MTH252 Midterm Assignment

Bao-Hiep Le

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1 Introduction

$$\frac{1}{P} \cdot \frac{dP}{dt} = aP + b$$

$$\int \frac{dP}{P(aP+b)} = \int dt \tag{1}$$

To evaluate the integral on the left side, we use partial fraction decomposition:

$$\frac{1}{P(aP+b)} = \frac{X}{P} + \frac{Y}{aP+b}$$

Multiplying both sides of this equation by $P(aP+b)$ and rearranging the terms, we have:

$$1 = (Xa + Y)P + Xb$$

The polynomials are identical, so their coefficients must be equal:

$$\begin{aligned} Xa + Y &= 0 \\ Xb &= 1 \end{aligned}$$

We get $X = \dfrac{1}{b}$ and $Y = \dfrac{-a}{b}$. Therefore:

$$\frac{1}{P(aP+b)} = \frac{1}{b} \cdot \frac{1}{P} - \frac{a}{b} \cdot \frac{1}{aP+b}$$

We can now rewrite equation 1 as:

$$\begin{aligned}
\int \left(\frac{1}{b} \cdot \frac{1}{P} - \frac{a}{b} \cdot \frac{1}{aP+b} \right) dP &= \int dt \\
\frac{1}{b} \ln|P| - \frac{1}{b} \ln|aP+b| &= t + C \\
\ln\left|\frac{P}{aP+b}\right| &= bt + bC \\
\left|\frac{P}{aP+b}\right| &= e^{bt+bC} = e^{bt}e^{bC} \\
\frac{P}{aP+b} &= Ae^{bt} \qquad (2)
\end{aligned}$$

where $A = \pm e^{bC}$. Solving for P, we get:

$$P = \frac{bAe^{bt}}{1 - aAe^{bt}}$$

We find the value of A by putting $t = 0$ in equation 2:

$$A = \frac{P_0}{aP_0 + b}$$

$$\frac{dP}{dt} = \frac{b^2Ae^{bt}}{(1 - aAe^{bt})^2}$$