP1) that is integral.

The other case involves the clusterability of the sign-balance graph. By the property, we can partition any value graph into two subsets, within which edges are all non-negative and across which edges are all negative. Then, one can simply extend a portion of Vohra's proof to this case with the following two modifications. First, for the n cardinality case, unlike the tree-valuation graph, my graph may not have a component of the maximal connected components that has a node with exactly one negative edge to a node in one of the other maximal connected components; rather, a node in such a component can be incident to multiple negative edges. Furthermore, within this component, there may be multiple nodes that are connected to other components with negative edges.

Lemma 1. *Let Assumption 3.1 and 3.2 hold. Then, (QP1) has an integral solution.*

Proof. Suppose $\boldsymbol{G}$ is a graph with node set $\mathcal{N}$ and suppose any $(i, j) \in \mathcal{N}$ such that for any $k \in \mathcal{N}$, $w_{ij}^k \neq 0$ introduces edge (i, j). By the sign consistency, we can label such edges as positive or negative based on the sign of w_{ij}^k. Let $\boldsymbol{E}^+ = \{(i, j) : w_{ij}^k \geq 0 \text{ for some } k \in \mathcal{N}\}$ and $\boldsymbol{E}^- = \{(i, j) : w_{ij}^k < 0 \text{ for some } k \in \mathcal{N}\}$. Notice that any edge (i, j) lies in the set of negative edges even when only one agent regards it as negative and the rest of agents regard it as zero. Now, by the clusterability of the sign-balance condition, we can partition the nodes of $\boldsymbol{G}$ into two subsets, $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$, called plus-sets, such that each subset only contains positive edges (including zero) and across the subsets, the edges are all strictly negative. If either $\boldsymbol{G}_1$ or $\boldsymbol{G}_2$ is the empty set, then the solution is easy. Every agent forms a relation with the rest, and we achieve the maximum, which implies that the solution is integeral. If both are the empty set, then a zero vector is the solution and therefore, the solution is integral.

Thus, suppose $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are non-empty sets. To solve this case, I first introduce a new variable, z_{ij}^k, to linearize the quadratic terms in (QP1):

$$
\begin{aligned}
\text{maximize} \sum_{k\in\mathcal{N}} \left(\sum_{i\neq k} w_{ik}^k x_i^k + \sum_{i\neq j\neq k} w_{ij}^k z_{ij}^k \right) & \\
\text{subject to } x_i^k \leq 1 \ \ \forall i,k \in \mathcal{N} & \\
x_i^k = x_k^i \ \ \forall i,k \in \mathcal{N} & \\
z_{ij}^k \leq x_i^k, x_j^k \ \ \forall k \in \mathcal{N}, (i,j) \in E_+^k & \\
z_{ij}^k \geq x_i^k + x_j^k - 1 \ \ \forall k \in \mathcal{N}, (i,j) \in E_-^k & \\
x_i^k, z_{ij}^k \geq 0 \ \ \forall i,j,k \in \mathcal{N} &
\end{aligned}
$$

Note that we can formulate this way, due to Bertsimas et al. (1999). We call this relaxed formulation (LP1). Let $\boldsymbol{P}_0$ be the polyhedron of feasible solutions to (LP1). The goal is to show the extreme points of $\boldsymbol{P}_0$ are integral, which implies there exists a feasible optimal solution that is integral. The way to do this is to use the fact that if the constraints matrix is totally unimodular, then the extreme points of $\boldsymbol{P}_0$ are integral.

Let $(\bar{z}, \bar{x})$ be an optimal solution to (LP1). We can choose it to be an extreme point of the corresponding polyhedron P_0 of (LP1). Also, let P be the polyhedron restricted to the nodes of $G1$ and let P' be that restricted to the vertices of $G2$. Then, consider any node p of G_1 that is connected to a proper subset of the members of the other group, say $Q \ni q$.

We know that the sign of the edge (p, q). By the logic from the case in which G_1 or G_2 is the empty set, both P and P' are integral polyhedrons. Now, let $X_1, ..., X_n$ be the set of extreme points of P for some natural number n while $Y_1, ..., Y_{n'}$ be that of P' for some natural number n'. Let $v(.)$ be the objective value of any extreme point X_r or Y_r.

Since a polyhedron is convex, we can express $(\bar{z}, \bar{x})$ restricted to P as $\sum_r \lambda_r X_r$ while $(\bar{z}, \bar{x})$ restricted to P' as $= \sum_r \zeta_r Y_r$. Let E_- as the set of negative edges restricted to those involving the vertices in $G1$. Then, we can rewrite (LP1) as:

$$
\begin{aligned}
\text{maximize} & \sum_r \lambda_r v(X_r) + \sum_r \zeta_r v(Y_r) - \sum_{k \in G1} \sum_{(p,q) \in E_-} |w_{pq}^k| y_{pq}^k \\
\text{subject to} & \sum_r \lambda_r = 1 \\
& \sum_r \zeta_r^q = 1 \\
& y_{pq}^k \leq 1 \quad \forall k \in G1, (p, q) \in E_- \\
& \lambda_r^p, \zeta_r^q, y_{pq}^k \geq 0 \; \forall r, k
\end{aligned}
$$

Notice that the constraint matrix of this linear program is again a network matrix, and thus totally unimodular. This is because each variable appears in at most one constraint with coefficient of 1. Therefore, there exists an integral solution in this program. ■

Note that as implied in Shapley and Scarf (1974) and discussed in Demange (2004), this is not enough to prove that the welfare-maximizing allocation does actually have the core property. With Lemma 1, my main theorem can be obtained by linearizing (QP1) and applying the primal-dual approach of a linear programming framework. My proof uses an equivalent formulation of the original primal problem whose dual does not immediately correspond to the core. I first make this step as a lemma. In particular, I claim that the following linear program named (P1) is an equivalent formulation to the linearly-relaxed formulation of the original problem, (LP1), when solutions are restricted to extreme points:

$$
\begin{aligned}
W(\mathcal{N}) = \max \sum_{S \subseteq [\mathcal{N}]^2} & v(S)\, x(S) \\
\text{subject to} \sum_{S \ni (i,j)} & x(S) \leq 1 \;\; \forall (i,j) \in [\mathcal{N}]^2 \\
& x(S) \geq 0 \;\; \forall S \subseteq [\mathcal{N}]^2,
\end{aligned}
$$

where S is a subset of the size-two order-free power set of $\mathcal{N}$, denoted by $[\mathcal{N}]^2$—i.e., $\{1,2\}$ and $\{2,1\}$ are considered equivalent sets and do not simultaneously lie in the power set—, $v(S) = \sum_{i \neq k \in S} w_{ik}^k + \sum_{i \neq j \neq k \in S} w_{ij}^k$ and $x(S)$ indicates an integral or fraction of S that form a relation. For example, if $S = \{\{1,2\},\{2,3\}\}$, and if $x(S) = 1$, then agent 1 and 2 form a relation and agent 2 and 3 form a relation, while agent 1 and 3 do not.

The equivalence is immediate by showing the one-to-one mapping between (P1) and (LP1) with extreme point solutions, and thus the proof is omitted from this paper.

Lemma 2. *Let Assumption 3.1 and 3.2 hold. Then, when solutions are restricted to extreme points, (P1) is an equivalent formulation to the linearly-relaxed formulation of (QP1).*

Note that the dual of (P1) does not immediately correspond to the core, either. To find the primal program of the dual that does correspond to the core, I apply balancing weights from Bondareva (1963) and Shapley (1967) to bridge (P1) to such a primal problem. In this sense, as far as I know, this is the first study to find a connection between sign-balanced graphs and balanced games, two different concepts. My technique provides a way for future research to find a point in the core when researchers study a network formation problem with complementarities that are beyond the existing class of complementarities.

Theorem 1. *Let Assumption 3.1 and 3.2 hold. Then, an integral solution to (QP1) lies in the core of both TU and NTU network formation games, implying that there is an efficient stable network in both TU and NTU games.*

Proof. Using the objective value of (P1), we can construct another linear program (P2) with its dual (DP2) as below:

P2

$$\max \sum_{T \subseteq \mathcal{N}} W(T)\, y(T)$$
$$\text{subject to} \sum_{T \ni i} y(T) = 1 \quad \forall i \in \mathcal{N}$$
$$y(T) \geq 0, \quad \forall T \subseteq \mathcal{N} \setminus \varnothing$$

DP2

$$Z(\mathcal{N}) = \min \sum_{i \in \mathcal{N}} \pi^k$$
$$\text{subject to} \sum_{i \in T} \pi_i \geq W(T)$$
$$\forall T \subseteq \mathcal{N} \setminus \varnothing$$
$$\pi_i \geq 0 \ \ \forall i \in \mathcal{N}$$

Notice that a solution with $y(T) = 1$ for $T = \mathcal{N} \setminus \varnothing$ and with $y(T') = 0$ for all $T' \subset T$ is a solution to (P2). Otherwise, there exists no extreme point solution in (P1). Similarly, for a game with any restricted subset $S \subset \mathcal{N} \setminus \varnothing$, $y(S) = 1$ and $y(S') = 0$ for all $S' \subset S$ is a solution to (P2) as well. Thus, the objective value of (P1) equals that of (P2). Note that this logic is valid only because we started with (P1) that is *not* an integer program.

Meanwhile, notice that (DP2) corresponds to the core. Take an optimal solution to (DP1), $(\pi^\star)$, and consider a subset of agents R. Denote by $(\pi^\star(R))$ an optimal solution to the dual when restricted to subset R. Now, we can compute the objective value of the dual for a subset of agents R, $Z(R) = \sum_{k\in R} \pi^{k^\star} \geq \sum_{T\subseteq R} W(T)y(T) = W(R)$ by weak duality and the equality of the objective values between (P1) and (P2). Now, by strong duality (coming from the integrality of a solution to (P2)), $\sum_{k\in\mathcal{N}} \pi^{k^\star} = \sum_{T\subseteq\mathcal{N}} W(T)y(T) = W(\mathcal{N})$. This implies that by Lemma 1 and 2, with an extreme point solution to (P1) that is also a solution to (LP1) and (QP1), there is a system of imputations to assign a payoff to every agent that results in the core of both TU and NTU games (for NTU games, use the system of imputations as the payoff vector $V(N)\setminus\bigcup_{S\subseteq\mathcal{N}} \text{int}V(S)$), given that this system of imputations comes from an extreme point solution. Then, one can construct a stable network that is also efficient from this solution. ■

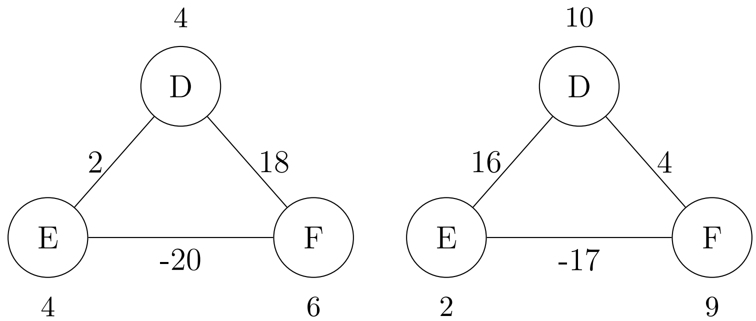
4 Example of Unstable Network

In this section, I shall show how the violation of the sign-balance condition will result in an unstable network with TU. Since transferable utility generally makes it easier to find a stable network, I shall only provide an example for TU games. Suppose we have three agents A, B, and C, and all these agents have relations with each other. A and B, and A and C are really good friends, while B and C do not like each other and form a relation only because

A has remarkably good relations with B and C. Suppose for simplicity, $w^k_{ij} = w^{k'}_{ij}$ for all $k, k' \in \{\mathrm{A, B, C}\}$ and all $(i, j) \in \{\{\mathrm{A, B}\}, \{\mathrm{A, C}\}, \{\mathrm{B, C}\}\}$, meaning that all the edge values remain constant across all the agents' point of view. Assume that $w^k_{\mathrm{AB}} = w^k_{\mathrm{AC}} = 100{,}000$ and $w^k_{\mathrm{BC}} = -1{,}000$. Furthermore, suppose there are thee more agents, D, E, and F with which agent B and C (but not A) are considering forming a relation—i.e, $w^{\mathrm{A}}_{\mathrm{AD}} = w^{\mathrm{A}}_{\mathrm{AE}} = w^{\mathrm{A}}_{\mathrm{AF}} = 0$ while $w^k_{k\mathrm{D}}, w^k_{k\mathrm{E}}, w^k_{k\mathrm{F}} > 0$ for $k \in \{\mathrm{B, C}\}$. Agent D, E, and F themselves are indifferent in forming relations with each other—i.e., $w^k_{\mathrm{DE}} = w^k_{\mathrm{DF}} = w^k_{\mathrm{EF}} = 0$ for $k \in \{\mathrm{D, E, F}\}$. Moreover, from these three agents' perspectives, there is no intrinsic benefit of forming a relation with A, B, or C—i.e., $w^k_{\mathrm{A}k} = w^k_{\mathrm{B}k} = w^k_{\mathrm{C}k} = 0$ for $k \in \{\mathrm{D, E, F}\}$. Thus, agent B and C have to make transfers to these three agents to form a relation. And yet, agent D, E, and F do not want to form a relation with *both* agent B and C simultaneously, and let us assume that $w^k_{BC} = -1{,}000$ for $k \in \{D, E, F\}$.

Finally, assume that w^k_{ki} for $k \in \{\mathrm{B, C}\}$ and $i \in \{\mathrm{D, E, F}\}$ and w^k_{ij} for $k \in \{\mathrm{B, C}\}$ and $(i, j) \in \{\{\mathrm{D, E}\}, \{\mathrm{D, F}\}, \{\mathrm{E, F}\}\}$ are depicted by Figure 2. The left graph corresponds to agent B's value graph when restricted to agent D, E, and F, and the right graph corresponds to that of C. The number above/below each node represents w^k_{ki} for $k \in \{\mathrm{B, C}\}$ and $i \in \{\mathrm{D, E, F}\}$, while the number above each edge represents w^k_{ij} for $k \in \{\mathrm{B, C}\}$ and $(i, j) \in \{\{\mathrm{D, E}\}, \{\mathrm{D, F}\}, \{\mathrm{E, F}\}\}$. This example does not have a stable network. Just to illustrate how the infinite loop of blocking occurs, let us look at an arbitrary start of this loop. Suppose agent B forms a relation with D and F together, while agent C forms a relation with agent E. At a glance, this seems to be an efficient outcome and thus achieves a stable network. Notice that if agent A pays agent D less than 26, then agent C forms a blocking coalition

Figure 2: Example without a stable network



The left graph corresponds to agent B's value graph when restricted to agent D, E, and F, and the right graph corresponds to that of C. The number above/below each node represents w_{ki}^k for $k \in \{\text{B, C}\}$ and $i \in \{\text{D, E, F}\}$, while the number above each edge represents w_{ij}^k for $k \in \{\text{B, C}\}$ and $(i, j) \in \{\{\text{D, E}\}, \{\text{D, F}\}, \{\text{E, F}\}\}$.

with agent D and E. So, agent B has to pay agent D 26, and pays agent F no more than 2 since otherwise, agent B would obtain a negative payoff. But then, agent C will leave agent E and form a blocking coalition with agent F, paying her any amount in (2, 7) (since agent C can get at most 2 from agent E).

Similar blocking processes will happen at any combination of network formation among B, C, D, E, and F, and thus this is an example with no stable network when the sign-balance condition is violated.

5 Conclusion

Restricting preferences to BQP preferences, this paper provides a new sufficient condition for the existence of a stable network with complementarities. I believe that my condition allows for a new class of interesting settings. Whether in school, the workplace, a community, a agent, or the globe, we tend to see that one joins a group of people with whom he or she finds comfortable, and across these groups, people fight.

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Appendices

This appendix introduces Example 3.2 of the Online Appendix of Candogan et al. (2015) that studies one-seller-many-buyer settings whose basis is closer than mine to Baldwin and Klemperer (2019). First, I summarize the main results of Baldwin and Klemperer (2019), and then, demonstrate an example that does not lie in their demand types but still constitutes a competitive equilibrium due to the smaller domain space for the number of each item.

In their paper, they define a concavity property of valuations and the notion of demand type that are used for the characterization of Walrasian equilibria. Suppose there is one seller, I buyers, and N goods. The concavity condition is satisfied by u if and only if for each bundle of goods S in the domain of valuations, there exists a price vector p such that the associated set of demand bundles $D_u(p)$ contains S. This concavity condition is satisfied by the monotonicity assumption that more goods are better, which is assumed in Candogan et al. (2015).

The demand type is defined by tropical hypersurfaces associated with demand sets. Tropical hypersurfaces are the set of prices, $T_v(p) = \left\{p \in \mathbb{R}^N \mid |D_u(p)| > 1\right\}$; in other words, the set of prices at which multiple bundles are demanded. This hypersurface defines a geometric object that separates different regions of the price space in which only a single bundle is demanded. The primitive integer normals corresponding to the facets of this geometric object capture how demand varies from one region to another. This set of normals characterizes an agent's demand type, formally defined as the following definition that is from Baldwin and Klemperer (2014) and is equivalent to Definition 3.1 from Baldwin and Klemperer (2019):

Definition .1. An agent has demand of type D if all the primitive integer normals to the facets of the tropical hypersurface of its demand lie in the set $\boldsymbol{D}$.

To make it more chewable, these normals basically capture how demand changes as prices change. For example, suppose there are three items i, j, k and suppose at price vector p, bundle $\{i\}$ is demanded. Furthermore, suppose as price of i increases, bundle $\{j, k\}$ starts being demanded. In this case, if at the price vector in which bundles $\{i\}$ and $\{j, k\}$ are both demanded, no other bundle is demanded, then the demand type involves vector $[-1, 1, 1]$, where the entries of this vector correspond to index i, j, k.

Now, we need to somehow aggregate individual valuation. The powerful existence results of Baldwin and Klemperer (2019) implicitly use a *strong* definition of existence of equilibrium for a class of valuations in a sense that their competitive equilibrium with aggregate valuation requires an equilibrium to exist for *any* choice of valuations and *any* number of copies of items consistent with the aggregate valuation function[12]. Note that the domain of the aggregate valuation captures the total demand by all buyers; for instance, if all buyers demand all items at a certain price, then this set will allow for $\boldsymbol{I}$ copies of each item). Then, if an equilibrium does not exist for some set of valuations with demand type $\boldsymbol{D}$ and some supplies of each item, their equilibrium definition suggests that an equilibrium does not exist. Notice that if there are restrictions on the number of copies of each item, then an equilibrium may exist. This is indeed the case for sign-consistent tree valuations.

Now, let us look at their formal definition of unimodular demand type:

Definition .2. A demand type $\boldsymbol{D}$ is unimodular if any linearly independent set of vectors in $\boldsymbol{D}$ is an integer basis for the subspace they span.

[12] Baldwin and Klemperer (2019) writes that the unimodular theorem "states that competitive equilibrium always exists, whatever is the market supply, if and only if [...]" (p.868).

With this definition, I introduce their main results:

Theorem 2. *(Unimodularity Theorem): An equilibrium exists for every pair of concave valuations of demand type $\boldsymbol{D}$, for all relevant supply bundles, iff $\boldsymbol{D}$ is unimodular.*

Corollary 1. *With $\boldsymbol{n}$ goods, if the vectors of $\boldsymbol{D}$ span $\mathbb{R}$, then an equilibrium exists for every finite set of concave valuations of demand type $\boldsymbol{D}$, for all relevant supplies, iff every subset of $\boldsymbol{n}$ vectors from $\boldsymbol{D}$ has determinant 0 or ± 1.*

Before introducing the example, I introduce a payoff function of buyers. Suppose the seller wants to maximize her revenue and buyer i has the following BQP preferences from obtaining a bundle of goods S:

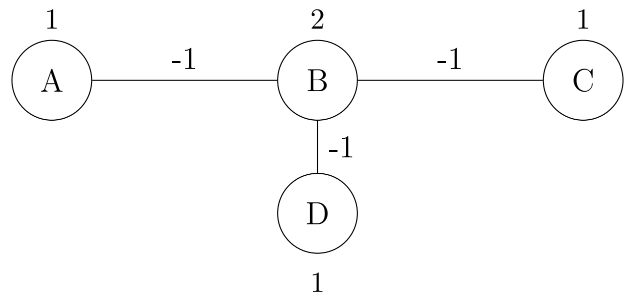
$$u_i(S) = \sum_{j \in S} (w_j^i - p_j) + \sum_{j,k \in S : j \neq k \neq i} w_{jk}^i, \tag{7}$$

where the difference from the main text is a uniform price of good j across buyers.

Now, consider a situation in which there are four goods, A, B, C, and D. To simplify the setting, suppose all the buyers have the same preferences over these four goods. Figure 3 demonstrates the buyers' preferences. Those numbers above or below the nodes represent individual surplus terms w_l from obtaining good l and those above or below the edges represent pairwise surplus terms w_{lk} from obtaining a pair of goods l and k.

Next, to check the demand type associated with the valuation function, I label the entries of the demand type vectors by A, B, C, and D, respectively. First, assume that the price of item D is arbitrarily high, and prices of the remaining goods are $p(\mathrm{A}) = 0.1, p(\mathrm{B}) = 0.5, p(\mathrm{C}) = 0.1$. Then, it follows that the corresponding demand $D(p) = \{\mathrm{A}, \mathrm{C}\}$. Now, change the price

Figure 3: Example 3.2 from the Online Appendix of Candogan et al. (2015)



Those numbers above or below the nodes represent individual surplus terms w_l from obtaining good l and those above or next to the edges represent pairwise surplus terms w_{lk} from obtaining a pair of goods l and k.

to $p'(\mathrm{A}) = 1, p'(\mathrm{B}) = 0.5, p'(\mathrm{C}) = 0.1$. It follows that the demand $\boldsymbol{D}(p') = \{\mathrm{B}\}$. This change in the demand set implies that vector $d_1 = [1, -1, 1, 0]$ lies in the demand type associated with the valuation. Using the symmetry between the nodes, we can deduce that vectors $d_2 = [1, -1, 0, 1]$ and $d_1 = [0, -1, 1, 1]$ also lie in the demand type. Lastly, consider another price vector $p'' = [2, 0, 2, 2]$ at which the only demanded bundle is $\{\mathrm{B}\}$. Notice that if we increase the price of B, the demand set switches to the empty set. Therefore, the demand type also contains $d_4 = [0, -1, 0, 0]$. Observe that the matrix with columns d_1, d_2, d_3, and d_4 has determinant 2. Hence, it follows from Corollary 1 that the demand type associated with the tree valuation in Figure 3 does not always have a Walrasian equilibrium *if any supply bundle is allowed*. Yet, restricting preferences to sign-consistent tree valuations and the supply of each item to be a single copy, Candogan et al. (2015) finds a Walrasian equilibrium. Note that the sign-consistent tree valuation class is a subset of my sign-consistent sign-balanced valuation class.