f(x) е дефинирано и ограничено в/у измер по Ж мн-во Ω

$$\tau = \{G_i\}_{i=1}^{n} - Razbivane - na - omega\Omega$$

$$1)G_i - izmerimo - po - Jordan$$

$$2)G_i \bigcap G_j = \emptyset, \forall i, j = 1, 2, ..n; i! = j$$

$$3) \bigcup_{i=1}^{n} = \Omega$$

$$m_i = \inf_{x \in G_i} f(x)$$

$$M_i = \sup_{x \in G_i} f(x)$$

$$s = \sum_{i=1}^{n} m_i m(G_i)$$

$$S = \sum_{i=1}^{n} M_i m(G_i)$$

Свойство 1:

$$\forall \tau = \{G_i\}_{i=1}^{n} \xi = \{\xi_i\}_{i=1}^{n}, \xi_i \in G_i \forall i = 1, 2, .., n$$

$$=> s\tau \leq \sum_{i=1}^{n} f(\xi_i) * m(G_i) \leq S_\tau$$

Дока-во:
$\forall i = 1, 2, ..., n \quad m_i \leq f(\xi_i) \leq M_i, (\xi_i \in G_i)$

$$=> \forall i = 1, 2, ..., n \quad m_i m(G_i) \leq f(\xi_i) m(G_i) \leq M_i m(G_i)$$

$$\sum_{i=1}^{n} m_i m(G_i) \leq \sum_{i=1}^{n} f(\xi_i) m(G_i) \leq \sum_{i=1}^{n} M_i m(G_i)$$

$$s_\tau \leq \sum_{i=1}^{n} f(\xi_i) m(G_i) \leq S_\tau$$

Свойство 2:

$$s_\tau = \inf_{\xi} \sigma_\tau (fi\xi)$$

$$S_\tau = \sup_{\xi} f(x)$$

Доказателство:

$$s_\tau = \inf_{\xi} \sigma_\tau (fi\xi)$$

$$\tau = \{G_i\}_{i=1}^n$$

$$s_\tau = \sum_{i=1}^n m_i m(G_i),\ m_i = \inf_{x\in G_i} f(x),\ \forall i = 1,2,..,n \quad \sigma_\tau(f_i\xi) = \sum_{i=1}^n f(\xi_i)m(G_i),\ \xi_i \in G_i\ \forall i = 1,2,...,n$$

$$\forall i = 1,2,...,n$$

$$1)mi <= f(\xi_i)$$

$$\forall \epsilon > 0 \exists \xi_i \in G_i : f(\xi_i) < m_i + \epsilon$$

$$f(\xi_i)m(G_i) < (m_i + \epsilon)m(G_i)$$

$$\sigma_\tau(f_i\xi) = \sum_{i=1}^n f(\xi_i)m(G_i) < \sum_{i=1}^n (m_i + \epsilon)m(G_i) = \sum_{i=1}^n (m_i)m(G_i) + \sum_{i=1}^n (\epsilon)m(G_i)$$

$$\exists \sigma_\tau(f_i\xi) < s_\tau + \epsilon \sum_{i=1}^n m(G_i) = s_\tau + \epsilon m(\Omega)$$

$$\sigma_\tau(f_i\xi) < s_{tau} + \epsilon m(\Omega) => s_{tau} = inf_\xi \sigma(f_i\xi)$$

Определение:
$\tau = \{G_i\}_{i=1}^n$ $\tau' = \{D_j\}_{j=1}^n$ - разбиване на Ω Казваме, че τ' следва τ, ако $\forall_j$=1,2,3...,p $\exists$ i = 1,2,...,n: $D_j \subset G_i$

Свойство 3: Ако τ= $\{G_i\}_i^n$, $\tau' = \{D_j\}_{j=1}^n$ $\tau' < \tau => s_\tau \leq s_{\tau'} \leq S_{\tau'} \leq S_\tau$

Доказателство:

$$\tau' < \tau => \forall_j = 1,2,...,p \exists i = 1,2,3,...,n : D_j \subset G_i$$

$$G_i = \sum_{D_j \subset G_i} D_j$$

$$s_\tau = \sum_{i=1}^n m_i m(G_i) = \sum_{i=1}^n m_i \sum_{D_j \subset G_i} m(D_j) = \sum_{i=1}^n \sum_{j=1}^p m'_j m(D_j)$$

$$m_i = \inf_{G_i} f(x)$$

$$m_{i'} = \inf_{D_j} f(x)$$

ако $D_j \subset G_i => m_i \leq m'_j$

$$\sum_{i=1}^n m_i \sum_{D_j \subset G_i} m(D_j) \leq \sum_{i=1}^n m_i \sum_{D_j \subset G_i} m(D_j) m'_j m(D_j) = \sum_{j=1}^p m'_j m(D_j) = s_{\tau'}$$

Свойство 4: $\forall$ τ, τ'- разбивания на измеримо по Ж. мн-во Ω => $s_\tau <= S_{\tau'}$
Доказателство:

$$\tau = \{G_i\}_{i=1}^n$$

$$\tau' = \{D_j\}_{j=1}^{p}$$

$$\tau = \{G_i \cap D_j\}, i = 1, 2, .., n; j = 1, 2, 3..., p$$

$$\tau'' < \tau$$

$$\tau'' < \tau'$$

$$s_\tau <= s_{\tau''} <= S_{\tau''} <= S_{\tau'}$$

$$s_\tau <= S_{\tau'}$$

Свойство 5:

$$\exists \sup_\tau s_\tau = \underline{I}$$

$$\exists \inf_\tau S_\tau = \overline{I}$$

$$s_\tau <= \underline{I} <= \overline{I} <= S_\tau, \forall \tau$$

Доказателство:

от миналото св-во имаме, че:

$$s_\tau <= S_{\tau'}$$

$$\forall \tau : s_\tau <= S_{\tau'}\{s_\tau : \tau\} - e - ogranicheno - otgore => \sup_\tau s_\tau = \underline{I}$$

$$=> \underline{I} <= s_{\tau'} => \{s_{\tau':\tau} e - ogranichena - otdole\}$$

$$=> \inf_\tau S_\tau = \overline{I}$$

$$=> \underline{I} <= \overline{I}$$

Критерий за интегрируемост - f(x) е интегрируема по Риман в/у изм. по Ж мн-во $\Omega \subset R^n <=> \forall \epsilon > 0 \exists \delta = \delta(\epsilon) > 0 : \forall \tau = \{G_i\}_i^n, \delta_\tau < \delta => S_\tau - s_\tau < \epsilon$

Доказателство:

В едната посока от ляво на дясно:

Нека f(x) е интегрируема по Риман в/у $\Omega => \exists I \in R : \forall \epsilon > 0 \exists \delta = \delta(\epsilon) > 0 : \forall \tau = \{G_i\}_{i=1}^n, \delta_\tau < \delta, \forall \xi = \{\xi_i\}_{i=1}^n, \xi_i \in G_i(i = 1, 2, 3, , ., n) => |I - \sigma_\tau(fi\xi)| < \epsilon$

$$I - \epsilon < \sigma_\tau(fi\xi) < I + \epsilon$$

$$I - \epsilon <= s_\tau = \inf_\xi \sigma_\tau(fi\xi) <= \sup_\xi \sigma_\tau(fi\xi) = S_\tau < I + \epsilon$$

$$I - \epsilon <= s_\tau <= S_\tau <= I + \epsilon$$

$$S_\tau <= I + \epsilon$$

$$s_\tau <= -I + \epsilon$$

$$S_\tau - s_\tau <= 2\epsilon$$

Доказателство в обратната посока: Нека $\forall \epsilon >= 0 \exists \delta = \delta(\epsilon) > 0 : \forall \tau = \{G_i\}_{i=1}^n, \delta_\tau < \delta => S_\tau - s_{\tau)} < \epsilon$

$$s_\tau <= \underline{I} <= \overline{I} <= S_\tau$$

$$0 <= \overline{I} - \underline{I} <= S_\tau - s_\tau < \epsilon$$

$$0 <= \overline{I} - \underline{I} < \epsilon$$

$$=> \overline{I} = \underline{I}$$

$$s_\tau <= \sigma_\tau(fi\xi) <= S_\tau$$

$$s_\tau <= I <= S_\tau$$

като от 1вото издавим 2рото:

$$\sigma_\tau(fi\xi) - I <= S_\tau - s_\tau$$

$\exists I : \forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0 \forall \tau = \{G_i\}_{i=1}^n, \delta_\tau < \delta, \forall \xi = \{\xi_i\}_{i=1}^n, \xi_i \in G_i, i = 1, 2, ..., n => |\sigma(fi\xi) - I| <= S_\tau - s_\tau < \epsilon =>$ f(x) е интегр. по Риман в/у Ω