The Enemy of My Enemy Is My Friend: A New Condition for Matching with Complementarities *

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Abstract

Team and group formation tend to involve peer effects. However, a stable coalition structure in the presence of complementarities between peers need not exist. I provide a new sufficient condition for the non-emptiness of the core of one-sided coalition formation games with pairwise complementarities between peers, in both transferable and non-transferable utility cases. My condition allows for a novel class of preferences that are relevant to team formation and diplomatic relation settings.

Keywords: Coalition Formation, Cooperative Games, Matching with Complementarities, Core

JEL Codes: C62, C68, C71, C78, D44, D47, D50

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The king who is situated anywhere immediately on the circumference of the conqueror's territory is termed the enemy. The king who is likewise situated close to the enemy, but separated from the conqueror only by the enemy, is termed the friend [of the conqueror].—Kautilya, Arthasastra

1 Introduction

It is well known that in the presence of general complementarities/substitutabilities, coalition formation games need not have a non-empty core[1], an indicator for the efficiency and stability of formed groups. Stability can be important since it keeps markets robust and supports their long-term sustainability (Roth (2002)). In both one-sided and two-sided markets with or without transferable utility, the recent matching literature provides an understanding of how restrictions on preferences over complementarities/substitutabilities ensure the non-emptiness of the core or the existence of a stable matching[2]. Focusing on one-sided coalition formation games with complementarities between peers, this paper provides a novel sufficient condition that allows for a new class of preferences relevant to team formation and diplomatic relations.

To model coalition formation with complementarities between peers, I restrict attention to pairwise complementarities. To explain this restriction, consider three agents, A, B, and

[1]See Shapley and Scarf (1974). A simple example in a cooperative game is as follows. Suppose a surplus of 2.9 will be generated when any pair of the three agents form a coalition, and a surplus of 3.0 will be generated when all three agents form a coalition. Notice that the grand coalition is welfare-maximizing but two of the three agents always form a blocking coalition.

[2]See, for example, Kelso and Crawford (1982); Roth (1984); Hatfield and Milgrom (2005); Bikhchandani and Mamer (1997); Gul et al. (1999); Gul et al. (2000); Milgrom (2000); Bikhchandani and Ostroy (2002); Bikhchandani et al. (2002); Ausubel et al. (2006); Sun and Yang (2006); Sun and Yang (2009); Ostrovsky (2008); Echenique and Oviedo (2006); Pycia (2012); Hatfield et al. (2013); Kojima et al. (2013); Azevedo et al. (2013); Sun and Yang (2014); Azevedo and Hatfield (2015); and Che et al. (2019).

C, who are thinking of forming a coalition. I assume that the total benefits for A from forming a team of size three with B and C can be decomposed into the gains from the individual relationships with B and C and the gain from the indirect synergy (pairwise complementarity) between B and C, in an additively separable manner. For instance, if A is a good writer, B is a good lab experimenter, and C is a good theorist, A can enjoy the benefits of teamwork not only through his/her individual collaboration with B and C, but also through the synergy between B and C such as the exchange of knowledge regarding the empirical and theoretical literature. Such preferences with additively separable pairwise complementarities are called *binary quadratic program* (BQP) preferences.

The implicit assumption here is that a coalition is *transparent* to every member in the coalition. That is, by forming a coalition with A, agents B and C agree to work together, which leads to the grand coalition of A, B, and C working as a team. I exclude the situation in which A can secretly build individual relationships with B and C and not form a team. If there are negative synergies or tension between B and C, then A must withstand the negativity they bring to the team. I believe that the transparency assumption is natural in settings such as team and political group formation.

To further understand political group formation settings, suppose A is the U.S., B is South Korea, and C is Japan. Partly for purposes of its national defense against countries such as North Korea, the U.S. maintains bilateral defensive alliances with South Korea and Japan. For the interoperability of militaristic cooperation among the three countries, the South Korea-Japan relation is important to the U.S. This exemplifies how the indirect relation between B and C matters for A if A has a relation with both B and C, and how

BQP preferences are applicable to political formation settings[3].

To analyze these settings, I study both non-transferable utility (NTU) and transferable utility (TU) cases. In the above example, while the U.S. enjoys the benefits of a good relationship between Japan and South Korea, the U.S. could also suffer from a negative relationship between the other two. In the TU case, the U.S. may maintain the trilateral relation by compensating for the disutilty that South Korea and Japan incur from such a relation. For example, when the tension between South Korea and Japan rose and South Korea almost withdrew from the General Security of Military Information Agreement with Japan in 2019, the U.S. made significant diplomatic efforts to reconcile the two sides since the withdrawal would have a negative effect on U.S. security interests[4].

With BQP preferences, there are two main conditions for the existence of a stable coalition structure. One is that all agents agree on which pair of agents have a good (bad) relation, which I call the *sign-consistency* condition. Second, a valuation graph of agents that specifies surplus from a match between two agents features the principle that the enemy of my enemy is my friend (and the friend of my friend is my friend). Mathematically, this principle is translated into the condition whereby the graph can be partitioned into a pair of subgraphs in which each of the subgraphs consists of positive edges, but the two subgraphs are connected by negative edges. I call this condition the *sign-balance* condition. I prove that these conditions are sufficient conditions for the non-emptiness of the core.

Although the sign-balance condition is mathematically technical, there is a chewable interpretation as encapsulated in the ancient principle, "the enemy of my enemy is my friend."

[3] Another example is that only when the South-North Korean relations improved in 2000 did the United Kingdom enter into a formal relation with North Korea.

[4] See, e.g., `https://www.japantimes.co.jp/news/2019/08/23/national/politics-diplomacy/japan-south-korea-gsomia-intelligence-pact/`.

It is not unusual in daily life to find examples in which two people or two nations get along with each other *and* have a common enemy. Indeed, it has been established empirically that in addition to the state in which everyone is friends with each other, this relational state is most commonly observed in social networks such as individual human relations in massive online game experiments (Szell et al. (2010)); international relations (Maoz et al. (2007)); inter-gang violence (Nakamura et al. (2019)); trust/distrust networks among the users of a product review website (Facchetti et al. (2011)); friend/foes networks of a technological news site (Facchetti et al. (2011)); and elections of Wikipedia administrators (Facchetti et al. (2011)).

Note that this paper does not intend to provide any *normative* arguments for such relations. For example, the West cooperated with Hitler, Mussolini, and Franco when its enemy of the 1930s was Stalin (Saperstein (2004)). Therefore, such a condition for stability does not justify any normative arguments for peace. Meanwhile, Maoz et al. (2007) empirically find that while there are many exceptions, enemies of enemies are three times more likely to become allies than by random chance.

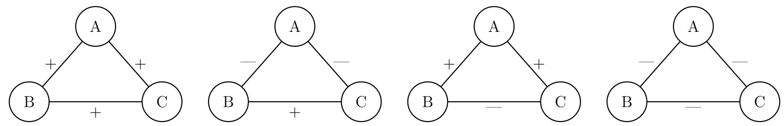
Also, note that the concept of stability is similar to that of Ostrovsky (2008). The concept is not strategic, and I analyze neither the dynamics of coalition formation nor "what-if" scenarios considered by agents who are deciding whether to temporarily form a coalition in the hopes of influencing the entire population in a way that is beneficial to them. Rather, the concept is "closer in spirit to general equilibrium models, where agents perceive conditions surrounding them as given, and optimize given those conditions" (Ostrovsky (2008), p. 899).

The positive result of this paper comes partly from the restriction to BQP preferences. The restriction allows me to explicitly express parameters on the positive or negative synergies between B and C from the perspective of A, and consequently allows me to identify how to group agents in order to prevent a blocking coalition. I believe that a BQP preference is a natural way to model team surplus from the perspective of an agent, since it uses the simple addition form. Ausubel et al. (1997), for example, uses a form of BQP preferences in spectrum auction settings with pairwise complementarities, while Bertsimas et al. (1999), Candogan et al. (2015), and Candogan et al. (2018) consider combinatorial auction problems. In addition, when it comes to the estimation of their socially optimal assignment problem with network externalities, Baccara et al. (2012) use BQP preferences that convert their problem into the class of quadratic assignment problems.

Meanwhile, I acknowledge the limitations of BQP preferences. One major limitation is that BQP preferences do not capture the influence of group-wise complementarities. For instance, an agent might consider whether or not another agent belongs to a group such as the Association of Southeast Asian Nations (ASEAN) when considering forming a relation with the agent. This study does not incorporate the effects of such club or group label effects.

Although it tends to be difficult to provide intuitive mathematical reasoning for approaches that use linear programming proofs, the intuition behind my main result is chewable and relatable. Consider the four possible triads among agent A, B, and C, as depicted in Figure 1. Plus means a positive synergy, and minus a negative synergy. The case with all positives (sign-balanced) or all negatives (sign-unbalanced) is easy to solve; everyone forms a relation with each other in the former, and no one forms a relation with each other in the latter. The second one from the left is the sign-balanced case with a common enemy. It is easy to imagine the stability of a formal relation between B and C; B and C will not form a relation with A and do not incur any damage from their negative synergies. Now, consider

Figure 1: Examples of a cycle in balanced and unbalanced graphs



The two graphs on the left are balanced, and the other ones on the right are unbalanced

the third triad, with two positives and one negative, which is sign-unbalanced. In this case, B and C hope to only work with A. Thus, if only A and B work together, C wants to block this coalition and form a team with only A; if just A and C work together, then B has an incentive to block this coalition, and so forth.

Meanwhile, the third case can be stable depending on the relative magnitude of the benefits from individual relationships to the damages from peer effects. If the benefits from individual relationships are larger than the damages from peer effects, then the coalition of interests may be stable. However, it is hard to precisely pin down such conditions. Consider a case in which an agent who already has 99 team members is trying to decide if she wants to bring in another member, with whom all of the other 99 members do not get along. Even if each of these negative peer effects is smaller than the positive direct value of forming a relationship with the 100th agent, the sum of all of the negative values can dominate the positive direct value.

The contributions of this paper are two-fold. The first is a technical contribution: the new sufficient condition contributes to the coalition formation and matching literature[5]. Despite the technicality, this new condition is easy to understand in a colloquial manner. The second contribution is that by exploiting BQP preferences, this paper studies an interesting set of one-sided coalition formation settings that involve complementarities between peers, such as

[5]Note that the strategic complementarity literature usually focuses on continuous divisible activities while the matching literature usually studies discrete indivisible activities.

team and political group formation, and yields a positive result for the condition that has empirical support.

1.1 Related Literature

The closest condition to mine is the gross substitutes and complements (GSC) condition suggested by Sun and Yang (2006)[6]. The GSC condition allows for preference structures more general than the preferences in this paper and is satisfied if goods can be divided into two groups, and within groups, goods are gross substitutes, and across groups, goods are gross complements[7]. However, in general, the condition would imply that a friend of my friend must be my enemy, which seems less plausible than mine in settings that involve humans (including countries and institutions).

Another set of papers close to mine are those by Candogan et al. (2015); Candogan et al. (2018); Nguyen and Vohra (2018); and Baldwin and Klemperer (2019). Nguyen and Vohra (2018) study two-sided matching markets with couples in the presence of capacity constraints, while allowing for general preferences in the context of nontransferable utility. In the presence of complementarities coupled with capacity constraints, their setting inevitably encounters potential emptiness of the core set, which they overcome by (possibly) minimally perturbing the capacity constraints. While their findings and algorithm are extremely powerful, direct application of their results to one-sided coalition formation is not immediate because of the lack of capacity constraints in my coalition formation problems.

Baldwin and Klemperer (2019) provide a novel and powerful characterization of classes

[6]The same-side substitutability and cross-side complementarity from Ostrovsky (2008) is similar.

[7]Note that while the GSC condition may sound *qualitatively* opposite to my condition, precisely speaking, it is not mathematically opposite in the class of preferences with additively separable pairwise complementarities.

of valuations that result in Walrasian equilibria. In the sphere of their demand types, they provide necessary and sufficient conditions for such an equilibrium to exist. One may think that an appropriate basis change might allow their results to be applied to my one-sided coalition formation setting. However, this does not need to be the case. As demonstrated by an example in the Appendix that is same as Example 3.2 in the Supplemental Appendix from Candogan et al. (2015), the results of Baldwin and Klemperer (2019) do not allow for establishing the existence of a Walrasian equilibrium for sign-consistent tree (graph) valuations. Note that the sign-consistent tree valuation class in Candogan et al. (2015) is a subset of my sign-consistent sign-balanced valuation class. Thus, there may not be such a basis change for the results of Baldwin and Klemperer (2019) to be applicable to my model. Note that the results of Candogan et al. (2015) do not contradict the necessary and sufficient condition of Baldwin and Klemperer (2019) "since for sign-consistent tree valuations, it is implicitly the case that each item has a single copy, and while the equilibrium need not exist in the sense of [Definition 4.2 from Baldwin and Klemperer (2019)], it always exists when we restrict attention to this single-copy setting" (p. 34, Candogan et al. (2015)) Online Appendix. Similarly, in my case, an agent cannot form multiple relationships with another agent, and implicitly, I assume no one agent is identical to another.

Candogan et al. (2015) provide necessary and sufficient conditions on agents' valuations to *guarantee* the existence of a Walrasian equilibrium in a case of one auctioneer and many bidders with multiple items[8]. They also exploit the BQP preferences and employ similar

[8]To avoid confusion, although Candogan et al. (2015) state that they "establish that the sign-consistency and tree graph assumptions are necessary and sufficient for our existence results," by providing examples in which violating one of the assumptions can lead to the non-existence of a Walrasian equilibrium, these assumptions are, strictly speaking, not technically necessary. This is because there can be many instances without the assumptions that have a Walrasian economy. What they really mean is that violating one of their conditions *can* lead to a lack of a Walrasian equilibrium.

proof strategies for existence results. Furthermore, their conditions—the sign-consistency and tree structure on agents' valuation graph for a bundle of commodities—are similar to my conditions. I employ the same sign-consistency assumption, while I expand their tree valuation graph restriction to a sign-balance valuation graph that is a superset of theirs. For instance, the tree condition requires that if there are three goods A, B, and C, and non-zero complementarity/substitutability value attached to {A, B} pair and {A, C} pair, then there cannot be a non-zero complementarity/substitutability value attached to {B, C} pair to ensure the absence of a cycle in the entire graph. On the other hand, my sign-balance condition allows for a cycle (and a tree of course), while it restricts the structure of each cycle.

Aside from the difference in the domain of valuation, the major difference is that Candogan et al. (2015) focus on welfare-maximizing Walrasian equilibria of one-seller-many-buyer economies, while I focus on the core of one-sided coaltiion formation games. In the presence of complementarities, as implied in Shapley and Scarf (1974), a Walrasian equilibrium may not have the core property. In coalition formation settings, the literature has paid greater attention to the possibility of a blocking coalition formation and thus the core/stability property. Therefore, in settings that concern team and political group formation, I believe the core is the most suitable property to analyze.

Candogan et al. (2018) provide powerful results whereby within BQP preferences and a more generalized version of these preferences, there always exists a certain pricing scheme to clear the market of a one-seller-many-buyer economy, as long as the pricing scheme is as complex as the preference structures. Note that this paper's one-sided coalition formation settings are outside the scope of their one-seller-many-buyer settings. Thus, their results are not directly applicable to my settings. Furthermore, my main result is applicable to both transferable utility and non-transferable utility games, and therefore, applicable to settings in which there is no market price (e.g., friendship, international relations, etc.).

2 Environment

I follow the notation and terminologies of Jackson and Wolinsky (1996) with some modifications, to facilitate exposition of my proof. Importantly, unlike Jackson and Wolinsky (1996), by "graphs," I mean "value graphs"[9]. In other words, a graph itself does not specify which agents actually form a relation with which other agents, but only specifies an exogenously given value for every pairwise coalition.

Let $\mathcal{N} = \{1, \dots, N\}$ be the finite set of agents. The exogenously given values of match surplus among these agents from the perspective of agent i are represented by the (complete) non-directed value graph $g^i = (\mathcal{N}, E^i)$, where edge $(i, j) \in E^i$ for $i \neq j$ captures the value of a match with agent j from the perspective of i, $w^i_{ij} \in \mathbb{R}$, while $(j, k) \in E^i$ for $i \neq j \neq k$ represents a pairwise complementarity between potential peers j and k for i. I assume that $w^i_{ii} = w^i_{jj} = 0$ for any i and j. Notice that the value of each pair of agents can be different among different agents, to account for heterogeneity in valuation and the costs of maintaining such relations. For existence, I will later impose a restriction on these values.

Furthermore, regarding pairwise complementarities between peers, for $i \neq j \neq k$, if $w^i_{jk} = 0$, there is no/zero complementarity between j and k, if $w^i_{jk} > 0$, there is a positive complementarity, and if $w^i_{jk} < 0$, there is a negative complementarity, from the perspective

[9]See, e.g., Candogan et al. (2018)

of agent i[10].

Let $N(g^i) = \{i | \exists j \text{ s.t. } w^i_{ij} \neq 0\}$. *A path in* g^i *connecting* i_1 *and* i_n *in value* is a set of distinct nodes $\{i_1, i_2, \dots, i_n\} \subset N(g^i)$ such that $w^i_{i_1 i_2}, w^i_{i_2 i_3}, \dots, w^i_{i_{n-1} i_n} \neq 0$. A *cycle* is a path in which no node except the first, which is also the last, appears more than once.

A matching function $\mu : \mathcal{N} \to 2^{\mathcal{N}}$ determines which agents form a bilateral relation with a specific agent. Following the convention, I assume that if $\mu(i) \ni j$, then $\mu(j) \ni i$. Given a set of agents to which agent i is matched, $\mu(i)$, a payoff function of agent i, u_i, lies in the class of BQP preferences, first *without transfer*:

$$u_i(\mu(i)) = \sum_{j \in \mu(i)} w^i_{ij} + \sum_{j,k \in \mu(i) : j \neq k \neq i} w^i_{jk}. \tag{1}$$

Utility is normalized to be zero for those agents who form no relation with any other agent under a matching. As stated in the introduction, by the transparency assumption, by forming a relation with agents j and k, agent i indirectly connects j and k to work as a team. Agent i benefits from positive synergies between them if $w^i_{jk} > 0$ and incurs negative synergies if $w^i_{jk} < 0$.

In contrast, the BQP-preference payoff function *without transfer* is the one without p:

$$u_i(\mu(i)) = \sum_{j \in \mu(i)} (w^i_{ij} - p_{ij}) + \sum_{j,k \in \mu(i) : j \neq k \neq i} w^i_{jk}, \tag{2}$$

where $p_{ij} \in \mathbb{R}$ is a transfer from agent i to j that satisfies the reciprocity, $p_{ij} = -p_{ji}$. In the TU case, agent i can pay j and k to compensate for the disutility they incur by forming a

[10]Substitutabilities do not correspond to negative complementarities in BQP preferences. See Koizumi (2019).

team. For example, for $i \neq j \neq k$, p_{ij} may be equal to $-w^j_{jk}$ for $w^j_{jk} < 0$ when i chooses to match with both j and k. Notice that unlike in the NTU case, a matching function does not specify a (unique) payoff vector in the TU case.

Finally, I denote by $v : 2^{\mathcal{N}} \rightarrow \mathbb{R}$ a characteristic function that specifies (exogenously given) surplus to every subset of $\mathcal{N}$ using the aforementioned value graphs. The total value of a matching function μ for any proper subset of agents $S \in \mathcal{N}$, denoted by $v(S)$, is defined as:

$$v(S) = \sum_{i \in S} u_i(\mu(i)). \tag{3}$$

That is, the total value is equivalent to the standard utilitarian welfare function. Note that by convention, $v(\emptyset) = 0$. To avoid confusion, I reiterate that by the transparency assumption, every member in coalition S is matched to every other member in S. This implies that from $i \in S$'s point of view, there is no secret team member (outside of S) who is matched to $j \in S$ but is not matched to i.

2.1 Stability And Core

Since an agent's payoff can be indirectly influenced by her peers, the standard pairwise stability concept is not suitable to my setting. Subsequently, I employ a groupwise stability notion. To do so, I adopt the standard blocking coalition definition.

Definition 2.1. A matching function μ is *blocked* by a coalition, $S \subseteq \mathcal{N}$, with a new matching function μ' if there exists at least one agent $i \in S$ that obtains a higher payoff under the new matching μ' than the original matching μ,

$$u_i\left(\mu'(i)\right) > u_i(\mu(i)) \quad \text{for some } i \in S, \tag{4}$$

while all the other agents in the coalition receive at least as good payoffs as before

$$u_j(\mu'(j)) \geq u_j(\mu(j)) \quad \forall j \in S. \tag{5}$$

A *stable* matching function is a matching function that is not blocked by any coalition under another matching function. To reiterate, unlike in the NTU case, a matching function does not specify a (unique) payoff vector in the TU case. Under a stable matching function, we know that there exists at least one payoff vector that prevents a blocking coalition, but we do not know a specific (bargaining) structure of this payoff vector.

The core, on the other hand, is the notion of stability with efficiency. I first define the core of the TU coalition formation game following the standard definition:

Definition 2.2. Denote by $(\mathcal{N}, v)$ a cooperative game with transferable utility (TU). Then, for any subset $T \subseteq \mathcal{N}$, the core is a set of *imputations* $\pi \in \mathbb{R}^N$ satisfying

(i) Efficiency: $\sum_{i \in N} \pi_i = v(\mathcal{N})$

(ii) Coalitional Rationality: $\sum_{i \in T} \pi_i \geq v(T)$

Notice that any point in the core induces a stable matching function by definition. The definition of the core without transfer is the following.

Definition 2.3. Denote by $(\mathcal{N}, V)$ a cooperative game with non-transferable utility (NTU), where $V \subset \mathbb{R}^n$ satisfies the following four conditions:

1. If $T \neq \varnothing$, then $V(T)$ is nonempty and closed; and $V(\varnothing) = \varnothing$

2. For every $i \in N$, there is a V_i such that for all $x \in \mathbb{R}^n, x \in V(i)$ if and only if $x_i \leq V_i$

3. If $x \in V(T)$ and $y \in \mathbb{R}^n$ with $y_i \leq x_i$ for all $i \in T$, then $y \in V(T)$

4. The set $\left\{x \in V(\mathcal{N}) : x_i \geq V_i\right\}$ is compact.

The core of the NTU game $(\mathcal{N}, V)$ is

$$V(N) \backslash \bigcup_{S \subseteq \mathcal{N}} \text{int} V(S), \tag{6}$$

where int$A$ indicates the set of all interior points of A.

In words, the core of an NTU game consists of all payoff vectors that are feasible for the grand coalition $\mathcal{N}$ and that cannot be improved on by any coalition, including $\mathcal{N}$ itself. Suppose $x \in V(\mathcal{N})$. Coalition S can improve upon x if there is $y \in V(S)$ with $y_i > x_i$ for all $i \in S$.

Furthermore, we need the concept of balancedness to invoke the celebrated results of Scarf (1967) later. Let $B(\mathcal{N})$ be the set of feasible solutions to the following system:

$$\textstyle\sum_{S : i \in S} y(S) = 1 \quad \forall i \in \mathcal{N}$$

$$y(S) \geq 0 \quad \forall S \subset \mathcal{N}$$

Notice that $B(N) \neq \varnothing$. Every $y \in B(\mathcal{N})$ is called the *balancing weights* and the collection of sets S such that $y(S) > 0$ is called a *balanced collection.*

Definition 2.4. (Balanced TU game) Let $(\mathcal{N}, v)$ be an arbitrary TU game. A TU game

$(\mathscr{N}, v)$ is *balanced* if for every balancing weights,

$$\sum_{S \subset \mathscr{N}} y(S)v(S) \leq v(\mathscr{N}). \tag{7}$$

As for NTU games, I adopt the following definition from Scarf (1967):

Definition 2.5. Let $(\mathscr{N}, V)$ be an arbitrary NTU game. An NTU game $(\mathscr{N}, V)$ is *balanced* if for any balanced collection $\mathscr{C}$ of subsets of $\mathscr{N}$,

$$\bigcap_{S \in \mathscr{C}} V(S) \subseteq V(\mathscr{N}) \tag{8}$$

Finally, following Peleg and Sudhölter (2007), with every TU game $(\mathscr{N}, v)$ I *associate* the NTU game $(\mathscr{N}, V_v)$ defined by

$$V_v(S) = \left\{ x^S \in \mathbb{R}^S \mid \sum_{i \in S} x^i \leq v(S) \right\} \text{ for every } \varnothing \neq S \subseteq \mathscr{N}.$$

Furthermore, I denote

$$V_0(S) = V(S) \times \left\{ 0^{N \setminus S} \right\} \subseteq \mathbb{R}^N \text{ for all } S \in 2^N \setminus \{\varnothing\},$$

and $V_0(\{\varnothing\}) = \{0\}$. With this notation, I define cardinal balanced games.

Definition 2.6. An NTU game $(\mathscr{N}, V)$ is *cardinal balanced* if for every balanced collection of coalitions $\mathscr{B}$ with a system $\left(\delta_S\right)_{S \in \mathscr{B}}$ of balancing coefficients,

$$\sum_{S \in \mathscr{B}} \delta_S V_0(S) \subseteq V(N).$$

3 Existence

I first introduce the so-called sign-consistency assumption introduced by Candogan et al. (2015). The idea is that roughly speaking, all agents agree on which pair of agents are good (bad) matches.

Assumption 3.1. *(Sign Consistency). For some* $(i, j) \in E^k$ *and* $k \in \mathcal{N}$*, if* $w_{ij}^k > 0$*, then* $w_{ij}^l \geq 0$ *for all* $l \in \mathcal{N}$*, and similarly, if* $w_{ij}^k < 0$*, then* $w_{ij}^l \leq 0$ *for all* $l \in \mathcal{N}$*.*

Next, I introduce the so-called sign-balance assumption[11]. Colloquially, the condition requires that the enemy of my enemy is my friend. The important property of a sign-balance (value) graph is so-called *clusterability* (Cartwright and Harary (1956)); one can regroup the nodes of the graph into two subgroups within which $w_{ij}^k > 0$ or $w_{ij}^k = 0$ and across which $w_{ij}^k < 0$. This seems qualitatively the opposite of the GSC condition, although the two conditions are mathematically not opposite due to the numerical restrictions of the GSC condition under BQP preferences[12]. Figure 1 shows examples of a sign-balance graph. The two graphs on the left are sign balanced, while the two on the right are not.

Assumption 3.2. *(Sign Balance). Let* $g^i = (\mathcal{N}, E^i)$ *be the value graph of agent* i*. For any* i*,* g^i *is a sign-balance value graph—i.e., any cycle in the graph contains an even number of negative edges.*

My proof for the main result exploits the primal-dual relation between welfare-maximizing solutions and the core. In particular, I first show that the following quadratic program (QP1) has an integer-value solution:

[11] This is sometimes called the "structural balance condition."

[12] See Murota (2003); Murota and Shioura (2004); Iwamasa (2018); and Koizumi (2019).

$$\begin{aligned} &\text{maximize} \sum_{k\in\mathcal{N}}\left(\sum_{i\neq k} w_{ik}^k x_i^k + \sum_{i\neq j\neq k} w_{ij}^k x_i^k x_j^k\right)\\ &\text{subject to } x_i^k = x_k^i \;\; \forall i,k\in\mathcal{N},\\ &\qquad 0\leq x_i^k \leq 1 \;\; \forall i,k\in\mathcal{N}, \end{aligned}$$

where $x_i^k = 1$ if agent i is matched to agent k, $0 < x_i^k < 1$ if *a fraction* of agent i is matched to agent k, and $x_i^k = 0$ if agent i does not form a relation with agent k. For convenience, define $x_k^k = 0$ for all $k \in \mathcal{N}$. The constraint, $\sum_{i\in\mathcal{N}} x_i^k \leq N$, ensures that agent k forms relations with no more than N agents including itself.

I prove the existence of an integral solution by extending the version of the proof for the tree-valuation graph from Candogan et al. (2015) written in one of Vohra's blog posts (2014)[13]. His proof uses induction, and in particular shows that an extreme point in the polyhedron of the welfare-maximizing problem formulated in the linear programming manner is integral for every natural number of the cardinality of the maximal connected components of the valuation graph after deleting negative edges. Exploiting the tree structure, he divides the graph into one connected component and the complement of the component, which allows him to formulate the original problem as the convex combination of the two parts of the graph, both of which have integral solutions.

The proof for the one-sided coalition formation setting turns out to be much simpler, since the setting does not involve typical constraints of one good to one agent from buyer-seller or

[13]https://theoryclass.wordpress.com/2014/02/10/combinatorial-auctions-and-binary-quadratic-valuations-postscript/

optimal assignment problems. Using the sign-consistency assumption, we can categorize each edge from each agent's value graph into either non-negative or strictly negative. Combined with the sign-balance assumptions, the absence of such constraints divides the induction problem into two simple cases, ignoring the degenerate case in which all of the edges are negative. The first case is that the sign of all the edges is all non-negative. In this case, the solution is easy since matching everyone to each other (i.e., the complete network) is the solution, and thus there exists a solution to (QP1) that is integral.

The second case involves the clusterability of the sign-balance graph. By this property, we can partition any value graph into two subsets, within which edges are all non-negative and across which the edges are all negative. Then, one can simply extend a portion of Vohra's proof to this case with the following two modifications. First, for the n cardinality case, unlike the tree-valuation graph, my graph may not have a component of the maximal connected components that has a node with exactly one negative edge to a node in one of the other maximal connected components; rather, a node in such a component can be incident to multiple negative edges. Furthermore, within this component, there may be multiple nodes that are connected to other components with negative edges. The proof for the following lemma can be found in the Appendix.

Lemma 1. *Let Assumption 3.1 and 3.2 hold. Then, (QP1) has an integral solution.*

Note that as implied in Shapley and Scarf (1974) and discussed in Demange (2004), this is not enough to prove that the welfare-maximizing allocation actually has the core property. With Lemma 1, my main theorem can be obtained by linearizing (QP1) and applying the primal-dual approach of a linear programming framework. My proof uses an equivalent formulation of the original primal problem whose dual does not immediately correspond to the core. I first render this step as a lemma. In particular, I claim that the following linear program named (P1) is an equivalent formulation to the linearly relaxed formulation of the original problem, (LP1), when solutions are restricted to extreme points:

$$W(\mathcal{N}) = \max \sum_{S \subseteq [\mathcal{N}]^2} v(S)\, x(S)$$

$$\text{subject to} \sum_{S \ni (i,j)} x(S) \leq 1 \;\; \forall (i,j) \in [\mathcal{N}]^2$$

$$x(S) \geq 0 \;\; \forall S \subseteq [\mathcal{N}]^2,$$

where S is a subset of the size-two order-free power set of $\mathcal{N}$, denoted by $[\mathcal{N}]^2$—i.e., $\{1,2\}$ and $\{2,1\}$ are considered to be equivalent sets and do not simultaneously lie in the power set—, $v(S) = \sum_{i \neq k \in S} w_{ik}^k + \sum_{i \neq j \neq k \in S} w_{ij}^k$ and $x(S)$ indicates an integral or fraction of S that form a relation. For example, if $S = \{\{1,2\},\{2,3\}\}$, and if $x(S) = 1$, then agents 1 and 2 form a relation and agents 2 and 3 form a relation, while agents 1 and 3 do not.

The equivalence is immediate by showing the one-to-one mapping between (P1) and (LP1) with extreme point solutions, and thus the proof is omitted from this paper.

Lemma 2. *Let Assumption 3.1 and 3.2 hold. Then, when solutions are restricted to extreme points, (P1) is an equivalent formulation to the linearly-relaxed formulation of (QP1).*

Note that the dual of (P1) does not immediately correspond to the core, either. To find the primal program of the dual that does correspond to the core, I apply balancing weights from Bondareva (1963) and Shapley (1967) to bridge (P1) to such a primal problem. In this sense, as far as I know, this is the first study to find a connection between sign-balanced graphs and balanced games, which are different concepts. My technique provides a way for future research to find a point in the core when researchers study one-sided formation problems with complementarities that are beyond the existing class of complementarities.

Theorem 1. *Let Assumption 3.1 and 3.2 hold. Then, an integral solution to (QP1) lies in the core of TU coalition formation games.*

Proof. Using the objective value of (P1), we can construct another linear program (P2) with its dual (DP2) as below:

P2

$$\max \sum_{T \subseteq \mathcal{N}} W(T)\, y(T)$$
$$\text{subject to} \sum_{T \ni i} y(T) = 1 \quad \forall i \in \mathcal{N}$$
$$y(T) \geq 0, \quad \forall T \subseteq \mathcal{N} \setminus \varnothing$$

DP2

$$Z(\mathcal{N}) = \min \sum_{i \in \mathcal{N}} \pi^k$$
$$\text{subject to} \sum_{i \in T} \pi_i \geq W(T)$$
$$\forall T \subseteq \mathcal{N} \setminus \varnothing$$
$$\pi_i \geq 0 \;\; \forall i \in \mathcal{N}$$

Notice that a solution with $y(T) = 1$ for $T = \mathcal{N} \setminus \varnothing$ and with $y(T') = 0$ for all $T' \subset T$ is a solution to (P2). Otherwise, there exists no extreme point solution in (P1). Similarly, for a game with any restricted subset $S \subset \mathcal{N} \setminus \varnothing$, $y(S) = 1$ and $y(S') = 0$ for all $S' \subset S$ is a solution to (P2) as well. Thus, the objective value of (P1) equals that of (P2). Note that this logic is valid only because we started with (P1) that is *not* an integer program.

Meanwhile, notice that (DP2) corresponds to the core. Take an optimal solution to

(DP1), $(\pi^\star)$, and consider a subset of agents $\boldsymbol{R}$. Denote by $(\pi^\star(\boldsymbol{R}))$ an optimal solution to the dual when restricted to subset $\boldsymbol{R}$. Now, we can compute the objective value of the dual for a subset of agents $\boldsymbol{R}$, $Z(\boldsymbol{R}) = \sum_{k\in R} \pi^{k^\star} \geq \sum_{T\subseteq R} W(T)y(T) = W(\boldsymbol{R})$ by weak duality and the equality of the objective values between (P1) and (P2). Now, by strong duality (coming from the integrality of a solution to (P2)), $\sum_{k\in\mathcal{N}} \pi^{k^\star} = \sum_{T\subseteq\mathcal{N}} W(T)y(T) = W(\mathcal{N})$. This implies that by Lemma 1 and 2, with an extreme point solution to (P1) that is also a solution to (LP1) and (QP1), there is a system of imputations to assign a payoff to every agent that results in the core of TU network formation games, given that this system of imputations comes from an extreme point solution. Then, one can construct a stable network that is also efficient from this solution. ■

Then, the following corollary is immediate by the Bondareva-Shapley theorem:

Corollary 1. *Let Assumption 3.1 and 3.2 hold. Then, a TU coalition formation game $(\mathcal{N}, v)$ is balanced.*

Given this result, using the connection between the TU and associated NTU games demonstrated by Peleg and Sudhölter (2007), we can invoke the celebrated result of Scarf (1967) to prove the non-emptiness of the core of the associated NTU coalition formation game.

Proposition 1. *Let Assumption 3.1 and 3.2 hold. Then, the core of a TU game $(\mathcal{N}, v)$ is non-empty, and thus the core of the associated NTU network formation game $(\mathcal{N}, V_v)$ is non-empty as well.*

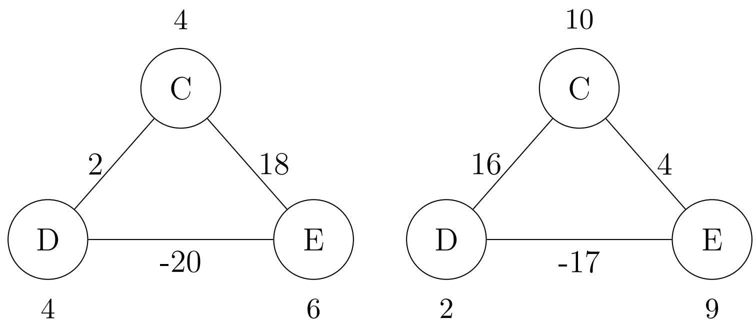
Proof. First, notice that a TU game is balanced if and only if the associated NTU game is cardial balanced (Peleg and Sudhölter (2007)). It is also immediate that a cardial balanced game is balanced (Peleg and Sudhölter (2007)). Therefore, by Corollary 1, the associated NTU formation game $(\mathcal{N}, V_v)$ is balanced. By Scarf (1967), the core of the associated NTU formation game is non-empty. ■

4 Example

In this section, I shall show how violation of the sign-balance condition will result in an unstable coalition formation game with TU. Since transferable utility generally makes it easier to find a stable coalition, I shall focus on an example for TU games. Suppose we have two agents A and B with $w^A_{AB} = w^B_{AB} = -100$. Furthermore, suppose there are thee more agents, C, D, and E with which agent A or B is considering forming a relation, while agents D, E, and F are indifferent in forming relations with each other—i.e., $w^k_{\mathrm{CD}} = w^k_{\mathrm{CE}} = w^k_{\mathrm{DE}} = 0$ for $k \in \{\mathrm{C, D, E}\}$. Moreover, from these three agents' perspectives, there is no intrinsic benefit of forming a relation with A or B—i.e., $w^k_{\mathrm{A}k} = w^k_{\mathrm{B}k} = 0$ for $k \in \{\mathrm{D, E, F}\}$. Finally, assume that w^k_{ki} for $k \in \{\mathrm{A, B}\}$ and $i \in \{\mathrm{C, D, E}\}$ and w^k_{ij} for $k \in \{\mathrm{A, B}\}$ and $(i, j) \in \{\{\mathrm{C, D}\}, \{\mathrm{C, E}\}, \{\mathrm{D, E}\}\}$ are depicted by Figure 2. The graph on the left corresponds to agents A's value graph when restricted to agent C, D, and E, and the graph on the right corresponds to that of B. The number above/below each node represents w^k_{ki} for $k \in \{\mathrm{A, B}\}$ and $i \in \{\mathrm{C, D, E}\}$, while the number above each edge represents w^k_{ij} for $k \in \{\mathrm{A, B}\}$ and $(i, j) \in \{\{\mathrm{C, D}\}, \{\mathrm{C, E}\}, \{\mathrm{D, E}\}\}$. In this example, any of the three agents C, D, and E will not form a coalition with *both* A and B, due to the prohibitively high negative synergy between A and B.

This example does not have a stable coalitional structure and thus have an empty core.

Figure 2: Example without a stable matching



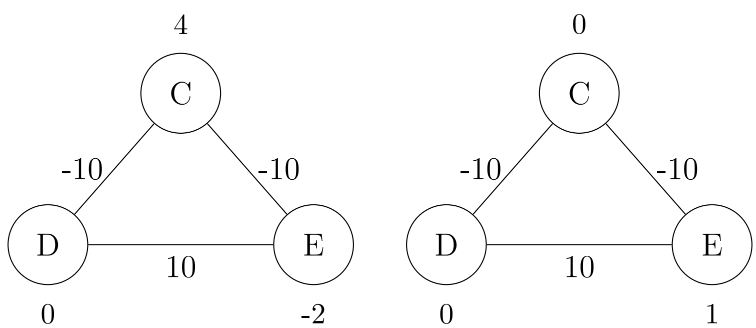
The left graph corresponds to agent A's value graph when restricted to agent C, D, and E, and the right graph corresponds to that of B. The number above/below each node represents w^k_{ki} for $k \in \{\text{A, B}\}$ and $i \in \{\text{C, D, E}\}$, while the number above each edge represents w^k_{ij} for $k \in \{\text{A, B}\}$ and $(i, j) \in \{\{\text{C, D}\}, \{\text{C, E}\}, \{\text{D, E}\}\}$.

Just to illustrate how the infinite loop of blocking occurs, let us look at an arbitrary start of this loop. Suppose agent A forms a relation with C and E together, while agent B forms a relation with agent D. At a glance, this seems to be an efficient outcome and thus achieves a stable coalitional structure. And yet, notice that if agent A pays agent C less than 26, then agent C forms a blocking coalition with agent B and D. So, agent A has to pay agent C 26, and pays agent E no more than 2 since otherwise, agent A would obtain a negative payoff. But then, agent B will leave agent D and form a blocking coalition with agent E, paying her any amount in (2, 7) (since agent B can get at most 2 from matching with agent D).

Similar blocking processes will happen at any combination of coalition formation among these five agents, and thus this is an example with no stable coalitional structure when the sign-balance condition is violated.

Next, I shall provide an example with a stable matching. Consider the value graphs of A and B as illustrated in Figure 4 that satisfy the sign-consistency and sign-balance conditions. In this case, A forms a coalition with C, while B forms a coalition with D and E.

Figure 3: Example with a stable matching



The left graph corresponds to agent A's value graph when restricted to agent C, D, and E, and the right graph corresponds to that of B. The number above/below each node represents w^k_{ki} for $k \in \{\text{A, B}\}$ and $i \in \{\text{C, D, E}\}$, while the number above each edge represents w^k_{ij} for $k \in \{\text{A, B}\}$ and $(i, j) \in \{\{\text{C, D}\}, \{\text{C, E}\}, \{\text{D, E}\}\}$.

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Appendices

A Proof of Lemma 1

Proof. Suppose $\boldsymbol{G}$ is a graph with node set $\mathcal{N}$ and suppose any $(i, j) \in \mathcal{N}$ such that for any $k \in \mathcal{N}$, $w_{ij}^k \neq 0$ introduces edge (i, j). By sign consistency, we can label such edges as positive or negative based on the sign of w_{ij}^k. Let $\boldsymbol{E}^+ = \{(i, j) : w_{ij}^k \geq 0 \text{ for some } k \in \mathcal{N}\}$ and $\boldsymbol{E}^- = \{(i, j) : w_{ij}^k < 0 \text{ for some } k \in \mathcal{N}\}$. Notice that any edge (i, j) lies in the set of negative edges even when only one agent regards it as negative and the remaining agents regard it as zero. Now, by the clusterability of the sign-balance condition, we can partition the nodes of $\boldsymbol{G}$ into two subsets, $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$, called plus-sets, such that each subset only contains positive edges (including zero) and across the subsets, the edges are all strictly negative. If either $\boldsymbol{G}_1$ or $\boldsymbol{G}_2$ is the empty set, then the solution is easy. Every agent forms a relation with the rest, and we achieve the maximum, which implies that the solution is integeral. If both are the empty set, then a zero vector is the solution and therefore, the solution is integral.

Thus, suppose $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are nonempty sets. To solve this case, I first introduce a new variable, z_{ij}^k, to linearize the quadratic terms in (QP1):

$$\begin{aligned}
\text{maximize} \sum_{k\in\mathcal{N}} &\left(\sum_{i\neq k} w_{ik}^k x_i^k + \sum_{i\neq j\neq k} w_{ij}^k z_{ij}^k\right) \\
\text{subject to } & x_i^k \leq 1 \;\; \forall i,k \in \mathcal{N} \\
& x_i^k = x_k^i \;\; \forall i,k \in \mathcal{N} \\
& z_{ij}^k \leq x_i^k, x_j^k \;\; \forall k \in \mathcal{N}, (i,j) \in E_+^k \\
& z_{ij}^k \geq x_i^k + x_j^k - 1 \;\; \forall k \in \mathcal{N}, (i,j) \in E_-^k \\
& x_i^k, z_{ij}^k \geq 0 \;\; \forall i,j,k \in \mathcal{N}
\end{aligned}$$

Note that we can formulate in this way, due to Bertsimas et al. (1999). We call this relaxed formulation (LP1). Let $\boldsymbol{P}_0$ be the polyhedron of feasible solutions to (LP1). The goal is to show the extreme points of $\boldsymbol{P}_0$ are integral, which implies there exists a feasible optimal solution that is integral. The way to do this is to use the fact that if the constraints matrix is totally unimodular, then the extreme points of $\boldsymbol{P}_0$ are integral.

Let $(\bar{z}, \bar{x})$ be an optimal solution to (LP1). We can choose it to be an extreme point of the corresponding polyhedron $\boldsymbol{P}_0$ of (LP1). Also, let $\boldsymbol{P}$ be the polyhedron restricted to the nodes of $\boldsymbol{G}1$ and let $\boldsymbol{P'}$ be that restricted to the vertices of $\boldsymbol{G}2$. Then, consider any node $\boldsymbol{p}$ of $\boldsymbol{G}_1$ that is connected to a proper subset of the members of the other group, say $\boldsymbol{Q} \ni \boldsymbol{q}$.

We know the sign of the edge $(\boldsymbol{p}, \boldsymbol{q})$. By the logic from the case in which $\boldsymbol{G}_1$ or $\boldsymbol{G}_2$ is the empty set, both $\boldsymbol{P}$ and $\boldsymbol{P'}$ are integral polyhedrons. Now, let $\boldsymbol{X}_1, ..., \boldsymbol{X}_n$ be the set of extreme points of $\boldsymbol{P}$ for some natural number $\boldsymbol{n}$ while $\boldsymbol{Y}_1, ..., \boldsymbol{Y}_{n'}$ be that of $\boldsymbol{P'}$ for some natural number $\boldsymbol{n'}$. Let $v(.)$ be the objective value of any extreme point $\boldsymbol{X}_r$ or $\boldsymbol{Y}_r$.

Since a polyhedron is convex, we can express $(\bar{z}, \bar{x})$ restricted to $\boldsymbol{P}$ as $\sum_r \lambda_r X_r$ while $(\bar{z}, \bar{x})$ restricted to $\boldsymbol{P'}$ as $= \sum_r \zeta_r Y_r$. Let $\boldsymbol{E_-}$ as the set of negative edges restricted to those involving the vertices in $\boldsymbol{G}1$. Then, we can rewrite (LP1) as:

$$
\begin{aligned}
\text{maximize} & \sum_r \lambda_r v(X_r) + \sum_r \zeta_r v(Y_r) - \sum_{k \in G1} \sum_{(p,q) \in E_-} |w^k_{pq}| y^k_{pq} \\
\text{subject to} & \sum_r \lambda_r = 1 \\
& \sum_r \zeta^q_r = 1 \\
& y^k_{pq} \leq 1 \quad \forall k \in G1, (p, q) \in E_- \\
& \lambda^p_r, \zeta^q_r, y^k_{pq} \geq 0 \ \forall r, k
\end{aligned}
$$

Notice that the constraint matrix of this linear program is again a network matrix, and thus totally unimodular. This is because each variable appears in at most one constraint with a coefficient of 1. Therefore, there exists an integral solution in this program. ■

B Example 3.2 of the Online Appendix of Candogan et al. (2015)

This section introduces Example 3.2 in the Online Appendix of Candogan et al. (2015) that studies one-seller-many-buyer settings whose basis is closer than mine to Baldwin and Klemperer (2019). First, I summarize the main results of Baldwin and Klemperer (2019), and then present an example that does not lie in their demand types but still constitutes a competitive equilibrium due to the smaller domain space for the number of each item.

In their paper, they define a concavity property of valuations and the notion of demand type that are used for the characterization of Walrasian equilibria. Suppose there is one seller, I buyers, and N goods. The concavity condition is satisfied by u if and only if for each bundle of goods S in the domain of valuations, there exists a price vector p such that the associated set of demand bundles $D_u(p)$ contains S. This concavity condition is satisfied by the monotonicity assumption that more goods are better, which is assumed in Candogan et al. (2015).

The demand type is defined by tropical hypersurfaces associated with demand sets. Tropical hypersurfaces are the set of prices, $T_v(p) = \left\{p \in \mathbb{R}^N || D_u(p)| > 1\right\}$; in other words, the set of prices at which multiple bundles are demanded. This hypersurface defines a geometric object that separates different regions of the price space in which only a single bundle is demanded. The primitive integer normals corresponding to the facets of this geometric object capture how demand varies from one region to another. This set of normals characterizes an agent's demand type, formally defined by Baldwin and Klemperer (2014) and equivalent to Definition 3.1 from Baldwin and Klemperer (2019), as follows:

Definition B.1. An agent has demand of type D if all of the primitive integer normals to the facets of the tropical hypersurface of its demand lie in the set D.

To make this more chewable, these normals basically capture how demand changes as prices change. For example, suppose there are three items i, j, k and suppose at price vector p, bundle $\{i\}$ is demanded. Furthermore, suppose as the price of i increases, bundle $\{j, k\}$ starts being demanded. In this case, if at the price vector at which bundles $\{i\}$ and $\{j, k\}$

are both demanded, no other bundle is demanded, then the demand type involves vector $[-1, 1, 1]$, where the entries of this vector correspond to index $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

Now, we need to somehow aggregate individual valuation. The powerful existence results of Baldwin and Klemperer (2019) implicitly use a *strong* definition of existence of equilibrium for a class of valuations, in the sense that their competitive equilibrium with aggregate valuation requires an equilibrium to exist for *any* choice of valuations and *any* number of copies of items consistent with the aggregate valuation function[14]. Note that the domain of the aggregate valuation captures the total demand by all buyers; for instance, if all buyers demand all items at a certain price, then this set will allow for $\boldsymbol{I}$ copies of each item). Then, if an equilibrium does not exist for some set of valuations with demand type $\boldsymbol{D}$ and some supplies of each item, their equilibrium definition suggests that an equilibrium does not exist. Notice that if there are restrictions on the number of copies of each item, then an equilibrium may exist. This is indeed the case for sign-consistent tree valuations.

Now, let us look at their formal definition of unimodular demand type:

Definition B.2. A demand type $\boldsymbol{D}$ is unimodular if any linearly independent set of vectors in $\boldsymbol{D}$ is an integer basis for the subspace they span.

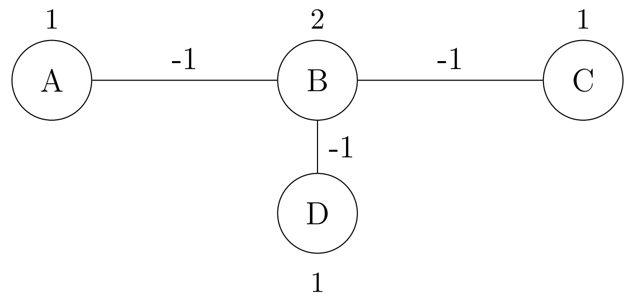
With this definition, I introduce their main results:

Theorem 2. *(Unimodularity Theorem): An equilibrium exists for every pair of concave valuations of demand type* $\boldsymbol{D}$*, for all relevant supply bundles, iff* $\boldsymbol{D}$ *is unimodular.*

Corollary 2. *With* $\boldsymbol{n}$ *goods, if the vectors of* $\boldsymbol{D}$ *span* $\mathbb{R}$*, then an equilibrium exists for every*

[14]Baldwin and Klemperer (2019) note that the unimodular theorem "states that competitive equilibrium always exists, whatever is the market supply, if and only if [...]" (p. 868).

Figure 4: Example 3.2 from the Online Appendix of Candogan et al. (2015)



Numbers above or below the nodes represent individual surplus terms w_l from obtaining good l and those above or next to the edges represent pairwise surplus terms w_{lk} from obtaining a pair of goods l and k.

finite set of concave valuations of demand type D*, for all relevant supplies, iff every subset of* n *vectors from* D *has determinant* 0 *or* ± 1.

Before introducing the example, I introduce a payoff function of buyers. Suppose the seller wants to maximize her revenue and buyer i has the following BQP preferences from obtaining a bundle of goods S:

$$u_i(S) = \sum_{j \in S} (w^i_j - p_j) + \sum_{j,k \in S : j \neq k \neq i} w^i_{jk}, \tag{9}$$

where the difference from the main text is a uniform price of good j across buyers.

Now, consider a situation in which there are four goods, A, B, C, and D. To simplify the setting, suppose all of the buyers have the same preferences over these four goods. Figure 3 demonstrates the buyers' preferences. Numbers above or below the nodes represent individual surplus terms w_l from obtaining good l and those above or below the edges represent pairwise surplus terms w_{lk} from obtaining a pair of goods l and k.

Next, to check the demand type associated with the valuation function, I label the entries of the demand type vectors by A, B, C, and D, respectively. First, assume that the price of item D is arbitrarily high, and prices of the remaining goods are $p(\mathrm{A}) = 0.1, p(\mathrm{B}) = 0.5, p(\mathrm{C}) = 0.1$. Then, it follows that the corresponding demand $D(p) = \{\mathrm{A}, \mathrm{C}\}$. Now, change the price to $p'(\mathrm{A}) = 1, p'(\mathrm{B}) = 0.5, p'(\mathrm{C}) = 0.1$. It follows that the demand $D(p') = \{\mathrm{B}\}$. This change in the demand set implies that vector $d_1 = [1, -1, 1, 0]$ lies in the demand type associated with the valuation. Using the symmetry between the nodes, we can deduce that vectors $d_2 = [1, -1, 0, 1]$ and $d_1 = [0, -1, 1, 1]$ also lie in the demand type. Lastly, consider another price vector $p'' = [2, 0, 2, 2]$ at which the only demanded bundle is $\{\mathrm{B}\}$. Notice that if we increase the price of B, the demand set switches to the empty set. Therefore, the demand type also contains $d_4 = [0, -1, 0, 0]$. Observe that the matrix with columns d_1, d_2, d_3, and d_4 has determinant 2. Hence, it follows from Corollary 1 that the demand type associated with the tree valuation in Figure 3 does not always have a Walrasian equilibrium *if any supply bundle is allowed*. Yet, restricting preferences to sign-consistent tree valuations and the supply of each item to be a single copy, Candogan et al. (2015) finds a Walrasian equilibrium. Note that the sign-consistent tree valuation class is a subset of my sign-consistent sign-balanced valuation class.