The Nexus Between Environmental Policy and Income Inequality: A Growth Approach*

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Abstract

This paper examines the impact of environmental policy in the form of emissions trading on income inequality in an endogenous growth model, an area that has received less attention thus far. We integrate pollution and environmental policy into a Schumpeterian growth framework to establish a theoretical relationship between the stringency of environmental policy and income inequality. Our analysis reveals that the nexus varies depending on the elasticity of labor supply. Specifically, a stricter environmental policy in the form of a lower cap for pollution permits increases income inequality when labor supply is inelastic. In the case of elastic labor supply, the relationship between environmental policy and income inequality becomes more involved and depends on parameter values. We show that under a reasonable calibration, a stricter environmental policy decreases income inequality.

Keywords: Emissions trading; Inequality; Schumpeterian growth
JEL codes: O41; O44; Q56; Q58

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1 Introduction

Our living standards have seen remarkable improvements over the past centuries. Since the dawn of the Industrial Revolution around 1760, we have achieved many breakthroughs not only in producing enough food to feed entire populations but also in developing innovations that would have been beyond our ancestors' wildest dreams. However, despite all these miracles, we have also created problems that persistently challenge us, even to this day. Among them, environmental degradation and income inequality have received considerable attention from both the general public and the research community.

Figure 1 shows annual anthropogenic CO_2 emissions during 1760-2021. It is observable that the CO_2 level has witnessed staggering growth during the post–World War II era. Even though the rate of emissions increase has slowed over time, it still took only around 40 years, 1980-2021, to double the already-excessive 1980 level. The annual emissions in 2021 surpassed the pre-pandemic level and reached the highest level ever. Carbon dioxide, together with other greenhouse gases, has contributed to global warming, causing many places on Earth to be unbearable to live in. Besides environmental degradation, income inequality is another major challenge we have been facing. According to the World Inequality Report 2022 (Chancel et al., 2022), global inequality, as measured by the ratio between the average income of the top 10% and the bottom 50% of the world population, doubled from 20 to 40 between 1820 and 1910 and has remained relatively constant around 40 since 1910.

Despite the extensive literature on income inequality and environmental degradation, many questions remain less studied. This paper contributes to the literature on the impact of environmental policy on income inequality, which has received even less attention in research, especially from theoretical economists. We extend a Schumpeterian growth model akin to the framework in Acemoglu (2009, chapter 14) since it provides a tractable framework for examining the relationship between environmental policy and income inequality.

We depart from Acemoglu (2009, chapter 14) along three dimensions. First, we incorporate pollution and emissions trading, that is, a cap-and-trade system for pollution permits. Pollution in the model is a by-product of the final good production process. The more final good firms produce, the more they contribute to pollution. A cap-and-trade system is introduced to make firms internalize the detrimental environmental consequences they cause. Excessive emissions mean that final good firms must purchase additional permits for the pollution that exceeds their distributed permit quotas. The firms can also devote part of their resources to costly abatement activities, which reduce the pollution flow they emit. Irrespective of what they choose to do, pollution costs firms money, and they have to take it into account when maximizing profits.

Second, our model is a scale-free model. It is well-known that most innovation-driven growth models feature scale effects in the sense that they predict a proportional increase in the economic growth rate with the number of working people or the number

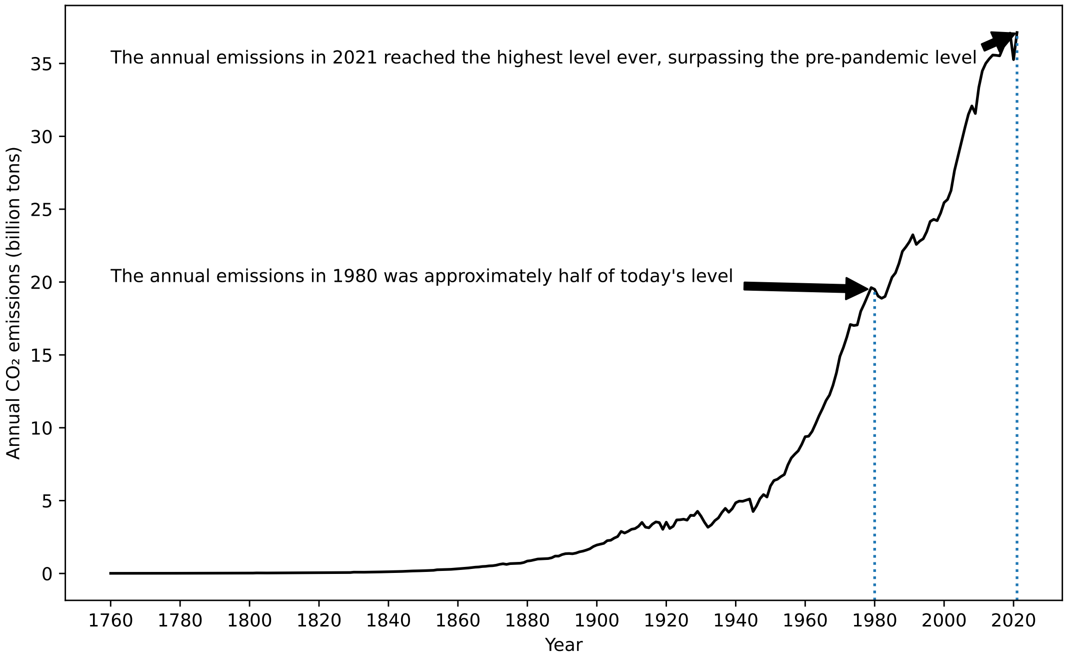

Figure 1: Annual CO_2 emissions (1760 − 2021). *Data source* Friedlingstein et al. (2022)

of R&D researchers. However, the time series evidence against scale effects is abundant. Perhaps one of the most classic examples is Jones (1995). A recent paper from Bloom et al. (2020) also finds that ideas are not increasing with the resources devoted to research. While the effective number of researchers in the U.S. has experienced a roughly exponential increase since 1930, the TFP growth has fallen constantly. All this evidence motivates us to consider a scale-free version of the standard model in Acemoglu (2009, chapter 14).

Third, we introduce heterogeneity into the household sector. Traditional growth models have a representative household, which renders the study of income inequality virtually impossible. Households in this model have identical preferences but hold different asset portfolios. They earn capital income from asset investments and labor income from working for final good firms. Furthermore, the typical assumption in standard models is that the representative household inelastically supplies a fixed amount of labor. We generalize this assumption by allowing households to supply labor elastically.

This study shows that a more restrictive environmental policy in the form of a lower cap for pollution permits affects income inequality through various channels, namely, the interest rate, the wage-asset, and the labor supply channels. In the model, a stricter environmental policy dampens aggregate activities, decreasing output, consumption, and wages. This reduction in final output leads to lower demand for intermediate goods. The declining profitability of the intermediate good sector reduces the incentives to innovate, which in turn results in a slowdown in economic growth. Less profitable investment in intermediate good firms also means a lower interest rate (rate of returns on savings) faces by households. These aggregate effects, in turn, impact income inequality as follows. First, a lower interest rate reduces the relative proportion of capital income in the households' income composition since the interest rate is the market rate of return on assets. As a result, households with more assets are affected more, and therefore, a lower interest rate decreases income inequality. Second, lower wages, relative to asset values, increase income inequality because households with fewer assets rely relatively more on labor income to finance their consumption. In our model, the wage-asset channel dominates the interest rate channel. Hence, in the case of inelastic labor supply, a stricter environmental policy increases income inequality.

The impact of a stricter policy is also channeled through labor supply adjustments when labor supply is elastic. This channel is muted in the case of inelastic labor supply since all households supply the same amount of labor. The households' intratemporal labor/leisure trade-off determines the magnitude of the response in labor supply. Given that households in the model share identical preferences, their optimal behavior should be the same, which results in the same growth rate of consumption and assets for all households. Consequently, individual and aggregate variables have the same growth rate. A lower economic growth rate due to constrained aggregate activities means lower savings are needed to support economic growth. Lower savings, in turn, translate into either higher consumption or lower labor supply, holding the interest rate and wages unchanged. The intratemporal labor/leisure trade-off determines how much the reduction in savings is attributed to the response from consumption or labor supply. Identical preferences result in the same division of savings reduction into higher consumption and lower labor supply for all households. Therefore, due to the same saving rate, households with more assets reduce savings more and thus reduce labor supply more compared to households with fewer assets. As a result, a more restrictive environmental policy decreases income inequality via the labor supply channel.

When labor supply is elastic, the interest rate and labor supply channels decrease income inequality, while the wage-asset channel increases income inequality in the presence of a more restrictive environmental policy. Since the wage-asset channel dominates the interest rate channel in this model, the overall effect of a policy change on income inequality depends on the labor supply channel whose magnitude is decided by the leisure parameter in this model. Our numerical exercise demonstrates that under a reasonable calibration, the combined effect through the interest rate and the labor supply channels dominates the wage-asset channel, and therefore, a stricter environmental policy reduces income inequality.

Our work relates to two strands of research. The first pertains to the nexus between income inequality and environmental issues. This strand of literature can be divided into two groups: the effects of income inequality on environmental issues and the effects of environmental issues on income inequality. While there is extensive literature on the impact of income inequality on environmental issues, the opposite direction remains less studied.

There is no clear consensus among theoretical and empirical economists on the impact of income inequality on environmental issues. Boyce (1994) is among the first theoretical papers that examine the impact of income inequality on environmental degradation. The author shows that greater income inequality degrades environmental quality. This result is supported by empirical evidence (e.g., Baek and Gweisah, 2013; Jorgenson, 2015; Hao et al., 2016; Uzar and Eyuboglu, 2019). However, Scruggs (1998) and Ravallion et al. (2000) oppose the result from Boyce (1994) from theoretical perspectives. According to Scruggs (1998), wealthy people prioritize environmental quality more, suggesting that income inequality could reduce environmental degradation. This conclusion is consistent with empirical findings from Heerink et al. (2001), Guo (2013), and Huang and Duan (2020).

There is a relatively small literature on the effects of environmental issues on income inequality. Jha et al. (2019) and Wei and Zhao (2022) are among the few empirical papers that investigate the impact of environmental policy on income inequality. While Jha et al. (2019) find evidence that a stricter environmental regulation might increase income inequality in the U.S, Wei and Zhao (2022) obtain a similar result for China. Our theoretical findings in this paper are consistent with these empirical papers.

Regarding theoretical research, there is a handful of papers that answer similar questions to our study. Li et al. (2020) study the impact of environmental pollution on skilled-unskilled wage inequality using a small open economy model where labor can flow between two countries: the source and the recipient country. The paper shows that environmental pollution in the source country increases the wage inequality between skilled and unskilled workers, where brain drain (i.e., migration of skilled workers to the recipient country) amplifies this effect. Likewise, Kuo et al. (2022) employ a Harris and Todaro (1970) rural-urban migration model to examine the impact of environmental policy in the form of a pollution tax on skilled-unskilled wage inequality. Their study concludes that if the elasticity of substitution between unskilled labor and pollution is small enough, a higher pollution tax increases skilled-unskilled wage inequality.

Dao (2022) analyzes the impact of environmental policy on wealth inequality in an overlapping generations framework with credit market imperfections and climate damage on the heterogeneous households' wealth. The government imposes a tax on the dirty sector and subsidizes the clean sector. The author shows that when the aggregate physical capital exceeds a certain threshold, a balanced budget environmental policy may widen the wealth inequality gap between the rich and poor. Our study differs from the aforementioned theoretical papers by considering the effect of environmental policy, specifically emissions trading, on income inequality instead of wage or wealth inequality. Income in our model consists of capital income and labor income.

The second strand of research that shares a similar methodology with this paper is the relatively small literature on the impact of various economic factors (e.g., patent policy, R&D subsidy, inflation) on income inequality. This group of theoretical research starts from García-Peñalosa and Turnovsky (2006) and employs different endogenous economic growth models to investigate possible effects on inequality (e.g., Chu and

Cozzi, 2018; Chu et al., 2019; Zheng et al., 2020; Chan et al., 2022). A crucial characteristic of these models is a stationary distribution of assets among heterogeneous households on the balanced growth path, which makes the analysis tractable. Our model is the first theoretical paper that applies a similar approach to examine the impact of environmental policy on income inequality. Additionally, in contrast to the papers in this second strand of literature, our study carefully considers the case of elastic labor supply.[1]

The remainder of the paper is organized as follows. Section 2 lays out the model. Section 3 characterizes the decentralized equilibrium and the balanced growth path of the model. Then, we explore the effects of environmental policy on income inequality in section 4. Section 5 discusses and concludes.

2 The model economy

The model in this study is based on the quality-ladder growth model in Acemoglu (2009, chapter 14). In this type of endogenous growth model, R&D activities lead to quality improvements of existing machines used in the production of a unique final good, rather than expanding the variety of machines as in the Romer (1990) model. This model class is often referred to as the Schumpeterian growth models. They are based on the concept of creative destruction originated from Schumpeter (1942) in the sense that newly invented goods replace obsolete vintages.

Regarding the extended model in this study, we include pollution and tradable pollution permits. Moreover, to examine income inequality, we incorporate heterogeneity into the household sector rather than the representative household we often observe in standard growth models. Our economic framework consists of a final good sector, an intermediate good sector, and an R&D sector. Final good firms compete in a perfectly competitive environment. They employ labor from households and machines (i.e., intermediate goods in this model) to produce the final good. Pollution is a by-product of this process of final good production and imposes a cost on the environment. More production means more pollution, and due to the presence of pollution permits, final good firms must internalize the environmental consequences of their production process. They must either pay for excessive pollution or participate in costly abatement activities to reduce emissions.

As in the standard quality-ladder model, R&D firms develop the design of a higher-quality version of machines in the intermediate good sector. Once innovating successfully, they then receive a perpetual patent and start producing the improved machine vintage. These entrants drive the incumbents out of the market and become new monopolists in the intermediate good sector until being replaced by new innovators. Firms in the R&D sector use research outlays as their sole input.

[1] Chu et al. (2021) is an exception. However, they only mention the case of elastic labor supply in the Appendix without detailed intuition about the mechanisms behind mathematical expressions.

2.1 Final good sector

There is a unit mass of perfectly competitive firms $i \in [0, 1]$ that produce the unique final good by the following production function:

$$Y(i,t) = \frac{1}{1-\beta} L(i,t)^{\beta} \int_0^1 q(\nu,t) x(i,\nu,t|q)^{1-\beta} \, d\nu, \tag{1}$$

where $Y(i,t)$ is the output of the final good at firm i at time t, $L(i,t)$ is labor hired by firm i at time t, and $\beta \in (0,1)$ captures the labor share of income. $x(i,\nu,t|q)$ is the quantity of machine ν of quality $q(\nu,t)$ used in production at firm i at time t, where $q(\nu,t)$ is the highest quality of machine ν at time t.[2] We follow Acemoglu (2009, chapter 14) to use "machines" to refer to intermediate goods. The machine quality evolves according to the following dynamics:

$$q(\nu,t) = q(\nu,0)\lambda^{n(\nu,t)},$$

where $q(\nu,0)$ is the quality level at time 0, $\lambda > 1$ denotes the quality step size of each innovation, and $n(\nu,t)$ is the number of innovations that occurred in the product line ν between time 0 and t. The production function in eq. (1) assumes that this model has a unit mass of machines: $\nu \in [0,1]$. Finally, we treat the final good as the numeraire.

Pollution is a by-product of the final good production process. Following Acemoglu et al. (2012), we assume that pollution flow is linearly increasing in output level. Furthermore, pollution can be reduced by the presence of costly abatement expenditure (e.g., Gradus and Smulders, 1993). Combining these assumptions, pollution flow $E(i,t)$ from firm i at time t is produced by the following linear mechanism:

$$E(i,t) = \alpha[Y(i,t) - D(i,t)], \tag{2}$$

with parameter $\alpha > 0$ and $D(i,t) < Y(i,t)$ is abatement expenditure by firm i at time t.

The authorities implement environmental policies to internalize the negative externalities of pollution. We consider a cap-and-trade system in this study. Each final good firm i is distributed permit quotas $\bar{E}Y(i,t)$ at time t. There is a competitive market for pollution permits where final good firms trade their distributed quotas. The unit price of the permits is denoted by $p_E(t)$. Any firm i that emits excessively (i.e., $E(i,t) > \bar{E}Y(i,t)$) must purchase permits of $E(i,t) - \bar{E}Y(i,t)$ at the market price $p_E(t)$. On the other hand, firms that emit less (i.e., $E(i,t) < \bar{E}Y(i,t)$) can sell permits of $\bar{E}Y(i,t) - E(i,t)$ to earn revenue. From the pollution function in eq. (2), we must have $\bar{E} < \alpha$. If we had $\bar{E} \geq \alpha$, the cap-and-trade would not impact final good firms' behavior since the emissions cap for each firm is no less than the pollution flow the firm emits. We call $\bar{E}$ the policy parameter in the model.

[2] Final good firms rent only leading-edge machines. The following section will give more details about this assumption.

Final good firm i chooses labor $L(i,t)$, quantity of machines $x(i,\nu,t|q)$, and abatement expenditure $D(i,t)$ to maximize its profit $\Pi(i,t)$ at time t as follows:

$$\Pi(i,t) = Y(i,t) - w(t)L(i,t) - \int_0^1 p(\nu,t|q)x(i,\nu,t|q)\,d\nu - D(i,t) - p_E(t)[E(i,t) - \bar{E}Y(i,t)], \tag{3}$$

where $w(t)$ denotes labor wages at time t and $p(\nu,t|q)$ refers to the price of machine variety ν with quality $q(\nu,t)$ at time t. The Appendix shows the detailed derivation of the firm's profit maximization problem. The first-order conditions of profit optimization are

$$w(t) = \left(\frac{\bar{E}}{\alpha}\right)\beta\frac{Y(i,t)}{L(i,t)}, \tag{4}$$

$$p(\nu,t|q) = \left(\frac{\bar{E}}{\alpha}\right)q(\nu,t)\left(\frac{L(i,t)}{x(i,\nu,t|q)}\right)^{\beta}, \tag{5}$$

$$p_E(t) = \frac{1}{\alpha}, \tag{6}$$

where eq. (4) is the labor demand function and eq. (5) shows the inverse demand function for machine variety $\nu \in [0,1]$. The equilibrium permit price is shown in eq. (6).

We sum up eqs. (4)-(5) for each firm within the final good sector to obtain aggregated equations:

$$w(t) = \left(\frac{\bar{E}}{\alpha}\right)\beta\frac{Y(t)}{L(t)}, \tag{7}$$

$$p(\nu,t|q) = \left(\frac{\bar{E}}{\alpha}\right)q(\nu,t)\left(\frac{L(t)}{x(\nu,t|q)}\right)^{\beta}, \tag{8}$$

where $Y(t) = \int_0^1 Y(i,t)di$, $L(t) = \int_0^1 L(i,t)di$, and $x(\nu,t|q) = \int_0^1 x(i,\nu,t|q)di$ denote the total final output, the aggregate labor and the final good sector's demand for machine ν, respectively.

2.2 Intermediate good sector

Monopolistic firms control the production of intermediate goods. Each monopolist starts as an R&D firm, developing an improved vintage of an existing machine. Upon achieving this, the innovator then receives a perpetual patent for the new vintage and replaces the incumbent in the product line of the machine to become the new monopolist. Figure 2 demonstrates the innovation process in machine lines $\nu \in [0,1]$. We assume that innovations are sufficiently drastic such that the innovator drives the incumbent out of the market.

Lemma 1. *If the quality step size λ satisfies*

$$\lambda \geq \left(\frac{1}{1-\beta}\right)^{\frac{1-\beta}{\beta}}, \tag{9}$$

entrants will always drive out incumbents even if they charge the monopolist's price.

Proof. See the Appendix.

Note that the production function in eq. (1) already assumes that only the highest quality machine of each line is utilized to produce the final good. Furthermore, we can imagine that each innovator picks a random machine line $\nu \in [0, 1]$ and puts effort into developing a higher-quality version of that machine. The innovation process in each line is, therefore, stochastic and independent. We later show that despite these stochastic innovations, aggregate variables are nonstochastic.

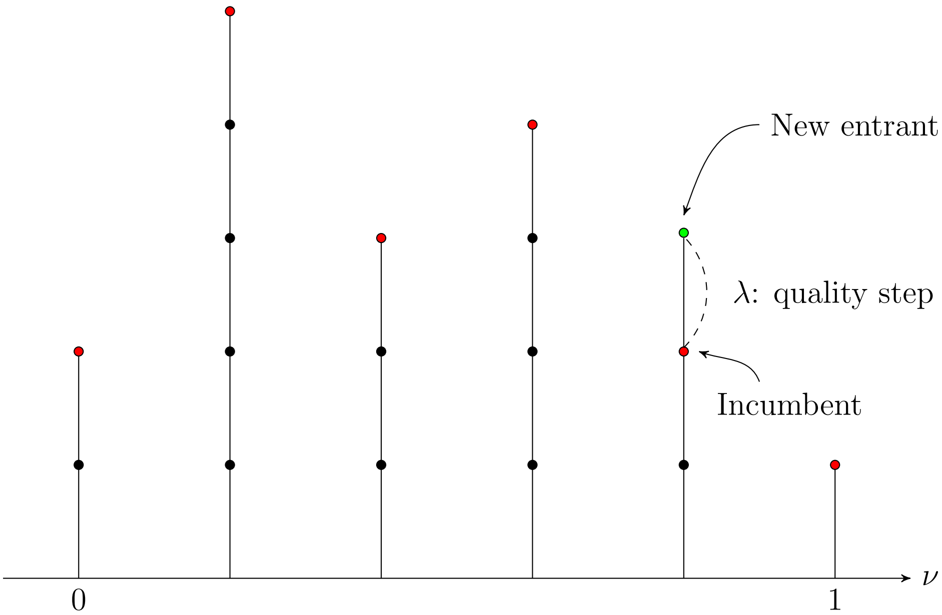

Figure 2: Innovation process in machine lines

We assume that the marginal cost of producing a unit of intermediate good ν at time t is $\psi q(\nu, t)$ units of the final good. This functional form captures the idea that more advanced machines are more expensive to produce. Each monopolist ν can charge the price $p(\nu, t|q)$ upon the demand $x(\nu, t|q)$ for the machine ν from final good firms. The profit optimization problem for machine ν's owner at time t is

$$\pi(\nu, t|q) = [p(\nu, t|q) - \psi q(\nu, t)]\, x(\nu, t|q), \tag{10}$$

subject to the demand function in eq. (8). The Appendix shows the derivation for the following first-order condition:

$$p(\nu, t|q) = \frac{1}{1-\beta}\psi q(\nu, t), \tag{11}$$

where $1/(1-\beta)$ is the markup of intermediate good firms. We normalize $\psi \equiv 1 - \beta$ and the optimal price becomes[3]

$$p(\nu, t|q) = q(\nu, t). \tag{12}$$

[3]The main purpose of this normalization is to keep expressions in the main text simple. The Appendix will show that it does not affect the main findings of this study.

The calculations of demand and profit for each intermediate good are straightforward from eqs. (5), (8), and (10):

$$x(i,\nu,t|q) = \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L(i,t), \tag{13}$$

$$x(\nu,t|q) = \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L(t), \tag{14}$$

$$\pi(\nu,t|q) = \beta\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} q(\nu,t)L(t), \tag{15}$$

which shows that the monopoly profits received by innovators on each machine line are increasing in machine quality.

Substituting the machine demand function in eq. (13) into the production function in eq. (1) to yield the expression for output at firm i

$$\begin{aligned} Y(i,t) &= \frac{1}{1-\beta}L(i,t)^{\beta}\int_0^1 q(\nu,t)\left[\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L(i,t)\right]^{1-\beta} d\nu \\ &= \frac{1}{1-\beta}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta-1} Q(t)L(i,t), \end{aligned} \tag{16}$$

where we define

$$Q(t) \equiv \int_0^1 q(\nu,t)d\nu$$

as the average total quality of machines at time t.

We utilize eqs. (4), (12)-(13), and (16) to show that

$$w(t)L(i,t) = \beta\frac{\bar{E}}{\alpha}Y(i,t), \tag{17}$$

$$\int_0^1 p(\nu,t|q)x(i,\nu,t|q)d\nu = (1-\beta)\frac{\bar{E}}{\alpha}Y(i,t). \tag{18}$$

Substituting eqs. (2) and (17)-(18) into (3) yields

$$\Pi(i,t) = \left(1-\frac{\bar{E}}{\alpha}\right)Y(i,t) - D(i,t) - \frac{1}{\alpha}\left\{\alpha[Y(i,t)-D(i,t)] - \bar{E}Y(i,t)\right\} = 0.$$

Hence, final good firms earn zero profit for any level of abatement expenditure $D(i,t)$. Moreover, if $D(i,t) < \left(1-\frac{\bar{E}}{\alpha}\right)Y(i,t)$, final good firm i must purchase additional permits. Firm i can sell permits if $D(i,t) > \left(1-\frac{\bar{E}}{\alpha}\right)Y(i,t)$.

2.3 R&D sector

Let us turn to the R&D sector, which is the growth engine in endogenous growth models. In reality, firms usually produce goods and carry out research simultaneously. However, it is a common practice in growth models to separate production and research sectors, which leads to a clearer exposition. R&D firms invent higher-quality machines to replace their previous vintages. This process is cumulative and new R&D knowledge builds on the technical know-how of the existing machines. We follow Annicchiarico et al. (2022) to assume that if a firm spends $Z(\nu, t|q)$ units of final good for research in product line ν at time t when the machine quality is $q(\nu, t)$, innovations arrive at the Poisson rate

$$z(\nu, t|q) = \frac{\eta Z(\nu, t|q)}{q(\nu, t)L(t)}, \tag{19}$$

where $\eta > 0$ is the parameter of research productivity. In other words, within a small time interval $[t, t+\Delta]$, an innovation occurs with the probability approximately equal to $\Delta z(\nu, t|q)$. The functional form in eq. (19) effectively removes the well-known scale effects in the literature of endogenous growth models. We will show later that the equilibrium rate of economic growth in this model does not depend on the size of the aggregate labor force.

Suppose the incumbent's quality in the machine line ν is $q(\nu, t)\lambda^{-1}$ and let $V(\nu, t|q)$ denote the value of a patent that an innovator obtains at time t when quality becomes $q = q(\nu, t)$. From eq. (19), we know that $Z(\nu, t|q\lambda^{-1})$ units of final good generate a flow rate of innovation $z(\nu, t|q\lambda^{-1}) = \dfrac{\eta Z(\nu, t|q\lambda^{-1})}{q(\nu, t)\lambda^{-1}L(t)}$. The expected profit of undertaking R&D is $z(\nu, t|q\lambda^{-1})V(\nu, t|q) - Z(\nu, t|q\lambda^{-1})$. If we assume free entry into research, the expected profit on R&D investment must be zero such that

$$\frac{\eta Z(\nu, t|q\lambda^{-1})}{q(\nu, t)\lambda^{-1}L(t)} V(\nu, t|q) - Z(\nu, t|q\lambda^{-1}) = 0 \Leftrightarrow V(\nu, t|q) = \frac{q(\nu, t)L(t)}{\eta\lambda}. \tag{20}$$

We refer to this as the free entry condition.

Lemma 2 (Arrow's replacement effect)**.** *All R&D is done by entrants.*

Proof. See the Appendix.

Lemma 3 (Hamilton-Jacobi-Bellman equation)**.**

$$r(t)V(\nu, t|q) = \pi(\nu, t|q) + \dot{V}(\nu, t|q) - z(\nu, t|q)V(\nu, t|q), \tag{21}$$

where $r(t)$ is the real interest rate prevailing at time t.

Proof. See the Appendix.

The investment return on a firm of any machine line ν at time t consists of the monopolistic profits $\pi(\nu, t|q)$, the capital gain $\dot{V}(\nu, t|q)$, and the loss due to the probability $z(\nu, t|q)$ of being replaced by a higher-quality machine vintage invented by a new entrant. This loss is often called the "business stealing" effect in the literature since the incumbent loses its position when innovation occurs. If there are riskless government bonds with the market rate of return $r(t)$, eq. (21) states that the expected return either on riskless assets or on firm investment must be the same. This is the reason why eq. (21) is also referred to as the no-arbitrage condition.

Note that for a given level of $q(\nu, t)$, which is the constant quality of the machine provided by the incumbent during the time interval $[t, t+\Delta]$ (until a new innovation arrives), the firm value in line ν must satisfy

$$\frac{\dot{V}(t)}{V(t)} = \frac{\dot{L}(t)}{L(t)}, \tag{22}$$

from the free entry condition in eq. (20) since $\dot{q}(\nu, t) = 0$ over the time interval. Hence, the no-arbitrage condition in eq. (21) implies

$$r(t) + z(\nu, t|q) = \frac{\pi(\nu, t|q)}{V(\nu, t|q)} + \frac{\dot{L}(t)}{L(t)} = \beta\eta\lambda \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \frac{\dot{L}(t)}{L(t)}, \tag{23}$$

where the last equality is the result of eqs. (15) and (20). Therefore, the innovation rate $z(\nu, t|q)$ is the same for all $\nu \in [0, 1]$: $z(\nu, t|q) = z(t)$. In other words, the probability of successful innovation per unit of time is the same for all machine lines, independent of the quality ladder position. This is the result of the functional form in eq. (19) for innovation rate.[4]

We already define

$$Q(t) \equiv \int_0^1 q(\nu, t) d\nu$$

as the average total quality of machines at time t. By definition of the Poisson innovation rate, during a small time interval Δ, a random fraction $z(t)\Delta$ of machine lines experience quality improvement by the step size λ. Therefore, the dynamics of average quality $Q(t)$ is

$$\begin{aligned} Q(t+\Delta) &= z(t)\Delta \int_0^1 \lambda q(\nu, t) d\nu + (1 - z(t)\Delta) \int_0^1 q(\nu, t) d\nu \\ &= Q(t)[1 + z(t)\Delta(\lambda - 1)]. \end{aligned}$$

We subtract $Q(t)$ from both sides, dividing by Δ, and taking the limit as $\Delta \to 0$ to obtain the growth rate of the average quality of machines

$$\frac{\dot{Q}(t)}{Q(t)} = (\lambda - 1) z(t), \tag{24}$$

where $\dot{Q}(t) = \lim_{\Delta \to 0} \dfrac{Q(t+\Delta) - Q(t)}{\Delta}$. If we assume that the economy starts at a given average quality level $Q(0)$, the average machine quality is non-stochastic even though the innovation process in each machine line is random.

2.4 Households

There is no population growth in this model, and we assume a unit mass of infinitely-lived households indexed by $h \in [0, 1]$. In other words, we normalize the whole

[4] Barro and Sala-i Martin (2004, chapter 7) explain why the equal probability of innovation across the intermediate good sector is a necessary condition for the constant growth rate of the economy.

population to one. Households have identical preferences but possess different asset portfolios. They value consumption and leisure and suffer from environmental pollution. They are endowed with L unit of labor each period which they can choose how much to allocate between work and leisure. The lifetime utility of household h is

$$U = \int_0^\infty e^{-\rho t} \left[\ln C(h,t) + \varphi \ln(L - L(h,t)) - \gamma E(t)^b \right] dt,$$

where ρ denotes the subjective discount rate. $C(h,t)$ is the household's consumption of the final good at time t, $L(h,t)$ is its labor supply at time t (i.e., $L - L(h,t)$ denoting leisure at time t). $\varphi \geq 0$ is the weight of leisure in utility function where $\varphi = 0$ corresponds to the case of inelastic labor supply. $E(t) = \int_0^1 E(i,t)di$ is the total pollution flow at time t. While $\gamma > 0$ is the weight of pollution in the utility function, the parameter $b > 1$ captures the idea that the marginal damage of pollution is increasing.

Household h owns the total amount of assets $A(h,t)$ at time t earning the market interest rate $r(t)$. We suppose households are allowed to hold both riskless bonds and ownership of intermediate good firms.[5] From the no-arbitrage condition in eq. (21), we can say that $r(t)$ is also the expected return on firm patents at time t. Hence, the instantaneous change in assets is

$$\dot{A}(h,t) = r(t)A(h,t) + w(t)L(h,t) - C(h,t), \tag{25}$$

where $\dot{A}(h,t)$ is effectively the household's savings at time t. Given the above flow budget constraint, the household chooses consumption and labor to maximize utility. The Appendix shows the detailed derivations of the household's optimization problem.

We obtain the intertemporal Euler equation:

$$\frac{\dot{C}(h,t)}{C(h,t)} = r(t) - \rho, \tag{26}$$

which characterizes the optimal path of household h's consumption for a given interest rate $r(t)$. A higher interest rate would mean a higher growth rate of consumption. This equation applies to every household h. In other words, consumption growth is the same for all households. Hence, aggregation yields

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho, \tag{27}$$

where $C(t) \equiv \int_0^1 C(h,t)dh$ denotes the aggregate consumption at time t.

Another optimality condition is the intratemporal condition (i.e., the labor supply condition):

$$w(t)[L - L(h,t)] = \varphi C(h,t). \tag{28}$$

[5] The ownership of final good and R&D firms is irrelevant to the household's problem since these firms earn zero profit.

It states that the marginal rate of substitution between leisure and consumption is equal to the real wage. Since the equation holds for all households, we can apply the above logic again. Aggregation gives

$$\begin{aligned} \int_0^1 w(t)[L - L(h,t)]dh &= \int_0^1 \varphi C(h,t)dh \\ w(t)[L - L(t)] &= \varphi C(t), \end{aligned} \tag{29}$$

where $L(t) \equiv \int_0^1 L(h,t)dh$ denotes the aggregate labor supply at time t.

Finally, the transversality condition must be satisfied as well:

$$\lim_{t\to\infty} [e^{-\rho t}\mu(h,t)A(h,t)] = 0, \tag{30}$$

where $\mu(h,t)$ is the costate variable in household h's utility maximization problem. The condition requires the present value of assets to converge to zero as the time horizon approaches infinity. This condition holds for all households h, and it implies the aggregate transversality condition:

$$\lim_{t\to\infty} [e^{-\rho t}\mu(t)A(t)] = 0, \tag{31}$$

where $A(t) \equiv \int_0^1 A(h,t)dh$ is the aggregate asset and $\mu(t) \equiv \int_0^1 A(t)dh$ is the aggregate costate variable at time t.

It is also worth mentioning the aggregate budget constraint, which can be obtained directly from the household's budget constraint in eq. (25) by aggregation:

$$\dot{A}(t) = r(t)A(t) + w(t)L(t) - C(t). \tag{32}$$

2.5 Aggregation

Aggregating the reduced-form production function in eq. (16) yields the aggregate final output:

$$Y(t) = \frac{1}{1-\beta}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta - 1} Q(t)L(t). \tag{33}$$

We then obtain the expression for labor wages from eq. (7):

$$w(t) = \left(\frac{\bar{E}}{\alpha}\right)\beta\frac{Y(t)}{L(t)} = \frac{\beta}{1-\beta}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t). \tag{34}$$

Denote $X(t) = \int_0^1 \psi q(\nu,t)x(\nu,t|q)\,d\nu$ as the aggregate expenditure on machines at time t. Employing eq. (14) again to obtain

$$X(t) = (1-\beta)\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t)L(t), \tag{35}$$

where we use the normalization $\psi \equiv 1 - \beta$. Next, we calculate the aggregate profits of the intermediate good sector at time t using eq. (15):

$$\pi(t) = \int_0^1 \pi(\nu, t|q)d\nu = \beta \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t)L(t). \tag{36}$$

Next, let us denote $V(t)$ as the aggregate market value of firms in the intermediate good sector at time t. From the free entry condition in eq. (20) we have

$$V(t) = \int_0^1 V(\nu, t|q)d\nu = \frac{Q(t)L(t)}{\eta\lambda}. \tag{37}$$

Denote $Z(t) = \int_0^1 Z(\nu, t|q)d\nu$ as the aggregate research expenditure at time t. From the functional form of innovation rate in eq. (19) and the fact that the innovation rate is equal across all machine lines, we derive a relationship between the aggregate R&D outlays and innovation rate:

$$z(t) = \frac{\eta Z(\nu, t|q)}{q(\nu, t)L(t)} = \frac{\int_0^1 \eta Z(\nu, t|q)d\nu}{\int_0^1 q(\nu, t)L(t)d\nu} = \frac{\eta Z(t)}{Q(t)L(t)}. \tag{38}$$

Finally, let us define the aggregate abatement expenditure:

$$D(t) = \int_0^1 D(i, t)di.$$

3 Equilibrium and balanced growth path (BGP)

3.1 Decentralized equilibrium

In this section, we define the decentralized equilibrium for the economy. The equilibrium consists of time paths of aggregate consumption, aggregate spending on abatement, machines, and R&D $\{C(t), D(t), X(t), Z(t)\}_{t=0}^{\infty}$; stochastic paths of prices and quantities for machines with the highest quality in their lines at that point $\{p(\nu, t|q), x(\nu, t|q)\}_{\nu\in[0,1],t=0}^{\infty}$; and time paths of aggregate machine quality $\{Q(t)\}_{t=0}^{\infty}$, interest rates $\{r(t)\}_{t=0}^{\infty}$, labor wage $\{w(t)\}_{t=0}^{\infty}$, and firm value functions $\{V(\nu, t|q)\}_{\nu\in[0,1],t=0}^{\infty}$ such that in each period

1. Household $h \in [0, 1]$ chooses $\{C(h, t), L(h, t)\}$ to maximize utility taking $\{r(t), w(t)\}$ as given. The optimal consumption and labor follow the Euler eq. (26) and intratemporal condition in eq. (28).

2. Final good firm $i \in [0, 1]$ chooses $\{L(i, t), x(i, \nu, t|q), D(i, t)\}$ to maximize profit in eq. (3) taking $\{w(t), p(\nu, t|q), p_E(t)\}$ as given. It earns zero profit under perfect competition and the optimality conditions are eqs. (4)-(6).

3. Intermediate good monopolist in the industry ν chooses $\{x(\nu, t|q), p(\nu, t|q)\}$ to maximize profit in eq. (10) given the demand from the final good firms in eq. (8).

4. R&D firm in machine line $\nu \in [0, 1]$ chooses $\{Z(\nu, t|q)\}$ to maximize its profit given $\{r(t), V(\nu, t|q)\}$ satisfying eqs. (20)-(21). It earns zero profit due to free entry into research.

5. The final good market clears such that all final output is used to finance consumption, machine expenditure, abatement expenditure, and R&D outlays:

$$Y(t) = C(t) + X(t) + D(t) + Z(t). \tag{39}$$

6. The labor market clears such that labor demand by final good firms is equal to labor supply by households:

$$\int_0^1 L(h, t)dh = L(t) = \int_0^1 L(i, t)di. \tag{40}$$

7. The asset market clears such that firm patents are the only type of assets in the model since bonds are in zero net supply in equilibrium:

$$\int_0^1 A(h, t)dh = A(t) = V(t). \tag{41}$$

8. The competitive market for pollution permits clears such that the aggregate pollution flow is equal to the total pollution permits:

$$E(t) = \int_0^1 E(i, t)di = \int_0^1 \alpha[Y(i, t) - D(i, t)]di = \int_0^1 \bar{E}Y(i, t)di.$$

Hence, we obtain the expression for the aggregate abatement expenditure:

$$\begin{aligned} D(t) &= \left(1 - \frac{\bar{E}}{\alpha}\right) Y(t) \\ &= \frac{1}{1-\beta} \left(1 - \frac{\bar{E}}{\alpha}\right) \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta - 1} Q(t)L(t), \end{aligned} \tag{42}$$

where the second equality employs eq. (33).

To derive the equilibrium aggregate consumption, combining the aggregate intratemporal condition in eq. (29) and the equilibrium wages in eq. (34) yields

$$C(t) = \frac{L - L(t)}{\varphi} w(t) = \frac{\beta}{1-\beta} \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t) \frac{L - L(t)}{\varphi}. \tag{43}$$

Finally, aggregate assets $A(t)$ are obtained from the aggregate firm value in eq. (37) and the asset market's clearing condition in eq. (41):

$$A(t) = \frac{Q(t)L(t)}{\eta\lambda}. \tag{44}$$

3.2 Balanced growth path (BGP)

We focus on *balanced growth path (BGP)* analysis. BGP is an equilibrium path where aggregate output $Y(t)$ and consumption $C(t)$ grow at a constant rate. We establish the following proposition:

Proposition 1. *For a given value $\bar{E}$, there exists a unique BGP where variables $Y(t), C(t), X(t), D(t), Z(t), V(t), A(t)$, and $w(t)$ grow at the same constant rate g with the aggregate quality $Q(t)$, and the interest rate $r(t)$, innovation rate $z(t)$, and aggregate labor $L(t)$ are constant along the BGP. The economy jumps immediately onto the BGP as there is no transition period.*

Proof. See the Appendix.

We pin down the equilibrium aggregate labor:[6]

$$L(t) = L\left(1 + \varphi + \frac{\rho}{\eta\lambda}\frac{1-\beta}{\beta}\varphi\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta}\right)^{-1}. \tag{45}$$

Our next task is to derive the equilibrium interest and economic growth rates. Combining eqs. (23)-(24) and the fact that equilibrium labor $L(t)$ is constant in eq. (45) yields

$$\frac{\dot{Q}(t)}{Q(t)} = (\lambda - 1)z(t) = (\lambda - 1)\left(\beta\eta\lambda\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} - r(t)\right),$$

which we refer to as the QQ-line. It shows that the growth rate of the average machine quality is a decreasing function of the interest rate $r(t)$. We call the graph of the Euler eq. (27) as the CC-line. Figure 3 shows the intersection of these two lines, which determines the interest and growth rates.

We pin down the interest and growth rates as follows:

$$g = \beta\eta(\lambda - 1)\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} - \frac{\lambda - 1}{\lambda}\rho, \tag{46}$$

$$r = \beta\eta(\lambda - 1)\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \frac{1}{\lambda}\rho. \tag{47}$$

In contrast to the standard quality-ladder models, the model in this study is scale-free: the growth rate does not depend on the size of the workforce.

Moreover, the Appendix shows that both the aggregate transversality condition in eq. (31) and the stronger transversality condition for individual households in eq. (30) are always satisfied along the BGP.

[6] Even though aggregate labor supply is constant along the BGP, we keep using $L(t)$ to avoid any confusion between aggregate labor supply $L(t)$ and endowed labor L.

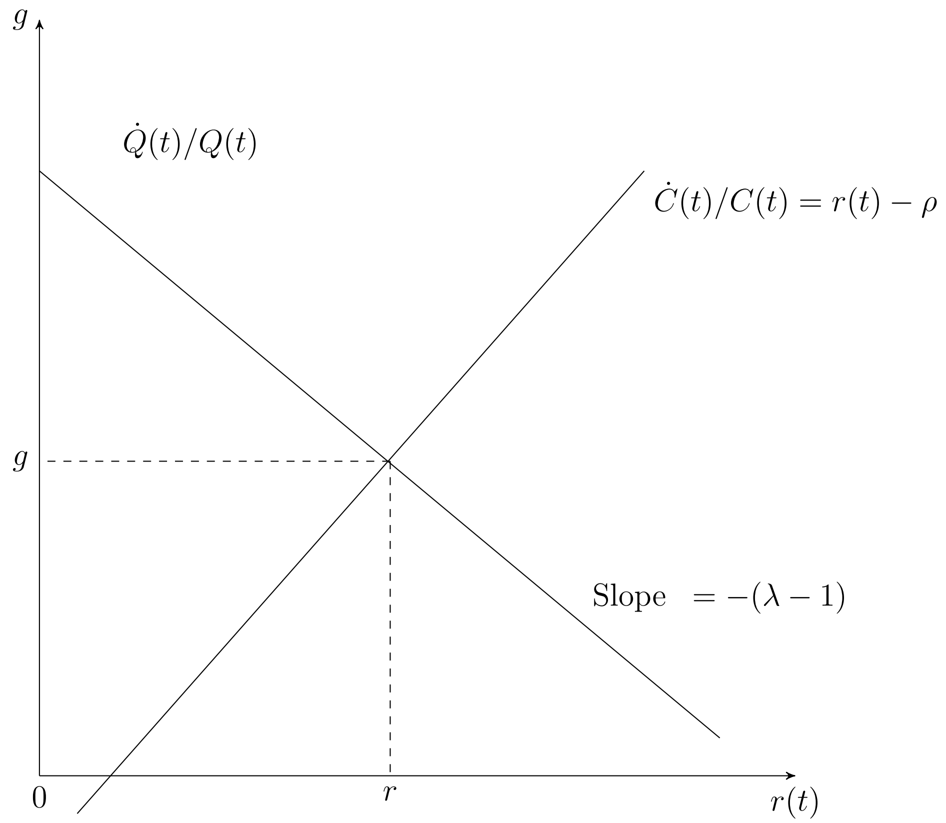

Figure 3: Equilibrium interest and growth rates

3.3 Effects of environmental policy in the BGP

To study the effects of environmental policy on the economy in the BGP, we do some comparative statics in this section. Suppose a stricter environmental regulation in the form of a lower $\bar{E}$. First, from the equilibrium labor in eq. (45), we observe that a more restrictive policy reduces the labor force in the final good sector. The economy jumps into a new BGP, and the aggregate labor remains constant thereafter until a structural or policy change occurs again.

What about the policy effects on other aggregate variables, such as output, consumption, and labor wages? To answer this question, we look at the expressions for equilibrium output, consumption, and wages in eqs. (33), (34), and (43). We decompose each variable into long-run (the average machine quality $Q(t)$) and short-run (the remaining factor) components. The effects of a lower $\bar{E}$ on each variable, therefore, can be split into long-run and short-run effects, respectively. It is straightforward from eq. (45) that a lower $\bar{E}$ reduces aggregate labor force $L(t)$ and thus decreases the short-run component of output and wages.[7] In other words, a more restrictive policy dampens output and wage levels at the point of new policy implementation. On the other hand, the policy also affects the economy via the long-run component, the average machine quality $Q(t)$. A more restrictive policy dampens the economic

[7] The short-run effect of environmental policy on consumption is more complex. See the Appendix for more details.

growth rate g as shown in eq. (46). Intuitively, stricter regulation limits production activities, depressing firms' labor demand and thus reducing output, wages, and consumption. Constrained economic activity also leads to lower demand for intermediate goods and, therefore, reduces the incentive to innovate in each product line due to lower profits. It results in a slowdown in economic growth.

Figure 4 shows two BGPs with two alternative environmental policies. A BGP with a lower cap for pollution permits is characterized by a lower initial level of output and consumption and a lower long-run growth rate. The slope of each line demonstrates the economic growth rate along each BGP.

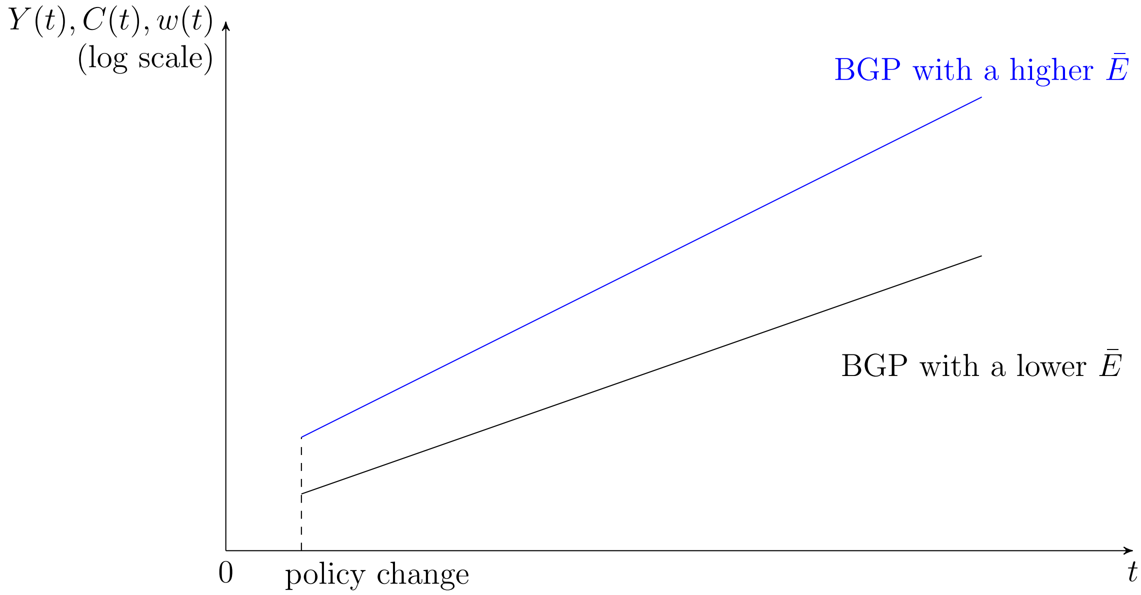

Figure 4: BGPs with different environmental policies

4 Income Inequality

4.1 Asset and income distribution

In this section, we first follow García-Peñalosa and Turnovsky (2006) and Chu and Cozzi (2018) to show that the asset distribution on the BGP is stationary in this economy. Define $\theta_A(h,t) \equiv A(h,t)/A(t)$ and $\theta_C(h,t) \equiv C(h,t)/C(t)$ as the share of assets and consumption by household h at time t where $-\infty < \theta_A(h,t) < +\infty, 0 < \theta_C(h,t) < +\infty$. It means that we allow borrowing in this model, and consumption cannot be negative. We assume that the economy starts with a given initial endowment of assets, that is, the initial asset share $\theta_A(h,0)$ is exogenously given.

We obtain the following lemma:

Lemma 4. *Both asset share $\theta_A(h,t)$ and consumption share $\theta_C(h,t)$ are time-invariant* *along the BGP:* $\theta_A(h,t) \equiv \theta_A(h), \theta_C(h,t) \equiv \theta_C(h)$, *and*

$$C(h,t) = [(r-g)A(h,t) + w(t)L]/(1+\varphi). \tag{48}$$

Proof. See the Appendix.

Next, define $I(h,t)$ as household h's real income at time t. The household's income consists of capital income (i.e., asset returns) from investing in intermediate good firms and labor income from working in the final good sector:

$$I(h,t) = r(t)A(h,t) + w(t)L(h,t) = rA(h,t) + w(t)L(h,t),$$

since the interest rate $r(t)$ is constant along the BGP. We employ the individual intratemporal condition in eq. (28) to transform the income expression as follows:

$$\begin{aligned} I(h,t) &= rA(h,t) + w(t)L - \varphi C(h,t) \\ &= rA(h,t) + w(t)L - \frac{\varphi}{1+\varphi}[(r-g)A(h,t) + w(t)L] \\ &= [(r+\varphi g)A(h,t) + w(t)L]/(1+\varphi), \end{aligned} \tag{49}$$

where the second equality applies the consumption function in eq. (48).

We now sequence households in the ascending order of assets. From eq. (49), income $I(h,t)$ is monotonically increasing in asset value $A(h,t)$. Thus, by ordering households with increasing assets, we already arrange them in the ascending order of income as well. Let $G_A(t)$ and $G_I(t)$ be the Gini coefficient of assets and income, respectively.

The Gini coefficient of assets is given by

$$G_A(t) = 1 - 2b_A(t) = 1 - 2\int_0^1 \mathscr{L}_A(h,t)dh,$$

where $b_A(t)$ is the area under the Lorenz curve and $\mathscr{L}_A(h,t)$ is the Lorenz function of assets. Because the economy starts with exogenous asset shares and asset shares are stationary along a BGP, the Lorenz function and Gini coefficient of assets are also stationary along a BGP:

$$\mathscr{L}_A(h,t) \equiv \mathscr{L}_A(h), G_A(t) \equiv G_A.$$

We establish the following proposition:

Proposition 2. *Income inequality remains constant along the BGP and*

$$G_I(t) \equiv G_I = \frac{r+\varphi g}{r+\varphi g + w(t)L/A(t)} G_A. \tag{50}$$

Proof. See the Appendix.

Therefore, studying the effects of environmental policy change on income inequality is equivalent to comparing income inequality on the two BGPs in figure 4 because income inequality remains constant along each BGP in this model. Note that in the special case when $G_A = 0$, eq. (50) implies that $G_I = 0$. This is understandable because if all households held the same amount of assets, their optimal choice would be exactly the same due to identical preferences. Consequently, their income level is the same forever. In the following sections, we only consider the general case when there is wealth inequality (i.e., $G_A > 0$).

4.2 Inelastic labor supply

We first consider $\varphi = 0$, corresponding to the case of inelastic labor supply. Eq. (50) becomes

$$G_I = \frac{r}{r + w(t)L/A(t)} G_A. \tag{51}$$

We employ a similar approach to Chu and Cozzi (2018) to decompose the effects of an environmental policy on income inequality into two channels via the interest rate (i.e., r) and the (relative) wage-asset value (i.e., $w(t)/A(t)$), respectively. In this part, we continue to suppose a more restrictive environmental policy in the form of a lower cap for pollution permits.

Regarding the interest rate channel, a more restrictive policy lowers the long-run growth rate and drives down the interest rate according to eqs. (46)-(47). A lower interest rate reduces the relative proportion of capital income in the households' income composition since the interest rate is the market rate of return on assets. Consequently, households with more assets are affected more compared to those with fewer assets. Thus, a stricter environmental policy decreases income inequality via the interest rate channel.

On the other hand, the policy effect also operates through the wage-asset channel. Section 3.3 showed that a lower cap for pollution permits reduces both the short-run wage level and its long-run growth rate. It is observable from eq. (44) that environmental policy only affects asset value via its long-run growth since $L(t) = L$ in the case of inelastic labor supply. Consequently, stricter regulation reduces the value of wages compared to assets. Relatively lower wages widen the income gap since households with fewer assets rely relatively more on labor income to finance their consumption. Therefore, a more restrictive policy worsens income inequality via this wage-asset channel. We combine the effects via these channels to obtain the following proposition:

Proposition 3. *A stricter environmental policy increases income inequality under inelastic labor supply.*

Proof. See the Appendix.

4.3 Elastic labor supply

From eq. (50), we realize that the case of elastic labor supply (i.e., $\varphi > 0$) introduces one additional channel via the term φg. We provide the meaning of this third effect as follows. A stricter environmental policy lowers the economic growth rate, as shown above. Since households have the same growth rate of consumption and assets, it would mean that individual and aggregate variables have the same growth rate. A lower economic growth rate means lower savings are needed to support economic growth. Lower savings, in turn, translate into either higher consumption or lower labor supply, holding the interest rate and wages unchanged. The intratemporal labor/leisure trade-off decides how much the reduction in savings comes from higher consumption or from lower labor work. From the individual intratemporal condition in eq. (28) we have

$$w(t)L(h,t) = w(t)L - \varphi C(h,t). \tag{52}$$

A worth-mentioning point is that in the case of inelastic labor supply when $\varphi = 0$, higher consumption is not associated with a decrease in labor supply since the households always supply all their labor endowment. We have two observations here. First, we realize from eq. (52) that the parameter φ plays a role as a "multiplier". When consumption $C(h,t)$ increases by ΔC units, labor supply would go down to the point that labor income decreases by $\varphi\Delta C$. Therefore, a fraction $1/(1+\varphi)$ of savings reduction comes from higher consumption, and the remaining fraction $\varphi/(1+\varphi)$ results from less labor supply. The consumption function in eq. (48) confirms our analysis with $\dot{A}(h,t) = gA(h,t)$ being the savings of household h at time t. Therefore, the higher the value of φ, the stronger the labor supply response. In other words, higher φ amplifies the policy effect via this channel. Second, the same saving rate would mean households with relatively more assets reduce savings more and thus reduce labor supply more, compared to households with fewer assets. Therefore, a more restrictive policy decreases income inequality via this channel.

Unlike the case of inelastic labor supply, stricter regulation now decreases the short-run component of assets through its impact on aggregate labor in eq. (45). To investigate if the policy has a greater impact on wages or assets, combining eqs. (34) and (44)-(45) yields

$$\begin{aligned}\frac{w(t)}{A(t)} &= \frac{\eta\lambda w(t)}{Q(t)L(t)}\\ &= \frac{\beta\eta\lambda}{1-\beta}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta}\left(1+\varphi+\frac{\rho}{\eta\lambda}\frac{1-\beta}{\beta}\varphi\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta}\right)L^{-1}\\ &= \frac{\beta\eta\lambda(1+\varphi)}{(1-\beta)L}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta}+\frac{\varphi\rho}{L},\end{aligned} \tag{53}$$

which is increasing in $\bar{E}$. Hence, stricter regulation negatively affects labor wages more than assets, thus reducing the wage-asset value ratio. Similar to when labor supply is inelastic, we conclude that a more restrictive policy increases income inequality through the wage-asset channel.

The situation is more complex than the case of inelastic labor supply. Stricter regulation reduces income inequality through the interest rate and the labor supply channels but increases it via the wage-asset channel. We establish the following proposition:

Proposition 4. *In the case of elastic labor supply, a more restrictive environmental policy increases (decreases) income inequality if* $\varphi < (>) \dfrac{1}{(2-\beta)(\lambda-1)}$. *Policy effects via various channels cancel each other out if* $\varphi = \dfrac{1}{(2-\beta)(\lambda-1)}$.

Proof. See the Appendix.

We can conduct a numerical exercise for the case of elastic labor supply here. Recall that β is the labor share of income in the economy. We follow the standard literature to choose a reasonable value $\beta = 0.6$. Note that we assume drastic innovation in this study, in the sense that entrants will always drive out incumbents even if they charge the monopolist's price. It means the quality step λ must satisfy the condition in eq. (9):

$$\lambda \geq \left(\frac{1}{1-\beta}\right)^{\frac{1-\beta}{\beta}} \approx 1.842.$$

If we use the lowest possible value for λ, the largest threshold value for φ will be $\dfrac{1}{(2-\beta)(\lambda-1)} \approx 0.848$. The widely used values of the leisure parameter φ in the literature (e.g., Azariadis et al., 2013; Chu et al., 2019; García-Peñalosa and Turnovsky, 2006; Hamaguchi, 2021) are higher than this threshold. Hence, it is most likely that a more restrictive environmental policy reduces income inequality in the case of elastic labor supply.

5 Discussion

In this study, we develop a Schumpeterian growth model incorporating pollution and environmental policy in the form of emissions trading. We model household heterogeneity by allowing them to hold different asset portfolios. The primary purpose of this paper is to establish a theoretical relationship between environmental policy and income inequality. A growth framework is relevant for this purpose since pollution has been an unavoidable by-product of economic growth throughout human history since the Industrial Revolution. There are people who greatly benefit from economic growth, whereas others are left behind. In the earlier literature, the growth aspect is absent from theoretical papers on the impact of environmental policy on either wage inequality or wealth inequality. They utilize different approaches from an OLG model (Dao, 2022), or a small open economy model (Li et al., 2020) to a Harris and Todaro (1970) model that focuses on rural-urban migration (Kuo et al., 2022). The small literature that employs endogenous economic growth models to examine possible effects of various economic factors on income inequality (e.g., Chu and Cozzi, 2018; Chu et al., 2019; Zheng et al., 2020; Chan et al., 2022) share a similar methodology to this paper. However, our paper is the first one that studies the impact of environmental policy on income inequality and considers carefully the case of elastic labor supply.

We find that under inelastic labor supply, a more restrictive environmental policy in the form of a lower cap for pollution permits increases income inequality. This is consistent with empirical evidence from Jha et al. (2019) and Wei and Zhao (2022). A stricter policy lowers the economic growth rate, interest rate, and labor wages. Households with more assets are affected more by a lower interest rate, whereas relatively lower wages have a greater impact on those with fewer assets. In our model, the impact of a stricter policy via lower wages dominates the impact via the interest rate channel. Hence, income inequality increases, and policymakers face a trade-off between a better environment and increased income inequality.

The case of elastic labor supply is more involved and depends on parameter values. Specifically, we find a threshold for the leisure weight in the households' utility function. This threshold is a function of the labor share of income and the innovation step. If leisure is sufficiently less important for a household's utility, a stricter policy increases income inequality. The existing literature on the impact on income inequality (e.g., Chu and Cozzi, 2018; Chu et al., 2019; Zheng et al., 2020; Chan et al., 2022) is different from our paper in the sense that they do not consider carefully the case when labor supply is elastic. Similar to the case of inelastic labor supply, the wage-asset channel dominates the interest rate channel. The case of elastic labor supply introduces an additional channel via households' labor supply. This channel decreases income inequality under a more restrictive environmental policy, and a higher value of the leisure weight amplifies the response of labor supply to environmental policy. We show that under a reasonable calibration, the combined effect via the labor supply channel and the interest rate channel is strong enough to dominate the wage-asset channel, and a stricter environmental policy decreases income inequality.

Standard endogenous growth models feature scale effects that predict a proportional increase in the economic growth rate with the number of working people or the number of R&D scientists. Researchers, however, have provided abundant evidence against scale effects (e.g., Jones, 1995; Bloom et al., 2020). Therefore, we utilize a scale-free model in this study. As a robustness check, the Appendix presents the model with scale effects and shows that our main findings in this paper are robust to those effects.

The model in this study also yields a trade-off between environmental quality and economic growth, which is not our main focus. It is worth mentioning that economists do not necessarily agree with this trade-off. There has been a growing literature on the so-called Porter hypothesis since, at least, 1991. This hypothesis states that a more restrictive environmental policy promotes innovation of cleaner technology, making production processes more efficient and potentially enhancing economic growth. Several theoretical papers (e.g., Nakada, 2004; Bianco and Salie, 2017) are able to seek validation of the hypothesis in their growth models. Although our paper does not validate the Porter hypothesis and the setup in the aforementioned papers is incompatible with our BGP analysis, income inequality in models that support the Porter hypothesis is still a potential agenda for future research.

A Derivations

A.1 Final good sector

A final good firm i chooses labor $L(i,t)$, quantity of intermediate goods $x(i,\nu,t|q)$, and abatement expenditure $D(i,t)$ to maximize its profit $\Pi(i,t)$ at time t as follows:

$$\Pi(i,t) = Y(i,t) - w(t)L(i,t) - \int_0^1 p(\nu,t|q)x(i,\nu,t|q)\,d\nu - D(i,t) - p_E(t)[E(i,t) - \bar{E}Y(i,t)], \tag{A.1}$$

where $w(t)$ denotes labor wages at time t and $p(\nu,t|q)$ refers to the price of machine variety ν with quality $q(\nu,t)$ at time t. The first-order conditions of profit optimization are straightforward:

$$\beta\frac{Y(i,t)}{L(i,t)} = w(t) + p_E(t)(\alpha - \bar{E})\beta\frac{Y(i,t)}{L(i,t)}, \tag{A.2}$$

$$q(\nu,t)\left(\frac{L(i,t)}{x(i,\nu,t|q)}\right)^{\beta} = p(\nu,t|q) + p_E(t)(\alpha - \bar{E})q(\nu,t)\left(\frac{L(i,t)}{x(i,\nu,t|q)}\right)^{\beta}, \tag{A.3}$$

$$\alpha p_E(t) \leq 1 \text{ and } [\alpha p_E(t) - 1]D(i,t) = 0. \tag{A.4}$$

Eqs. (A.2)-(A.3) state that firms employ labor and machines until their marginal products are equal to their costs. The second term on the right-hand side of eqs. (A.2)-(A.3) captures the (net) marginal environmental cost of an additional unit of labor and machines, respectively.

Regarding abatement expenditure, eq. (A.4) says that if final good firms devote a positive amount of resources to abatement activity (i.e., $D(i,t) > 0$), the optimality condition is that the cost of one additional expenditure unit (which is equal to one) equals its marginal environmental benefit (which is equal to $\alpha p_E(t)$). In this case, we have the unit price of permits constant, that is, $p_E(t) = 1/\alpha$, and final good firms are indifferent between different levels of abatement expenditure.

If $D(i,t) = 0$ for all final good firms i, the permit price must satisfy $p_E(t) < 1/\alpha$. In that case, no trade indeed occurs since devoting resources to abatement activities is not beneficial and every final good firm would be happy to purchase additional permits instead. The government can always intervene and set the permit price at $p_E(t) = 1/\alpha + \varepsilon$ where $\varepsilon > 0$. The intervention would encourage final good firms to engage in abatement activities. The intervention ends quickly and the free market mechanism will bring the permit price back to $p_E(t) = 1/\alpha$. At this price, trade does occur and firms are indifferent between different levels of abatement expenditure. Therefore, we can safely say that the equilibrium permit price is $p_E(t) = 1/\alpha$ and at least some final good firms trade their distributed permits.

We substitute $p_E(t) = 1/\alpha$ into eqs. (A.2)-(A.3) to yield

$$
\begin{aligned}
w(t) &= \left(\frac{\bar{E}}{\alpha}\right)\beta\frac{Y(i,t)}{L(i,t)}, \\
p(\nu,t|q) &= \left(\frac{\bar{E}}{\alpha}\right) q(\nu,t)\left(\frac{L(i,t)}{x(i,\nu,t|q)}\right)^{\beta}.
\end{aligned}
$$

A.2 Intermediate good sector

A.2.1 Proof of Lemma 1

We consider a more general production function as follows:

$$
Y(i,t) = \frac{1}{1-\beta}L(i,t)^{\beta}\int_0^1\left(\sum_{\tau} q_{\tau}(\nu,t)^{\frac{1}{1-\beta}}x_{\tau}(i,\nu,t|q)\right)^{1-\beta} d\nu,
$$

where τ denotes machine vintages. We will find the final good firm i's optimal choice of machines across vintages for a given variety.

Consider adjacent vintages τ and $\tau+1$ with $\tau+1$ is the leading-edge machine vintage so we have: $q_{\tau+1}(\nu,t) = \lambda q_{\tau}(\nu,t)$. Next, we define

$$
\tilde{X}(i,\nu,t) = \sum_{\tau} q_{\tau}(\nu,t)^{\frac{1}{1-\beta}}x_{\tau}(i,\nu,t|q).
$$

The profit maximization becomes

$$
\begin{aligned}
\max \Pi(i,t) = &\frac{1}{1-\beta}L(i,t)^{\beta}\int_0^1\left(\tilde{X}(i,\nu,t)\right)^{1-\beta} d\nu - w(t)L(i,t) \\
&- \int_0^1\sum_{\tau} p_{\tau}(\nu,t|q)x_{\tau}(i,\nu,t|q)\, d\nu - D(i,t) - p_E(t)(E(i,t) - \bar{E}Y(i,t)).
\end{aligned}
$$

In this part, we only care about the firm's choice of machine vintages. Take the first order conditions with respect to $x_{\tau}(i,\nu,t|q)$ and $x_{\tau+1}(i,\nu,t|q)$ as follows:

$\partial\Pi(i,t)/\partial x_{\tau}(i,\nu,t|q) \le 0$, with equality if $x_{\tau}(i,\nu,t|q) > 0$:

$$
\left(\frac{\bar{E}}{\alpha}\right) L(i,t)^{\beta}\tilde{X}(i,\nu,t)^{-\beta}q_{\tau}(\nu,t)^{\frac{1}{1-\beta}} \le p_{\tau}(\nu,t|q),
$$

and $\partial\Pi(i,t)/\partial x_{\tau+1}(i,\nu,t|q) \le 0$, with equality if $x_{\tau+1}(i,\nu,t|q) > 0$:

$$
\left(\frac{\bar{E}}{\alpha}\right) L(i,t)^{\beta}\tilde{X}(i,\nu,t)^{-\beta}q_{\tau+1}(\nu,t)^{\frac{1}{1-\beta}} \le p_{\tau+1}(\nu,t|q).
$$

Note that this formula of first-order conditions allows for a corner solution. We assume that final good firms hire only the leading-edge machine if they are indifferent between vintages. We have that $x_\tau(i,\nu,t|q) = 0$ and $x_{\tau+1}(i,\nu,t|q) > 0$ will be satisfied if

$$\frac{\partial \Pi(i,t)}{\partial x_\tau(i,\nu,t|q)} \le 0$$
$$\frac{\partial \Pi(i,t)}{\partial x_{\tau+1}(i,\nu,t|q)} = 0$$

Combining and simplifying these conditions yield

$$p_{\tau+1}(\nu,t|q) \le \lambda^{\frac{1}{1-\beta}} p_\tau(\nu,t|q). \tag{A.5}$$

The following Appendix A.2.2 shows the profit maximization of the machines' owners. We obtain the optimality condition called the "unconstrained monopoly price" as follows:

$$p_{\tau+1}(\nu,t|q) = \frac{1}{1-\beta}\psi q_{\tau+1}(\nu,t) = \frac{\psi}{1-\beta}\lambda q_\tau(\nu,t), \tag{A.6}$$

where $\psi q_{\tau+1}(\nu,t)$ is the marginal cost of producing one unit of the machine vintage $\tau+1$ and $1/(1-\beta)$ is the monopoly markup.

On the other hand, the lowest possible price the owner of the τ-vintage can charge is simply the marginal cost. It is the break-even price for the owner:

$$p_\tau(\nu,t|q) = \psi q_\tau(\nu,t). \tag{A.7}$$

By combining (A.5), (A.6), and (A.7), the owner of the $\tau+1$-vintage can drive the incumbent out of the market even if he charges the unconstrained monopoly price and the owner of the τ-vintage charges the lowest possible price iff

$$\frac{\psi}{1-\beta}\lambda q_\tau(\nu,t) \le \lambda^{\frac{1}{1-\beta}}\psi q_\tau(\nu,t)$$
$$\frac{1}{1-\beta} \le \lambda^{\frac{1}{1-\beta}-1}$$
$$\lambda \ge \left(\frac{1}{1-\beta}\right)^{\frac{1-\beta}{\beta}},$$

which is what we need to show.

A.2.2 Optimal monopoly price

We rearrange the inverse demand function in eq. (8) to obtain the demand function for machine ν as follows:

$$x(\nu,t|q) = \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} q(\nu,t)^{1/\beta} p(\nu,t|q)^{-1/\beta} L(t). \tag{A.8}$$

Substituting the demand function in eq. (A.8) into the profit function in eq. (10) to yield

$$\pi(\nu,t|q) = [p(\nu,t|q) - \psi q(\nu,t)] \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} q(\nu,t)^{1/\beta} p(\nu,t|q)^{-1/\beta} L(t). \tag{A.9}$$

Intermediate good monopolists choose their price to maximize this profit function. Since the factor $\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} q(\nu,t)^{1/\beta}L(t)$ is irrelevant in this optimization problem, we find the price $p(\nu,t|q)$ to maximize the following expression:

$$[p(\nu,t|q) - \psi q(\nu,t)]\, p(\nu,t|q)^{-1/\beta}.$$

Take the first order condition with respect to the price $p(\nu,t|q)$

$$p(\nu,t|q)^{-1/\beta} + [p(\nu,t|q) - \psi q(\nu,t)] \left(-\frac{1}{\beta}\right) p(\nu,t|q)^{-1/\beta-1} = 0.$$

Simplify the equation to yield the optimal price:

$$p(\nu,t|q) - \frac{1}{\beta}[p(\nu,t|q) - \psi q(\nu,t)] = 0$$
$$p(\nu,t|q) = \frac{1}{1-\beta}\psi q(\nu,t).$$

A.3 R&D sector

A.3.1 Proof of Lemma 2

Let the incumbent's quality be q^I. Free entry condition means $V(\nu,t|q^I\lambda) = q^I L(t)/\eta$. If the incumbent invested one unit of resources into R&D, he would generate additional value

$$\frac{\eta}{q^I}\left(V(\nu,t|q^I\lambda) - V(\nu,t|q^I)\right) = \frac{\eta}{q^I L(t)}\left(\frac{q^I L(t)}{\eta} - \frac{q^I L(t)}{\eta\lambda}\right) = 1 - \frac{1}{\lambda} < 1,$$

where $\eta/(q^I L(t))$ is the innovation rate generated by one unit of the final good when quality is q^I. Hence, the incumbent has no incentive to innovate since it would replace its own machine and thus destroy existing profits. Consequently, all R&D is undertaken by entrants.

A.3.2 Proof of Lemma 3

We can determine the firm value $V(\nu,t|q)$ through a Bellman equation. During a small time interval $[t, t+\Delta]$, the intermediate good firm ν collects $\Delta\pi(\nu,t|q)$. At the end of the time interval, either it is replaced by a new entrant with the probability $\Delta z(\nu,t|q)$ and gets zero profits thereafter, or the firm remains its monopoly power and $V(\nu,t+\Delta|q)$. Hence, we have the following Bellman equation:

$$\begin{aligned} V(\nu,t|q) &= \Delta\pi(\nu,t|q) + e^{-r(t)\Delta}\begin{bmatrix} \Delta z(\nu,t|q)\times 0 + \\ (1-\Delta z(\nu,t|q))\times V(\nu,t+\Delta|q)\end{bmatrix} \\ &= \Delta\pi(\nu,t|q) + e^{-r(t)\Delta}(1-\Delta z(\nu,t|q))V(\nu,t+\Delta|q),\end{aligned}$$

where $e^{-r(t)\Delta}$ is the time discount factor. Subtracting $e^{-r(t)\Delta}V(\nu,t|q)$ from both sides then dividing by Δ to obtain

$$\begin{aligned}\frac{1-e^{-r(t)\Delta}}{\Delta}V(\nu,t|q) =& \pi(\nu,t|q) - e^{-r(t)\Delta}z(\nu,t|q)V(\nu,t+\Delta|q) \\ &+ e^{-r(t)\Delta}\frac{V(\nu,t+\Delta|q)-V(\nu,t|q)}{\Delta}.\end{aligned}$$

Apply the L'Hôpital's rule

$$\lim_{\Delta\to 0}\frac{1-e^{-r(t)\Delta}}{\Delta} = \lim_{\Delta\to 0}\frac{r(t)e^{-r(t)\Delta}}{1} = r(t).$$

Taking the limit $\Delta \to 0$ yields

$$r(t)V(\nu,t|q) = \pi(\nu,t|q) - z(\nu,t|q)V(\nu,t|q) + \dot{V}(\nu,t|q).$$

This is the no-arbitrage condition we need to show.

A.4 Households

Each household h chooses consumption and labor to maximize its utility given its flow budget constraint. Households do not have control over pollution even though pollution affects their utility directly.

We set up the following current-value Hamiltonian for the household's optimization problem:

$$\mathscr{H} = \ln C(h,t) + \varphi\ln(L - L(h,t)) - \gamma E(t)^b + \mu(h,t)[r(t)A(h,t) + w(t)L(h,t) - C(h,t)],$$

where $\mu(h,t)$ denotes the Hamiltonian costate variable. The first-order conditions are as follows:

$$\frac{\partial\mathscr{H}}{\partial C(h,t)} = \frac{1}{C(h,t)} - \mu(h,t) = 0, \tag{A.10}$$

$$\frac{\partial\mathscr{H}}{\partial L(h,t)} = \frac{\varphi}{L - L(h,t)} - w(t)\mu(h,t) = 0, \tag{A.11}$$

$$\frac{\partial\mathscr{H}}{\partial A(h,t)} = r(t)\mu(h,t) = -\dot{\mu}(h,t) + \rho\mu(h,t). \tag{A.12}$$

Combining eqs. (A.10)-(A.11) yields

$$w(t)[L - L(h,t)] = \varphi C(h,t),$$

which can be referred to as the intratemporal condition or the labor supply condition. Denote $C(t) \equiv \int_0^1 C(h,t)dh$ and $L(t) \equiv \int_0^1 L(h,t)dh$ as the aggregate consumption and labor, respectively. We aggregate the intratemporal condition over the whole population to obtain

$$w(t)[L - L(t)] = \varphi C(t).$$

Taking logarithm and then taking the time derivative of eq. (A.10) yield

$$\frac{\dot{C}(h,t)}{C(h,t)} = -\frac{\dot{\mu}(h,t)}{\mu(h,t)}. \tag{A.13}$$

From eq. (A.12)

$$-\frac{\dot{\mu}(h,t)}{\mu(h,t)} = r(t) - \rho. \tag{A.14}$$

We combine eqs. (A.13)-(A.14) to obtain the following intertemporal condition for consumption (i.e., the Euler equation):

$$\frac{\dot{C}(h,t)}{C(h,t)} = r(t) - \rho.$$

A.5 Balanced growth path (BGP)

A.5.1 Proof of Proposition 1

We apply the good market clearing condition in eq. (39), the equilibrium output in eq. (33), aggregate machine expenditure in eq. (35), abatement expenditure in eq. (42), and aggregate consumption in eq. (43) to obtain

$$\begin{aligned} Z(t) &= Y(t) - X(t) - D(t) - C(t) \\ &= \frac{\beta(2-\beta)}{1-\beta}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t)L(t) - \frac{\beta}{1-\beta}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t)\frac{\ell(t)}{\varphi}, \end{aligned} \tag{A.15}$$

where $\ell(t) = L - L(t)$ is the aggregate leisure at time t. Combining eq. (A.15) with eq. (38) to yield

$$z(t) = \frac{\beta\eta(2-\beta)}{1-\beta}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} - \frac{\beta\eta}{(1-\beta)\varphi}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta}\chi(t), \tag{A.16}$$

where $\chi(t) = \dfrac{\ell(t)}{L(t)}$, which is the aggregate leisure-labor ratio at time t.

Moreover, eq. (23) yields

$$\begin{aligned} z(t) &= \beta\eta\lambda\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \frac{\dot{L}(t)}{L(t)} - r(t) \\ &= \beta\eta\lambda\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \frac{\dot{L}(t)}{L(t)} - \rho - \frac{\dot{C}(t)}{C(t)}, \end{aligned} \tag{A.17}$$

which is the result of the Euler eq. (27). We take the logarithm and then take the time derivative of the equilibrium consumption in eq. (43)

$$\begin{aligned} \frac{\dot{C}(t)}{C(t)} &= \frac{\dot{Q}(t)}{Q(t)} + \frac{\dot{\ell}(t)}{\ell(t)} \\ &= (\lambda-1)z(t) + \frac{\dot{\ell}(t)}{\ell(t)}, \end{aligned} \tag{A.18}$$

where we employ eq. (24) for the growth rate of average machine quality. Combining eqs. (A.17)-(A.18) yields

$$z(t) = \beta\eta\lambda \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \frac{\dot{L}(t)}{L(t)} - \rho - (\lambda - 1)z(t) - \frac{\dot{\ell}(t)}{\ell(t)},$$

or

$$z(t) = \beta\eta \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} - \frac{\rho}{\lambda} - \frac{1}{\lambda}\left(\frac{\dot{\ell}(t)}{\ell(t)} - \frac{\dot{L}(t)}{L(t)}\right). \tag{A.19}$$

Note that $\chi(t) = \frac{\ell(t)}{L(t)}$. Taking logarithm and then taking time derivative yield

$$\frac{\dot{\chi}(t)}{\chi(t)} = \frac{\dot{\ell}(t)}{\ell(t)} - \frac{\dot{L}(t)}{L(t)}, \tag{A.20}$$

which can be utilized to transform eq. (A.19) into

$$z(t) = \beta\eta \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} - \frac{\rho}{\lambda} - \frac{1}{\lambda}\frac{\dot{\chi}(t)}{\chi(t)}. \tag{A.21}$$

We combine eqs. (A.16) and (A.21) to obtain

$$\frac{1}{\lambda}\frac{\dot{\chi}(t)}{\chi(t)} = \zeta_1\chi(t) - \zeta_2, \tag{A.22}$$

where $\zeta_1 = \frac{\beta\eta}{(1-\beta)\varphi}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} > 0, \zeta_2 = \frac{\beta\eta}{1-\beta}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \frac{\rho}{\lambda} > 0$. This is an ordinary differential equation (ODE) for the leisure-labor ratio $\chi(t)$.

1. If $\chi(t) > \zeta_2/\zeta_1$, from eq. (A.22) the leisure-labor ratio $\chi(t)$ grows unboundedly. Since $\ell(t) + L(t) = L$, we must have aggregate leisure increasing and aggregate labor decreasing over time. From (33) and (43)

   $$\frac{\dot{C}(t)}{C(t)} = \frac{\dot{Q}(t)}{Q(t)} + \frac{\dot{\ell}(t)}{\ell(t)} > \frac{\dot{Q}(t)}{Q(t)} > \frac{\dot{Q}(t)}{Q(t)} + \frac{\dot{L}(t)}{L(t)} = \frac{\dot{Y}(t)}{Y(t)}.$$

   This path is obviously unsustained since the consumption-output ratio is increasing over time.

2. If $\chi(t) < \zeta_2/\zeta_1$, from (A.22) the leisure-labor ratio $\chi(t)$ keeps decreasing and asymptotically approaches zero. In contrast to the previous case, aggregate leisure keeps decreasing, and aggregate labor increases over time. This path is not optimal since consumption grows slower than output $Y(t)$, from eqs. (33) and (43).

3. If $\chi(t) = \zeta_2/\zeta_1$, both aggregate leisure $\ell(t)$ and labor $L(t)$ are constant. As a result, the innovation rate $z(t)$ also remains unchanged over time from eq. (A.16) and so does the growth rate of the average machine quality $Q(t)$ since $\dot{Q}(t)/Q(t) = (\lambda - 1)z(t)$.

   From equilibrium conditions in eqs. (33)-(35) and (42)-(44), we conclude that $\{Y(t), X(t), D(t), C(t), w(t), A(t)\}$ grow at the same constant rate with $Q(t)$.

We establish that there exists a unique BGP where aggregate variables $\{Y(t), X(t), D(t), C(t), w(t), A(t)\}$ grow at the same constant rate g as the average quality of machines, and the aggregate labor is constant along the BGP. The above argument also shows that the BGP is the only optimal equilibrium path in this model.

The constant leisure-labor ratio is

$$\frac{\ell(t)}{L(t)} = \frac{\zeta_2}{\zeta_1} = \varphi + \frac{\rho}{\eta\lambda}\frac{1-\beta}{\beta}\varphi\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta},$$

which can be combined with the equation $\ell(t)+L(t) = L$ to pin down the equilibrium aggregate labor:

$$L(t) = L\left(1+\varphi+\frac{\rho}{\eta\lambda}\frac{1-\beta}{\beta}\varphi\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta}\right)^{-1}.$$

It is simple to confirm that in the case of inelastic labor supply when $\varphi = 0$, we have $L(t) = L$.

From eqs. (A.16) and (A.21), we observe that the innovation rate depends solely on the leisure-labor ratio and

$$\begin{aligned} z(t) &= \beta\eta\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} - \frac{\rho}{\lambda} - \frac{1}{\lambda}\frac{\dot{\chi}(t)}{\chi(t)} \\ &= \beta\eta\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} - \frac{\rho}{\lambda}, \end{aligned}$$

since $\dot{\chi}(t) = 0$. Calculations of the equilibrium growth and interest rates are straightforward

$$\begin{aligned} g &= (\lambda-1)z(t) \\ &= \beta\eta(\lambda-1)\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} - \frac{\lambda-1}{\lambda}\rho, \\ r &= g+\rho \\ &= \beta\eta(\lambda-1)\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \frac{1}{\lambda}\rho. \end{aligned}$$

Our final task in this Appendix is to show that the aggregate transversality condition in eq. (31) is always satisfied on the BGP. Given the growth rate of individual costate variable $\mu(h,t)$ in (A.14), the aggregate costate $\mu(t)$ must also grow at the rate of $-r+\rho$

$$\frac{\dot{\mu}(t)}{\mu(t)} = -r+\rho \quad \Longleftrightarrow \quad \mu(t) = \mu(0)e^{(-r+\rho)t}.$$

We already established that the aggregate asset $A(t)$ grows at the rate of g on the BGP

$$A(t) = A(0)e^{gt}.$$

Therefore, we can rewrite the transversality condition in eq. (31) as follows:

$$\lim_{t\to\infty}[e^{-\rho t}\mu(0)e^{(-r+\rho)t}A(0)e^{gt}] = 0$$
$$\lim_{t\to\infty}[\mu(0)A(0)e^{-(r-g)t}] = 0.$$

Hence, the aggregate transversality condition is always satisfied on the BGP since $r - g = \rho > 0$ from the Euler equation.[8]

A.6 Effects of environmental policy

We examine the short-run effect of stricter environmental policy on aggregate consumption. From eq. (43), the short-run component of consumption is equal to

$$\begin{aligned}&\frac{w(t)}{Q(t)}\frac{L-L(t)}{\varphi}\\ =&\frac{\beta}{1-\beta}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta}\frac{\ell(t)}{\varphi}\\ =&\frac{\beta}{(1-\beta)\varphi}L\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta}\left[1-\left(1+\varphi+\frac{\rho}{\eta\lambda}\frac{1-\beta}{\beta}\varphi\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta}\right)^{-1}\right],\end{aligned}$$

which is increasing in $\bar{E}$. Stricter regulation lowers labor wages but increases leisure. However, the latter effect is not strong enough to offset the former one as we have shown above; therefore, short-run consumption decreases.

A.7 Income inequality

A.7.1 Proof of Lemma 4.

First, we prove that the share of consumption $\theta_C(h,t)$ is time-invariant. From individual and aggregate Euler eqs. (26)-(27), we realize that individual and aggregate consumption always grow at the same rate over time. Since $\theta_C(h,t) = C(h,t)/C(t)$, we take the logarithm of this expression and then take the time derivative

$$\frac{\dot{\theta}_C(h,t)}{\theta_C(h,t)} = \frac{\dot{C}(h,t)}{C(h,t)} - \frac{\dot{C}(t)}{C(t)} = 0.$$

It means that the consumption share of each household must be constant. In other words, $\theta_C(h) = \theta_C(h,t) = C(h,t)/C(t)$ is time-invariant.

Because $\theta_A(h,t) = A(h,t)/A(t)$ we take the logarithm of this expression and then take the time derivative to obtain

$$\frac{\dot{\theta}_A(h,t)}{\theta_A(h,t)} = \frac{\dot{A}(h,t)}{A(h,t)} - \frac{\dot{A}(t)}{A(t)}. \tag{A.23}$$

[8] Note that at this point, we cannot say if the stronger transversality condition in eq. (30) for individual households also holds along the BGP.

We next employ the individual and aggregate budget constraint in eqs. (25) and (32)

$$\begin{aligned}
\frac{\dot{A}(h,t)}{A(h,t)} &= r(t) + \frac{w(t)L(h,t) - C(h,t)}{A(h,t)}, \\
\frac{\dot{A}(t)}{A(t)} &= r(t) + \frac{w(t)L(t) - C(t)}{A(t)},
\end{aligned}$$

which we substitute into eq. (A.23) to yield

$$\begin{aligned}
\frac{\dot{\theta}_A(h,t)}{\theta_A(h,t)} &= \frac{C(t) - w(t)L(t)}{A(t)} - \frac{C(h,t) - w(t)L(h,t)}{A(h,t)} \\
&= \frac{(1+\varphi)C(t) - w(t)L}{A(t)} - \frac{(1+\varphi)C(h,t) - w(t)L}{A(h,t)},
\end{aligned}$$

where the individual and aggregate intratemporal condition in eqs. (28)-(29) are used to derive the second equality. We can go further from this

$$\begin{aligned}
\dot{\theta}_A(h,t) &= \frac{(1+\varphi)C(t) - w(t)L}{A(t)}\theta_A(h,t) - \frac{(1+\varphi)C(h,t) - w(t)L}{A(h,t)}\theta_A(h,t) \\
&= \frac{(1+\varphi)C(t) - w(t)L}{A(t)}\theta_A(h,t) - \frac{(1+\varphi)C(t)\theta_C(h,t) - w(t)L}{A(t)} \\
&= \frac{(1+\varphi)C(t) - w(t)L}{A(t)}\theta_A(h,t) - \frac{(1+\varphi)C(t)\theta_C(h) - w(t)L}{A(t)}, \qquad \text{(A.24)}
\end{aligned}$$

where we use the definition of $\theta_A(h,t)$ and $\theta_C(h,t)$, and the fact that $\theta_C(h,t)$ is time-invariant. Since all variables $C(t), w(t)$, and $A(t)$ share the same growth rate along the BGP, the coefficient on $\theta_A(h,t)$ and the independent term in eq. (A.24) are constant on the BGP. We can even show that the coefficient on $\theta_A(h,t)$ is positive. To do that, we first use the fact that aggregate asset $A(t)$ grows at a constant rate of g along the BGP to substitute into the aggregate budget constraint in eq. (32)

$$\begin{aligned}
gA(t) &= rA(t) + w(t)L(t) - C(t) \\
(r-g)A(t) &= C(t) - w(t)L(t) \\
(1+\varphi)C(t) - w(t)L &= (r-g)A(t),
\end{aligned}$$

where we employ the aggregate intratemporal condition in eq. (29) to obtain the last equation. Therefore, the coefficient on $\theta_A(h,t)$ equals $r - g = \rho > 0$ from the Euler condition.

Denote $B = \dfrac{(1+\varphi)C(t)\theta_C(h) - w(t)L}{A(t)}$ and B is constant along the BGP since $C(t), w(t)$, and $A(t)$ grow at the same rate g. The differential equation (A.24) becomes

$$\dot{\theta}_A(h,t) = \rho\theta_A(h,t) - B. \qquad \text{(A.25)}$$

If $\theta_A(h,t) > B/\rho$, the asset share of household h keeps increasing forever and goes to positive infinity. Similarly, if $\theta_A(h,t) < B/\rho$, the asset share of household h keeps decreasing forever and goes to negative infinity. Both cases are unfeasible since asset shares must be finite by definition. Therefore, the only solution to this ODE is

$$\theta_A(h,t) = B/\rho. \qquad \text{(A.26)}$$

In other words, the asset share of each household must be constant along the BGP: $\theta_A(h,t) \equiv \theta_A(h) = B/\rho$. Figure 5 shows the dynamics of eq. (A.25). By mathematical language, $\theta_A(h) = B/\rho$ is an unstable steady-state. The system always diverges once $\theta_A(h,t)$ deviates from this value as shown in figure 5.

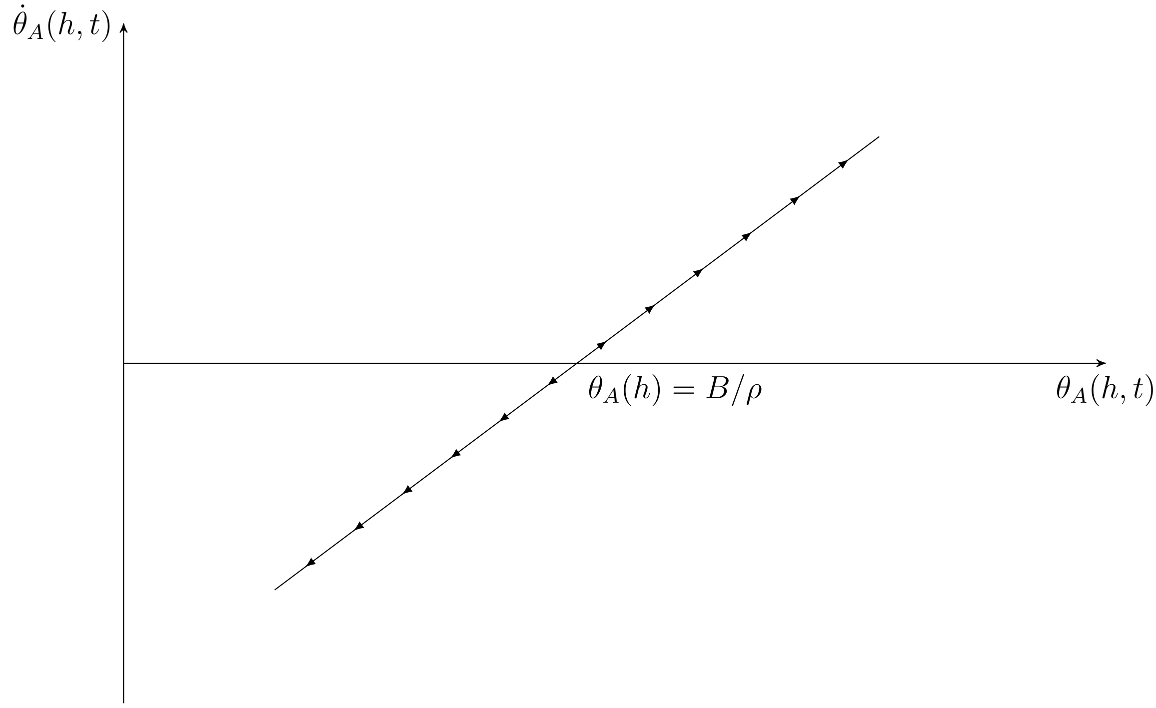

Figure 5: Dynamics of the asset share $\theta_A(h,t)$

For a more formal proof, eq. (A.25) is a separable ODE. We transform it as follows:

$$\begin{aligned}
\frac{\dot{\theta}_A(h,t)}{\theta_A(h,t) - B/\rho} &= \rho \\
\frac{d\theta_A(h,t)}{\theta_A(h,t) - B/\rho} &= \rho dt \\
\int \frac{d\theta_A(h,t)}{\theta_A(h,t) - B/\rho} &= \int \rho dt \\
\ln|\theta_A(h,t) - B/\rho| &= \rho t + c_1 \\
|\theta_A(h,t) - B/\rho| &= e^{\rho t + c_1} \\
\theta_A(h,t) &= B/\rho + c_2 e^{\rho t},
\end{aligned}$$

where c_1 is constant and $c_2 = \pm e^{c_1}$. The trivial (steady-state) solution $\theta_A(h) = B/\rho$ corresponds to $c_2 = 0$. We realize that, except for this special case, the solution always diverges to infinity, which is unfeasible for the asset share. We have again that $\theta_A(h) = B/\rho$ is the only stable solution.

We can transform eq. (A.26) further as follows:

$$\begin{aligned}
(r-g)A(h,t)/A(t) = B &= \frac{(1+\varphi)C(h,t) - w(t)L}{A(t)} \\
C(h,t) &= [(r-g)A(h,t) + w(t)L]/(1+\varphi),
\end{aligned}$$

and this is the individual consumption function we need to show.

Using the fact that asset share is time-invariant, we can show that the individual transversality condition in eq. (30) is always satisfied along the BGP. Appendix A.5.1 already proved a similar statement that the aggregate transversality condition

$$\lim_{t\to\infty}[e^{-\rho t}\mu(t)A(t)] = 0,$$

is always satisfied along the BGP. We can rewrite it as

$$\lim_{t\to\infty}[e^{-\rho t}\mu(t)A(h,t)\theta_A(h)] = 0,$$

where we utilize the fact that the household h's asset share is constant on the BGP. Hence,

$$\lim_{t\to\infty}[e^{-\rho t}\mu(t)A(h,t)] = 0.$$

Because both aggregate costate $\mu(t)$ and individual costate $\mu(h,t)$ grow at the same rate $-r+\rho$, we have

$$\mu(t) = \mu(0)e^{(-r+\rho)t} = \frac{\mu(0)}{\mu(h,0)}\mu(h,0)e^{(-r+\rho)t} = \frac{\mu(0)}{\mu(h,0)}\mu(h,t).$$

Since $\mu(0)/\mu(h,0)$ is finite, we obtain that the individual transversality condition

$$\lim_{t\to\infty}[e^{-\rho t}\mu(h,t)A(h,t)] = 0.$$

is always satisfied along the BGP.

A.7.2 Proof of Proposition 2

The Gini coefficient of income is given by

$$G_I(t) = 1 - 2b_I(t) = 1 - 2\int_0^1 \mathscr{L}_I(h,t)dh, \tag{A.27}$$

where $b_I(t)$ is the area under the Lorenz curve and $\mathscr{L}_I(h,t)$ is the Lorenz function of income

$$\begin{aligned}\mathscr{L}_I(h,t) &= \frac{\int_0^h I(j,t)dj}{\int_0^1 I(j,t)dj} \\ &= \frac{(r+\varphi g)\int_0^h \theta_A(j)dj + w(t)Lh/A(t)}{r+\varphi g + w(t)L/A(t)},\end{aligned} \tag{A.28}$$

where the second equality utilizes the income function (49).

Note that

$$\int_0^h \theta_A(j)dj = \frac{\int_0^h A(j,t)dj}{A(t)} = \frac{\int_0^h A(j,t)dj}{\int_0^1 A(j,t)dj} = \mathscr{L}_A(h,t) \tag{A.29}$$

which is the Lorenz function of assets.

Substituting (A.28) into (A.27) yields

$$
\begin{aligned}
G_I(t) &= 1 - 2\int_0^1 \frac{(r+\varphi g)\mathscr{L}_A(h) + w(t)Lh/A(t)}{r+\varphi g + w(t)L/A(t)} dh \\
&= 1 - 2\frac{(r+\varphi g)\int_0^1 \mathscr{L}_A(h)dh + w(t)L/A(t)\int_0^1 hdh}{r+\varphi g + w(t)L/A(t)} \\
&= 1 - \frac{(r+\varphi g)(1-G_A) + w(t)L/A(t)}{r+\varphi g + w(t)L/A(t)} \\
&= \frac{r+\varphi g}{r+\varphi g + w(t)L/A(t)} G_A,
\end{aligned}
$$

which shows that the Gini coefficient of income is constant along a BGP (i.e., $G_I(t) \equiv G_I$) since labor wages $w(t)$ and aggregate assets $A(t)$ always grow at the same rate on a BGP.

A.7.3 Proof of Proposition 3

Before we calculate the combined effects of a restrictive policy on income inequality, let us use a simple trick to express the Gini coefficient of income G_I from eq. (51) as follows:

$$
G_I = \frac{r}{r + w(t)L/A(t)} G_A = \frac{rA(t)/(w(t)L)}{rA(t)/(w(t)L) + 1} G_A.
$$

The purpose of this trick is for calculation ease. We consider the composite term $\xi_1 = rA(t)/(w(t)L)$ and ξ_1 is constant along a BGP since $A(t)$ and $w(t)$ grow at the same constant rate g. The Gini coefficient G_I is increasing in this term. From eqs. (34), (44), and (47), and the fact that $L(t) = L$ when labor supply is inelastic we obtain

$$
\begin{aligned}
\xi_1 &= \left(\beta\eta(\lambda-1)\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \frac{1}{\lambda}\rho\right)\frac{Q(t)L}{\eta\lambda}\left(\frac{\beta}{1-\beta}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t)L\right)^{-1} \\
&= \left(\beta\eta(\lambda-1)\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \frac{1}{\lambda}\rho\right)\frac{1-\beta}{\beta\eta\lambda}\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta} \\
&= \frac{(1-\beta)(\lambda-1)}{\lambda} + \frac{(1-\beta)\rho}{\beta\eta\lambda^2}\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta}.
\end{aligned}
$$

Hence,

$$
\text{sign}\left(\frac{\partial G_I}{\partial \bar{E}}\right) = \text{sign}\left(\frac{\partial \xi_1}{\partial \bar{E}}\right) = -1.
$$

Therefore, a more restrictive environmental policy, which lowers pollution permits to firms, increases income inequality in the case of inelastic labor supply.

A.7.4 Proof of Proposition 4

Substituting eqs. (46)-(47) and (53) into eq. (50) yields

$$G_I = \frac{\xi_2 \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \xi_3}{\xi_4 \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \xi_5} G_A,$$

where

$$\begin{cases} \xi_2 = \beta\eta(\lambda-1)(1+\varphi) \\ \xi_3 = \frac{1+\varphi-\lambda\varphi}{\lambda}\rho \\ \xi_4 = \frac{\beta\eta(1+\varphi)[1+(2-\beta)(\lambda-1)]}{1-\beta} \\ \xi_5 = \frac{1+\varphi}{\lambda}\rho \end{cases}$$

We apply the chain rule to obtain

$$\begin{aligned} \text{sign}\left(\frac{\partial G_I}{\partial \bar{E}}\right) &= \text{sign}\left(\frac{\partial G_I}{\partial \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta}}\right) . \underbrace{\text{sign}\left(\frac{\partial \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta}}{\partial \bar{E}}\right)}_{=1} \\ &= \text{sign}(\xi_2\xi_5 - \xi_3\xi_4) \\ &= \text{sign}\left((\lambda-1)(1+\varphi) - \frac{(1+\varphi-\lambda\varphi)[1+(2-\beta)(\lambda-1)]}{1-\beta}\right) \\ &= \text{sign}(\varphi(2-\beta)(\lambda-1)-1) \\ &= \text{sign}\left(\varphi - \frac{1}{(2-\beta)(\lambda-1)}\right). \end{aligned}$$

We have three separate cases from here

1. If $\varphi > \dfrac{1}{(2-\beta)(\lambda-1)}$, a more restrictive environmental policy, which lowers the cap for pollution permits to firms, reduces income inequality.
2. If $\varphi < \dfrac{1}{(2-\beta)(\lambda-1)}$, a more restrictive environmental policy increases income inequality.
3. If $\varphi = \dfrac{1}{(2-\beta)(\lambda-1)}$, various channels cancel each other out and eliminate the effects of environmental policy on income inequality.

A.8 Scale effects

Instead of eq. (19), innovations arrive at the rate

$$z(\nu, t|q) = \frac{\eta Z(\nu, t)}{q(\nu, t)}. \tag{A.30}$$

Eq. (A.30) captures the increasing research complexity, in the sense that a more advanced machine is harder to improve upon. The free entry condition now becomes

$$V(\nu, t|q) = \frac{q(\nu, t)}{\eta\lambda}, \tag{A.31}$$

which implies that $\dot{V}(\nu, t|q) = 0$ during the time interval $[t, t + \Delta]$ for a given level of $q(\nu, t)$. Hence, the no-arbitrage condition in eq. (21) implies

$$\begin{aligned} r(t) + z(\nu, t|q) &= \frac{\pi(\nu, t|q)}{V(\nu, t|q)} \\ &= \beta\eta\lambda \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L(t), \end{aligned} \tag{A.32}$$

which again implies that the innovation rate is the same across machine lines: $z(\nu, t|q) = z(t)$.

Instead of eq. (44), the aggregate assets become

$$A(t) = V(t) = \int_0^1 V(\nu, t|q) d\nu = \frac{Q(t)}{\eta\lambda}. \tag{A.33}$$

From the functional form of innovation rate in eq. (A.30) and the fact that the innovation rate is equal across all machine lines

$$z(t) = \frac{\eta Z(\nu, t|q)}{q(\nu, t)} = \frac{\int_0^1 \eta Z(\nu, t|q) d\nu}{\int_0^1 q(\nu, t) d\nu} = \frac{\eta Z(t)}{Q(t)}. \tag{A.34}$$

Next, we utilize the good market clearing condition in eq. (39), the equilibrium output in eq. (33), aggregate machine expenditure in eq. (35), abatement expenditure in eq. (42), and aggregate consumption in eq. (43) to obtain

$$\begin{aligned} Z(t) &= Y(t) - X(t) - D(t) - C(t) \\ &= \frac{\beta(2-\beta)}{1-\beta} \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t)L(t) - \frac{\beta}{1-\beta} \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t)\frac{L - L(t)}{\varphi} \\ &= \frac{\beta(2-\beta)}{1-\beta} \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t)L - \frac{\beta}{1-\beta}\frac{1+\varphi(2-\beta)}{\varphi} \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t)\ell(t), \end{aligned} \tag{A.35}$$

We combine eqs. (A.34) and (A.35)

$$z(t) = \frac{\beta\eta(2-\beta)}{1-\beta} \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L - \frac{\beta\eta}{1-\beta}\frac{1+\varphi(2-\beta)}{\varphi} \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} \ell(t). \tag{A.36}$$

Moreover, eq. (A.32) yields

$$\begin{aligned} z(t) &= \beta\eta\lambda \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L(t) - r(t) \\ &= \beta\eta\lambda \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L(t) - \rho - \frac{\dot{C}(t)}{C(t)} \\ &= \beta\eta\lambda \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L(t) - \rho - (\lambda - 1)z(t) - \frac{\dot{\ell}(t)}{\ell(t)} \end{aligned} \tag{A.37}$$

which is the result of the Euler eq.(27) and eq. (A.18). Hence,

$$z(t) = \beta\eta \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L - \frac{\rho}{\lambda} - \beta\eta \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} \ell(t) - \frac{1}{\lambda}\frac{\dot{\ell}(t)}{\ell(t)} \tag{A.38}$$

From eqs. (A.36) and (A.38)

$$\frac{1}{\lambda}\frac{\dot{\ell}(t)}{\ell(t)} = \zeta_1 \ell(t) - \zeta_2, \tag{A.39}$$

where $\zeta_1 = \frac{\beta\eta(1+\varphi)}{(1-\beta)\varphi}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} > 0, \zeta_2 = \frac{\beta\eta}{1-\beta}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L + \frac{\rho}{\lambda} > 0$. This is an ODE for leisure $\ell(t)$.

1. If $\ell(t) = \zeta_2/\zeta_1$, the aggregate labor $L(t) = L - \ell(t)$ is constant as well. As a result, the innovation rate $z(t)$ also remains unchanged over time from eq. (A.36) and so does the growth rate of the average machine quality $Q(t)$ since $\dot{Q}(t)/Q(t) = (\lambda - 1)z(t)$. The interest rate is also constant.

   From equilibrium conditions in eqs. (33)-(35), (42)-(43), and (A.33) we conclude that aggregate variables $\{Y(t), X(t), D(t), C(t), w(t), A(t)\}$ grow at the same constant rate with $Q(t)$.

2. If $\ell(t) > \zeta_2/\zeta_1$, eq. (A.22) implies that leisure $\ell(t)$ grows unboundedly. This is impossible for a scarce factor since $\ell(t) \leq L$.

3. If $\ell(t) < \zeta_2/\zeta_1$, from eq. (A.22) leisure $\ell(t)$ keep decreasing forever and asymptotically approaches zero. Therefore, the aggregate labor $L(t)$ is increasing and asymptotically approaches L. This path is not optimal since consumption grows slower than both the average quality $Q(t)$ and total output $Y(t)$, from eqs. (33) and (43)

$$\frac{\dot{C}(t)}{C(t)} = \frac{\dot{Q}(t)}{Q(t)} + \frac{\dot{\ell}(t)}{\ell(t)} < \frac{\dot{Q}(t)}{Q(t)} < \frac{\dot{Q}(t)}{Q(t)} + \frac{\dot{L}(t)}{L(t)} = \frac{\dot{Y}(t)}{Y(t)}.$$

Therefore, we establish that there exists a unique BGP where aggregate variables grow at the same constant rate g as the average quality of machines, and the aggregate labor is constant along the BGP. The above argument also shows that the BGP is the only optimal equilibrium path. We pin down the equilibrium leisure and labor

$$\ell(t) = \frac{\zeta_2}{\zeta_1} = \frac{\varphi}{1+\varphi}L + \frac{\rho}{\eta\lambda}\frac{1-\beta}{\beta}\frac{\varphi}{1+\varphi}\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta},$$

$$L(t) = L - \ell(t) = \frac{1}{1+\varphi}L - \frac{\rho}{\eta\lambda}\frac{1-\beta}{\beta}\frac{\varphi}{1+\varphi}\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta}.$$

It is straightforward to pin down equilibrium interest and growth rates as follows:

$$\begin{aligned} g &= \frac{\beta\eta(\lambda-1)}{1+\varphi}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L - \frac{1+\varphi(2-\beta)}{1+\varphi}\frac{\lambda-1}{\lambda}\rho, \\ r &= \frac{\beta\eta(\lambda-1)}{1+\varphi}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L + \frac{1+\varphi-\varphi(1-\beta)(\lambda-1)}{(1+\varphi)\lambda}\rho. \end{aligned} \tag{A.40}$$

Case 1: Inelastic labor supply ($\varphi = 0$)

The interest rate becomes

$$r = \beta\eta(\lambda - 1)\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L + \frac{\rho}{\lambda}, \tag{A.41}$$

Using the same trick in Appendix A.7.3, we rewrite the Gini coefficient G_I

$$G_I = \frac{\xi_1}{\xi_1 + 1} G_A,$$

where $\xi_1 = rA(t)/(w(t)L)$. From eqs. (34) and (A.33) we have

$$\frac{A(t)}{w(t)L} = \frac{\frac{Q(t)}{\eta\lambda}}{\frac{\beta}{1-\beta}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t)L} = \frac{1-\beta}{\beta\eta\lambda L}\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta}, \tag{A.42}$$

which combines with eq. (A.41) to yield

$$\begin{aligned}
\xi_1 &= \left[\beta\eta(\lambda - 1)\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L + \frac{\rho}{\lambda}\right] \frac{1-\beta}{\beta\eta\lambda L}\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta} \\
&= \frac{(1-\beta)(\lambda - 1)}{\lambda} + \frac{(1-\beta)\rho}{\beta\eta\lambda^2 L}\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta}.
\end{aligned}$$

Hence,

$$\operatorname{sign}\left(\frac{\partial G_I}{\partial \bar{E}}\right) = \operatorname{sign}\left(\frac{\partial \xi_1}{\partial \bar{E}}\right) = -1$$

Therefore, a more restrictive environmental policy, which lowers pollution permits to firms, increases income inequality in the case of inelastic labor supply. We obtain the same conclusion as in proposition 3.

Case 2: Elastic labor supply ($\varphi \neq 0$)

We have

$$G_I = \frac{\xi_2}{\xi_2 + 1} G_A,$$

where $\xi_2 = (r + \varphi g)A(t)/(w(t)L)$. From eq. (A.40) we obtain

$$r + \varphi g = (1+\varphi)r - \varphi\rho = \beta\eta(\lambda - 1)\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L + \frac{1 - \varphi(2-\beta)(\lambda - 1)}{\lambda}\rho,$$

which can be combined with eq. (A.42) to obtain

$$\xi_2 = \frac{(1-\beta)(\lambda - 1)}{\lambda} + \frac{(1-\beta)\rho}{\beta\eta\lambda^2 L}\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta}\left[1 - \varphi(2-\beta)(\lambda - 1)\right].$$

Hence,

$$\begin{aligned}
\operatorname{sign}\left(\frac{\partial G_I}{\partial \bar{E}}\right) &= \operatorname{sign}\left(\frac{\partial \xi_2}{\partial \bar{E}}\right) \\
&= -\operatorname{sign}\left(1-\varphi(2-\beta)(\lambda-1)\right) \\
&= \operatorname{sign}\left(\varphi - \frac{1}{(2-\beta)(\lambda-1)}\right).
\end{aligned}$$

We obtain a similar result to proposition 4. Therefore, the main findings of this study are robust to scale effects.

A.9 Model without the normalization

This Appendix presents the model without the normalization $\psi \equiv 1-\beta$ we made in the main model.

A.9.1 Intermediate good sector

For $\psi \neq 1-\beta$, the optimal price condition is now in eq. (11) instead of eq. (12)

$$p(\nu, t|q) = \frac{\psi}{1-\beta} q(\nu, t),$$

which can be substituted into eq. (8) to yield the optimal demand

$$x(\nu, t|q) = \left(\frac{1-\beta}{\psi}\right)^{1/\beta} \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} L(t).$$

The profit function is straightforward

$$\pi(\nu, t|q) = \beta \left(\frac{1-\beta}{\psi}\right)^{1/\beta - 1} \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} q(\nu, t) L(t).$$

A.9.2 R&D sector

From the no-arbitrage condition in eq. (21) we obtain

$$\begin{aligned}
r(t) + z(\nu, t|q) &= \frac{\pi(\nu, t|q)}{V(\nu, t|q)} + \frac{\dot{L}(t)}{L(t)} \\
&= \beta\eta\lambda \left(\frac{1-\beta}{\psi}\right)^{1/\beta - 1} \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \frac{\dot{L}(t)}{L(t)}.
\end{aligned}$$

The free-entry condition in eq. (20) and eq. (24) for the growth rate of the average machine quality are still valid.

A.9.3 Aggregation and equilibrium

The equilibrium aggregate output

$$Y(t) = \frac{1}{1-\beta}\left(\frac{1-\beta}{\psi}\right)^{1/\beta-1}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta-1} Q(t)L(t).$$

The first-order condition in eq. (7) pins down labor wages

$$\begin{aligned} w(t) &= \frac{\beta Y(t)}{L(t)}\left(\frac{\bar{E}}{\alpha}\right) \\ &= \frac{\beta}{1-\beta}\left(\frac{1-\beta}{\psi}\right)^{1/\beta-1}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t). \end{aligned} \tag{A.43}$$

The derivation of aggregate machine spending $X(t)$ is straightforward from its definition

$$X(t) = (1-\beta)\left(\frac{1-\beta}{\psi}\right)^{1/\beta-1}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t)L(t).$$

Abatement expenditure still accounts for the same proportion of the output

$$\begin{aligned} D(t) &= \left(1-\frac{\bar{E}}{\alpha}\right) Y(t) \\ &= \frac{1}{1-\beta}\left(1-\frac{\bar{E}}{\alpha}\right)\left(\frac{1-\beta}{\psi}\right)^{1/\beta-1}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta-1} Q(t)L(t). \end{aligned}$$

Aggregate consumption again is pinned down from the intratemporal condition in eq. (29)

$$\begin{aligned} C(t) &= w(t)\frac{L-L(t)}{\varphi} \\ &= \frac{\beta}{1-\beta}\left(\frac{1-\beta}{\psi}\right)^{1/\beta-1}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t)\frac{L-L(t)}{\varphi}. \end{aligned}$$

Finally, the expression for aggregate assets in eq. (44) still holds. We realize that variables $\{Y(t), X(t), D(t), C(t), w(t)\}$ in equilibrium are different from their counterparts in the main model only by a factor of $\left(\frac{1-\beta}{\psi}\right)^{1/\beta-1}$.

A.9.4 Balanced growth path (BGP)

The analysis in Appendix A.5.1 is still valid, except for the steady-state values of variables.

Instead of eq. (45), the aggregate labor supply equals

$$L(t) = L\left(1+\varphi+\frac{\rho}{\eta\lambda}\frac{1-\beta}{\beta}\varphi\left(\frac{1-\beta}{\psi}\right)^{1-1/\beta}\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta}\right)^{-1}. \tag{A.44}$$

The no-arbitrage condition pins down the equilibrium innovation rate

$$z(t) = \beta\eta \left(\frac{1-\beta}{\psi}\right)^{1/\beta-1} \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} - \frac{\rho}{\lambda}.$$

Derivations of the growth and interest rate are straightforward

$$\begin{aligned} g &= (\lambda-1)z(t) \\ &= \beta\eta(\lambda-1) \left(\frac{1-\beta}{\psi}\right)^{1/\beta-1} \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} - \frac{\lambda-1}{\lambda}\rho, \quad \text{(A.45)} \\ r &= g+\rho \\ &= \beta\eta(\lambda-1) \left(\frac{1-\beta}{\psi}\right)^{1/\beta-1} \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \frac{1}{\lambda}\rho. \quad \text{(A.46)} \end{aligned}$$

Compared to eqs. (46)-(47), it is observable that the first terms are different from their counterparts in the main model by a factor of $\left(\frac{1-\beta}{\psi}\right)^{1/\beta-1}$. The coefficients of ρ are the same.

A.9.5 Income inequality

Case 1: Inelastic labor supply ($\varphi = 0$)

We write down again the established relationship between income and wealth inequality in eq. (51)

$$G_I = \frac{r}{r + w(t)L/A(t)} G_A = \frac{\xi_1}{\xi_1+1} G_A,$$

where $\xi_1 = rA(t)/(w(t)L)$. Combining eqs. (A.43) and (44) to obtain

$$\begin{aligned} \frac{A(t)}{w(t)L} &= \frac{\frac{Q(t)L}{\eta\lambda}}{\frac{\beta}{1-\beta}\left(\frac{1-\beta}{\psi}\right)^{1/\beta-1}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} Q(t)L} \\ &= \frac{1-\beta}{\beta\eta\lambda}\left(\frac{1-\beta}{\psi}\right)^{1-1/\beta}\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta}. \quad \text{(A.47)} \end{aligned}$$

Therefore,

$$\begin{aligned} \xi_1 &= \left[\beta\eta(\lambda-1)\left(\frac{1-\beta}{\psi}\right)^{1/\beta-1}\left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \frac{1}{\lambda}\rho\right] \frac{1-\beta}{\beta\eta\lambda}\left(\frac{1-\beta}{\psi}\right)^{1-1/\beta}\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta} \\ &= \frac{(1-\beta)(\lambda-1)}{\lambda} + \frac{(1-\beta)\rho}{\beta\eta\lambda^2}\left(\frac{1-\beta}{\psi}\right)^{1-1/\beta}\left(\frac{\bar{E}}{\alpha}\right)^{-1/\beta}. \end{aligned}$$

Hence,

$$\text{sign}\left(\frac{\partial G_I}{\partial \bar{E}}\right) = \text{sign}\left(\frac{\partial \xi_1}{\partial \bar{E}}\right) = -1,$$

which confirms that the proposition 3 still holds.

Case 2: Elastic labor supply ($\varphi \neq 0$)

Substituting eqs. (44) and (A.43)-(A.46) into eq. (50) yields

$$G_I = \left(\frac{1-\beta}{\psi}\right)^{1/\beta - 1} \frac{\xi_2 \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \xi_3}{\xi_4 \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta} + \xi_5} G_A,$$

where

$$\begin{cases} \xi_2 = \beta\eta(\lambda - 1)(1+\varphi)\left(\frac{1-\beta}{\psi}\right)^{1/\beta - 1} \\ \xi_3 = \frac{1+\varphi-\lambda\varphi}{\lambda}\rho \\ \xi_4 = \frac{\beta\eta(1+\varphi)[1+(2-\beta)(\lambda-1)]}{1-\beta}\left(\frac{1-\beta}{\psi}\right)^{1/\beta - 1} \\ \xi_5 = \frac{1+\varphi}{\lambda}\rho \end{cases}$$

We again apply the chain rule to obtain

$$\begin{aligned} \text{sign}\left(\frac{\partial G_I}{\partial \bar{E}}\right) &= \text{sign}\left(\frac{\partial G_I}{\partial \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta}}\right) . \underbrace{\text{sign}\left(\frac{\partial \left(\frac{\bar{E}}{\alpha}\right)^{1/\beta}}{\partial \bar{E}}\right)}_{=1} \\ &= \text{sign}(\xi_2\xi_5 - \xi_3\xi_4) \\ &= \text{sign}\left(\varphi - \frac{1}{(2-\beta)(\lambda-1)}\right). \end{aligned}$$

We confirm that proposition 4 is still valid.

To sum up, the normalization $\psi \equiv 1 - \beta$ does not affect the main findings of this study.

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