Franz W. Peren

Statistics for Business and Economics

Compendium of Essential Formulas

Second Edition

Springer

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For my mother, Maria.

Preface

Preface to the 2nd edition

The 2nd edition of this compendium of formulas for economic statistics has been supplemented with numerous practical examples, especially in chapters 3 and 4. My valuable student and research assistants Steven Dyla, Nawid Schahab, and Paula Schmidt have contributed to the current edition. They deserve my thanks. Should any mistakes remain, such errors shall be exclusively at the expense of the author. The author is thankful in advance to all users of this formulary for any constructive comments or suggestions.

Bonn, June 2022 Franz W. Peren

Preface to the 1st edition

The following book is based on the author's expertise in the field of business statistics. After completing his studies in business administration and mathematics, he started his career working for a global bank and the German government. Later he became a professor of business administration, specialising in quantitative methods. He has been a professor at the Bonn-Rhein-Sieg University in Sankt Augustin, Germany since 1995, where he is mainly teaching business mathematics, business statistics, and operations research. He has also previously taught and conducted research at the University of Victoria in Victoria, BC, Canada and at Columbia University in New York City, New York, USA. To the author's best knowledge and beliefs, this formulary presents its statistical contents in a practical manner, as they are needed for meaningful and relevant application in global business, as well as in universities and economic practice.

The author would like to thank his academic colleagues who have contributed to this work and to many other projects with creativity, knowledge and dedication for more than 25 years. In particular, he would

like to thank Ms. Eva Siebertz and Mr. Nawid Schahab, who were instrumental in managing and creating this formulary. Special thanks are given to Ms. Camilla Demuth, Ms. Linh Hoang, Ms. Michelle Jarsen, and Ms. Paula Schmidt. Should any mistakes remain, such errors shall be exclusively at the expense of the author. The author is thankful in advance to all users of this formulary for any constructive comments or suggestions.

Bonn, June 2021 Franz W. Peren

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About the Author

Prof. Dr. rer. pol. Franz W. Peren, Ph.D., is a professor of business administration, specializing in quantitative methods. He has been teaching business mathematics, business statistics and quantitative methods in planning, taxation and controlling within operational and strategic management since 1993, mainly at German universities of applied sciences. He has also taught and conducted research as a visiting professor at the University of Victoria in Victoria, BC, Canada and at Columbia University in New York City, USA.

List of Abbreviations

bln	billion
cm	centimetre(s)
CV	coefficient of variation
e	Euler’s number
ed.	edition, editor
e.g.	exempli gratia
Fig.	figure
G	geometric mean
GDP	gross domestic product
H	harmonic mean
i.e.	id est
kWh	kilowatt-hour(s)
lb/lbs	libra
lim	limit
max	maximum, maximise
Me	median
min	minimum, minimise
Mo	mode
mph	miles per hour
PCI	Peren-Clement index

QU	quantity units
R	range
rep.	repetition
Tab.	table
Var	variance
Vol.	volume
w/	with
w/out	without

Chapter 1

Statistical Signs and Symbols

General

Signs/Symbols	Meaning
$\mathbb{N}$	set of natural numbers $\{0, 1, 2, \ldots\}$ (formerly $\mathbb{N}_0$)
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$a \geq b$	a is greater than or equal to b
$a \approx b$	a is approximately equal to b
$\sum_{i=1}^{n} a_i$	$a_1 + a_2 + \ldots + a_n$
$\prod_{i=1}^{n} a_i$	$a_1 \cdot a_2 \cdot \ldots \cdot a_n$
$\frac{\mathrm{d}y}{\mathrm{d}x} = y'(x)$	1st derivative of the function $y = y(x)$ with respect to the variable x
$\frac{\partial y}{\partial x}$	1st partial derivative of the function y with respect to the variable x
$\int$	integral
$\|a\|$	absolute value of a
$\lim_{x \to a} f(x)$	limit of the function $f(x)$, with x converging towards a

F. W. Peren, *Statistics for Business and Economics*,
https://doi.org/10.1007/978-3-662-65846-8_1

Set Theory

Symbols	Meaning
A'	transposed matrix for A
$sgn(x)$	algebraic sign of x
$\{a_1,\ a_2,\ ...,\ a_n\}$	set of the elements $a_1,\ a_2,\ ...,\ a_n$
$\{x \mid B(x)\}$	set of all x, to which $B(x)$ applies
$\varnothing$, also $\{\}$	empty set (includes no elements)
$a \in A$	a is element of A
$a \notin A$	a is not element of A
$A = B$	A equals B
$A \subseteq B$, also $A \subset B$	A is subset of B
$A \subsetneq B$	A is proper subset of B
$A \supseteq B$, also $A \supset B$	A is superset of B
$A \cap B$	intersection of A and B
$A \cup B$	union of A and B
$A \backslash B$	relative complement of A and B
$\bar{A}$	complement of A
$A \times B$	cartesian product of A and B
$\phi(A)$	power set of A

Chapter 2

Descriptive Statistics

2.1 Empirical Distributions

2.1.1 Frequencies

The frequency distribution is a clear and meaningful summary, sorted by frequencies of results in the form of tables, graphs and statistical measurement figures (e.g. mean values, measures of dispersion).

If a statistical characteristic exists in k different characteristic values, $x_1, x_2, ..., x_k$, for which, given a population of N elements or a sample of n observations, the

absolute frequencies, h_i $\quad h_1, h_2, ..., h_k \quad$ with $\quad 0 \leq h_i \leq N$

and $\quad \sum_{i=1}^{k} h_i = N$

or $\quad \sum_{i=1}^{k} h_i = n$

are given, this results in the corresponding

relative frequencies, f_i $\quad f_1, f_2, ..., f_k \quad$ with $\quad 0 \leq f_i \leq 1$

and $\quad \sum_{i=1}^{k} f_i = 1$

$$f_i = \frac{h_i}{N} \quad \text{or} \quad f_i = \frac{h_i}{n}$$

F. W. Peren, *Statistics for Business and Economics*,
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Example: Height of 100 students

i	Height [cm]	h_i	f_i
1	under 160	9	$0.09 = \frac{9}{100}$
2	$[160 - 170[$	28	0.28
3	$[170 - 180[$	35	0.35
4	$[180 - 190[$	24	0.24
5	$190 \leq$	4	0.04
$\sum$	-	100	1.0

2.1.2 Cumulative Frequencies

By continuous summation (cumulation) of the absolute frequencies h_j, one obtains the *absolute* cumulative frequencies H_i.

$$\begin{aligned} H_i &= h_1 + h_2 + ... + h_i \qquad\qquad j = 1, ..., i \\ &= \sum_{j=1}^{i} h_j \end{aligned}$$

By continuous summation of the relative frequencies f_j, one obtains the *relative* cumulative frequencies F_i.

$$\begin{aligned} F_i &= f_1 + f_2 + ... + f_i \\ &= \sum_{j=1}^{i} f_j \qquad\qquad j = 1, ..., i \\ &= \frac{H_i}{N} \qquad \Rightarrow \text{ for the population} \\ &= \frac{H_i}{n} \qquad \Rightarrow \text{ for the sample} \end{aligned}$$

Cumulative Frequency Function for Ungrouped Data

$$F(x) = \begin{cases} 0 & \text{for} \quad x < x_1 \\ F_i & \text{for} \quad x_i \leq x \leq x_{i+1} \qquad \text{with} \quad i = 1, \ldots, k-1 \\ 1 & \text{for} \quad x \geq x_k \end{cases}$$

Example: Number of newspapers regularly read by students

i	x_i	h_i	f_i	H_i	F_i
1	0	200	0.160	200	0.160
2	1	510	0.407	710	0.567
3	2	253	0.202	963	0.769
4	3	163	0.130	1,126	0.899
5	4	127	0.101	1,253	1.0
$\sum$	-	1,253	1.0	-	-

Cumulative Frequency Function for Grouped Data

The accumulated (absolute or relative) cumulative frequencies are each assigned to the ends of the class interval.

Example: Height of 100 students

i	Height [cm]	h_i	f_i	H_i	F_i
1	under 160	9	0.09	9	0.09
2	$[160 - 170[$	28	0.28	37	0.37
3	$[170 - 180[$	35	0.35	72	0.72
4	$[180 - 190[$	24	0.24	96	0.96
5	$190 \leq$	4	0.04	100	1.0
$\sum$	-	100	1.0	-	-

2.2 Mean Values and Measures of Dispersion

2.2.1 Mean Values

Arithmetic Mean (μ or $\bar{x}$)

Definition for the population N with x_1, x_2, ..., x_N

$$\mu = \frac{1}{N}(x_1 + ... + x_N) = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Definition for a sample in the range of n with x_1, x_2, ..., x_n

$$\bar{x} = \frac{1}{n}(x_1 + ... + x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Frequency Distributions

- absolute frequency distributions

$$\mu = \frac{1}{N}\sum_{i=1}^{k} x_i h_i = \frac{1}{N}(x_1 h_1 + x_2 h_2 + ... + x_k h_k)$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{k} x_i h_i = \frac{1}{n}(x_1 h_1 + x_2 h_2 + ... + x_k h_k)$$

- relative frequency distributions

$$\mu = \sum_{i=1}^{k} x_i f_i \qquad \text{with} \qquad f_i = \frac{h_i}{N}$$

$$\bar{x} = \sum_{i=1}^{k} x_i f_i \qquad \text{with} \qquad f_i = \frac{h_i}{n}$$

- with a frequency distribution of *grouped data*, the following applies

for the population:

$$\mu = \frac{1}{N}\sum_{i=1}^{k} x_i' h_i \;=\; \sum x_i' f_i$$

Usually the centre of the class interval is chosen.

for the sample:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{k} x_i' h_i \;=\; \sum x_i' f_i$$

Usually the centre of the class interval is chosen.

Example:

i	x_i hours of sleep per night	h_i people	f_i	F_i	H_i
1	8	3	0.176	0.176	3
2	6	1	0.059	0.235	4
3	7	7	0.412	0.647	11
4	10	4	0.235	0.882	15
5	4	2	0.118	1	17
Σ	-	17	-	-	-

arithmetic mean

$$\mu = \frac{1}{17}\,(8\cdot 3 + 6\cdot 1 + 7\cdot 7 + 10\cdot 4 + 4\cdot 2)$$

$$= 7.471 \text{ hours}$$

On average, the respondents slept 7.471 hours per night.

Median (Me)

The single values $x_1, x_2, ..., x_N$ are ordered, so that the following applies:

$x_{[1]} \leq x_{[2]} \leq ... \leq x_{[N]}$ with $x_{[j]} =$ the element x at the j^{th} position; $j = 1, ..., N$

Median for **uneven** N: $$Me = x_{\left[\frac{N+1}{2}\right]}$$

Example:

Height of five children in cm: 120, 150, 110, 124, 132

Sorting the values: 110, 120, 124, 132, 150

Calculating median: $$\frac{N+1}{2}$$

$$\Rightarrow \frac{5+1}{2} = 3^{rd} \text{ observation}$$

$$\Rightarrow x_3 = 124$$

The median is 124 cm.

Median for **even** N: $$Me = \frac{1}{2}\left(x_{\left[\frac{N}{2}\right]} + x_{\left[\frac{N}{2}+1\right]}\right)$$

Example:

Height of six children in cm: 131, 124, 135, 115, 119, 126

Sorting the values: 115, 119, 124, 126, 131, 135

Calculating median: $$\frac{1}{2}\left(\frac{6}{2} + \frac{6}{2} + 1\right)$$

$$= \frac{1}{2}(x_3 + x_4)$$

$$= \frac{1}{2}(124 + 126)$$

$$= \frac{1}{2} \cdot 250$$

$$= 125$$

The median is 125 cm.

<u>Frequency Distributions</u>

For *ungrouped data*, the median is equal to the characteristic value x_i, for which the cumulative frequency function $F(x)$ exceeds the value 0.5.

For *grouped data*, the median is calculated using the class interval's lower limit x_i^l and the class interval's upper limit x_i^u of the class, in which the cumulative frequency function $F(x)$ exceeds the value 0.5.

$$Me = x_i^l + \alpha$$

$$\Rightarrow \quad \frac{\alpha}{0.5 - F(x_i^l)} = \frac{x_i^u - x_i^l}{F(x_i^u) - F(x_i^l)}$$

$$\Rightarrow \quad \alpha = \frac{x_i^u - x_i^l}{F(x_i^u) - F(x_i^l)} \cdot (0.5 - F(x_i^l))$$

$$Me = x_i^l + \frac{x_i^u - x_i^l}{F(x_i^u) - F(x_i^l)} \cdot (0.5 - F(x_i^l))$$

Mode (Mo)

The mode is defined as the most frequent characteristic value.

<u>Example</u>: newspapers read regularly

0 newspapers $\Rightarrow$ 19 people

$\boxed{1 \text{ newspaper} \Rightarrow 45 \text{ people}}$

2 newspapers $\Rightarrow$ 24 people

3 newspapers $\Rightarrow$ 8 people

$\Rightarrow$ $Mo = 1$ newspaper, as 45 people form the largest (absolute) frequency

For *grouped data*, the class interval with the highest frequency density is first selected as the modal class. The centre of this class interval is then defined as the mode.

$$\text{density} = \frac{h_i}{\text{class width}} \cdot \text{standard class width}$$

Example: Height of 100 students

i	Height $[cm]$	h_i	x_i'	Δx	Density
1	$[140-160[$	9	150	20	0.45
2	$[160-170[$	28	165	10	2.8
3	$[170-180[$	35	175	10	3.5
4	$[180-190[$	24	185	10	2.4
5	$[190-210[$	4	200	20	0.2
Σ	-	100	-	-	-

The highest frequency density is located in the third class interval:

$$x_3 = \frac{35}{10} = 3.5 \text{ students per } 10\text{-cm-interval height}$$

The centre of this class is $175\ cm$. $\Rightarrow\ Mo = 175\ cm$

Geometric Mean (G)

Geometric mean for **single values**:

$$G = \sqrt[N]{x_1 \cdot x_2 \cdot \ldots \cdot x_N}$$

Example:

Percentage p	$7\ \%$	$3\ \%$	$-5\ \%$	$4\ \%$
$x_i = 1 + \frac{p}{100}$	1.07	1.03	0.95	1.04

$G = \sqrt[4]{1.07 \cdot 1.03 \cdot 0.95 \cdot 1.04} = 1.022$

Geometric mean for **frequency distributions**:

$$G = \sqrt[N]{x_1^{h_1} \cdot x_2^{h_2} \cdot \ldots \cdot x_k^{h_k}} \qquad \text{with} \quad i = 1, \ldots, k$$

Example:

Percentage p	-3 %	-2 %	1 %	3 %
$x_i = 1 + \frac{p}{100}$	0.97	0.98	1.01	1.03
absolute frequencies H_i	3	2	4	1

$n = 3 + 2 + 4 + 1 = 10$

$G = \sqrt[10]{0.97^3 \cdot 0.98^2 \cdot 1.01^4 \cdot 1.03^1} = 0.994 = -0.6\ \%$

Tab. 2.1 shows at which scale level the application of the corresponding mean values is possible.

Mean value	Scale			
	Nominal scale	Ordinal scale	Interval scale	Ratio scale
Mode	×	×	×	×
Median		×	×	×
Arithmetic mean			×	×
Geometric mean				×
Harmonic mean				×

Tab. 2.1: Mean Values in Correspondence to Scale Levels[1]

[1] Cf. Bleymüller, J. & Gehlert, G. (2011), p. 13.

Harmonic Mean (H)

$$H = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$$

The dimension of the respective characteristic under consideration and the resulting harmonic mean corresponds to a quotient.

Example: A car drives 12 miles: a) 6 miles at 6 mph and b) 6 miles at 60 mph.

What is the average speed?

$$H = \frac{2}{\frac{1}{6mph} + \frac{1}{60mph}} = \frac{2}{\frac{10}{60} + \frac{1}{60}} = \frac{2}{\frac{11}{60}} = \frac{2 \cdot 60}{11} = 10.91\,mph$$

Remark: The dimension of the characteristic considered here corresponds to the quotient mph.

2.2.2 Measures of Dispersion

Variance σ^2 / Standard Deviation σ

For **single values**:

Definition for the population N

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i^2) - \mu^2$$

$$\sigma = \sqrt{\sigma^2}$$

Example:

US shoe sizes of four people: 9, 8, 10, 11

$$\mu = \frac{9 + 8 + 10 + 11}{4} = 9.5$$

$$\sigma^2 = \frac{(9-9.5)^2 + (8-9.5)^2 + (10-9.5)^2 + (11-9.5)^2}{4} =$$

$$= 1.25$$

$$\sigma = \sqrt{1.25} = 1.12$$

Definition for a sample of n observations with $x_1, x_2, ..., x_n$

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i^2) - \bar{x}^2$$

$$s = \sqrt{s^2}$$

For **frequency distributions**:

- for **absolute** frequency distributions

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{k}(x_i - \mu)^2 h_i = \frac{1}{N}\sum_{i=1}^{k}(x_i^2\, h_i) - \mu^2$$

$$s^2 = \frac{1}{n}\sum_{i=1}^{k}(x_i - \bar{x})^2 h_i = \frac{1}{n}\sum_{i=1}^{k}(x_i^2\, h_i) - \bar{x}^2$$

Example of frequency distribution with absolute frequencies:

i	x_i Number of sold books of a particular book	h_i Number of days	$x_i h_i$	$x_i^2 h_i$
1	0	19	0	0
2	1	34	34	34
3	2	17	34	68
4	3	6	18	54
5	4	12	48	192
$\sum$	-	88	-	348

$$\begin{aligned}
\mu &= \frac{1}{N}(x_1 \cdot h_1 + \ldots + x_N \cdot h_N) = \frac{1}{N}\sum_{i=1}^{N} x_i h_i = \\
&= \frac{1}{88}(0 \cdot 19 \;+\; 1 \cdot 34 \;+\; 2 \cdot 17 \;+\; 3 \cdot 6 \;+\; 4 \cdot 12) = \\
&= 1.523
\end{aligned}$$

$$\begin{aligned}
\sigma^2 &= \frac{348}{88} \;-\; 1.523^2 = \\
&= 1.635 \text{ books}^2
\end{aligned}$$

$$\sigma = \sqrt{1.635} = 1.279 \text{ books}$$

- for **relative** frequency distributions

$$\sigma^2 = \sum_{i=1}^{k}(x_i - \mu)^2 f_i = \sum_{i=1}^{k}(x_i^2 f_i) - \mu^2$$

$$s^2 = \sum_{i=1}^{k}(x_i - \bar{x})^2 f_i = \sum_{i=1}^{k}(x_i^2 f_i) - \bar{x}^2$$

Example of a frequency distribution for relative frequencies:

i	x_i Number of sold books of a particular books	h_i Number of days	$x_i h_i$	$x_i^2 h_i$	f_i	$x_i f_i$	$x_i^2 f_i$
1	0	19	0	0	0.216	0	0
2	1	34	34	34	0.386	0.386	0.386
3	2	17	34	68	0.193	0.386	0.772
4	3	6	18	54	0.068	0.204	0.612
5	4	12	48	192	0.136	0.544	2.176
Σ	-	88	-	348	1.000	-	3.946

$$\begin{aligned}\mu &= \frac{1}{N}(x_1 \cdot h_1 + ... + x_N \cdot h_N) = \frac{1}{N}\sum_{i=1}^{N} x_i h_i = \\ &= \frac{1}{88}(0 \cdot 19 \ + \ 1 \cdot 34 \ + \ 2 \cdot 17 \ + \ 3 \cdot 6 \ + \ 4 \cdot 12) = \\ &= 1.523\end{aligned}$$

$$\begin{aligned}\sigma^2 &= 3.946 \ - \ 1.523^2 = \\ &= 1.626 \text{ books}^2\end{aligned}$$

$$\sigma = \sqrt{1.626} = 1.275 \text{ books}$$

With a frequency distribution of *grouped data*, the variance/standard deviation are approximately calculated using the centres of the class intervals x_i'.

- for absolute frequency distributions of grouped data

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{k}(x_i' - \mu)^2 h_i = \frac{1}{N}\sum_{i=1}^{k}(x_i'^2 h_i) - \mu^2$$

$$s^2 = \frac{1}{n}\sum_{i=1}^{k}(x_i' - \bar{x})^2 h_i = \frac{1}{n}\sum_{i=1}^{k}(x_i'^2 h_i) - \bar{x}^2$$

Example: Height of 100 students

i	Height $[cm]$	h_i	x'_i	Δx	Density	$x'_i h_i$	$x'^2_i h_i$
1	$[140-160[$	9	150	20	0.45	1,350	202,500
2	$[160-170[$	28	165	10	2.8	4,620	762,300
3	$[170-180[$	35	175	10	3.5	6,125	1,071,875
4	$[180-190[$	24	185	10	2.4	4,440	821,400
5	$[190-210[$	4	200	20	0.2	800	160,000
Σ	-	100	-	-	-	-	3,018,075

with $x'_i =$ class midpoint,
$\Delta x =$ class width,
density of the elements $= \dfrac{h_i}{\Delta x} =$ elements per unit

$$\begin{aligned}\mu &= \frac{1}{N}(x'_1 \cdot h_1 + \ldots + x'_N \cdot h_N) = \frac{1}{N}\sum_{i=1}^{N} x'_i h_i = \\ &= \frac{1}{100}(150\cdot 9 \ + \ 165\cdot 28 \ + \ 175\cdot 35 \ + \ 185\cdot 24 \ + \ 200\cdot 4) = \\ &= 173.35\end{aligned}$$

$$\begin{aligned}\sigma^2 &= \frac{3,018,075}{100} \ - \ 173.35^2 = \\ &= 130.5275 \text{ cm}^2\end{aligned}$$

$$\sigma = \sqrt{130.5275} = 11.425 \text{ cm}$$

- for relative frequency distributions of grouped data

$$\sigma^2 = \sum_{i=1}^{k}(x'_i-\mu)^2 f_i = \sum_{i=1}^{k}(x'^2_i f_i) - \mu^2$$

$$s^2 = \sum_{i=1}^{k}(x'_i-\bar{x})^2 f_i = \sum_{i=1}^{k}(x'^2_i f_i) - \bar{x}^2$$

Example: Height of 100 students

i	Height $[cm]$	h_i	x'_i	Δx	Density	f_i	$x'_i f_i$	$x'^2_i f_i$
1	$[140-160[$	9	150	20	0.45	0.09	13.5	2,025
2	$[160-170[$	28	165	10	2.8	0.28	46.2	7,623
3	$[170-180[$	35	175	10	3.5	0.35	61.25	10,718.75
4	$[180-190[$	24	185	10	2.4	0.24	44.4	8,214
5	$[190-210[$	4	200	20	0.2	0.04	8	1,600
Σ	-	100	-	-	-	1	-	30,180.75

with $x'_i =$ class midpoint,
$\Delta x =$ class width,
density of the elements $= \dfrac{h_i}{\Delta x} =$ elements per unit

$$\mu = \frac{1}{N}(x'_1 \cdot h_1 + ... + x'_N \cdot h_N) = \frac{1}{N}\sum_{i=1}^{N} x'_i h_i =$$
$$= \frac{1}{100}(150 \cdot 9 \;+\; 165 \cdot 28 \;+\; 175 \cdot 35 \;+\; 185 \cdot 24 \;+\; 200 \cdot 4) =$$
$$= 173.35$$

$$\sigma^2 = 30,180.75 \;-\; 173.35^2 =$$
$$= 130.5275 \text{ cm}^2$$

$$\sigma = \sqrt{130.5275} = 11.425 \text{ cm}$$

If the distribution of the characteristic values is unimodal and the class widths Δx are constant, Sheppard's correction leads to a better approximate value:

$$\sigma^2_{corr.} = \sigma^2 - \frac{(\Delta x)^2}{12}$$

Example: Height of 100 students

i	Height $[cm]$	h_i	x'_i	Δx	Density	$x'_i h_i$	$x'^2_i h_i$	f_i	$x'_i f_i$	$x'^2_i f_i$
1	$[140-150[$	9	145	10	0.9	1,305	189,225	0.09	13.05	1,892.25
2	$[150-160[$	28	155	10	2.8	4,340	672,700	0.28	43.4	6,727
3	$[160-170[$	35	165	10	3.5	5,775	952,875	0.35	57.75	9,528.75
4	$[170-180[$	24	175	10	2.4	4,200	735,000	0.24	42	7,350
5	$[180-190[$	4	185	10	0.4	740	136,900	0.04	7.4	1,369
Σ	-	100	-	-	-	-	2,686,700	1	-	26,867

$$\begin{aligned}\mu &= \frac{1}{N}\left(x'_1 \cdot h_1 + \ldots + x'_N \cdot h_N\right) = \frac{1}{N}\sum_{i=1}^{N} x'_i h_i = \\ &= \frac{1}{100}\left(145\cdot 9\ +\ 155\cdot 28\ +\ 165\cdot 35\ +\ 175\cdot 24\ +\ 185\cdot 4\right) = \\ &= 163.6\end{aligned}$$

$$\begin{aligned}\sigma^2 &= 26,867\ -\ 163.6^2 = \\ &= 102.04\ \text{cm}^2\end{aligned}$$

$$\sigma = \sqrt{102.04} = 10.10\ \text{cm}$$

Sheppard's correction:

$$\begin{aligned}\sigma^2_{korr.} &= \sigma^2 - \frac{(\Delta x)^2}{12} = \\ &= 102.04\ -\ \frac{(10)^2}{12} = \\ &= 93.706\end{aligned}$$

The absolute frequencies h_i show that this distribution is unimodal. The maximum of this distribution is within the class $i = 3$.

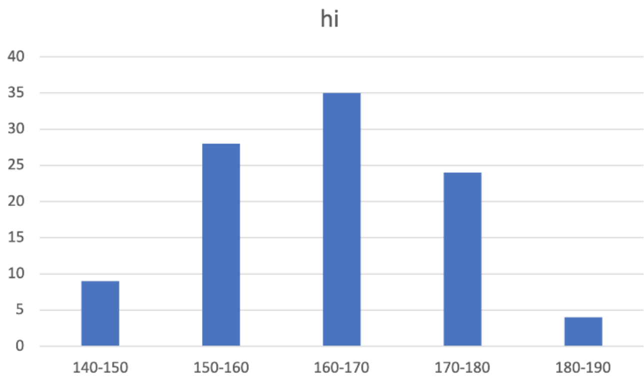

Fig. 2.1: Example of a unimodal frequency distribution with constant class width

Coefficient of Variation (*CV*)

$$CV = \frac{\sigma}{\mu}\,(\cdot\,100\,\%) \qquad \text{or} \qquad CV = \frac{s}{\bar{x}}\,(\cdot\,100\,\%)$$

Example:

Height of six people in cm: 174, 168, 151, 160, 171, 147

$$\mu = \frac{1}{6}\,(174+168+151+160+171+147) = $$
$$= 161.83 \text{ cm}$$

$$\sigma^2 = \frac{(-14.83)^2+(-10.83)^2+(-1.83)^2+6.17^2+9.17^2+12.17^2}{6} =$$
$$= 101.806 \text{ cm}^2$$

$$\sigma = \sqrt{101.806} = 10.09 \text{ cm}$$

$$CV = \frac{\sigma}{\mu \text{ or } \bar{x}} = \frac{10.09}{161.83} = 0.062$$

Range (R)

If the single values x_1, x_2, ..., x_N are arranged according to size, so that:

$$x_{[1]} \leq x_{[2]} \leq ... \leq x_{[N]},$$

the following applies:

$$R = x_{[N]} - x_{[1]} \quad \text{or}$$
$$R = x_{maximum} - x_{minimum}$$

<u>Example</u>:

Person	1	2	3	4	5	6
Age in years	17	24	12	42	60	11

Range: $60 - 11 = 49$ years

- for **grounded data**

  $R =$ upper limit of the largest class interval minus lower limit of the smallest class interval

<u>Example</u>: Height of 100 students

i	Height $[cm]$	h_i	x'_i	Δx	Density
1	$[140-160[$	9	150	20	0.45
2	$[160-170[$	28	165	10	2.8
3	$[170-180[$	35	175	10	3.5
4	$[180-190[$	24	185	10	2.4
5	$[190-210[$	4	200	20	0.2
$\sum$	-	100	-	-	-

$R = 210 - 140 = 70\ cm$

Tab. 2.2 shows at which scale level the calculation of the corresponding measure of dispersion is possible.

Measures of dispersion	Scale			
	Nominal scale	Ordinal scale	Interval scale	Ratio scale
Range		×	×	×
Mean absolute deviation			×	×
Variance, standard deviation			×	×
Coefficient of variation				×

Tab. 2.2: Measures of Dispersion in Correspondence to Scale Levels[2]

[2] Cf. Bleymüller, J. & Gehlert, G. (2011), p. 17.

2.3 Ratios and Index Figures

2.3.1 Ratios

Ratios are key figures that are formed as quotients (Fig. 2.2).

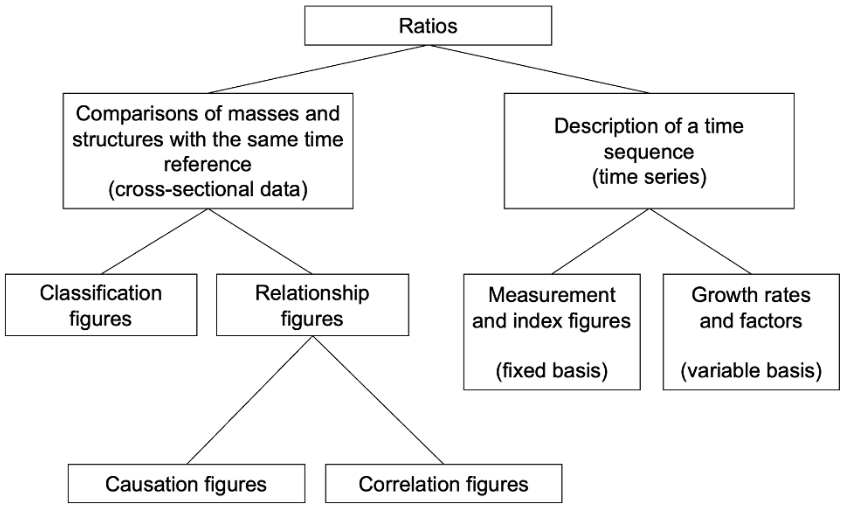

Fig. 2.2: Ratio Figures[3]

<u>Examples</u>:

Classification figures

These relate a partial quantity to a corresponding total quantity

$$\text{e.g.:} \quad \text{consumption rate} = \frac{\text{consumption}}{\text{income}^4} \cdot 100\,\%$$

[3] Cf. Voß, W. (2000), p. 209.

[4] Generally, the disposable income is chosen here.

Relationship figures

These relativise two measures that belong to different sets, i.e. the numerator of the quotient is not a subset of the denominator. In the case of relationship figures, a distinction is made between causation figures and correlation figures.

In the case of *causation figures*, the subset measured in the numerator is “caused” by the mass shown in the denominator:

e.g.: $$\text{productivity} = \frac{\text{output}}{\text{input}}$$

$$\text{profitability} = \frac{\text{profit}}{\text{capital}}$$

In the case of *correlation figures*, there is no causality between the masses shown in the numerator and denominator:

e.g.: $$\text{population density} = \frac{\text{population}}{\text{area}}$$

$$\text{density of doctors} = \frac{\text{number of doctors}}{\text{population}}$$

A **measurement figure** m_{0t} describes the ratio of a (usually current) value x_t to the base value x_0, where t is the reporting period or reporting time and 0 is the base period (reference period) or base time (reference time):

$$m_{0t} = \frac{x_t}{x_0} \quad \text{or} \quad \frac{x_t}{x_0} \cdot 100\,\%$$

with $t =$ reporting period or reporting time

$0 =$ base period or base time

Properties of Measurement Figures

(1) If the base and reporting periods or base and reporting time are equal, the following applies: $m_{0t} = 1$.

(2) Measurement figures are dimensionless. The same dimensions of x_t and x_0 cancel each other out.

(3) If base and reporting periods or base and reporting time are swapped reciprocally, the following applies: $m_{t0} = \dfrac{1}{m_{0t}}$.

(4) Several periods $(0, s$ and $t)$ can be concatenated or uniformly based (see: Operations with Measurement Figures).

(5) If the value W is the product of P and Q for all periods, the following applies analogously to the measurement figures: $m^W_{0t} = m^P_{0t} \cdot m^Q_{0t}$ (factor reversal sampling).

Operations with Measurement Figures

Concatenation: $m_{0t} = m_{0s} \cdot m_{st}$

Rebasing: $m_{st} = \dfrac{m_{0t}}{m_{0s}}$

Growth rate: $w_t = \dfrac{x_t - x_{t-1}}{x_{t-1}} \cdot 100\,\%$ or $w_t = \left(\dfrac{x_t}{x_{t-1}} - 1\right) \cdot 100\,\%$

Examples for Measurement Figures:

1. Index figures (see chapter 2.3.2)

2. Growth rate

The (nominal) gross domestic product (GDP) of an economy in three subsequent years t_i with $i = 1, ..., 3$ is:

$t_1 = \$3,700$ bln, $t_2 = \$3,800$ bln, $t_3 = \$3,900$ bln.

The annual economic growth, measured as the rate of change in nominal GDP, is calculated as follows:

$$w_{t_1-t_2} = \frac{3,800 - 3,700}{3,700} \cdot 100\,\% = 2.70\,\%$$

t_1 corresponds to the base period (reference period)

$$w_{t_2-t_3} = \frac{3,900 - 3,800}{3,800} \cdot 100\,\% = 2.63\,\%$$

t_2 corresponds to the base period (reference period)

2.3.2 Index Figures

Index figures measure aggregated changes.

Symbols for Prices and Quantities

$p_0^{(j)}$ ... price of good j at base period or at base time

$p_t^{(j)}$ ... price of good j at reporting period or at reporting time

$q_0^{(j)}$ ... quantity of good j at base period or at base time

$q_t^{(j)}$ ... quantity of good j at reporting period or at reporting time

Example:

Goods j	Price		Quantity		$p_0 q_0$	$p_t q_t$	$p_0 q_t$	$p_t q_0$
	Base period p_0	Reporting period p_t	Base period q_0	Reporting period q_t				
1	5	7	3	6	15	42	30	21
2	7	9	6	13	42	117	91	54
3	8	4	10	15	80	60	120	40
4	10	12	7	19	70	228	190	96
Σ	-	-	-	-	207	447	431	211

Tab. 2.3: Example with one base period and one reporting period

Sales Index/Value Index

$$U_{0t} = \frac{\sum p_t q_t}{\sum p_0 q_0} \cdot 100\,\%$$

Example:

$$\begin{aligned} U_{0t} &= \frac{447}{207} \cdot 100\,\% = \\ &= 2.1594 \cdot 100\,\% = \\ &= 215.94 \end{aligned}$$

The sales generated within the reporting period more than doubled those generated at the base period, i.e. the sales generated within the reporting period correspond to $215.94\,\%$ of those generated at the base period, i.e. 2.1594 times.

Laspeyres[5] Price Index

$$P_{0t}^{L} = \frac{\sum\limits_{j=1}^{n} \frac{p_t^{(j)}}{p_0^{(j)}} \cdot p_0^{(j)} q_0^{(j)}}{\sum\limits_{j=1}^{n} p_0^{(j)} q_0^{(j)}} \cdot 100\ \% =$$

$$= \frac{\sum\limits_{j=1}^{n} p_t^{(j)} q_0^{(j)}}{\sum\limits_{j=1}^{n} p_0^{(j)} q_0^{(j)}} \cdot 100\ \% =$$

$$= \frac{\sum p_t\, q_0}{\sum p_0 q_0} \cdot 100\ \%$$

Example:

$$P_{0t}^{L} = \frac{211}{207} \cdot 100\ \% =$$
$$= 1.0193 \cdot 100\ \% =$$
$$= 101.93\ \%$$

The prices of this shopping cart within the reporting period increased by 1.93 percent compared to those at the base period, i.e. during the reporting period according to Laspeyres, they amounted to 1.0193 times the average price level within the base period.

Laspeyres Volume Index

$$Q_{0t}^{L} = \frac{\sum q_t\, p_0}{\sum q_0\, p_0} \cdot 100\,\% = \frac{U_{0t}}{P_{0t}^{P}}$$

Example:

$$Q_{0t}^{L} = \frac{431}{207} \cdot 100\ \% = 208.21\ \%$$

[5] Ernst Louis Étienne Laspeyres (1834 - 1913) was a German national economist and statistician.

The quantities of this shopping cart within the reporting period more than doubled those at the base period, i.e. during the reporting period according to Laspeyres, they amounted to 2.0821 times the average quantity level within the base period.

Paasche[6] Price Index

$$P^P_{0t} = \frac{\sum_{j=1}^{n} \frac{p_t^{(j)}}{p_0^{(j)}} \cdot p_0^{(j)}\, q_t^{(j)}}{\sum_{j=1}^{n} p_0^{(j)}\, q_t^{(j)}} \cdot 100\ \% =$$

$$= \frac{\sum_{j=1}^{n} p_t^{(j)}\, q_t^{(j)}}{\sum_{j=1}^{n} p_0^{(j)}\, q_t^{(j)}} \cdot 100\ \% =$$

$$= \frac{\sum p_t\, q_t}{\sum p_0\, q_t} \cdot 100\ \%$$

<u>Example</u>:

$$P^P_{0t} = \frac{447}{431} \cdot 100\ \% =$$

$$= 1.0371 \cdot 100\ \% =$$

$$= 103.71\ \%$$

The prices of this shopping cart within the reporting period increased by 17.81 percent compared to those at the base period, i.e. during the reporting period according to Paasche, they amounted to 1.1781 times the average price level within the base period.

Paasche Volume Index

$$Q^P_{0t} = \frac{\sum q_t\, p_t}{\sum q_0\, p_t} \cdot 100\,\% = \frac{U_{0t}}{P^L_{0t}}$$

[6] Hermann Paasche (1851 - 1925) was a German statistician.

Example:

$$Q^P_{0t} = \frac{447}{211} \cdot 100\ \% = 211.85\ \%$$

The volumes of this shopping cart within the reporting period more than doubled those at the base period, i.e., during the reporting period according to Paasche, they amounted to 2.1185 times the average volume level within the base period.

Sales Index/Value Index as Index Product

$$U_{0t} = P^L_{0t}\, Q^P_{0t} = P^P_{0t}\, Q^L_{0t}$$

Example:

$$U_{0t} = \frac{447}{431} \cdot \frac{431}{207} \cdot 100\ \% =$$

$$= 215.94\ \%$$

or

$$U_{0t} = \frac{101.93 \cdot 211.85}{100} = \frac{103.71 \cdot 208.21}{100} = 215.94\ \%$$

Fisher[7] Price Index

$$P^F_{0t} = \sqrt{P^L_{0t} \cdot P^P_{0t}} \cdot 100\,\%$$

Example:

$$P^F_{0t} = \sqrt{\frac{211}{207} \cdot \frac{447}{431}} \cdot 100\ \% =$$

$$= \sqrt{1.0193 \cdot 1.0371} \cdot 100\ \% =$$

$$= 102.82\ \%$$

[7] Irving Fisher (1867 - 1947) was an American economist.

Fisher Volume Index

$$Q_{0t}^{F} = \sqrt{Q_{0t}^{L} \cdot Q_{0t}^{P}} \cdot 100\ \%$$

Example:

$$Q_{0t}^{F} = \sqrt{\frac{431}{207} \cdot \frac{447}{211}} \cdot 100\ \% =$$
$$= \sqrt{2.0821 \cdot 2.1185} \cdot 100\ \% =$$
$$= 210.02\ \%$$

Stuvel Price Index[8]

$$P_{0t}^{ST} = \left(\frac{P_{0t}^{L} - Q_{0t}^{L}}{2} + \sqrt{\left(\frac{P_{0t}^{L} - Q_{0t}^{L}}{2} \right)^{2} + U_{0t}} \right) \cdot 100\,\%$$

with $\quad U_{0t} = \dfrac{\sum p_t\, q_t}{\sum p_0 q_0}$ (sales-/value index)

Example:

$$P_{0t}^{ST} = \left(\frac{1.0193 - 2.0821}{2} + \sqrt{\left(\frac{1.0193 - 2.0821}{2} \right)^{2} + 2.1594} \right) \cdot 100\ \% =$$
$$= 1.0312 \cdot 100\ \% =$$
$$= 103.12\ \%$$

Stuvel Volume Index[9]

$$Q_{0t}^{ST} = \left(\frac{Q_{0t}^{L} - P_{0t}^{L}}{2} + \sqrt{\left(\frac{Q_{0t}^{L} - P_{0t}^{L}}{2} \right)^{2} + U_{0t}} \right) \cdot 100\,\%$$

---

[8] The *Stuvel Price Index* can also be explained as a special case of the *Banerjee* approach. Cf. Banerjee, K.S. (1977): On the Factorial Approach Providing the True Cost of Living Index, Göttingen.

[9] The *Stuvel Volume Index* can also be explained as a special case of the *Banerjee* approach. Cf. Banerjee, K.S. (1977): On the Factorial Approach Providing the True Cost of Living Index, Göttingen.

$$\text{with} \quad U_{0t} = \frac{\sum p_t\, q_t}{\sum p_0 q_0} \quad \text{(sales-/value index)}$$

Example:

$$Q_{0t}^{ST} = \left(\frac{2.0821 - 1.0193}{2} + \sqrt{\left(\frac{2.0821 - 1.0193}{2}\right)^2 + 2.1594} \right) \cdot 100\,\% =$$

$$= 2.094 \cdot 100\,\% =$$

$$= 209.4\,\%$$

Example:

A household consumed in the years 2014 - 2020 products A, B and C in the following volumes:

k	Year	Product A		Product B		Product C	
		volumes	prices per unit	volumes	prices per unit	volumes	prices per unit
0	2014	300	0.25	180	0.70	96	0.80
1	2015	250	0.30	145	0.71	100	1.10
2	2016	340	0.41	290	0.73	142	0.95
3	2017	170	0.28	242	0.72	200	0.96
4	2018	190	0.21	311	0.69	170	0.87
5	2019	245	0.19	196	0.68	164	0.91
6	2020	320	0.31	215	0.74	171	1.02

Tab. 2.4: Example with one base period (year 2014) and five subsequent periods

Lowe[10] Price Index

$$P_{0t}^{LO} = \frac{\sum p_i^{(t)} q_i}{\sum p_i^{(0)} q_i} \cdot 100\,\% \quad \text{with} \quad q_i = \frac{1}{t+1}\sum_{k=0}^{t} q_i^k$$

q_i is the arithmetic mean of the values within the periods 0 to t.

Example:

$$q_1 = \frac{1}{7}\,(300+250+340+170+190+245+320) = 259.29$$
$$q_2 = \frac{1}{7}\,(180+145+290+242+311+196+215) = 225.57$$
$$q_3 = \frac{1}{7}\,(96+100+142+200+170+164+171) = 149.00$$

$$P_{06}^{LO} = \frac{0.31\cdot 259.29+0.74\cdot 225.57+1.02\cdot 149.00}{0.25\cdot 259.29+0.70\cdot 225.57+0.80\cdot 149.00} \cdot 100\,\% =$$
$$= \frac{399.28}{341.92} \cdot 100\,\% =$$
$$= 116.78\,\%$$

Lowe Volume Index

$$Q_{0t}^{LO} = \frac{\sum q_i^{(t)} p_i}{\sum q_i^{(0)} p_i} \cdot 100\,\% \quad \text{with} \quad p_i = \frac{1}{t+1}\sum_{k=0}^{t} p_i^k$$

p_i is the arithmetic mean of the values within the periods *0* to t.

Example:

$$p_1 = \frac{1}{7}\,(0.25+0.30+0.41+0.28+0.21+0.19+0.31) = 0.28$$
$$p_2 = \frac{1}{7}\,(0.70+0.71+0.73+0.72+0.69+0.68+0.74) = 0.71$$

[10] Adolph Lowe, born Adolf Löwe, (1893 - 1995) was a German sociologist and national economist.

$$p_3 = \frac{1}{7}\,(0.80+1.10+0.95+0.96+0.87+0.91+1.02) = 0.94$$

$$\begin{aligned} Q_{06}^{LO} &= \frac{320\cdot 0.28+215\cdot 0.71+171\cdot 0.94}{300\cdot 0.28+180\cdot 0.71+96\cdot 0.94}\cdot 100\,\% = \\ &= \frac{402.99}{302.04}\cdot 100\,\% = \\ &= 133.42\,\% \end{aligned}$$

Laspeyres Price Index

Example:

$$\begin{aligned} P_{06}^{L} &= \frac{\sum p_6 q_0}{\sum p_0 q_0}\cdot 100\,\% = \\ &= \frac{(0.31\cdot 300)+(0.74\cdot 180)+(1.02\cdot 96)}{(0.25\cdot 300)+(0.70\cdot 180)+(0.80\cdot 96)}\cdot 100\,\% = \\ &= \frac{324.12}{277.80}\cdot 100\,\% = \\ &= 116.67\,\% \end{aligned}$$

Laspeyres Volume Index

Example:

$$\begin{aligned} Q_{06}^{L} &= \frac{\sum p_0 q_6}{\sum p_0 q_0}\cdot 100\,\% = \\ &= \frac{(0.25\cdot 320)+(0.70\cdot 215)+(0.80\cdot 171)}{(0.25\cdot 300)+(0.70\cdot 180)+(0.80\cdot 96)}\cdot 100\,\% = \\ &= \frac{367.30}{277.80}\cdot 100\,\% = \\ &= 132.22\,\% \end{aligned}$$

Alternative calculation using the sales/value index U_{06} and the price index according to Paasche P_{06}^{P}:

$$
\begin{aligned}
Q_{06}^{L} &= \frac{U_{06}}{P_{06}^{P}} = \\
&= \frac{432.72\,/\,277.80}{432.72\,/\,367.30} \cdot 100\ \% = \frac{1.5577}{1.1781} \cdot 100\ \% = \\
&= 132.22\ \%
\end{aligned}
$$

with

$$
\begin{aligned}
U_{06} &= \frac{\sum p_6 q_6}{\sum p_0 q_0} = \\
&= \frac{(0.31 \cdot 320) + (0.74 \cdot 215) + (1.02 \cdot 171)}{(0.25 \cdot 300) + (0.70 \cdot 180) + (0.80 \cdot 96)} = \\
&= \frac{432.72}{277.80} = 1.5577
\end{aligned}
$$

$$
\begin{aligned}
P_{06}^{P} &= \frac{\sum p_6 q_6}{\sum p_0 q_6} = \\
&= \frac{(0.31 \cdot 320) + (0.74 \cdot 215) + (1.02 \cdot 171)}{(0.25 \cdot 320) + (0.70 \cdot 215) + (0.80 \cdot 171)} = \\
&= \frac{432.72}{367.30} = 1.1781
\end{aligned}
$$

Paasche Price Index

Example:

$$
\begin{aligned}
P_{06}^{P} &= \frac{\sum p_6 q_6}{\sum p_0 q_6} \cdot 100\ \% = \\
&= \frac{(0.31 \cdot 320) + (0.74 \cdot 215) + (1.02 \cdot 171)}{(0.25 \cdot 320) + (0.70 \cdot 215) + (0.80 \cdot 171)} \cdot 100\ \% = \\
&= \frac{432.72}{367.30} \cdot 100\ \% = \\
&= 117.81\ \%
\end{aligned}
$$

Paasche Volume Index

Example:

$$
\begin{aligned}
Q_{06}^{P} &= \frac{\sum p_6 q_6}{\sum p_6 q_0} \cdot 100\ \% = \\
&= \frac{(0.31 \cdot 320) + (0.74 \cdot 215) + (1.02 \cdot 171)}{(0.31 \cdot 300) + (0.74 \cdot 180) + (1.02 \cdot 96)} \cdot 100\ \% = \\
&= \frac{432.72}{324.12} \cdot 100\ \% = \\
&= 133.51\ \%
\end{aligned}
$$

Alternative calculation using the sales/value index U_{06} and the price index according to Laspeyres P_{06}^{L}:

$$
\begin{aligned}
Q_{06}^{P} &= \frac{U_{06}}{P_{06}^{L}} = \\
&= \frac{432.72/277.80}{324.12/277.80} \cdot 100\ \% = \frac{1.5577}{1.1667} \cdot 100\ \% = \\
&= 133.51\ \%
\end{aligned}
$$

with

$$
\begin{aligned}
U_{06} &= \frac{\sum p_6 q_6}{\sum p_0 q_0} = \\
&= \frac{(0.31 \cdot 320) + (0.74 \cdot 215) + (1.02 \cdot 171)}{(0.25 \cdot 300) + (0.70 \cdot 180) + (0.80 \cdot 96)} = \\
&= \frac{432.72}{277.80} = 1.5577
\end{aligned}
$$

$$
\begin{aligned}
P_{06}^{L} &= \frac{\sum p_6 q_0}{\sum p_0 q_0} = \\
&= \frac{(0.31 \cdot 300) + (0.74 \cdot 180) + (1.02 \cdot 96)}{(0.25 \cdot 300) + (0.70 \cdot 180) + (0.80 \cdot 96)} =
\end{aligned}
$$

$$= \frac{324.12}{277.80} = 1.1667$$

Fisher Price Index

Example:

$$P_{06}^{F} = \sqrt{P_{06}^{L} \cdot P_{06}^{P}} \cdot 100\ \% =$$
$$= \sqrt{\frac{324.12}{277.80} \cdot \frac{432.72}{367.30}} \cdot 100\ \% =$$
$$= \sqrt{1.1667 \cdot 1.1781} \cdot 100\ \% =$$
$$= 117.24\ \%$$

Fisher Volume Index

Example:

$$Q_{06}^{F} = \sqrt{Q_{06}^{L} \cdot Q_{06}^{P}} \cdot 100\ \% =$$
$$= \sqrt{\frac{367.30}{277.80} \cdot \frac{432.72}{324.12}} \cdot 100\ \% =$$
$$= \sqrt{1.3222 \cdot 1.3351} \cdot 100\ \% =$$
$$= 132.86\ \%$$

Marshall[11] Edgeworth[12] Price Index

$$P_{0t}^{ME} = \frac{\sum p_{it}(q_{i0} + q_{it})}{\sum p_{i0}(q_{i0} + q_{it})} \cdot 100\,\%$$

[11] Alfred Marshall (1842 - 1924) was a British national economist.

[12] Francis Ysidro Edgeworth (1845 - 1926) was an Irish economist.

Example:

$$\begin{aligned}
P_{06}^{ME} &= \frac{\sum p_{i6}(q_{i0}+q_{i6})}{\sum p_{i0}(q_{i0}+q_{i6})} \cdot 100\ \% = \\
&= \frac{0.31\ (300+320)+0.74\ (180+215)+1.02\ (96+171)}{0.25\ (300+320)+0.70\ (180+215)+0.80\ (96+171)} \cdot 100\ \% = \\
&= \frac{756.84}{645.10} \cdot 100\ \% = \\
&= 117.32\ \%
\end{aligned}$$

Walsh Price Index

$$P_{0t}^{W} = \frac{\sum p_{it}(q_{i0}\,q_{it})^{0.5}}{\sum p_{i0}(q_{i0}\,q_{it})^{0.5}} \cdot 100\,\%$$

Example:

$$\begin{aligned}
P_{06}^{W} &= \frac{\sum p_{i6}(q_{i0}\,q_{i6})^{0.5}}{\sum p_{i0}(q_{i0}\,q_{i6})^{0.5}} \cdot 100\,\% = \\
&= \frac{0.31(300\cdot 320)^{0.5}+0.74(180\cdot 215)^{0.5}+1.02(96\cdot 171)^{0.5}}{0.25(300\cdot 320)^{0.5}+0.70(180\cdot 215)^{0.5}+0.80(96\cdot 171)^{0.5}} \cdot 100\,\% = \\
&= \frac{372.31}{317.67} \cdot 100\,\% = \\
&= 117.20\ \%
\end{aligned}$$

2.3.3 Peren-Clement Index (PCI)[13]

The PCI[14] is a risk index for the assessment of country risks in *Foreign Direct Investments*.

The PCI is determined by three factors:

- cross-company factors,
- cost and production-oriented factors, and
- sales-oriented factors.

Cross-company factors include:

- political and social stability,
- state influence on business decisions and bureaucratic obstacles,
- general economic policy,
- investment incentives,
- enforceability of contractual agreements, and
- the respect of property rights in the transfer of technology and know-how

Cost and production-oriented factors include:

- legal restrictions on production,
- cost of capital in the host country and possibilities of capital import,
- availability and cost of acquiring land and property,
- availability and cost of labour,
- availability and cost of fixed assets, raw materials and supplies in the host country,
- trade barriers to the import of goods, and
- availability and quality of infrastructure and government services

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[13] Reiner Clement (1958 - 2017) was a German economist.

[14] Cf. Pakusch, C.; Peren, F.W. & Shakoor, M.A. (2016): The PCI - A Global Risk Index for the Simultaneous Assessment of Macro and Company Individual Investment Risks. In: Journal of Business Strategies, 33(2), p. 154-173; Clement, R. & Peren, F.W. (2017): Peren-Clement-Index: Bewertung von Direktinvestitionen durch eine simultane Erfassung von Makroebene und Unternehmensebene, Wiesbaden; Pakusch, C.; Peren, F.W. & Shakoor, M.A. (2018): Peren-Clement-Index – eine exemplarische Fallstudie. In: Gadatsch, A. et al. (Ed.): Nachhaltiges Wirtschaften im digitalen Zeitalter: Innovation - Steuerung - Compliance, Wiesbaden, p. 105-117.

Sales-oriented factors include:

- size and dynamics of the market,
- competitive situation,
- reliability, quality of local contractors,
- quality and sales opportunities, and
- trade barriers to exports from the host country

Depending on the type of investment or motives of the respective companies, there are different location factors or differing weightings of the various factors.

PCI | Case Study

Environment	**Points**	**Weight**	**Sum**
Political and social stability	...	4	...
Bureaucratic obstacles	...	2	...
Economic policy	...	3	...
Legal security	...	3	...
Solvency	...	3	...
Sum		**15**	**...**

Localisation	**Points**	**Weight**	**Sum**
Human capital	...	4	...
Transport connections	...	2	...
Management skills	...	2	...
Access to markets	...	2	...
Quality of life	...	3	...
Sum		**13**	**...**

Production	**Points**	**Weight**	**Sum**
Economic and property rights	...	2	...
Manufacturing costs	...	2	...
Capital procurement	...	3	...
Complementary industries	...	2	...
Investment incentives	...	2	...
Sum		**11**	**...**

Sales	**Points**	**Weight**	**Sum**
Size and dynamic of the market	...	3	...
Per capita income	...	2	...
Avoidance of tariff barriers	...	2	...
Reliability of local contractors	...	2	...
Distribution structures	...	2	...
Sum		**11**	**...**

Total score PCI	**...**

Tab. 2.5: Structural Arrangement of PCI

Company A and Company B want to internationalise. Both Countries/ Regions X and Y are eligible for direct investment abroad. Both companies use the index presented here (Tab. 2.5) to measure and make a comparative quantitative assessment of country risks. The PCI shows a total score of 71 points for Country X (Tab. 2.6). This value implies a relatively low investment risk.

Investment Alternative 1: Country/Region X PCI = 71

Environment	**Points**	**Weight**	**Sum**
Political and social stability	1.5	4	6
Bureaucratic obstacles	2	2	4
Economic policy	2	3	6
Legal security	1.5	3	4.5
Solvency	1.5	3	4.5
Sum		**15**	**25**

Localisation	**Points**	**Weight**	**Sum**
Human capital	0.5	4	2
Transport connections	2	2	4
Management skills	0	2	0
Access to markets	1.5	2	3
Quality of life	2	3	6
Sum		**13**	**15**

Production	**Points**	**Weight**	**Sum**
Economic and property rights	2	2	4
Manufacturing costs	2	2	4
Capital procurement	2	3	6
Complementary industries	2	2	4
Investment incentives	2	2	4
Sum		**11**	**22**

Sales	**Points**	**Weight**	**Sum**
Size and dynamic of the market	0	3	0
Per capita income	0.5	2	1
Avoidance of tariff barriers	2	2	4
Reliability of local contractors	0.5	2	1
Distribution structures	1.5	2	3
Sum		**11**	**9**

Total score PCI	**71**

Tab. 2.6: Risk Assessment for Country/Region X

The calculation for the alternative Country/Region Y also results in a total score of 71 points (Tab. 2.7). Company A and Company B now know that both direct investments would be associated with a relatively low risk and thus tend to be assessed positively. However, which region would be the more suitable location for the company cannot be derived from the macroeconomic risk assessment carried out so far.

Investment Alternative 2: Country/Region Y PCI = 71

Environment	Points	Weight	Sum
Political and social stability	2	4	8
Bureaucratic obstacles	1	2	2
Economic policy	1	3	3
Legal security	2	3	6
Solvency	1.5	3	4.5
Sum		**15**	**23.5**

Localisation	Points	Weight	Sum
Human capital	1.5	4	6
Transport connections	2	2	4
Management skills	1.5	2	3
Access to markets	2	2	4
Quality of life	1.5	3	4.5
Sum		**13**	**21.5**

Production	Points	Weight	Sum
Economic and property rights	2	2	4
Manufacturing costs	0.5	2	1
Capital procurement	0	3	0
Complementary industries	0	2	0
Investment incentives	0.5	2	1
Sum		**11**	**6**

Sales	**Points**	**Weight**	**Sum**
Size and dynamic of the market	2	3	6
Per capita income	2	2	4
Avoidance of tariff barriers	1.5	2	3
Reliability of local contractors	1.5	2	3
Distribution structures	2	2	4
Sum		**11**	**20**

Total score PCI	**71**

Tab. 2.7: Risk Assessment for Country/Region Y

The evaluation model is now extended to include another company-specific dimension. The companies pursue different company-specific goals with internationalisation (Tab. 2.8). While for Company A it is most important to access the foreign market and to expand the existing resources through additional resources of the foreign location, Company B focuses on the cost advantages that can be generated with production at the foreign location.

Internationalisation aims of Company A:	Target weight
1. Sales in the foreign market	50 %
2. Expand resources through localisation	30 %
3. Secure foreign location/strategic importance (macro environment)	15 %
4. Realise cost advantages (production)	5 %
	100 %
Internationalisation aims of Company B:	**Target weight**
1. Realise cost advantages (production)	70 %
2. Secure foreign location/strategic importance (macro environment)	15 %
3. Expand resources through localisation	10 %
4. Sales in the foreign market	5 %
	100 %

Tab. 2.8: Company-Specific Internationalisation Aims

Accordingly, the companies now assign the individual factors of the PCI in a second dimension with the weights of their individual goals. The factors *macro environment*, *localisation*, *production* and *sales* are to be distributed to 100 % (or 1.0 target weight overall) on a company-specific basis. The result is a company-specific, two-dimensional total score, which now enables a company-specific, target-oriented decision (Fig. 2.3).

In contrast to all currently existing risk indices, the two-dimensional cumulative scores measured in PECLE[15] now clearly show which region is best suited for which investor for a direct investment at the current time. Company A should opt for a direct investment in Country/Region X, as this is where it can best realise its individual company goals at the moment. Country/Region X achieves a company-specific total score of 20.275 PECLE for Company A, while Country/Region Y is only rated with a total of 13.85 PECLE for the individual company. The reason for the now apparent distinction is that Country/Region A offers the better

[15] PECLE as a one-dimensional measurement of a two-dimensional cumulative score.

starting position for the factors *sales* and *localisation*, which are most important to Company A when choosing a location (“sales in the foreign market: 50 % and “expand recources through localisation: 30 %“).

Company B primarily wants to internationalise in order to realise cost advantages in production (“realise cost advantages in production: 70 %“). Company B should therefore choose Country/Region Y. Country/Region Y, with a company-specific total score of 21.1 PECLE, is much more suitable for a direct investment for Company B than Country/Region X (total score of 10.875 PECLE).

The two-dimensional graphical representation clearly shows the advantage of such a combinatorial view of the macroeconomic level on the one hand and company-specific objectives on the other, illustrated by the example of Country/Region X versus Country/Region Y (Fig. 2.4).

Company A

Country X

Factor	PCI	Target weights	Sum in PECLE
Macro environment (M)	23.5	0.15	3.525
Localisation (L)	21.5	0.30	6.450
Production (P)	6	0.05	0.300
Sales (S)	20	0.05	10.000
Total Score PCI	**71**	**1.00**	**20.275**

Country Y

Factor	PCI	Target weights	Sum in PECLE
Macro environment (M)	25	0.15	3.75
Localisation (L)	15	0.30	4.5
Production (P)	22	0.05	1.1
Sales (S)	6	0.50	4.5
Total Score PCI	**71**	**1.00**	**13.85**

Company B

Country X

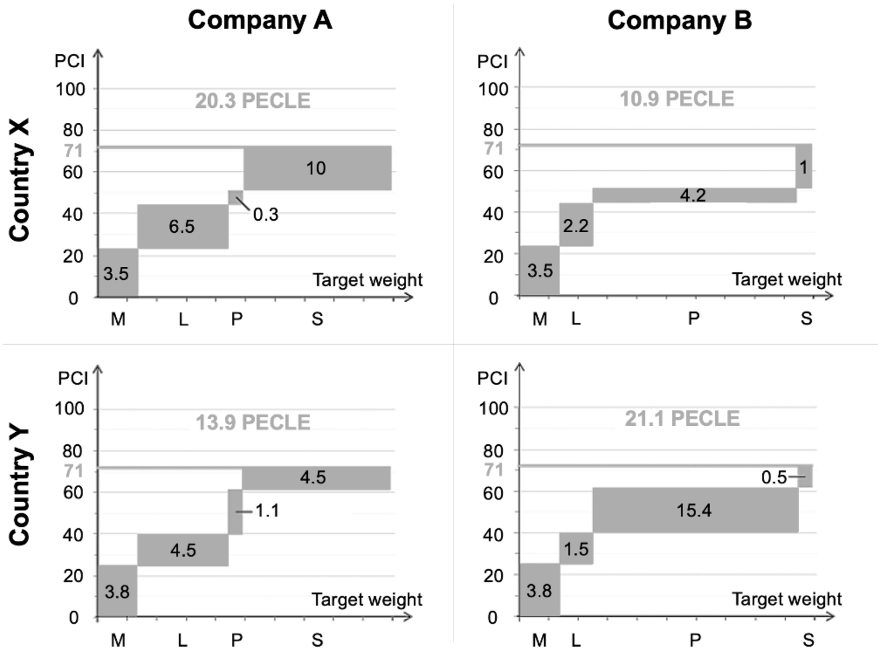
Factor	PCI	Target weights	Sum in PECLE
Macro environment (M)	23.5	0.15	3.525
Localisation (L)	21.5	0.10	2.15
Production (P)	6	0.70	4.2
Sales (S)	20	0.05	1
Total Score PCI	**71**	**1.00**	**10.875**

Country Y

Factor	PCI	Target weights	Sum in PECLE
Macro environment (M)	25	0.15	3.75
Localisation (L)	15	0.10	1.5
Production (P)	22	0.70	15.4
Sales (S)	6	0.05	0.45
Total Score PCI	**71**	**1.00**	**21.1**

Fig. 2.3: Individual Assessment of Country Risks

Fig. 2.4: PCI – Simultaneous Assessment of Macroeconomic



Macro Level and Individual Company Objectives in Direct Investments

2.4 Correlation Analysis

Simple Linear Coefficient of Determination (r^2)

$$r^2 = \frac{SSE}{SST} = \frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} = 1 - \frac{SSR}{SST} \qquad 0 \leq r^2 \leq 1$$

with

SST = sum of the squared deviations
= $SSE + SSR$

SSE = sum of the deviation squares explained (by the regression function)

SSR = sum of the deviation squares unexplained (by the regression function)

Simple Linear Correlation Coefficient (r)

$$r = sgn(b_2)\sqrt{r^2} \qquad -1 \leq r \leq 1$$

Pearson's[16] Correlation Coefficient (r)

$$r = \frac{s_{xy}}{s_x \cdot s_y} \qquad -1 \leq r \leq 1$$

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})\,(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\,\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}} =$$

[16] Karl Pearson (1857 - 1936) was a British mathematician.

$$= \frac{\sum_{i=1}^{n} x_i y_i - \frac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n}}{\sqrt{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}\sqrt{\sum_{i=1}^{n} y_i^2 - \frac{\left(\sum_{i=1}^{n} y_i\right)^2}{n}}}$$

2.5 Regression Analysis

2.5.1 Simple Linear Regression

Description of the relationship (dependence) between two variables x and y using a linear function.

Symbols:

x_i value of the independent variables x at the i^{th} observation

y_i value of the dependent variables y at the i^{th} observation

$\hat{y}_i$ value of the dependent variables y estimated by the regression function (regression line) at the point x_i

b_1, b_2 regression coefficients sought, specifying the regression line (b_1: ordinate intercept; b_2: slope of the regression line)

e_i residual ($e_i = y_i - \hat{y}_i$); deviation of the value estimated by the regression function from the observed (true) value of the dependent variables y

with $i = 1, ..., n$

Least Squares Method (= LS-Method)

The sum of the squared deviation (SAQ) must be minimised:

$$\begin{aligned} SAQ &= \sum_{i=1}^{n} e_i^2 \qquad \rightarrow min \\ &= \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \qquad \rightarrow min \\ &= \sum_{i=1}^{n} (y_1 - b_1 - b_2 \cdot x_i)^2 \qquad \rightarrow min \end{aligned}$$

$$\Rightarrow \quad \frac{\delta SAQ}{\delta b_1} = \frac{\delta SAQ}{\delta b_2} \stackrel{!}{=} 0$$ (necessary condition of a local minimum)

Regression coefficients:

$$b_1 = \frac{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$b_2 = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \quad \text{or} \quad b_2 = \frac{s_{xy}}{s_{x^2}} \quad \text{or} \quad b_2 = r \cdot \frac{s_y}{s_x}$$

Regression function:

$\hat{y}_i = b_1 + b_2 x_i$ with $i = 1, ..., n$
or $\hat{y} = b_1 + b_2 x$

$\hat{y}_i$ = estimated values of y_i with $i = 1, ..., n$
$\hat{y}$ = estimated regression function

Example:

Following points (observations) are given:

x_i	1	2	3	4	5	6	7	8
y_i	6.36	7.12	8.22	9.55	10.40	11.51	12.58	13.67

with

$$\bar{x} = \frac{1}{8}\sum_{i=1}^{8} x_i = \frac{1}{8}\cdot 36 = 4.5$$

$$\bar{y} = \frac{1}{8}\sum_{i=1}^{8} y_i = \frac{1}{8}\cdot 79.41 = 9.93$$

$$\sum_{i=1}^{8} x_i = 36 \qquad \sum_{i=1}^{8} y_i = 79.41$$

$$\sum_{i=1}^{8} x_i^2 = 204 \qquad \sum_{i=1}^{8} y_i^2 = 835.68$$

$$\sum_{i=1}^{8} x_i y_i = 401.94$$

$$\sum_{i=1}^{8} (x_i - \bar{x})^2 = 12.25 + 6.25 + 2.25 + 0.25 + 0.25 + 2.25 + 6.25 + 12.25 = 42$$

$$b_1 = \frac{204\cdot 79.41 - 36\cdot 401.94}{8\cdot 204 - (36)^2} = 5.1482$$

$$b_2 = \frac{8\cdot 401.94 - 36\cdot 79.41}{8\cdot 204 - (36)^2} = 1.0618$$

$$\hat{y} = 5.1482 + 1.0618x$$

The ordinate value (y-value with $x = 0$) of the estimated regression function $\hat{y}$ is 5.1482 and the slope of $\hat{y}$ is 1.0618.

Properties of the Regression Function

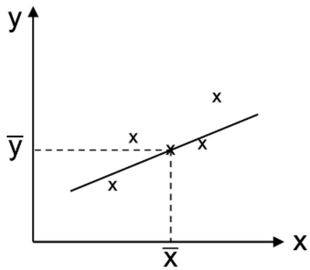
(1) $\sum\limits_{i=1}^{n} e_i = 0$

(2) $\sum\limits_{i=1}^{n} x_i e_i = 0$

(3) $\frac{1}{n} \sum\limits_{i=1}^{n} y_i = \frac{1}{n} \sum\limits_{i=1}^{n} \hat{y}_i$

(4) The regression line passes through the centre of gravity $\bar{P}(\bar{x};\bar{y})$ of the corresponding cluster of points

$\bar{x} = \frac{1}{n} \sum x_i$ or $\bar{y} = \frac{1}{n} \sum y_i$



(5) $s_E =$ standard deviation of the residuals e_i with $i = 1, ..., n$

$s_E^2 =$ variance of the residuals e_i with $i = 1, ..., n$

$$s_E^2 = \frac{1}{n-2} \sum_{i=1}^{n} e_i^2 =$$

$$= \frac{1}{n-2} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 =$$

$$= \frac{1}{n-2} \left[\sum_{i=1}^{n} y_i^2 - b_1 \sum_{i=1}^{n} y_i - b_2 \sum_{i=1}^{n} x_i y_i \right]$$

2.5.1.1 Confidence Intervals for the Regression Coefficients of a Simple Linear Regression Function

For a simple linear regression function, the confidence intervals for the regression coefficients are shown in Tab. 2.9.

Parameter	Confidence Interval	Standard Error	Applicable Distribution
$\hat{b}_1$	$b_1 - ts_{B_1} \leq \hat{b}_1 \leq b_1 + ts_{B_1}$	$s_{B_1} = s_E \sqrt{\dfrac{\sum\limits_{i=1}^{n} x_i^2}{n \sum\limits_{i=1}^{n} (x_i - \bar{x})^2}}$	**Student's Distribution** with $v = n - 2$ Requirement: validity of the model assumptions
$\hat{b}_2$	$b_2 - ts_{B_2} \leq \hat{b}_2 \leq b_2 + ts_{B_2}$	$s_{B_2} = \dfrac{s_E}{\sqrt{\sum\limits_{i=1}^{n} (x_i - \bar{x})^2}}$	

Tab. 2.9: Confidence Intervals for the Regression Coefficients of a Simple Linear Regression Function[17]

Example:

If $\hat{y} = 5.1482 + 1.0618x$ by considering a confidence level of 95%, the confidence intervals for the regression coefficients b_1 and b_2 are calculated as follows:

When $(1 - \alpha) = 0.95$ and $v = n - 2 = 8 - 2 = 6$, the t-value would be 2.447 (Student's t-distribution, two-sided symmetric confidence interval; see appendix A, statistical tables).

Confidence intervals

(1) for the regression coefficient $\hat{b}_1$:

$$b_1 - ts_{B_1} \leq \hat{b}_1 \leq b_1 + ts_{B_1}$$

$$5.1482 - 2.447 \cdot s_{B_1} \leq \hat{b}_1 \leq 5.1482 + 2.447 \cdot s_{B_1}$$

[17] Cf. Bleymüller, J. & Gehlert, G. (2011), p. 54.

$$s_{B_1} = s_E \sqrt{\frac{\sum\limits_{i=1}^{n} x_i^2}{n \sum\limits_{i=1}^{n} (x_i - \bar{x})^2}}$$

with

$$\begin{aligned}
s_E^2 &= \frac{1}{n-2}\left[\sum_{i=1}^{n} y_i^2 - b_1 \sum_{i=1}^{n} y_i - b_2 \sum_{i=1}^{n} x_i y_i\right] = \\
&= \frac{1}{8-2}\,[835.68 - 5.1482 \cdot 79.41 - 1.0618 \cdot 401.94] = \\
&= 0.0136 \\
s_E &= \sqrt{0.0136} = 0.1166 \\
s_{B_1} &= 0.1166 \cdot \sqrt{\frac{204}{8 \cdot 42}} = 0.0909
\end{aligned}$$

$$\begin{aligned}
5.1482 - 2.447 \cdot 0.0909 \leq \hat{b}_1 &\leq 5.1482 + 2.447 \cdot 0.0909 \\
+4.9258 \leq \hat{b}_1 &\leq +5.3706
\end{aligned}$$

With a probability of $95\,\%$ (the selected confidence level), the unknown regression coefficient $\hat{b}_1$, which determines the ordinate value (y-value with $x = 0$) of the estimated (unknown) regression function, assumes a value between $+4.9258$ and $+5.3706$.

(2) for the regression coefficient $\hat{b}_2$:

$$b_2 - ts_{B_2} \leq \hat{b}_2 \leq b_2 + ts_{B_2}$$

$$1.0618 - 2.447 \cdot s_{B_2} \leq \hat{b}_2 \leq 1.0618 + 2.447 \cdot s_{B_2}$$

$$
\begin{aligned}
s_{B_2} &= \frac{s_E}{\sqrt{\sum\limits_{i=1}^{n}(x_i-\bar{x})^2}} \\
&= \frac{0.1166}{\sqrt{42}} = \\
&= 0.0180
\end{aligned}
$$

$$
\begin{aligned}
1.0618 - 2.447 \cdot 0.018 \leq \hat{b}_2 &\leq 1.0618 + 2.447 \cdot 0.018 \\
+1.0178 \leq \hat{b}_2 &\leq +1.1058
\end{aligned}
$$

With a probability of $95\,\%$ (the selected confidence level), the unknown regression coefficient $\hat{b}_2$, which determines the slope of the estimated (unknown) regression function, assumes a value between $+1.0178$ and $+1.1058$.

2.5.1.2 Student's t-Tests for the Regression Coefficients of a Simple Linear Regression Function

For a simple linear regression function, the Student's t-tests for the regression coefficients are shown in Tab. 2.10.

Parameter	Standard Error			Applicable Distribution
$\hat{b}_1 = 0$	$t = \frac{b_1}{s_{B_1}}$	with	$s_{B_1} = s_E \sqrt{\frac{\sum\limits_{i=1}^{n} x_i^2}{n \sum\limits_{i=1}^{n}(x_i-\bar{x})^2}}$	**Student's Distribution** with $v = n-2$ Requirement: validity of the model assumptions
$\hat{b}_2 = 0$	$t = \frac{b_2}{s_{B_2}}$	with	$s_{B_2} = \frac{s_E}{\sqrt{\sum\limits_{i=1}^{n}(x_i-\bar{x})^2}}$	

Tab. 2.10: Student's t-Tests for the Regression Coefficients of a Simple Linear Regression Function[18]

[18] Cf. Bleymüller, J. & Gehlert, G. (2011), p. 54.

For testing hypotheses of stochastic parameters like regression coefficients, the practical procedure is as follows:

a. Definition of null hypothesis (H_0) and alternative hypothesis (H_A) as well as significance level (α)

b. Determination of the test statistic

c. Determination of the test distribution

d. Identification of the critical range

e. Calculation of the value of the test statistic

f. Decision and interpretation

(1) Test of the regression coefficient $\hat{b}_1$

a. H_0: $\hat{b}_1 = 0$
   H_A: $\hat{b}_1 \neq 0$
   $\alpha = 0.05 \quad (1-\alpha) = 0.95$ (in the example above)

   H_0 means that the ordinate value (y-value with $x = 0$) of the estimated regression function would be zero. For example, in a cost function, the fixed costs are assumed to be zero.

   H_A can be $\hat{b}_1 \neq 0$, $\hat{b}_1 > 0$ or $\hat{b}_1 < 0$. For example, in a cost function, H_A would be defined as $\hat{b}_1 > 0$.

b. Test statistic

   $$t = \frac{b_1}{s_{B_1}} \quad \text{(table 2.10)}$$

   with $$s_{B_1} = s_E \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n \sum_{i=1}^{n} (x_i - \bar{x})^2}}$$

e.g. $s_{B_1} = 0.0909$ (in the example above)
with $b_1 = 5.1482$

c. Determination of the test distribution

Student's t-distribution, two-sided symmetric confidence interval with $(1-\alpha)$ and $\nu = n-2$ (see appendix A, statistical tables)

e.g. with $(1-\alpha) = 0.95$ and $\nu = 8-2 = 6$
(in the example above)

d. Identification of the critical range

For $(1-\alpha) = 0.95$ and $\nu = 6$, the critical t-value, t_c, is 2.447 (Student's t-distribution, two-sided symmetric confidence interval; see appendix A, statistical tables).

If $t = \dfrac{b_1}{s_{B_1}} > 2.447$, the null hypothesis H_0 has to be rejected.

If $t = \dfrac{b_1}{s_{B_1}} \leq 2.447$, the null hypothesis H_0 cannot be rejected.

e. Calculation of the value of the test statistic

$t = \dfrac{5.1482}{0.0909} = 56.636$ (in the example above)

f. Decision and interpretation

$t > t_c \quad (56.636 > 2.447)$

H_0 has to be rejected.

The observed value for b_1 $(b_1 = 5.1482)$ is statistically valid with a significance level of 0.05 $(= 5\ \%)$.

In case of a cost function, there would be a $95\,\%$ probability that the fixed costs are positive at a level of approximately 5.1482 monetary units (e.g. $\$5.1482$ bln).

(2) Test of the regression coefficient $\hat{b}_2$

a. H_0: $\hat{b}_2 = 0$

H_A: $\hat{b}_2 \neq 0$

$\alpha = 0.05 \quad (1-\alpha) = 0.95$ (in the example above)

H_0 implies that the slope of the real function would be zero, which means there is no correlation between the tested variables (y and x_2 in the example above).

H_A can be $\hat{b}_2 \neq 0$, $\hat{b}_2 > 0$ (positive correlation between x and y) or $\hat{b}_2 < 0$ (negative correlation between x_2 and y).

b. Test statistic

$$t = \frac{b_2}{s_{B_2}} \quad \text{(table 2.10)}$$

with $$s_{B_2} = \frac{s_E}{\sqrt{\sum\limits_{i=1}^{n}(x_i - \bar{x})^2}}$$

e.g. $s_{B_2} = 0.0180$ (in the example above)

with $b_2 = 1.0618$

c. Determination of the test distribution

Student's t-distribution, two-sided symmetric confidence interval with $(1-\alpha)$ and $\nu = n-2$ (see appendix A, statistical tables)

e.g. with $(1-\alpha) = 0.95$ and $\nu = 8-2 = 6$
(in the example above)

d. Identification of the critical range

For $(1-\alpha) = 0.95$ and $\nu = 6$, the critical t-value, t_c, is 2.447 (Student's t-distribution, two-sided symmetric confidence interval; see appendix A, statistical tables).

If $t = \dfrac{b_2}{s_{B_2}} > 2.447$, the null hypothesis H_0 has to be rejected.

If $t = \dfrac{b_2}{s_{B_2}} \leq 2.447$, the null hypothesis H_0 cannot be rejected.

e. Calculation of the value of the test statistic

$$t = \frac{1.0618}{0.0180} = 58.989 \quad \text{(in the example above)}$$

f. Decision and interpretation

$t > t_c \quad (58.989 > 2.447)$

H_0 has to be rejected.

The observed value for b_2 $(b_2 = 1.0618)$ is statistically valid with a significance level of 0.05 $(= 5\ \%)$.

There is a significant correlation between the tested variables (y and x_2 in the example above).

2.5.2 Multiple Linear Regression

Description of the relationship (dependence) between a dependent variable y and multiple independent variables x_i, with $i = 1, ..., n$ using a linear function.

Symbols:

x_{ji} fixed (non-random) value of the independent random variable X_j ($j = 2, ..., k$) at the i^{th} observation ($i = 1, ..., n$) with $x_{1i} = 1$ for all i

y_i stochastic (random) value of the dependent variable y at the i^{th} observation

$\hat{y}_i$ the estimated value provided by the sampling regression line for y_i

b_j the regression coefficient sought for the independent variable x_j ($j = 2, ..., k$) for the regression function of the population

e_i residual ($e_i = y_i - \hat{y}_i$); value of the deviation between the observed value y_i and the value $\hat{y}_i$ estimated by the regression function at the point x_i

$[\hat{y}]$ column vector of dimension $n \times 1$ containing the estimated values of the dependent variables y

$[b]$ column vector of dimension $k \times 1$ containing the sought sample regression coefficients

$[X]$ matrix of the dimension $n \times k$ (n: observations, k: characteristic values considered in the estimation), which contains the observed values of the independent variables

Remark: $x_{1i} = 1$ for all i with $i = 1, ..., n$

$$[\hat{y}] = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \vdots \\ \hat{y}_n \end{bmatrix}_{n\times 1} \qquad [b] = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_k \end{bmatrix}_{k\times 1} \qquad [X] = \begin{bmatrix} 1 & x_{21} & x_{31} & \dots & x_{k1} \\ 1 & x_{22} & x_{32} & \dots & x_{k2} \\ 1 & x_{23} & x_{33} & \dots & x_{k3} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{2n} & x_{3n} & \dots & x_{kn} \end{bmatrix}_{n\times k}$$

Least Squares Method

Minimise the sum of squared deviation (SAQ):

$$
\begin{aligned}
SAQ &= \sum_{i=1}^{n} e_i^2 \qquad \rightarrow min \\
&= \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \qquad \rightarrow min \\
&= \sum_{i=1}^{n} (y_i - b_1 - b_2 x_{2i} - b_3 x_{3i} - \ldots - b_k x_{ki})^2 \qquad \rightarrow min \\
&\Rightarrow \quad \frac{\delta SAQ}{\delta b_1} = \frac{\delta SAQ}{\delta b_2} = \frac{\delta SAQ}{\delta b_3} = \ldots = \frac{\delta SAQ}{\delta b_k} \stackrel{!}{=} 0
\end{aligned}
$$

Regression coefficients:

$[b]_{k\times 1} = ([X]' \cdot [X])^{-1}_{k\times k} \cdot [X]'_{k\times n} \cdot [y]_{n\times 1}$ with $b = (b_1, \ldots, b_k)'$

Regression function:

- Using the normal equation

  $\hat{y}_i = b_1 + b_2 x_{2i} + b_3 x_{3i} + \ldots + b_k x_{ki}$ with $i = 1, \ldots, n$

- Using the matrix notation

  $[\hat{y}]_{n\times 1} = [X]_{n\times k} \cdot [b]_{k\times 1}$

Partial Linear Coefficient of Determination

$$SST = \sum_{i=1}^{n} (y_i - \bar{y}_i)^2 = \sum_{i=1}^{n} y_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} y_i\right)^2$$

$$SSE = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2$$

Multiple Linear Coefficient of Determination

$$r^2_{Y\cdot 2,3,\ldots,k} = \frac{SSE}{SST} = \frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} = 1 - \frac{SSR}{SST} = 1 - \frac{\sum_{i=1}^{n} e_i^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}$$

$$0 \le r^2_{Y\cdot 2,3,\ldots,k} \le 1$$

with

$$SST = \sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n} y_i^2 - \frac{1}{n}\left(\sum y_i\right)^2$$

$$SSE = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2$$

$$SSR = \sum_{i=1}^{n}(y_i - \hat{y})^2 = \sum_{i=1}^{n} e_i^2$$

$$= \sum_{i=1}^{n} y_i^2 - b_1 \sum_{i=1}^{n} y_i - b_2 \sum_{i=1}^{n} x_{2i} y_i - \ldots - b_k \sum_{i=1}^{n} x_{ki} y_i$$

Multiple Linear Correlation Coefficient

$$r_{Y\cdot 2,3,\ldots,k} = \sqrt{r^2_{Y\cdot 2,3,\ldots,k}} \qquad 0 \le r_{Y\cdot 2,3,\ldots,k} \le 1$$

2.5.2.1 Confidence Intervals for the Regression Coefficients of a Multiple Linear Regression Function

For a multiple linear regression function, the confidence intervals for the regression coefficients are shown in Tab. 2.11.

Parameter	Confidence Interval	Standard Error	Applicable Distribution
$\hat{b}_j$ $j = 1, ..., k$	$b_j - ts_{B_j} \le \hat{b}_j \le b_j + ts_{B_j}$	$s_{B_j} = \sqrt{\widehat{CV}_{jj}}$ $\widehat{CV} = s_E^2 \left([X]' \cdot [X]\right)^{-1}$ s_{Bj}^2 with $j = 1, ... k$ correspond to the elements at the main diagonal of the estimated covariance matrix $\widehat{CV}$.	**Student's Distribution** with $v = n - k$ Requirement: validity of the model assumptions

Tab. 2.11: Confidence Intervals for the Regression Coefficients of a Multiple Linear Regression Function[19]

A covariance matrix (also known as auto-covariance matrix or variance-covariance matrix) is a square matrix which shows the covariances between each pair of elements of a given random vector (of a multiple random variable). Any covariance matrix is symmetric and positive semi-definite. Its main diagonal contains the covariances (the variances of each element with itself).

The estimated cavariance matrix can be calculated as follows:

$$\widehat{CV} = s_E^2 \left([X]'_{n\times n} \cdot [X]_{n\times n}\right)^{-1}$$

with $s_E^2 = \frac{1}{n-k} \sum e_i^2$

$n =$ number of observations (elements)
$k =$ number of variables

$$s_E^2 = \frac{1}{n-k} \left[\sum_{i=1}^{n} y_i^2 - b_1 \sum_{i=1}^{n} y_i - b_2 \sum_{i=1}^{n} x_{2i} y_i - ... - b_k \sum_{i=1}^{n} x_{ki} y_i \right]$$

[19] Cf. Bleymüller, J. & Gehlert, G. (2011), p. 61.

2.5.2.2 Student's t-Tests for the Regression Coefficients of a Multiple Linear Regression Function

For a multiple linear regression function, the Student's t-tests for the regression coefficients are shown in Tab. 2.12.

Hypothesis	Test Statistic	Applicable Distribution
$\hat{b}_j = 0$ $j = 1, ..., k$	$t = \frac{b_j}{s_{B_j}}$ with $s_{B_j} = \sqrt{\widehat{CV}_{jj}}$ $\widehat{CV} = s_E^2 \left([X]' \cdot [X] \right)^{-1}$ s_{Bj}^2 with $j = 1, ... k$ correspond to the elements at the main diagonal of the estimated covariance matrix $\widehat{CV}$.	**Student's Distribution** with $v = n - k$ Requirement: validity of the model assumptions

Tab. 2.12: Student's t-Tests for the Regression Coefficients of a Multiple Linear Regression Function[20]

The Student's t-Tests for the Regression Coefficients of a Multiple Regression Function take place analogously as the Student's t-Tests for the Regression Coefficients of a Linear Regression Function (chapter 2.5.1.2).

2.5.3 Double Linear Regression

Regression function:

- Regression function for the population (normal equation)

$$\hat{y} = b_1 + b_2 x_{2i} + b_3 x_{3i} \qquad i = 1, ..., n$$

- Regression function for the population (matrix notation)

$$[\hat{y}]_{n\times 1} = [X]_{n\times 3} \cdot [b]_{3\times 1}$$

[20] Cf. Bleymüller, J. & Gehlert, G. (2011), p. 62.

Regression coefficients:

- Regression coefficients by using equations

$$\begin{aligned}
b_1 n &+ b_2 \sum x_{2i} &+ b_3 \sum x_{3i} &= \sum y_i \\
b_1 \sum x_{2i} &+ b_2 \sum x_{2i}^2 &+ b_3 \sum x_{2i} x_{3i} &= \sum x_{2i} y_i \\
b_1 \sum x_{3i} &+ b_2 \sum x_{3i} x_{2i} &+ b_3 \sum x_{3i}^2 &= \sum x_{3i} y_i
\end{aligned}$$

- Regression coefficients by using a matrix notation

$$[b]_{3\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad [b]_{3\times 1} = ([X]'\,[X])^{-1}_{3\times 3} \cdot [X]'_{3\times n} \cdot [y]_{n\times 1}$$

Example:

Following revenues (observations) are given:

i	y_i revenue in $\$1,000$	x_{1i}	x_{2i} price of an offered good USD/unit	x_{3i} sales area in square metres
1	2,512	1	112	1,980
2	1,810	1	108	1,400
3	1,635	1	104	1,420
4	1,487	1	100	1,160
5	2,270	1	110	1,750
6	1,805	1	108	1,400
7	1,984	1	109	1,560
8	2,043	1	110	1,590
9	1,943	1	108	1,500
10	2,170	1	111	1,700
11	1,820	1	108	1,430
12	1,440	1	102	1,110

Tab. 2.13: Example of a double linear regression

$$[X]' \cdot [X] = \begin{pmatrix} 12 & 1290 & 18000 \\ 1290 & 138822 & 1943640 \\ 18000 & 1943640 & 27643600 \end{pmatrix}$$

$$([X]' \cdot [X])^{-1} = \begin{pmatrix} 249.685 & -2.8170 & 0.03549 \\ -2.8170 & 0.03224 & -0.00043 \\ 0.03549 & -0.00043 & 0.0000074 \end{pmatrix}$$

$$[X]' \cdot [Y] = \begin{pmatrix} 22919 \\ 2475344 \\ 35201790 \end{pmatrix}$$

$$\begin{aligned}
b_1 n \quad &+ b_2 \sum x_{2i} &+ b_3 \sum x_{3i} &= \sum y_i \\
b_1 \sum x_{2i} &+ b_2 \sum x_{2i}^2 &+ b_3 \sum x_{2i} x_{3i} &= \sum x_{2i} y_i \\
b_1 \sum x_{3i} &+ b_2 \sum x_{3i} x_{2i} &+ b_3 \sum x_{3i}^2 &= \sum x_{3i} y_i
\end{aligned}$$

$$\begin{aligned}
12\ b_1 + \quad 1290\ b_2 + \quad 18000\ b_3 &= 22919 \\
1290\ b_1 + \quad 138822\ b_2 + \quad 1943640\ b_3 &= 2475344 \\
18000\ b_1 + \quad 1943640\ b_2 + \quad 27643600\ b_3 &= 35201790
\end{aligned}$$

This is a linear system of equations with three equations and three variables. A linear system of equations can be solved with the substitution method, the equalization method or the addition method.[21] A definite determination of n variables is only possible if n independent equations exist (unambiguously determinable equation system).

$$\begin{aligned}
b_1 &= -1415.294 \\
b_2 &= 16.099 \\
b_3 &= 1.0631
\end{aligned}$$

$$\hat{y} = -1415.294 + 16.099\,x_2 + 1.0631\,x_3$$

with $i = 1, \dots, 12$

2.5.3.1 Confidence Intervals for the Regression Coefficients of a Double Linear Regression Function

$$s_E^2 = \frac{1}{n-k} \sum e_i^2$$

$n =$ number of observations (elements)
$k =$ number of variables

[21] Cf. Peren, F.W. (2021), pp. 57-60.

$$
\begin{aligned}
s_E^2 &= \frac{1}{n-3}\left[\sum y_i^2 - b_1 \sum y_i - b_2 \sum x_{2i} y_i - b_3 \sum x_{3i} y_i\right] \\
&= \frac{1}{12-3} \cdot (44861817 + 1415.294 \cdot 22919 - 16.099 \cdot 2475344 - \\
&\qquad - 1.0631 \cdot 35201990) \\
&= \frac{1}{9} \cdot 25354.181 \\
&= 2817.13 \\
s_E &= \sqrt{2817.13} = 53.08
\end{aligned}
$$

At a multiple linear regression function, it is recommendable to identify the variances respectively the standard deviations of the regression coefficients by using the (estimated) covariance matrix:

(a) $\widehat{CV} = s_E^2 \left([X]'_{n\times n} \cdot [X]_{n\times n}\right)^{-1}$

$$
= 2817.13 \cdot \begin{pmatrix} 249.685 & -2.8170 & 0.03549 \\ -2.8170 & 0.03224 & -0.00043 \\ 0.03549 & -0.00043 & 0.0000074 \end{pmatrix}
$$

$$
= \begin{pmatrix} 703395.10 & -7935.86 & 99.98 \\ -7935.86 & 90.82 & -1.2114 \\ 99.98 & -1.2114 & 0.02085 \end{pmatrix}
$$

(b) s_{Bj}^2 with $j = 1, \ldots k$ correspond to the elements at the main diagonal of the estimated covariance matrix $\widehat{CV}$.

$s_{B1}^2 = 703395.10$ $\qquad s_{B1} = \sqrt{703395.10} = 838.69$

$s_{B2}^2 = 90.82$ $\qquad s_{B2} = 9.530$

$s_{B3}^2 = 0.02085$ $\qquad s_{B3} = 0.1444$

Confidence intervals when $(1-\alpha) = 0.95$

(1) for the regression coefficient $\hat{b}_1$:

$$b_1 - ts_{B_1} \leq \hat{b}_1 \leq b_1 + ts_{B_1}$$

$$-1415.294 - 2.228 \cdot 838.69 \leq \hat{b}_1 \leq -1415.294 + 2.228 \cdot 838.69$$

$$-3283.90 \leq \hat{b}_1 \leq +453.31$$

(2) for the regression coefficient $\hat{b}_2$:

$$b_2 - ts_{B_2} \leq \hat{b}_2 \leq b_2 + ts_{B_2}$$

$$16.099 - 2.228 \cdot 9.530 \leq \hat{b}_2 \leq 16.099 + 2.228 \cdot 9.530$$

$$-5.134 \leq \hat{b}_2 \leq +37.332$$

(3) for the regression coefficient $\hat{b}_3$:

$$b_3 - ts_{B_3} \leq \hat{b}_3 \leq b_3 + ts_{B_3}$$

$$1.0631 - 2.228 \cdot 0.1444 \leq \hat{b}_3 \leq 1.0631 + 2.228 \cdot 0.1444$$

$$+0.7414 \leq \hat{b}_3 \leq +1.3848$$

2.5.3.2 Student's t-Tests for the Regression Coefficients of a Double Linear Regression Function

For testing hypotheses of stochastic parameters like regression coefficients, the practical procedure is as follows:

a. Definition of null hypothesis (H_0) and alternative hypothesis (H_A) as well as significance level (α)

b. Determination of the test statistic

c. Determination of the test distribution

d. Identification of the critical range

e. Calculation of the value of the test statistic

f. Decision and interpretation

(1) Test of the regression coefficient $\hat{b}_1$

a. H_0: $\hat{b}_1 = 0$
H_A: $\hat{b}_1 \neq 0$
$\alpha = 0.05 \quad (1-\alpha) = 0.95$ (in the example above)

H_0 means that the ordinate value of the estimated regression function would be zero.

H_A can be $\hat{b}_1 \neq 0$, $\hat{b}_1 > 0$ or $\hat{b}_1 < 0$.

b. Test statistic

$$t = \frac{b_1}{s_{B_1}} \quad \text{(table 2.12)}$$

with $s_{B_1} = 838.69$ (in the example above)

c. Determination of the test distribution

Student's t-distribution, two-sided symmetric confidence interval with $(1-\alpha)$ and $\nu = n-2$ (see appendix A, statistical tables)

e.g. with $(1-\alpha) = 0.95$ and $\nu = 12-2 = 10$
(in the example above)

d. Identification of the critical range

For $(1-\alpha) = 0.95$ and $\nu = 10$, the critical t-value, t_c, is 2.228 (Student's t-distribution, two-sided symmetric confidence interval; see appendix A, statistical tables).

If $t = \dfrac{b_1}{s_{B_1}} > 2.228$, the null hypothesis H_0 has to be rejected.

If $t = \dfrac{b_1}{s_{B_1}} \leq 2.228$, the null hypothesis H_0 cannot be rejected.

e. Calculation of the value of the test statistic

$$t = \frac{-1415.294}{838.69} = -1,6875 \quad \text{(in the example above)}$$

f. Decision and interpretation

$t < t_c \quad (-1,6875 < 2.228)$

H_0 cannot be rejected.

The observed value for b_1 $(b_1 = -1415.294)$ is statistically invalid with a significance level of 0.05 $(= 5\ \%)$.

(2) Test of the regression coefficient $\hat{b}_2$

a. H_0: $\hat{b}_2 = 0$
H_A: $\hat{b}_2 \neq 0$

$\alpha = 0.05 \quad (1-\alpha) = 0.95$ (in the example above)

H_0 implies that there would be no correlation between the tested variables (y and x_2 in the example above).

H_A can be $\hat{b}_2 \neq 0$, $\hat{b}_2 > 0$ (positive correlation between x_2 and y) or $\hat{b}_2 < 0$ (negative correlation between x_2 and y).

b. Test statistic

$$t = \frac{b_2}{s_{B_2}}$$ (table 2.12)

with $s_{B_2} = 9.530$ (in the example above)

c. Determination of the test distribution

Student's t-distribution, two-sided symmetric confidence interval with $(1-\alpha)$ and $\nu = n-2$ (see appendix A, statistical tables)

e.g. with $(1-\alpha) = 0.95$ and $\nu = 12-2 = 10$
(in the example above)

d. Identification of the critical range

For $(1-\alpha) = 0.95$ and $\nu = 6$, the critical t-value, t_c, is 2.228 (Student's t-distribution, two-sided symmetric confidence interval; see appendix A, statistical tables).

If $t = \dfrac{b_2}{s_{B_2}} > 2.228$, the null hypothesis H_0 has to be rejected.

If $t = \dfrac{b_2}{s_{B_2}} \leq 2.228$, the null hypothesis H_0 cannot be rejected.

e. Calculation of the value of the test statistic

$$t = \frac{16.099}{9.530} = 1.6893$$ (in the example above)

f. Decision and interpretation

$t < t_c \quad (1.6893 < 2.228)$

H_0 cannot be rejected.

The observed value for b_2 $(b_2 = 16.099)$ is statistically invalid with a significance level of 0.05 $(= 5\ \%)$.

There is no significant correlation between the tested variables y (revenue) and x_2 (price) in the example above.

(3) Test of the regression coefficient $\hat{b}_3$

a. H_0: $\hat{b}_3 = 0$

H_A: $\hat{b}_3 > 0$ (a positive correlation between y (revenue) and x_3 (sales area) can be expected)

$\alpha = 0.05 \quad (1-\alpha) = 0.95$ (in the example above)

b. Test statistic

$$t = \frac{b_3}{s_{B_3}} \quad \text{(table 2.12)}$$

with $s_{B_3} = 0.1444$ (in the example above)

c. Determination of the test distribution

Student's t-distribution function with $(1-\alpha) = 0.95$ and $\nu = 12 - 2 = 10$ (in the example above)

d. Identification of the critical range

For $(1-\alpha) = 0.95$ and $\nu = 10$, the critical t-value, t_c, is 1.812 (Student's t-distribution, two-sided symmetric confidence interval; see appendix A, statistical tables).

If $t = \dfrac{b_3}{s_{B_3}} > 1.812$, the null hypothesis H_0 has to be rejected.

If $t = \dfrac{b_3}{s_{B_3}} \leq 1.812$, the null hypothesis H_0 cannot be rejected.

e. Calculation of the value of the test statistic

$$t = \frac{1.0631}{0.1444} = 7.3622 \quad \text{(in the example above)}$$

f. Decision and interpretation

$t > t_c \quad (7.3622 > 1.812)$

H_0 has to be rejected.

The observed value for b_3 $(b_3 = 1.0631)$ is statistically valid with a significance level of 0.05 $(= 5\ \%)$.

There is a significant correlation between the tested variables y (revenue) and x_3 (sales area) in the example above.

Chapter 3

Inferential Statistics

3.1 Probability Calculation

3.1.1 Fundamental Terms/Definitions

Random Experiment

A process that can be repeated as often as desired and whose result depends on chance.

Elementary Event e_i

An elementary event e_i (also called a sample point) is an event which contains only the single outcome e_i in the sample space S.

Sample Space S

The sample space S (of an experiment or random trial) is the set of all elementary events or possible outcomes or results (of that experiment or that random trial):

$$S = \{e_1, e_2, \ldots, e_i, \ldots, e_n\}$$

Event A

Any subset of the sample space S.

F. W. Peren, *Statistics for Business and Economics*,
https://doi.org/10.1007/978-3-662-65846-8_3

Laplace's[1] Definition of Probability

If all elementary events are equally possible, then

$$P(A) = \frac{\text{number of favourable cases}}{\text{number of all cases}}$$

Examples: For the event A that a coin lands on heads in a toss, $P(A) = \frac{1}{2}$

For the event B that a die lands on a two in a roll, $P(B) = \frac{1}{6}$

Von Mises'[2] Definition of Probability

$$P(A) = \lim_{n\to\infty} f_n(A) = \lim_{n\to\infty} \frac{h_n(A)}{n}$$

Example: For the event A that a die lands on a two in infinite rolls, $P(A) = \frac{1}{6}$

The more often a die is rolled, the probability that this die lands on the number two approaches $\frac{1}{6}$.

[1] Pierre-Simon Laplace (1749 - 1827) was a French mathematician, physicist and astronomer.

[2] Richard Edler von Mises (1883 - 1953) was an Austrian-American mathematician.

Kolmogorov's[3] Definition of Probability

Axiom 1:	$0 \leq P(A) \leq 1$	for $A \subset S$	(non-negativity)
Axiom 2:	$P(S) = 1$		(standardisation)
Axiom 3:	$P(A \cup B) = P(A) + P(B)$	for $A \cap B = \varnothing$	(additivity)

Axiom (3) results in the relation

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

for $\quad A_i \cap A_j = \varnothing \quad (i \neq j)$

Rules of Calculation for Probabilities

(1) Complementary Probability (Counter Probability)

For $\overline{A}$, the complementary event of A, the following applies:

$P(\overline{A}) = 1 - P(A)$

Example:

The probability of rolling a six in one roll of a die is $\frac{1}{6}$. What is the probability of rolling a six when rolling a die four times?

$\overline{A}$ = rolling at least a six
A = not rolling any sixes

[3] Andrej Nikolaevič Kolmogorov (1903 - 1987) was a Soviet mathematician.

The probability of not rolling any sixes in four rolls is:

$$\frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \;=\; \frac{625}{1,296} \;=\; 0.482 \;=\; 48.2\,\%$$

The counter probability is therefore:

$$1 \;-\; \frac{625}{1,296} \;=\; 0.518 \;=\; 51.8\,\%$$

(2) Probability of an Impossible Event

$P(\varnothing) = 0$

<u>Example</u>:

The probability of getting an eight in a single roll of a die is zero.
$\varnothing$: “getting eight points in a single roll of a die”

Indeed, a normal die only has the numbers one to six.
$\varnothing$: “number of dots is equal to eight”

$P(\varnothing) = 0$

(3) Addition Theorem for Two Arbitrary Events

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

<u>Example</u>:

A = {rolling a number < 4}
B = {rolling an odd number}

$P(A) \;=\; \frac{3}{6}$ and $P(B) \;=\; \frac{3}{6}$

Probability of $A \cap B$:

$$\begin{aligned} P(A \cap B) &= P(\{\text{rolling an odd number less than } 4\}) = \\ &= P(\{\text{rolling } 1 \text{ or } 3\}) = \\ &= \frac{2}{6} = \frac{1}{3} \end{aligned}$$

Accordingly:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) = \\ &= \frac{3}{6} + \frac{3}{6} - \frac{2}{6} = \frac{4}{6} = \\ &= \frac{2}{3} = 66.67\ \% \end{aligned}$$

(4) Condition of Probability for a Sub-Event

$P(A) \leq P(B)$ for $A \subset B$

Example:

$A = \{\text{rolling an odd number} < 4\}$

$B = \{\text{rolling a number} < 4\}$

$P(A) = \frac{2}{6}$ and $P(B) = \frac{3}{6}$

$P(A) < P(B) \qquad \frac{2}{6} < \frac{3}{6}$

3.1.2 Theorems of Probability Theory

Multiplication Theorem

For two stochastically independent events A and B, the following applies:

$$P(A \cap B) = P(A) \cdot P(B)$$

Example:

What is the probability of getting "heads" twice if a coin is tossed twice?

For each toss, the probability of getting "heads" is $\frac{1}{2}$, so the following applies:

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B) = \\ &= \frac{1}{2} \cdot \frac{1}{2} = \\ &= \frac{1}{4} = 25\,\% \end{aligned}$$

For two stochastically dependent events A and B, the following applies:

$$P(A \cap B) = P(A) \cdot P(B/A) = P(B) \cdot P(A/B)$$

Example:

The probability of being a motorcyclist in a traffic accident last year was $31\,\%$. Of these, $46\,\%$ were not wearing a helmet. What is the probability of being a motorcyclist in a traffic accident who is not wearing a helmet?

$$\begin{aligned} P(A) &= P(\text{accident as a motorcyclist}) = 0.31 \\ P(B/A) &= P(\{\text{without helmet}\} / \{\text{accident as a motorcyclist}\}) = 0.46 \end{aligned}$$

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B/A) = \\ &= 0.31 \cdot 0.46 = \\ &= 0.1426 = 14.26\,\% \end{aligned}$$

Conditional Probability

For $P(A) > 0$, the conditional probability of event B under the condition A is defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

Example:

Out of 20 students, 4 own a car. 12 of these 20 students are male. 3 of the 12 male students own a car and the remaining 9 do not. What is the probability that a randomly selected student in this group who owns a car is male?

A: The selected student has a car.
B: The selected student is male.

Fourfold table:

	B	$\overline{B}$	
A	3	1	4
$\overline{A}$	9	7	16
	12	8	20

Calculating probabilities:

	B	$\overline{B}$	
A	$\frac{3}{20} = 0.15$	$\frac{1}{20} = 0.05$	$\frac{4}{20} = 0.2$
$\overline{A}$	$\frac{9}{20} = 0.45$	$\frac{7}{20} = 0.35$	$\frac{16}{20} = 0.8$
	$\frac{12}{20} = 0.6$	$\frac{8}{20} = 0.4$	$\frac{20}{20} = 1$

Conditional probability:

$$\begin{aligned} P(B/A) &= \frac{P(A \cap B)}{P(A)} = \\ &= \frac{0.15}{0.2} = \\ &= 0.75 = 75\ \% \end{aligned}$$

The probability of a randomly selected student in this group who owns a car being male is $75\ \%$.

$$\begin{aligned} \text{with} \quad P(A \cap B) &= P(A) \cdot P(B/A) = P(B) \cdot P(A/B) = \\ &= 0.2 \cdot (0.15/0.2) = 0.6 \cdot (0.15/0.6) = \\ &= 0.15 = 15\ \% \end{aligned}$$

The proportion of students who A) own a car and B) are male is $15\ \%$.

Stochastic Independence

Two events A and B are stochastically independent if the following applies:

$$P(A/B) = P(A/\overline{B}) \quad \vee \quad P(B/A) = P(B/\overline{A})$$

or $\quad P(A \cap B) = P(A) \cdot P(B)$

Example:

A die is rolled once. Event A is “even number of dots” and event B is “number of dots less than 5”.

A: 2, 4, 6
B: 1, 2, 3, 4

$$P(A) = \frac{3}{6} = \frac{1}{2} \qquad P(\overline{A}) = \frac{3}{6} = \frac{1}{2}$$

$$P(B) = \frac{4}{6} = \frac{2}{3} \qquad P(\overline{B}) = \frac{2}{6} = \frac{1}{3}$$

$$P(A \cap B) = \text{events } 2 \text{ and } 4 = \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{6} = \frac{1}{3}$$

$$P(\overline{A} \cap B) = \text{events } 1 \text{ and } 3 = \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{6} = \frac{1}{3}$$

Furthermore, the following applies in the case of stochastic independence:

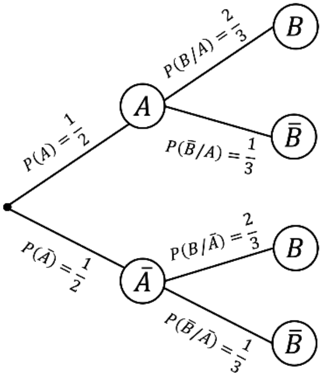
$$P(B/A) = P(B/\overline{A})$$

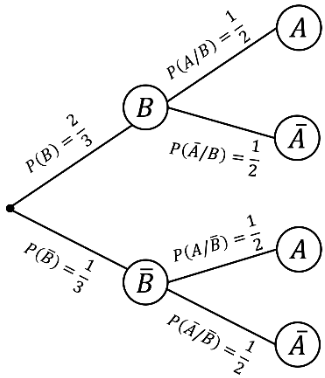
with

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{3} \Big/ \frac{1}{2} = \frac{2}{3}$$

$$P(B/\overline{A}) = \frac{P(\overline{A} \cap B)}{P(A)} = \frac{1}{3} \Big/ \frac{1}{2} = \frac{2}{3}$$

$$P(B/A) = P(B/\overline{A}) \qquad\qquad P(A/B) = P(A/\overline{B})$$
$$\frac{2}{3} = \frac{2}{3} \qquad\qquad \frac{1}{2} = \frac{1}{2}$$





Law of Total Probability

If $A_1 \cup A_2 \cup ... \cup A_n = S$ and $A_i \cap A_j = \varnothing$ for $i \neq j$, the following is true for $E \subset S$:

$$P(E) = \sum_{i=1}^{n} P(A_i) \cdot P(E/A_i) \qquad \text{with} \qquad i = 1, ..., n$$

Example:

A company manufactures $2,000$ units of a product daily. Out of these, the following machines produce:

M_1 500 units with a scrap rate of $5\ \%$,
M_2 800 units with a scrap rate of $4\ \%$,
M_3 700 units with a scrap rate of $2\ \%$.

One unit is randomly selected from a daily production. What is the probability that the selected unit is defective?

A_i : The selected unit was produced by M_i.
E : The selected unit is defective.

$$P(A_1) = \frac{500}{2{,}000} = 0.25 \quad \text{and} \quad P(E/A_1) = \frac{5}{100} = 0.05$$

$$P(A_2) = \frac{800}{2{,}000} = 0.4 \quad \text{and} \quad P(E/A_2) = \frac{4}{100} = 0.04$$

$$P(A_3) = \frac{700}{2{,}000} = 0.35 \quad \text{and} \quad P(E/A_3) = \frac{2}{100} = 0.02$$

Searched probability:

$$\begin{aligned} P(E) &= \sum_{i=1}^{3} P(A_i) \cdot P(E/A_i) = \\ &= 0.25 \cdot 0.05 + 0.4 \cdot 0.04 + 0.35 \cdot 0.02 = \\ &= 0.0355 \end{aligned}$$

The probability of a unit being defective is approximately $3.55\ \%$.

Bayes'[4] Theorem

If $A_1 \cup A_2 \cup ... \cup A_n = S$ and $A_i \cap A_j = \varnothing$ for $i \neq j$, the following is true for $E \subset S$:

$$P(A_j/E) = \frac{P(A_j) \cdot P(E/A_j)}{\sum\limits_{i=1}^{n} P(A_i) \cdot P(E/A_i)} \qquad \text{with} \qquad i, j = 1, ..., n$$

[4] Thomas Bayes (1701 - 1761) was an English mathematician and statistician.

Example:

The example of the “Law of Total Probability” is considered again. One unit is selected from a daily production again. In this case, it is a defective unit.

What is the probability that this unit comes from M_1, M_2 or M_3?

$$
\begin{aligned}
P(A_1/E) &= \frac{P(A_1) \cdot P(E/A_1)}{\sum\limits_{i=1}^{3} P(A_i) \cdot P(E/A_i)} = \\
&= \frac{0.25 \cdot 0.05}{0.25 \cdot 0.05 \,+\, 0.4 \cdot 0.04 \,+\, 0.35 \cdot 0.02} = \\
&= \frac{0.0125}{0.0355} = 0.35
\end{aligned}
$$

$$P(A_2/E) = \frac{0.4 \cdot 0.04}{0.0355} = 0.45$$

$$P(A_3/E) = \frac{0.35 \cdot 0.02}{0.0355} = 0.20$$

The probability of the randomly selected unit being from machine 1 is $35\ \%$, of it being from machine 2 is $45\ \%$, and of it being from machine 3 is $20\ \%$.

3.2 Probability Distributions

3.2.1 Concept of Random Variables

To express or process the result of a random experiment quantitatively, it must be transformed into a real number, if possible.

The random variable X comprises a certain number of n elementary events e_j with $j = 1, 2, ..., n$ in the sample space S.

Domain: sample space S
Codomain: set of real numbers

A distinction must be made:

1. *Discrete random variables:* Each possible event can be assigned a specific probability of occurrence (e.g. rolling a die).
2. *Continuous random variables:* There is an infinite number of possible manifestations of a characteristic. The possibility of determining the probability of occurrence is almost zero.

3.2.2 Probability, Distribution and Density Function

3.2.2.1 Discrete Random Variables

Probability Function

$$f(x_i) = P(X = x_i) \qquad \text{with} \qquad i = 1, 2, ..., n$$

Characteristics of a probability function:

(1) $f(x_i) \geq 0$ with $i = 1, 2, ..., n$

(2) $\sum\limits_i f(x_i) = 1$

Distribution Function

$$F(x) = P(X \leq x)$$

Characteristics of a distribution function:

(1) $F(x)$ is monotonically increasing

(2) $F(x)$ is continuous

(3) $\lim\limits_{x \to -\infty} F(x) = 0$

(4) $\lim\limits_{x \to +\infty} F(x) = 1$

Example:

A die is rolled. The set of events Ω that can occur, is:

$\Omega = \{1,2,3,4,5,6\}$

Probability Function

The probability that one of the events e_i with $i = 1, 2, ..., 6$, will occur, is in each case $\frac{1}{6}$.

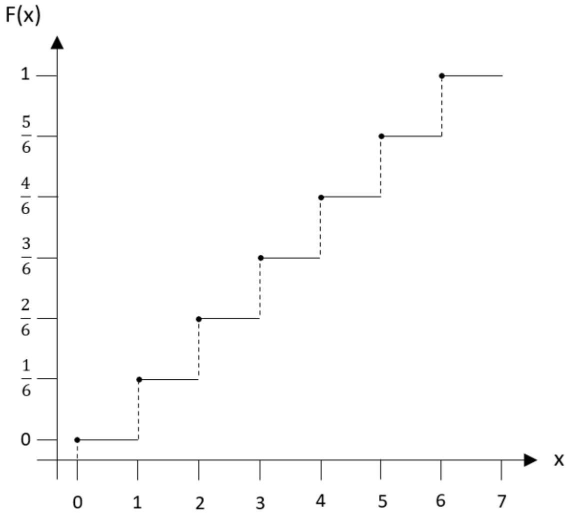
$$P(x=1) = \frac{1}{6}$$

$$P(x=2) = \frac{1}{6} \qquad \text{and so forth}$$

$$f(x_i) = \frac{1}{6} \qquad \text{with} \qquad i = 1, ..., 6$$

Distribution Function

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{6} & \text{for } 1 \leq x < 2 \\ \frac{2}{6} & \text{for } 2 \leq x < 3 \\ \frac{3}{6} & \text{for } 3 \leq x < 4 \\ \frac{4}{6} & \text{for } 4 \leq x < 5 \\ \frac{5}{6} & \text{for } 5 \leq x < 6 \\ 1 & \text{for } x \geq 6 \end{cases}$$



3.2.2.2 Continuous Random Variables

Instead of probabilities, so-called densities are given. This is done in the form of density functions.

Probability Function

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

Characteristics of each probability density:

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{+\infty} f(x)dx = 1$

Distribution Function

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(q)dq$$

$$\Rightarrow F'(x) = f(x)$$

Properties of each distribution function of continuous random variables:

(1) $0 \leq F(x) \leq 1$

(2) $F(x)$ is monotonically increasing,

for $x_1 < x_2$ applies $F(x_1) \leq F(x_2)$

(3) $\lim\limits_{x \to -\infty} F(x) = 0$

(4) $\lim\limits_{x \to +\infty} F(x) = 1$

(5) $F(x)$ is continuous throughout the entire domain.

<u>Example</u>:

The random variable X describes the delay of a cab at a certain stop, measured in minutes. This results in the following density function (dimension: minutes):

$$f(x) = \begin{cases} 0.4 - 0.125x & \text{for } 0 \leq x \leq 4 \\ 0 & \text{for all other x} \end{cases}$$

Probability Function

The probability for X to take a value between one and three minutes is:

$$
\begin{aligned}
P(1 \leq x \leq 3) &= \int_{1}^{3} f(x)dx = \\
&= \int_{1}^{3} (0.4 - 0.125x)dx = \\
&= \left[0.4x - \frac{0.125}{2} x^2 \right]_1^3 = \\
&= \frac{51}{80} - \frac{27}{80} = 0.3
\end{aligned}
$$

Distribution Function

$$
F(x) = P(X \leq x) = \int_{-\infty}^{x} f(q)dq
$$

$$
\Rightarrow F'(x) = f(x)
$$

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(q)dq = \\
&= \int_{0}^{x} (0.4 - 0.125q)dq = \\
&= \left[0.4q - \frac{0.125}{2} q^2 \right]_0^x = \\
&= 0.4x - 0.0625x^2
\end{aligned}
$$

3.2.3 Parameters for Probability Distributions

Expected Value and Variance of Random Variables

Discrete Random Variables

$$E(X) = \sum_i x_i \, f(x_i)$$

$$\begin{aligned} Var(X) &= E\left[[X - E(X)]^2\right] = \\ &= \sum_i [x_i - E(X)]^2 \, f(x_i) = \\ &= \sum_i x_i^2 \cdot f(x_i) - [E(X)]^2 \end{aligned}$$

Example:

The random variable X describes the odd number of a die rolled once.

There are three possible realizations:

$x_1 = 1 \quad x_2 = 3 \quad x_3 = 5$

The probability for each realization is $\frac{1}{3}$.

$$E(X) = \sum_i x_i \, f(x_i) =$$

$$E(X) = 1 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} + 5 \cdot \frac{1}{3} = 3$$

$$\begin{aligned} Var(X) &= \sum_i x_i^2 \cdot f(x_i) - [E(X)]^2 = \\ &= 1^2 \cdot \frac{1}{3} + 3^2 \cdot \frac{1}{3} + 5^2 \cdot \frac{1}{3} - 3^2 = \\ &= \frac{8}{3} \approx 2.6667 \end{aligned}$$

Continuous Random Variables

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx$$

$$Var(X) = E\left[[X - E(X)]^2\right] =$$

$$= \int_{-\infty}^{+\infty} [x - E(X)]^2 f(x) dx =$$

$$= \int_{-\infty}^{+\infty} x^2 \cdot f(x) dx - [E(X)]^2$$

<u>Example</u>:

Given a continuous random variable X with the following density function:

$$f(x) = \begin{cases} 0.4 - 0.125x & \text{für } 0 \leq x \leq 4 \\ 0 & \text{for all other x} \end{cases}$$

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx$$

$$E(X) = \int_{0}^{4} x(0.4 - 0.125x) dx =$$

$$= \int_{0}^{4} (0.4x - 0.125x^2) dx =$$

$$= \left[\frac{0.4}{2} x^2 - \frac{0.125}{3} x^3\right]_0^4 =$$

$$= \frac{8}{15} \approx 0.5333$$

$$\begin{aligned}
Var(X) &= \int_{-\infty}^{+\infty} x^2 \cdot f(x)dx - [E(X)]^2 = \\
&= \int_0^4 x^2(0.4x - 0.125x)dx - 0.5333^2 = \\
&= \int_0^4 (0.4x^2 - 0.125x^3)dx - 0.5333^2 = \\
&= \left[\frac{0.4}{3}\, x^3 - \frac{0.125}{4}\, x^4\right]_0^4 - 0.5333^2 = \\
&= 0.2489
\end{aligned}$$

3.3 Theoretical Distributions

3.3.1 Discrete Distributions

Binomial Distribution

Probability Function

$$f_B(x/n;\theta) = \begin{cases} \binom{n}{x}\,\theta^x(1-\theta)^{n-x} & \text{for } x = 0, 1, ..., n \\ 0 & \text{for } x \neq 0, 1, ..., n \end{cases}$$

Distribution Function

$$F_B(x/n;\theta) = \sum_{v=0}^{x} \binom{n}{v}\theta^v\,(1-\theta)^{n-v}$$

Expected Value

$$E(X) = n \cdot \theta$$

Variance

$$Var(X) = n \cdot \theta(1-\theta)$$

Recursive Formula

$$f_B(x+1/n;\,\theta) = f_B(x/n;\,\theta) \cdot \frac{n-x}{x+1} \cdot \frac{\theta}{1-\theta}$$

Example:

Probability Function

The probability of tossing "heads" exactly $x=2$ times in $n=4$ coin tosses is $P(X=2)$.

$$f_B(x/n;\,\theta) = \binom{n}{x}\theta^x\,(1-\theta)^{n-x}$$

$$f_B(2/4;\,0.5) = \binom{4}{2}0.5^2\,(1-0.5)^{4-2} = 0.3750$$

See Appendix A, Binomial Distribution – Probability Mass Function for $n=4$, $x=2$ and $\theta=0.5$.

Distribution Function

The probability of tossing "heads" at most $x=2$ times in $n=4$ coin tosses is $P(X=2)$.

$$F_B(x/n;\,\theta) = \sum_{v=0}^{x}\binom{n}{v}\theta^v\,(1-\theta)^{n-v}$$

$$= F_B(2/4;\,0.5) = f_B(0/4;\,0.5) + f_B(1/4;\,0.5) + f_B(2/4;\,0.5) =$$
$$= 0.0625 + 0.2500 + 0.3750 = 0.6875$$

See Appendix A, Binomial Distribution – Probability Mass Function.
See also Appendix A, Binomial Distribution – Cumulative Distribution Function for $n = 4$, $x = 2$ and $\theta = 0.5$.

Expected Value

$$E(X) = n \cdot \theta$$
$$E(X) = 4 \cdot 0.5 = 2$$

Variance

$$Var(X) = n \cdot \theta(1 - \theta)$$
$$Var(X) = 4 \cdot 0.5(1 - 0.5) = 1$$

Recursive Formula

$$f_B(x+1/n;\, \theta) = f_B(x/n;\, \theta) \cdot \frac{n-x}{x+1} \cdot \frac{\theta}{1-\theta}$$

$$f_B(x+1/n;\, \theta) = f_B(2+1/4;\, 0.5) = f_B(3/4;\, 0.5) =$$
$$= 0.3750 \cdot \frac{4-2}{2+1} \cdot \frac{0.5}{1-0.5} = 0.25$$

See Appendix A, Binomial Distribution – Probability Mass Function for $n = 4$, $x = 3$ and $\theta = 0.5$.

The binomial distribution is equal to the one-dimensional case of a multidimensional distribution, the so-called *multinomial distribution* with the probability function

$$f_B^{mult}(x_1, x_2, ..., x_k/n; \theta_1; \theta_2; ...; \theta_k) =$$

$$= \frac{n!}{x_1!x_2!\dots x_k!} \cdot \theta_1^{x_1} \cdot \theta_2^{x_2} \cdot ... \cdot \theta_k^{x_k}$$

with $\sum_{i=1}^{k} x_i = n$ and $\sum_{i=1}^{k} \theta_i = 1$

For $k = 2$ the probability function of the *multinomial distribution* is equal to the probability function of the *binomial distribution* with $x_2 = n - x_1$ and $\theta = 1 - \theta_1$.

Example:

An imported delivery of oranges includes the following qualities:

$50\ \%$ correspond to grade I,

$30\ \%$ correspond to grade II and

$20\ \%$ of the delivery are unusable.

A sample with replacement of ten oranges is randomly taken. What is the probability that six of these oranges are of grade I and four are of grade II, i.e. that all ten oranges are for sale?

$X_1 =$ random variable "Number of oranges of grade I" in the sample
$X_2 =$ random variable "Number of oranges of grade II" in the sample
$X_3 =$ random variable "Number of unusable oranges" in the sample

The multinomial probability looked for is calculated as follows:

$$f_B^{mult}(6,4,0/10;0.5;0.3;0.2) =$$
$$= \frac{10!}{6!4!0!} \cdot 0.5^6 \cdot 0.3^4 \cdot 0.2^0 =$$
$$= 210 \cdot 0.00013 \approx 0.02668 \approx 2.66\,\%$$

Hypergeometric Distribution

Probability Function

$$f_H(x/N;n;M) = \begin{cases} \dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} & \text{for } x = 0, 1, ..., n \\ 0 & \text{for } x \neq 0, 1, ..., n \end{cases}$$

Distribution Function

$$F_H(x/N;n;M) = \sum_{v=0}^{x} \frac{\binom{M}{v}\binom{N-M}{n-v}}{\binom{N}{n}}$$

Expected Value

$$E(X) = n \cdot \frac{M}{N}$$

Variance

$$Var(X) = n \cdot \frac{M}{N} \cdot \frac{N-M}{N} \cdot \frac{N-n}{N-1}$$

Recursive Formula

$$f_H(x+1/N;n;M) = f_H(x/N;n;M) \cdot \frac{(M-x)(n-x)}{(x+1)(N-M-n+x+1)}$$

Example:

Probability Function

$$f_H(x/N;n;M) = \begin{cases} \dfrac{\dbinom{M}{x}\dbinom{N-M}{n-x}}{\dbinom{N}{n}} & \text{for } x = 0, 1, ..., n \\ 0 & \text{for } x \neq 0, 1, ..., n \end{cases}$$

A total of $N = 10$ balls includes $M = 4$ green balls and $N - M = 6$ non-green balls. A random sample of $n = 4$ balls is drawn without returning any already drawn ball to the population, meaning the number of balls still available is reduced by 1 from draw to draw. Since the sample is drawn "without replacement", each element (each individual ball) can occur at most only once in the drawn sample. The order of the selected elements (drawn balls) does not matter.

What is the probability of drawing $x = 2$ green balls at $n = 4$ (sample size)?

$$f_H(2/10;4;4)$$

$$P(X=2) = f_H(2/10;4;4) = \frac{\dbinom{4}{2}\dbinom{10-4}{4-2}}{\dbinom{10}{4}} = \frac{\dbinom{4}{2}\dbinom{6}{2}}{\dbinom{10}{4}} =$$

$$= \frac{6 \cdot 15}{210} = 0.4286$$

The probability of drawing two green balls in a sample size of $n = 4$ is $42\ \%$.

See Appendix A, Hypergeometric Distribution – Probability Mass Function for $N = 10$, $n = 4$, $M = 4$ and $x = 2$.

Distribution Function

$$F_H(x/N;n;M) = \sum_{v=0}^{x} \frac{\binom{M}{v}\binom{N-M}{n-v}}{\binom{N}{n}}$$

If the probability is looked for that at most $x = 2$ balls should be green within the conducted sample in the extent of $n = 4$, then the distribution function is to be used:

$$\begin{aligned} F_H(2/10;4;4) &= f_H(0/10;4;4) + f_H(1/10;4;4) + f_H(2/10;4;4) = \\ &= 0.0174 + 0.3810 + 0.4286 = 0.8810 \end{aligned}$$

See Appendix A, Hypergeometric Distribution – Probability Mass Function. See also Appendix A, Hypergeometric Distribution – Cumulative Distribution Function for $N = 10$, $n = 4$, $M = 4$ and $x = 2$.

Expected Value

$$E(X) = n \cdot \frac{M}{N}$$

$$E(X) = 4 \cdot \frac{4}{10} = 1.6$$

Variance

$$Var(X) = n \cdot \frac{M}{N} \cdot \frac{N-M}{N} \cdot \frac{N-n}{N-1}$$

$$Var(X) = 4 \cdot \frac{4}{10} \cdot \frac{10-4}{10} \cdot \frac{10-4}{10-1} = 0.64$$

Recursive Formula

$$f_H(x+1/N;n;M) = f_H(x/N;n;M) \cdot \frac{(M-x)(n-x)}{(x+1)(N-M-n+x+1)}$$

$$f_H(3/10;4;4) = \frac{\binom{4}{3}\binom{10-4}{4-3}}{\binom{10}{4}} = \frac{\binom{4}{3}\binom{6}{1}}{\binom{10}{4}} = \frac{4 \cdot 6}{210} = 0.1143$$

$$f_H(2/10;4;4) \cdot \frac{(4-2)(4-2)}{(2+1)(10-4-4+2+1)} =$$

$$= 0.4286 \cdot 0.2667 =$$

$$= 0.1143$$

See Appendix A, Hypergeometric Distribution – Probability Mass Function for $N = 10$, $n = 4$, $M = 4$ and $x = 2$.

Poisson[5] Distribution

Probability Function

$$f_p(x/\mu) = \begin{cases} \dfrac{\mu^x e^{-\mu}}{x!} & \text{for } x = 0, 1, ..., n \\ 0 & \text{for } x \neq 0, 1, ..., n \end{cases}$$

$e = 2.71828...$ (Euler's Number)

5 Siméon Denis Poisson (1781 - 1840) was a French physicist and mathematician.

Distribution Function

$$F_p(x/\mu) = \sum_{v=0}^{x} \frac{\mu^v e^{-\mu}}{v!}$$

Expected Value and Variance

$$E(X) = Var(X) = \mu$$

Recursive Formula

$$f_p(x+1/\mu) = f_p(x/\mu)\frac{\mu}{x+1}$$

Example:

Probability Function

A company manufactures motor vehicles in an assembly line. The proportion of motor vehicles produced that do not meet the desired standard of quality is $\theta = 0.001$; i.e. one per mill. It is looked for the probability that exactly two motor vehicles are defective, meaning they do not meet the required quality standards, during a random inspection in which $n = 1,500$ motor vehicles are randomly selected.

$$f_p(x/\mu) = \begin{cases} \dfrac{\mu^x e^{-\mu}}{x!} & \text{for } x = 0, 1, ..., n \qquad \text{with } \mu = n \cdot \theta \\ 0 & \text{for } x \neq 0, 1, ..., n \end{cases}$$

$$f_p(2/\mu) = f_p(2/1.5) = \frac{1.5^2 e^{-1.5}}{2!} = 0.2510$$

with $\mu = 1,500 \cdot 0.001 = 1.5$

See Appendix A, Poisson Distribution – Probability Mass Function for $\mu = 1.5$ and $x = 2$.

Distribution Function

$$F_p(x/\mu) = \sum_{v=0}^{x} \frac{\mu^v e^{-\mu}}{v!}$$

Now it is looked for the probability that, in a random sample in which $n = 1,500$ motor vehicles are randomly selected, a at most two motor vehicles are defective, meaning they do not meet the required quality standards.

$$F_p(2/1.5) = \frac{1.5^0 e^{-1.5}}{0!} + \frac{1.5^1 e^{-1.5}}{1!} + \frac{1.5^2 e^{-1.5}}{2!} =$$

$$= 0.2231 + 0.3347 + 0.2510 = 0.8088$$

See Appendix A, Poisson Distribution – Probability Mass Function. See also Appendix A, Poisson Distribution – Cumulative Distribution Function for $\mu = 1.5$ and $x = 2$.

Expected Value and Variance

$$E(X) = Var(X) = \mu = n \cdot \theta = 1,500 \cdot 0.001 = 1.5$$

Recursive Formula

$$f_p(x+1/\mu) = f_p(x/\mu)\frac{\mu}{x+1}$$

$$f_p(3/1.5) = \frac{1.5^3 e^{-1.5}}{3!} = 0.1255$$

alternative calculation:

$$f_p(3/1.5) = \frac{1.5}{2+1} = 0.1255 =$$

$$= 0.2510\ \frac{1.5}{2+1} =$$

$$= 0.1255$$

See Appendix A, Poisson Distribution – Probability Mass Function for $\mu = 1.5$ and $x = 3$.

3.3.2 Continuous Distributions

Normal Distribution

Probability Function

$$f_n(x/\mu;\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Distribution Function

$$F_n(x/\mu;\sigma^2) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{q-\mu}{\sigma}\right)^2} dq$$

Expected Value

$$E(X) = \mu$$

Variance

$$Var(X) = \sigma^2$$

Standard Normal Distribution

If the random variable X is normally distributed with $E(X) = \mu$ and $Var(X) = \sigma^2$, then X becomes the standardised random variable Z with

$$Z = \frac{X - \mu}{\sigma}$$

and the expected value $E(Z) = 0$ and the variance $Var(Z) = 1$.

Probability Function

$$f_N(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

Distribution Function

$$F_N(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}q^2}\, dq$$

Expected Value

$$E(Z) = 0$$

Variance

$$Var(Z) = 1$$

Example 1:

Probability Function

The height of a given tree, X, is assumed to be normally distributed over the entire Earth with an expected value $E(X)$ of $10m$ and a variance $Var(X)$ of $1.44m^2$.

What is the probability that a randomly selected tree is exactly

a) $9.90m$
b) $11.40m$
c) $14.10m$

tall?

$$f_n(x/\mu;\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

a) $f_n(9.90/10;1.44) =$
$$= \frac{1}{1.2\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{9.90-10}{1.2}\right)^2} =$$
$$= 0.3325 \cdot 0.9965 = 0.3313 = 33.13\%$$

b) $f_n(11.40/10;1.44) =$
$$= \frac{1}{1.2\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{11.40-10}{1.2}\right)^2} =$$
$$= 0.3325 \cdot 0.5063 = 0.1684 = 16.84\%$$

c) $f_n(14.10/10;1.44) =$
$$= \frac{1}{1.2\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{14.10-10}{1.2}\right)^2} =$$
$$= 0.3325 \cdot 0.0029 = 0.00097 \approx 0.001 =$$
$$\approx 0.1\% \approx 1‰$$

What is the probability that a randomly selected tree is

a) smaller than $9m$,
b) at least $10.80m$ tall,
c) between $9.20m$ and $11.20m$ tall?

Here it is a case of a normal distribution with the probability function (density function) of

$$f_n(x/\mu;\sigma^2) = \frac{1}{1.0954 \cdot \sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-10}{1.0954}\right)^2}$$

with

a) $W\,(X < 9)$

b) $W\,(X \geq 10,8)$

c) $W\,(9.20 < X < 11.20)$

Since the values in the probability function are given according to a left-sided integral (area under the normally distributed curve), the relations are all to be written in "less than" or "less than/equal to" directions (see Appendix A, Standard Normal Distribution).

a) $W\,(X < 9)$

b) $1 - W\,(X < 10.8)$

c) $W\,(X < 11.20) - W(X \leq 9.20)$

Since these are integrals (area segments), it is of no significance whether it is called "$<$" or "$\leq$", so that usually the following applies:

a) $W\,(X \leq 9)$

b) $1 - W\,(X \leq 10.8)$

c) $W\,(X \leq 11.20) - W(X \leq 9.20)$

If the random variable X is normally distributed with $E(X) = \mu$ and $Var(X) = \sigma^2$, then X becomes the standardized random variable Z with $Z = \dfrac{X-\mu}{\sigma}$ and $E(Z) = 0$ and $Var(Z) = 1$.

a) $W\,(Z \leq \frac{9-10}{1.2}) = W(Z \leq -0.833)$

b) $1 - W\,(Z \leq \frac{10.8-10}{1.2}) = 1 - W(Z \leq 0.667)$

c) $W\,(Z \leq \frac{11.20-10}{1.2}) - W(Z \leq \frac{9.20-10}{1.2}) =$
$= W(Z \leq 1) - W(Z \leq -0.667)$

a) $W\,(Z \leq -0.833) = z\,(-0.833) = 1 - z\,(0.833)$

b) $1 - W\,(Z \leq 0.667) = 1 - z\,(0.667)$

c) $W\,(Z \leq 1) - W(Z \leq -0.667) = z(1) - z\,(-0.667) =$
$= z\,(1) - (1 - z\,(0.667)) =$
$= z\,(1) - 1 + z\,(0.667)$

Due to the symmetry of the (standard) normal distribution $z(-g) = 1 - z(g)$ can be applied. The z is also often written as the Greek letter Phi ϕ.

The z-values can be taken from Appendix A, Standard Normal Distribution - Distribution Function, so that the following applies:

a) $1 - z\,(0.833) = 1 - 0.7976 = 0.2024 = 20.24\%$

b) $1 - z\,(0.667) = 1 - 0.7476 = 0.2524 = 25.24\%$

c) $z\,(1) - 1 + z\,(0.667) = 0.8413 - 1 + 0.7476 =$
$= 0.5889 = 58.89\%$

Example 2:

What height do 30% of the trees of example 1 exceed?

$$W(X \geq x_o) = 0.3$$

$$\Leftrightarrow 1 - W(X \leq x_o) = 0.3$$

$$\Leftrightarrow -W(X \leq x_o) = -0.7$$

$$\Leftrightarrow W(X \leq x_o) = 0.7$$

$$\Rightarrow W(Z \leq z_o) = 0.7$$

$$\Leftrightarrow W(Z \leq \frac{X - \mu}{\sigma}) = W(Z \leq \frac{X - 10}{1,2}) = 0.7$$

In Appendix A, Standard Normal Distribution - Distribution Function, the value of 0.7 is first exceeded at a z-value of 0.525.

$$\Rightarrow \frac{x_o - 10}{1.2} = 0.525$$
$$x_o = 0.525 \cdot 1.2 + 10 = 10.63\,m$$

Chi-Squared Distribution

Probability Function

$$f_{Ch}(\chi^2/\nu) = \begin{cases} \dfrac{1}{2^{\nu/2}\,\Gamma\left(\dfrac{\nu}{2}\right)}\, e^{-\frac{\chi^2}{2}}\, \left(\chi^2\right)^{\left(\frac{\nu}{2}-1\right)} & \text{for } \chi^2 \geq 0 \\ 0 & \text{for } \chi^2 < 0 \end{cases}$$

Distribution Function

$$F_{Ch}(\chi^2/\nu) = \frac{1}{2^{\nu/2}\,\Gamma\left(\dfrac{\nu}{2}\right)} \int_0^{\chi^2} e^{-\frac{q}{2}}\, q^{\left(\frac{\nu}{2}-1\right)}\, dq$$

Expected Value

$$E(\chi^2) = v$$

Variance

$$Var(\chi^2) = 2v$$

Example:

A bus line is served by a bus once a day. The variances of the arrival times x_i with $i = 1, ...n$ are normally distributed with an expected value of $E(x_i) = 0$ hours and a variance of $Var(x_i) = 1$ hour^2 ($\sigma = 1$ hour). The (total) costs in \$ caused by an early or late arrival of the bus (measured in hours) are calculated by the following total cost function:

$$K(x) = 20 \sum_{i=1}^{n} x_i^2 \quad \text{with} \quad i = 1, ..., n \ \text{ rides}$$

The daily varying deviations occur independently of each other.

a) What are the total costs to be calculated that will be generated as a result of a **single** time deviation of this bus with a probability of 90%?

b) With what probability can costs of more than $\$100$ per ride generated as a result of a **single** time deviation of this bus be excluded?

c) This bus is now to be observed for one month, i.e. over 30 days. What is the probability that, if this bus continues to make one trip per day, the cost will be no more than $\$800$ per month as a result of time deviations?

d) Now the bus is to be observed for two months, i.e. over 60 days. What is the probability that at most $\$800$ costs will be generated each month as a result of time deviations (measured in hours)?

a) $K(x) = 20 \sum_{i=1}^{n} x_i^2$ with $i = 1, ..., n$ rides

x is normally distributed over $i = 1, ..., n$ with $E(x_i) = 0$ and $Var(x_i) = 1$

That means that $\sum_{i=1}^{n} x_i^2$ as the sum of the squared deviations (sum of squares of distances) from the scheduled arrival time is Chi-squared distributed.

The Chi-squared distribution can be derived from the normal distribution. Given n random variables X_i with $i = 1, ..., n$, that are independently and standardly normally distributed, the Chi-squared distribution is $\chi^2 = \sum_{i=1}^{n} x_i^2$ with n degrees of freedom.

The exact formulas for the Chi-squared distribution are:

Probability Function

$$f_{Ch}(\chi^2/v) = \begin{cases} \dfrac{1}{2^{v/2}\,\Gamma\left(\dfrac{v}{2}\right)}\, e^{-\frac{\chi^2}{2}}\, \left(\chi^2\right)^{\left(\frac{v}{2}-1\right)} & \text{for } \chi^2 \geq 0 \\ 0 & \text{for } \chi^2 < 0 \end{cases}$$

Distribution Function

$$F_{Ch}(\chi^2/v) = \frac{1}{2^{v/2}\,\Gamma\left(\dfrac{v}{2}\right)} \int_0^{\chi^2} e^{-\frac{q}{2}}\, q^{\left(\frac{v}{2}-1\right)}\, dq$$

For the present case, the cost function $K(x) = \sum_{i=1}^{n} x_i^2$ can be represented accordingly by $K(\chi^2) = 20\chi^2$. Investigated is a **single** ride, so that the following can be applied:

$\nu = 1$ at $1 - \alpha = 0.90$

$W(\chi^2 < \chi_o^2) = 0.90$ with $\nu = 1$

$\Rightarrow \chi^2 = 2.706$

See Appendix A, Chi-Squared Distribution.

$\Rightarrow K(2.706) = 20 \cdot 2.706 = \$54.12$

The operator of this bus has to calculate $\$54.12$ in costs **per ride** for possible delays.

b) Again, a **single** ride is considered, i.e. $\nu = 1$.

$K(\chi^2) = 20 \cdot \chi^2 \leq \$100$

$\Rightarrow \chi^2 \leq 5$

See Appendix A, Chi-Squared Distribution.

For $\nu = 1$, a Chi-squared value is first exceeded at a $(1 - \alpha)$-value of 0.975. The probability is approximately $97.5\,\%$.

c) $n = 30$

Since the daily deviations occur independentlyf from each other, the Chi-squared distribution here includes $\nu = 30$ degrees of freedom.

$K(\chi^2) = 20\chi^2 \leq \$800$

$\Rightarrow \chi^2 \leq 40$

For $\nu = 30$, the Chi-squared value of 40 is reached at a $(1 - \alpha)$-value of approximately 0.9000 (see Appendix A, Chi-Squared Distribution). The probability that the time deviations cost at most $\$800$ per month is approximately $90\,\%$. Higher costs can be expected with a probability of about $10\,\%$.

d) If the sample size consists of more than 30 elements or observations, the Chi-squared distribution can be approximated by the standard normal distribution as follows:

$$\chi^2 = \frac{1}{2}(z+\sqrt{2n-1})^2 \qquad \text{for} \quad n > 30$$

$$\Rightarrow z = \sqrt{\chi^2 \cdot 2} - \sqrt{2n-1}$$

monthly ($= 30$ days): $\quad K = 20\chi^2 \leq \$800$

two months ($= 60$ days): $\quad K = 20\chi^2 \leq \$1,600$

$$\Rightarrow \chi^2 \leq 80$$

Since $n = 60 > 30$, an approximation of the Chi-squared distribution can be made by the standard normal distribution with

$$\begin{aligned} z &= \sqrt{\chi^2 \cdot 2} - \sqrt{2n-1} = \\ &= \sqrt{80 \cdot 2} - \sqrt{2 \cdot 60 - 1} = \\ &= 12.6491 - 10.9087 = 1.7404 \end{aligned}$$

For $z = 1.7404$, a $(1-\alpha)$-value of approximately 0.9591 is calculated (see Appendix A, Standard Normal Distribution – Distribution Function).[6]

The probability that the time deviations cost at most $\$800$ per month increases compared to task c) with a now higher sample size of $n = 60$ to approximately $\$95.91$. Higher costs can be expected with a probability of about $4.09\,\%$.

Student's t-Distribution[7]

Probability Function

$$f_S(t/v) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)} \cdot \frac{1}{\left(1+\frac{t^2}{v}\right)^{(v+1)/2}} \qquad -\infty < t < +\infty$$

[6] In Appendix A, Standard Normal Distribution - Distribution Function, a $(1-\alpha)$-value of 0.9591 is calculated for $z = 1.740$ and a $(1-\alpha)$-value of 0.9592 is calculated for $z = 1.741$.

[7] Developed by William Sealy Gosset (1876 - 1937). Gosset was an English statistician.

Distribution Function

$$F_S(t/v) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)} \cdot \int_{-\infty}^{t} \frac{dq}{\left(1+\frac{q^2}{v}\right)^{(v+1)/2}}$$

Expected Value

$$E(T) = 0 \qquad \text{for } v > 1$$

Variance

$$Var(T) = \frac{v}{v-2} \qquad \text{for } v > 2$$

The Student's t-distribution is relevant when the variance needed to standardize the normal distribution is unknown. The sample variance s^2 is used for this case.

If $n > 30$, the Student's t-distribution can be approximated by the standardized normal distribution. The larger the sample size (= number of degrees of freedom) is selected, the closer the Student's t-distribution approaches the (standardized) normal distribution until it becomes congruent with the (standardized) normal distribution at $v = \infty$.

Example:

A healthy kidney excretes about $60-70\,\%$ of the phosphate ingested with food. To evaluate the functioning of a patient's kidneys, six measurements are taken at different times, which indicate the concentrations of phosphate in the blood, measured in mg/dl (milligrams/deciliter):

t_i	1	2	3	4	5	6
Phosphate in mg/dl	5.4	4.8	5.7	6.4	4.5	5.0

Both the mean μ and the variance σ^2 of the population are unknown, so to estimate these two parameters a sample (here of six measurements) is used and the Student's t-distribution, which, like the standardized normal distribution, is symmetric.

a) It is looked for a symmetrical confidence interval, which shows the distribution of phosphate concentrations within the blood of the patient analyzed above at a significance level (= probability of error) of $1\,\%$.

Accordingly, it is searched for a $99\,\%$ confidence interval of the following structure:

$$\bar{x} - t_{1-\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{1-\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$$

or $\quad (\bar{x} - t\frac{s}{\sqrt{n}} \;\; ; \;\; \bar{x} + t\frac{s}{\sqrt{n}})$

with $\quad \bar{x} = \frac{1}{6}(5.4+4.8+5.7+6.4+4.5+5.0) =$

$$= 5.3$$

$$s^2 = \frac{1}{6}\big[(5.4-5.3)^2 + (4.8-5.3)^2 + (5.7-5.3)^2 +$$

$$+ (6.4-5.3)^2 + (4.5-5.3)^2 + (5.0-5.3)^2\big] =$$

$$= \frac{1}{6}(0.01+0.25+0.16+1.21+0.64+0.09) =$$

$$= \frac{1}{6} \cdot 2.36 = 0.3933\,(mg/dl)^2$$

$$\nu = n - 1 = 5$$

$$\Rightarrow \bar{x} - t_{1-\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{1-\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$$

or $\quad (5.3 - t \cdot \frac{\sqrt{0.3933}}{\sqrt{6}} \; ; \; 5.3 + t \cdot \frac{\sqrt{0.3933}}{\sqrt{6}}) \; mg$

with $\quad t = 4.032$

See Appendix A, Student's t-Distribution – Two-sided, Symmetric Confidence Intervals.

$$\Rightarrow (5.3 - 4.032 \cdot \frac{\sqrt{0.3933}}{\sqrt{6}} \; ; \; 5.3 + 4.032 \cdot \frac{\sqrt{0.3933}}{\sqrt{6}}) =$$

$$= \; (4.268 \; mg/dl \; ; \; 6.332 \; mg/dl)$$

With a probability of $99\,\%$, the true average phosphate value μ lies between $4.268\; mg/dl$ and $6.332\; mg/dl$. With a (residual) probability of $1\,\%$, the true average phosphate value μ may also lie outside (below or above) this confidence interval.

b) If the sample size increases to $n > 30$, the Student's t-distribution can be approximated by the standardized normal distribution.

For example, if $n = 100$ with unchanged parameters, i.e., with $\bar{x} = 5.3$ and $s^2 = 0.3933$, then the following applies:

$$\bar{x} - z_{1-\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{1-\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$$

or $\quad (5.3 - z \cdot \frac{\sqrt{0.3933}}{\sqrt{6}} \; ; \; 5.3 + z \cdot \frac{\sqrt{0.3933}}{\sqrt{6}}) \; mg$

with $\quad z = 2.58$

See Appendix A, Standard Normal Distribution –

One-sided Confidence Intervals with $\frac{0.99}{2} = 0.495$.

$$\Rightarrow (5.3 - 2.58 \cdot \frac{\sqrt{0.3933}}{\sqrt{6}} \;\; ; \;\; 5.3 + 2.58 \cdot \frac{\sqrt{0.3933}}{\sqrt{6}}) =$$

$$= (4.639\ mg/dl \;\; ; \;\; 5.961\ mg/dl)$$

If unchanged parameters apply, i.e. $\bar{x} = 5.3$ and $s^2 = 0.3933$, with a now significantly higher sample size of $n = 100$, the confidence interval looked for is shortened compared with task a), so that it can now be assumed with a probability of $99\,\%$ that the true average phosphate value μ of the examined patient lies between $4.639\,mg/dl$ and $5.961\,mg/dl$. With a (residual) probability of $1\,\%$, the true average phosphate value μ can also lie outside (below or above) this confidence interval.

c) Searched is the probability that under the conditions of task a), i.e., at $\bar{x} = 5.3$, $s^2 = 0.3933$ and $\nu = 6 - 1 = 5$, the mean μ lies below $5.68\,mg/dl$?

$$\Rightarrow P(\mu < 5.68) = ?$$

$$\Rightarrow 5.68 = 5.3 + t \cdot \frac{\sqrt{0.3933}}{\sqrt{6}}$$

$$\Leftrightarrow t = \frac{(5.68 - 5.3) \cdot \sqrt{6}}{\sqrt{0.3933}}$$

$$\Leftrightarrow t = 1.4842 \qquad \text{with} \quad \nu = 5$$

See Appendix A, Student's t-Distribution – Distribution Function.

For $\nu = 5$, a t-value of 1.476 is given at a probability of 0.900.

$1.4842 > 1.476$, so it can be assumed that the probability looked for is at least $90\,\%$.

d) Searched is the probability that under the conditions of task a), i.e., at $\bar{x} = 5.3$, $s^2 = 0.3933$ and $\nu = 6-1 = 5$, the mean μ lies below $5.68\ mg/dl$ and above $5.37\ mg/dl$?

$$P(\mu < 5.68) \approx 90 \quad \text{according to task c)}$$

$$\Rightarrow P(\mu < 5.37) = \ ?$$

$$\Rightarrow 5.37 = 5.3 + t \cdot \frac{\sqrt{0.3933}}{\sqrt{6}}$$

$$\Leftrightarrow t = \frac{(5.37-5.3)\cdot\sqrt{6}}{\sqrt{0.3933}} = 0.2734 \qquad \text{with} \qquad \nu = 5$$

See Appendix A, Student's t-Distribution – Distribution Function.

For $\nu = 5$, a t-value of 0.267 is given at a probability of 0.6000.

$0.2734 > 0.267$, so it can be assumed that the partial probability looked for is slightly more than $60\,\%$.

The probability looked for between the two values given, $P(5.37 < \mu < 5.68)$, should therefore be somewhat less than $0.90 - 0.60 = 0.30$, i.e., $< 30\,\%$.

e) Searched is the probability that under the conditions of task a), i.e., at $\bar{x} = 5.3$, $s^2 = 0.3933$ and $\nu = 6-1 = 5$, the mean μ lies below $5.68\ mg/dl$ and above $5.16\ mg/dl$?

$$P(\mu < 5.68) \approx 90 \quad \text{according to task c)}$$

$$\Rightarrow P(\mu < 5.16) = \ ?$$

$$\Rightarrow 5.16 = 5.3 - t \cdot \frac{\sqrt{0.3933}}{\sqrt{6}}$$

Note:

There is now **minus** $t \cdot \frac{s}{\sqrt{n}}$ to calculate because $5.16 < 5.3$, i.e., the given value of $5.16\,mg/dl$ is to the left of μ, i.e., in the left half of the confidence interval.

$$\Leftrightarrow -t = \frac{(5.16-5.3)\cdot\sqrt{6}}{\sqrt{0.3933}}$$

$$\Leftrightarrow -t = -0.5468$$

$$\Leftrightarrow t = 0.5468$$

See Appendix A, Student's t-Distribution – Distribution Function.

For $\nu = 5$ and $t = 0.559$, a probability of 0.7000 is obtained.

Since $5.16 < 5.3$, i.e., it is left of the (symmetric) center, i.e., left of μ ($\bar{x} = 5.3$), the probability is not $70\,\%$, but $30\,\%$ for $t = 0.559$.

The calculated t-value is 0.5468 and is slightly smaller than the table value 0.559 from Appendix A, i.e., the probability looked for is slightly larger than $30\,\%$.

The probability $P(5.16 < \mu < 5.68)$ is calculated approximately from the difference of the two individual probabilities.

$P(\mu < 5.68) \approx 90$ according to task c)

$P(\mu < 5.16) \approx 30$ see above

Thus, the sought probability that the mean value of the concentration of phosphate in blood is above $5.16 mg/dl$ and below $5.68 mg/dl$, $P(5.16 < \mu < 5.68)$, is about 60%. In praxi it would be advisable to increase the sample size n (at least to the extent of $n > 30$), so that also in the present case the Student's t-distribution could be approximated by the standardized normal distribution (as in task b)).

F-Distribution[8]

Probability Function

$$f_F(f/v_1;v_2) = \begin{cases} \dfrac{\Gamma\left(\dfrac{v_1+v_2}{2}\right)}{\Gamma\left(\dfrac{v_1}{2}\right)\Gamma\left(\dfrac{v_2}{2}\right)} \quad \dfrac{\left(\dfrac{v_1}{v_2}\right)^{\frac{v_1}{2}} f^{\frac{v_1}{2}-1}}{\left(1+\dfrac{v_1}{v_2}f\right)^{\frac{v_1+v_2}{2}}} & \text{for } f > 0 \\ 0 & \text{for } f \leq 0 \end{cases}$$

Distribution Function

$$F_F(f/v_1;v_2) = \begin{cases} \dfrac{\Gamma\left(\dfrac{v_1+v_2}{2}\right)}{\Gamma\left(\dfrac{v_1}{2}\right)\Gamma\left(\dfrac{v_2}{2}\right)} \left(\dfrac{v_1}{v_2}\right)^{\frac{v_1}{2}} \displaystyle\int_0^f \dfrac{q^{\frac{v_1}{2}-1}}{\left(1+\dfrac{v_1}{v_2}q\right)^{\frac{v_1+v_2}{2}}}\, dq & \text{for } f > 0 \\ 0 & \text{for } f \leq 0 \end{cases}$$

Expected Value

$$E(F) = \frac{v_2}{v_2-2} \qquad \text{for} \quad v_2 > 2$$

Variance

$$Var(F) = \frac{2\,{v_2}^2\,(v_1+v_2-2)}{v_1(v_2-2)^2\,(v_2-4)} \qquad \text{for} \quad v_2 > 4$$

[8] Ronald Aylmer Fisher (1890 - 1962) was an English statistician.

The F-distribution was constructed for testing purposes. It describes the quotient of two Chi-squared distributed random variables, V_1 and V_2, each divided by the number of their degrees of freedom:

$$F = \frac{\frac{V_1}{\nu_1}}{\frac{V_2}{\nu_2}} \qquad \text{with} \qquad \begin{aligned} \nu_1 &= n_1 - 1 \\ \nu_2 &= n_2 - 1 \end{aligned}$$

$$F = \frac{\frac{(n_1 - 1) \cdot S_1^2}{(n_1 - 1) \cdot \sigma_1^2}}{\frac{(n_2 - 1) \cdot S_2^2}{(n_2 - 1) \cdot \sigma_2^2}} = \frac{S_1^2 / \sigma_1^2}{S_2^2 / \sigma_2^2}$$

The F-distribution is mainly used to compare the variances of two samples from normally distributed populations and for variance analysis to compare sample mean values.

For n independent standard normally distributed random variables $X_1, ..., X_{n_1}$ and m independent standard normally distributed random variables $Y_1, ..., Y_{n_2}$, the two Chi-squared distributed random variables, V_1 and V_2, can be formed from the respective sums of squares:

$$V_1 = \sum_{i=1}^{n_1} X_i^2 \qquad \text{with} \qquad i = 1, ..., n_1$$

$$V_2 = \sum_{j=1}^{n_2} Y_j^2 \qquad \text{with} \qquad j = 1, ..., n_2$$

<u>Example</u>: Testing for equality of two variances

Two samples (of different sizes) are drawn from two normally distributed populations. The aim is to test whether both populations differ significantly in their variances.

The aim is to test statistically whether the price fluctuations of airline tickets for a specific route in travel agencies are significantly higher than the price fluctuations of online offers on the Internet.

Two contemporary samples are used for this purpose:

Travel agency offers i	1	2	3	4	5
Prices x_i in \$	98.90	109.50	102.90	108.00	96.00

with $$\bar{x} = \frac{1}{5}(98.90 + 109.50 + 102.90 + 108.00 + 96.00) = \$103.06$$

$$s_1^2 = \frac{1}{5}\big[(98.90 - 103.06)^2 + (109.50 - 103.06)^2 +$$

$$+ (102.90 - 103.06)^2 + (108.00 - 103.06)^2 +$$

$$+ (96.00 - 103.06)^2\big] = \$^2\, 26.6104$$

Internet offers i	1	2	3	4
Prices y_j in \$	104.50	99.90	106.10	110.50

with $$\bar{y} = \frac{1}{4}(104.50 + 99.90 + 106.10 + 110.50) = \$105.25$$

$$s_2^2 = \frac{1}{4}\big[(104.50 - 105.25)^2 + (99.90 - 105.25)^2 +$$

$$+ (106.10 - 105.25)^2 + (110.50 - 105.25)^2\big] =$$

$$= \frac{1}{4}(0.5625 + 28.6225 + 0.7225 + 27.5625) = \$^2\, 14.3675$$

It is to be assumed that both distributions are standard normally distributed. The test is to be conducted at a significance level of $\alpha = 0.05$ (probability of error).

This is a so-called **parameter test**. Cf. chapter 3.6.1.

(1) null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$

(2) alternative hypothesis $H_A : \sigma_1^2 > \sigma_2^2$

with $n_1 = 5 \Rightarrow \quad \nu = 5 - 1 = 4$

$n_2 = 4 \Rightarrow \quad \nu = 4 - 1 = 3$

Test distribution is the F-distribution.

Calculation of the **test size**:

$$F = \frac{s_1^2 / \nu_1}{s_2^2 / \nu_2} = \frac{26.6104 / 4}{14.3675 / 3} = 1.3891$$

This F-value is to be compared with the so-called **critical value** of the F-distribution for the selected significance level of $\alpha = 0.05$.

$\nu_1 = 4$ and $\nu_2 = 3$ at $\alpha = 0.05$

See Appendix A, F-distribution with $\alpha = 0.05$.

$$\Rightarrow F_{critical} = 9.12$$

Since $F = 1.3891$ is smaller than $F_{critical} = 9.12$, the null hypothesis must be rejected.

The variance of the first distribution (price fluctuations in travel agencies) is not significantly higher than the variance of the second distribution (price fluctuations of online offers), i.e., at a significance level of $\alpha = 0.05$, the observed price fluctuations suggest that they are independent of the type of distribution. Price fluctuations occur regardless of whether these airline tickets are offered online on the Internet or terrestrially in travel agencies.

Notes:

- Reciprocal symmetry applies to the F-distribution:

$$F_{(1-\alpha)}(\nu_{V_1};\nu_{V_2}) = \frac{1}{F_{(\alpha)}(\nu_{V_2};\nu_{V_1})}$$

- If $n_1 > 30$ and $n_2 > 30$, the F-distribution can be approximated by the standardized normal distribution.

3.4 Statistical Estimation Methods (Confidence Intervals)

3.4.1 Confidence Interval for the Arithmetic Mean of the Population μ

(a) for samples of $n \geq 30$ and known variance $\sigma_{\bar{x}}^2$

with $\bar{x} =$ arithmetic mean of the sample

Confidence interval: $\left[\bar{x} - z_{(1-\frac{\alpha}{2})} \cdot \sigma_{\bar{x}}\ ;\ \bar{x} + z_{(1-\frac{\alpha}{2})} \cdot \sigma_{\bar{x}}\right]$

or $\bar{x} - z_{(1-\frac{\alpha}{2})} \cdot \sigma_{\bar{x}} \leq \mu \leq \bar{x} + z_{(1-\frac{\alpha}{2})} \cdot \sigma_{\bar{x}}$

with $z_{(1-\frac{\alpha}{2})} =$ value of the distribution function of the standard normal distribution at $1 - \frac{\alpha}{2}$

in case of normally distributed population.

The standard error $\sigma_{\bar{x}}$ is calculated with the known variance σ :

$\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$ when sampling with replacement (w/ rep.) or

$\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}} \sqrt{\dfrac{N-n}{N}}$ when sampling without replacement (w/o rep.)

If $\dfrac{n}{N} < 0.05$, $\sqrt{\dfrac{N-n}{N}}$ can be omitted.

(b) for samples of $n < 30$ and unknown variance $\sigma_{\bar{x}}^2$

with $\bar{x} =$ arithmetic mean of the sample

Confidence interval: $\left[\bar{x} - t_{(1-\frac{\alpha}{2};\nu)} \cdot \hat{\sigma}_{\bar{x}}\ ;\ \bar{x} + t_{(1-\frac{\alpha}{2};\nu)} \cdot \hat{\sigma}_{\bar{x}}\right]$

or $\bar{x} - t_{(1-\frac{\alpha}{2};\nu)} \cdot \hat{\sigma}_{\bar{x}} \leq \mu \leq \bar{x} + t_{(1-\frac{\alpha}{2};\nu)} \cdot \hat{\sigma}_{\bar{x}}$

with $t_{(1-\frac{\alpha}{2};\nu)} =$ value of the distribution function of the Student's t-distribution at $1 - \frac{\alpha}{2}$ and $\nu = n - 1$

in case of normally distributed population.

The standard error $\hat{\sigma}_{\bar{x}}$ is calculated with the known variance s:

$\hat{\sigma}_{\bar{x}} = \dfrac{s}{\sqrt{n}}$ sampling with replacement (w/ rep.) or

$\hat{\sigma}_{\bar{x}} = \dfrac{s}{\sqrt{n}} \sqrt{\dfrac{N-n}{N}}$ when sampling without replacement (w/o rep.)

If $\dfrac{n}{N} < 0.05$, $\sqrt{\dfrac{N-n}{N}}$ can be omitted.

Example 1:

To estimate the average monthly expenditures on rent, μ, of a total of 3,347 students in a given faculty, a sample of over $n = 100$ students is surveyed on this in the form of sampling without replacement (w/o rep.). It can be assumed that the monthly expenditures for rent are distributed approximately normally within the population for all 3,347 students, i.e., that they are normally distributed.

The confidence level of a confidence interval to be estimated is to cover $95\,\%$, i.e. with a probability of error of $5\,\%$ the parameter μ to be estimated can also lie outside the resulting confidence interval.

The survey of $n = 100$ students results in an arithmetic mean of $\bar{x} = \$356$ with a standard deviation of $\sigma_{\bar{x}} = \$34.17$.

Since $n = 100 > 30$, the following applies:

$$\bar{x} - z_{(1-\frac{\alpha}{2})} \cdot \sigma_{\bar{x}} \leq \mu \leq \bar{x} + z_{(1-\frac{\alpha}{2})} \cdot \sigma_{\bar{x}}$$

A confidence level of $95\,\%$ results in $(1 - \alpha) = 0.95$ and $\alpha = 0.05$, respectively. This results in a so-called quantile of
$(1 - \frac{\alpha}{2}) = (1 - 0.025) = 0.975$.

$$\Rightarrow z_{0.975} = 1.960$$

See Appendix A, Standard Normal Distribution – Distribution Function.

$\Rightarrow$ confidence interval

$$356 - 1.96 \cdot 34.17 \leq \mu \leq 356 + 1.96 \cdot 34.17$$
$$289.03 \leq \mu \leq 422.97$$

Since $\dfrac{n}{N} = \dfrac{100}{3.347} = 0.0299 < 0.05$, the so-called correction factor for finite totals $\sqrt{\dfrac{N-n}{N}}$ can be omitted for the standard deviation.

At a confidence interval of 0.95, i.e., of $95\,\%$, it can be assumed that the average expenditure on rent of all 3,347 students will be between $\$289.03$ and $\$422.97$. The risk that the estimated parameter μ could fall outside this interval is $5\,\%$.

Example 2:

A machine fills 250g packs of coffee. From past quality tests it can be assumed that the filling weight is approximately normally distributed. From a daily production, a sample is taken in the form of sampling with replacement (w/ rep.) in the amount of $n = 6$, which shows the following weights, in g:

$$245, 256, 251, 244, 252, 250.$$

Looked for is a $99\,\%$ confidence interval for the average filling weight of all coffee packages of this type filled on that day.

$$\bar{x} = \frac{1}{6}(245 + 256 + 251 + 244 + 252 + 250) =$$

$$= \frac{1,498}{6} = 249.67g$$

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}) =$$

$$= \frac{1}{6} \cdot \Big[(245 - 249.67)^2 + (256 - 249.67)^2 +$$

$$+ (251 - 249.67)^2 + (244 - 249.67)^2 +$$

$$+ (252 - 249.67)^2 + (250 - 249.67)^2\Big] =$$

$$= \frac{1}{6} \cdot 101.3334 = 16.8889g^2$$

$$\hat{\sigma}_{\bar{x}} = \frac{s}{\sqrt{n}} = \frac{\sqrt{16.8889}}{\sqrt{6}} = 1.6777g$$

At a confidence level of $99\,\%$, i.e., a significance level of $\alpha = 0.01$, and a degree of freedom of $\nu = n - 1 = 5$

$$\Rightarrow t_{(0.99\,;5)} = 4.032$$

See Appendix A, Student's t-Distribution – Two-sided, Symmetric Confidence Intervals.

$\Rightarrow$ confidence interval

$$249.67 - 4.032 \cdot 1.6777 \leq \mu \leq 249.67 + 4.032 \cdot 1.6777$$
$$242.91 \leq \mu \leq 256.43$$

At a confidence interval of 0.99, i.e. of $99\,\%$, it can be assumed that the average filling weight of the daily production observed is between $242.91g$ and $256.43g$. The risk that the estimated parameter μ could lie outside this interval is $1\,\%$.

3.4.2 Confidence Interval for the Variance of the Population σ^2

Confidence interval: $$\left[\frac{(n-1)s^2}{\chi^2_{(1-\frac{\alpha}{2}\,;\,n-1)}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{(\frac{\alpha}{2}\,;\,n-1)}}\right]$$

or $$\frac{(n-1)s^2}{\chi^2_{(1-\frac{\alpha}{2}\,;\,n-1)}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{(\frac{\alpha}{2}\,;\,n-1)}}$$

with $s^2 =$ variance of the sample

$n =$ sample size

$\chi^2 =$ Chi-squared distribution with $\nu = n - 1$

sampling without replacement (w/o rep.) $=$ sampling with replacement (w/ rep.), i.e. there is no case distinction for normally distributed population

Example:

In a randomly selected sample of a group of $n = 24$ people, the average age is $\bar{x} = 44.2$ years with a standard deviation of $s = 5.6$ years. Let the ages of all individuals, i.e. within the population, be normally distributed.

Searched is a $98\,\%$ confidence interval for the unknown variance σ^2 of all people, i.e. of the population.

$$(1 - \alpha) = 0.98 \quad \text{or} \quad \alpha = 0.02$$

$$\Rightarrow \chi^2_{(1-\frac{\alpha}{2}\,;\,n-1)} = \chi^2_{(0.99\,;\,23)} = 41.638$$

$$\chi^2_{(\frac{\alpha}{2}\,;\,n-1)} = \chi^2_{(0.01\,;\,23)} = 10.196$$

See Appendix A, Chi-Squared Distribution – Distribution Function.

$$\frac{(n-1)s^2}{\chi^2_{(0.99\,;\,23)}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{(0.01\,;\,23)}}$$

with $n - 1 = 24 - 1 = 23$

$s^2 = 5.6^2 = 31.36$ years^2

$$\frac{23 \cdot 31.36}{41.638} \le \sigma^2 \le \frac{23 \cdot 31.36}{10.196}$$

$$17.3226 \le \sigma^2 \le 70.7415$$

$$\sqrt{17.3226} \le \sigma \le \sqrt{70.7415}$$

$$4.162 \le \sigma \le 8.411$$

At a confidence interval of 0.98, i.e. $98\,\%$, the standard deviation in the population can be expected to be between 4.162 and 8.411 years. The risk that the estimated parameter σ or σ^2 might lie outside this interval is $2\,\%$.

3.4.3 Confidence Interval for the Share Value in the Population θ

The share value p within a sample drawn from a dichotomous population is normally distributed with the expected value $E(P) = \theta$ and the variance $Var(P) = \sigma_p^2$ for a sufficiently large sample size n, with at least $n \geq 30$.

Confidence interval:
$$\left[p - z_{(1-\frac{\alpha}{2})} \cdot \hat{\sigma}_p \,;\, p + z_{(1-\frac{\alpha}{2})} \cdot \hat{\sigma}_p\right]$$

or
$$p - z_{(1-\frac{\alpha}{2})} \cdot \hat{\sigma}_p \leq \theta \leq p + z_{(1-\frac{\alpha}{2})} \cdot \hat{\sigma}_p$$

with $p =$ share value of the sample

$z_{(1-\frac{\alpha}{2})} =$ value of the distribution function of the standard normal distribution at $1 - \frac{\alpha}{2}$ for normally distributed population

The (estimated) standard error $\hat{\sigma}_p$ is calculated as follows:

sampling without replacement

$$\hat{\sigma}_p = \sqrt{\frac{p(1-p)}{n-1}}\sqrt{\frac{N-n}{N}}$$

sampling with replacement

$$\hat{\sigma}_p = \sqrt{\frac{p(1-p)}{n-1}}$$

If $\frac{n}{N} < 0.05$, $\sqrt{\frac{N-n}{N}}$ can be omitted.

Example:

For a given geography, e.g., a given city, with a population of about 240,000, the share of the population that is vaccinated against COVID-19 should be estimated. For this purpose, a sample of $n = 1,000$ inhabitants is used in the form of sampling without replacement (w/o rep.). The share of those vaccinated out of these 1,000 inhabitants is $p = 0.64$ $(64$ percent).

Searched is a 99% confidence interval for the corresponding share value in the population θ, i.e., for the entire urban population.

$$\hat{\sigma} p = \sqrt{\frac{p\,(1-p)}{n-1}} =$$

$$= \sqrt{\frac{0.64\,(1-0.64)}{1,000-1}} =$$

$$= 0.01519$$

Since $\dfrac{n}{N} = \dfrac{1,000}{240,000} \approx 0.0042 < 0.05$, the so-called correction factor for finite populations $\sqrt{\dfrac{N-n}{N}}$ can be omitted from the standard deviation.

At a confidence interval of 99%, $(1-\alpha) = 0.99$ or $\alpha = 0.01$

This results in a so-called quantile of $(1-\alpha) = (1-0.005) = 0.995$

$$\Rightarrow = z_{0.995} = 2.58$$

See Appendix A, Standard Normal Distribution – Cumulative Distribution Function.

$\Rightarrow$ confidence interval

$$0.64 - 2.58 \cdot 0.01519 \leq \theta \leq 0.64 + 2.58 \cdot 0.01519$$

$$0.6008 \leq \theta \leq 0.6792$$

At a confidence interval of 0.99, i.e. of $99\,\%$, it can be assumed that the investigated vaccination rate of the entire population of the investigated city is between $60.08\,\%$ and $67.92\,\%$. The risk that the estimated parameter θ could lie outside this interval is $1\,\%$.

3.4.4 Confidence Interval for the Difference of the Mean Values of Two Populations μ_1 and μ_2

Samples of size n_1 and n_2 are drawn from two populations. These two independent samples result in the sample means $\bar{x}_1$ and $\bar{x}_2$ as well as their difference $\bar{x}_1 - \bar{x}_2$. If, in addition to the independence of these two samples, sufficiently large sample sizes of $n_1 \geq 30$ and $n_2 \geq 30$ can be guaranteed, it can be assumed that the two random variables $\bar{X}_1$ and $\bar{X}_2$ are normally distributed.

The expected value of the difference of these two random variables $\bar{X}_1$ and $\bar{X}_2$ is then given by

$$E(\bar{X}_1 - \bar{X}_2) = \mu_1 - \mu_2$$

with the variance

$$Var(\bar{X}_1 - \bar{X}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

μ_1 and μ_2 correspond to the arithmetic means and σ_1^2 and σ_2^2 to the variances of the two populations relevant here.

Confidence Interval: $$\left[(\bar{x}_1 - \bar{x}_2) - t\,\hat{\sigma}_D\,;\,(\bar{x}_1 + \bar{x}_2) - t\,\hat{\sigma}_D\right]$$

or $$(\bar{x}_1 - \bar{x}_2) - t\,\hat{\sigma}_D \leq \mu_1 - \mu_2 \leq (\bar{x}_1 - \bar{x}_2) + t\,\hat{\sigma}_D$$

with $\bar{x}_1 - \bar{x}_2 =$ difference of the means of the two samples

$t =$ Student's distribution with $\nu = \dfrac{\left[\dfrac{{s_1}^2}{n_1} + \dfrac{{s_2}^2}{n_2}\right]^2}{\dfrac{\left[\dfrac{{s_1}^2}{n_1}\right]^2}{n_1 - 1} + \dfrac{\left[\dfrac{{s_2}^2}{n_2}\right]^2}{n_2 - 1}}$

$$\hat{\sigma}_D = \hat{\sigma}_{\mu_1 - \mu_2} = \sqrt{\frac{{s_1}^2}{n_1} + \frac{{s_2}^2}{n_2}}$$

with ${s_1}^2 =$ variance of the first sample

${s_2}^2 =$ variance of the second sample

sampling without replacement (w/o rep.) $\widehat{=}$ sampling with replacement (w/ rep.), i.e. w/o rep. corresponds to w/ rep.

Example 1:

It is to be clarified whether within a certain occupational group significant differences exist in the remuneration between men and women. For this purpose, two random samples of equal size are drawn:

men: $n_1 = 100$ with $\bar{x}_1 = \$\,42{,}400$
as an annual gross average salary with a
standard deviation of $s_1 = \$\,10{,}400$

women: $n_2 = 100$ mit $\bar{x}_2 = \$\,40{,}100$
as an annual gross average salary with a
standard deviation of $s_2 = \$\,14{,}800$

It is looked for a $95\,\%$ confidence interval for the difference of the average gross salaries $\mu_1 - \mu_2$.

$$\hat{\sigma}_D = \hat{\sigma}_{\mu_1 - \mu_2} = \sqrt{\frac{10{,}400^2}{100} + \frac{14{,}800^2}{100}} = 1{,}808.87$$

Since $n_1, n_2 = 100 > 30$, the standard normal distribution with $z = 1.960$ can be used. At a confidence interval of $95\,\%$, $(1-\alpha) = 0.95$ and $\alpha = 0.05$, respectively. This results in a so-called quartile of

$$(1-\tfrac{\alpha}{2}) = (1-0.025) = 0.975 \;\Rightarrow\; z_{0.975} = 1.960$$

See Appendix A, Standard Normal Distribution – Distribution Function.

Confidence interval:

$$(\bar{x}_1 - \bar{x}_2) - z\hat{\sigma}_D \le \mu_1 - \mu_2 \le (\bar{x}_1 - \bar{x}_2) + z\hat{\sigma}_D$$

with $(\bar{x}_1 - \bar{x}_2) = 42{,}400 - 40{,}100 = \$\,2{,}300$

$$2{,}300 - 1.96 \cdot 1{,}808.47 \le \mu_1 - \mu_2 \le 2{,}300 + 1.96 \cdot 1{,}808.87$$

$$-1{,}245.39 \le \mu_1 - \mu_2 \le 5{,}845.39$$

At a confidence level of 0.95, i.e., of $95\,\%$, the difference in annual gross average salaries between men and women can be assumed to range from $-\$\,1{,}245.39$ to $+\$\,5{,}845.39$, i.e., within the populations being evaluated, men tend to be predominantly paid more than their female counterparts (so-called right skewed distribution) in favor of male employees.

Example 2:

The portfolios offered by two leading producers of sports wagering are to be compared with each other in terms of their addictiveness. will be compared. For this purpose, the addiction potentials of $n_1 = 12$ products of the sports wagering producer A and $n_2 = 15$ products of the sports wagering producer B are measured with the help of the world's leading tool for the measurement and Assessment of addiction risk potentials of gambling products, ASTERIG[9].

[9] Cf. Wikimedia Foundation Inc. (Ed.): https:// en.wikipedia.org/wiki/ASTERIG [2022-03-10]

For the portfolio of producer A, the average score is $\bar{x}_1 = 90$ with a standard deviation of $s_1 = 8$ points. The portfolio of producer B has an average score of $\bar{x}_2 = 78$ points with a standard deviation of $s_2 = 6$ points.

Searched for is a $98\,\%$ confidence interval for the difference of the average score according to ASTERIG $\mu_1 - \mu_2$.

$$\hat{\sigma}_D \;=\; \hat{\sigma}_{\mu_1 - \mu_2} \;=\; \sqrt{\frac{8^2}{12} + \frac{6^2}{15}} \;=\; 2.7809$$

Since $n_1 = 12$ and $n_2 = 15$ are smaller than 30, here the Student's t-distribution must be used with

$$v = \frac{\left[\dfrac{8^2}{12} + \dfrac{6^2}{15}\right]^2}{\dfrac{\left[\dfrac{8^2}{12}\right]^2}{12-1} + \dfrac{\left[\dfrac{6^2}{15}\right]^2}{15-1}} \;=\; 19.953 \approx 20$$

At a confidence interval of $98\,\%$ $(1-\alpha) \;=\; 0.98$ or $\alpha \;=\; 0.02$. This results in a so-called quartile of $(1 - \frac{\alpha}{2}) \;=\; (1 - 0.01) \;=\; 0.99$

$$\Rightarrow\; t_{0.99} \;=\; 2.528 \quad \text{at} \quad v \;=\; 20$$

See Appendix A, Student's t-Distribution – Distribution Function.

Confidence interval:

$$(\bar{x}_1 - \bar{x}_2) - t\hat{\sigma}_D \;\le\; \mu_1 - \mu_2 \;\le\; (\bar{x}_1 - \bar{x}_2) + t\hat{\sigma}_D$$

with $(\bar{x}_1 - \bar{x}_2) = 90 - 78 = \$\,12$

$$12 - 2.528 \cdot 2.7809 \;\le\; \mu_1 - \mu_2 \;\le\; 12 + 2.528 \cdot 2.7809$$

$$4.9699 \;\le\; \mu_1 - \mu_2 \;\le\; 19.0301$$

At a confidence interval of 0.98, i.e. $98\,\%$, it can be assumed that the differences in the average rating points according to ASTERIG between sports betting wagering producer A and sports wagering producer B range approximately from 4.97 to 19.03 points, i.e. it can be assumed with an probability of error of $2\,\%$ that the portfolio of producer A is significantly more addictive than the portfolio of producer B.

3.4.5 Conficence Interval for the Difference of the Share Values of Two Populations θ_1 and θ_2

Samples of n_1 and n_2 are drawn from two populations. These two independent samples result in the share values p_1 and p_2 as well as their difference $p_1 - p_2$. If, in addition to the independence of these two samples, sufficiently large sample sizes of $n_1\, p_1\,(1-p_1) \geq 9$ and $n_2\, p_2\,(1-p_2) \geq 9$ can be ensured, it can be assumed that the two random variables P_1 and P_2 are normally distributed.

If the conditions $\frac{n_1}{N_1} < 0.05$ and $\frac{n_2}{N_2} < 0.05$ do also apply, the so-called correction factor $\sqrt{\frac{N-n}{N}}$ can be omitted.

The expected value of the difference of these two random variables P_1 and P_2 is then given by

$$E\,(P_1 - P_2) \;=\; \theta_1 - \theta_2$$

with the variance

$$Var\,(P_1 - P_2) \;=\; \frac{\theta_1(1-\theta_1)}{n_1} + \frac{\theta_2(1-\theta_2)}{n_2}$$

θ_1 and θ_2 correspond to the share values within the two here relevant populations.

Confidence interval: $\left[(p_1 - p_2) - z\hat{\sigma}_D \; ; \; (p_1 - p_2) + z\hat{\sigma}_D\right]$

or $(p_1 - p_2) - z\hat{\sigma}_D \leq \theta_1 - \theta_2 \leq (p_1 - p_2) + z\hat{\sigma}_D$

with $p_1 - p_2 =$ difference of the share values within the two samples

$z =$ standard normal distribution

$$\hat{\sigma}_D = \hat{\sigma}_{\theta_1 - \theta_2} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

with $p_1 =$ share value in the first sample

$p_2 =$ share value in the second sample

<u>Example 2</u>:

Within a pandemic, $n_1 = 400$ residents of a city (group 1) and $n_2 = 350$ residents of a rural neighboring region (group 2) are asked about their approval of a vaccination intended by the government. In the city, the proportion of those agreeing is $68\,\%$, whereas in the countryside only $42\,\%$ agree.

$$p_1 = 0.68$$
$$p_2 = 0.42$$

Searched for is a $98\,\%$ confidence interval for the difference in the proportional consents $q_1 - q_2$.

$$\hat{\sigma}_D = \hat{\sigma}_{\theta_1 - \theta_2} = \sqrt{\frac{0.68(1-0.68)}{400} + \frac{0.42(1-0.42)}{350}} = 0.0352$$

Since $\quad n_1 \cdot p_1\,(1-p_1) \;=\; 400 \cdot 0.68\,(1-0.68) \;=\; 87.04 > 9$

and $\quad n_2 \cdot p_2\,(1-p_2) \;=\; 350 \cdot 0.42\,(1-0.42) \;=\; 85.26 > 9,$

the standard normal distribution can be used with $z \;=\; 2.325$

At a confidence interval of $98\,\%\,(1-\alpha) = 0.98$ or $\alpha \;=\; 0.02$. This results in a so-called quartile of $(1-\frac{a}{2}) \;=\; (1-0.01) = 0.99 \;\;\Rightarrow\;\; z_{0.99} = 2.325$

See Appendix A, Standard Normal Distribution – Distribution Function.

$\Rightarrow$ Confidence interval

$$(p_1 - p_2) - z\sigma_D \leq \theta_1 \,-\, \theta_2 \leq (p_1 - p_2) + z\sigma_D$$

with $\quad (p_1 - p_2) \;=\; 0.68 - 0.42 \;=\; 0.26$

$$0.26 - 2.325 \cdot 0.0352 \leq \theta_1 \,-\, \theta_2 \leq 0.26 + 2.325 \cdot 0.0352$$

$$0.1782 \leq \theta_1 \,-\, \theta_2 \leq 0.3418$$

At a confidence interval of 0.98, i.e., $98\,\%$, it can be assumed that the difference in the share values of the two investigated populations is between 0.1782 and 0.3418, i.e., that in the investigated city between $17.82\,\%$ and $34.18\,\%$ more agreement with the planned vaccination prevails than in the investigated rural region.

3.5 Determination of the Required Sample Size

3.5.1 Determination of the Required Sample Size for an Estimation of the Arithmetic Mean μ

Estimation of the required sample size n

$$n = \frac{z^2 \cdot \sigma^2}{(\Delta\mu)^2}$$ sampling w/ rep. or if $\frac{n}{N} < 0.05$

$$n = \frac{z^2 \cdot N \cdot \sigma^2}{(\Delta\mu)^2(N-1) + z^2 \cdot \sigma^2}$$ sampling w/o rep.

$\Delta\mu$ is defined as the absolute error of the arithmetic mean which is a measure of the accuracy of the estimate of the arithmetic mean μ. The range of the confidence interval estimated for μ is determined by the difference between the upper and lower limits of this confidence interval:

$$2\Delta\mu = 2z\sigma_{\overline{x}}$$

or $$\Delta\mu = z \cdot \sigma_{\overline{x}} = z \cdot \frac{\sigma}{\sqrt{n}}$$

when sampling w/ rep. or if $\frac{n}{N} < 0.05$

When sampling w/o rep. the absolute error is $\Delta\mu = z \cdot \frac{\sigma}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}}$

If the variance of the population σ^2 or its standard deviation σ is unknown, in practice known values from previous surveys or expert interviews (Delphi studies) are used or σ^2 or σ respectively is estimated from a preliminary sample, which then generates s^2 or s.

Example:

The average expenditure on a given good within a defined geography where $20,511$ consumers of that good live is to be estimated at a confidence interval of $95\,\%$, i.e., at a significance level of $5\,\%$, and with an accepted absolute error of $\Delta\mu = \$2.5$. A preliminary sample in the size of $n = 100$ consumers results in a standard deviation of $s = \$28$.

$$\frac{n}{N} = \frac{100}{20,511} = 0.0049 < 0.05 \Rightarrow \quad \text{sampling w/ rep.}$$

$$(1-\alpha) = 0.99 \quad \text{or} \quad \alpha = 0.01$$

This results in a so-called quartile of $(1 - \frac{\alpha}{2}) = (1 - 0.005) = 0.995$

$$\Rightarrow z_{0.995} = 2.58$$

See Appendix A, Standard Normal Distribution – Distribution Function.

$$\Rightarrow n = \frac{z^2 \cdot \sigma^2}{(\Delta\mu)^2} = \frac{z^2 \cdot s^2}{(\Delta\mu)^2} =$$

$$= \frac{2.58^2 \cdot 28^2}{(2.5)^2} = 834.98 \approx 835$$

The required sample size equals $n = 835$ consumers.

3.5.2 Determination of the Required Sample Size for an Estimation of the Share Value θ

Estimation of the required sample size n

$$n = \frac{z^2 \cdot \theta(1-\theta)}{(\Delta\theta)^2}$$ sampling w/ rep. or if $\frac{n}{N} < 0.05$

$$n = \frac{z^2 \cdot N \cdot \theta(1-\theta)}{(\Delta\theta)^2(N-1)+z^2 \cdot \theta(1-\theta)}$$ sampling w/o rep.

$\Delta\theta$ is defined as the absolute error of the share value which is a measure of the precision of the estimate of the share value θ. The range of the confidence interval estimated for θ is determined by the difference between the upper and lower limits of this confidence interval:

$$2\Delta\theta = 2z\sigma_p$$

or $$\Delta\theta = z \cdot \sigma_p = z \cdot \sqrt{\frac{\theta\,(1-\theta)}{n}}$$

when sampling w/ rep. or if $\frac{n}{N} < 0.05$

When sampling w/o rep. the absolute error is

$$\Delta\theta = z\sqrt{\frac{\theta\,(1-\theta)}{n}}\ \sqrt{\frac{N-n}{N-1}}$$

If the share value of the population θ is unknown, in practice known values from earlier surveys or expert interviews (Delphi studies) are used or θ is estimated from a preliminary sample, which then generates p. It turns out that $\theta\,(1-\theta)$ can at most have the value 0.25.

Example:

The proportion vaccinated against COVID-19 within a given city of $231,509$ residents at a given time is to be estimated at a confidence interval of $98\,\%$, i.e., at a significance level of $2\,\%$, and with an accepted absolute error of $\Delta\theta = 0.02$. A preliminary sample in the size of $n = 500$ residents of this city results in a proportion vaccinated against COVID-19 of $p = 0.68$.

$$\frac{n}{N} = \frac{500}{231,509} = 0.0022 < 0.05 \;\Rightarrow\; \text{samlpling w/ rep.}$$

$$(1 - \alpha) = 0.98 \quad \text{or} \quad \alpha = 0.02$$

This results in a so-called quartile of $(1 - \frac{\alpha}{2}) = (1 - 0.01) = 0.99$

$$\Rightarrow z_{0.99} = 2.325$$

See Appendix A, Standard Normal Distribution – Distribution Function.

$$\Rightarrow n = \frac{z^2 \cdot \theta(1-\theta)}{(\Delta\theta)^2} = \frac{z^2 \cdot p(1-p)}{(\Delta\theta)^2} =$$

$$= \frac{(2.325)^2 \cdot 0.68 \cdot 0.32}{0.02^2} = 2,940.66 \approx 2,941$$

The required sample size equals $n = 2,941$ people.

3.6 Statistical Testing Methods

Random sampling is used to test certain assumptions (hypotheses) about unknown populations.

3.6.1 Parameter Tests

Testing hypotheses of unknown parameters of one or two populations.

Basic Scheme | Practical Procedure

a. Definition of null hypothesis (H_0) and alternative hypothesis (H_A) as well as significance level (α)

b. Determination of the test statistic

c. Determination of the test distribution

d. Identification of the critical range

e. Calculation of the value of the test statistic

f. Decision and interpretation

3.6.1.1 Arithmetic Mean with Known Variance of the Population | One Sample Test

a. Null hypothesis $\mu = \mu_o$ (σ known)

   Testing a specific mean value μ_o using a single sample (one sample test).

b. Test statistic

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

c. Test distribution

$z =$ standard normal distribution under the condition that the population is normally distributed or $n > 30$.

Example:

The average electric power consumption of stoves using ceramic cooktops of 500 randomly selected 4-person households has historically averaged $360\ kWh$ (kilowatt-hours) per month total, i.e., including the use of an oven, with a standard deviation of $\sigma = 42\ kWh$.

After all cooktops are replaced with induction cooktops, a sample size of $n = 68$ households results in an average electric power consumption of $292\ kWh$ per month.

Based on the result of this sample, can it be concluded that the average electric power consumption in the population has changed assuming that the standard deviation σ has remained unchanged at a significance level of $\alpha = 0.01$?

a. Definition of null and alternative hypothesis

$$H_0: \ \mu = 360\ kWh$$
$$H_A: \ \mu \neq 360\ kWh$$
$$\alpha = 0.01$$

b. Determination of the test statistic

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

c. Determination of the test distribution

$z =$ standard normal distribution as $n = 68 > 30$

d. Identification of the critical range

$\alpha = 0.01$ or $(1 - \alpha) = 0.99$

$\Rightarrow \; z = 2.58$

See Appendix A, Standard Normal Distribution – Two-sided, Symmetric Confidence Intervals.

If the test statistic is between $z = -2.58$ and $z = +2.58$, then the null hypothesis H_0 cannot be rejected, i.e., the following applies:

rejection of H_0 if $z < -2.58$ or
if $z > +2.58$

no rejection of H_0 if $-2.58 \leq z \leq +2.58$

e. Calculation of the value of the test statistic

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{292 - 360}{\frac{42}{\sqrt{68}}} = -13.351$$

f. Decision and interpretation

Since $z = -13.351 < -2.58$, the null hypothesis H_0 must be rejected, i.e., at a significance level of $\alpha = 0.01$ and a sample size of $n = 68$, it can be assumed that the use of induction cooktops results in significantly lower average electric power consumption.

3.6.1.2 Arithmetic Mean with Unknown Variance of the Population | One Sample Test

a. Null hypothesis $\mu = \mu_o$ (σ unknown)

Testing a specific mean value μ_o using a single sample (one sample test).

b. Test statistic

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$$

c. Test distribution

$t =$ Student's t-distribution with $\nu = n - 1$
under the condition that the population is normally distributed

Example:

Within a production of a certain car, the engine hood of this type of car is also produced. These punched hoods must not be too thin, otherwise they would endanger the safety of the passengers inside this car. On the other hand, they must not be too thick, as this would negatively affect the consumption of the car.

In the practice of the investigated plant, the thickness of the hoods produced there is normally distributed with a nominal value (mean value) of $\mu = 0.073\, inches$.

Within a sample of $n = 15$ engine hoods, the arithmetic mean is $\overline{x} = 0.071\,inches$ with a standard deviation of $s = 0.0025\,inches$. Based on a significance level of $\alpha = 0.01$, it is to be tested whether the machine on which the engine hoods are manufactured is working accurately.

a. Definition of null and alternative hypothesis

$$H_0: \ \mu = 0.073\,inches$$
$$H_A: \ \mu \neq 0.073\,inches$$
$$\alpha = 0.01$$

b. Determination of the test statistic

$$t = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}$$

c. Determination of the test distribution

$t =$ Student's t-distribution

with $\nu = n - 1 = 15 - 1 = 14$ degrees of freedom

d. Identification of the critical range

$\alpha = 0.01$ bzw. $(1 - \alpha) = 0.99$ und $\nu = 14$

$\Rightarrow \ t = 2.977$

See Appendix A, Student's t-Distribution – Two-sided, Symmetric Confidence Intervals.

If the test statistic is between $t = -2.977$ and $t = +2.977$, then the null hypothesis H_0 cannot be rejected, i.e., the following applies:

rejection of H_0 if $t < -2.977$ or
if $t > +2.977$

no rejection of H_0 if $-2.977 \leq t \leq +2.977$

e. Calculation of the value of the test statistic

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{0.071 - 0.073}{\frac{0.0025}{\sqrt{15}}} = -3.0984$$

f. Decision and interpretation

Since $t = -3.0984 < -2.977$ the null hypothesis H_0 must be rejected, i.e., at a significance level of $\alpha = 0.01$ and a sample size of $n = 15$, it can be assumed that the investigated machine does not work accurately.

3.6.1.3 Share Value | One Sample Test

a. Null hypothesis $\theta = \theta_0$

Testing a specific share value θ_0 using a single sample (one sample test).

b. Test statistic

$$z = \frac{p - \theta_0}{\sqrt{\frac{\theta_0(1 - \theta_0)}{n}}}$$

c. Test distribution

$z =$ standard normal distribution

under the following condition: $n\,\theta_0\,(1 - \theta_0) \geq 9$

<u>Example</u>:

A manufacturer of kitchen furniture has the fittings for furniture supplied externally. For this purpose, he contractually agrees with this supplier that the share of defective parts, the so-called share of rejects, may not exceed five percent $(\theta \leq 0.05)$.

During a quality control, a random sample without replacement (w/o rep.) is drawn to the extent of $n = 250$, of which 15 are found to be defective. To be examined is whether this may violate the terms of the underlying contract based on a significance level of $\alpha = 0.01$.

a. Definition of null and alternative hypothesis

$$H_0: \;\; \theta \leq \theta_0 = 0.05$$
$$H_A: \;\; \theta > \theta_0 = 0.05$$
$$\alpha = 0.01$$

b. Determination of the test statistic

$$z = \frac{p - \theta_0}{\sqrt{\dfrac{\theta_0(1-\theta_0)}{n}}}$$

c. Determination of the test distribution

$z =$ standard normal distribution
as $\;n\,\theta_0\,(1 - \theta_0) = 250 \cdot 0.05\,(1 - 0.05) = 11.875 > 9$

d. Identification of the critical range

$\alpha = 0.01 \;$ or $\; (1 - \alpha) = 0.99$
$\Rightarrow \; z = 2.325$

See Appendix A, Standard Normal Distribution – Distribution Function, since one-sided question.

If the test statistic is below $z = 2.325$, then the null hypothesis H_0 cannot be rejected, i.e., the following applies:

rejection of H_0 if $z > 2.325$
no rejection of H_0 if $z \leq 2.325$

e. Calculation of the value of the test statistic

$$z = \frac{p - \theta_0}{\sqrt{\frac{\theta_0(1-\theta_0)}{n}}} = \frac{0.06 - 0.05}{\sqrt{\frac{0.05\,(1-0.05)}{250}}} = 0.7255$$

with $p = \frac{15}{250} = 0.06$

f. Decision and interpretation

Since $z = 0.7255 < 2.325$ the null hypothesis H_0 cannot be rejected. Based on a significance level of $\alpha = 0.01$ and a sample size of $n = 250$, the sample result suggests that the contract is fulfilled by the supplier.

3.6.1.4 Variance | One Sample Test

a. Null hypothesis $\sigma^2 = \sigma_0^2$

Testing a specific variance σ_0^2 using a single sample (one sample test).

b. Test statistic

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

c. Test distribution

$\chi^2 =$ Chi-squared distribution with $v = n-1$ under the condition, that the population is normally distributed

Example:

With prescribed storage, the standard deviation of the shelf life of a particular drug in the past was $\sigma = 1.5$ years.

The shelf life of this drug always behaved normally distributed. Using random sampling by drawing without replacement (w/o rep.), it will be tested whether the variance of the shelf life of this drug increases as a result of changes in storage conditions.

Within a test under the altered storage conditions, a sample of $n = 46$ drugs results in a standard deviation of $s = 2.1$ years.

The test of the altered storage conditions is to be based on a significance level of $\alpha = 0.01$.

a. Definition of null and alternative hypothesis

$H_0: \ \sigma^2 = \sigma_0^2 = 2.25\, years^2$

$H_A: \ \sigma^2 > \sigma_0^2 = 2.25\, years^2$

$\alpha = 0.01$

$\underline{Remark}: \sigma = 1.5\, years \ \Rightarrow \ \sigma^2 = 2.25\, years^2$

b. Determination of the test statistic

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

c. Determination of the test distribution

$\chi^2 =$ Chi-squared distribution
with $\nu = n - 1 = 45$ degrees of freedom

d. Identification of the critical range
$\alpha = 0.01$ or $(1-\alpha) = 0.99$ and $\nu = 45$
$\Rightarrow \chi^2 = 69.957$

See Appendix A, Chi-Squared Distribution – Distribution Function, since one-sided question.

If the test statistic is below $\chi^2 = 69.957$, then the null hypothesis H_0 cannot be rejected, i.e., the following applies:

rejection of H_0 if $\chi^2 > 69.957$
no rejection of H_0 if $\chi^2 \leq 69.957$

e. Calculation of the value of the test statistic

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(46-1)\cdot 2.1^2}{1.5^2} = 88.2$$

f. Decision and interpretation

Since $\chi^2 = 88.2 > 69.957$, the null hypothesis H_0 has to be rejected. Based on a significance level of $\alpha = 0.01$ and a sample size of $n = 46$ the sample result suggests that the altered storage conditions increase the variance of the shelf life of the investigated drug significantly.

3.6.1.5 Difference of Two Arithmetic Means with Known Variances of the Population | Two Samples Test

a. Null hypothesis $\mu_1 = \mu_2$ with σ_1, σ_2 known

Testing the equality of the mean values of two populations using two samples (two samples test).

b. Test statistic

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

c. Test distribution

$z =$ standard normal distribution under the conditions that

- the two investigated populations are normally distributed or the sizes of both samples are sufficiently large with $n_1 > 30$ and $n_2 > 30$,
- the two samples are indpendent of each other.

Example:

Two samples are drawn from a production of vendor parts for the manufacture of cars from different machines with which these parts are manufactured.

From machine I $n_1 = 50$ parts are drawn which on average weigh $\bar{x}_1 = 6.3\,lbs$. It is known from regular quality checks that the standard deviation of all units produced on machine I so far is $0.6\,lbs$.

From machine II $n_2 = 55$ parts are drawn, which on average weigh $\bar{x}_2 = 6.6\,lbs$. Here, the standard deviation of all units produced on Machine II so far is $0.5\,lbs$.

Based on a significance level of $\alpha = 0.05$, it is to be tested whether there is a significant difference between the (average) output qualities of both machines.

a. Definition of null and alternative hypothesis

$$H_0: \ \mu_1 = \mu_2$$
$$H_A: \ \mu_1 \neq \mu_2$$
$$\alpha = 0.05$$

b. Determination of the the test statistic

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

c. Determination of the test distribution

$z =$ standard normal distribution
as $n_1 = 50 > 30$ and $n_2 = 55 > 30$

d. Identification of the critical range

$\alpha = 0.05$ or $(1 - \alpha) = 0.95$
$\Rightarrow \ z = 1.96$

See Appendix A, Standard Normal Distribution – Two-sided, Symmetric Confidence Intervals.

If the test statistic is between $z = -1.96$ and $z = +1.96$, then the null hypothesis H_0 cannot be rejected, i.e., the following applies:

rejection of H_0 if $z < -1{,}96$ or
if $z > +1.96$
no rejection of H_0 if $-1.96 \leq z \leq +1.96$

e. Calculation of the value of the test statistic

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{6.3 - 6.6}{\sqrt{\frac{3.6}{50} + \frac{2.5}{55}}} = -0.8754$$

f. Decision and interpretation

Since $z = -0.8754 > -1.96$ and $< +1.96$, the null hypothesis H_0 cannot be rejected. Based on a significance level of $\alpha = 0.05$ and the two sample sizes of $n_1 = 50$ and $n_2 = 55$ it can be assumed, that significantly equal qualities are produced on the two machines.

3.6.1.6 Difference of Two Arithmetic Means with Unknown Variances of the Populations under the Assumption that their Variances are Unequal | Two Samples Test

a. Null hypothesis $\mu_1 = \mu_2$ with σ_1, σ_2 unknown and $\sigma_1 \neq \sigma_2$

Testing the equality of the mean values of two populations using two samples (two samples test).

b. Test statistic

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

c. Test distribution

$z =$ standard normal distribution under the conditions that

- the two investigated populations are normally distributed or the sizes of both samples are sufficiently large with $n_1 > 30$ and $n_2 > 30$,

- the two samples are independent of each other.

Example:

A family business bottles tea into $30\,oz$ bags using two machines.

To guarantee the fill weight marked on the bags, $n_1 = n_2 = 50$ bags are taken from both machines per week and the tea contained in each is weighed.

The bags filled on machine I contained an average of $\bar{x}_1 = 32\,oz$ tea with a standard deviation of $s_1 = 3\,oz$ in the last sample taken.

The bags filled on machine II meanwhile, contained an average of $\bar{x}_2 = 33\,oz$ tea with a standard deviation of $s_2 = 4\,oz$.

Based on a significance level of $\alpha = 0.03$, it is to be tested whether there is a significant difference between the average filling weights of the two machines.

a. Definition of null and alternative hypothesis

$H_0: \ \mu_1 = \mu_2$

$H_A: \ \mu_1 \neq \mu_2$

$\alpha = 0.03$

b. Determiantion of the test statistic

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

c. Determination of the test distribution

$z =$ standard normal distribution
as $n_1 = n_2 = 50 > 30$

d. Identification of the critical range

$\alpha = 0.03$ or $(1 - \alpha) = 0.97$
$\Rightarrow z = 2.17$

See Appendix A, Standard Normal Distribution – Two-sided, Symmetric Confidence Intervals.

If the test statistic is between $z = -2.17$ and $z = +2.17$, then the null hypothesis H_0 cannot be rejected, i.e., the following applies:

rejection of H_0 if $z < -2.17$ or
if $z > +2.17$
no rejection of H_0 if $-2.17 \leq z \leq +2.17$

e. Calculation of the value of the test statistic

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{32 - 33}{\sqrt{\frac{9}{50} + \frac{16}{50}}} = -1,4142$$

f. Decision and interpretation

Since $z = -1.4142 > -2.17$ and $< +2.17$, the null hypothesis H_0 cannot be rejected. Based on a significance level of $\alpha = 0.03$ and the two sample sizes of $n_1 = n_2 = 50$ it can be assumed, that the average fill weights are significantly equal for both machines.

3.6.1.7 Difference of Two Arithmetic Means with Unknown Variances of the Populations under the Assumption that their Variances are Equal | Two Samples Test

a. Null hypothesis $\mu_1 = \mu_2$ with σ_1, σ_2 unknown and $\sigma_1 = \sigma_2$ (homogeneity of variance)

Testing the equality of the mean values of two populations using two samples (two samples test).

b. Test statistic

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s \cdot \sqrt{\dfrac{n_1 + n_2}{n_1 \, n_2}}}$$

with $s = \sqrt{\dfrac{(n_1 - 1)\, s_1^2 + (n_2 - 1)\, s_2^2}{n_1 + n_2 - 2}}$

c. Test distribution

$t =$ Student's t-distribution with $\nu = n_1 + n_2 - 2$ under the conditions that

- the two investigated populations are normally distributed or the sizes of both samples are sufficiently large with $n_1 > 30$ and $n_2 > 30$,

- the two samples are independet of each other.

Example:

Within a milk processing factory, a special energy drink based on milk is bottled to a third of a $gallon$ each on two different machines.

For quality control of the filling weight, $n_1 = n_2 = 51$ bottles are taken from both machines weekly and their contents are measured.

In the past, it was noticed that machine I tends to fill a larger quantity than machine II, which will now be tested based on a significance level of $\alpha = 0.01$.

The two current samples provide the following measurement results:

machine I : $\bar{x}_1 = 0.3352\, gallon$ with a standard deviation of $s_1 = 0.0040\, gallon$

machine II : $\bar{x}_2 = 0.3320\, gallon$ with a standard deviation of $s_2 = 0.0040\, gallon$

a. Definition of null and alternative hypothesis

$$H_0: \ \mu_1 = \mu_2$$
$$H_A: \ \mu_1 > \mu_2$$
$$\alpha = 0.01$$

b. Determination of the test statistic

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s \cdot \sqrt{\dfrac{n_1 + n_2}{n_1 n_2}}}$$

with $s = \sqrt{\dfrac{(n_1 - 1)\, s_1^2 + (n_2 - 1)\, s_2^2}{n_1 + n_2 - 2}}$

c. Determination of the test distribution

$t =$ Student's t-distribution
with $\nu = n_1 + n_2 - 2 = 51 + 51 - 2 = 100$ degrees of freedom

d. Identificaiton of the critical range

$\alpha = 0.01$ or $(1 - \alpha) = 0.99$ and $\nu = 100$
$\Rightarrow t = 2.364$

See Appendix A, Student's t-Distribution – Distribution Function, since one-sided question.

If the test statistic is smaller than $t = -2.364$, then the null hypthesis H_0 cannot be rejected, i.e., the following applies:

rejection of H_0 if $t > 2.364$
no rejection of H_0 if $t \leq 2.364$

e. Calculation of the value of the test statistic

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s \cdot \sqrt{\frac{n_1 + n_2}{n_1 n_2}}} = \frac{0.3352 - 0.3320}{0.004 \cdot \sqrt{\frac{51 + 51}{51 \cdot 51}}} = 4.0398$$

with $$s = \sqrt{\frac{(n_1 - 1)\, s_1^2 + (n_2 - 1)\, s_2^2}{n_1 + n_2 - 2}} =$$

$$= \sqrt{\frac{(51 - 1) \cdot 0.004^2 + (51 - 1) \cdot 0.004^2}{51 + 51 - 2}} = 0.004$$

f. Decision and interpretation

Since $t = 4.0398 > 2.364$, the null hypothesis H_0 can be rejected. Based on a significance level of $\alpha = 0.01$ and the two sample sizes of $n_1 = n_2 = 51$ it can be assumed, that the average fill weight of machine I is significantly higher than that of machine II.

3.6.1.8 Difference of Two Share Values | Two Samples Test

a. Null hypothesis $\theta_1 = \theta_2$

Testing the equality of equal share values of two populations using two samples (two samples test).

b. Test statistic

$$z = \frac{p_1 - p_2}{\sqrt{p(1-p)}\sqrt{\frac{n_1 + n_2}{n_1 n_2}}}$$

with $p = \dfrac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$

c. Test distribution

$z =$ standard normal distribution under the conditions that

- the following applies: $n_1 \theta_1 (1 - \theta_1) \geq 9$ and $n_2 \theta_2 (1 - \theta_2) \geq 9$,

- the two samples are independent of each other.

Example:

A producer of kitchen furniture has the fittings for furniture supplied by two different producers. In order to check the quality, two random samples are drawn from the supplied fittings for furniture without replacement (w/o rep.) in the size of $n_1 = 250$ and $n_2 = 300$.

The sample $n_1 = 250$ drawn from the supplied fittings for furniture of supplier I shows 15 defective parts, which corresponds to a share of rejects of $p_1 = \dfrac{15}{250} = 0.06\ (6\,\%)$.

The sample $n_2 = 300$, drawn from the supplied fittings for furniture of supplier II, shows 17 defective parts, which corresponds to a share of rejects of $p_2 = \dfrac{17}{300} = 0.0567\ (5.67\,\%)$.

To be checked is whether the shares of defective fittings for furniture (shares of rejects) differ significantly between the two suppliers. The test is to be based on a significance level of $\alpha = 0.01$.

a. Definition of null and alternative hypothesis

$$H_0: \ \theta_1 = \theta_2$$
$$H_A: \ \theta_1 \neq \theta_2$$
$$\alpha = 0.01$$

b. Determination of the test statistic

$$z = \frac{p_1 - p_2}{\sqrt{p\,(1-p)}\sqrt{\dfrac{n_1 + n_2}{n_1\,n_2}}}$$

with $\quad p = \dfrac{n_1\,p_1 + n_2\,p_2}{n_1 + n_2}$

c. Determination of the test distribution

$z =$ standard normal distribution

as: $\ n_1\,p_1\,(1 - p_1) = 250 \cdot 0.06 \cdot (1 - 0.06) = 14.1$

which suggests that also $\ n_1\,\theta_1\,(1 - \theta_1) \geq 9\ $ (condition for the population) is fulfilled.

Furthermore: $n_2\, p_2\,(1 - p_2) = 300 \cdot 0.0567 \cdot (1 - 0.0567) = 16.05$ which suggests that also $n_2\, \theta_2\,(1 - \theta_2) \geq 9$ (condition for the population) is fulfilled.

d. Identification of the critical range

$\alpha = 0.01$ or $(1 - \alpha) = 0.99$
$\Rightarrow\ z = 2.58$

See Appendix A, Standard Normal Distribution – Two-sided, Symmetric Confidence Intervals.

If the test statistic is between $z = -2.58$ and $z = +2.58$, then the null hypothesis H_0 cannot be rejected, i.e., the following applies:

rejection of H_0 if $z < -2{,}58$
or if $z > +2{,}58$
no rejection of H_0 if $-2{,}58 \leq z \leq +2{,}58$

e. Calculation of the value of test statistic

$$z = \frac{p_1 - p_2}{\sqrt{p\,(1-p)}\sqrt{\dfrac{n_1 + n_2}{n_1\, n_2}}} = \frac{0.06 - 0.0567}{\sqrt{0.0582\,(1 - 0.0582)}\sqrt{\dfrac{250 + 300}{250 \cdot 300}}} = $$
$$= 0.1646$$

with $p = \dfrac{n_1\, p_1 + n_2\, p_2}{n_1 + n_2} = \dfrac{250 \cdot 0.06 + 300 \cdot 0.0567}{250 + 300} = 0.0582$

f. Decision and interpretation

Since $z = 0.1646 > -2.58$ and $< +2.58$, the null hypothesis H_0 cannot be rejeceted. Based on a significance level of $\alpha = 0.01$ and the two sample sizes of $n_1 = 250$ and $n_2 = 300$ it can be assumed that the shares of defective fittings for furniture (share of rejects) is significantly equal for both suppliers.

3.6.1.9 Quotients of Two Variances | Two Samples Test

a. Null hypothesis $\sigma_1^2 = \sigma_2^2$

   Testing the equality of the variance of two populations
   using <u>two</u> samples (two samples test).

b. Test statistic

   $$F = \frac{s_1^2}{s_2^2}$$

c. Test distribution

   $F = F$-ditribution with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$
   under the conditions that

   - the two investigated populations are normally distributed
     or the sizes of both samples are sufficiently large
     with $n_1 > 30$ and $n_2 > 30$,

   - the two samples are independent of each other.

<u>Example</u>:

To measure a possible correlation between the use of a particular pesticide within agriculture on species extinction, the statistical spread of the lifespan of a particular ant species will be examined as an example

a) within an agriculture A, where this pesticide is not used
   in comparison to
b) an agriculture B where this pesticide is used

Empirical data from past measurements show that in in both investigated populations the life span of these ants is normally distributed in the sense of a Gaussian[10] distribution, i.e., that in the case of additive superposition of many small independent random effects to a total effect, a normal distribution according to Gauss is approximated, insofar as no single effect has a dominant influence on the variance (central limit theorem according to Lindeberg[11] and Lévy[12]).

A sample of $n_1 = 101$ ants from agriculture A results in a standard deviation of $s_1 = 2.6$ years of life, while a sample of $n_2 = 151$ ants from agriculture B results in a standard deviation of $s_2 = 2.1$ years.

Based on a significance level of $\alpha = 0.05$, it is to be investigated whether the dispersion of the lifespan of the investigated ants is larger within agriculture A than in agriculture B.

a. Definition of null and alternative hypothesis

$$H_0: \ \sigma_1^2 = \sigma_2^2$$
$$H_A: \ \sigma_1^2 > \sigma_2^2$$
$$\alpha = 0.05$$

b. Determination of the test statistic

$$F = \frac{s_1^2}{s_2^2}$$

c. Determination of the test distribution

$F = F$-distribution

with $\nu_1 = n_1 - 1 = 101 - 1 = 100$ degrees of freedom and
$\nu_2 = n_2 - 1 = 151 - 1 = 150$ degrees of freedom

[10] Johann Carl Friedrich Gauß (1777-1855) was a German mathematician.
[11] Jarl Waldemar Lindeberg (1876-1932) was a Finnish mathematician.
[12] Paul Pierre Lévy (1886-1971) was a French mathematician.

d. Identification of the critical range

$$\alpha = 0.05 \quad \text{or} \quad (1-\alpha) = 0.95$$
$$\Rightarrow F = 1.34$$

See Appendix A, F-Distribution – Distribution Function with $\alpha = 0.05$.

If the test statistic is below $F = 1.34$, the the null hypothesis H_0 cannot be rejected, i.e., the following applies:

rejection of H_0 if $F > 1.34$
no rejection of H_0 if $F < 1.34$

e. Calculation of the value of the test statistic

$$F = \frac{s_1^2}{s_2^2} = \frac{2.6^2}{2.1^2} = 1.5329$$

f. Decision and interpretation

Since $F = 1.5329 > 1.34$, the null hypothesis H_0 is to be rejected. Based on a significance level of $\alpha = 0.05$ and the two sample sizes of $n_1 = 101$ and $n_2 = 151$ it can be assumed that the dispersion of the life span of the investigated ant species is significantly shortened by the use of this pesticide.

3.6.2 Distribution Tests (Chi-Squared Tests)

Testing hypotheses of unknown distributions of a population.

3.6.2.1 Chi-Squared Goodness of Fit Test

Goodness of fit tests check whether empirical data are (approximately) fitted with a theoretical distribution, i.e. whether the distribution observed on the basis of a sample (approximately) conforms to or contradicts the assumed (unknown) distribution of the population.

Goodness of fit tests check the goodness of fit of an empirical distribution based on a sample to a theoretical (unknown) distribution assumed for the population.

3.6.2.1.1 Chi-Squared Goodness of Fit Test for a Discrete Distribution of the Population

a. Null hypothesis

   Testing a sample that is drawn from a population
   with unknown distribution.

b. Test statistc

$$\chi^2 = \sum_{i=1}^{k} \frac{(h_i^o - h_i^e)^2}{h_i^e}$$

   with

   h_i^o observed absolute frequency of a characteristic in its i^{th} occurrence $(i = 1, ..., k)$

   h_i^e expected absolute frequency of a characteristic in its i^{th} occurrence $(i = 1, ..., k)$

$k =$ number of classes

c. Test distribution

$\chi^2 =$ Chi-squared distribution with $v = k - m - 1$

$k =$ number of classes

$m =$ number of estimated parameters

Condition: $h_i^e \geq 5 \qquad i = 1, ..., k$

Example:

Four elevators exist inside an office building. In order to optimize the control of these elevators, the people who enter this office building and want to use an elevator within an interval of two minutes are recorded.

The discrete distribution of the corresponding measurement in the form of a random sampling by drawing without replacement (w/o rep.) is shown in the following (absolute) frequency distribution:

i	number of people x_i	class widths $x_i^{'}$	number of 2-minute-intervals with x_i relevant people; h_i^o
1	0	0	5
2	1	1	8
3	2	2	12
4	3	3	16
5	4	4	8
6	$5-7$	6	4
7	≥ 8	$-$	0

Based on a significance level of $\alpha = 0.01$ it is to be checked whether the arrivals of people who want to use an elevator are poisson distributed.

a. Definition of null and alternative hypothesis

H_0 : The arrivals of people who want to use an elevator are poisson distributed.

H_A : The arrivals of people who want to use an elevator are not poisson distributed.

$\alpha = 0.01$

b. Determiantion of the test statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(h_i^o - h_i^e)^2}{h_i^e}$$

c. Determination of the test distribution

$\chi^2 =$ Chi-squared distribution

with $\nu = k - m - 1 =$ degrees of freedom

To determine the expected absolute frequencies h_i^e, the parameter μ of the Poisson distribution has to be estimated (see chapter 3.3.1). For this purpose, the arithmetic mean of the sample $\bar{x}$ is appropriate, which forms an unbiased estimate $\hat{\mu}$ for μ:

$$\hat{\mu} = \bar{x} = \frac{\sum_{i=1}^{k} x_i h_i^o}{\sum_{i=1}^{k} h_i^o} = \frac{136}{53} = 2.566$$

with $\sum_{i=1}^{k} x_i h_o = (0 \cdot 5) + (1 \cdot 8) + (2 \cdot 12) + (3 \cdot 16) + (4 \cdot 8) + (6 \cdot 4) =$

$= 136$

At class "$5 - 7$" using the class center $x_i^{'}$ (cf. Chapter 2.2.1).

$$\sum_{i=1}^{k} h_i^o = 5+8+12+16+8+4+0 = 53$$
$$\Rightarrow \hat{\mu} = \bar{x} = \frac{136}{53} = 2.566$$

Using the probability function of the Poisson distribution (see Appendix A), $\hat{\mu} = 2.566 \approx 2.6$ the individual probabilities can be estimated (approximately) (see following table):

i	x_i	x_i'	h_i^o	$f_p(x_i/2,6)$
1	0	0	5	0.0743
2	1	1	8	0.1931
3	2	2	12	0.2510
4	3	3	16	0.2176
5	4	4	8	0.1414
6	$5-7$	6	4	$0.0735+0.0319+0.0118 = = 0.1172$
7	≥ 8	–	0	$0.0038+0.0011+0.0003+ +0.0001 = 0.0053$[13]

Interpreting the $f_p(x_i/\mu)$ values as relative frequencies, multiplication by $n = \sum_{i=1}^{k} h_i^o = 53$ produces the searched expected absolute frequencies h_i^e :

[13] $1-0.0743-0.1931-0.2510-0.2176-0.1414-0.1172 = 0.0054 \approx 0.0053$ (rounding error). See Appendix A, Poisson Distribution – Cumulative Distribution Function. For $\mu = 2.6$ and $x = 7$ the resulting value is $0.9947 \cdot (1-0.9947) = 0.0053$.

i	x_i	h_i^o	$h_i^e = 53 \cdot f_p(x_i/2.6)$
1	0	5	3.9379
2	1	8	10.2343
3	2	12	13.3030
4	3	16	11.5328
5	4	8	7.4942
6	$5-7$	4	6.2116
7	≥ 8	0	0.2809
Σ	–	53	≈ 53[14]

Since the use of the Chi-squared distribution as a test distribution requires that $h_i^e \geq 5$ must be valid for all $1, ..., k$ (see above), the classes 1 and 2 as well as 6 and 7 are grouped together:

i	x_i	h_i^o	$h_i^e = 53$
1	≤ 1	$5+8=13$	$3.9379+10.2343=14.1722$
2	2	12	13.3030
3	3	16	11.5328
4	4	8	7.4942
5	≥ 5	$4+0=4$	$6.2116+0.2809=6.4925$
Σ	–	53	≈ 53[15]

d. Identification of the critical range

$\alpha = 0.01$ or $(1-\alpha) = 0.99$ and $\nu = 5-1-1 = 3$

Remarks:

- By grouping classes together due to the condition $h_i^e \geq 5$ eventually the result is not seven classes, but five classes,

[14] rounding error
[15] rounding error

i.e. $k = 5$.

- $m = 1$, because only the parameter μ has to be estimated due to the Poisson distribution.
- Since $\nu = 3 > 1$, no continuity correction (Yates's correction) is to be applied.

$\Rightarrow \chi^2 = 11.345$

See Appendix A, Chi-Squared Distribution – Distribution Function.

If the test statistic is below $\chi^2 = 11.345$, then the null hypothesis H_0 cannot be rejected, i.e., the following applies:

rejection of H_0 if $\chi^2 > 11.345$
no rejection of H_0 if $\chi^2 \leq 11.345$

e. Calculation of the value of the test statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(h_i^o - h_i^e)^2}{h_i^e} =$$

$$= \frac{(13 - 14.1722)^2}{14.1722} + \frac{(12 - 13.3030)^2}{13.3030} + \frac{(16 - 11.5328)^2}{11.5328} +$$

$$+ \frac{(8 - 7.4942)^2}{7.4942} + \frac{(4 - 6.4925)^2}{6.4925} = 2.9460$$

f. Decision and interpretation

Since $\chi^2 = 2.9460 < 11.345$, the null hypothesis H_0 cannot be rejected. Based on a significance level of $\alpha = 0.01$ and the sample size of $n = 53$ it can be assumed that the people entering the office building who want to use an elevator are poisson distributed.

3.6.2.1.2 Chi-Squared Goodness of Fit Test for a Continous Distribution of the Population

If the feature is continuous or if a discrete feature occurs in a very large number of characteristics, the characteristic values are to be classified in k intervals.

Example:

Based on a significance level of $\alpha = 0.05$, it is to be tested whether the lifespan of an endangered butterfly is normally distributed. For this purpose, $n = 100$ butterflies are observed. The following table shows the test result:

i	lifespan in days, x_i	number of butterflys, h_i^o
1	≤ 14	8
2	> 14 bis ≤ 18	17
3	> 18 bis ≤ 22	25
4	> 22 bis ≤ 26	30
5	> 26 bis ≤ 30	18
6	> 30	2
Σ	–	100

Close observation within this sample of $n = 100$ butterflies indicates an average lifespan of $\bar{x} = 21.48$ days with a standard deviation of $s = 5.26$ days.

a. Definition of null and alternative hypothesis

H_0 : The lifespan of this butterfly is normally distributed.

H_A : The lifespan of this butterfly is not normally distributed.

$\alpha = 0.05$

b. Determination of the test statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(h_i^o - h_i^e)^2}{h_i^e}$$

c. Determiantion of the test distribution

$\chi^2 =$ Chi-squared distribution

with $\nu = k - m - 1 =$ degrees of freedom

The arithmetic mean from the sampling $\bar{x}$ and the variance s^2 serve as unbiased estimates for the two unknown parameters μ and σ^2 of the normal distribution (cf. chapter 3.3.2):

$$\hat{\mu} = \bar{x} = 21.48\,days$$

$$\hat{\sigma}^2 = s^2 = (5.26)^2 = 27.6676\,days^2$$

By using the distribution function of the normal distribution (see Appendix A, Standard Normal Distribution – Cumulative Distribution Function), $\hat{\mu} = 21.48$ and $\hat{\sigma}^2 = 27.6676$ the values of the distribution function of the normal distribution for the upper class limits x_i^o, i.e. $F_n\,(x_i^o\,/\,21.48\,;\,27.6676)$ can be determined.

The following relationship applies to the use of the standard normal distribution (cf. Chapter 3.3.2):

$$F_n\,(x\,/\,\mu\,;\,\sigma^2) = F_N\,(z)$$

with $z = \dfrac{x - \mu}{\sigma}$

or $F_n\,(x_i^o\,/\,\mu\,;\,\sigma^2) = F_N\,(z_i^o)$

with $z_i^o = \dfrac{x_i^o - \mu}{\sigma}$ for $i = 1, ..., k$

i	x_i^o	$z_i^o = \dfrac{x_i^o - 21.48}{5.26}$	$F_N(z_i^o)$
1	14	-1.422	$1 - 0.9225 = 0.0775$
2	18	-0.662	$1 - 0.7460 = 0.2540$
3	22	0.099	0.5394
4	26	0.859	0.8048
5	30	1.620	0.9474
6	∞	∞	1.0000

The expected relative frequencies f_i^e result from the formation of the following differences:

$$f_i^e = F_N(z_i^o) - F_N(z_{i-1}^o)$$
with $F_N(z_{i-1}^o) = F_N(z_0^o) = 0$

The expected absolute frequencies h_i^e are calculated as follows:

$$h_i^e = n \cdot f_i^e$$
with $n =$ sample size

For this example, the following applies:

i	h_i^o	f_i^e	$h_i^e = 100 \cdot f_i^e$
1	8	$0.0775 - 0 = 0.0775$	7.75
2	17	$0.2540 - 0.0775 = 0.1765$	17.65
3	25	$0.5394 - 0.2540 = 0.2854$	28.54
4	30	$0.8048 - 0.5394 = 0.2654$	26.54
5	18	$0.9474 - 0.8048 = 0.1426$	14.26
6	2	$1.0000 - 0.9474 = 0.0526$	5.26
Σ	100	1.0000	100.00

$h_i^e \geq 5$ is fulfilled for all classes i with $i = 1, ..., 6$ so that no classes need to be grouped together.

d. Identification of the critical range

$\alpha = 0.05$ or $(1 - \alpha) = 0.95$ and $\nu = 6 - 2 - 1 = 3$

Remarks:

- $m = 2$ because here, as a result of the (standard) normal distribution, the two parameters μ and σ^2 are estimated.
- Since $\nu = 3 > 1$, no continuity correction (Yates's correction) is to be applied.

$\Rightarrow \chi^2 = 7.815$

See Appendix A, Chi-Squared Distribution – Distribution Function.

If the test statistic is below $\chi^2 = 7.815$, then the null hypothesis H_0 cannot be rejected, i.e., the following applies:

rejection of H_0 if $\chi^2 > 7.815$
no rejection of H_0 if $\chi^2 \leq 7.815$

e. Calculation of the value of the test statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(h_i^o - h_i^e)^2}{h_i^e} =$$

$$= \frac{(8 - 7.75)^2}{7.75} + \frac{(17 - 17.65)^2}{17.65} + \frac{(25 - 28.54)^2}{28.54} +$$

$$+ \frac{(30 - 26.54)^2}{26.54} + \frac{(18 - 14.26)^2}{14.26} + \frac{(2 - 5.26)^2}{5.26} =$$

$$= 3.9235$$

f. Decision and interpretation

Since $\chi^2 = 3.9235 < 7.815$, the null hypothesis H_0 cannot be rejected. Based on a significance level of $\alpha = 0,05$ and a sample size of $n = 100$, it can be assumed that the lifespan of this butterfly is normally distributed.

3.6.2.2 Chi-Squared Independence Test

This test can be used to check whether two characteristics are stochastically independent of each other. It also applies to qualitative (nominally or ordinally scaled) characteristics.

a. Null hypothesis

   Testing two characteristics of two samples drawn from one or from two different populations.

b. Test statistic

$$\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{(h_{ij}^o - h_{ij}^e)^2}{h_{ij}^e}$$

with

h_{ij}^o observed absolute frequency of the combination of two characteristics. The first characteristic in its i^{th} occurrence ($i = 1, ..., r$) and the second characteristic in its j^{th} occurrence ($j = 1, ..., s$).

$$h_{i.}^o = \sum_{j=1}^{s} h_{ij}^o \qquad i = 1, ..., r$$

$$h_{.j}^o = \sum_{i=1}^{r} h_{ij}^o \qquad j = 1, ..., s$$

$$h_{..}^o = \sum_{i=1}^{r} h_{i.}^o = \sum_{j=1}^{s} h_{.j}^o = n \qquad i = 1, ..., r \text{ and } j = 1, ..., s$$

h^e_{ij} expected absolute frequency of the combination of two characteristics. The first characteristic in its i^{th} occurrence ($i = 1, ..., r$) and the second characteristic in its j^{th} occurrence ($j = 1, ..., s$).

$$h^e_{ij} = \frac{h^o_{i.} \cdot h^o_{.j}}{n} \qquad i = 1, ..., r \text{ und } j = 1, ..., s$$

c. Test distribution

χ^2 = Chi-squared distribution with $v = (r-1)(s-1)$
r: number of the i^{th} characteristic value
s: number of the j^{th} characteristic value
Condition: $h^e_{ij} \geq 5 \quad i = 1, ..., r$ and $j = 1, ..., s$

Example:

It is to be tested whether the demand for a certain product is related to gender by the color of the packaging of this product. Both characteristics are nominally scaled. The following observations concerning the buying behavior of $n = 100$ subjects are summarized in the following contingency table[16]:

gender (B) / colour (A)	male (B1)	female (B2)	diverse (B3)	Σ
red (A1)	$h^o_{11} = 10$	$h^o_{12} = 16$	$h^o_{13} = 2$	$h^o_{1.} = 28$
blue (A2)	$h^o_{21} = 22$	$h^o_{22} = 9$	$h^o_{23} = 2$	$h^o_{2.} = 33$
green (A3)	$h^o_{31} = 8$	$h^o_{32} = 14$	$h^o_{33} = 2$	$h^o_{3.} = 24$
yellow (A4)	$h^o_{41} = 6$	$h^o_{42} = 9$	$h^o_{43} = 0$	$h^o_{4.} = 15$
Σ	$h^o_{.1} = 46$	$h^o_{.2} = 48$	$h^o_{.3} = 6$	$h^o_{..} = n = 100$

[16] Contingency tables are tables that show the absolute or relative frequencies of combinations of certain characteristic values.

The significance level of this Chi-squared independence test should be $\alpha = 0.05$.

a. Definition of null and alternative hypothesis

H_0 : The two characteristics "color of the packaging of the investigated product" (A) and "gender of the demander" (B) are independent of each other.

H_A : The two characteristics "color of the packaging of the investigated product" (A) and "gender of the demander" (B) are dependent on each other.

$\alpha = 0.05$

b. Determination of the test statistic

$$\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{(h_{ij}^o - h_{ij}^e)^2}{h_{ij}^e}$$

c. Determination of the test distribution

$\chi^2 =$ Chi-squared distribution

with $\nu = (r-1)(s-1)$ degrees of freedom

To determine the expected absolute frequencies h_i^e, it is known that the **share** of the elements with the characteristic value is

$$A_i \quad \text{with} \quad i = 1, ..., 3 \quad f(A_i) = \frac{h_{i.}^o}{n} \quad \text{with} \quad i = 1, ..., r$$

and the **share** of elements with characteristic value is

$$B_j \quad \text{with} \quad j = 1, ..., 4 \quad f(B_j) = \frac{h_{.j}^o}{n} \quad \text{with} \quad j = 1, ..., s.$$

The relative frequencies $f(A_i)$ and $f(B_j)$ can be interpreted as probabilities, so that if the two investigated characteristics are independent, according to the multiplication theorem for two stochastically independent events A and B the following applies (cf. chapter 3.1.2):

$$W(A \cap B) = W(A) \cdot W(B)$$

or

$$W(A \cap B) = f(A_i \cap B_j) =$$
$$= f(A_i) \cdot f(B_j) = \frac{h_{i.}^o}{n} \cdot \frac{h_{.j}^o}{n}$$
$$\Rightarrow \ h_{ij}^e = n \cdot W(A \cap B) = n \cdot f(A_i \cap B_j) =$$
$$= n \cdot \frac{h_{i.}^o \, h_{.j}^o}{n^2} = \frac{h_{i.}^o \, h_{.j}^o}{n}$$

with $i = 1, ..., r$ and $j = 1, ..., s$

gender (B) / colour (A)	male (B1)	female (B2)	diverse (B3)	Σ
red (A1)	$h_{11}^e = \frac{28 \cdot 46}{100} = 12.88$	$h_{12}^e = \frac{28 \cdot 48}{100} = 13.44$	$h_{13}^e = \frac{28 \cdot 6}{100} = 1.68$	$h_{1.}^e = 28$
blue (A2)	$h_{21}^e = \frac{33 \cdot 46}{100} = 15.18$	$h_{22}^e = \frac{33 \cdot 48}{100} = 15.84$	$h_{23}^e = \frac{33 \cdot 6}{100} = 1.98$	$h_{2.}^e = 33$
green (A3)	$h_{31}^e = \frac{24 \cdot 46}{100} = 11.04$	$h_{32}^e = \frac{24 \cdot 48}{100} = 11.52$	$h_{33}^e = \frac{24 \cdot 6}{100} = 1.44$	$h_{3.}^e = 24$
yellow (A4)	$h_{41}^e = \frac{15 \cdot 46}{100} = 6.90$	$h_{42}^e = \frac{15 \cdot 48}{100} = 7.20$	$h_{43}^e = \frac{15 \cdot 6}{100} = 0.90$	$h_{4.}^e = 15$
Σ	$h_{.1}^e = 46$	$h_{.2}^e = 48$	$h_{.3}^e = 6$	$h_{..}^e = 100$

Since using the Chi-squared distribution as a test distribution requires $h_{ij}^e \geq 5$ for all $i = 1, ..., r$ and $j = 1, ..., s$ (see above), rows and/or columns may need to be grouped appropriately.

In the present case, all h_{i3}^e-values with $i = 1, ..., 4$ are smaller than 5, so here the condition $h_{ij}^e \geq 5$ is violated. Since there are 48 female participants and 46 male participants in the study, the diverse individuals will be equally distributed between the male and female participants in the following:

colour (A) \ gender (B)	male (B1) + diverse (B3)	female (B2)	Σ
red (A1)	$h_{11}^o = 11$ $h_{11}^e = 13.72$	$h_{12}^o = 17$ $h_{12}^e = 14.28$	28
blue (A2)	$h_{21}^o = 23$ $h_{21}^e = 16.17$	$h_{22}^o = 10$ $h_{22}^e = 16.83$	33
green (A3)	$h_{31}^o = 9$ $h_{31}^e = 11.76$	$h_{32}^o = 15$ $h_{32}^e = 12.24$	24
yellow (A4)	$h_{41}^o = 6$ $h_{41}^e = 7.35$	$h_{42}^o = 9$ $h_{42}^e = 7.65$	15
Σ	49	51	100

d. Identification of the critical range

$\alpha = 0.05$ or $(1 - \alpha) = 0.95$

and $\nu = (r-1)(s-1) = (2-1)(4-1) = 1 \cdot 3 = 3$

Remarks:

- By grouping male and diverse participants due to the condition $h_{ij}^e \geq 5$, the characteristic "gender" finally statistically does not result in three expressions, but in two, i.e. $r = 2$.
- Since $v = 3 > 1$ no continuity correction (Yates's correction) is to be applied.

$\Rightarrow \ \chi^2 = 7.815$

See Appendix A, Chi-Squared Distribution – Distribution Function.

If the test statistic is below $\chi^2 = 7{,}815$, then the null hypothesis H_0 cannot be rejected, i.e., the following applies:

rejection of H_0 if $\chi^2 > 7{,}815$
no rejection of H_0 if $\chi^2 \leq 7{,}815$

e. Calculation of the value of the test statistic

$$\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{(h_{ij}^o - h_{ij}^e)^2}{h_{ij}^e} =$$

$$= \frac{(11-13.72)^2}{13.72} + \frac{(17-14.28)^2}{14.28} + \frac{(23-16.17)^2}{16.17} +$$

$$+ \frac{(10-16.83)^2}{16.83} + \frac{(9-11.76)^2}{11.76} + \frac{(15-12.24)^2}{12.24} + \frac{(6-7.35)^2}{7.35} +$$

$$+ \frac{(9-7.65)^2}{7.65} = 8.4704$$

f. Decision and interpretation

Since $\chi^2 = 8.4704 > 7.815$, the null hypothesis H_0 is to be rejected. Based on a significance level of $\alpha = 0.05$ and a sample size of $n = 100$, it can be assumed that the gender-specific demand for this product does not depend on the color of its packaging.

3.6.2.3 Chi-Squared Homogeneity Test

This test can be used to check whether two or more random samples of discrete characteristics originate from the same distribution respectively a homogeneous population.

a. Null hypothesis

Testing two or more samples (two samples or multiple samples test) which hypothetically come from a (homogeneous) population.

b. Test statistic

$$\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{\left(h_{ij}^o - \frac{h_{i.}^o h_{.j}^o}{n}\right)^2}{\frac{h_{i.}^o h_{.j}^o}{n}}$$

with

h_{ij}^o observed absolute frequency of the combination of two characteristics.The first characteristic in its i^{th} occurrence $(i = 1, ..., r)$ and the second characteristic in its j^{th} occurrence $(j = 1, ..., s)$.

$$h_{i.}^o = \sum_{j=1}^{s} h_{ij}^o \qquad i = 1, ..., r$$

$$h_{.j}^o = \sum_{i=1}^{r} h_{ij}^o \qquad j = 1, ..., s$$

$$h_{..}^o = \sum_{i=1}^{r} h_{i.}^o = \sum_{j=1}^{s} h_{.j}^o = n \qquad i = 1, ..., r \text{ and } j = 1, ..., s$$

h^e_{ij} expected absolute frequency of the combination of two characteristics.The first characteristic in its i^{th} occurrence $(i = 1, ..., r)$ and the second characteristic in its j^{th} occurrence $(j = 1, ..., s)$.

$$h^e_{ij} = \frac{h^o_{i.} \cdot h^o_{.j}}{n} \qquad i = 1, ..., r \text{ and } j = 1, ..., s$$

c. Test distribution

$\chi^2 =$ Chi-squared distribution with $\nu = (r-1)(s-1)$
r: number of the i^{th} characteristic value
s: number of the j^{th} characteristic value
Condition: $h^e_{ij} \geq 5 \quad i = 1, ..., r$ and $j = 1, ..., s$

Example:

To monitor the populations of birds, the occurrence of five specific endangered birds is also monitored by two world-leading biological institutes. Within two samples of these two institutes during the same period, the following observed absolute frequencies were obtained:

bird species i	observation sample of institute A	observation sample of institute B	Σ
1	$h^o_{11} = 18$	$h^o_{12} = 12$	$h^o_{1.} = 30$
2	$h^o_{21} = 9$	$h^o_{22} = 10$	$h^o_{2.} = 19$
3	$h^o_{31} = 9$	$h^o_{32} = 8$	$h^o_{3.} = 17$
4	$h^o_{41} = 12$	$h^o_{42} = 11$	$h^o_{4.} = 23$
5	$h^o_{51} = 15$	$h^o_{52} = 17$	$h^o_{5.} = 32$
Σ	$h^o_{.1} = 63$	$h^o_{.2} = 58$	$n = 121$

At a significance level of $\alpha = 0.05$, it is to be tested whether the study results differ significantly. In other words, it is to be examined whether the two study results of the two institutes are homogeneous, i.e. whether they can be regarded as observation samples from the same population.

a. Definition of null and alternative hypothesis

H_0 : The study results are homogeneous.

H_A : The study results are heterogeneous.

$\alpha = 0.05$

b. Determination of the test statistc

$$\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{\left(h^o_{ij} - \frac{h^o_{i.}\, h^o_{.j}}{n}\right)^2}{\frac{h^o_{i.}\, h^o_{.j}}{n}}$$

c. Determination of the test distribution

$\chi^2 =$ Chi-squared distribution

with $\nu = (r-1)(s-1)$ degrees of freedom

To determine the expected absolute frequencies h^e_i the sums per considered bird species i, $h^o_{i.}$, can be put into perspective as follows:

$$h^o_{.1} : h^o_{.2} = 63 : 58 = 1.0862 : 1$$

Alternatively, the already known formula (see chapter 3.6.2.2) can be used:

$$h^e_{ij} = \frac{h^o_{i.}\, h^o_{.j}}{n}$$

with $i = 1, ..., r$ and $j = 1, ..., s$

i	observations of institute A	observations of institute B	Σ
1	$h_{11}^e = \frac{30 \cdot 63}{121} = = 15.62$	$h_{12}^e = \frac{30 \cdot 58}{121} = = 14.38$	$h_{1.}^e = 30$
2	$h_{21}^e = \frac{19 \cdot 63}{121} = = 9.89$	$h_{22}^e = \frac{19 \cdot 58}{121} = = 9.11$	$h_{2.}^e = 19$
3	$h_{31}^e = \frac{17 \cdot 63}{121} = = 8.85$	$h_{32}^e = \frac{17 \cdot 58}{121} = = 8.15$	$h_{3.}^e = 17$
4	$h_{41}^e = \frac{23 \cdot 63}{121} = = 11.98$	$h_{42}^e = \frac{23 \cdot 58}{121} = = 11.02$	$h_{4.}^e = 23$
5	$h_{51}^e = \frac{32 \cdot 63}{121} = = 16.66$	$h_{52}^e = \frac{32 \cdot 58}{121} = = 15.34$	$h_{5.}^e = 32$
Σ	$h_{.1}^e = 63$	$h_{.2}^e = 58$	$h_{..}^e = 121$

The condition $h_{ij}^e \geq 5$ is fulfilled for all $i = 1, ..., r$ and $j = 1, ..., s$, so that the Chi-squared distribution may be (directly) applied as a test distribution.

Remark:

- Since $\nu = 4 > 1$, no continuity correction (Yates's correction) is to be applied.

d. Identification of the critical range

$\alpha = 0.05$ or $(1-\alpha) = 0.95$

and $\nu = (r-1)(s-1) = (5-1)(2-1) = 4 \cdot 1 = 4$

$\Rightarrow \chi^2 = 9.488$

See Appendix A, Chi-Squared Distribution – Distribution Function.

If the test statistic is below $\chi^2 = 9.488$, then the null hypothesis H_0 cannot be rejected, i.e., the following applies:

rejection of H_0 if $\chi^2 > 9.488$

no rejection of H_0 if $\chi^2 \leq 9.488$

e. Calculation of the value of the test statistic

$$\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{\left(h_{ij}^o - \frac{h_{i.}^o h_{.j}^o}{n}\right)^2}{\frac{h_{i.}^o h_{.j}^o}{n}} = \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{(h_{ij}^o - (h_{ij}^e)^2}{h_{ij}^e} =$$

$$= \frac{(18-15.62)^2}{15.62} + \frac{(12-14.38)^2}{14.38} + \frac{(9-9.89)^2}{9.89} +$$

$$+ \frac{(10-9.11)^2}{9.11} + \frac{(9-8.85)^2}{8.85} + \frac{(8-8.15)^2}{8.15} +$$

$$+ \frac{(12-11.98)^2}{11.98} + \frac{(11-11.02)^2}{11.02} + \frac{(15-16.66)^2}{16.66} +$$

$$+ \frac{(17-15.34)^2}{15.34} = 1.2738$$

f. Decision and interpretation

Since $\chi^2 = 1.2738 < 9.488$, the null hypothesis H_0 cannot be rejected. Based on a significance level of $\alpha = 0.05$ and a sample size of $n = 121$, it can be assumed that that the two independently obtained study results are homogeneous, i.e. they do not differ significantly from each other.

3.6.3 Yates's Correction

At $v = 1$ degree of freedom a continuity correction (Yates's[17] correction) is to be applied:

$$\chi^2_{korr} = \sum_{i=1}^{k} \frac{(|h_i^o - h_i^e| - 0.5)^2}{h_i^e}$$

or

$$\chi^2_{korr} = \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{\left(|h_{ij}^o - h_{ij}^e| - 0.5\right)^2}{h_{ij}^e}$$

<u>Example</u>:

... according to the example of chapter 3.6.2.3 reduced to two bird species:

bird species i	observation sample of institute A	observation sample of institute B	Σ
1	$h_{11}^o = 18$	$h_{12}^o = 12$	$h_{1.}^o = 30$
2	$h_{21}^o = 9$	$h_{22}^o = 10$	$h_{2.}^o = 19$
Σ	$h_{1.}^o = 27$	$h_{2.}^o = 22$	$n = 49$

[17] Frank Yates (1902 - 1994) was an English statistician.

b. and e. Determination and calculation of the test statistic

$$\chi^2_{korr} = \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{\left(|h^o_{ij} - h^e_{ij}| - 0.5\right)^2}{h^e_{ij}}$$

using Yates's correction,

since $\nu = (r-1)(s-1) = (2-1)(2-1) = 1$.

$$h^e_{ij} = \frac{h^o_{i.}\, h^o_{.j}}{n}$$

with $i = 1, ..., r$ and $j = 1, ..., s$

i	observations of institute A	observations of institute B	Σ
1	$h^e_{11} = \frac{30 \cdot 27}{49} = = 16.53$	$h^e_{12} = \frac{30 \cdot 22}{49} = = 13.47$	$h^e_{1.} = 30$
2	$h^e_{21} = \frac{19 \cdot 27}{49} = = 10.47$	$h^e_{22} = \frac{19 \cdot 22}{49} = = 8.53$	$h^e_{2.} = 19$
Σ	$h^e_{.1} = 27$	$h^e_{.2} = 22$	$h^e_{..} = 49$

with $h^e_{ij} \geq 5$ for all $i = 1, ..., r$ and $j = 1, ..., s$

$$\chi^2_{korr} = \sum_{i=1}^{r}\sum_{j=1}^{s} \frac{\left(|h^o_{ij} - h^e_{ij}| - 0.5\right)^2}{h^e_{ij}}$$

$$= \frac{(|18 - 16.53| - 0,5)^2}{16.53} + \frac{(|12 - 13.47| - 0,5)^2}{13.47} +$$

$$+ \frac{(|9 - 10.47| - 0.5)^2}{10.47} + \frac{(|10 - 8.53| - 0.5)^2}{8.53} =$$

$$= 0.0569 + 0.0699 + 0.0899 + 0.1103 = 0.3270$$

d. Identification of the critical range

$\alpha = 0.05$ bzw. $(1 - \alpha) = 0.95$
and $\nu = (r-1)(s-1) = (2-1)(2-1) = 1$
$\Rightarrow \chi^2 = 3,841$

See Appendix A, Chi-Squared Distribution – Distribution Function.

rejection of H_0 if $\chi^2 > 3.841$
no rejection of H_0 if $\chi^2 \leq 3.841$

f. Decision and interpretation

Since $\chi^2 = 0.3270 < 3.841$, the null hypothesis H_0 cannot be rejected. Given the test conditions ($\alpha = 0.05$ and $n = 49$), the sample result suggests that the two independently obtained test results are homogeneous, i.e., they are not significantly different from each other.

Chapter 4

Probability Calculation

4.1 Terms and Definitions

Definition:

Probability calculation forms the basis of inferential statistics.

Many events of economic decisions are not strictly determined (predictable), but are stochastic, i.e. determined by chance.

Fundamental Terms

(1) *Random Experiment*

A process which is carried out according to specific regulations and can be repeated as often as desired, but whose result is determined by chance, i.e. it cannot be clearly determined in advance.

Example:

Games of chance like lottery or roulette.

(2) *Elementary Event / Event / Sample space*

In every random experiment, a set of possible elementary events exists, so-called elementary events or realisations. An event comprises any subset of the sample space S.

F. W. Peren, *Statistics for Business and Economics*,
https://doi.org/10.1007/978-3-662-65846-8_4

<u>Example:</u>

Throwing a die once.

This random experiment is comprised of six possible *elementary events*: $e_1 = 1,\ e_2 = 2,\ ...,\ e_6 = 6$

The set of all elementary events comprises the so-called sample space S.

4.2 Definitions of Probability

Probability is a measure to quantify the certainty or uncertainty of the occurrence of a certain event in the context of a random experiment.

4.2.1 The Classical Definition of Probabilty

If all elementary events are equally probable, the probability of event A occuring in a given random experiment is determined by the following quotient:

$$P(A) = \frac{\text{number of favourable cases}}{\text{number of all equally probable cases}}$$

with $A = \{$set of all favourable cases$\}$

<u>Example:</u>

What is the probability of event A, that you land on the number 6 when throwing an (untainted) die?

$P(A) = \frac{1}{6}$

Remark:

The practical significance of the classical definition of probability is limited since it can only be applied to equally probable events.

4.2.2 The Statistical Definition of Probability

The statistical definition of probability assumes a random experiment consisting of a sequence of independent trials tending towards infinity.

$$P(A) = \lim_{n \to \infty} \frac{h_n(A)}{n} = \lim_{n \to \infty} f_n(A)$$

with: n = number of tries/observations

h_n = absolute frequency of event A

f_n = relative frequency of event A

Example:

What is the probability of tossing the event "heads" with a coin that has two sides ("heads" and "tails")?

It can be assumed that $f_n(A)$ approaches increasingly the value 0.5, if the tosses were continued infinitely. If $f_n(A)$ was found to deviate increasingly less from the value 0.5, it would be an "ideal coin" here. For this coin, the probability of the event "heads" is the same as the probability of the event "tails".

4.2.3 The Subjective Definition of Probability

Especially in practical decision making, probabilities are determined neither by using the classical nor the statistcal definition of probability. In most cases, the subjective probability of experts are used, usually by using the so-called *Delphi method*. In economics, subjective probabilities are often used in decision models with uncertainty.

4.2.4 Axioms of Probability Calculation

The axiomatic definition of probability does not seek to explain the essence of probability, but rather defines the mathematical properties of probability in terms of three axioms.

Axiom 1:

The probability $P(A)$ of event A of a random experiment is a uniquely determined, real, non-negative number between zero and one:

$$0 \leq P(A) \leq 1 \text{ with } P(A) \in \text{real numbers and } A \subset S$$

Remark:

$P(A)$ is not negative.

Example:

What is the probability of rolling the number 5 when rolling the dice once?

$$W(5) = \frac{1}{6} = 0.166$$

Axiom 2:

The probability of the occurrence of all events within a random experiment, i.e. for the entire sample space S, is one:

$$\boxed{P(S) = 1}$$

Example:

Consider a die roll within the sample space S with $S = \{1, 2, 3, 4, 5, 6\}$ where $e_1 = 1, e_2 = 2, e_3 = 3, e_4 = 4, e_5 = 5$ and $e_6 = 6$.

$$W(S) = \frac{6}{6} = 1$$

Axiom 3:

If two events A and B of a random experiment are mutually exclusive, the following applies:

$$\boxed{P(A \cup B) = P(A) + P(B) \text{ with } A \cap B = \{\ \}}$$

Example:

What is the probability of rolling a "2" or a "5" when rolling the dice once?

$$W(e_1 \cup e_2) \text{ with } e_1 = 2 \text{ and } e_2 = 5 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

$$W(e_1 \cap e_2) = \{\}$$

4.3 Theorems of Probability Calculation

4.3.1 Theorem of Complementary Events

The sum of probabilities of the random event A and the event complementary to A, $\overline{A}$ ($=$ event not-A), is equal to one:

$$\boxed{P(A \cup \overline{A}) = P(A) + P(\overline{A}) = 1}$$

<u>Example:</u>

What is the probability that in a roll of two dice the sum of both dice is not 4?

$\Rightarrow$ 2 dice of 6 values each, i.e. there are 36 equally probable elementary events (i, j), whereby $i = 1, ..., 6$ and $j = 1, ..., 6$.

Since the elementary events (i, j) are equally probable, the following applies:

$$P(i,\, j) = \frac{1}{36} \quad \text{with} \quad \sum_{i=1}^{6} \sum_{j=1}^{6} P(i,\, j) = 1$$

Event A, that the sum of both dice equals 4, is composed of three elementary events:

$$A = \{(1,\, 3) \cup (2,\, 2) \cup (3,\, 1)\}$$

$$\Rightarrow \quad P(A) = P(1, 3) + P(2, 2) + P(3, 1) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{1}{12}$$

$$\Rightarrow \quad P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{12} = \frac{11}{12}$$

4.3.2 The Multiplication Theorem with Independence of Events

If events A and B occur independently from another, the following applies:

$$\boxed{P(A \cap B) = P(A) \cdot P(B)}$$

Example:

A coin with tails on one side is tossed twice. What is the probability of getting tails both times?

Each toss is new and therefore independent of one other.

$\Rightarrow \quad P(A) = 0.5 \quad$ with $\; A =$ tails and
$\overline{A} =$ not-tails

$\Rightarrow \quad P(A_1 \cap A_2) = P(A_1) \cdot P(A_2) = 0.5 \cdot 0.5 = 0.25$

The probability of getting tails twice in a row is 0.25; the counter probability is 0.75.

4.3.3 The Addition Theorem

(1) Events A and B are not mutually exclusive.

If A and B are any two events of a random experiment that are not mutually exclusive, the additive (joint) probability of the event $P(A \cup B)$ is calculated as:

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

The probability that either A or B or $A \cap B$, i.e. A and B occur together, is obtained by adding $P(A)$ and $P(B)$ minus the probability of the joint occurrence of A and B.

(2) A and B are mutually exclusive.

In the case where events A and B are mutually exclusive, the following applies:

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

Example:

What is the probability that, if two identical coins are tossed with heads on one side and tails on the other, at least one coin will show tails?

Step 1: Sample space S

$S = \{tt, th, ht, hh\}$ with $t =$ tails and $h =$ heads

$P(tt) = P(th) = P(ht) = P(hh) = \dfrac{1}{4}$

All events are equally probable.

Step 2: Exclusivity / non-exclusivity:

$A =$ event that the 1st coin lands on tails $= \{tt, th\}$

$B =$ event that the 2nd coin lands on tails $= \{tt, ht\}$

$\Rightarrow$ Events A and B are not mutually exclusive if the 1st and 2nd coin shows tails $\{tt\}$.

Step 3: Addition theorem

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Step 4: Calculation

$$P(A \cup B) = \underbrace{\frac{1}{4}+\frac{1}{4}}_{P(A)} + \underbrace{\frac{1}{4}+\frac{1}{4}}_{P(B)} - \underbrace{\frac{1}{4}}_{P(tt)} = \frac{3}{4} = 0.75$$

4.3.4 Conditional Probability

The probability of an event A occuring is often *conditional*, i.e. dependent on the occurrence of another event B. The probability of A given the occurrence of (another) event B is called the *conditional probability* of event A given condition B, $P(A/B)$, with:

$$\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)}}$$

Example:

A dice roll is considered. The sample space $A = \{1, 3, 5\}$ includes all odd numbers and the sample space B includes the following set of numbers $B = \{3, 5, 6\}$.

$$W(A) = \frac{3}{6} = \frac{1}{2}$$

$$W(B) = \frac{3}{6} = \frac{1}{2}$$

$$A \cap B = \{3, 5\}$$

$$W(A \cap B) = \frac{2}{6} = \frac{1}{3}$$

$$W(A/B) = \frac{W(A \cap B)}{W(B)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

4.3.5 Stochastic Independence

An event A is *stochastically* (taking chance into account) *independent* of event B if the occurrence of A does not depend on the occurrence or non-occurrence of event B.

The following applies:

$$\boxed{P(A/B) = P(A/\overline{B})}$$

Example:

A dice roll is considered. The event A includes all dice rolls with even numbers and the event B describes all dice rolls with numbers smaller than 5. It is to be tested whether the event A is stochastically independent of the event B.

Event $A = \{2, 4, 6\}$ $\qquad$ $\overline{A} = \{1, 3, 5\}$

$$W(A) = \frac{3}{6} = \frac{1}{2} \qquad W(\overline{A}) = \frac{3}{6} = \frac{1}{2}$$

Event $B = \{1, 2, 3, 4\}$ $\qquad$ $\overline{B} = \{5, 6\}$

$$W(B) = \frac{4}{6} = \frac{2}{3} \qquad W(\overline{B}) = \frac{2}{6} = \frac{1}{3}$$

$$A \cap B = \{2, 4\}$$

$$W(A \cap B) = \frac{2}{6} = \frac{1}{3}$$

$$W(A/B) = \frac{W(A \cap B)}{W(B)} = \frac{2/6}{2/3} = \frac{3}{6} = \frac{1}{2}$$

$$W(A/\overline{B}) = \frac{W(A \cap \overline{B})}{W(\overline{B})} = \frac{1/6}{1/3} = \frac{1}{2}$$

$\Rightarrow$ The two events A and B are stochastically independent of each other.

4.3.6 The Multiplication Theorem in General Form

If the multiplication theorem in section 4.3.2 is to be generally applied, i.e. event B is conditioned by event A, the following applies:

$$P(A \cap B) = P(A) \cdot P(B/A) = P(B) \cdot P(A/B)$$

Example:

The probability of drawing a king from a pack of cards (52 cards with 4 kings) is: $P(A) = \frac{4}{52} = \frac{1}{13}$. If the drawn card is a king, the probability of drawing a king again is: $P(B) = \frac{3}{51}$. The probability of drawing two kings with two cards is therefore:

$$P(A \cap B) = P(A) \cdot P(B/A) = \frac{1}{13} \cdot \frac{3}{51} \approx 0.0045 \mathrel{\widehat{=}} 0.45\,\% = 4.5\%$$

4.3.7 The Theorem of Total Probability

Let $A_1, A_2, ..., A_n$ be mutually exclusive events that completely fill a sample space S, so that:

$$A_1 \cup A_2 \cup ... \cup A_n = S \text{ and } A_i \cap A_j = \{\ \} \text{ with } i, j = 1, ..., n;\ i \neq j$$

then any event E can be expressed as a union of mutually exclusive events:

$$E = (E \cap A_1) \cup (E \cap A_2) \cup ... \cup (E \cap A_n)$$

(1) Application of the addition theorem for mutually exclusive events:

$$\Rightarrow P(E) = P(E \cap A_1) + ... + P(E \cap A_n)$$

(2) Application of the general multiplication theorem in the case of dependency of events:

$$\Rightarrow \quad P(E) = P(A_1) \cdot P(E/A_1) + ... + P(A_n) \cdot P(E/A_n)$$
$$\Leftrightarrow \quad P(E) = \sum_{i=1}^{n} P(A_i) \cdot P(E/A_i)$$

Example:

A plant produces $2,000$ pieces of a product per day. Of this, machine M_1 delivers 800 pieces with $9\,\%$ share of rejects, M_2 delivers 700 pieces with $7\,\%$ share of rejects, and M_3 delivers 500 pieces with $4\,\%$ share of rejects.

Randomly, a unit of quantity is selected. What is the probability that the selected unit of quantity is defective.

$A_i\,(i = 1, 2, 3)$ describes the event that a piece was produced by machine M_i.

E_i describes the number of pieces produced on machine M_i.

The following probabilities can be calculated:

$$W_1(A_1) = \frac{800}{2{,}000} = 0.40 \text{ and } W_1(E_1/A_1) = \frac{9}{100} = 0.09$$
$$W_2(A_2) = \frac{700}{2{,}000} = 0.35 \text{ and } W_2(E_2/A_2) = \frac{7}{100} = 0.07$$
$$W_3(A_3) = \frac{500}{2{,}000} = 0.25 \text{ and } W_3(E_3/A_3) = \frac{4}{100} = 0.04$$

Event E is calculated as:

$$E = (E \cap A_1) \cup (E \cap A_2) \cup (E \cap A_3)$$

$$W(E) = \sum_{i=1}^{3} W_i(A_i) \cdot W_i(E_i/A_i) =$$
$$= 0.4 \cdot 0.09 + 0.35 \cdot 0.07 + 0.25 \cdot 0.04 =$$
$$= 0.0705$$

A: The probability that a selected piece is defective is $7.05\,\%$.

4.3.8 Bayes' Theorem (Bayes' Rule)

With the help of Bayes' theorem, the conditional probability for any event A_j can be determined from the possible events $A_1, A_2, ..., A_n$ of the sample space S under the condition that a certain event E has occurred:

$$P(A_j/E) = \frac{P(A_j \cap E)}{P(E)} = \frac{P(A_j) \cdot P(E/A_j)}{\sum_{i=1}^{n} P(A_i) \cdot P(E/A_i)}$$

with:

$$A_1 \cup A_2 \cup ... \cup A_n = S \text{ and } A_i \cap A_j = \{\} \text{ with } i, j = 1, ..., n;\ i \neq j$$

Since the probability for the occurrence of event A_j refers to an event that has occurred before, $P(A_j/E)$, i.e. can in fact only be calculated after the event E, the probability $P(A_i/E)$ is also called the a posteriori probability. The non-conditional probability $P(A_j)$ is accordingly called the a priori probability.

Example:

A daily production of $1{,}000$ quantity units (QU) of a product is distributed among three machines as follows: M_1 100 QU, M_2 400 QU and M_3 500 QU. The relative error frequencies (error rates) are 0.05 ($5\,\%$) for M_1, 0.04 ($4\,\%$) for M_2 and 0.02 ($2\,\%$) for M_3.

One unit is randomly selected from a daily production. This unit is found to be defective.

What is the probability that this defective unit was produced on machine M_1 or M_2 or M_3?

A_j: QU was produced on machine M_j; $j = 1, 2, 3$

E: QU is defective.

Bayes' Theorem

$$P(A_j/E) = \frac{P(A_j \cap E)}{P(E)} = \frac{P(A_j) \cdot P(E/A_j)}{\sum_{i=1}^{n} P(A_i) \cdot P(E/A_i)}$$

$$P(A_1) = \frac{100}{1,000} = 0.1$$

$$P(A_2) = \frac{400}{1,000} = 0.4$$

$$P(A_3) = \frac{500}{1,000} = 0.5$$

$$\Rightarrow P(A_1/E) = \frac{0.1 \cdot 0.05}{0.1 \cdot 0.05 + 0.4 \cdot 0.04 + 0.5 \cdot 0.02} = \frac{0.1 \cdot 0.05}{0.031} \approx 0.16$$

$$\Rightarrow P(A_2/E) = \frac{0.4 \cdot 0.04}{0.031} \approx 0.52$$

$$\Rightarrow P(A_3/E) = \frac{0.5 \cdot 0.02}{0.031} \approx 0.32$$

The probability that the defective unit was drawn from machine M_2 is highest with approximately 0.52 ($= 52\,\%$).

4.3.9 Overview of the Probability Calculation of Mutually Exclusive and Non-Exclusive Events

	$\cup \Rightarrow +$ or	$\cap \Rightarrow \times$ and
Events A and B are mutually exclusive $\Rightarrow$ no common elements $\Rightarrow$ **independence** of events	$P(A \cup B) =$ $= P(A) + P(B)$ see 4.3.3 (2)	$P(A \cap B) =$ $= P(A) \cdot P(B)$ see 4.3.2
Events A and B are not mutually exclusive $\Rightarrow$ there are common elements $\Rightarrow$ **dependence** of events	$P(A \cup B) =$ $= P(A) + P(B)$ $-P(A \cap B)$ see 4.3.3 (1)	$P(A \cap B) =$ $= P(A) \cdot P(B/A)$ $= P(B) \cdot P(A/B)$ see 4.3.6

4.4 Random Variable

4.4.1 The Concept of Random Variables

Definition:

If a variable changes in an unpredictable way, i.e. if the variable takes on its values only depending on chance, it is called a *random variable*. Random variables are usually symbolised with capital letters X, Y, Z, while their values are correspondingly marked with lowercase letters.

Example:

In the random experiment of tossing a coin with tails on one side twice, the frequency of the “tails”-event depends on chance. The values of this random variable X, $X =$ number of tails when a coin is tossed twice, can be $x = 0 \lor x = 1 \lor x = 2$.

4.4.2 The Probability Function of Discrete Random Variables

A probability $P(X = x_i)$ can be assigned to each specific value x_i of a random variable X. The function $f(x_i)$, which indicates the probability of realisation for each value of the random variable, is referred to as the probability function of the random variable X:

$$\boxed{f(x_i) = P(X = x_i)}$$

It has the properties: $f(x_i) \geq 0$ and $\sum_{i=1}^{n} f(x_i) = 1$

with $i = 1, ..., n$

Example:

A coin with two sides "heads" (H) and "tails" (T) is tossed twice in a row. What is the probability that heads will be tossed at least once?

The different possibilities resulting from tossing a coin twice are: HH, HT, TH, HH.

elementary event e_i with $i = 1, ..., 4$	probability $W(e_i)$	number of heads x	probability $W(X = x_i) = f(x)$
$e_1 = TT$	$W(e_1) = \frac{1}{4} = 0.25$	$x_1 = 0$	$f(x_1) = 0.25$
$e_2 = HT$ $e_3 = TH$	$W(e_2) = \frac{1}{4} = 0.25$ $W(e_3) = \frac{1}{4} = 0.25$	$x_2 = 1$	$f(x_2) = 0.50$
$e_4 = HH$	$W(e_4) = \frac{1}{4} = 0.25$	$x_3 = 2$	$f(x_3) = 0.25$
			$f(x_1) + f(x_2) +$ $+ f(x_3) = 1$

The probability of tossing heads at least once is:

$$W(e_2) + W(e_3) + W(e_4) = 0.25 + 0.25 + 0.25 = 0.75$$

4.4.3 The Distribution Function of Discrete Random Variables

The function value of a distribution function $F(x)$ of a random variable X indicates the probability that the random variable X is at most the value x_i:

$$F(x) = P(X \leq x_i)$$

The distribution function of discrete random variables has the following properties:

(1) for an arbitrary x: $\quad 0 \leq F(x) \leq 1$

(2) impossible event: $\quad \lim\limits_{\substack{x \to -\infty \\ \text{or } x \to 0}} F(x) = 0$

(3) certain event: $\quad \lim\limits_{x \to \infty} F(x) = 1$

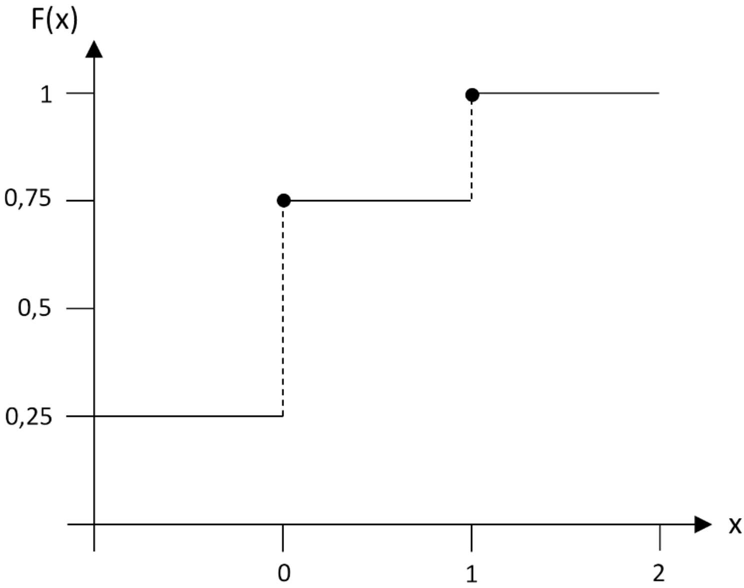
Example:

Regarding the example from 4.4.2:

$F(x) = W(X \leq x_i)$

x_i	$F(x) = W(X \leq x)$
$x_1 = 0$	0.25
$x_2 = 1$	0.75
$x_3 = 2$	1.00

graphical distribution function



4.4.4 Probability Density and Distribution Function of Continuous Random Variables

(1) The Probability Density (The Density Function)

If the observed random variable X takes not only on discrete values but (in a certain interval) any value from the range of real numbers, then the probability distribution becomes a so-called probability density or, in other words, a density function.

The density function has the properties:

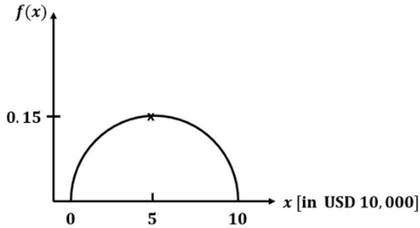
$$f(x) \geq 0 \text{ and } \int_{x_{min}}^{x_{max}} f(x)dx = 1$$

Example:

The daily cash receipts of a retailer fluctuate between $x_{min} = \$0$ and $x_{max} = \$10,000$. By definition, the variable x has 1 million [in cents] values and thus corresponds de facto to a continuous random variable. The probability distribution of the random variable X can be represented by the following density function:

$$f(x) = -0.006x^2 + 0.06x$$
$$f'(x) = -0.012x + 0.06 = 0$$
$$\Rightarrow x_{max} = 5$$

$$f(5) = 0.15$$
$$f(x) = 0$$
$$\Rightarrow x = 0 \;\vee\; x = 10$$



The function $f(x) = -0.006x^2 + 0.06x$ is a density function if the area under the function between x_{min} and x_{max} is equal to one.

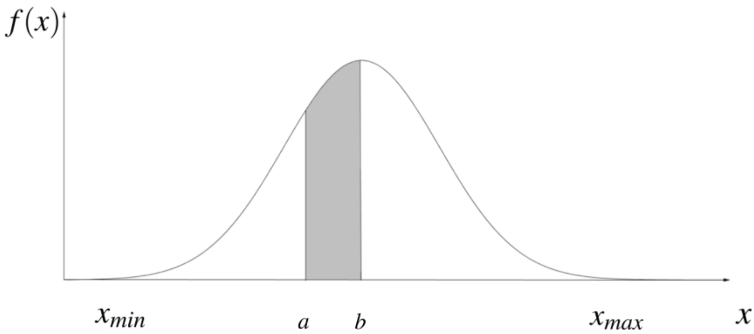
$$\int_{x_{min}}^{x_{max}} f(x)dx = \int_{0}^{10} (-0.006x^2 + 0.06x)dx =$$

$$= \left[-0.006 \cdot \frac{1}{3}x^3 + 0.06 \cdot \frac{1}{2} \cdot x^2\right]_0^{10} =$$

$$= (-0.002 \cdot 10^3 + 0.03 \cdot 10^2) - 0 = -2 + 3 = 1$$

The probability that the continuous random variable X will take on a value x that lies in the interval $[a, b]$ corresponds to the area, which locates under the considered density function $f(x)$ within the limits a and b:

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$



Remark: For continuous random variables, the probability that the random variable X takes on any specific value x is always equal to zero:

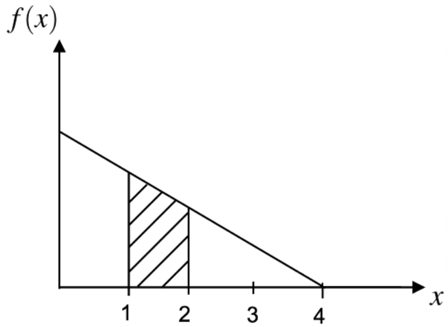
$P(X = x) = 0$

so that: $P(a \leq X \leq b) = P(a < x < b)$

Example:

The continuous random variable X records the delay of a subway at a certain stop and has the following density function in minutes:

$$f(x) = \begin{cases} 0.5 - 0.125x & \text{for } 0 \le x \le 4 \text{ [minutes]} \\ 0 & \text{for } \quad x > 4 \text{ [minutes]} \end{cases}$$



$f(x)$ is a density function, because:

(1) $f(x) \ge 0$ for all x

(2) $\int_{x_{min}}^{x_{max}} f(x)dx = \int_0^4 (0.5 - 0.125x)dx =$

$$= \left[0.5x - 0.0625 \cdot x^2\right]_0^4 =$$

$$= 2 - 1 = 1$$

What is the probability that the subway will be delayed by one to two minutes?

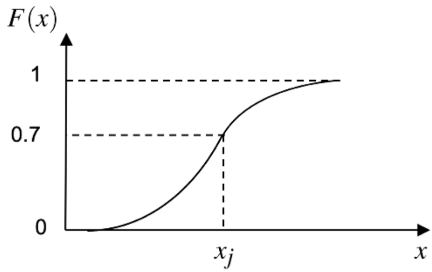
$P(1 \le x \le 2) =$ corresponding area under the density function

$$f(x) = \int_1^2 f(x)dx = \int_1^2 (0.5 - 0.125x)dx =$$

$$= \left[0.5x - 0.125 \cdot \frac{1}{2}x^2\right]_1^2 = 0.75 - 0.4375 = 0.3125 = 31.25\,\%$$

(2) The Distribution Function

The distribution function of a continuous random variable X, $F(x)$, is a continuous function:



<u>Interpretation</u>: $x_j : F(x_j) = 0.7$ indicates the probability that the random variable takes on the realisation x_j <u>at most</u>.

<u>Properties of the Distribution Function $F(x)$</u>

(1) $0 \leq F(x) \leq 1$

(2) $F(x)$ is monotonically increasing, i.e. for $x_1 \leq x_2$: $F(x_1) \leq F(x_2)$

(3) $\lim\limits_{x \to \infty} F(x) = 1$

(4) $\lim\limits_{\substack{x \to -\infty \\ \text{or } x \to 0}} F(x) = 0$

(5) $F(x)$ is continuous for all x

(6) The first derivative of the distribution function $F(x)$ yields the density function (= probability function) $f(x)$.

For the density function from the example "delay of a subway" (see section 4.4.4)

$$f(x) = \begin{cases} 0.5 - 0.125x & \text{for } 0 \leq x \leq 4 \text{ [minutes]} \\ 0 & \text{for } \quad x > 4 \text{ [minutes]} \end{cases}$$

the following distribution function is obtained

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 0.5x - 0.125 \cdot \frac{1}{2}x^2 & \text{for } 0 \leq x \leq 4 \\ 1 & \text{for } \quad x > 4 \end{cases}$$

Example: Random variable $X =$ delay of a subway

What is the probability that the subway is delayed by one to two minutes?

$$\begin{aligned} P(1 \leq x \leq 2) &= F(2) - F(1) = \\ &= (0.5 \cdot 2 - 0.0625 \cdot 2^2) - (0.5 \cdot 1 - 0.0625 \cdot 1^2) = \\ &= 0.75 - 0.4375 = 0.3125 = 31.25\,\% \end{aligned}$$

4.4.5 Expected Value and Variance of Random Variables

Like the frequency distributions of descriptive statistics, the probability distributions of random variables can be characterised by measures (parameters).

(1) Expected Value $E(X)$

- for <u>discrete</u> random variables:

$$E(X) = \sum_i x_i \cdot P(X = x_i) = \sum_i x_i \cdot f(x_i)$$

- for <u>continuous</u> random variables:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx$$

or for an interval $[x_u \; ; \; x_o]$: $E(X) = \int_{x_u}^{x_o} x \cdot f(x)dx$

(2) Variance $Var(X)$

- for <u>discrete</u> random variables:

$$Var(X) = \sum_i [x_i - E(X)]^2 \; f(x_i) =$$

$$= \sum_i {x_i}^2 \; f(x_i) - [E(X)]^2$$

- for <u>continuous</u> random variables:

$$Var(X) = \int_{-\infty}^{\infty} [x - E(X)]^2 \; f(x)dx =$$

$$= \int_{-\infty}^{\infty} x^2 \; f(x)dx - [E(X)]^2$$

or for an interval $[x_u \; ; \; x_o]$:

$$\int_{x_u}^{x_o} [x - E(X)]^2 \; f(x)dx =$$

$$= \int_{x_u}^{x_o} x^2 \; f(x)dx - [E(X)]^2$$

Example 1:

Random experiment “tossing a coin three times” with tails on one side:

$X =$ number of tails $\Rightarrow$ discrete random variable

$E(X) =$ the number of tails to be expected on average for a larger number of trials

$$\begin{aligned} E(X) &= \sum_i x_i f(x_i) = \\ &= 0 \cdot 0.125 + 1 \cdot 0.375 + 2 \cdot 0.375 + 3 \cdot 0.125 = 1.5 \end{aligned}$$

On average, each attempt, i.e. “tossing a coin three times”, is expected to result in tails 1.5 times.

$$\begin{aligned} Var(X) &= \sum_i {x_i}^2 f(x_i) - [E(X)]^2 = \\ &= 0^2 \cdot 0.125 + 1^2 \cdot 0.375 + 2^2 \cdot 0.375 + 3^2 \cdot 0.125 - 1.5^2 = 0.75 \end{aligned}$$

Example 2:

Delay of a subway at a certain stop in minutes.

$X =$ delay in minutes $\Rightarrow$ continuous random variable

(1) $f(x) = \begin{cases} 0.5 - 0.125x & \text{for } 0 \leq x \leq 4 \\ 0 & \text{for } \quad x > 4 \end{cases}$

(2) $$\begin{aligned} E(X) &= \int_{x_u}^{x_o} x \cdot f(x)dx = \int_0^4 x \cdot (0.5 - 0.125x)dx = \\ &= \int_0^4 (0.5x - 0.125x^2)dx = \left[0.5 \cdot \frac{1}{2}x^2 - 0.125 \cdot \frac{1}{3}x^3\right]_0^4 = \\ &= F(4) - F(0) = 1.\overline{3} \ \text{[minutes]} \end{aligned}$$

On average, a delay of about 1.33 minutes can be expected.

$$\begin{aligned}
Var(x) &= \int_{x_u}^{x_o} x^2 \cdot f(x)dx - [E(X)]^2 = \\
&= \int_0^4 x^2 \cdot (0.5 - 0.125x)dx - (1.3333)^2 = \\
&= \int_0^4 (0.5x^2 - 0.125x^3)dx - (1.3333)^2 = \\
&= \left[0.5 \cdot \frac{1}{3}x^3 - 0.125 \cdot \frac{1}{4}x^4\right]_0^4 - 1.3333^2 = \\
&= F(4) - F(0) - 1.3333^2 = 0.889 \ \ \text{[minutes}^2\text{]}
\end{aligned}$$

Standard deviation $= \sqrt{0.889} = 0.9429$ [minutes]

Appendix A

Statistical Tables

F. W. Peren, *Statistics for Business and Economics*,
https://doi.org/10.1007/978-3-662-65846-8

Binomial Distribution – Probability Mass Function

$$f_B(x \mid n;\ \theta) = \begin{cases} \binom{n}{x}\ \theta^x\ (1-\theta)^{n-x} & \text{for}\quad x = 0,\ 1,\ \ldots,\ n \qquad 0 < \theta < 1 \\ 0 & \text{others} \end{cases}$$

n	x	θ							
		0.01	**0.05**	**0.10**	**0.15**	**0.20**	**0.30**	**0.40**	**0.50**
1	0	0.9900	0.9500	0.9000	0.8500	0.8000	0.7000	0.6000	0.5000
	1	0.0100	0.0500	0.1000	0.1500	0.2000	0.3000	0.4000	0.5000
2	0	0.9801	0.9025	0.8100	0.7225	0.6400	0.4900	0.3600	0.2500
	1	0.0198	0.0950	0.1800	0.2550	0.3200	0.4200	0.4800	0.5000
	2	0.0001	0.0025	0.0100	0.0225	0.0400	0.0900	0.1600	0.2500
3	0	0.9703	0.8574	0.7290	0.6141	0.5120	0.3430	0.2160	0.1250
	1	0.0294	0.1354	0.2430	0.3251	0.3840	0.4410	0.4320	0.3750
	2	0.0003	0.0071	0.0270	0.0574	0.0960	0.1890	0.2880	0.3750
	3	0.0000	0.0001	0.0010	0.0034	0.0080	0.0270	0.0640	0.1250
4	0	0.9606	0.8145	0.6561	0.5220	0.4096	0.2401	0.1296	0.0625
	1	0.0388	0.1715	0.2916	0.3685	0.4096	0.4116	0.3456	0.2500
	2	0.0006	0.0135	0.0486	0.0975	0.1536	0.2646	0.3456	0.3750
	3	0.0000	0.0005	0.0036	0.0115	0.0256	0.0756	0.1536	0.2500
	4	0.0000	0.0000	0.0001	0.0005	0.0016	0.0081	0.0256	0.0625
5	0	0.9510	0.7738	0.5905	0.4437	0.3277	0.1681	0.0778	0.0313
	1	0.0480	0.2036	0.3281	0.3915	0.4096	0.3602	0.2592	0.1563
	2	0.0010	0.0214	0.0729	0.1382	0.2048	0.3087	0.3456	0.3125
	3	0.0000	0.0011	0.0081	0.0244	0.0512	0.1323	0.2304	0.3125
	4	0.0000	0.0000	0.0005	0.0022	0.0064	0.0284	0.0768	0.1563
	5	0.0000	0.0000	0.0000	0.0001	0.0003	0.0024	0.0102	0.0313
6	0	0.9415	0.7351	0.5314	0.3771	0.2621	0.1176	0.0467	0.0156
	1	0.0571	0.2321	0.3543	0.3993	0.3932	0.3025	0.1866	0.0938
	2	0.0014	0.0305	0.0984	0.1762	0.2458	0.3241	0.3110	0.2344
	3	0.0000	0.0021	0.0146	0.0415	0.0819	0.1852	0.2765	0.3125
	4	0.0000	0.0001	0.0012	0.0055	0.0154	0.0595	0.1382	0.2344
	5	0.0000	0.0000	0.0001	0.0004	0.0015	0.0102	0.0369	0.0938
	6	0.0000	0.0000	0.0000	0.0000	0.0001	0.0007	0.0041	0.0156
7	0	0.9321	0.6983	0.4783	0.3206	0.2097	0.0824	0.0280	0.0078
	1	0.0659	0.2573	0.3720	0.3960	0.3670	0.2471	0.1306	0.0547
	2	0.0020	0.0406	0.1240	0.2097	0.2753	0.3177	0.2613	0.1641
	3	0.0000	0.0036	0.0230	0.0617	0.1147	0.2269	0.2903	0.2734
	4	0.0000	0.0002	0.0026	0.0109	0.0287	0.0972	0.1935	0.2734

Binomial Distribution – Probability Mass Function

n	x	θ							
		0.60	**0.70**	**0.75**	**0.80**	**0.85**	**0.90**	**0.95**	**0.99**
1	0	0.4000	0.3000	0.2500	0.2000	0.1500	0.1000	0.0500	0.0100
	1	0.6000	0.7000	0.7500	0.8000	0.8500	0.9000	0.9500	0.9900
2	0	0.1600	0.0900	0.0625	0.0400	0.0225	0.0100	0.0025	0.0001
	1	0.4800	0.4200	0.3750	0.3200	0.2550	0.1800	0.0950	0.0198
	2	0.3600	0.4900	0.5625	0.6400	0.7225	0.8100	0.9025	0.9801
3	0	0.0640	0.0270	0.0156	0.0080	0.0034	0.0010	0.0001	0.0000
	1	0.2880	0.1890	0.1406	0.0960	0.0574	0.0270	0.0071	0.0003
	2	0.4320	0.4410	0.4219	0.3840	0.3251	0.2430	0.1354	0.0294
	3	0.2160	0.3430	0.4219	0.5120	0.6141	0.7290	0.8574	0.9703
4	0	0.0256	0.0081	0.0039	0.0016	0.0005	0.0001	0.0000	0.0000
	1	0.1536	0.0756	0.0469	0.0256	0.0115	0.0036	0.0005	0.0000
	2	0.3456	0.2646	0.2109	0.1536	0.0975	0.0486	0.0135	0.0006
	3	0.3456	0.4116	0.4219	0.4096	0.3685	0.2916	0.1715	0.0388
	4	0.1296	0.2401	0.3164	0.4096	0.5220	0.6561	0.8145	0.9606
5	0	0.0102	0.0024	0.0010	0.0003	0.0001	0.0000	0.0000	0.0000
	1	0.0768	0.0284	0.0146	0.0064	0.0022	0.0005	0.0000	0.0000
	2	0.2304	0.1323	0.0879	0.0512	0.0244	0.0081	0.0011	0.0000
	3	0.3456	0.3087	0.2637	0.2048	0.1382	0.0729	0.0214	0.0010
	4	0.2592	0.3602	0.3955	0.4096	0.3915	0.3281	0.2036	0.0480
	5	0.0778	0.1681	0.2373	0.3277	0.4437	0.5905	0.7738	0.9510
6	0	0.0041	0.0007	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000
	1	0.0369	0.0102	0.0044	0.0015	0.0004	0.0001	0.0000	0.0000
	2	0.1382	0.0595	0.0330	0.0154	0.0055	0.0012	0.0001	0.0000
	3	0.2765	0.1852	0.1318	0.0819	0.0415	0.0146	0.0021	0.0000
	4	0.3110	0.3241	0.2966	0.2458	0.1762	0.0984	0.0305	0.0014
	5	0.1866	0.3025	0.3560	0.3932	0.3993	0.3543	0.2321	0.0571
	6	0.0467	0.1176	0.1780	0.2621	0.3771	0.5314	0.7351	0.9415
7	0	0.0016	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0172	0.0036	0.0013	0.0004	0.0001	0.0000	0.0000	0.0000
	2	0.0774	0.0250	0.0115	0.0043	0.0012	0.0002	0.0000	0.0000
	3	0.1935	0.0972	0.0577	0.0287	0.0109	0.0026	0.0002	0.0000
	4	0.2903	0.2269	0.1730	0.1147	0.0617	0.0230	0.0036	0.0000

Binomial Distribution – Probability Mass Function

n	x	θ							
		0.01	**0.05**	**0.10**	**0.15**	**0.20**	**0.30**	**0.40**	**0.50**
7	5	0.0000	0.0000	0.0002	0.0012	0.0043	0.0250	0.0774	0.1641
	6	0.0000	0.0000	0.0000	0.0001	0.0004	0.0036	0.0172	0.0547
	7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0016	0.0078
8	0	0.9227	0.6634	0.4305	0.2725	0.1678	0.0576	0.0168	0.0039
	1	0.0746	0.2793	0.3826	0.3847	0.3355	0.1977	0.0896	0.0313
	2	0.0026	0.0515	0.1488	0.2376	0.2936	0.2965	0.2090	0.1094
	3	0.0001	0.0054	0.0331	0.0839	0.1468	0.2541	0.2787	0.2188
	4	0.0000	0.0004	0.0046	0.0185	0.0459	0.1361	0.2322	0.2734
	5	0.0000	0.0000	0.0004	0.0026	0.0092	0.0467	0.1239	0.2188
	6	0.0000	0.0000	0.0000	0.0002	0.0011	0.0100	0.0413	0.1094
	7	0.0000	0.0000	0.0000	0.0000	0.0001	0.0012	0.0079	0.0313
	8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0007	0.0039
9	0	0.9135	0.6302	0.3874	0.2316	0.1342	0.0404	0.0101	0.0020
	1	0.0830	0.2985	0.3874	0.3679	0.3020	0.1556	0.0605	0.0176
	2	0.0034	0.0629	0.1722	0.2597	0.3020	0.2668	0.1612	0.0703
	3	0.0001	0.0077	0.0446	0.1069	0.1762	0.2668	0.2508	0.1641
	4	0.0000	0.0006	0.0074	0.0283	0.0661	0.1715	0.2508	0.2461
	5	0.0000	0.0000	0.0008	0.0050	0.0165	0.0735	0.1672	0.2461
	6	0.0000	0.0000	0.0001	0.0006	0.0028	0.0210	0.0743	0.1641
	7	0.0000	0.0000	0.0000	0.0000	0.0003	0.0039	0.0212	0.0703
	8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0035	0.0176
	9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0020
10	0	0.9044	0.5987	0.3487	0.1969	0.1074	0.0282	0.0060	0.0010
	1	0.0914	0.3151	0.3874	0.3474	0.2684	0.1211	0.0403	0.0098
	2	0.0042	0.0746	0.1937	0.2759	0.3020	0.2335	0.1209	0.0439
	3	0.0001	0.0105	0.0574	0.1298	0.2013	0.2668	0.2150	0.1172
	4	0.0000	0.0010	0.0112	0.0401	0.0881	0.2001	0.2508	0.2051
	5	0.0000	0.0001	0.0015	0.0085	0.0264	0.1029	0.2007	0.2461
	6	0.0000	0.0000	0.0001	0.0012	0.0055	0.0368	0.1115	0.2051
	7	0.0000	0.0000	0.0000	0.0001	0.0008	0.0090	0.0425	0.1172
	8	0.0000	0.0000	0.0000	0.0000	0.0001	0.0014	0.0106	0.0439
	9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0016	0.0098
	10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0010
11	0	0.8953	0.5688	0.3138	0.1673	0.0859	0.0198	0.0036	0.0005
	1	0.0995	0.3293	0.3835	0.3248	0.2362	0.0932	0.0266	0.0054
	2	0.0050	0.0867	0.2131	0.2866	0.2953	0.1998	0.0887	0.0269
	3	0.0002	0.0137	0.0710	0.1517	0.2215	0.2568	0.1774	0.0806
	4	0.0000	0.0014	0.0158	0.0536	0.1107	0.2201	0.2365	0.1611

Binomial Distribution – Probability Mass Function

n	x	θ							
		0.60	**0.70**	**0.75**	**0.80**	**0.85**	**0.90**	**0.95**	**0.99**
7	5	0.2613	0.3177	0.3115	0.2753	0.2097	0.1240	0.0406	0.0020
	6	0.1306	0.2471	0.3115	0.3670	0.3960	0.3720	0.2573	0.0659
	7	0.0280	0.0824	0.1335	0.2097	0.3206	0.4783	0.6983	0.9321
8	0	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0079	0.0012	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000
	2	0.0413	0.0100	0.0038	0.0011	0.0002	0.0000	0.0000	0.0000
	3	0.1239	0.0467	0.0231	0.0092	0.0026	0.0004	0.0000	0.0000
	4	0.2322	0.1361	0.0865	0.0459	0.0185	0.0046	0.0004	0.0000
	5	0.2787	0.2541	0.2076	0.1468	0.0839	0.0331	0.0054	0.0001
	6	0.2090	0.2965	0.3115	0.2936	0.2376	0.1488	0.0515	0.0026
	7	0.0896	0.1977	0.2670	0.3355	0.3847	0.3826	0.2793	0.0746
	8	0.0168	0.0576	0.1001	0.1678	0.2725	0.4305	0.6634	0.9227
9	0	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0035	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0212	0.0039	0.0012	0.0003	0.0000	0.0000	0.0000	0.0000
	3	0.0743	0.0210	0.0087	0.0028	0.0006	0.0001	0.0000	0.0000
	4	0.1672	0.0735	0.0389	0.0165	0.0050	0.0008	0.0000	0.0000
	5	0.2508	0.1715	0.1168	0.0661	0.0283	0.0074	0.0006	0.0000
	6	0.2508	0.2668	0.2336	0.1762	0.1069	0.0446	0.0077	0.0001
	7	0.1612	0.2668	0.3003	0.3020	0.2597	0.1722	0.0629	0.0034
	8	0.0605	0.1556	0.2253	0.3020	0.3679	0.3874	0.2985	0.0830
	9	0.0101	0.0404	0.0751	0.1342	0.2316	0.3874	0.6302	0.9135
10	0	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0016	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0106	0.0014	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000
	3	0.0425	0.0090	0.0031	0.0008	0.0001	0.0000	0.0000	0.0000
	4	0.1115	0.0368	0.0162	0.0055	0.0012	0.0001	0.0000	0.0000
	5	0.2007	0.1029	0.0584	0.0264	0.0085	0.0015	0.0001	0.0000
	6	0.2508	0.2001	0.1460	0.0881	0.0401	0.0112	0.0010	0.0000
	7	0.2150	0.2668	0.2503	0.2013	0.1298	0.0574	0.0105	0.0001
	8	0.1209	0.2335	0.2816	0.3020	0.2759	0.1937	0.0746	0.0042
	9	0.0403	0.1211	0.1877	0.2684	0.3474	0.3874	0.3151	0.0914
	10	0.0060	0.0282	0.0563	0.1074	0.1969	0.3487	0.5987	0.9044
11	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0007	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0052	0.0005	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0234	0.0037	0.0011	0.0002	0.0000	0.0000	0.0000	0.0000
	4	0.0701	0.0173	0.0064	0.0017	0.0003	0.0000	0.0000	0.0000

Binomial Distribution – Probability Mass Function

n	x	θ							
		0.01	**0.05**	**0.10**	**0.15**	**0.20**	**0.30**	**0.40**	**0.50**
11	5	0.0000	0.0001	0.0025	0.0132	0.0388	0.1321	0.2207	0.2256
	6	0.0000	0.0000	0.0003	0.0023	0.0097	0.0566	0.1471	0.2256
	7	0.0000	0.0000	0.0000	0.0003	0.0017	0.0173	0.0701	0.1611
	8	0.0000	0.0000	0.0000	0.0000	0.0002	0.0037	0.0234	0.0806
	9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.0052	0.0269
	10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0007	0.0054
	11	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005
12	0	0.8864	0.5404	0.2824	0.1422	0.0687	0.0138	0.0022	0.0002
	1	0.1074	0.3413	0.3766	0.3012	0.2062	0.0712	0.0174	0.0029
	2	0.0060	0.0988	0.2301	0.2924	0.2835	0.1678	0.0639	0.0161
	3	0.0002	0.0173	0.0852	0.1720	0.2362	0.2397	0.1419	0.0537
	4	0.0000	0.0021	0.0213	0.0683	0.1329	0.2311	0.2128	0.1208
	5	0.0000	0.0002	0.0038	0.0193	0.0532	0.1585	0.2270	0.1934
	6	0.0000	0.0000	0.0005	0.0040	0.0155	0.0792	0.1766	0.2256
	7	0.0000	0.0000	0.0000	0.0006	0.0033	0.0291	0.1009	0.1934
	8	0.0000	0.0000	0.0000	0.0001	0.0005	0.0078	0.0420	0.1208
	9	0.0000	0.0000	0.0000	0.0000	0.0001	0.0015	0.0125	0.0537
	10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0025	0.0161
	11	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0029
	12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002
13	0	0.8775	0.5133	0.2542	0.1209	0.0550	0.0097	0.0013	0.0001
	1	0.1152	0.3512	0.3672	0.2774	0.1787	0.0540	0.0113	0.0016
	2	0.0070	0.1109	0.2448	0.2937	0.2680	0.1388	0.0453	0.0095
	3	0.0003	0.0214	0.0997	0.1900	0.2457	0.2181	0.1107	0.0349
	4	0.0000	0.0028	0.0277	0.0838	0.1535	0.2337	0.1845	0.0873
	5	0.0000	0.0003	0.0055	0.0266	0.0691	0.1803	0.2214	0.1571
	6	0.0000	0.0000	0.0008	0.0063	0.0230	0.1030	0.1968	0.2095
	7	0.0000	0.0000	0.0001	0.0011	0.0058	0.0442	0.1312	0.2095
	8	0.0000	0.0000	0.0000	0.0001	0.0011	0.0142	0.0656	0.1571
	9	0.0000	0.0000	0.0000	0.0000	0.0001	0.0034	0.0243	0.0873
	10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0006	0.0065	0.0349
	11	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0012	0.0095
	12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0016
	13	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
14	0	0.8687	0.4877	0.2288	0.1028	0.0440	0.0068	0.0008	0.0001
	1	0.1229	0.3593	0.3559	0.2539	0.1539	0.0407	0.0073	0.0009
	2	0.0081	0.1229	0.2570	0.2912	0.2501	0.1134	0.0317	0.0056
	3	0.0003	0.0259	0.1142	0.2056	0.2501	0.1943	0.0845	0.0222

Binomial Distribution – Probability Mass Function

n	x	θ							
		0.60	0.70	0.75	0.80	0.85	0.90	0.95	0.99
11	5	0.1471	0.0566	0.0268	0.0097	0.0023	0.0003	0.0000	0.0000
	6	0.2207	0.1321	0.0803	0.0388	0.0132	0.0025	0.0001	0.0000
	7	0.2365	0.2201	0.1721	0.1107	0.0536	0.0158	0.0014	0.0000
	8	0.1774	0.2568	0.2581	0.2215	0.1517	0.0710	0.0137	0.0002
	9	0.0887	0.1998	0.2581	0.2953	0.2866	0.2131	0.0867	0.0050
	10	0.0266	0.0932	0.1549	0.2362	0.3248	0.3835	0.3293	0.0995
	11	0.0036	0.0198	0.0422	0.0859	0.1673	0.3138	0.5688	0.8953
12	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0025	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0125	0.0015	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000
	4	0.0420	0.0078	0.0024	0.0005	0.0001	0.0000	0.0000	0.0000
	5	0.1009	0.0291	0.0115	0.0033	0.0006	0.0000	0.0000	0.0000
	6	0.1766	0.0792	0.0401	0.0155	0.0040	0.0005	0.0000	0.0000
	7	0.2270	0.1585	0.1032	0.0532	0.0193	0.0038	0.0002	0.0000
	8	0.2128	0.2311	0.1936	0.1329	0.0683	0.0213	0.0021	0.0000
	9	0.1419	0.2397	0.2581	0.2362	0.1720	0.0852	0.0173	0.0002
	10	0.0639	0.1678	0.2323	0.2835	0.2924	0.2301	0.0988	0.0060
	11	0.0174	0.0712	0.1267	0.2062	0.3012	0.3766	0.3413	0.1074
	12	0.0022	0.0138	0.0317	0.0687	0.1422	0.2824	0.5404	0.8864
13	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0012	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0065	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.0243	0.0034	0.0009	0.0001	0.0000	0.0000	0.0000	0.0000
	5	0.0656	0.0142	0.0047	0.0011	0.0001	0.0000	0.0000	0.0000
	6	0.1312	0.0442	0.0186	0.0058	0.0011	0.0001	0.0000	0.0000
	7	0.1968	0.1030	0.0559	0.0230	0.0063	0.0008	0.0000	0.0000
	8	0.2214	0.1803	0.1258	0.0691	0.0266	0.0055	0.0003	0.0000
	9	0.1845	0.2337	0.2097	0.1535	0.0838	0.0277	0.0028	0.0000
	10	0.1107	0.2181	0.2517	0.2457	0.1900	0.0997	0.0214	0.0003
	11	0.0453	0.1388	0.2059	0.2680	0.2937	0.2448	0.1109	0.0070
	12	0.0113	0.0540	0.1029	0.1787	0.2774	0.3672	0.3512	0.1152
	13	0.0013	0.0097	0.0238	0.0550	0.1209	0.2542	0.5133	0.8775
14	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0033	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Binomial Distribution – Probability Mass Function

n	x	θ							
		0.01	**0.05**	**0.10**	**0.15**	**0.20**	**0.30**	**0.40**	**0.50**
14	4	0.0000	0.0037	0.0349	0.0998	0.1720	0.2290	0.1549	0.0611
	5	0.0000	0.0004	0.0078	0.0352	0.0860	0.1963	0.2066	0.1222
	6	0.0000	0.0000	0.0013	0.0093	0.0322	0.1262	0.2066	0.1833
	7	0.0000	0.0000	0.0002	0.0019	0.0092	0.0618	0.1574	0.2095
	8	0.0000	0.0000	0.0000	0.0003	0.0020	0.0232	0.0918	0.1833
	9	0.0000	0.0000	0.0000	0.0000	0.0003	0.0066	0.0408	0.1222
	10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0014	0.0136	0.0611
	11	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0033	0.0222
	12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.0056
	13	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0009
	14	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
15	0	0.8601	0.4633	0.2059	0.0874	0.0352	0.0047	0.0005	0.0000
	1	0.1303	0.3658	0.3432	0.2312	0.1319	0.0305	0.0047	0.0005
	2	0.0092	0.1348	0.2669	0.2856	0.2309	0.0916	0.0219	0.0032
	3	0.0004	0.0307	0.1285	0.2184	0.2501	0.1700	0.0634	0.0139
	4	0.0000	0.0049	0.0428	0.1156	0.1876	0.2186	0.1268	0.0417
	5	0.0000	0.0006	0.0105	0.0449	0.1032	0.2061	0.1859	0.0916
	6	0.0000	0.0000	0.0019	0.0132	0.0430	0.1472	0.2066	0.1527
	7	0.0000	0.0000	0.0003	0.0030	0.0138	0.0811	0.1771	0.1964
	8	0.0000	0.0000	0.0000	0.0005	0.0035	0.0348	0.1181	0.1964
	9	0.0000	0.0000	0.0000	0.0001	0.0007	0.0116	0.0612	0.1527
	10	0.0000	0.0000	0.0000	0.0000	0.0001	0.0030	0.0245	0.0916
	11	0.0000	0.0000	0.0000	0.0000	0.0000	0.0006	0.0074	0.0417
	12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0016	0.0139
	13	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0032
	14	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005
	15	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
20	0	0.8179	0.3585	0.1216	0.0388	0.0115	0.0008	0.0000	0.0000
	1	0.1652	0.3774	0.2702	0.1368	0.0576	0.0068	0.0005	0.0000
	2	0.0159	0.1887	0.2852	0.2293	0.1369	0.0278	0.0031	0.0002
	3	0.0010	0.0596	0.1901	0.2428	0.2054	0.0716	0.0123	0.0011
	4	0.0000	0.0133	0.0898	0.1821	0.2182	0.1304	0.0350	0.0046
	5	0.0000	0.0022	0.0319	0.1028	0.1746	0.1789	0.0746	0.0148
	6	0.0000	0.0003	0.0089	0.0454	0.1091	0.1916	0.1244	0.0370
	7	0.0000	0.0000	0.0020	0.0160	0.0545	0.1643	0.1659	0.0739
	8	0.0000	0.0000	0.0004	0.0046	0.0222	0.1144	0.1797	0.1201
	9	0.0000	0.0000	0.0001	0.0011	0.0074	0.0654	0.1597	0.1602
	10	0.0000	0.0000	0.0000	0.0002	0.0020	0.0308	0.1171	0.1762

Binomial Distribution – Probability Mass Function

n	x	θ 0.60	0.70	0.75	0.80	0.85	0.90	0.95	0.99
14	4	0.0136	0.0014	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.0408	0.0066	0.0018	0.0003	0.0000	0.0000	0.0000	0.0000
	6	0.0918	0.0232	0.0082	0.0020	0.0003	0.0000	0.0000	0.0000
	7	0.1574	0.0618	0.0280	0.0092	0.0019	0.0002	0.0000	0.0000
	8	0.2066	0.1262	0.0734	0.0322	0.0093	0.0013	0.0000	0.0000
	9	0.2066	0.1963	0.1468	0.0860	0.0352	0.0078	0.0004	0.0000
	10	0.1549	0.2290	0.2202	0.1720	0.0998	0.0349	0.0037	0.0000
	11	0.0845	0.1943	0.2402	0.2501	0.2056	0.1142	0.0259	0.0003
	12	0.0317	0.1134	0.1802	0.2501	0.2912	0.2570	0.1229	0.0081
	13	0.0073	0.0407	0.0832	0.1539	0.2539	0.3559	0.3593	0.1229
	14	0.0008	0.0068	0.0178	0.0440	0.1028	0.2288	0.4877	0.8687
15	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0016	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.0074	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.0245	0.0030	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000
	6	0.0612	0.0116	0.0034	0.0007	0.0001	0.0000	0.0000	0.0000
	7	0.1181	0.0348	0.0131	0.0035	0.0005	0.0000	0.0000	0.0000
	8	0.1771	0.0811	0.0393	0.0138	0.0030	0.0003	0.0000	0.0000
	9	0.2066	0.1472	0.0917	0.0430	0.0132	0.0019	0.0000	0.0000
	10	0.1859	0.2061	0.1651	0.1032	0.0449	0.0105	0.0006	0.0000
	11	0.1268	0.2186	0.2252	0.1876	0.1156	0.0428	0.0049	0.0000
	12	0.0634	0.1700	0.2252	0.2501	0.2184	0.1285	0.0307	0.0004
	13	0.0219	0.0916	0.1559	0.2309	0.2856	0.2669	0.1348	0.0092
	14	0.0047	0.0305	0.0668	0.1319	0.2312	0.3432	0.3658	0.1303
	15	0.0005	0.0047	0.0134	0.0352	0.0874	0.2059	0.4633	0.8601
20	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	6	0.0049	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	7	0.0146	0.0010	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
	8	0.0355	0.0039	0.0008	0.0001	0.0000	0.0000	0.0000	0.0000
	9	0.0710	0.0120	0.0030	0.0005	0.0000	0.0000	0.0000	0.0000
	10	0.1171	0.0308	0.0099	0.0020	0.0002	0.0000	0.0000	0.0000

Binomial Distribution – Probability Mass Function

n	x	θ							
		0.01	**0.05**	**0.10**	**0.15**	**0.20**	**0.30**	**0.40**	**0.50**
20	11	0.0000	0.0000	0.0000	0.0000	0.0005	0.0120	0.0710	0.1602
	12	0.0000	0.0000	0.0000	0.0000	0.0001	0.0039	0.0355	0.1201
	13	0.0000	0.0000	0.0000	0.0000	0.0000	0.0010	0.0146	0.0739
	14	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0049	0.0370
	15	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0013	0.0148
	16	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0046
	17	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0011
	18	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002
	19	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
30	0	0.7397	0.2146	0.0424	0.0076	0.0012	0.0000	0.0000	0.0000
	1	0.2242	0.3389	0.1413	0.0404	0.0093	0.0003	0.0000	0.0000
	2	0.0328	0.2586	0.2277	0.1034	0.0337	0.0018	0.0000	0.0000
	3	0.0031	0.1270	0.2361	0.1703	0.0785	0.0072	0.0003	0.0000
	4	0.0002	0.0451	0.1771	0.2028	0.1325	0.0208	0.0012	0.0000
	5	0.0000	0.0124	0.1023	0.1861	0.1723	0.0464	0.0041	0.0001
	6	0.0000	0.0027	0.0474	0.1368	0.1795	0.0829	0.0115	0.0006
	7	0.0000	0.0005	0.0180	0.0828	0.1538	0.1219	0.0263	0.0019
	8	0.0000	0.0001	0.0058	0.0420	0.1106	0.1501	0.0505	0.0055
	9	0.0000	0.0000	0.0016	0.0181	0.0676	0.1573	0.0823	0.0133
	10	0.0000	0.0000	0.0004	0.0067	0.0355	0.1416	0.1152	0.0280
	11	0.0000	0.0000	0.0001	0.0022	0.0161	0.1103	0.1396	0.0509
	12	0.0000	0.0000	0.0000	0.0006	0.0064	0.0749	0.1474	0.0806
	13	0.0000	0.0000	0.0000	0.0001	0.0022	0.0444	0.1360	0.1115
	14	0.0000	0.0000	0.0000	0.0000	0.0007	0.0231	0.1101	0.1354
	15	0.0000	0.0000	0.0000	0.0000	0.0002	0.0106	0.0783	0.1445
	16	0.0000	0.0000	0.0000	0.0000	0.0000	0.0042	0.0489	0.1354
	17	0.0000	0.0000	0.0000	0.0000	0.0000	0.0015	0.0269	0.1115
	18	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.0129	0.0806
	19	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0054	0.0509
	20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0020	0.0280
	21	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0006	0.0133
	22	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0055
	23	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0019
	24	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0006
	25	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
	26	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Binomial Distribution – Probability Mass Function

n	x	θ							
		0.60	**0.70**	**0.75**	**0.80**	**0.85**	**0.90**	**0.95**	**0.99**
20	11	0.1597	0.0654	0.0271	0.0074	0.0011	0.0001	0.0000	0.0000
	12	0.1797	0.1144	0.0609	0.0222	0.0046	0.0004	0.0000	0.0000
	13	0.1659	0.1643	0.1124	0.0545	0.0160	0.0020	0.0000	0.0000
	14	0.1244	0.1916	0.1686	0.1091	0.0454	0.0089	0.0003	0.0000
	15	0.0746	0.1789	0.2023	0.1746	0.1028	0.0319	0.0022	0.0000
	16	0.0350	0.1304	0.1897	0.2182	0.1821	0.0898	0.0133	0.0000
	17	0.0123	0.0716	0.1339	0.2054	0.2428	0.1901	0.0596	0.0010
	18	0.0031	0.0278	0.0669	0.1369	0.2293	0.2852	0.1887	0.0159
	19	0.0005	0.0068	0.0211	0.0576	0.1368	0.2702	0.3774	0.1652
	20	0.0000	0.0008	0.0032	0.0115	0.0388	0.1216	0.3585	0.8179
30	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	8	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	9	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	10	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	11	0.0054	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	12	0.0129	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	13	0.0269	0.0015	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
	14	0.0489	0.0042	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000
	15	0.0783	0.0106	0.0019	0.0002	0.0000	0.0000	0.0000	0.0000
	16	0.1101	0.0231	0.0054	0.0007	0.0000	0.0000	0.0000	0.0000
	17	0.1360	0.0444	0.0134	0.0022	0.0001	0.0000	0.0000	0.0000
	18	0.1474	0.0749	0.0291	0.0064	0.0006	0.0000	0.0000	0.0000
	19	0.1396	0.1103	0.0551	0.0161	0.0022	0.0001	0.0000	0.0000
	20	0.1152	0.1416	0.0909	0.0355	0.0067	0.0004	0.0000	0.0000
	21	0.0823	0.1573	0.1298	0.0676	0.0181	0.0016	0.0000	0.0000
	22	0.0505	0.1501	0.1593	0.1106	0.0420	0.0058	0.0001	0.0000
	23	0.0263	0.1219	0.1662	0.1538	0.0828	0.0180	0.0005	0.0000
	24	0.0115	0.0829	0.1455	0.1795	0.1368	0.0474	0.0027	0.0000
	25	0.0041	0.0464	0.1047	0.1723	0.1861	0.1023	0.0124	0.0000
	26	0.0012	0.0208	0.0604	0.1325	0.2028	0.1771	0.0451	0.0002

Binomial Distribution – Probability Mass Function

n	x	θ 0.01	0.05	0.10	0.15	0.20	0.30	0.40	0.50
50	0	0.6050	0.0769	0.0052	0.0003	0.0000	0.0000	0.0000	0.0000
	1	0.3056	0.2025	0.0286	0.0026	0.0002	0.0000	0.0000	0.0000
	2	0.0756	0.2611	0.0779	0.0113	0.0011	0.0000	0.0000	0.0000
	3	0.0122	0.2199	0.1386	0.0319	0.0044	0.0000	0.0000	0.0000
	4	0.0015	0.1360	0.1809	0.0661	0.0128	0.0001	0.0000	0.0000
	5	0.0001	0.0658	0.1849	0.1072	0.0295	0.0006	0.0000	0.0000
	6	0.0000	0.0260	0.1541	0.1419	0.0554	0.0018	0.0000	0.0000
	7	0.0000	0.0086	0.1076	0.1575	0.0870	0.0048	0.0000	0.0000
	8	0.0000	0.0024	0.0643	0.1493	0.1169	0.0110	0.0002	0.0000
	9	0.0000	0.0006	0.0333	0.1230	0.1364	0.0220	0.0005	0.0000
	10	0.0000	0.0001	0.0152	0.0890	0.1398	0.0386	0.0014	0.0000
	11	0.0000	0.0000	0.0061	0.0571	0.1271	0.0602	0.0035	0.0000
	12	0.0000	0.0000	0.0022	0.0328	0.1033	0.0838	0.0076	0.0001
	13	0.0000	0.0000	0.0007	0.0169	0.0755	0.1050	0.0147	0.0003
	14	0.0000	0.0000	0.0002	0.0079	0.0499	0.1189	0.0260	0.0008
	15	0.0000	0.0000	0.0001	0.0033	0.0299	0.1223	0.0415	0.0020
	16	0.0000	0.0000	0.0000	0.0013	0.0164	0.1147	0.0606	0.0044
	17	0.0000	0.0000	0.0000	0.0005	0.0082	0.0983	0.0808	0.0087
	18	0.0000	0.0000	0.0000	0.0001	0.0037	0.0772	0.0987	0.0160
	19	0.0000	0.0000	0.0000	0.0000	0.0016	0.0558	0.1109	0.0270
	20	0.0000	0.0000	0.0000	0.0000	0.0006	0.0370	0.1146	0.0419
	21	0.0000	0.0000	0.0000	0.0000	0.0002	0.0227	0.1091	0.0598
	22	0.0000	0.0000	0.0000	0.0000	0.0001	0.0128	0.0959	0.0788
	23	0.0000	0.0000	0.0000	0.0000	0.0000	0.0067	0.0778	0.0960
	24	0.0000	0.0000	0.0000	0.0000	0.0000	0.0032	0.0584	0.1080
	25	0.0000	0.0000	0.0000	0.0000	0.0000	0.0014	0.0405	0.1123
	26	0.0000	0.0000	0.0000	0.0000	0.0000	0.0006	0.0259	0.1080
	27	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0154	0.0960
	38	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0084	0.0788
	29	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0043	0.0598
	30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0020	0.0419
	31	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0009	0.0270
	32	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0160
	33	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0087
	34	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0044
	35	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0020
	40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	45	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Binomial Distribution – Probability Mass Function

n	x	θ							
		0.60	**0.70**	**0.75**	**0.80**	**0.85**	**0.90**	**0.95**	**0.99**
50	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	11	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	13	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	14	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	15	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	16	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	17	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	18	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	19	0.0009	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	20	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	21	0.0043	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	22	0.0084	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	23	0.0154	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	24	0.0259	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	25	0.0405	0.0014	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	26	0.0584	0.0032	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
	27	0.0778	0.0067	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000
	38	0.0959	0.0128	0.0016	0.0001	0.0000	0.0000	0.0000	0.0000
	29	0.1091	0.0227	0.0036	0.0002	0.0000	0.0000	0.0000	0.0000
	30	0.1146	0.0370	0.0077	0.0006	0.0000	0.0000	0.0000	0.0000
	31	0.1109	0.0558	0.0148	0.0016	0.0000	0.0000	0.0000	0.0000
	32	0.0987	0.0772	0.0264	0.0037	0.0001	0.0000	0.0000	0.0000
	33	0.0808	0.0983	0.0432	0.0082	0.0005	0.0000	0.0000	0.0000
	34	0.0606	0.1147	0.0648	0.0164	0.0013	0.0000	0.0000	0.0000
	35	0.0415	0.1223	0.0888	0.0299	0.0033	0.0001	0.0000	0.0000
	40	0.0014	0.0386	0.0985	0.1398	0.0890	0.0152	0.0001	0.0000
	45	0.0000	0.0006	0.0049	0.0295	0.1072	0.1849	0.0658	0.0001
	50	0.0000	0.0000	0.0000	0.0000	0.0003	0.0052	0.0769	0.6050

Binomial Distribution – Cumulative Distribution Function

$$F_B(x \mid n;\ \theta) = \sum_{\nu=0}^{x} \binom{n}{\nu} \theta^{\nu} (1-\theta)^{n-\nu} \qquad 0 < \theta < 1$$

n	x	θ							
		0.01	**0.05**	**0.10**	**0.15**	**0.20**	**0.30**	**0.40**	**0.50**
1	0	0.9900	0.9500	0.9000	0.8500	0.8000	0.7000	0.6000	0.5000
	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	0	0.9801	0.9025	0.8100	0.7225	0.6400	0.4900	0.3600	0.2500
	1	0.9999	0.9975	0.9900	0.9775	0.9600	0.9100	0.8400	0.7500
	2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3	0	0.9703	0.8574	0.7290	0.6141	0.5120	0.3430	0.2160	0.1250
	1	0.9997	0.9928	0.9720	0.9393	0.8960	0.7840	0.6480	0.5000
	2	1.0000	0.9999	0.9990	0.9966	0.9920	0.9730	0.9360	0.8750
	3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4	0	0.9606	0.8145	0.6561	0.5220	0.4096	0.2401	0.1296	0.0625
	1	0.9994	0.9860	0.9477	0.8905	0.8192	0.6517	0.4752	0.3125
	2	1.0000	0.9995	0.9963	0.9880	0.9728	0.9163	0.8208	0.6875
	3	1.0000	1.0000	0.9999	0.9995	0.9984	0.9919	0.9744	0.9375
	4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	0	0.9510	0.7738	0.5905	0.4437	0.3277	0.1681	0.0778	0.0313
	1	0.9990	0.9774	0.9185	0.8352	0.7373	0.5282	0.3370	0.1875
	2	1.0000	0.9988	0.9914	0.9734	0.9421	0.8369	0.6826	0.5000
	3	1.0000	1.0000	0.9995	0.9978	0.9933	0.9692	0.9130	0.8125
	4	1.0000	1.0000	1.0000	0.9999	0.9997	0.9976	0.9898	0.9688
	5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
6	0	0.9415	0.7351	0.5314	0.3771	0.2621	0.1176	0.0467	0.0156
	1	0.9985	0.9672	0.8857	0.7765	0.6554	0.4202	0.2333	0.1094
	2	1.0000	0.9978	0.9842	0.9527	0.9011	0.7443	0.5443	0.3438
	3	1.0000	0.9999	0.9987	0.9941	0.9830	0.9295	0.8208	0.6563
	4	1.0000	1.0000	0.9999	0.9996	0.9984	0.9891	0.9590	0.8906
	5	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993	0.9959	0.9844
	6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
7	0	0.9321	0.6983	0.4783	0.3206	0.2097	0.0824	0.0280	0.0078
	1	0.9980	0.9556	0.8503	0.7166	0.5767	0.3294	0.1586	0.0625
	2	1.0000	0.9962	0.9743	0.9262	0.8520	0.6471	0.4199	0.2266
	3	1.0000	0.9998	0.9973	0.9879	0.9667	0.8740	0.7102	0.5000
	4	1.0000	1.0000	0.9998	0.9988	0.9953	0.9712	0.9037	0.7734
	5	1.0000	1.0000	1.0000	0.9999	0.9996	0.9962	0.9812	0.9375
	6	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9984	0.9922
	7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Binomial Distribution – Cumulative Distribution Function

n	x	θ							
		0.60	**0.70**	**0.75**	**0.80**	**0.85**	**0.90**	**0.95**	**0.99**
1	0	0.4000	0.3000	0.2500	0.2000	0.1500	0.1000	0.0500	0.0100
	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	0	0.1600	0.0900	0.0625	0.0400	0.0225	0.0100	0.0025	0.0001
	1	0.6400	0.5100	0.4375	0.3600	0.2775	0.1900	0.0975	0.0199
	2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3	0	0.0640	0.0270	0.0156	0.0080	0.0034	0.0010	0.0001	0.0000
	1	0.3520	0.2160	0.1563	0.1040	0.0608	0.0280	0.0073	0.0003
	2	0.7840	0.6570	0.5781	0.4880	0.3859	0.2710	0.1426	0.0297
	3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4	0	0.0256	0.0081	0.0039	0.0016	0.0005	0.0001	0.0000	0.0000
	1	0.1792	0.0837	0.0508	0.0272	0.0120	0.0037	0.0005	0.0000
	2	0.5248	0.3483	0.2617	0.1808	0.1095	0.0523	0.0140	0.0006
	3	0.8704	0.7599	0.6836	0.5904	0.4780	0.3439	0.1855	0.0394
	4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	0	0.0102	0.0024	0.0010	0.0003	0.0001	0.0000	0.0000	0.0000
	1	0.0870	0.0308	0.0156	0.0067	0.0022	0.0005	0.0000	0.0000
	2	0.3174	0.1631	0.1035	0.0579	0.0266	0.0086	0.0012	0.0000
	3	0.6630	0.4718	0.3672	0.2627	0.1648	0.0815	0.0226	0.0010
	4	0.9222	0.8319	0.7627	0.6723	0.5563	0.4095	0.2262	0.0490
	5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
6	0	0.0041	0.0007	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000
	1	0.0410	0.0109	0.0046	0.0016	0.0004	0.0001	0.0000	0.0000
	2	0.1792	0.0705	0.0376	0.0170	0.0059	0.0013	0.0001	0.0000
	3	0.4557	0.2557	0.1694	0.0989	0.0473	0.0159	0.0022	0.0000
	4	0.7667	0.5798	0.4661	0.3446	0.2235	0.1143	0.0328	0.0015
	5	0.9533	0.8824	0.8220	0.7379	0.6229	0.4686	0.2649	0.0585
	6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
7	0	0.0016	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0188	0.0038	0.0013	0.0004	0.0001	0.0000	0.0000	0.0000
	2	0.0963	0.0288	0.0129	0.0047	0.0012	0.0002	0.0000	0.0000
	3	0.2898	0.1260	0.0706	0.0333	0.0121	0.0027	0.0002	0.0000
	4	0.5801	0.3529	0.2436	0.1480	0.0738	0.0257	0.0038	0.0000
	5	0.8414	0.6706	0.5551	0.4233	0.2834	0.1497	0.0444	0.0020
	6	0.9720	0.9176	0.8665	0.7903	0.6794	0.5217	0.3017	0.0679
	7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Binomial Distribution – Cumulative Distribution Function

n	x	θ							
		0.01	**0.05**	**0.10**	**0.15**	**0.20**	**0.30**	**0.40**	**0.50**
8	0	0.9227	0.6634	0.4305	0.2725	0.1678	0.0576	0.0168	0.0039
	1	0.9973	0.9428	0.8131	0.6572	0.5033	0.2553	0.1064	0.0352
	2	0.9999	0.9942	0.9619	0.8948	0.7969	0.5518	0.3154	0.1445
	3	1.0000	0.9996	0.995	0.9786	0.9437	0.8059	0.5941	0.3633
	4	1.0000	1.0000	0.9996	0.9971	0.9896	0.9420	0.8263	0.6367
	5	1.0000	1.0000	1.0000	0.9998	0.9988	0.9887	0.9502	0.8555
	6	1.0000	1.0000	1.0000	1.0000	0.9999	0.9987	0.9915	0.9648
	7	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993	0.9961
	8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
9	0	0.9135	0.6302	0.3874	0.2316	0.1342	0.0404	0.0101	0.0020
	1	0.9966	0.9288	0.7748	0.5995	0.4362	0.1960	0.0705	0.0195
	2	0.9999	0.9916	0.9470	0.8591	0.7382	0.4628	0.2318	0.0898
	3	1.0000	0.9994	0.9917	0.9661	0.9144	0.7297	0.4826	0.2539
	4	1.0000	1.0000	0.9991	0.9944	0.9804	0.9012	0.7334	0.5000
	5	1.0000	1.0000	0.9999	0.9994	0.9969	0.9747	0.9006	0.7461
	6	1.0000	1.0000	1.0000	1.0000	0.9997	0.9957	0.9750	0.9102
	7	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9962	0.9805
	8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9980
	9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
10	0	0.9044	0.5987	0.3487	0.1969	0.1074	0.0282	0.0060	0.0010
	1	0.9957	0.9139	0.7361	0.5443	0.3758	0.1493	0.0464	0.0107
	2	0.9999	0.9885	0.9298	0.8202	0.6778	0.3828	0.1673	0.0547
	3	1.0000	0.9990	0.9872	0.9500	0.8791	0.6496	0.3823	0.1719
	4	1.0000	0.9999	0.9984	0.9901	0.9672	0.8497	0.6331	0.3770
	5	1.0000	1.0000	0.9999	0.9986	0.9936	0.9527	0.8338	0.6230
	6	1.0000	1.0000	1.0000	0.9999	0.9991	0.9894	0.9452	0.8281
	7	1.0000	1.0000	1.0000	1.0000	0.9999	0.9984	0.9877	0.9453
	8	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9983	0.9893
	9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9990
	10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
11	0	0.8953	0.5688	0.3138	0.1673	0.0859	0.0198	0.0036	0.0005
	1	0.9948	0.8981	0.6974	0.4922	0.3221	0.1130	0.0302	0.0059
	2	0.9998	0.9848	0.9104	0.7788	0.6174	0.3127	0.1189	0.0327
	3	1.0000	0.9984	0.9815	0.9306	0.8389	0.5696	0.2963	0.1133
	4	1.0000	0.9999	0.9972	0.9841	0.9496	0.7897	0.5328	0.2744
	5	1.0000	1.0000	0.9997	0.9973	0.9883	0.9218	0.7535	0.5000
	6	1.0000	1.0000	1.0000	0.9997	0.9980	0.9784	0.9006	0.7256
	7	1.0000	1.0000	1.0000	1.0000	0.9998	0.9957	0.9707	0.8867
	8	1.0000	1.0000	1.0000	1.0000	1.0000	0.9994	0.9941	0.9673
	9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9993	0.9941
	10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995
	11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Binomial Distribution – Cumulative Distribution Function

n	x	θ 0.60	0.70	0.75	0.80	0.85	0.90	0.95	0.99
8	0	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0085	0.0013	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000
	2	0.0498	0.0113	0.0042	0.0012	0.0002	0.0000	0.0000	0.0000
	3	0.1737	0.0580	0.0273	0.0104	0.0029	0.0004	0.0000	0.0000
	4	0.4059	0.1941	0.1138	0.0563	0.0214	0.0050	0.0004	0.0000
	5	0.6846	0.4482	0.3215	0.2031	0.1052	0.0381	0.0058	0.0001
	6	0.8936	0.7447	0.6329	0.4967	0.3428	0.1869	0.0572	0.0027
	7	0.9832	0.9424	0.8999	0.8322	0.7275	0.5695	0.3366	0.0773
	8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
9	0	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0038	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0250	0.0043	0.0013	0.0003	0.0000	0.0000	0.0000	0.0000
	3	0.0994	0.0253	0.0100	0.0031	0.0006	0.0001	0.0000	0.0000
	4	0.2666	0.0988	0.0489	0.0196	0.0056	0.0009	0.0000	0.0000
	5	0.5174	0.2703	0.1657	0.0856	0.0339	0.0083	0.0006	0.0000
	6	0.7682	0.5372	0.3993	0.2618	0.1409	0.0530	0.0084	0.0001
	7	0.9295	0.8040	0.6997	0.5638	0.4005	0.2252	0.0712	0.0034
	8	0.9899	0.9596	0.9249	0.8658	0.7684	0.6126	0.3698	0.0865
	9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
10	0	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0017	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0123	0.0016	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000
	3	0.0548	0.0106	0.0035	0.0009	0.0001	0.0000	0.0000	0.0000
	4	0.1662	0.0473	0.0197	0.0064	0.0014	0.0001	0.0000	0.0000
	5	0.3669	0.1503	0.0781	0.0328	0.0099	0.0016	0.0001	0.0000
	6	0.6177	0.3504	0.2241	0.1209	0.0500	0.0128	0.0010	0.0000
	7	0.8327	0.6172	0.4744	0.3222	0.1798	0.0702	0.0115	0.0001
	8	0.9536	0.8507	0.7560	0.6242	0.4557	0.2639	0.0861	0.0043
	9	0.9940	0.9718	0.9437	0.8926	0.8031	0.6513	0.4013	0.0956
	10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
11	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0007	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0059	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0293	0.0043	0.0012	0.0002	0.0000	0.0000	0.0000	0.0000
	4	0.0994	0.0216	0.0076	0.0020	0.0003	0.0000	0.0000	0.0000
	5	0.2465	0.0782	0.0343	0.0117	0.0027	0.0003	0.0000	0.0000
	6	0.4672	0.2103	0.1146	0.0504	0.0159	0.0028	0.0001	0.0000
	7	0.7037	0.4304	0.2867	0.1611	0.0694	0.0185	0.0016	0.0000
	8	0.8811	0.6873	0.5448	0.3826	0.2212	0.0896	0.0152	0.0002
	9	0.9698	0.8870	0.8029	0.6779	0.5078	0.3026	0.1019	0.0052
	10	0.9964	0.9802	0.9578	0.9141	0.8327	0.6862	0.4312	0.1047
	11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Binomial Distribution – Cumulative Distribution Function

n	x	θ							
		0.01	**0.05**	**0.10**	**0.15**	**0.20**	**0.30**	**0.40**	**0.50**
12	0	0.8864	0.5404	0.2824	0.1422	0.0687	0.0138	0.0022	0.0002
	1	0.9938	0.8816	0.6590	0.4435	0.2749	0.0850	0.0196	0.0032
	2	0.9998	0.9804	0.8891	0.7358	0.5583	0.2528	0.0834	0.0193
	3	1.0000	0.9978	0.9744	0.9078	0.7946	0.4925	0.2253	0.0730
	4	1.0000	0.9998	0.9957	0.9761	0.9274	0.7237	0.4382	0.1938
	5	1.0000	1.0000	0.9995	0.9954	0.9806	0.8822	0.6652	0.3872
	6	1.0000	1.0000	0.9999	0.9993	0.9961	0.9614	0.8418	0.6128
	7	1.0000	1.0000	1.0000	0.9999	0.9994	0.9905	0.9427	0.8062
	8	1.0000	1.0000	1.0000	1.0000	0.9999	0.9983	0.9847	0.9270
	9	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9972	0.9807
	10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9968
	11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
13	0	0.8775	0.5133	0.2542	0.1209	0.0550	0.0097	0.0013	0.0001
	1	0.9928	0.8646	0.6213	0.3983	0.2336	0.0637	0.0126	0.0017
	2	0.9997	0.9755	0.8661	0.6920	0.5017	0.2025	0.0579	0.0112
	3	1.0000	0.9969	0.9658	0.8820	0.7473	0.4206	0.1686	0.0461
	4	1.0000	0.9997	0.9935	0.9658	0.9009	0.6543	0.3530	0.1334
	5	1.0000	1.0000	0.9991	0.9925	0.9700	0.8346	0.5744	0.2905
	6	1.0000	1.0000	0.9999	0.9987	0.9930	0.9376	0.7712	0.5000
	7	1.0000	1.0000	1.0000	0.9998	0.9988	0.9818	0.9023	0.7095
	8	1.0000	1.0000	1.0000	1.0000	0.9998	0.9960	0.9679	0.8666
	9	1.0000	1.0000	1.0000	1.0000	1.0000	0.9993	0.9922	0.9539
	10	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9987	0.9888
	11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9983
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
14	0	0.8687	0.4877	0.2288	0.1028	0.0440	0.0068	0.0008	0.0001
	1	0.9916	0.8470	0.5846	0.3567	0.1979	0.0475	0.0081	0.0009
	2	0.9997	0.9699	0.8416	0.6479	0.4481	0.1608	0.0398	0.0065
	3	1.0000	0.9958	0.9559	0.8535	0.6982	0.3552	0.1243	0.0287
	4	1.0000	0.9996	0.9908	0.9533	0.8702	0.5842	0.2793	0.0898
	5	1.0000	1.0000	0.9985	0.9885	0.9561	0.7805	0.4859	0.2120
	6	1.0000	1.0000	0.9998	0.9978	0.9884	0.9067	0.6925	0.3953
	7	1.0000	1.0000	1.0000	0.9997	0.9976	0.9685	0.8499	0.6047
	8	1.0000	1.0000	1.0000	1.0000	0.9996	0.9917	0.9417	0.7880
	9	1.0000	1.0000	1.0000	1.0000	1.0000	0.9983	0.9825	0.9102
	10	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9961	0.9713
	11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9994	0.9935
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9991
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Binomial Distribution – Cumulative Distribution Function

n	x	θ							
		0.60	**0.70**	**0.75**	**0.80**	**0.85**	**0.90**	**0.95**	**0.99**
12	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0028	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0153	0.0017	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000
	4	0.0573	0.0095	0.0028	0.0006	0.0001	0.0000	0.0000	0.0000
	5	0.1582	0.0386	0.0143	0.0039	0.0007	0.0001	0.0000	0.0000
	6	0.3348	0.1178	0.0544	0.0194	0.0046	0.0005	0.0000	0.0000
	7	0.5618	0.2763	0.1576	0.0726	0.0239	0.0043	0.0002	0.0000
	8	0.7747	0.5075	0.3512	0.2054	0.0922	0.0256	0.0022	0.0000
	9	0.9166	0.7472	0.6093	0.4417	0.2642	0.1109	0.0196	0.0002
	10	0.9804	0.9150	0.8416	0.7251	0.5565	0.3410	0.1184	0.0062
	11	0.9978	0.9862	0.9683	0.9313	0.8578	0.7176	0.4596	0.1136
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
13	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0013	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0078	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.0321	0.0040	0.0010	0.0002	0.0000	0.0000	0.0000	0.0000
	5	0.0977	0.0182	0.0056	0.0012	0.0002	0.0000	0.0000	0.0000
	6	0.2288	0.0624	0.0243	0.0070	0.0013	0.0001	0.0000	0.0000
	7	0.4256	0.1654	0.0802	0.0300	0.0075	0.0009	0.0000	0.0000
	8	0.6470	0.3457	0.2060	0.0991	0.0342	0.0065	0.0003	0.0000
	9	0.8314	0.5794	0.4157	0.2527	0.1180	0.0342	0.0031	0.0000
	10	0.9421	0.7975	0.6674	0.4983	0.3080	0.1339	0.0245	0.0003
	11	0.9874	0.9363	0.8733	0.7664	0.6017	0.3787	0.1354	0.0072
	12	0.9987	0.9903	0.9762	0.9450	0.8791	0.7458	0.4867	0.1225
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
14	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0039	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.0175	0.0017	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.0583	0.0083	0.0022	0.0004	0.0000	0.0000	0.0000	0.0000
	6	0.1501	0.0315	0.0103	0.0024	0.0003	0.0000	0.0000	0.0000
	7	0.3075	0.0933	0.0383	0.0116	0.0022	0.0002	0.0000	0.0000
	8	0.5141	0.2195	0.1117	0.0439	0.0115	0.0015	0.0000	0.0000
	9	0.7207	0.4158	0.2585	0.1298	0.0467	0.0092	0.0004	0.0000
	10	0.8757	0.6448	0.4787	0.3018	0.1465	0.0441	0.0042	0.0000
	11	0.9602	0.8392	0.7189	0.5519	0.3521	0.1584	0.0301	0.0003
	12	0.9919	0.9525	0.8990	0.8021	0.6433	0.4154	0.1530	0.0084
	13	0.9992	0.9932	0.9822	0.9560	0.8972	0.7712	0.5123	0.1313
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Binomial Distribution – Cumulative Distribution Function

n	x	θ							
		0.01	**0.05**	**0.10**	**0.15**	**0.20**	**0.30**	**0.40**	**0.50**
15	0	0.8601	0.4633	0.2059	0.0874	0.0352	0.0047	0.0005	0.0000
	1	0.9904	0.8290	0.5490	0.3186	0.1671	0.0353	0.0052	0.0005
	2	0.9996	0.9638	0.8159	0.6042	0.3980	0.1268	0.0271	0.0037
	3	1.0000	0.9945	0.9444	0.8227	0.6482	0.2969	0.0905	0.0176
	4	1.0000	0.9994	0.9873	0.9383	0.8358	0.5155	0.2173	0.0592
	5	1.0000	0.9999	0.9978	0.9832	0.9389	0.7216	0.4032	0.1509
	6	1.0000	1.0000	0.9997	0.9964	0.9819	0.8689	0.6098	0.3036
	7	1.0000	1.0000	1.0000	0.9994	0.9958	0.9500	0.7869	0.5000
	8	1.0000	1.0000	1.0000	0.9999	0.9992	0.9848	0.9050	0.6964
	9	1.0000	1.0000	1.0000	1.0000	0.9999	0.9963	0.9662	0.8491
	10	1.0000	1.0000	1.0000	1.0000	1.0000	0.9993	0.9907	0.9408
	11	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9981	0.9824
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9963
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
20	0	0.8179	0.3585	0.1216	0.0388	0.0115	0.0008	0.0000	0.0000
	1	0.9831	0.7358	0.3917	0.1756	0.0692	0.0076	0.0005	0.0000
	2	0.9990	0.9245	0.6769	0.4049	0.2061	0.0355	0.0036	0.0002
	3	1.0000	0.9841	0.8670	0.6477	0.4114	0.1071	0.0160	0.0013
	4	1.0000	0.9974	0.9568	0.8298	0.6296	0.2375	0.0510	0.0059
	5	1.0000	0.9997	0.9887	0.9327	0.8042	0.4164	0.1256	0.0207
	6	1.0000	1.0000	0.9976	0.9781	0.9133	0.6080	0.2500	0.0577
	7	1.0000	1.0000	0.9996	0.9941	0.9679	0.7723	0.4159	0.1316
	8	1.0000	1.0000	0.9999	0.9987	0.9900	0.8867	0.5956	0.2517
	9	1.0000	1.0000	1.0000	0.9998	0.9974	0.9520	0.7553	0.4119
	10	1.0000	1.0000	1.0000	1.0000	0.9994	0.9829	0.8725	0.5881
	11	1.0000	1.0000	1.0000	1.0000	0.9999	0.9949	0.9435	0.7483
	12	1.0000	1.0000	1.0000	1.0000	1.0000	0.9987	0.9790	0.8684
	13	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9935	0.9423
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9984	0.9793
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9941
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9987
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998
	18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Binomial Distribution – Cumulative Distribution Function

n	x	θ							
		0.60	**0.70**	**0.75**	**0.80**	**0.85**	**0.90**	**0.95**	**0.99**
15	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0019	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.0093	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.0338	0.0037	0.0008	0.0001	0.0000	0.0000	0.0000	0.0000
	6	0.0950	0.0152	0.0042	0.0008	0.0001	0.0000	0.0000	0.0000
	7	0.2131	0.0500	0.0173	0.0042	0.0006	0.0000	0.0000	0.0000
	8	0.3902	0.1311	0.0566	0.0181	0.0036	0.0003	0.0000	0.0000
	9	0.5968	0.2784	0.1484	0.0611	0.0168	0.0022	0.0001	0.0000
	10	0.7827	0.4845	0.3135	0.1642	0.0617	0.0127	0.0006	0.0000
	11	0.9095	0.7031	0.5387	0.3518	0.1773	0.0556	0.0055	0.0000
	12	0.9729	0.8732	0.7639	0.6020	0.3958	0.1841	0.0362	0.0004
	13	0.9948	0.9647	0.9198	0.8329	0.6814	0.4510	0.1710	0.0096
	14	0.9995	0.9953	0.9866	0.9648	0.9126	0.7941	0.5367	0.1399
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
20	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.0016	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	6	0.0065	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	7	0.0210	0.0013	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
	8	0.0565	0.0051	0.0009	0.0001	0.0000	0.0000	0.0000	0.0000
	9	0.1275	0.0171	0.0039	0.0006	0.0000	0.0000	0.0000	0.0000
	10	0.2447	0.0480	0.0139	0.0026	0.0002	0.0000	0.0000	0.0000
	11	0.4044	0.1133	0.0409	0.0100	0.0013	0.0001	0.0000	0.0000
	12	0.5841	0.2277	0.1018	0.0321	0.0059	0.0004	0.0000	0.0000
	13	0.7500	0.3920	0.2142	0.0867	0.0219	0.0024	0.0000	0.0000
	14	0.8744	0.5836	0.3828	0.1958	0.0673	0.0113	0.0003	0.0000
	15	0.9490	0.7625	0.5852	0.3704	0.1702	0.0432	0.0026	0.0000
	16	0.9840	0.8929	0.7748	0.5886	0.3523	0.1330	0.0159	0.0000
	17	0.9964	0.9645	0.9087	0.7939	0.5951	0.3231	0.0755	0.0010
	18	0.9995	0.9924	0.9757	0.9308	0.8244	0.6083	0.2642	0.0169
	19	1.0000	0.9992	0.9968	0.9885	0.9612	0.8784	0.6415	0.1821
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Binomial Distribution – Cumulative Distribution Function

n	x	θ							
		0.01	**0.05**	**0.10**	**0.15**	**0.20**	**0.30**	**0.40**	**0.50**
30	0	0.7397	0.2146	0.0424	0.0076	0.0012	0.0000	0.0000	0.0000
	1	0.9639	0.5535	0.1837	0.0480	0.0105	0.0003	0.0000	0.0000
	2	0.9967	0.8122	0.4114	0.1514	0.0442	0.0021	0.0000	0.0000
	3	0.9998	0.9392	0.6474	0.3217	0.1227	0.0093	0.0003	0.0000
	4	1.0000	0.9844	0.8245	0.5245	0.2552	0.0302	0.0015	0.0000
	5	1.0000	0.9967	0.9268	0.7106	0.4275	0.0766	0.0057	0.0002
	6	1.0000	0.9994	0.9742	0.8474	0.6070	0.1595	0.0172	0.0007
	7	1.0000	0.9999	0.9922	0.9302	0.7608	0.2814	0.0435	0.0026
	8	1.0000	1.0000	0.9980	0.9722	0.8713	0.4315	0.0940	0.0081
	9	1.0000	1.0000	0.9995	0.9903	0.9389	0.5888	0.1763	0.0214
	10	1.0000	1.0000	0.9999	0.9971	0.9744	0.7304	0.2915	0.0494
	11	1.0000	1.0000	1.0000	0.9992	0.9905	0.8407	0.4311	0.1002
	12	1.0000	1.0000	1.0000	0.9998	0.9969	0.9155	0.5785	0.1808
	13	1.0000	1.0000	1.0000	1.0000	0.9991	0.9599	0.7145	0.2923
	14	1.0000	1.0000	1.0000	1.0000	0.9998	0.9831	0.8246	0.4278
	15	1.0000	1.0000	1.0000	1.0000	0.9999	0.9936	0.9029	0.5722
	16	1.0000	1.0000	1.0000	1.0000	1.0000	0.9979	0.9519	0.7077
	17	1.0000	1.0000	1.0000	1.0000	1.0000	0.9994	0.9788	0.8192
	18	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9917	0.8998
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9971	0.9506
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9991	0.9786
	21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9919
	22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9974
	23	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9993
	24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998
	25	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	26	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
50	0	0.6050	0.0769	0.0052	0.0003	0.0000	0.0000	0.0000	0.0000
	1	0.9106	0.2794	0.0338	0.0029	0.0002	0.0000	0.0000	0.0000
	2	0.9862	0.5405	0.1117	0.0142	0.0013	0.0000	0.0000	0.0000
	3	0.9984	0.7604	0.2503	0.0460	0.0057	0.0000	0.0000	0.0000
	4	0.9999	0.8964	0.4312	0.1121	0.0185	0.0002	0.0000	0.0000
	5	1.0000	0.9622	0.6161	0.2194	0.0480	0.0007	0.0000	0.0000
	6	1.0000	0.9882	0.7702	0.3613	0.1034	0.0025	0.0000	0.0000
	7	1.0000	0.9968	0.8779	0.5188	0.1904	0.0073	0.0001	0.0000
	8	1.0000	0.9992	0.9421	0.6681	0.3073	0.0183	0.0002	0.0000
	9	1.0000	0.9998	0.9755	0.7911	0.4437	0.0402	0.0008	0.0000
	10	1.0000	1.0000	0.9906	0.8801	0.5836	0.0789	0.0022	0.0000

Binomial Distribution – Cumulative Distribution Function

n	x	θ							
		0.60	**0.70**	**0.75**	**0.80**	**0.85**	**0.90**	**0.95**	**0.99**
30	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	8	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	9	0.0009	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	10	0.0029	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	11	0.0083	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	12	0.0212	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	13	0.0481	0.0021	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
	14	0.0971	0.0064	0.0008	0.0001	0.0000	0.0000	0.0000	0.0000
	15	0.1754	0.0169	0.0027	0.0002	0.0000	0.0000	0.0000	0.0000
	16	0.2855	0.0401	0.0082	0.0009	0.0000	0.0000	0.0000	0.0000
	17	0.4215	0.0845	0.0216	0.0031	0.0002	0.0000	0.0000	0.0000
	18	0.5689	0.1593	0.0507	0.0095	0.0008	0.0000	0.0000	0.0000
	19	0.7085	0.2696	0.1057	0.0256	0.0029	0.0001	0.0000	0.0000
	20	0.8237	0.4112	0.1966	0.0611	0.0097	0.0005	0.0000	0.0000
	21	0.9060	0.5685	0.3264	0.1287	0.0278	0.0020	0.0000	0.0000
	22	0.9565	0.7186	0.4857	0.2392	0.0698	0.0078	0.0001	0.0000
	23	0.9828	0.8405	0.6519	0.3930	0.1526	0.0258	0.0006	0.0000
	24	0.9943	0.9234	0.7974	0.5725	0.2894	0.0732	0.0033	0.0000
	25	0.9985	0.9698	0.9021	0.7448	0.4755	0.1755	0.0156	0.0000
	26	0.9997	0.9907	0.9626	0.8773	0.6783	0.3526	0.0608	0.0002
50	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Binomial Distribution – Cumulative Distribution Function

n	x	θ							
		0.01	**0.05**	**0.10**	**0.15**	**0.20**	**0.30**	**0.40**	**0.50**
50	11	1.0000	1.0000	0.9968	0.9372	0.7107	0.1390	0.0057	0.0000
	12	1.0000	1.0000	0.9990	0.9699	0.8139	0.2229	0.0133	0.0002
	13	1.0000	1.0000	0.9997	0.9868	0.8894	0.3279	0.0280	0.0005
	14	1.0000	1.0000	0.9999	0.9947	0.9393	0.4468	0.0540	0.0013
	15	1.0000	1.0000	1.0000	0.9981	0.9692	0.5692	0.0955	0.0033
	16	1.0000	1.0000	1.0000	0.9993	0.9856	0.6839	0.1561	0.0077
	17	1.0000	1.0000	1.0000	0.9998	0.9937	0.7822	0.2369	0.0164
	18	1.0000	1.0000	1.0000	0.9999	0.9975	0.8594	0.3356	0.0325
	19	1.0000	1.0000	1.0000	1.0000	0.9991	0.9152	0.4465	0.0595
	20	1.0000	1.0000	1.0000	1.0000	0.9997	0.9522	0.5610	0.1013
	21	1.0000	1.0000	1.0000	1.0000	0.9999	0.9749	0.6701	0.1611
	22	1.0000	1.0000	1.0000	1.0000	1.0000	0.9877	0.7660	0.2399
	23	1.0000	1.0000	1.0000	1.0000	1.0000	0.9944	0.8438	0.3359
	24	1.0000	1.0000	1.0000	1.0000	1.0000	0.9976	0.9022	0.4439
	25	1.0000	1.0000	1.0000	1.0000	1.0000	0.9991	0.9427	0.5561
	26	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9686	0.6641
	27	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9840	0.7601
	38	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9924	0.8389
	29	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9966	0.8987
	30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9986	0.9405
	31	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9675
	32	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9836
	33	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9923
	34	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9967
	35	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9987
	40	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	45	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Binomial Distribution – Cumulative Distribution Function

n	x	θ							
		0.60	**0.70**	**0.75**	**0.80**	**0.85**	**0.90**	**0.95**	**0.99**
50	11	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	13	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	14	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	15	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	16	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	17	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	18	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	19	0.0014	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	20	0.0034	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	21	0.0076	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	22	0.0160	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	23	0.0314	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	24	0.0573	0.0009	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	25	0.0978	0.0024	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	26	0.1562	0.0056	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000
	27	0.2340	0.0123	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000
	38	0.3299	0.0251	0.0026	0.0001	0.0000	0.0000	0.0000	0.0000
	29	0.4390	0.0478	0.0063	0.0003	0.0000	0.0000	0.0000	0.0000
	30	0.5535	0.0848	0.0139	0.0009	0.0000	0.0000	0.0000	0.0000
	31	0.6644	0.1406	0.0287	0.0025	0.0001	0.0000	0.0000	0.0000
	32	0.7631	0.2178	0.0551	0.0063	0.0002	0.0000	0.0000	0.0000
	33	0.8439	0.3161	0.0983	0.0144	0.0007	0.0000	0.0000	0.0000
	34	0.9045	0.4308	0.1631	0.0308	0.0019	0.0000	0.0000	0.0000
	35	0.9460	0.5532	0.2519	0.0607	0.0053	0.0001	0.0000	0.0000
	40	0.9992	0.9598	0.8363	0.5563	0.2089	0.0245	0.0002	0.0000
	45	1.0000	0.9998	0.9979	0.9815	0.8879	0.5688	0.1036	0.0001
	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Hypergeometric Distribution – Probability Mass Function

$$f_H(x\,|\,N;\,n;\,M) = \begin{cases} \dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} & \text{for} \quad x = 0,\ 1,\ ...,\ n \\ 0 & \text{others} \end{cases}$$

For $M > n$ the following applies: $f_H\,(x\,|\,N;\,n;\,M) = f_H\,(x\,|\,N;\,M;\,n)$

N	n	M	x	f_H
2	1	1	0	0.5000
2	1	1	1	0.5000
3	1	1	0	0.6667
3	1	1	1	0.3333
3	2	1	0	0.3333
3	2	1	1	0.6667
3	2	2	1	0.6667
3	2	2	2	0.3333
4	1	1	0	0.7500
4	1	1	1	0.2500
4	2	1	0	0.5000
4	2	1	1	0.5000
4	2	2	0	0.1667
4	2	2	1	0.6667
4	2	2	2	0.1667
4	3	1	0	0.2500
4	3	1	1	0.7500
4	3	2	1	0.5000
4	3	2	2	0.5000
4	3	3	2	0.7500
4	3	3	3	0.2500
5	1	1	0	0.8000
5	1	1	1	0.2000
5	2	1	0	0.6000
5	2	1	1	0.4000
5	2	2	0	0.3000
5	2	2	1	0.6000
5	2	2	2	0.1000
5	3	1	0	0.4000
5	3	1	1	0.6000
5	3	2	0	0.1000
5	3	2	1	0.6000
5	3	2	2	0.3000
5	3	3	1	0.3000
5	3	3	2	0.6000
5	3	3	3	0.1000
5	4	1	0	0.2000
5	4	1	1	0.8000
5	4	2	1	0.4000
5	4	2	2	0.6000
5	4	3	2	0.6000
5	4	3	3	0.4000
5	4	4	3	0.8000
5	4	4	4	0.2000
6	1	1	0	0.8333
6	1	1	1	0.1667
6	2	1	0	0.6667
6	2	1	1	0.3333
6	2	2	0	0.4000
6	2	2	1	0.5333
6	2	2	2	0.0667
6	3	1	0	0.5000
6	3	1	1	0.5000
6	3	2	0	0.2000
6	3	2	1	0.6000
6	3	2	2	0.2000
6	3	3	0	0.0500
6	3	3	1	0.4500
6	3	3	2	0.4500
6	3	3	3	0.0500
6	4	1	0	0.3333
6	4	1	1	0.6667
6	4	2	0	0.0667
6	4	2	1	0.5333
6	4	2	2	0.4000
6	4	3	1	0.2000
6	4	3	2	0.6000
6	4	3	3	0.2000
6	4	4	2	0.4000
6	4	4	3	0.5333
6	4	4	4	0.0667
6	5	1	0	0.1667
6	5	1	1	0.8333
6	5	2	1	0.3333
6	5	2	2	0.6667
6	5	3	2	0.5000
6	5	3	3	0.5000
6	5	4	3	0.6667
6	5	4	4	0.3333
6	5	5	4	0.8333
6	5	5	5	0.1667
7	1	1	0	0.8571
7	1	1	1	0.1429
7	2	1	0	0.7143
7	2	1	1	0.2857
7	2	2	0	0.4762
7	2	2	1	0.4762
7	2	2	2	0.0476
7	3	1	0	0.5714
7	3	1	1	0.4286

Hypergeometric Distribution – Probability Mass Function

N	n	M	x	f_H
7	3	2	0	0.2857
7	3	2	1	0.5714
7	3	2	2	0.1429
7	3	3	0	0.1143
7	3	3	1	0.5143
7	3	3	2	0.3429
7	3	3	3	0.0286
7	4	1	0	0.4286
7	4	1	1	0.5714
7	4	2	0	0.1429
7	4	2	1	0.5714
7	4	2	2	0.2857
7	4	3	0	0.0286
7	4	3	1	0.3429
7	4	3	2	0.5143
7	4	3	3	0.1143
7	4	4	1	0.1143
7	4	4	2	0.5143
7	4	4	3	0.3429
7	4	4	4	0.0286
7	5	1	0	0.2857
7	5	1	1	0.7143
7	5	2	0	0.0476
7	5	2	1	0.4762
7	5	2	2	0.4762
7	5	3	1	0.1429
7	5	3	2	0.5714
7	5	3	3	0.2857
7	5	4	2	0.2857
7	5	4	3	0.5714
7	5	4	4	0.1429
7	5	5	3	0.4762
7	5	5	4	0.4762
7	5	5	5	0.0476
7	6	1	0	0.1429
7	6	1	1	0.8571
7	6	2	1	0.2857
7	6	2	2	0.7143
7	6	3	2	0.4286
7	6	3	3	0.5714
7	6	4	3	0.5714
7	6	4	4	0.4286
7	6	5	4	0.7143
7	6	5	5	0.2857
7	6	6	5	0.8571

N	n	M	x	f_H
7	6	6	6	0.1429
8	1	1	0	0.8750
8	1	1	1	0.1250
8	2	1	0	0.7500
8	2	1	1	0.2500
8	2	2	0	0.5357
8	2	2	1	0.4286
8	2	2	2	0.0357
8	3	1	0	0.6250
8	3	1	1	0.3750
8	3	2	0	0.3571
8	3	2	1	0.5357
8	3	2	2	0.1071
8	3	3	0	0.1786
8	3	3	1	0.5357
8	3	3	2	0.2679
8	3	3	3	0.0179
8	4	1	0	0.5000
8	4	1	1	0.5000
8	4	2	0	0.2143
8	4	2	1	0.5714
8	4	2	2	0.2143
8	4	3	0	0.0714
8	4	3	1	0.4286
8	4	3	2	0.4286
8	4	3	3	0.0714
8	4	4	0	0.0143
8	4	4	1	0.2286
8	4	4	2	0.5143
8	4	4	3	0.2286
8	4	4	4	0.0143
8	5	1	0	0.3750
8	5	1	1	0.6250
8	5	2	0	0.1071
8	5	2	1	0.5357
8	5	2	2	0.3571
8	5	3	0	0.0179
8	5	3	1	0.2679
8	5	3	2	0.5357
8	5	3	3	0.1786
8	5	4	1	0.0714
8	5	4	2	0.4286
8	5	4	3	0.4286
8	5	4	4	0.0714
8	5	5	2	0.1786

N	n	M	x	f_H
8	5	5	3	0.5357
8	5	5	4	0.2679
8	5	5	5	0.0179
8	6	1	0	0.2500
8	6	1	1	0.7500
8	6	2	0	0.0357
8	6	2	1	0.4286
8	6	2	2	0.5357
8	6	3	1	0.1071
8	6	3	2	0.5357
8	6	3	3	0.3571
8	6	4	2	0.2143
8	6	4	3	0.5714
8	6	4	4	0.2143
8	6	5	3	0.3571
8	6	5	4	0.5357
8	6	5	5	0.1071
8	6	6	4	0.5357
8	6	6	5	0.4286
8	6	6	6	0.0357
8	7	1	0	0.1250
8	7	1	1	0.8750
8	7	2	1	0.2500
8	7	2	2	0.7500
8	7	3	2	0.3750
8	7	3	3	0.6250
8	7	4	3	0.5000
8	7	4	4	0.5000
8	7	5	4	0.6250
8	7	5	5	0.3750
8	7	6	5	0.7500
8	7	6	6	0.2500
8	7	7	6	0.8750
8	7	7	7	0.1250
9	1	1	0	0.8889
9	1	1	1	0.1111
9	2	1	0	0.7778
9	2	1	1	0.2222
9	2	2	0	0.5833
9	2	2	1	0.3889
9	2	2	2	0.0278
9	3	1	0	0.6667
9	3	1	1	0.3333
9	3	2	0	0.4167
9	3	2	1	0.5000

Hypergeometric Distribution – Probability Mass Function

N	n	M	x	f_H
9	3	2	2	0.0833
9	3	3	0	0.2381
9	3	3	1	0.5357
9	3	3	2	0.2143
9	3	3	3	0.0119
9	4	1	0	0.5556
9	4	1	1	0.4444
9	4	2	0	0.2778
9	4	2	1	0.5556
9	4	2	2	0.1667
9	4	3	0	0.1190
9	4	3	1	0.4762
9	4	3	2	0.3571
9	4	3	3	0.0476
9	4	4	0	0.0397
9	4	4	1	0.3175
9	4	4	2	0.4762
9	4	4	3	0.1587
9	4	4	4	0.0079
9	5	1	0	0.4444
9	5	1	1	0.5556
9	5	2	0	0.1667
9	5	2	1	0.5556
9	5	2	2	0.2778
9	5	3	0	0.0476
9	5	3	1	0.3571
9	5	3	2	0.4762
9	5	3	3	0.1190
9	5	4	0	0.0079
9	5	4	1	0.1587
9	5	4	2	0.4762
9	5	4	3	0.3175
9	5	4	4	0.0397
9	5	5	1	0.0397
9	5	5	2	0.3175
9	5	5	3	0.4762
9	5	5	4	0.1587
9	5	5	5	0.0079
9	6	1	0	0.3333
9	6	1	1	0.6667
9	6	2	0	0.0833
9	6	2	1	0.5000
9	6	2	2	0.4167
9	6	3	0	0.0119
9	6	3	1	0.2143

N	n	M	x	f_H
9	6	3	2	0.5357
9	6	3	3	0.2381
9	6	4	1	0.0476
9	6	4	2	0.3571
9	6	4	3	0.4762
9	6	4	4	0.1190
9	6	5	2	0.1190
9	6	5	3	0.4762
9	6	5	4	0.3571
9	6	5	5	0.0476
9	6	6	3	0.2381
9	6	6	4	0.5357
9	6	6	5	0.2143
9	6	6	6	0.0119
9	7	1	0	0.2222
9	7	1	1	0.7778
9	7	2	0	0.0278
9	7	2	1	0.3889
9	7	2	2	0.5833
9	7	3	1	0.0833
9	7	3	2	0.5000
9	7	3	3	0.4167
9	7	4	2	0.1667
9	7	4	3	0.5556
9	7	4	4	0.2778
9	7	5	3	0.2778
9	7	5	4	0.5556
9	7	5	5	0.1667
9	7	6	4	0.4167
9	7	6	5	0.5000
9	7	6	6	0.0833
9	7	7	5	0.5833
9	7	7	6	0.3889
9	7	7	7	0.0278
9	8	1	0	0.1111
9	8	1	1	0.8889
9	8	2	1	0.2222
9	8	2	2	0.7778
9	8	3	2	0.3333
9	8	3	3	0.6667
9	8	4	3	0.4444
9	8	4	4	0.5556
9	8	5	4	0.5556
9	8	5	5	0.4444
9	8	6	5	0.6667

N	n	M	x	f_H
9	8	6	6	0.3333
9	8	7	6	0.7778
9	8	7	7	0.2222
9	8	8	7	0.8889
9	8	8	8	0.1111
10	1	1	0	0.9000
10	1	1	1	0.1000
10	2	1	0	0.8000
10	2	1	1	0.2000
10	2	2	0	0.6222
10	2	2	1	0.3556
10	2	2	2	0.0222
10	3	1	0	0.7000
10	3	1	1	0.3000
10	3	2	0	0.4667
10	3	2	1	0.4667
10	3	2	2	0.0667
10	3	3	0	0.2917
10	3	3	1	0.5250
10	3	3	2	0.1750
10	3	3	3	0.0083
10	4	1	0	0.6000
10	4	1	1	0.4000
10	4	2	0	0.3333
10	4	2	1	0.5333
10	4	2	2	0.1333
10	4	3	0	0.1667
10	4	3	1	0.5000
10	4	3	2	0.3000
10	4	3	3	0.0333
10	4	4	0	0.0714
10	4	4	1	0.3810
10	4	4	2	0.4286
10	4	4	3	0.1143
10	4	4	4	0.0048
10	5	1	0	0.5000
10	5	1	1	0.5000
10	5	2	0	0.2222
10	5	2	1	0.5556
10	5	2	2	0.2222
10	5	3	0	0.0833
10	5	3	1	0.4167
10	5	3	2	0.4167
10	5	3	3	0.0833
10	5	4	0	0.0238

Hypergeometric Distribution – Probability Mass Function

N	n	M	x	f_H
10	5	4	1	0.2381
10	5	4	2	0.4762
10	5	4	3	0.2381
10	5	4	4	0.0238
10	5	5	0	0.0040
10	5	5	1	0.0992
10	5	5	2	0.3968
10	5	5	3	0.3968
10	5	5	4	0.0992
10	5	5	5	0.0040
10	6	1	0	0.4000
10	6	1	1	0.6000
10	6	2	0	0.1333
10	6	2	1	0.5333
10	6	2	2	0.3333
10	6	3	0	0.0333
10	6	3	1	0.3000
10	6	3	2	0.5000
10	6	3	3	0.1667
10	6	4	0	0.0048
10	6	4	1	0.1143
10	6	4	2	0.4286
10	6	4	3	0.3810
10	6	4	4	0.0714
10	6	5	1	0.0238
10	6	5	2	0.2381
10	6	5	3	0.4762
10	6	5	4	0.2381
10	6	5	5	0.0238
10	6	6	2	0.0714
10	6	6	3	0.3810
10	6	6	4	0.4286
10	6	6	5	0.1143
10	6	6	6	0.0048
10	7	1	0	0.3000
10	7	1	1	0.7000
10	7	2	0	0.0667
10	7	2	1	0.4667
10	7	2	2	0.4667
10	7	3	0	0.0083
10	7	3	1	0.1750
10	7	3	2	0.5250
10	7	3	3	0.2917
10	7	4	1	0.0333
10	7	4	2	0.3000
10	7	4	3	0.5000
10	7	4	4	0.1667
10	7	5	2	0.0833
10	7	5	3	0.4167
10	7	5	4	0.4167
10	7	5	5	0.0833
10	7	6	3	0.1667
10	7	6	4	0.5000
10	7	6	5	0.3000
10	7	6	6	0.0333
10	7	7	4	0.2917
10	7	7	5	0.5250
10	7	7	6	0.1750
10	7	7	7	0.0083
10	8	1	0	0.2000
10	8	1	1	0.8000
10	8	2	0	0.0222
10	8	2	1	0.3556
10	8	2	2	0.6222
10	8	3	1	0.0667
10	8	3	2	0.4667
10	8	3	3	0.4667
10	8	4	2	0.1333
10	8	4	3	0.5333
10	8	4	4	0.3333
10	8	5	3	0.2222
10	8	5	4	0.5556
10	8	5	5	0.2222
10	8	6	4	0.3333
10	8	6	5	0.5333
10	8	6	6	0.1333
10	8	7	5	0.4667
10	8	7	6	0.4667
10	8	7	7	0.0667
10	8	8	6	0.6222
10	8	8	7	0.3556
10	8	8	8	0.0222
10	9	1	0	0.1000
10	9	1	1	0.9000
10	9	2	1	0.2000
10	9	2	2	0.8000
10	9	3	2	0.3000
10	9	3	3	0.7000
10	9	4	3	0.4000
10	9	4	4	0.6000
10	9	5	4	0.5000
10	9	5	5	0.5000
10	9	6	5	0.6000
10	9	6	6	0.4000
10	9	7	6	0.7000
10	9	7	7	0.3000
10	9	8	7	0.8000
10	9	8	8	0.2000
10	9	9	8	0.9000
10	9	9	9	0.1000
11	1	1	0	0.9091
11	1	1	1	0.0909
11	2	1	0	0.8182
11	2	1	1	0.1818
11	2	2	0	0.6545
11	2	2	1	0.3273
11	2	2	2	0.0182
11	3	1	0	0.7273
11	3	1	1	0.2727
11	3	2	0	0.5091
11	3	2	1	0.4364
11	3	2	2	0.0545
11	3	3	0	0.3394
11	3	3	1	0.5091
11	3	3	2	0.1455
11	3	3	3	0.0061
11	4	1	0	0.6364
11	4	1	1	0.3636
11	4	2	0	0.3818
11	4	2	1	0.5091
11	4	2	2	0.1091
11	4	3	0	0.2121
11	4	3	1	0.5091
11	4	3	2	0.2545
11	4	3	3	0.0242
11	4	4	0	0.1061
11	4	4	1	0.4242
11	4	4	2	0.3818
11	4	4	3	0.0848
11	4	4	4	0.0030
11	5	1	0	0.5455
11	5	1	1	0.4545
11	5	2	0	0.2727
11	5	2	1	0.5455
11	5	2	2	0.1818

Hypergeometric Distribution – Probability Mass Function

N	n	M	x	f_H
11	5	3	0	0.1212
11	5	3	1	0.4545
11	5	3	2	0.3636
11	5	3	3	0.0606
11	5	4	0	0.0455
11	5	4	1	0.3030
11	5	4	2	0.4545
11	5	4	3	0.1818
11	5	4	4	0.0152
11	5	5	0	0.0130
11	5	5	1	0.1623
11	5	5	2	0.4329
11	5	5	3	0.3247
11	5	5	4	0.0649
11	5	5	5	0.0022
11	6	1	0	0.4545
11	6	1	1	0.5455
11	6	2	0	0.1818
11	6	2	1	0.5455
11	6	2	2	0.2727
11	6	3	0	0.0606
11	6	3	1	0.3636
11	6	3	2	0.4545
11	6	3	3	0.1212
11	6	4	0	0.0152
11	6	4	1	0.1818
11	6	4	2	0.4545
11	6	4	3	0.3030
11	6	4	4	0.0455
11	6	5	0	0.0022
11	6	5	1	0.0649
11	6	5	2	0.3247
11	6	5	3	0.4329
11	6	5	4	0.1623
11	6	5	5	0.0130
11	6	6	1	0.0130
11	6	6	2	0.1623
11	6	6	3	0.4329
11	6	6	4	0.3247
11	6	6	5	0.0649
11	6	6	6	0.0022
11	7	1	0	0.3636
11	7	1	1	0.6364
11	7	2	0	0.1091
11	7	2	1	0.5091

N	n	M	x	f_H
11	7	2	2	0.3818
11	7	3	0	0.0242
11	7	3	1	0.2545
11	7	3	2	0.5091
11	7	3	3	0.2121
11	7	4	0	0.0030
11	7	4	1	0.0848
11	7	4	2	0.3818
11	7	4	3	0.4242
11	7	4	4	0.1061
11	7	5	1	0.0152
11	7	5	2	0.1818
11	7	5	3	0.4545
11	7	5	4	0.3030
11	7	5	5	0.0455
11	7	6	2	0.0455
11	7	6	3	0.3030
11	7	6	4	0.4545
11	7	6	5	0.1818
11	7	6	6	0.0152
11	7	7	3	0.1061
11	7	7	4	0.4242
11	7	7	5	0.3818
11	7	7	6	0.0848
11	7	7	7	0.0030
11	8	1	0	0.2727
11	8	1	1	0.7273
11	8	2	0	0.0545
11	8	2	1	0.4364
11	8	2	2	0.5091
11	8	3	0	0.0061
11	8	3	1	0.1455
11	8	3	2	0.5091
11	8	3	3	0.3394
11	8	4	1	0.0242
11	8	4	2	0.2545
11	8	4	3	0.5091
11	8	4	4	0.2121
11	8	5	2	0.0606
11	8	5	3	0.3636
11	8	5	4	0.4545
11	8	5	5	0.1212
11	8	6	3	0.1212
11	8	6	4	0.4545
11	8	6	5	0.3636

N	n	M	x	f_H
11	8	6	6	0.0606
11	8	7	4	0.2121
11	8	7	5	0.5091
11	8	7	6	0.2545
11	8	7	7	0.0242
11	8	8	5	0.3394
11	8	8	6	0.5091
11	8	8	7	0.1455
11	8	8	8	0.0061
11	9	1	0	0.1818
11	9	1	1	0.8182
11	9	2	0	0.0182
11	9	2	1	0.3273
11	9	2	2	0.6545
11	9	3	1	0.0545
11	9	3	2	0.4364
11	9	3	3	0.5091
11	9	4	2	0.1091
11	9	4	3	0.5091
11	9	4	4	0.3818
11	9	5	3	0.1818
11	9	5	4	0.5455
11	9	5	5	0.2727
11	9	6	4	0.2727
11	9	6	5	0.5455
11	9	6	6	0.1818
11	9	7	5	0.3818
11	9	7	6	0.5091
11	9	7	7	0.1091
11	9	8	6	0.5091
11	9	8	7	0.4364
11	9	8	8	0.0545
11	9	9	7	0.6545
11	9	9	8	0.3273
11	9	9	9	0.0182
11	10	1	0	0.0909
11	10	1	1	0.9091
11	10	2	1	0.1818
11	10	2	2	0.8182
11	10	3	2	0.2727
11	10	3	3	0.7273
11	10	4	3	0.3636
11	10	4	4	0.6364
11	10	5	4	0.4545
11	10	5	5	0.5455

Hypergeometric Distribution – Probability Mass Function

N	n	M	x	f_H
11	10	6	5	0.5455
11	10	6	6	0.4545
11	10	7	6	0.6364
11	10	7	7	0.3636
11	10	8	7	0.7273
11	10	8	8	0.2727
11	10	9	8	0.8182
11	10	9	9	0.1818
11	10	10	9	0.9091
11	10	10	10	0.0909
12	1	1	0	0.9167
12	1	1	1	0.0833
12	2	1	0	0.8333
12	2	1	1	0.1667
12	2	2	0	0.6818
12	2	2	1	0.3030
12	2	2	2	0.0152
12	3	1	0	0.7500
12	3	1	1	0.2500
12	3	2	0	0.5455
12	3	2	1	0.4091
12	3	2	2	0.0455
12	3	3	0	0.3818
12	3	3	1	0.4909
12	3	3	2	0.1227
12	3	3	3	0.0045
12	4	1	0	0.6667
12	4	1	1	0.3333
12	4	2	0	0.4242
12	4	2	1	0.4848
12	4	2	2	0.0909
12	4	3	0	0.2545
12	4	3	1	0.5091
12	4	3	2	0.2182
12	4	3	3	0.0182
12	4	4	0	0.1414
12	4	4	1	0.4525
12	4	4	2	0.3394
12	4	4	3	0.0646
12	4	4	4	0.0020
12	5	1	0	0.5833
12	5	1	1	0.4167
12	5	2	0	0.3182
12	5	2	1	0.5303
12	5	2	2	0.1515
12	5	3	0	0.1591
12	5	3	1	0.4773
12	5	3	2	0.3182
12	5	3	3	0.0455
12	5	4	0	0.0707
12	5	4	1	0.3535
12	5	4	2	0.4242
12	5	4	3	0.1414
12	5	4	4	0.0101
12	5	5	0	0.0265
12	5	5	1	0.2210
12	5	5	2	0.4419
12	5	5	3	0.2652
12	5	5	4	0.0442
12	5	5	5	0.0013
12	6	1	0	0.5000
12	6	1	1	0.5000
12	6	2	0	0.2273
12	6	2	1	0.5455
12	6	2	2	0.2273
12	6	3	0	0.0909
12	6	3	1	0.4091
12	6	3	2	0.4091
12	6	3	3	0.0909
12	6	4	0	0.0303
12	6	4	1	0.2424
12	6	4	2	0.4545
12	6	4	3	0.2424
12	6	4	4	0.0303
12	6	5	0	0.0076
12	6	5	1	0.1136
12	6	5	2	0.3788
12	6	5	3	0.3788
12	6	5	4	0.1136
12	6	5	5	0.0076
12	6	6	0	0.0011
12	6	6	1	0.0390
12	6	6	2	0.2435
12	6	6	3	0.4329
12	6	6	4	0.2435
12	6	6	5	0.0390
12	6	6	6	0.0011
12	7	1	0	0.4167
12	7	1	1	0.5833
12	7	2	0	0.1515
12	7	2	1	0.5303
12	7	2	2	0.3182
12	7	3	0	0.0455
12	7	3	1	0.3182
12	7	3	2	0.4773
12	7	3	3	0.1591
12	7	4	0	0.0101
12	7	4	1	0.1414
12	7	4	2	0.4242
12	7	4	3	0.3535
12	7	4	4	0.0707
12	7	5	0	0.0013
12	7	5	1	0.0442
12	7	5	2	0.2652
12	7	5	3	0.4419
12	7	5	4	0.2210
12	7	5	5	0.0265
12	7	6	1	0.0076
12	7	6	2	0.1136
12	7	6	3	0.3788
12	7	6	4	0.3788
12	7	6	5	0.1136
12	7	6	6	0.0076
12	7	7	2	0.0265
12	7	7	3	0.2210
12	7	7	4	0.4419
12	7	7	5	0.2652
12	7	7	6	0.0442
12	7	7	7	0.0013
12	8	1	0	0.3333
12	8	1	1	0.6667
12	8	2	0	0.0909
12	8	2	1	0.4848
12	8	2	2	0.4242
12	8	3	0	0.0182
12	8	3	1	0.2182
12	8	3	2	0.5091
12	8	3	3	0.2545
12	8	4	0	0.0020
12	8	4	1	0.0646
12	8	4	2	0.3394
12	8	4	3	0.4525
12	8	4	4	0.1414
12	8	5	1	0.0101
12	8	5	2	0.1414

Hypergeometric Distribution – Probability Mass Function

N	n	M	x	f_H
12	8	5	3	0.4242
12	8	5	4	0.3535
12	8	5	5	0.0707
12	8	6	2	0.0303
12	8	6	3	0.2424
12	8	6	4	0.4545
12	8	6	5	0.2424
12	8	6	6	0.0303
12	8	7	3	0.0707
12	8	7	4	0.3535
12	8	7	5	0.4242
12	8	7	6	0.1414
12	8	7	7	0.0101
12	8	8	4	0.1414
12	8	8	5	0.4525
12	8	8	6	0.3394
12	8	8	7	0.0646
12	8	8	8	0.0020
12	9	1	0	0.2500
12	9	1	1	0.7500
12	9	2	0	0.0455
12	9	2	1	0.4091
12	9	2	2	0.5455
12	9	3	0	0.0045
12	9	3	1	0.1227
12	9	3	2	0.4909
12	9	3	3	0.3818
12	9	4	1	0.0182
12	9	4	2	0.2182
12	9	4	3	0.5091
12	9	4	4	0.2545
12	9	5	2	0.0455
12	9	5	3	0.3182
12	9	5	4	0.4773
12	9	5	5	0.1591
12	9	6	3	0.0909
12	9	6	4	0.4091
12	9	6	5	0.4091
12	9	6	6	0.0909
12	9	7	4	0.1591
12	9	7	5	0.4773
12	9	7	6	0.3182
12	9	7	7	0.0455
12	9	8	5	0.2545
12	9	8	6	0.5091

N	n	M	x	f_H
12	9	8	7	0.2182
12	9	8	8	0.0182
12	9	9	6	0.3818
12	9	9	7	0.4909
12	9	9	8	0.1227
12	9	9	9	0.0045
12	10	1	0	0.1667
12	10	1	1	0.8333
12	10	2	0	0.0152
12	10	2	1	0.3030
12	10	2	2	0.6818
12	10	3	1	0.0455
12	10	3	2	0.4091
12	10	3	3	0.5455
12	10	4	2	0.0909
12	10	4	3	0.4848
12	10	4	4	0.4242
12	10	5	3	0.1515
12	10	5	4	0.5303
12	10	5	5	0.3182
12	10	6	4	0.2273
12	10	6	5	0.5455
12	10	6	6	0.2273
12	10	7	5	0.3182
12	10	7	6	0.5303
12	10	7	7	0.1515
12	10	8	6	0.4242
12	10	8	7	0.4848
12	10	8	8	0.0909
12	10	9	7	0.5455
12	10	9	8	0.4091
12	10	9	9	0.0455
12	10	10	8	0.6818
12	10	10	9	0.3030
12	10	10	10	0.0152
12	11	1	0	0.0833
12	11	1	1	0.9167
12	11	2	1	0.1667
12	11	2	2	0.8333
12	11	3	2	0.2500
12	11	3	3	0.7500
12	11	4	3	0.3333
12	11	4	4	0.6667
12	11	5	4	0.4167
12	11	5	5	0.5833

N	n	M	x	f_H
12	11	6	5	0.5000
12	11	6	6	0.5000
12	11	7	6	0.5833
12	11	7	7	0.4167
12	11	8	7	0.6667
12	11	8	8	0.3333
12	11	9	8	0.7500
12	11	9	9	0.2500
12	11	10	9	0.8333
12	11	10	10	0.1667
12	11	11	10	0.9167
12	11	11	11	0.0833

Hypergeometric Distribution – Cumulative Distribution Function

$$F_H(x \mid N;\, n;\, M) = \sum_{v=0}^{x} \frac{\binom{M}{v}\binom{N-M}{n-v}}{\binom{N}{n}} \qquad \text{for} \quad x = 0, 1, ..., n$$

For $M > n$ the following applies: $F_H\,(x \mid N;\, n;\, M) = F_H\,(x \mid N;\, M;\, n)$

N	n	M	x	F_H
2	1	1	0	0.5000
2	1	1	1	1.0000
3	1	1	0	0.6667
3	1	1	1	1.0000
3	2	1	0	0.3333
3	2	1	1	1.0000
3	2	2	1	0.6667
3	2	2	2	1.0000
4	1	1	0	0.7500
4	1	1	1	1.0000
4	2	1	0	0.5000
4	2	1	1	1.0000
4	2	2	0	0.1667
4	2	2	1	0.8333
4	2	2	2	1.0000
4	3	1	0	0.2500
4	3	1	1	1.0000
4	3	2	1	0.5000
4	3	2	2	1.0000
4	3	3	2	0.7500
4	3	3	3	1.0000
5	1	1	0	0.8000
5	1	1	1	1.0000
5	2	1	0	0.6000
5	2	1	1	1.0000
5	2	2	0	0.3000
5	2	2	1	0.9000
5	2	2	2	1.0000
5	3	1	0	0.4000
5	3	1	1	1.0000

N	n	M	x	F_H
5	3	2	0	0.1000
5	3	2	1	0.7000
5	3	2	2	1.0000
5	3	3	1	0.3000
5	3	3	2	0.9000
5	3	3	3	1.0000
5	4	1	0	0.2000
5	4	1	1	1.0000
5	4	2	1	0.4000
5	4	2	2	1.0000
5	4	3	2	0.6000
5	4	3	3	1.0000
5	4	4	3	0.8000
5	4	4	4	1.0000
6	1	1	0	0.8333
6	1	1	1	1.0000
6	2	1	0	0.6667
6	2	1	1	1.0000
6	2	2	0	0.4000
6	2	2	1	0.9333
6	2	2	2	1.0000
6	3	1	0	0.5000
6	3	1	1	1.0000
6	3	2	0	0.2000
6	3	2	1	0.8000
6	3	2	2	1.0000
6	3	3	0	0.0500
6	3	3	1	0.5000
6	3	3	2	0.9500
6	3	3	3	1.0000

N	n	M	x	F_H
6	4	1	0	0.3333
6	4	1	1	1.0000
6	4	2	0	0.0667
6	4	2	1	0.6000
6	4	2	2	1.0000
6	4	3	1	0.2000
6	4	3	2	0.8000
6	4	3	3	1.0000
6	4	4	2	0.4000
6	4	4	3	0.9333
6	4	4	4	1.0000
6	5	1	0	0.1667
6	5	1	1	1.0000
6	5	2	1	0.3333
6	5	2	2	1.0000
6	5	3	2	0.5000
6	5	3	3	1.0000
6	5	4	3	0.6667
6	5	4	4	1.0000
6	5	5	4	0.8333
6	5	5	5	1.0000
7	1	1	0	0.8571
7	1	1	1	1.0000
7	2	1	0	0.7143
7	2	1	1	1.0000
7	2	2	0	0.4762
7	2	2	1	0.9524
7	2	2	2	1.0000
7	3	1	0	0.5714
7	3	1	1	1.0000

Hypergeometric Distribution – Cumulative Distribution Function

N	n	M	x	F_H
7	3	2	0	0.2857
7	3	2	1	0.8571
7	3	2	2	1.0000
7	3	3	0	0.1143
7	3	3	1	0.6286
7	3	3	2	0.9714
7	3	3	3	1.0000
7	4	1	0	0.4286
7	4	1	1	1.0000
7	4	2	0	0.1429
7	4	2	1	0.7143
7	4	2	2	1.0000
7	4	3	0	0.0286
7	4	3	1	0.3714
7	4	3	2	0.8857
7	4	3	3	1.0000
7	4	4	1	0.1143
7	4	4	2	0.6286
7	4	4	3	0.9714
7	4	4	4	1.0000
7	5	1	0	0.2857
7	5	1	1	1.0000
7	5	2	0	0.0476
7	5	2	1	0.5238
7	5	2	2	1.0000
7	5	3	1	0.1429
7	5	3	2	0.7143
7	5	3	3	1.0000
7	5	4	2	0.2857
7	5	4	3	0.8571
7	5	4	4	1.0000
7	5	5	3	0.4762
7	5	5	4	0.9524
7	5	5	5	1.0000
7	6	1	0	0.1429
7	6	1	1	1.0000
7	6	2	1	0.2857
7	6	2	2	1.0000
7	6	3	2	0.4286
7	6	3	3	1.0000
7	6	4	3	0.5714
7	6	4	4	1.0000
7	6	5	4	0.7143
7	6	5	5	1.0000
7	6	6	5	0.8571
7	6	6	6	1.0000
8	1	1	0	0.8750
8	1	1	1	1.0000
8	2	1	0	0.7500
8	2	1	1	1.0000
8	2	2	0	0.5357
8	2	2	1	0.9643
8	2	2	2	1.0000
8	3	1	0	0.6250
8	3	1	1	1.0000
8	3	2	0	0.3571
8	3	2	1	0.8929
8	3	2	2	1.0000
8	3	3	0	0.1786
8	3	3	1	0.7143
8	3	3	2	0.9821
8	3	3	3	1.0000
8	4	1	0	0.5000
8	4	1	1	1.0000
8	4	2	0	0.2143
8	4	2	1	0.7857
8	4	2	2	1.0000
8	4	3	0	0.0714
8	4	3	1	0.5000
8	4	3	2	0.9286
8	4	3	3	1.0000
8	4	4	0	0.0143
8	4	4	1	0.2429
8	4	4	2	0.7571
8	4	4	3	0.9857
8	4	4	4	1.0000
8	5	1	0	0.3750
8	5	1	1	1.0000
8	5	2	0	0.1071
8	5	2	1	0.6429
8	5	2	2	1.0000
8	5	3	0	0.0179
8	5	3	1	0.2857
8	5	3	2	0.8214
8	5	3	3	1.0000
8	5	4	1	0.0714
8	5	4	2	0.5000
8	5	4	3	0.9286
8	5	4	4	1.0000
8	5	5	2	0.1786
8	5	5	3	0.7143
8	5	5	4	0.9821
8	5	5	5	1.0000
8	6	1	0	0.2500
8	6	1	1	1.0000
8	6	2	0	0.0357
8	6	2	1	0.4643
8	6	2	2	1.0000
8	6	3	1	0.1071
8	6	3	2	0.6429
8	6	3	3	1.0000
8	6	4	2	0.2143
8	6	4	3	0.7857
8	6	4	4	1.0000
8	6	5	3	0.3571
8	6	5	4	0.8929
8	6	5	5	1.0000
8	6	6	4	0.5357
8	6	6	5	0.9643
8	6	6	6	1.0000
8	7	1	0	0.1250
8	7	1	1	1.0000
8	7	2	1	0.2500
8	7	2	2	1.0000
8	7	3	2	0.3750
8	7	3	3	1.0000
8	7	4	3	0.5000
8	7	4	4	1.0000
8	7	5	4	0.6250
8	7	5	5	1.0000
8	7	6	5	0.7500
8	7	6	6	1.0000
8	7	7	6	0.8750
8	7	7	7	1.0000
9	1	1	0	0.8889
9	1	1	1	1.0000
9	2	1	0	0.7778
9	2	1	1	1.0000
9	2	2	0	0.5833
9	2	2	1	0.9722
9	2	2	2	1.0000
9	3	1	0	0.6667
9	3	1	1	1.0000
9	3	2	0	0.4167
9	3	2	1	0.9167

Hypergeometric Distribution – Cumulative Distribution Function

N	n	M	x	F_H
9	3	2	2	1.0000
9	3	3	0	0.2381
9	3	3	1	0.7738
9	3	3	2	0.9881
9	3	3	3	1.0000
9	4	1	0	0.5556
9	4	1	1	1.0000
9	4	2	0	0.2778
9	4	2	1	0.8333
9	4	2	2	1.0000
9	4	3	0	0.1190
9	4	3	1	0.5952
9	4	3	2	0.9524
9	4	3	3	1.0000
9	4	4	0	0.0397
9	4	4	1	0.3571
9	4	4	2	0.8333
9	4	4	3	0.9921
9	4	4	4	1.0000
9	5	1	0	0.4444
9	5	1	1	1.0000
9	5	2	0	0.1667
9	5	2	1	0.7222
9	5	2	2	1.0000
9	5	3	0	0.0476
9	5	3	1	0.4048
9	5	3	2	0.8810
9	5	3	3	1.0000
9	5	4	0	0.0079
9	5	4	1	0.1667
9	5	4	2	0.6429
9	5	4	3	0.9603
9	5	4	4	1.0000
9	5	5	1	0.0397
9	5	5	2	0.3571
9	5	5	3	0.8333
9	5	5	4	0.9921
9	5	5	5	1.0000
9	6	1	0	0.3333
9	6	1	1	1.0000
9	6	2	0	0.0833
9	6	2	1	0.5833
9	6	2	2	1.0000
9	6	3	0	0.0119
9	6	3	1	0.2262

N	n	M	x	F_H
9	6	3	2	0.7619
9	6	3	3	1.0000
9	6	4	1	0.0476
9	6	4	2	0.4048
9	6	4	3	0.8810
9	6	4	4	1.0000
9	6	5	2	0.1190
9	6	5	3	0.5952
9	6	5	4	0.9524
9	6	5	5	1.0000
9	6	6	3	0.2381
9	6	6	4	0.7738
9	6	6	5	0.9881
9	6	6	6	1.0000
9	7	1	0	0.2222
9	7	1	1	1.0000
9	7	2	0	0.0278
9	7	2	1	0.4167
9	7	2	2	1.0000
9	7	3	1	0.0833
9	7	3	2	0.5833
9	7	3	3	1.0000
9	7	4	2	0.1667
9	7	4	3	0.7222
9	7	4	4	1.0000
9	7	5	3	0.2778
9	7	5	4	0.8333
9	7	5	5	1.0000
9	7	6	4	0.4167
9	7	6	5	0.9167
9	7	6	6	1.0000
9	7	7	5	0.5833
9	7	7	6	0.9722
9	7	7	7	1.0000
9	8	1	0	0.1111
9	8	1	1	1.0000
9	8	2	1	0.2222
9	8	2	2	1.0000
9	8	3	2	0.3333
9	8	3	3	1.0000
9	8	4	3	0.4444
9	8	4	4	1.0000
9	8	5	4	0.5556
9	8	5	5	1.0000
9	8	6	5	0.6667

N	n	M	x	F_H
9	8	6	6	1.0000
9	8	7	6	0.7778
9	8	7	7	1.0000
9	8	8	7	0.8889
9	8	8	8	1.0000
10	1	1	0	0.9000
10	1	1	1	1.0000
10	2	1	0	0.8000
10	2	1	1	1.0000
10	2	2	0	0.6222
10	2	2	1	0.9778
10	2	2	2	1.0000
10	3	1	0	0.7000
10	3	1	1	1.0000
10	3	2	0	0.4667
10	3	2	1	0.9333
10	3	2	2	1.0000
10	3	3	0	0.2917
10	3	3	1	0.8167
10	3	3	2	0.9917
10	3	3	3	1.0000
10	4	1	0	0.6000
10	4	1	1	1.0000
10	4	2	0	0.3333
10	4	2	1	0.8667
10	4	2	2	1.0000
10	4	3	0	0.1667
10	4	3	1	0.6667
10	4	3	2	0.9667
10	4	3	3	1.0000
10	4	4	0	0.0714
10	4	4	1	0.4524
10	4	4	2	0.8810
10	4	4	3	0.9952
10	4	4	4	1.0000
10	5	1	0	0.5000
10	5	1	1	1.0000
10	5	2	0	0.2222
10	5	2	1	0.7778
10	5	2	2	1.0000
10	5	3	0	0.0833
10	5	3	1	0.5000
10	5	3	2	0.9167
10	5	3	3	1.0000
10	5	4	0	0.0238

Hypergeometric Distribution – Cumulative Distribution Function

N	n	M	x	F_H
10	5	4	1	0.2619
10	5	4	2	0.7381
10	5	4	3	0.9762
10	5	4	4	1.0000
10	5	5	0	0.0040
10	5	5	1	0.1032
10	5	5	2	0.5000
10	5	5	3	0.8968
10	5	5	4	0.9960
10	5	5	5	1.0000
10	6	1	0	0.4000
10	6	1	1	1.0000
10	6	2	0	0.1333
10	6	2	1	0.6667
10	6	2	2	1.0000
10	6	3	0	0.0333
10	6	3	1	0.3333
10	6	3	2	0.8333
10	6	3	3	1.0000
10	6	4	0	0.0048
10	6	4	1	0.1190
10	6	4	2	0.5476
10	6	4	3	0.9286
10	6	4	4	1.0000
10	6	5	1	0.0238
10	6	5	2	0.2619
10	6	5	3	0.7381
10	6	5	4	0.9762
10	6	5	5	1.0000
10	6	6	2	0.0714
10	6	6	3	0.4524
10	6	6	4	0.8810
10	6	6	5	0.9952
10	6	6	6	1.0000
10	7	1	0	0.3000
10	7	1	1	1.0000
10	7	2	0	0.0667
10	7	2	1	0.5333
10	7	2	2	1.0000
10	7	3	0	0.0083
10	7	3	1	0.1833
10	7	3	2	0.7083
10	7	3	3	1.0000
10	7	4	1	0.0333
10	7	4	2	0.3333
10	7	4	3	0.8333
10	7	4	4	1.0000
10	7	5	2	0.0833
10	7	5	3	0.5000
10	7	5	4	0.9167
10	7	5	5	1.0000
10	7	6	3	0.1667
10	7	6	4	0.6667
10	7	6	5	0.9667
10	7	6	6	1.0000
10	7	7	4	0.2917
10	7	7	5	0.8167
10	7	7	6	0.9917
10	7	7	7	1.0000
10	8	1	0	0.2000
10	8	1	1	1.0000
10	8	2	0	0.0222
10	8	2	1	0.3778
10	8	2	2	1.0000
10	8	3	1	0.0667
10	8	3	2	0.5333
10	8	3	3	1.0000
10	8	4	2	0.1333
10	8	4	3	0.6667
10	8	4	4	1.0000
10	8	5	3	0.2222
10	8	5	4	0.7778
10	8	5	5	1.0000
10	8	6	4	0.3333
10	8	6	5	0.8667
10	8	6	6	1.0000
10	8	7	5	0.4667
10	8	7	6	0.9333
10	8	7	7	1.0000
10	8	8	6	0.6222
10	8	8	7	0.9778
10	8	8	8	1.0000
10	9	1	0	0.1000
10	9	1	1	1.0000
10	9	2	1	0.2000
10	9	2	2	1.0000
10	9	3	2	0.3000
10	9	3	3	1.0000
10	9	4	3	0.4000
10	9	4	4	1.0000
10	9	5	4	0.5000
10	9	5	5	1.0000
10	9	6	5	0.6000
10	9	6	6	1.0000
10	9	7	6	0.7000
10	9	7	7	1.0000
10	9	8	7	0.8000
10	9	8	8	1.0000
10	9	9	8	0.9000
10	9	9	9	1.0000
11	1	1	0	0.9091
11	1	1	1	1.0000
11	2	1	0	0.8182
11	2	1	1	1.0000
11	2	2	0	0.6545
11	2	2	1	0.9818
11	2	2	2	1.0000
11	3	1	0	0.7273
11	3	1	1	1.0000
11	3	2	0	0.5091
11	3	2	1	0.9455
11	3	2	2	1.0000
11	3	3	0	0.3394
11	3	3	1	0.8485
11	3	3	2	0.9939
11	3	3	3	1.0000
11	4	1	0	0.6364
11	4	1	1	1.0000
11	4	2	0	0.3818
11	4	2	1	0.8909
11	4	2	2	1.0000
11	4	3	0	0.2121
11	4	3	1	0.7212
11	4	3	2	0.9758
11	4	3	3	1.0000
11	4	4	0	0.1061
11	4	4	1	0.5303
11	4	4	2	0.9121
11	4	4	3	0.9970
11	4	4	4	1.0000
11	5	1	0	0.5455
11	5	1	1	1.0000
11	5	2	0	0.2727
11	5	2	1	0.8182
11	5	2	2	1.0000

Hypergeometric Distribution – Cumulative Distribution Function

N	n	M	x	F_H
11	5	3	0	0.1212
11	5	3	1	0.5758
11	5	3	2	0.9394
11	5	3	3	1.0000
11	5	4	0	0.0455
11	5	4	1	0.3485
11	5	4	2	0.8030
11	5	4	3	0.9848
11	5	4	4	1.0000
11	5	5	0	0.0130
11	5	5	1	0.1753
11	5	5	2	0.6082
11	5	5	3	0.9329
11	5	5	4	0.9978
11	5	5	5	1.0000
11	6	1	0	0.4545
11	6	1	1	1.0000
11	6	2	0	0.1818
11	6	2	1	0.7273
11	6	2	2	1.0000
11	6	3	0	0.0606
11	6	3	1	0.4242
11	6	3	2	0.8788
11	6	3	3	1.0000
11	6	4	0	0.0152
11	6	4	1	0.1970
11	6	4	2	0.6515
11	6	4	3	0.9545
11	6	4	4	1.0000
11	6	5	0	0.0022
11	6	5	1	0.0671
11	6	5	2	0.3918
11	6	5	3	0.8247
11	6	5	4	0.9870
11	6	5	5	1.0000
11	6	6	1	0.0130
11	6	6	2	0.1753
11	6	6	3	0.6082
11	6	6	4	0.9329
11	6	6	5	0.9978
11	6	6	6	1.0000
11	7	1	0	0.3636
11	7	1	1	1.0000
11	7	2	0	0.1091
11	7	2	1	0.6182
11	7	2	2	1.0000
11	7	3	0	0.0242
11	7	3	1	0.2788
11	7	3	2	0.7879
11	7	3	3	1.0000
11	7	4	0	0.0030
11	7	4	1	0.0879
11	7	4	2	0.4697
11	7	4	3	0.8939
11	7	4	4	1.0000
11	7	5	1	0.0152
11	7	5	2	0.1970
11	7	5	3	0.6515
11	7	5	4	0.9545
11	7	5	5	1.0000
11	7	6	2	0.0455
11	7	6	3	0.3485
11	7	6	4	0.8030
11	7	6	5	0.9848
11	7	6	6	1.0000
11	7	7	3	0.1061
11	7	7	4	0.5303
11	7	7	5	0.9121
11	7	7	6	0.9970
11	7	7	7	1.0000
11	8	1	0	0.2727
11	8	1	1	1.0000
11	8	2	0	0.0545
11	8	2	1	0.4909
11	8	2	2	1.0000
11	8	3	0	0.0061
11	8	3	1	0.1515
11	8	3	2	0.6606
11	8	3	3	1.0000
11	8	4	1	0.0242
11	8	4	2	0.2788
11	8	4	3	0.7879
11	8	4	4	1.0000
11	8	5	2	0.0606
11	8	5	3	0.4242
11	8	5	4	0.8788
11	8	5	5	1.0000
11	8	6	3	0.1212
11	8	6	4	0.5758
11	8	6	5	0.9394
11	8	6	6	1.0000
11	8	7	4	0.2121
11	8	7	5	0.7212
11	8	7	6	0.9758
11	8	7	7	1.0000
11	8	8	5	0.3394
11	8	8	6	0.8485
11	8	8	7	0.9939
11	8	8	8	1.0000
11	9	1	0	0.1818
11	9	1	1	1.0000
11	9	2	0	0.0182
11	9	2	1	0.3455
11	9	2	2	1.0000
11	9	3	1	0.0545
11	9	3	2	0.4909
11	9	3	3	1.0000
11	9	4	2	0.1091
11	9	4	3	0.6182
11	9	4	4	1.0000
11	9	5	3	0.1818
11	9	5	4	0.7273
11	9	5	5	1.0000
11	9	6	4	0.2727
11	9	6	5	0.8182
11	9	6	6	1.0000
11	9	7	5	0.3818
11	9	7	6	0.8909
11	9	7	7	1.0000
11	9	8	6	0.5091
11	9	8	7	0.9455
11	9	8	8	1.0000
11	9	9	7	0.6545
11	9	9	8	0.9818
11	9	9	9	1.0000
11	10	1	0	0.0909
11	10	1	1	1.0000
11	10	2	1	0.1818
11	10	2	2	1.0000
11	10	3	2	0.2727
11	10	3	3	1.0000
11	10	4	3	0.3636
11	10	4	4	1.0000
11	10	5	4	0.4545
11	10	5	5	1.0000

Hypergeometric Distribution – Cumulative Distribution Function

N	n	M	x	F_H
11	10	6	5	0.5455
11	10	6	6	1.0000
11	10	7	6	0.6364
11	10	7	7	1.0000
11	10	8	7	0.7273
11	10	8	8	1.0000
11	10	9	8	0.8182
11	10	9	9	1.0000
11	10	10	9	0.9091
11	10	10	10	1.0000
12	1	1	0	0.9167
12	1	1	1	1.0000
12	2	1	0	0.8333
12	2	1	1	1.0000
12	2	2	0	0.6818
12	2	2	1	0.9848
12	2	2	2	1.0000
12	3	1	0	0.7500
12	3	1	1	1.0000
12	3	2	0	0.5455
12	3	2	1	0.9545
12	3	2	2	1.0000
12	3	3	0	0.3818
12	3	3	1	0.8727
12	3	3	2	0.9955
12	3	3	3	1.0000
12	4	1	0	0.6667
12	4	1	1	1.0000
12	4	2	0	0.4242
12	4	2	1	0.9091
12	4	2	2	1.0000
12	4	3	0	0.2545
12	4	3	1	0.7636
12	4	3	2	0.9818
12	4	3	3	1.0000
12	4	4	0	0.1414
12	4	4	1	0.5939
12	4	4	2	0.9333
12	4	4	3	0.9980
12	4	4	4	1.0000
12	5	1	0	0.5833
12	5	1	1	1.0000
12	5	2	0	0.3182
12	5	2	1	0.8485
12	5	2	2	1.0000

N	n	M	x	F_H
12	5	3	0	0.1591
12	5	3	1	0.6364
12	5	3	2	0.9545
12	5	3	3	1.0000
12	5	4	0	0.0707
12	5	4	1	0.4242
12	5	4	2	0.8485
12	5	4	3	0.9899
12	5	4	4	1.0000
12	5	5	0	0.0265
12	5	5	1	0.2475
12	5	5	2	0.6894
12	5	5	3	0.9545
12	5	5	4	0.9987
12	5	5	5	1.0000
12	6	1	0	0.5000
12	6	1	1	1.0000
12	6	2	0	0.2273
12	6	2	1	0.7727
12	6	2	2	1.0000
12	6	3	0	0.0909
12	6	3	1	0.5000
12	6	3	2	0.9091
12	6	3	3	1.0000
12	6	4	0	0.0303
12	6	4	1	0.2727
12	6	4	2	0.7273
12	6	4	3	0.9697
12	6	4	4	1.0000
12	6	5	0	0.0076
12	6	5	1	0.1212
12	6	5	2	0.5000
12	6	5	3	0.8788
12	6	5	4	0.9924
12	6	5	5	1.0000
12	6	6	0	0.0011
12	6	6	1	0.0400
12	6	6	2	0.2835
12	6	6	3	0.7165
12	6	6	4	0.9600
12	6	6	5	0.9989
12	6	6	6	1.0000
12	7	1	0	0.4167
12	7	1	1	1.0000
12	7	2	0	0.1515

N	n	M	x	F_H
12	7	2	1	0.6818
12	7	2	2	1.0000
12	7	3	0	0.0455
12	7	3	1	0.3636
12	7	3	2	0.8409
12	7	3	3	1.0000
12	7	4	0	0.0101
12	7	4	1	0.1515
12	7	4	2	0.5758
12	7	4	3	0.9293
12	7	4	4	1.0000
12	7	5	0	0.0013
12	7	5	1	0.0455
12	7	5	2	0.3106
12	7	5	3	0.7525
12	7	5	4	0.9735
12	7	5	5	1.0000
12	7	6	1	0.0076
12	7	6	2	0.1212
12	7	6	3	0.5000
12	7	6	4	0.8788
12	7	6	5	0.9924
12	7	6	6	1.0000
12	7	7	2	0.0265
12	7	7	3	0.2475
12	7	7	4	0.6894
12	7	7	5	0.9545
12	7	7	6	0.9987
12	7	7	7	1.0000
12	8	1	0	0.3333
12	8	1	1	1.0000
12	8	2	0	0.0909
12	8	2	1	0.5758
12	8	2	2	1.0000
12	8	3	0	0.0182
12	8	3	1	0.2364
12	8	3	2	0.7455
12	8	3	3	1.0000
12	8	4	0	0.0020
12	8	4	1	0.0667
12	8	4	2	0.4061
12	8	4	3	0.8586
12	8	4	4	1.0000
12	8	5	1	0.0101
12	8	5	2	0.1515

Hypergeometric Distribution – Cumulative Distribution Function

N	n	M	x	F_H
12	8	5	3	0.5758
12	8	5	4	0.9293
12	8	5	5	1.0000
12	8	6	2	0.0303
12	8	6	3	0.2727
12	8	6	4	0.7273
12	8	6	5	0.9697
12	8	6	6	1.0000
12	8	7	3	0.0707
12	8	7	4	0.4242
12	8	7	5	0.8485
12	8	7	6	0.9899
12	8	7	7	1.0000
12	8	8	4	0.1414
12	8	8	5	0.5939
12	8	8	6	0.9333
12	8	8	7	0.9980
12	8	8	8	1.0000
12	9	1	0	0.2500
12	9	1	1	1.0000
12	9	2	0	0.0455
12	9	2	1	0.4545
12	9	2	2	1.0000
12	9	3	0	0.0045
12	9	3	1	0.1273
12	9	3	2	0.6182
12	9	3	3	1.0000
12	9	4	1	0.0182
12	9	4	2	0.2364
12	9	4	3	0.7455
12	9	4	4	1.0000
12	9	5	2	0.0455
12	9	5	3	0.3636
12	9	5	4	0.8409
12	9	5	5	1.0000
12	9	6	3	0.0909
12	9	6	4	0.5000
12	9	6	5	0.9091
12	9	6	6	1.0000
12	9	7	4	0.1591
12	9	7	5	0.6364
12	9	7	6	0.9545
12	9	7	7	1.0000
12	9	8	5	0.2545
12	9	8	6	0.7636
12	9	8	7	0.9818
12	9	8	8	1.0000
12	9	9	6	0.3818
12	9	9	7	0.8727
12	9	9	8	0.9955
12	9	9	9	1.0000
12	10	1	0	0.1667
12	10	1	1	1.0000
12	10	2	0	0.0152
12	10	2	1	0.3182
12	10	2	2	1.0000
12	10	3	1	0.0455
12	10	3	2	0.4545
12	10	3	3	1.0000
12	10	4	2	0.0909
12	10	4	3	0.5758
12	10	4	4	1.0000
12	10	5	3	0.1515
12	10	5	4	0.6818
12	10	5	5	1.0000
12	10	6	4	0.2273
12	10	6	5	0.7727
12	10	6	6	1.0000
12	10	7	5	0.3182
12	10	7	6	0.8485
12	10	7	7	1.0000
12	10	8	6	0.4242
12	10	8	7	0.9091
12	10	8	8	1.0000
12	10	9	7	0.5455
12	10	9	8	0.9545
12	10	9	9	1.0000
12	10	10	8	0.6818
12	10	10	9	0.9848
12	10	10	10	1.0000
12	11	1	0	0.0833
12	11	1	1	1.0000
12	11	2	1	0.1667
12	11	2	2	1.0000
12	11	3	2	0.2500
12	11	3	3	1.0000
12	11	4	3	0.3333
12	11	4	4	1.0000
12	11	5	4	0.4167
12	11	5	5	1.0000
12	11	6	5	0.5000
12	11	6	6	1.0000
12	11	7	6	0.5833
12	11	7	7	1.0000
12	11	8	7	0.6667
12	11	8	8	1.0000
12	11	9	8	0.7500
12	11	9	9	1.0000
12	11	10	9	0.8333
12	11	10	10	1.0000
12	11	11	10	0.9167
12	11	11	11	1.0000

Poisson Distribution – Probability Mass Function

$$f_P(x\,|\,\mu) = \begin{cases} \dfrac{\mu^x e^{-\mu}}{x!} & \text{for } x = 0,\ 1,\ \ldots \quad \text{with } \mu > 0 \qquad e = 2.71828\ldots \\ 0 & \text{others} \end{cases}$$

x	μ								
	0.001	**0.005**	**0.010**	**0.015**	**0.020**	**0.025**	**0.030**	**0.035**	**0.040**
0	0.9990	0.9950	0.9900	0.9851	0.9802	0.9753	0.9704	0.9656	0.9608
1	0.0010	0.0050	0.0099	0.0148	0.0196	0.0244	0.0291	0.0338	0.0384
2	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003	0.0004	0.0006	0.0008

x	μ								
	0.045	**0.050**	**0.055**	**0.060**	**0.065**	**0.070**	**0.075**	**0.080**	**0.085**
0	0.9560	0.9512	0.9465	0.9418	0.9371	0.9324	0.9277	0.9231	0.9185
1	0.0430	0.0476	0.0521	0.0565	0.0609	0.0653	0.0696	0.0738	0.0781
2	0.0010	0.0012	0.0014	0.0017	0.0020	0.0023	0.0026	0.0030	0.0033
3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001	0.0001

x	μ								
	0.090	**0.100**	**0.150**	**0.200**	**0.300**	**0.400**	**0.500**	**0.600**	**0.700**
0	0.9139	0.9048	0.8607	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966
1	0.0823	0.0905	0.1291	0.1637	0.2222	0.2681	0.3033	0.3293	0.3476
2	0.0037	0.0045	0.0097	0.0164	0.0333	0.0536	0.0758	0.0988	0.1217
3	0.0001	0.0002	0.0005	0.0011	0.0033	0.0072	0.0126	0.0198	0.0284
4	0.0000	0.0000	0.0000	0.0001	0.0003	0.0007	0.0016	0.0030	0.0050
5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0004	0.0007
6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001

x	μ								
	0.800	**0.900**	**1.000**	**1.100**	**1.200**	**1.300**	**1.400**	**1.500**	**1.600**
0	0.4493	0.4066	0.3679	0.3329	0.3012	0.2725	0.2466	0.2231	0.2019
1	0.3595	0.3659	0.3679	0.3662	0.3614	0.3543	0.3452	0.3347	0.3230
2	0.1438	0.1647	0.1839	0.2014	0.2169	0.2303	0.2417	0.2510	0.2584
3	0.0383	0.0494	0.0613	0.0738	0.0867	0.0998	0.1128	0.1255	0.1378
4	0.0077	0.0111	0.0153	0.0203	0.0260	0.0324	0.0395	0.0471	0.0551
5	0.0012	0.0020	0.0031	0.0045	0.0062	0.0084	0.0111	0.0141	0.0176
6	0.0002	0.0003	0.0005	0.0008	0.0012	0.0018	0.0026	0.0035	0.0047
7	0.0000	0.0000	0.0001	0.0001	0.0002	0.0003	0.0005	0.0008	0.0011
8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001	0.0002

Poisson Distribution – Probability Mass Function

x	μ								
	1.700	**1.800**	**1.900**	**2.000**	**2.100**	**2.200**	**2.300**	**2.400**	**2.500**
0	0.1827	0.1653	0.1496	0.1353	0.1225	0.1108	0.1003	0.0907	0.0821
1	0.3106	0.2975	0.2842	0.2707	0.2572	0.2438	0.2306	0.2177	0.2052
2	0.2640	0.2678	0.2700	0.2707	0.2700	0.2681	0.2652	0.2613	0.2565
3	0.1496	0.1607	0.1710	0.1804	0.1890	0.1966	0.2033	0.2090	0.2138
4	0.0636	0.0723	0.0812	0.0902	0.0992	0.1082	0.1169	0.1254	0.1336
5	0.0216	0.0260	0.0309	0.0361	0.0417	0.0476	0.0538	0.0602	0.0668
6	0.0061	0.0078	0.0098	0.0120	0.0146	0.0174	0.0206	0.0241	0.0278
7	0.0015	0.0020	0.0027	0.0034	0.0044	0.0055	0.0068	0.0083	0.0099
8	0.0003	0.0005	0.0006	0.0009	0.0011	0.0015	0.0019	0.0025	0.0031
9	0.0001	0.0001	0.0001	0.0002	0.0003	0.0004	0.0005	0.0007	0.0009
10	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001	0.0002	0.0002

x	μ								
	2.600	**2.700**	**2.800**	**2.900**	**3.000**	**3.100**	**3.200**	**3.300**	**3.400**
0	0.0743	0.0672	0.0608	0.0550	0.0498	0.0450	0.0408	0.0369	0.0334
1	0.1931	0.1815	0.1703	0.1596	0.1494	0.1397	0.1304	0.1217	0.1135
2	0.2510	0.2450	0.2384	0.2314	0.2240	0.2165	0.2087	0.2008	0.1929
3	0.2176	0.2205	0.2225	0.2237	0.2240	0.2237	0.2226	0.2209	0.2186
4	0.1414	0.1488	0.1557	0.1622	0.1680	0.1733	0.1781	0.1823	0.1858
5	0.0735	0.0804	0.0872	0.0940	0.1008	0.1075	0.1140	0.1203	0.1264
6	0.0319	0.0362	0.0407	0.0455	0.0504	0.0555	0.0608	0.0662	0.0716
7	0.0118	0.0139	0.0163	0.0188	0.0216	0.0246	0.0278	0.0312	0.0348
8	0.0038	0.0047	0.0057	0.0068	0.0081	0.0095	0.0111	0.0129	0.0148
9	0.0011	0.0014	0.0018	0.0022	0.0027	0.0033	0.0040	0.0047	0.0056
10	0.0003	0.0004	0.0005	0.0006	0.0008	0.0010	0.0013	0.0016	0.0019
11	0.0001	0.0001	0.0001	0.0002	0.0002	0.0003	0.0004	0.0005	0.0006
12	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0002

x	μ								
	3.500	**3.600**	**3.700**	**3.800**	**3.900**	**4.000**	**4.500**	**5.000**	**5.500**
0	0.0302	0.0273	0.0247	0.0224	0.0202	0.0183	0.0111	0.0067	0.0041
1	0.1057	0.0984	0.0915	0.0850	0.0789	0.0733	0.0500	0.0337	0.0225
2	0.1850	0.1771	0.1692	0.1615	0.1539	0.1465	0.1125	0.0842	0.0618
3	0.2158	0.2125	0.2087	0.2046	0.2001	0.1954	0.1687	0.1404	0.1133
4	0.1888	0.1912	0.1931	0.1944	0.1951	0.1954	0.1898	0.1755	0.1558
5	0.1322	0.1377	0.1429	0.1477	0.1522	0.1563	0.1708	0.1755	0.1714
6	0.0771	0.0826	0.0881	0.0936	0.0989	0.1042	0.1281	0.1462	0.1571
7	0.0385	0.0425	0.0466	0.0508	0.0551	0.0595	0.0824	0.1044	0.1234

Poisson Distribution – Probability Mass Function

x	μ								
	3.500	**3.600**	**3.700**	**3.800**	**3.900**	**4.000**	**4.500**	**5.000**	**5.500**
8	0.0169	0.0191	0.0215	0.0241	0.0269	0.0298	0.0463	0.0653	0.0849
9	0.0066	0.0076	0.0089	0.0102	0.0116	0.0132	0.0232	0.0363	0.0519
10	0.0023	0.0028	0.0033	0.0039	0.0045	0.0053	0.0104	0.0181	0.0285
11	0.0007	0.0009	0.0011	0.0013	0.0016	0.0019	0.0043	0.0082	0.0143
12	0.0002	0.0003	0.0003	0.0004	0.0005	0.0006	0.0016	0.0034	0.0065
13	0.0001	0.0001	0.0001	0.0001	0.0002	0.0002	0.0006	0.0013	0.0028
14	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0005	0.0011
15	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0004
16	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001

x	μ								
	6.000	**6.500**	**7.000**	**7.500**	**8.000**	**8.500**	**9.000**	**9.500**	**10.000**
0	0.0025	0.0015	0.0009	0.0006	0.0003	0.0002	0.0001	0.0001	0.0000
1	0.0149	0.0098	0.0064	0.0041	0.0027	0.0017	0.0011	0.0007	0.0005
2	0.0446	0.0318	0.0223	0.0156	0.0107	0.0074	0.0050	0.0034	0.0023
3	0.0892	0.0688	0.0521	0.0389	0.0286	0.0208	0.0150	0.0107	0.0076
4	0.1339	0.1118	0.0912	0.0729	0.0573	0.0443	0.0337	0.0254	0.0189
5	0.1606	0.1454	0.1277	0.1094	0.0916	0.0752	0.0607	0.0483	0.0378
6	0.1606	0.1575	0.1490	0.1367	0.1221	0.1066	0.0911	0.0764	0.0631
7	0.1377	0.1462	0.1490	0.1465	0.1396	0.1294	0.1171	0.1037	0.0901
8	0.1033	0.1188	0.1304	0.1373	0.1396	0.1375	0.1318	0.1232	0.1126
9	0.0688	0.0858	0.1014	0.1144	0.1241	0.1299	0.1318	0.1300	0.1251
10	0.0413	0.0558	0.0710	0.0858	0.0993	0.1104	0.1186	0.1235	0.1251
11	0.0225	0.0330	0.0452	0.0585	0.0722	0.0853	0.0970	0.1067	0.1137
12	0.0113	0.0179	0.0263	0.0366	0.0481	0.0604	0.0728	0.0844	0.0948
13	0.0052	0.0089	0.0142	0.0211	0.0296	0.0395	0.0504	0.0617	0.0729
14	0.0022	0.0041	0.0071	0.0113	0.0169	0.0240	0.0324	0.0419	0.0521
15	0.0009	0.0018	0.0033	0.0057	0.0090	0.0136	0.0194	0.0265	0.0347
16	0.0003	0.0007	0.0014	0.0026	0.0045	0.0072	0.0109	0.0157	0.0217
17	0.0001	0.0003	0.0006	0.0012	0.0021	0.0036	0.0058	0.0088	0.0128
18	0.0000	0.0001	0.0002	0.0005	0.0009	0.0017	0.0029	0.0046	0.0071
19	0.0000	0.0000	0.0001	0.0002	0.0004	0.0008	0.0014	0.0023	0.0037
20	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003	0.0006	0.0011	0.0019
21	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0003	0.0005	0.0009
22	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0002	0.0004
23	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002
24	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001

Poisson Distribution – Cumulative Distribution Function

$$F_P(x \mid \mu) = \sum_{\nu=0}^{x} \frac{\mu^\nu e^{-\mu}}{\nu!} \qquad \text{with } e = 2.71828...$$

x	μ								
	0.001	**0.005**	**0.010**	**0.015**	**0.020**	**0.025**	**0.030**	**0.035**	**0.040**
0	0.9990	0.9950	0.9900	0.9851	0.9802	0.9753	0.9704	0.9656	0.9608
1	1.0000	1.0000	1.0000	0.9999	0.9998	0.9997	0.9996	0.9994	0.9992
2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

x	μ								
	0.045	**0.050**	**0.055**	**0.060**	**0.065**	**0.070**	**0.075**	**0.080**	**0.085**
0	0.9560	0.9512	0.9465	0.9418	0.9371	0.9324	0.9277	0.9231	0.9185
1	0.9990	0.9988	0.9985	0.9983	0.9980	0.9977	0.9973	0.9970	0.9966
2	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999
3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

x	μ								
	0.090	**0.100**	**0.150**	**0.200**	**0.300**	**0.400**	**0.500**	**0.600**	**0.700**
0	0.9139	0.9048	0.8607	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966
1	0.9962	0.9953	0.9898	0.9825	0.9631	0.9384	0.9098	0.8781	0.8442
2	0.9999	0.9998	0.9995	0.9989	0.9964	0.9921	0.9856	0.9769	0.9659
3	1.0000	1.0000	1.0000	0.9999	0.9997	0.9992	0.9982	0.9966	0.9942
4	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9992
5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

x	μ								
	0.800	**0.900**	**1.000**	**1.100**	**1.200**	**1.300**	**1.400**	**1.500**	**1.600**
0	0.4493	0.4066	0.3679	0.3329	0.3012	0.2725	0.2466	0.2231	0.2019
1	0.8088	0.7725	0.7358	0.6990	0.6626	0.6268	0.5918	0.5578	0.5249
2	0.9526	0.9371	0.9197	0.9004	0.8795	0.8571	0.8335	0.8088	0.7834
3	0.9909	0.9865	0.9810	0.9743	0.9662	0.9569	0.9463	0.9344	0.9212
4	0.9986	0.9977	0.9963	0.9946	0.9923	0.9893	0.9857	0.9814	0.9763
5	0.9998	0.9997	0.9994	0.9990	0.9985	0.9978	0.9968	0.9955	0.9940
6	1.0000	1.0000	0.9999	0.9999	0.9997	0.9996	0.9994	0.9991	0.9987
7	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9998	0.9997
8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Poisson Distribution – Cumulative Distribution Function

x	μ								
	1.700	**1.800**	**1.900**	**2.000**	**2.100**	**2.200**	**2.300**	**2.400**	**2.500**
0	0.1827	0.1653	0.1496	0.1353	0.1225	0.1108	0.1003	0.0907	0.0821
1	0.4932	0.4628	0.4337	0.4060	0.3796	0.3546	0.3309	0.3084	0.2873
2	0.7572	0.7306	0.7037	0.6767	0.6496	0.6227	0.5960	0.5697	0.5438
3	0.9068	0.8913	0.8747	0.8571	0.8386	0.8194	0.7993	0.7787	0.7576
4	0.9704	0.9636	0.9559	0.9473	0.9379	0.9275	0.9162	0.9041	0.8912
5	0.9920	0.9896	0.9868	0.9834	0.9796	0.9751	0.9700	0.9643	0.9580
6	0.9981	0.9974	0.9966	0.9955	0.9941	0.9925	0.9906	0.9884	0.9858
7	0.9996	0.9994	0.9992	0.9989	0.9985	0.9980	0.9974	0.9967	0.9958
8	0.9999	0.9999	0.9998	0.9998	0.9997	0.9995	0.9994	0.9991	0.9989
9	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9998	0.9997
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

x	μ								
	2.600	**2.700**	**2.800**	**2.900**	**3.000**	**3.100**	**3.200**	**3.300**	**3.400**
0	0.0743	0.0672	0.0608	0.0550	0.0498	0.0450	0.0408	0.0369	0.0334
1	0.2674	0.2487	0.2311	0.2146	0.1991	0.1847	0.1712	0.1586	0.1468
2	0.5184	0.4936	0.4695	0.4460	0.4232	0.4012	0.3799	0.3594	0.3397
3	0.7360	0.7141	0.6919	0.6696	0.6472	0.6248	0.6025	0.5803	0.5584
4	0.8774	0.8629	0.8477	0.8318	0.8153	0.7982	0.7806	0.7626	0.7442
5	0.9510	0.9433	0.9349	0.9258	0.9161	0.9057	0.8946	0.8829	0.8705
6	0.9828	0.9794	0.9756	0.9713	0.9665	0.9612	0.9554	0.9490	0.9421
7	0.9947	0.9934	0.9919	0.9901	0.9881	0.9858	0.9832	0.9802	0.9769
8	0.9985	0.9981	0.9976	0.9969	0.9962	0.9953	0.9943	0.9931	0.9917
9	0.9996	0.9995	0.9993	0.9991	0.9989	0.9986	0.9982	0.9978	0.9973
10	0.9999	0.9999	0.9998	0.9998	0.9997	0.9996	0.9995	0.9994	0.9992
11	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

x	μ								
	3.500	**3.600**	**3.700**	**3.800**	**3.900**	**4.000**	**4.500**	**5.000**	**5.500**
0	0.0302	0.0273	0.0247	0.0224	0.0202	0.0183	0.0111	0.0067	0.0041
1	0.1359	0.1257	0.1162	0.1074	0.0992	0.0916	0.0611	0.0404	0.0266
2	0.3208	0.3027	0.2854	0.2689	0.2531	0.2381	0.1736	0.1247	0.0884
3	0.5366	0.5152	0.4942	0.4735	0.4532	0.4335	0.3423	0.2650	0.2017
4	0.7254	0.7064	0.6872	0.6678	0.6484	0.6288	0.5321	0.4405	0.3575
5	0.8576	0.8441	0.8301	0.8156	0.8006	0.7851	0.7029	0.6160	0.5289

Poisson Distribution – Cumulative Distribution Function

x	μ								
	3.500	**3.600**	**3.700**	**3.800**	**3.900**	**4.000**	**4.500**	**5.000**	**5.500**
6	0.9347	0.9267	0.9182	0.9091	0.8995	0.8893	0.8311	0.7622	0.6860
7	0.9733	0.9692	0.9648	0.9599	0.9546	0.9489	0.9134	0.8666	0.8095
8	0.9901	0.9883	0.9863	0.9840	0.9815	0.9786	0.9597	0.9319	0.8944
9	0.9967	0.9960	0.9952	0.9942	0.9931	0.9919	0.9829	0.9682	0.9462
10	0.9990	0.9987	0.9984	0.9981	0.9977	0.9972	0.9933	0.9863	0.9747
11	0.9997	0.9996	0.9995	0.9994	0.9993	0.9991	0.9976	0.9945	0.9890
12	0.9999	0.9999	0.9999	0.9998	0.9998	0.9997	0.9992	0.9980	0.9955
13	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9997	0.9993	0.9983
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9994
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998
16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

x	μ								
	6.000	**6.500**	**7.000**	**7.500**	**8.000**	**8.500**	**9.000**	**9.500**	**10.000**
0	0.0025	0.0015	0.0009	0.0006	0.0003	0.0002	0.0001	0.0001	0.0000
1	0.0174	0.0113	0.0073	0.0047	0.0030	0.0019	0.0012	0.0008	0.0005
2	0.0620	0.0430	0.0296	0.0203	0.0138	0.0093	0.0062	0.0042	0.0028
3	0.1512	0.1118	0.0818	0.0591	0.0424	0.0301	0.0212	0.0149	0.0103
4	0.2851	0.2237	0.1730	0.1321	0.0996	0.0744	0.0550	0.0403	0.0293
5	0.4457	0.3690	0.3007	0.2414	0.1912	0.1496	0.1157	0.0885	0.0671
6	0.6063	0.5265	0.4497	0.3782	0.3134	0.2562	0.2068	0.1649	0.1301
7	0.7440	0.6728	0.5987	0.5246	0.4530	0.3856	0.3239	0.2687	0.2202
8	0.8472	0.7916	0.7291	0.6620	0.5925	0.5231	0.4557	0.3918	0.3328
9	0.9161	0.8774	0.8305	0.7764	0.7166	0.6530	0.5874	0.5218	0.4579
10	0.9574	0.9332	0.9015	0.8622	0.8159	0.7634	0.7060	0.6453	0.5830
11	0.9799	0.9661	0.9467	0.9208	0.8881	0.8487	0.8030	0.7520	0.6968
12	0.9912	0.9840	0.9730	0.9573	0.9362	0.9091	0.8758	0.8364	0.7916
13	0.9964	0.9929	0.9872	0.9784	0.9658	0.9486	0.9261	0.8981	0.8645
14	0.9986	0.9970	0.9943	0.9897	0.9827	0.9726	0.9585	0.9400	0.9165
15	0.9995	0.9988	0.9976	0.9954	0.9918	0.9862	0.9780	0.9665	0.9513
16	0.9998	0.9996	0.9990	0.9980	0.9963	0.9934	0.9889	0.9823	0.9730
17	0.9999	0.9998	0.9996	0.9992	0.9984	0.9970	0.9947	0.9911	0.9857
18	1.0000	0.9999	0.9999	0.9997	0.9993	0.9987	0.9976	0.9957	0.9928
19	1.0000	1.0000	1.0000	0.9999	0.9997	0.9995	0.9989	0.9980	0.9965
20	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9991	0.9984
21	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9993
22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9997
23	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999
24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Standard Normal Distribution – Probability Density Function

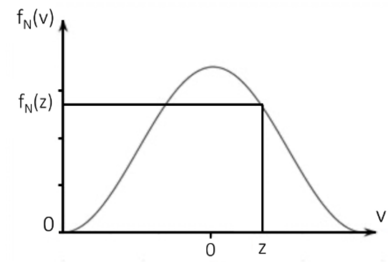
$$f_N(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$$

with $\pi = 3.14159...$ $\quad e = 2.71828...$

The following applies:

$$f_N(-z) = f_N(z)$$

$$-\infty < z < +\infty$$



z	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.00	0.39894	0.39894	0.39894	0.39894	0.39894	0.39894	0.39894	0.39893	0.39893	0.39893
0.01	0.39892	0.39892	0.39891	0.39891	0.39890	0.39890	0.39889	0.39888	0.39888	0.39887
0.02	0.39886	0.39885	0.39885	0.39884	0.39883	0.39882	0.39881	0.39880	0.39879	0.39877
0.03	0.39876	0.39875	0.39874	0.39873	0.39871	0.39870	0.39868	0.39867	0.39865	0.39864
0.04	0.39862	0.39861	0.39859	0.39857	0.39856	0.39854	0.39852	0.39850	0.39848	0.39846
0.05	0.39844	0.39842	0.39840	0.39838	0.39836	0.39834	0.39832	0.39829	0.39827	0.39825
0.06	0.39822	0.39820	0.39818	0.39815	0.39813	0.39810	0.39807	0.39805	0.39802	0.39799
0.07	0.39797	0.39794	0.39791	0.39788	0.39785	0.39782	0.39779	0.39776	0.39773	0.39770
0.08	0.39767	0.39764	0.39760	0.39757	0.39754	0.39750	0.39747	0.39744	0.39740	0.39737
0.09	0.39733	0.39729	0.39726	0.39722	0.39718	0.39715	0.39711	0.39707	0.39703	0.39699
0.10	0.39695	0.39691	0.39687	0.39683	0.39679	0.39675	0.39671	0.39667	0.39662	0.39658
0.11	0.39654	0.39649	0.39645	0.39640	0.39636	0.39631	0.39627	0.39622	0.39617	0.39613
0.12	0.39608	0.39603	0.39598	0.39594	0.39589	0.39584	0.39579	0.39574	0.39569	0.39564
0.13	0.39559	0.39553	0.39548	0.39543	0.39538	0.39532	0.39527	0.39522	0.39516	0.39511
0.14	0.39505	0.39500	0.39494	0.39488	0.39483	0.39477	0.39471	0.39466	0.39460	0.39454
0.15	0.39448	0.39442	0.39436	0.39430	0.39424	0.39418	0.39412	0.39406	0.39399	0.39393
0.16	0.39387	0.39381	0.39374	0.39368	0.39361	0.39355	0.39348	0.39342	0.39335	0.39329
0.17	0.39322	0.39315	0.39308	0.39302	0.39295	0.39288	0.39281	0.39274	0.39267	0.39260
0.18	0.39253	0.39246	0.39239	0.39232	0.39225	0.39217	0.39210	0.39203	0.39195	0.39188
0.19	0.39181	0.39173	0.39166	0.39158	0.39151	0.39143	0.39135	0.39128	0.39120	0.39112
0.20	0.39104	0.39096	0.39089	0.39081	0.39073	0.39065	0.39057	0.39049	0.39041	0.39032
0.21	0.39024	0.39016	0.39008	0.38999	0.38991	0.38983	0.38974	0.38966	0.38957	0.38949
0.22	0.38940	0.38932	0.38923	0.38915	0.38906	0.38897	0.38888	0.38880	0.38871	0.38862
0.23	0.38853	0.38844	0.38835	0.38826	0.38817	0.38808	0.38799	0.38789	0.38780	0.38771
0.24	0.38762	0.38752	0.38743	0.38734	0.38724	0.38715	0.38705	0.38696	0.38686	0.38676
0.25	0.38667	0.38657	0.38647	0.38638	0.38628	0.38618	0.38608	0.38598	0.38588	0.38578
0.26	0.38568	0.38558	0.38548	0.38538	0.38528	0.38518	0.38508	0.38497	0.38487	0.38477
0.27	0.38466	0.38456	0.38445	0.38435	0.38424	0.38414	0.38403	0.38393	0.38382	0.38371
0.28	0.38361	0.38350	0.38339	0.38328	0.38317	0.38306	0.38296	0.38285	0.38274	0.38263
0.29	0.38251	0.38240	0.38229	0.38218	0.38207	0.38196	0.38184	0.38173	0.38162	0.38150

Standard Normal Distribution – Probability Density Function

z	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.30	0.38139	0.38127	0.38116	0.38104	0.38093	0.38081	0.38070	0.38058	0.38046	0.38034
0.31	0.38023	0.38011	0.37999	0.37987	0.37975	0.37963	0.37951	0.37939	0.37927	0.37915
0.32	0.37903	0.37891	0.37879	0.37867	0.37854	0.37842	0.37830	0.37817	0.37805	0.37793
0.33	0.37780	0.37768	0.37755	0.37743	0.37730	0.37717	0.37705	0.37692	0.37679	0.37667
0.34	0.37654	0.37641	0.37628	0.37615	0.37602	0.37589	0.37576	0.37563	0.37550	0.37537
0.35	0.37524	0.37511	0.37498	0.37484	0.37471	0.37458	0.37445	0.37431	0.37418	0.37405
0.36	0.37391	0.37378	0.37364	0.37351	0.37337	0.37323	0.37310	0.37296	0.37282	0.37269
0.37	0.37255	0.37241	0.37227	0.37213	0.37199	0.37186	0.37172	0.37158	0.37144	0.37129
0.38	0.37115	0.37101	0.37087	0.37073	0.37059	0.37044	0.37030	0.37016	0.37002	0.36987
0.39	0.36973	0.36958	0.36944	0.36929	0.36915	0.36900	0.36886	0.36871	0.36856	0.36842
0.40	0.36827	0.36812	0.36797	0.36783	0.36768	0.36753	0.36738	0.36723	0.36708	0.36693
0.41	0.36678	0.36663	0.36648	0.36633	0.36618	0.36603	0.36587	0.36572	0.36557	0.36542
0.42	0.36526	0.36511	0.36496	0.36480	0.36465	0.36449	0.36434	0.36418	0.36403	0.36387
0.43	0.36371	0.36356	0.36340	0.36324	0.36309	0.36293	0.36277	0.36261	0.36245	0.36229
0.44	0.36213	0.36198	0.36182	0.36166	0.36150	0.36133	0.36117	0.36101	0.36085	0.36069
0.45	0.36053	0.36036	0.36020	0.36004	0.35988	0.35971	0.35955	0.35938	0.35922	0.35906
0.46	0.35889	0.35873	0.35856	0.35839	0.35823	0.35806	0.35789	0.35773	0.35756	0.35739
0.47	0.35723	0.35706	0.35689	0.35672	0.35655	0.35638	0.35621	0.35604	0.35587	0.35570
0.48	0.35553	0.35536	0.35519	0.35502	0.35485	0.35468	0.35450	0.35433	0.35416	0.35399
0.49	0.35381	0.35364	0.35347	0.35329	0.35312	0.35294	0.35277	0.35259	0.35242	0.35224
0.50	0.35207	0.35189	0.35171	0.35154	0.35136	0.35118	0.35100	0.35083	0.35065	0.35047
0.51	0.35029	0.35011	0.34993	0.34975	0.34958	0.34940	0.34922	0.34904	0.34885	0.34867
0.52	0.34849	0.34831	0.34813	0.34795	0.34777	0.34758	0.34740	0.34722	0.34703	0.34685
0.53	0.34667	0.34648	0.34630	0.34612	0.34593	0.34575	0.34556	0.34538	0.34519	0.34500
0.54	0.34482	0.34463	0.34445	0.34426	0.34407	0.34388	0.34370	0.34351	0.34332	0.34313
0.55	0.34294	0.34276	0.34257	0.34238	0.34219	0.34200	0.34181	0.34162	0.34143	0.34124
0.56	0.34105	0.34085	0.34066	0.34047	0.34028	0.34009	0.33990	0.33970	0.33951	0.33932
0.57	0.33912	0.33893	0.33874	0.33854	0.33835	0.33815	0.33796	0.33777	0.33757	0.33738
0.58	0.33718	0.33698	0.33679	0.33659	0.33640	0.33620	0.33600	0.33581	0.33561	0.33541
0.59	0.33521	0.33502	0.33482	0.33462	0.33442	0.33422	0.33402	0.33382	0.33362	0.33342
0.60	0.33322	0.33302	0.33282	0.33262	0.33242	0.33222	0.33202	0.33182	0.33162	0.33142
0.61	0.33121	0.33101	0.33081	0.33061	0.33040	0.33020	0.33000	0.32980	0.32959	0.32939
0.62	0.32918	0.32898	0.32878	0.32857	0.32837	0.32816	0.32796	0.32775	0.32754	0.32734
0.63	0.32713	0.32693	0.32672	0.32651	0.32631	0.32610	0.32589	0.32569	0.32548	0.32527
0.64	0.32506	0.32485	0.32465	0.32444	0.32423	0.32402	0.32381	0.32360	0.32339	0.32318
0.65	0.32297	0.32276	0.32255	0.32234	0.32213	0.32192	0.32171	0.32150	0.32129	0.32108
0.66	0.32086	0.32065	0.32044	0.32023	0.32002	0.31980	0.31959	0.31938	0.31916	0.31895
0.67	0.31874	0.31852	0.31831	0.31810	0.31788	0.31767	0.31745	0.31724	0.31702	0.31681
0.68	0.31659	0.31638	0.31616	0.31595	0.31573	0.31551	0.31530	0.31508	0.31487	0.31465
0.69	0.31443	0.31421	0.31400	0.31378	0.31356	0.31334	0.31313	0.31291	0.31269	0.31247

Standard Normal Distribution – Probability Density Function

z	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.70	0.31225	0.31204	0.31182	0.31160	0.31138	0.31116	0.31094	0.31072	0.31050	0.31028
0.71	0.31006	0.30984	0.30962	0.30940	0.30918	0.30896	0.30874	0.30852	0.30829	0.30807
0.72	0.30785	0.30763	0.30741	0.30719	0.30696	0.30674	0.30652	0.30630	0.30607	0.30585
0.73	0.30563	0.30540	0.30518	0.30496	0.30473	0.30451	0.30429	0.30406	0.30384	0.30361
0.74	0.30339	0.30316	0.30294	0.30272	0.30249	0.30227	0.30204	0.30181	0.30159	0.30136
0.75	0.30114	0.30091	0.30069	0.30046	0.30023	0.30001	0.29978	0.29955	0.29933	0.29910
0.76	0.29887	0.29865	0.29842	0.29819	0.29796	0.29774	0.29751	0.29728	0.29705	0.29682
0.77	0.29659	0.29637	0.29614	0.29591	0.29568	0.29545	0.29522	0.29499	0.29476	0.29453
0.78	0.29431	0.29408	0.29385	0.29362	0.29339	0.29316	0.29293	0.29270	0.29246	0.29223
0.79	0.29200	0.29177	0.29154	0.29131	0.29108	0.29085	0.29062	0.29039	0.29015	0.28992
0.80	0.28969	0.28946	0.28923	0.28900	0.28876	0.28853	0.28830	0.28807	0.28783	0.28760
0.81	0.28737	0.28714	0.28690	0.28667	0.28644	0.28620	0.28597	0.28574	0.28550	0.28527
0.82	0.28504	0.28480	0.28457	0.28433	0.28410	0.28387	0.28363	0.28340	0.28316	0.28293
0.83	0.28269	0.28246	0.28223	0.28199	0.28176	0.28152	0.28129	0.28105	0.28081	0.28058
0.84	0.28034	0.28011	0.27987	0.27964	0.27940	0.27917	0.27893	0.27869	0.27846	0.27822
0.85	0.27798	0.27775	0.27751	0.27728	0.27704	0.27680	0.27657	0.27633	0.27609	0.27586
0.86	0.27562	0.27538	0.27514	0.27491	0.27467	0.27443	0.27419	0.27396	0.27372	0.27348
0.87	0.27324	0.27301	0.27277	0.27253	0.27229	0.27205	0.27182	0.27158	0.27134	0.27110
0.88	0.27086	0.27063	0.27039	0.27015	0.26991	0.26967	0.26943	0.26919	0.26896	0.26872
0.89	0.26848	0.26824	0.26800	0.26776	0.26752	0.26728	0.26704	0.26680	0.26656	0.26632
0.90	0.26609	0.26585	0.26561	0.26537	0.26513	0.26489	0.26465	0.26441	0.26417	0.26393
0.91	0.26369	0.26345	0.26321	0.26297	0.26273	0.26249	0.26225	0.26201	0.26177	0.26153
0.92	0.26129	0.26105	0.26081	0.26056	0.26032	0.26008	0.25984	0.25960	0.25936	0.25912
0.93	0.25888	0.25864	0.25840	0.25816	0.25792	0.25768	0.25744	0.25719	0.25695	0.25671
0.94	0.25647	0.25623	0.25599	0.25575	0.25551	0.25527	0.25502	0.25478	0.25454	0.25430
0.95	0.25406	0.25382	0.25358	0.25333	0.25309	0.25285	0.25261	0.25237	0.25213	0.25189
0.96	0.25164	0.25140	0.25116	0.25092	0.25068	0.25044	0.25019	0.24995	0.24971	0.24947
0.97	0.24923	0.24899	0.24874	0.24850	0.24826	0.24802	0.24778	0.24754	0.24729	0.24705
0.98	0.24681	0.24657	0.24633	0.24608	0.24584	0.24560	0.24536	0.24512	0.24487	0.24463
0.99	0.24439	0.24415	0.24391	0.24366	0.24342	0.24318	0.24294	0.24270	0.24245	0.24221
1.00	0.24197	0.24173	0.24149	0.24124	0.24100	0.24076	0.24052	0.24028	0.24003	0.23979
1.01	0.23955	0.23931	0.23907	0.23883	0.23858	0.23834	0.23810	0.23786	0.23762	0.23737
1.02	0.23713	0.23689	0.23665	0.23641	0.23616	0.23592	0.23568	0.23544	0.23520	0.23496
1.03	0.23471	0.23447	0.23423	0.23399	0.23375	0.23351	0.23326	0.23302	0.23278	0.23254
1.04	0.23230	0.23206	0.23181	0.23157	0.23133	0.23109	0.23085	0.23061	0.23036	0.23012
1.05	0.22988	0.22964	0.22940	0.22916	0.22892	0.22868	0.22843	0.22819	0.22795	0.22771
1.06	0.22747	0.22723	0.22699	0.22675	0.22651	0.22626	0.22602	0.22578	0.22554	0.22530
1.07	0.22506	0.22482	0.22458	0.22434	0.22410	0.22386	0.22362	0.22338	0.22313	0.22289
1.08	0.22265	0.22241	0.22217	0.22193	0.22169	0.22145	0.22121	0.22097	0.22073	0.22049
1.09	0.22025	0.22001	0.21977	0.21953	0.21929	0.21905	0.21881	0.21857	0.21833	0.21809

Standard Normal Distribution – Probability Density Function

z	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
1.10	0.21785	0.21761	0.21737	0.21713	0.21689	0.21665	0.21642	0.21618	0.21594	0.21570
1.11	0.21546	0.21522	0.21498	0.21474	0.21450	0.21426	0.21402	0.21379	0.21355	0.21331
1.12	0.21307	0.21283	0.21259	0.21235	0.21212	0.21188	0.21164	0.21140	0.21116	0.21092
1.13	0.21069	0.21045	0.21021	0.20997	0.20973	0.20950	0.20926	0.20902	0.20878	0.20855
1.14	0.20831	0.20807	0.20783	0.20760	0.20736	0.20712	0.20688	0.20665	0.20641	0.20617
1.15	0.20594	0.20570	0.20546	0.20523	0.20499	0.20475	0.20452	0.20428	0.20404	0.20381
1.16	0.20357	0.20334	0.20310	0.20286	0.20263	0.20239	0.20216	0.20192	0.20168	0.20145
1.17	0.20121	0.20098	0.20074	0.20051	0.20027	0.20004	0.19980	0.19957	0.19933	0.19910
1.18	0.19886	0.19863	0.19839	0.19816	0.19793	0.19769	0.19746	0.19722	0.19699	0.19675
1.19	0.19652	0.19629	0.19605	0.19582	0.19559	0.19535	0.19512	0.19489	0.19465	0.19442
1.20	0.19419	0.19395	0.19372	0.19349	0.19325	0.19302	0.19279	0.19256	0.19232	0.19209
1.21	0.19186	0.19163	0.19140	0.19116	0.19093	0.19070	0.19047	0.19024	0.19001	0.18977
1.22	0.18954	0.18931	0.18908	0.18885	0.18862	0.18839	0.18816	0.18793	0.18770	0.18747
1.23	0.18724	0.18701	0.18678	0.18654	0.18631	0.18609	0.18586	0.18563	0.18540	0.18517
1.24	0.18494	0.18471	0.18448	0.18425	0.18402	0.18379	0.18356	0.18333	0.18311	0.18288
1.25	0.18265	0.18242	0.18219	0.18196	0.18174	0.18151	0.18128	0.18105	0.18083	0.18060
1.26	0.18037	0.18014	0.17992	0.17969	0.17946	0.17924	0.17901	0.17878	0.17856	0.17833
1.27	0.17810	0.17788	0.17765	0.17743	0.17720	0.17697	0.17675	0.17652	0.17630	0.17607
1.28	0.17585	0.17562	0.17540	0.17517	0.17495	0.17472	0.17450	0.17427	0.17405	0.17383
1.29	0.17360	0.17338	0.17315	0.17293	0.17271	0.17248	0.17226	0.17204	0.17181	0.17159
1.30	0.17137	0.17115	0.17092	0.17070	0.17048	0.17026	0.17003	0.16981	0.16959	0.16937
1.31	0.16915	0.16893	0.16870	0.16848	0.16826	0.16804	0.16782	0.16760	0.16738	0.16716
1.32	0.16694	0.16672	0.16650	0.16628	0.16606	0.16584	0.16562	0.16540	0.16518	0.16496
1.33	0.16474	0.16452	0.16430	0.16408	0.16386	0.16365	0.16343	0.16321	0.16299	0.16277
1.34	0.16256	0.16234	0.16212	0.16190	0.16168	0.16147	0.16125	0.16103	0.16082	0.16060
1.35	0.16038	0.16017	0.15995	0.15973	0.15952	0.15930	0.15909	0.15887	0.15866	0.15844
1.36	0.15822	0.15801	0.15779	0.15758	0.15737	0.15715	0.15694	0.15672	0.15651	0.15629
1.37	0.15608	0.15587	0.15565	0.15544	0.15523	0.15501	0.15480	0.15459	0.15437	0.15416
1.38	0.15395	0.15374	0.15352	0.15331	0.15310	0.15289	0.15268	0.15246	0.15225	0.15204
1.39	0.15183	0.15162	0.15141	0.15120	0.15099	0.15078	0.15057	0.15036	0.15015	0.14994
1.40	0.14973	0.14952	0.14931	0.14910	0.14889	0.14868	0.14847	0.14826	0.14806	0.14785
1.41	0.14764	0.14743	0.14722	0.14701	0.14681	0.14660	0.14639	0.14618	0.14598	0.14577
1.42	0.14556	0.14536	0.14515	0.14494	0.14474	0.14453	0.14433	0.14412	0.14392	0.14371
1.43	0.14350	0.14330	0.14309	0.14289	0.14268	0.14248	0.14228	0.14207	0.14187	0.14166
1.44	0.14146	0.14126	0.14105	0.14085	0.14065	0.14044	0.14024	0.14004	0.13984	0.13963
1.45	0.13943	0.13923	0.13903	0.13882	0.13862	0.13842	0.13822	0.13802	0.13782	0.13762
1.46	0.13742	0.13722	0.13702	0.13682	0.13662	0.13642	0.13622	0.13602	0.13582	0.13562
1.47	0.13542	0.13522	0.13502	0.13482	0.13462	0.13442	0.13423	0.13403	0.13383	0.13363
1.48	0.13344	0.13324	0.13304	0.13284	0.13265	0.13245	0.13225	0.13206	0.13186	0.13166
1.49	0.13147	0.13127	0.13108	0.13088	0.13069	0.13049	0.13030	0.13010	0.12991	0.12971

Standard Normal Distribution – Probability Density Function

z	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
1.50	0.12952	0.12932	0.12913	0.12894	0.12874	0.12855	0.12835	0.12816	0.12797	0.12778
1.51	0.12758	0.12739	0.12720	0.12701	0.12681	0.12662	0.12643	0.12624	0.12605	0.12586
1.52	0.12566	0.12547	0.12528	0.12509	0.12490	0.12471	0.12452	0.12433	0.12414	0.12395
1.53	0.12376	0.12357	0.12338	0.12320	0.12301	0.12282	0.12263	0.12244	0.12225	0.12207
1.54	0.12188	0.12169	0.12150	0.12132	0.12113	0.12094	0.12075	0.12057	0.12038	0.12020
1.55	0.12001	0.11982	0.11964	0.11945	0.11927	0.11908	0.11890	0.11871	0.11853	0.11834
1.56	0.11816	0.11797	0.11779	0.11761	0.11742	0.11724	0.11705	0.11687	0.11669	0.11651
1.57	0.11632	0.11614	0.11596	0.11578	0.11559	0.11541	0.11523	0.11505	0.11487	0.11469
1.58	0.11450	0.11432	0.11414	0.11396	0.11378	0.11360	0.11342	0.11324	0.11306	0.11288
1.59	0.11270	0.11253	0.11235	0.11217	0.11199	0.11181	0.11163	0.11145	0.11128	0.11110
1.60	0.11092	0.11074	0.11057	0.11039	0.11021	0.11004	0.10986	0.10968	0.10951	0.10933
1.61	0.10915	0.10898	0.10880	0.10863	0.10845	0.10828	0.10810	0.10793	0.10775	0.10758
1.62	0.10741	0.10723	0.10706	0.10688	0.10671	0.10654	0.10637	0.10619	0.10602	0.10585
1.63	0.10567	0.10550	0.10533	0.10516	0.10499	0.10482	0.10464	0.10447	0.10430	0.10413
1.64	0.10396	0.10379	0.10362	0.10345	0.10328	0.10311	0.10294	0.10277	0.10260	0.10243
1.65	0.10226	0.10210	0.10193	0.10176	0.10159	0.10142	0.10126	0.10109	0.10092	0.10075
1.66	0.10059	0.10042	0.10025	0.10009	0.09992	0.09975	0.09959	0.09942	0.09926	0.09909
1.67	0.09893	0.09876	0.09860	0.09843	0.09827	0.09810	0.09794	0.09777	0.09761	0.09745
1.68	0.09728	0.09712	0.09696	0.09679	0.09663	0.09647	0.09630	0.09614	0.09598	0.09582
1.69	0.09566	0.09550	0.09533	0.09517	0.09501	0.09485	0.09469	0.09453	0.09437	0.09421
1.70	0.09405	0.09389	0.09373	0.09357	0.09341	0.09325	0.09309	0.09293	0.09278	0.09262
1.71	0.09246	0.09230	0.09214	0.09199	0.09183	0.09167	0.09151	0.09136	0.09120	0.09104
1.72	0.09089	0.09073	0.09057	0.09042	0.09026	0.09011	0.08995	0.08980	0.08964	0.08949
1.73	0.08933	0.08918	0.08902	0.08887	0.08872	0.08856	0.08841	0.08826	0.08810	0.08795
1.74	0.08780	0.08764	0.08749	0.08734	0.08719	0.08703	0.08688	0.08673	0.08658	0.08643
1.75	0.08628	0.08613	0.08598	0.08583	0.08567	0.08552	0.08537	0.08522	0.08508	0.08493
1.76	0.08478	0.08463	0.08448	0.08433	0.08418	0.08403	0.08388	0.08374	0.08359	0.08344
1.77	0.08329	0.08315	0.08300	0.08285	0.08270	0.08256	0.08241	0.08227	0.08212	0.08197
1.78	0.08183	0.08168	0.08154	0.08139	0.08125	0.08110	0.08096	0.08081	0.08067	0.08052
1.79	0.08038	0.08024	0.08009	0.07995	0.07981	0.07966	0.07952	0.07938	0.07923	0.07909
1.80	0.07895	0.07881	0.07867	0.07852	0.07838	0.07824	0.07810	0.07796	0.07782	0.07768
1.81	0.07754	0.07740	0.07726	0.07712	0.07698	0.07684	0.07670	0.07656	0.07642	0.07628
1.82	0.07614	0.07600	0.07587	0.07573	0.07559	0.07545	0.07531	0.07518	0.07504	0.07490
1.83	0.07477	0.07463	0.07449	0.07436	0.07422	0.07408	0.07395	0.07381	0.07368	0.07354
1.84	0.07341	0.07327	0.07314	0.07300	0.07287	0.07273	0.07260	0.07247	0.07233	0.07220
1.85	0.07206	0.07193	0.07180	0.07167	0.07153	0.07140	0.07127	0.07114	0.07100	0.07087
1.86	0.07074	0.07061	0.07048	0.07035	0.07022	0.07008	0.06995	0.06982	0.06969	0.06956
1.87	0.06943	0.06930	0.06917	0.06904	0.06892	0.06879	0.06866	0.06853	0.06840	0.06827
1.88	0.06814	0.06802	0.06789	0.06776	0.06763	0.06751	0.06738	0.06725	0.06712	0.06700
1.89	0.06687	0.06674	0.06662	0.06649	0.06637	0.06624	0.06612	0.06599	0.06587	0.06574

Standard Normal Distribution – Probability Density Function

z	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
1.90	0.06562	0.06549	0.06537	0.06524	0.06512	0.06499	0.06487	0.06475	0.06462	0.06450
1.91	0.06438	0.06425	0.06413	0.06401	0.06389	0.06376	0.06364	0.06352	0.06340	0.06328
1.92	0.06316	0.06304	0.06291	0.06279	0.06267	0.06255	0.06243	0.06231	0.06219	0.06207
1.93	0.06195	0.06183	0.06171	0.06159	0.06148	0.06136	0.06124	0.06112	0.06100	0.06088
1.94	0.06077	0.06065	0.06053	0.06041	0.06029	0.06018	0.06006	0.05994	0.05983	0.05971
1.95	0.05959	0.05948	0.05936	0.05925	0.05913	0.05902	0.05890	0.05879	0.05867	0.05856
1.96	0.05844	0.05833	0.05821	0.05810	0.05798	0.05787	0.05776	0.05764	0.05753	0.05742
1.97	0.05730	0.05719	0.05708	0.05697	0.05685	0.05674	0.05663	0.05652	0.05641	0.05629
1.98	0.05618	0.05607	0.05596	0.05585	0.05574	0.05563	0.05552	0.05541	0.05530	0.05519
1.99	0.05508	0.05497	0.05486	0.05475	0.05464	0.05453	0.05442	0.05432	0.05421	0.05410
2.00	0.05399	0.05388	0.05378	0.05367	0.05356	0.05345	0.05335	0.05324	0.05313	0.05303
2.01	0.05292	0.05281	0.05271	0.05260	0.05250	0.05239	0.05228	0.05218	0.05207	0.05197
2.02	0.05186	0.05176	0.05165	0.05155	0.05145	0.05134	0.05124	0.05113	0.05103	0.05093
2.03	0.05082	0.05072	0.05062	0.05052	0.05041	0.05031	0.05021	0.05011	0.05000	0.04990
2.04	0.04980	0.04970	0.04960	0.04950	0.04939	0.04929	0.04919	0.04909	0.04899	0.04889
2.05	0.04879	0.04869	0.04859	0.04849	0.04839	0.04829	0.04819	0.04810	0.04800	0.04790
2.06	0.04780	0.04770	0.04760	0.04750	0.04741	0.04731	0.04721	0.04711	0.04702	0.04692
2.07	0.04682	0.04673	0.04663	0.04653	0.04644	0.04634	0.04624	0.04615	0.04605	0.04596
2.08	0.04586	0.04577	0.04567	0.04558	0.04548	0.04539	0.04529	0.04520	0.04510	0.04501
2.09	0.04491	0.04482	0.04473	0.04463	0.04454	0.04445	0.04435	0.04426	0.04417	0.04408
2.10	0.04398	0.04389	0.04380	0.04371	0.04362	0.04352	0.04343	0.04334	0.04325	0.04316
2.11	0.04307	0.04298	0.04289	0.04280	0.04271	0.04261	0.04252	0.04243	0.04235	0.04226
2.12	0.04217	0.04208	0.04199	0.04190	0.04181	0.04172	0.04163	0.04154	0.04146	0.04137
2.13	0.04128	0.04119	0.04110	0.04102	0.04093	0.04084	0.04075	0.04067	0.04058	0.04049
2.14	0.04041	0.04032	0.04023	0.04015	0.04006	0.03998	0.03989	0.03981	0.03972	0.03964
2.15	0.03955	0.03947	0.03938	0.03930	0.03921	0.03913	0.03904	0.03896	0.03887	0.03879
2.16	0.03871	0.03862	0.03854	0.03846	0.03837	0.03829	0.03821	0.03813	0.03804	0.03796
2.17	0.03788	0.03780	0.03771	0.03763	0.03755	0.03747	0.03739	0.03731	0.03722	0.03714
2.18	0.03706	0.03698	0.03690	0.03682	0.03674	0.03666	0.03658	0.03650	0.03642	0.03634
2.19	0.03626	0.03618	0.03610	0.03602	0.03595	0.03587	0.03579	0.03571	0.03563	0.03555
2.20	0.03547	0.03540	0.03532	0.03524	0.03516	0.03509	0.03501	0.03493	0.03485	0.03478
2.21	0.03470	0.03462	0.03455	0.03447	0.03440	0.03432	0.03424	0.03417	0.03409	0.03402
2.22	0.03394	0.03387	0.03379	0.03372	0.03364	0.03357	0.03349	0.03342	0.03334	0.03327
2.23	0.03319	0.03312	0.03305	0.03297	0.03290	0.03283	0.03275	0.03268	0.03261	0.03253
2.24	0.03246	0.03239	0.03232	0.03224	0.03217	0.03210	0.03203	0.03195	0.03188	0.03181
2.25	0.03174	0.03167	0.03160	0.03153	0.03146	0.03138	0.03131	0.03124	0.03117	0.03110
2.26	0.03103	0.03096	0.03089	0.03082	0.03075	0.03068	0.03061	0.03054	0.03047	0.03041
2.27	0.03034	0.03027	0.03020	0.03013	0.03006	0.02999	0.02993	0.02986	0.02979	0.02972
2.28	0.02965	0.02959	0.02952	0.02945	0.02939	0.02932	0.02925	0.02918	0.02912	0.02905
2.29	0.02898	0.02892	0.02885	0.02879	0.02872	0.02865	0.02859	0.02852	0.02846	0.02839

Standard Normal Distribution – Probability Density Function

z	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
2.30	0.02833	0.02826	0.02820	0.02813	0.02807	0.02800	0.02794	0.02787	0.02781	0.02775
2.31	0.02768	0.02762	0.02755	0.02749	0.02743	0.02736	0.02730	0.02724	0.02717	0.02711
2.32	0.02705	0.02699	0.02692	0.02686	0.02680	0.02674	0.02667	0.02661	0.02655	0.02649
2.33	0.02643	0.02636	0.02630	0.02624	0.02618	0.02612	0.02606	0.02600	0.02594	0.02588
2.34	0.02582	0.02576	0.02570	0.02564	0.02558	0.02552	0.02546	0.02540	0.02534	0.02528
2.35	0.02522	0.02516	0.02510	0.02504	0.02498	0.02492	0.02486	0.02481	0.02475	0.02469
2.36	0.02463	0.02457	0.02452	0.02446	0.02440	0.02434	0.02428	0.02423	0.02417	0.02411
2.37	0.02406	0.02400	0.02394	0.02389	0.02383	0.02377	0.02372	0.02366	0.02360	0.02355
2.38	0.02349	0.02344	0.02338	0.02332	0.02327	0.02321	0.02316	0.02310	0.02305	0.02299
2.39	0.02294	0.02288	0.02283	0.02277	0.02272	0.02266	0.02261	0.02256	0.02250	0.02245
2.40	0.02239	0.02234	0.02229	0.02223	0.02218	0.02213	0.02207	0.02202	0.02197	0.02192
2.41	0.02186	0.02181	0.02176	0.02170	0.02165	0.02160	0.02155	0.02150	0.02144	0.02139
2.42	0.02134	0.02129	0.02124	0.02119	0.02113	0.02108	0.02103	0.02098	0.02093	0.02088
2.43	0.02083	0.02078	0.02073	0.02068	0.02063	0.02058	0.02053	0.02048	0.02043	0.02038
2.44	0.02033	0.02028	0.02023	0.02018	0.02013	0.02008	0.02003	0.01998	0.01993	0.01989
2.45	0.01984	0.01979	0.01974	0.01969	0.01964	0.01960	0.01955	0.01950	0.01945	0.01940
2.46	0.01936	0.01931	0.01926	0.01921	0.01917	0.01912	0.01907	0.01903	0.01898	0.01893
2.47	0.01888	0.01884	0.01879	0.01875	0.01870	0.01865	0.01861	0.01856	0.01851	0.01847
2.48	0.01842	0.01838	0.01833	0.01829	0.01824	0.01820	0.01815	0.01811	0.01806	0.01802
2.49	0.01797	0.01793	0.01788	0.01784	0.01779	0.01775	0.01770	0.01766	0.01762	0.01757

Standard Normal Distribution – Probability Density Function

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.5	0.01753	0.01709	0.01667	0.01625	0.01585	0.01545	0.01506	0.01468	0.01431	0.01394
2.6	0.01358	0.01323	0.01289	0.01256	0.01223	0.01191	0.01160	0.01130	0.01100	0.01071
2.7	0.01042	0.01014	0.00987	0.00961	0.00935	0.00909	0.00885	0.00861	0.00837	0.00814
2.8	0.00792	0.00770	0.00748	0.00727	0.00707	0.00687	0.00668	0.00649	0.00631	0.00613
2.9	0.00595	0.00578	0.00562	0.00545	0.00530	0.00514	0.00499	0.00485	0.00470	0.00457
3.0	0.00443	0.00430	0.00417	0.00405	0.00393	0.00381	0.00370	0.00358	0.00348	0.00337
3.1	0.00327	0.00317	0.00307	0.00298	0.00288	0.00279	0.00271	0.00262	0.00254	0.00246
3.2	0.00238	0.00231	0.00224	0.00216	0.00210	0.00203	0.00196	0.00190	0.00184	0.00178
3.3	0.00172	0.00167	0.00161	0.00156	0.00151	0.00146	0.00141	0.00136	0.00132	0.00127
3.4	0.00123	0.00119	0.00115	0.00111	0.00107	0.00104	0.00100	0.00097	0.00094	0.00090
3.5	0.00087	0.00084	0.00081	0.00079	0.00076	0.00073	0.00071	0.00068	0.00066	0.00063
3.6	0.00061	0.00059	0.00057	0.00055	0.00053	0.00051	0.00049	0.00047	0.00046	0.00044
3.7	0.00042	0.00041	0.00039	0.00038	0.00037	0.00035	0.00034	0.00033	0.00031	0.00030
3.8	0.00029	0.00028	0.00027	0.00026	0.00025	0.00024	0.00023	0.00022	0.00021	0.00021
3.9	0.00020	0.00019	0.00018	0.00018	0.00017	0.00016	0.00016	0.00015	0.00014	0.00014

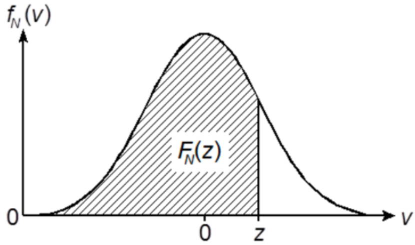
Standard Normal Distribution – Cumulative Distribution Function

$$F_N(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{\frac{-v^2}{2}} \, dv$$

with $\pi = 3.14159...$
$e = 2.71828...$

The following applies:

$$F_N(-z) = 1 - F_N(z)$$
$$-\infty < z < +\infty$$



z	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.00	0.5000	0.5004	0.5008	0.5012	0.5016	0.5020	0.5024	0.5028	0.5032	0.5036
0.01	0.5040	0.5044	0.5048	0.5052	0.5056	0.5060	0.5064	0.5068	0.5072	0.5076
0.02	0.5080	0.5084	0.5088	0.5092	0.5096	0.5100	0.5104	0.5108	0.5112	0.5116
0.03	0.5120	0.5124	0.5128	0.5132	0.5136	0.5140	0.5144	0.5148	0.5152	0.5156
0.04	0.5160	0.5164	0.5168	0.5171	0.5175	0.5179	0.5183	0.5187	0.5191	0.5195
0.05	0.5199	0.5203	0.5207	0.5211	0.5215	0.5219	0.5223	0.5227	0.5231	0.5235
0.06	0.5239	0.5243	0.5247	0.5251	0.5255	0.5259	0.5263	0.5267	0.5271	0.5275
0.07	0.5279	0.5283	0.5287	0.5291	0.5295	0.5299	0.5303	0.5307	0.5311	0.5315
0.08	0.5319	0.5323	0.5327	0.5331	0.5335	0.5339	0.5343	0.5347	0.5351	0.5355
0.09	0.5359	0.5363	0.5367	0.5370	0.5374	0.5378	0.5382	0.5386	0.5390	0.5394
0.10	0.5398	0.5402	0.5406	0.5410	0.5414	0.5418	0.5422	0.5426	0.5430	0.5434
0.11	0.5438	0.5442	0.5446	0.5450	0.5454	0.5458	0.5462	0.5466	0.5470	0.5474
0.12	0.5478	0.5482	0.5486	0.5489	0.5493	0.5497	0.5501	0.5505	0.5509	0.5513
0.13	0.5517	0.5521	0.5525	0.5529	0.5533	0.5537	0.5541	0.5545	0.5549	0.5553
0.14	0.5557	0.5561	0.5565	0.5569	0.5572	0.5576	0.5580	0.5584	0.5588	0.5592
0.15	0.5596	0.5600	0.5604	0.5608	0.5612	0.5616	0.5620	0.5624	0.5628	0.5632
0.16	0.5636	0.5640	0.5643	0.5647	0.5651	0.5655	0.5659	0.5663	0.5667	0.5671
0.17	0.5675	0.5679	0.5683	0.5687	0.5691	0.5695	0.5699	0.5702	0.5706	0.5710
0.18	0.5714	0.5718	0.5722	0.5726	0.5730	0.5734	0.5738	0.5742	0.5746	0.5750
0.19	0.5753	0.5757	0.5761	0.5765	0.5769	0.5773	0.5777	0.5781	0.5785	0.5789
0.20	0.5793	0.5797	0.5800	0.5804	0.5808	0.5812	0.5816	0.5820	0.5824	0.5828
0.21	0.5832	0.5836	0.5839	0.5843	0.5847	0.5851	0.5855	0.5859	0.5863	0.5867
0.22	0.5871	0.5875	0.5878	0.5882	0.5886	0.5890	0.5894	0.5898	0.5902	0.5906
0.23	0.5910	0.5913	0.5917	0.5921	0.5925	0.5929	0.5933	0.5937	0.5941	0.5944
0.24	0.5948	0.5952	0.5956	0.5960	0.5964	0.5968	0.5972	0.5975	0.5979	0.5983
0.25	0.5987	0.5991	0.5995	0.5999	0.6003	0.6006	0.6010	0.6014	0.6018	0.6022
0.26	0.6026	0.6030	0.6033	0.6037	0.6041	0.6045	0.6049	0.6053	0.6057	0.6060
0.27	0.6064	0.6068	0.6072	0.6076	0.6080	0.6083	0.6087	0.6091	0.6095	0.6099
0.28	0.6103	0.6106	0.6110	0.6114	0.6118	0.6122	0.6126	0.6129	0.6133	0.6137
0.29	0.6141	0.6145	0.6149	0.6152	0.6156	0.6160	0.6164	0.6168	0.6171	0.6175

Standard Normal Distribution – Cumulative Distribution Function

z	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.30	0.6179	0.6183	0.6187	0.6191	0.6194	0.6198	0.6202	0.6206	0.6210	0.6213
0.31	0.6217	0.6221	0.6225	0.6229	0.6232	0.6236	0.6240	0.6244	0.6248	0.6251
0.32	0.6255	0.6259	0.6263	0.6267	0.6270	0.6274	0.6278	0.6282	0.6285	0.6289
0.33	0.6293	0.6297	0.6301	0.6304	0.6308	0.6312	0.6316	0.6319	0.6323	0.6327
0.34	0.6331	0.6334	0.6338	0.6342	0.6346	0.6350	0.6353	0.6357	0.6361	0.6365
0.35	0.6368	0.6372	0.6376	0.6380	0.6383	0.6387	0.6391	0.6395	0.6398	0.6402
0.36	0.6406	0.6410	0.6413	0.6417	0.6421	0.6424	0.6428	0.6432	0.6436	0.6439
0.37	0.6443	0.6447	0.6451	0.6454	0.6458	0.6462	0.6465	0.6469	0.6473	0.6477
0.38	0.6480	0.6484	0.6488	0.6491	0.6495	0.6499	0.6503	0.6506	0.6510	0.6514
0.39	0.6517	0.6521	0.6525	0.6528	0.6532	0.6536	0.6539	0.6543	0.6547	0.6551
0.40	0.6554	0.6558	0.6562	0.6565	0.6569	0.6573	0.6576	0.6580	0.6584	0.6587
0.41	0.6591	0.6595	0.6598	0.6602	0.6606	0.6609	0.6613	0.6617	0.6620	0.6624
0.42	0.6628	0.6631	0.6635	0.6639	0.6642	0.6646	0.6649	0.6653	0.6657	0.6660
0.43	0.6664	0.6668	0.6671	0.6675	0.6679	0.6682	0.6686	0.6689	0.6693	0.6697
0.44	0.6700	0.6704	0.6708	0.6711	0.6715	0.6718	0.6722	0.6726	0.6729	0.6733
0.45	0.6736	0.6740	0.6744	0.6747	0.6751	0.6754	0.6758	0.6762	0.6765	0.6769
0.46	0.6772	0.6776	0.6780	0.6783	0.6787	0.6790	0.6794	0.6798	0.6801	0.6805
0.47	0.6808	0.6812	0.6815	0.6819	0.6823	0.6826	0.6830	0.6833	0.6837	0.6840
0.48	0.6844	0.6847	0.6851	0.6855	0.6858	0.6862	0.6865	0.6869	0.6872	0.6876
0.49	0.6879	0.6883	0.6886	0.6890	0.6893	0.6897	0.6901	0.6904	0.6908	0.6911
0.50	0.6915	0.6918	0.6922	0.6925	0.6929	0.6932	0.6936	0.6939	0.6943	0.6946
0.51	0.6950	0.6953	0.6957	0.6960	0.6964	0.6967	0.6971	0.6974	0.6978	0.6981
0.52	0.6985	0.6988	0.6992	0.6995	0.6999	0.7002	0.7006	0.7009	0.7013	0.7016
0.53	0.7019	0.7023	0.7026	0.7030	0.7033	0.7037	0.7040	0.7044	0.7047	0.7051
0.54	0.7054	0.7057	0.7061	0.7064	0.7068	0.7071	0.7075	0.7078	0.7082	0.7085
0.55	0.7088	0.7092	0.7095	0.7099	0.7102	0.7106	0.7109	0.7112	0.7116	0.7119
0.56	0.7123	0.7126	0.7129	0.7133	0.7136	0.7140	0.7143	0.7146	0.7150	0.7153
0.57	0.7157	0.7160	0.7163	0.7167	0.7170	0.7174	0.7177	0.7180	0.7184	0.7187
0.58	0.7190	0.7194	0.7197	0.7201	0.7204	0.7207	0.7211	0.7214	0.7217	0.7221
0.59	0.7224	0.7227	0.7231	0.7234	0.7237	0.7241	0.7244	0.7247	0.7251	0.7254
0.60	0.7257	0.7261	0.7264	0.7267	0.7271	0.7274	0.7277	0.7281	0.7284	0.7287
0.61	0.7291	0.7294	0.7297	0.7301	0.7304	0.7307	0.7311	0.7314	0.7317	0.7320
0.62	0.7324	0.7327	0.7330	0.7334	0.7337	0.7340	0.7343	0.7347	0.7350	0.7353
0.63	0.7357	0.7360	0.7363	0.7366	0.7370	0.7373	0.7376	0.7379	0.7383	0.7386
0.64	0.7389	0.7392	0.7396	0.7399	0.7402	0.7405	0.7409	0.7412	0.7415	0.7418
0.65	0.7422	0.7425	0.7428	0.7431	0.7434	0.7438	0.7441	0.7444	0.7447	0.7451
0.66	0.7454	0.7457	0.7460	0.7463	0.7467	0.7470	0.7473	0.7476	0.7479	0.7483
0.67	0.7486	0.7489	0.7492	0.7495	0.7498	0.7502	0.7505	0.7508	0.7511	0.7514
0.68	0.7517	0.7521	0.7524	0.7527	0.7530	0.7533	0.7536	0.7540	0.7543	0.7546
0.69	0.7549	0.7552	0.7555	0.7558	0.7562	0.7565	0.7568	0.7571	0.7574	0.7577

Standard Normal Distribution – Cumulative Distribution Function

z	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.70	0.7580	0.7583	0.7587	0.7590	0.7593	0.7596	0.7599	0.7602	0.7605	0.7608
0.71	0.7611	0.7615	0.7618	0.7621	0.7624	0.7627	0.7630	0.7633	0.7636	0.7639
0.72	0.7642	0.7645	0.7649	0.7652	0.7655	0.7658	0.7661	0.7664	0.7667	0.7670
0.73	0.7673	0.7676	0.7679	0.7682	0.7685	0.7688	0.7691	0.7694	0.7697	0.7700
0.74	0.7704	0.7707	0.7710	0.7713	0.7716	0.7719	0.7722	0.7725	0.7728	0.7731
0.75	0.7734	0.7737	0.7740	0.7743	0.7746	0.7749	0.7752	0.7755	0.7758	0.7761
0.76	0.7764	0.7767	0.7770	0.7773	0.7776	0.7779	0.7782	0.7785	0.7788	0.7791
0.77	0.7794	0.7796	0.7799	0.7802	0.7805	0.7808	0.7811	0.7814	0.7817	0.7820
0.78	0.7823	0.7826	0.7829	0.7832	0.7835	0.7838	0.7841	0.7844	0.7847	0.7849
0.79	0.7852	0.7855	0.7858	0.7861	0.7864	0.7867	0.7870	0.7873	0.7876	0.7879
0.80	0.7881	0.7884	0.7887	0.7890	0.7893	0.7896	0.7899	0.7902	0.7905	0.7907
0.81	0.7910	0.7913	0.7916	0.7919	0.7922	0.7925	0.7927	0.7930	0.7933	0.7936
0.82	0.7939	0.7942	0.7945	0.7947	0.7950	0.7953	0.7956	0.7959	0.7962	0.7964
0.83	0.7967	0.7970	0.7973	0.7976	0.7979	0.7981	0.7984	0.7987	0.7990	0.7993
0.84	0.7995	0.7998	0.8001	0.8004	0.8007	0.8009	0.8012	0.8015	0.8018	0.8021
0.85	0.8023	0.8026	0.8029	0.8032	0.8034	0.8037	0.8040	0.8043	0.8046	0.8048
0.86	0.8051	0.8054	0.8057	0.8059	0.8062	0.8065	0.8068	0.8070	0.8073	0.8076
0.87	0.8078	0.8081	0.8084	0.8087	0.8089	0.8092	0.8095	0.8098	0.8100	0.8103
0.88	0.8106	0.8108	0.8111	0.8114	0.8117	0.8119	0.8122	0.8125	0.8127	0.8130
0.89	0.8133	0.8135	0.8138	0.8141	0.8143	0.8146	0.8149	0.8151	0.8154	0.8157
0.90	0.8159	0.8162	0.8165	0.8167	0.8170	0.8173	0.8175	0.8178	0.8181	0.8183
0.91	0.8186	0.8189	0.8191	0.8194	0.8196	0.8199	0.8202	0.8204	0.8207	0.8210
0.92	0.8212	0.8215	0.8217	0.8220	0.8223	0.8225	0.8228	0.8230	0.8233	0.8236
0.93	0.8238	0.8241	0.8243	0.8246	0.8248	0.8251	0.8254	0.8256	0.8259	0.8261
0.94	0.8264	0.8266	0.8269	0.8272	0.8274	0.8277	0.8279	0.8282	0.8284	0.8287
0.95	0.8289	0.8292	0.8295	0.8297	0.8300	0.8302	0.8305	0.8307	0.8310	0.8312
0.96	0.8315	0.8317	0.8320	0.8322	0.8325	0.8327	0.8330	0.8332	0.8335	0.8337
0.97	0.8340	0.8342	0.8345	0.8347	0.8350	0.8352	0.8355	0.8357	0.8360	0.8362
0.98	0.8365	0.8367	0.8370	0.8372	0.8374	0.8377	0.8379	0.8382	0.8384	0.8387
0.99	0.8389	0.8392	0.8394	0.8396	0.8399	0.8401	0.8404	0.8406	0.8409	0.8411
1.00	0.8413	0.8416	0.8418	0.8421	0.8423	0.8426	0.8428	0.8430	0.8433	0.8435
1.01	0.8438	0.8440	0.8442	0.8445	0.8447	0.8449	0.8452	0.8454	0.8457	0.8459
1.02	0.8461	0.8464	0.8466	0.8468	0.8471	0.8473	0.8476	0.8478	0.8480	0.8483
1.03	0.8485	0.8487	0.8490	0.8492	0.8494	0.8497	0.8499	0.8501	0.8504	0.8506
1.04	0.8508	0.8511	0.8513	0.8515	0.8518	0.8520	0.8522	0.8525	0.8527	0.8529
1.05	0.8531	0.8534	0.8536	0.8538	0.8541	0.8543	0.8545	0.8547	0.8550	0.8552
1.06	0.8554	0.8557	0.8559	0.8561	0.8563	0.8566	0.8568	0.8570	0.8572	0.8575
1.07	0.8577	0.8579	0.8581	0.8584	0.8586	0.8588	0.8590	0.8593	0.8595	0.8597
1.08	0.8599	0.8602	0.8604	0.8606	0.8608	0.8610	0.8613	0.8615	0.8617	0.8619
1.09	0.8621	0.8624	0.8626	0.8628	0.8630	0.8632	0.8635	0.8637	0.8639	0.8641

Standard Normal Distribution – Cumulative Distribution Function

z	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
1.10	0.8643	0.8646	0.8648	0.8650	0.8652	0.8654	0.8656	0.8659	0.8661	0.8663
1.11	0.8665	0.8667	0.8669	0.8671	0.8674	0.8676	0.8678	0.8680	0.8682	0.8684
1.12	0.8686	0.8689	0.8691	0.8693	0.8695	0.8697	0.8699	0.8701	0.8703	0.8706
1.13	0.8708	0.8710	0.8712	0.8714	0.8716	0.8718	0.8720	0.8722	0.8724	0.8726
1.14	0.8729	0.8731	0.8733	0.8735	0.8737	0.8739	0.8741	0.8743	0.8745	0.8747
1.15	0.8749	0.8751	0.8753	0.8755	0.8757	0.8760	0.8762	0.8764	0.8766	0.8768
1.16	0.8770	0.8772	0.8774	0.8776	0.8778	0.8780	0.8782	0.8784	0.8786	0.8788
1.17	0.8790	0.8792	0.8794	0.8796	0.8798	0.8800	0.8802	0.8804	0.8806	0.8808
1.18	0.8810	0.8812	0.8814	0.8816	0.8818	0.8820	0.8822	0.8824	0.8826	0.8828
1.19	0.8830	0.8832	0.8834	0.8836	0.8838	0.8840	0.8842	0.8843	0.8845	0.8847
1.20	0.8849	0.8851	0.8853	0.8855	0.8857	0.8859	0.8861	0.8863	0.8865	0.8867
1.21	0.8869	0.8871	0.8872	0.8874	0.8876	0.8878	0.8880	0.8882	0.8884	0.8886
1.22	0.8888	0.8890	0.8891	0.8893	0.8895	0.8897	0.8899	0.8901	0.8903	0.8905
1.23	0.8907	0.8908	0.8910	0.8912	0.8914	0.8916	0.8918	0.8920	0.8921	0.8923
1.24	0.8925	0.8927	0.8929	0.8931	0.8933	0.8934	0.8936	0.8938	0.8940	0.8942
1.25	0.8944	0.8945	0.8947	0.8949	0.8951	0.8953	0.8954	0.8956	0.8958	0.8960
1.26	0.8962	0.8963	0.8965	0.8967	0.8969	0.8971	0.8972	0.8974	0.8976	0.8978
1.27	0.8980	0.8981	0.8983	0.8985	0.8987	0.8988	0.8990	0.8992	0.8994	0.8996
1.28	0.8997	0.8999	0.9001	0.9003	0.9004	0.9006	0.9008	0.9010	0.9011	0.9013
1.29	0.9015	0.9016	0.9018	0.9020	0.9022	0.9023	0.9025	0.9027	0.9029	0.9030
1.30	0.9032	0.9034	0.9035	0.9037	0.9039	0.9041	0.9042	0.9044	0.9046	0.9047
1.31	0.9049	0.9051	0.9052	0.9054	0.9056	0.9057	0.9059	0.9061	0.9062	0.9064
1.32	0.9066	0.9067	0.9069	0.9071	0.9072	0.9074	0.9076	0.9077	0.9079	0.9081
1.33	0.9082	0.9084	0.9086	0.9087	0.9089	0.9091	0.9092	0.9094	0.9096	0.9097
1.34	0.9099	0.9100	0.9102	0.9104	0.9105	0.9107	0.9108	0.9110	0.9112	0.9113
1.35	0.9115	0.9117	0.9118	0.9120	0.9121	0.9123	0.9125	0.9126	0.9128	0.9129
1.36	0.9131	0.9132	0.9134	0.9136	0.9137	0.9139	0.9140	0.9142	0.9143	0.9145
1.37	0.9147	0.9148	0.9150	0.9151	0.9153	0.9154	0.9156	0.9157	0.9159	0.9161
1.38	0.9162	0.9164	0.9165	0.9167	0.9168	0.9170	0.9171	0.9173	0.9174	0.9176
1.39	0.9177	0.9179	0.9180	0.9182	0.9183	0.9185	0.9186	0.9188	0.9189	0.9191
1.40	0.9192	0.9194	0.9195	0.9197	0.9198	0.9200	0.9201	0.9203	0.9204	0.9206
1.41	0.9207	0.9209	0.9210	0.9212	0.9213	0.9215	0.9216	0.9218	0.9219	0.9221
1.42	0.9222	0.9223	0.9225	0.9226	0.9228	0.9229	0.9231	0.9232	0.9234	0.9235
1.43	0.9236	0.9238	0.9239	0.9241	0.9242	0.9244	0.9245	0.9246	0.9248	0.9249
1.44	0.9251	0.9252	0.9253	0.9255	0.9256	0.9258	0.9259	0.9261	0.9262	0.9263
1.45	0.9265	0.9266	0.9267	0.9269	0.9270	0.9272	0.9273	0.9274	0.9276	0.9277
1.46	0.9279	0.9280	0.9281	0.9283	0.9284	0.9285	0.9287	0.9288	0.9289	0.9291
1.47	0.9292	0.9294	0.9295	0.9296	0.9298	0.9299	0.9300	0.9302	0.9303	0.9304
1.48	0.9306	0.9307	0.9308	0.9310	0.9311	0.9312	0.9314	0.9315	0.9316	0.9318
1.49	0.9319	0.9320	0.9322	0.9323	0.9324	0.9325	0.9327	0.9328	0.9329	0.9331

Standard Normal Distribution – Cumulative Distribution Function

z	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
1.50	0.9332	0.9333	0.9335	0.9336	0.9337	0.9338	0.9340	0.9341	0.9342	0.9344
1.51	0.9345	0.9346	0.9347	0.9349	0.9350	0.9351	0.9352	0.9354	0.9355	0.9356
1.52	0.9357	0.9359	0.9360	0.9361	0.9362	0.9364	0.9365	0.9366	0.9367	0.9369
1.53	0.9370	0.9371	0.9372	0.9374	0.9375	0.9376	0.9377	0.9379	0.9380	0.9381
1.54	0.9382	0.9383	0.9385	0.9386	0.9387	0.9388	0.9389	0.9391	0.9392	0.9393
1.55	0.9394	0.9395	0.9397	0.9398	0.9399	0.9400	0.9401	0.9403	0.9404	0.9405
1.56	0.9406	0.9407	0.9409	0.9410	0.9411	0.9412	0.9413	0.9414	0.9416	0.9417
1.57	0.9418	0.9419	0.9420	0.9421	0.9423	0.9424	0.9425	0.9426	0.9427	0.9428
1.58	0.9429	0.9431	0.9432	0.9433	0.9434	0.9435	0.9436	0.9437	0.9439	0.9440
1.59	0.9441	0.9442	0.9443	0.9444	0.9445	0.9446	0.9448	0.9449	0.9450	0.9451
1.60	0.9452	0.9453	0.9454	0.9455	0.9456	0.9458	0.9459	0.9460	0.9461	0.9462
1.61	0.9463	0.9464	0.9465	0.9466	0.9467	0.9468	0.9470	0.9471	0.9472	0.9473
1.62	0.9474	0.9475	0.9476	0.9477	0.9478	0.9479	0.9480	0.9481	0.9482	0.9483
1.63	0.9484	0.9486	0.9487	0.9488	0.9489	0.9490	0.9491	0.9492	0.9493	0.9494
1.64	0.9495	0.9496	0.9497	0.9498	0.9499	0.9500	0.9501	0.9502	0.9503	0.9504
1.65	0.9505	0.9506	0.9507	0.9508	0.9509	0.9510	0.9511	0.9512	0.9513	0.9514
1.66	0.9515	0.9516	0.9517	0.9518	0.9519	0.9520	0.9521	0.9522	0.9523	0.9524
1.67	0.9525	0.9526	0.9527	0.9528	0.9529	0.9530	0.9531	0.9532	0.9533	0.9534
1.68	0.9535	0.9536	0.9537	0.9538	0.9539	0.9540	0.9541	0.9542	0.9543	0.9544
1.69	0.9545	0.9546	0.9547	0.9548	0.9549	0.9550	0.9551	0.9552	0.9552	0.9553
1.70	0.9554	0.9555	0.9556	0.9557	0.9558	0.9559	0.9560	0.9561	0.9562	0.9563
1.71	0.9564	0.9565	0.9566	0.9566	0.9567	0.9568	0.9569	0.9570	0.9571	0.9572
1.72	0.9573	0.9574	0.9575	0.9576	0.9576	0.9577	0.9578	0.9579	0.9580	0.9581
1.73	0.9582	0.9583	0.9584	0.9585	0.9585	0.9586	0.9587	0.9588	0.9589	0.9590
1.74	0.9591	0.9592	0.9592	0.9593	0.9594	0.9595	0.9596	0.9597	0.9598	0.9599
1.75	0.9599	0.9600	0.9601	0.9602	0.9603	0.9604	0.9605	0.9605	0.9606	0.9607
1.76	0.9608	0.9609	0.9610	0.9610	0.9611	0.9612	0.9613	0.9614	0.9615	0.9616
1.77	0.9616	0.9617	0.9618	0.9619	0.9620	0.9621	0.9621	0.9622	0.9623	0.9624
1.78	0.9625	0.9625	0.9626	0.9627	0.9628	0.9629	0.9630	0.9630	0.9631	0.9632
1.79	0.9633	0.9634	0.9634	0.9635	0.9636	0.9637	0.9638	0.9638	0.9639	0.9640
1.80	0.9641	0.9641	0.9642	0.9643	0.9644	0.9645	0.9645	0.9646	0.9647	0.9648
1.81	0.9649	0.9649	0.9650	0.9651	0.9652	0.9652	0.9653	0.9654	0.9655	0.9655
1.82	0.9656	0.9657	0.9658	0.9658	0.9659	0.9660	0.9661	0.9662	0.9662	0.9663
1.83	0.9664	0.9664	0.9665	0.9666	0.9667	0.9667	0.9668	0.9669	0.9670	0.9670
1.84	0.9671	0.9672	0.9673	0.9673	0.9674	0.9675	0.9676	0.9676	0.9677	0.9678
1.85	0.9678	0.9679	0.9680	0.9681	0.9681	0.9682	0.9683	0.9683	0.9684	0.9685
1.86	0.9686	0.9686	0.9687	0.9688	0.9688	0.9689	0.9690	0.9690	0.9691	0.9692
1.87	0.9693	0.9693	0.9694	0.9695	0.9695	0.9696	0.9697	0.9697	0.9698	0.9699
1.88	0.9699	0.9700	0.9701	0.9701	0.9702	0.9703	0.9704	0.9704	0.9705	0.9706
1.89	0.9706	0.9707	0.9708	0.9708	0.9709	0.9710	0.9710	0.9711	0.9712	0.9712

Standard Normal Distribution – Cumulative Distribution Function

z	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
1.90	0.9713	0.9713	0.9714	0.9715	0.9715	0.9716	0.9717	0.9717	0.9718	0.9719
1.91	0.9719	0.9720	0.9721	0.9721	0.9722	0.9723	0.9723	0.9724	0.9724	0.9725
1.92	0.9726	0.9726	0.9727	0.9728	0.9728	0.9729	0.9729	0.9730	0.9731	0.9731
1.93	0.9732	0.9733	0.9733	0.9734	0.9734	0.9735	0.9736	0.9736	0.9737	0.9737
1.94	0.9738	0.9739	0.9739	0.9740	0.9741	0.9741	0.9742	0.9742	0.9743	0.9744
1.95	0.9744	0.9745	0.9745	0.9746	0.9746	0.9747	0.9748	0.9748	0.9749	0.9749
1.96	0.9750	0.9751	0.9751	0.9752	0.9752	0.9753	0.9754	0.9754	0.9755	0.9755
1.97	0.9756	0.9756	0.9757	0.9758	0.9758	0.9759	0.9759	0.9760	0.9760	0.9761
1.98	0.9761	0.9762	0.9763	0.9763	0.9764	0.9764	0.9765	0.9765	0.9766	0.9766
1.99	0.9767	0.9768	0.9768	0.9769	0.9769	0.9770	0.9770	0.9771	0.9771	0.9772
2.00	0.9772	0.9773	0.9774	0.9774	0.9775	0.9775	0.9776	0.9776	0.9777	0.9777
2.01	0.9778	0.9778	0.9779	0.9779	0.9780	0.9780	0.9781	0.9782	0.9782	0.9783
2.02	0.9783	0.9784	0.9784	0.9785	0.9785	0.9786	0.9786	0.9787	0.9787	0.9788
2.03	0.9788	0.9789	0.9789	0.9790	0.9790	0.9791	0.9791	0.9792	0.9792	0.9793
2.04	0.9793	0.9794	0.9794	0.9795	0.9795	0.9796	0.9796	0.9797	0.9797	0.9798
2.05	0.9798	0.9799	0.9799	0.9800	0.9800	0.9801	0.9801	0.9802	0.9802	0.9803
2.06	0.9803	0.9803	0.9804	0.9804	0.9805	0.9805	0.9806	0.9806	0.9807	0.9807
2.07	0.9808	0.9808	0.9809	0.9809	0.9810	0.9810	0.9811	0.9811	0.9811	0.9812
2.08	0.9812	0.9813	0.9813	0.9814	0.9814	0.9815	0.9815	0.9816	0.9816	0.9816
2.09	0.9817	0.9817	0.9818	0.9818	0.9819	0.9819	0.9820	0.9820	0.9820	0.9821
2.10	0.9821	0.9822	0.9822	0.9823	0.9823	0.9824	0.9824	0.9824	0.9825	0.9825
2.11	0.9826	0.9826	0.9827	0.9827	0.9827	0.9828	0.9828	0.9829	0.9829	0.9830
2.12	0.9830	0.9830	0.9831	0.9831	0.9832	0.9832	0.9832	0.9833	0.9833	0.9834
2.13	0.9834	0.9835	0.9835	0.9835	0.9836	0.9836	0.9837	0.9837	0.9837	0.9838
2.14	0.9838	0.9839	0.9839	0.9839	0.9840	0.9840	0.9841	0.9841	0.9841	0.9842
2.15	0.9842	0.9843	0.9843	0.9843	0.9844	0.9844	0.9845	0.9845	0.9845	0.9846
2.16	0.9846	0.9847	0.9847	0.9847	0.9848	0.9848	0.9848	0.9849	0.9849	0.9850
2.17	0.9850	0.9850	0.9851	0.9851	0.9851	0.9852	0.9852	0.9853	0.9853	0.9853
2.18	0.9854	0.9854	0.9854	0.9855	0.9855	0.9856	0.9856	0.9856	0.9857	0.9857
2.19	0.9857	0.9858	0.9858	0.9858	0.9859	0.9859	0.9860	0.9860	0.9860	0.9861
2.20	0.9861	0.9861	0.9862	0.9862	0.9862	0.9863	0.9863	0.9863	0.9864	0.9864
2.21	0.9864	0.9865	0.9865	0.9866	0.9866	0.9866	0.9867	0.9867	0.9867	0.9868
2.22	0.9868	0.9868	0.9869	0.9869	0.9869	0.9870	0.9870	0.9870	0.9871	0.9871
2.23	0.9871	0.9872	0.9872	0.9872	0.9873	0.9873	0.9873	0.9874	0.9874	0.9874
2.24	0.9875	0.9875	0.9875	0.9876	0.9876	0.9876	0.9876	0.9877	0.9877	0.9877
2.25	0.9878	0.9878	0.9878	0.9879	0.9879	0.9879	0.9880	0.9880	0.9880	0.9881
2.26	0.9881	0.9881	0.9882	0.9882	0.9882	0.9882	0.9883	0.9883	0.9883	0.9884
2.27	0.9884	0.9884	0.9885	0.9885	0.9885	0.9885	0.9886	0.9886	0.9886	0.9887
2.28	0.9887	0.9887	0.9888	0.9888	0.9888	0.9888	0.9889	0.9889	0.9889	0.9890
2.29	0.9890	0.9890	0.9890	0.9891	0.9891	0.9891	0.9892	0.9892	0.9892	0.9892

Standard Normal Distribution – Cumulative Distribution Function

z	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
2.30	0.9893	0.9893	0.9893	0.9894	0.9894	0.9894	0.9894	0.9895	0.9895	0.9895
2.31	0.9896	0.9896	0.9896	0.9896	0.9897	0.9897	0.9897	0.9897	0.9898	0.9898
2.32	0.9898	0.9899	0.9899	0.9899	0.9899	0.9900	0.9900	0.9900	0.9900	0.9901
2.33	0.9901	0.9901	0.9901	0.9902	0.9902	0.9902	0.9903	0.9903	0.9903	0.9903
2.34	0.9904	0.9904	0.9904	0.9904	0.9905	0.9905	0.9905	0.9905	0.9906	0.9906
2.35	0.9906	0.9906	0.9907	0.9907	0.9907	0.9907	0.9908	0.9908	0.9908	0.9908
2.36	0.9909	0.9909	0.9909	0.9909	0.9910	0.9910	0.9910	0.9910	0.9911	0.9911
2.37	0.9911	0.9911	0.9912	0.9912	0.9912	0.9912	0.9912	0.9913	0.9913	0.9913
2.38	0.9913	0.9914	0.9914	0.9914	0.9914	0.9915	0.9915	0.9915	0.9915	0.9916
2.39	0.9916	0.9916	0.9916	0.9916	0.9917	0.9917	0.9917	0.9917	0.9918	0.9918
2.40	0.9918	0.9918	0.9918	0.9919	0.9919	0.9919	0.9919	0.9920	0.9920	0.9920
2.41	0.9920	0.9920	0.9921	0.9921	0.9921	0.9921	0.9922	0.9922	0.9922	0.9922
2.42	0.9922	0.9923	0.9923	0.9923	0.9923	0.9923	0.9924	0.9924	0.9924	0.9924
2.43	0.9925	0.9925	0.9925	0.9925	0.9925	0.9926	0.9926	0.9926	0.9926	0.9926
2.44	0.9927	0.9927	0.9927	0.9927	0.9927	0.9928	0.9928	0.9928	0.9928	0.9928
2.45	0.9929	0.9929	0.9929	0.9929	0.9929	0.9930	0.9930	0.9930	0.9930	0.9930
2.46	0.9931	0.9931	0.9931	0.9931	0.9931	0.9931	0.9932	0.9932	0.9932	0.9932
2.47	0.9932	0.9933	0.9933	0.9933	0.9933	0.9933	0.9934	0.9934	0.9934	0.9934
2.48	0.9934	0.9934	0.9935	0.9935	0.9935	0.9935	0.9935	0.9936	0.9936	0.9936
2.49	0.9936	0.9936	0.9936	0.9937	0.9937	0.9937	0.9937	0.9937	0.9938	0.9938

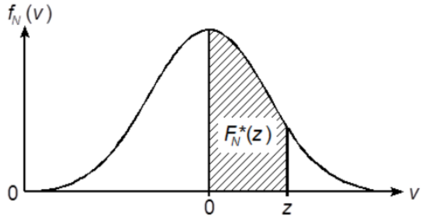
Standard Normal Distribution – Cumulative Distribution Function

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999

Standard Normal Distribution – One-sided Confidence Intervals

The following applies:

$$\begin{aligned} F_N^*(z) &= F_N(z) - 0.5 \\ &= F_N^*(-z) \\ F_N^*(0) &= 0 \\ 0 \le z &< +\infty \end{aligned}$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986

Standard Normal Distribution – One-sided Confidence Intervals

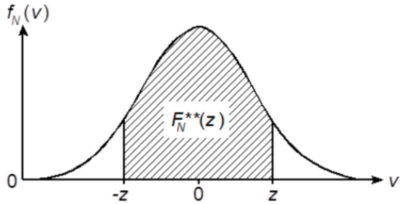
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998
3.5	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4998	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.9	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000

Standard Normal Distribution – Two-sided, Symmetric Confidence Intervals

The following applies:

$$F_N^{**}(z) = F_N(z) - F_N(-z) = 2F_N(z) - 1 = 2F_N^*(z)$$

$-\infty < z < +\infty$



$F_N^*(z)$ see also standard normal distribution, one-sided confidence intervals.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0080	0.0160	0.0239	0.0319	0.0399	0.0478	0.0558	0.0638	0.0717
0.1	0.0797	0.0876	0.0955	0.1034	0.1113	0.1192	0.1271	0.1350	0.1428	0.1507
0.2	0.1585	0.1663	0.1741	0.1819	0.1897	0.1974	0.2051	0.2128	0.2205	0.2282
0.3	0.2358	0.2434	0.2510	0.2586	0.2661	0.2737	0.2812	0.2886	0.2961	0.3035
0.4	0.3108	0.3182	0.3255	0.3328	0.3401	0.3473	0.3545	0.3616	0.3688	0.3759
0.5	0.3829	0.3899	0.3969	0.4039	0.4108	0.4177	0.4245	0.4313	0.4381	0.4448
0.6	0.4515	0.4581	0.4647	0.4713	0.4778	0.4843	0.4907	0.4971	0.5035	0.5098
0.7	0.5161	0.5223	0.5285	0.5346	0.5407	0.5467	0.5527	0.5587	0.5646	0.5705
0.8	0.5763	0.5821	0.5878	0.5935	0.5991	0.6047	0.6102	0.6157	0.6211	0.6265
0.9	0.6319	0.6372	0.6424	0.6476	0.6528	0.6579	0.6629	0.6680	0.6729	0.6778
1.0	0.6827	0.6875	0.6923	0.6970	0.7017	0.7063	0.7109	0.7154	0.7199	0.7243
1.1	0.7287	0.7330	0.7373	0.7415	0.7457	0.7499	0.7540	0.7580	0.7620	0.7660
1.2	0.7699	0.7737	0.7775	0.7813	0.7850	0.7887	0.7923	0.7959	0.7995	0.8029
1.3	0.8064	0.8098	0.8132	0.8165	0.8198	0.8230	0.8262	0.8293	0.8324	0.8355
1.4	0.8385	0.8415	0.8444	0.8473	0.8501	0.8529	0.8557	0.8584	0.8611	0.8638
1.5	0.8664	0.8690	0.8715	0.8740	0.8764	0.8789	0.8812	0.8836	0.8859	0.8882
1.6	0.8904	0.8926	0.8948	0.8969	0.8990	0.9011	0.9031	0.9051	0.9070	0.9090
1.7	0.9109	0.9127	0.9146	0.9164	0.9181	0.9199	0.9216	0.9233	0.9249	0.9265
1.8	0.9281	0.9297	0.9312	0.9328	0.9342	0.9357	0.9371	0.9385	0.9399	0.9412
1.9	0.9426	0.9439	0.9451	0.9464	0.9476	0.9488	0.9500	0.9512	0.9523	0.9534
2.0	0.9545	0.9556	0.9566	0.9576	0.9586	0.9596	0.9606	0.9615	0.9625	0.9634
2.1	0.9643	0.9651	0.9660	0.9668	0.9676	0.9684	0.9692	0.9700	0.9707	0.9715
2.2	0.9722	0.9729	0.9736	0.9743	0.9749	0.9756	0.9762	0.9768	0.9774	0.9780
2.3	0.9786	0.9791	0.9797	0.9802	0.9807	0.9812	0.9817	0.9822	0.9827	0.9832
2.4	0.9836	0.9840	0.9845	0.9849	0.9853	0.9857	0.9861	0.9865	0.9869	0.9872
2.5	0.9876	0.9879	0.9883	0.9886	0.9889	0.9892	0.9895	0.9898	0.9901	0.9904
2.6	0.9907	0.9909	0.9912	0.9915	0.9917	0.9920	0.9922	0.9924	0.9926	0.9929
2.7	0.9931	0.9933	0.9935	0.9937	0.9939	0.9940	0.9942	0.9944	0.9946	0.9947
2.8	0.9949	0.9950	0.9952	0.9953	0.9955	0.9956	0.9958	0.9959	0.9960	0.9961
2.9	0.9963	0.9964	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972

Standard Normal Distribution – Two-sided, Symmetric Confidence Intervals

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
3.0	0.9973	0.9974	0.9975	0.9976	0.9976	0.9977	0.9978	0.9979	0.9979	0.9980
3.1	0.9981	0.9981	0.9982	0.9983	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986
3.2	0.9986	0.9987	0.9987	0.9988	0.9988	0.9988	0.9989	0.9989	0.9990	0.9990
3.3	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.4	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995	0.9995
3.5	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	0.9997
3.6	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	0.9998	0.9998
3.7	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999

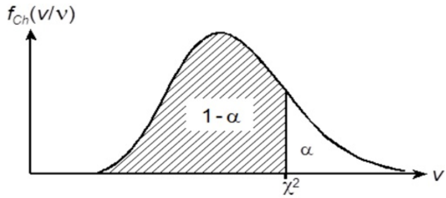
Chi-Squared Distribution – Distribution Function

The corresponding values of χ^2 for the parameters v and $(1-a)$ in form of a distribution function are given.

The following is applicable for χ^2:

$$W(0 < X^2 \leq \chi^2) = F_{CH}\left(\frac{\chi^2}{v}\right) = 1-\alpha$$

with $X^2 =$ random variable



v	$1-\alpha$								
	0.001	**0.005**	**0.010**	**0.025**	**0.050**	**0.100**	**0.250**	**0.400**	**0.500**
1	0.000	0.000	0.000	0.001	0.004	0.016	0.102	0.275	0.455
2	0.002	0.010	0.020	0.051	0.103	0.211	0.575	1.022	1.386
3	0.024	0.072	0.115	0.216	0.352	0.5840	1.213	1.869	2.366
4	0.091	0.207	0.297	0.484	0.711	1.064	1.923	2.753	3.357
5	0.210	0.412	0.554	0.831	1.145	1.610	2.675	3.655	4.351
6	0.381	0.676	0.872	1.237	1.635	2.204	3.455	4.570	5.348
7	0.598	0.989	1.239	1.690	2.167	2.833	4.255	5.493	6.346
8	0.857	1.344	1.646	2.180	2.733	3.490	5.071	6.423	7.344
9	1.152	1.735	2.088	2.700	3.325	4.168	5.899	7.357	8.343
10	1.479	2.156	2.558	3.247	3.940	4.865	6.737	8.295	9.342
11	1.834	2.603	3.053	3.816	4.575	5.578	7.584	9.237	10.341
12	2.214	3.074	3.571	4.404	5.226	6.304	8.438	10.182	11.340
13	2.617	3.565	4.107	5.009	5.892	7.042	9.299	11.129	12.340
14	3.041	4.075	4.660	5.629	6.571	7.790	10.165	12.078	13.339
15	3.483	4.601	5.229	6.262	7.261	8.547	11.037	13.030	14.339
16	3.942	5.142	5.812	6.908	7.962	9.312	11.912	13.983	15.338
17	4.416	5.697	6.408	7.564	8.672	10.085	12.792	14.937	16.338
18	4.905	6.265	7.015	8.231	9.390	10.865	13.675	15.893	17.338
19	5.407	6.844	7.633	8.907	10.117	11.651	14.562	16.850	18.338
20	5.921	7.434	8.260	9.591	10.851	12.443	15.452	17.809	19.337
21	6.447	8.034	8.897	10.283	11.591	13.240	16.344	18.768	20.337
22	6.983	8.643	9.542	10.982	12.338	14.041	17.240	19.729	21.337
23	7.529	9.260	10.196	11.689	13.091	14.848	18.137	20.690	22.337
24	8.085	9.886	10.856	12.401	13.848	15.659	19.037	21.652	23.337
25	8.649	10.520	11.524	13.120	14.611	16.473	19.939	22.616	24.337

Chi-Squared Distribution – Distribution Function

ν	$1-\alpha$								
	0.001	**0.005**	**0.010**	**0.025**	**0.050**	**0.100**	**0.250**	**0.400**	**0.500**
30	11.588	13.787	14.953	16.791	18.493	20.599	24.478	27.442	29.336
35	14.688	17.192	18.509	20.569	22.465	24.797	29.054	32.282	34.336
40	17.916	20.707	22.164	24.433	26.509	29.051	33.660	37.134	39.335
45	21.251	24.311	25.901	28.366	30.612	33.350	38.291	41.995	44.335
50	24.674	27.991	29.707	32.357	34.764	37.689	42.942	46.864	49.335
60	31.738	35.534	37.485	40.482	43.188	46.459	52.294	56.620	59.335
70	39.036	43.275	45.442	48.758	51.739	55.329	61.698	66.396	69.334
80	46.520	51.172	53.540	57.153	60.391	64.278	71.145	76.188	79.334
90	54.155	59.196	61.754	65.647	69.126	73.291	80.625	85.993	89.334
100	61.918	67.328	70.065	74.222	77.929	82.358	90.133	95.808	99.334

Chi-Squared Distribution – Distribution Function

ν	$1-\alpha$								
	0.600	**0.750**	**0.800**	**0.850**	**0.900**	**0.950**	**0.975**	**0.980**	**0.990**
1	0.708	1.323	1.642	2.072	2.706	3.841	5.024	5.412	6.635
2	1.833	2.773	3.219	3.794	4.605	5.991	7.378	7.824	9.210
3	2.946	4.108	4.642	5.317	6.251	7.815	9.348	9.837	11.345
4	4.045	5.385	5.989	6.745	7.779	9.488	11.143	11.668	13.277
5	5.132	6.626	7.289	8.115	9.236	11.070	12.833	13.388	15.086
6	6.211	7.841	8.558	9.446	10.645	12.592	14.449	15.033	16.812
7	7.283	9.037	9.803	10.748	12.017	14.067	16.013	16.622	18.475
8	8.351	10.219	11.030	12.027	13.362	15.507	17.535	18.168	20.090
9	9.414	11.389	12.242	13.288	14.684	16.919	19.023	19.679	21.666
10	10.473	12.549	13.442	14.534	15.987	18.307	20.483	21.161	23.209
11	11.530	13.701	14.631	15.767	17.275	19.675	21.920	22.618	24.725
12	12.584	14.845	15.812	16.989	18.549	21.026	23.337	24.054	26.217
13	13.636	15.984	16.985	18.202	19.812	22.362	24.736	25.472	27.688
14	14.685	17.117	18.151	19.406	21.064	23.685	26.119	26.873	29.141
15	15.733	18.245	19.311	20.603	22.307	24.996	27.488	28.259	30.578
16	16.780	19.369	20.465	21.793	23.542	26.296	28.845	29.633	32.000
17	17.824	20.489	21.615	22.977	24.769	27.587	30.191	30.995	33.409
18	18.868	21.605	22.760	24.155	25.989	28.869	31.526	32.346	34.805
19	19.910	22.718	23.900	25.329	27.204	30.144	32.852	33.687	36.191
20	20.951	23.828	25.038	26.498	28.412	31.410	34.170	35.020	37.566
21	21.991	24.935	26.171	27.662	29.615	32.671	35.479	36.343	38.932
22	23.031	26.039	27.301	28.822	30.813	33.924	36.781	37.659	40.289
23	24.069	27.141	28.429	29.979	32.007	35.172	38.076	38.968	41.638
24	25.106	28.241	29.553	31.132	33.196	36.415	39.364	40.270	42.980
25	26.143	29.339	30.675	32.282	34.382	37.652	40.646	41.566	44.314
30	31.316	34.800	36.250	37.990	40.256	43.773	46.979	47.962	50.892
35	36.475	40.223	41.778	43.640	46.059	49.802	53.203	54.244	57.342
40	41.622	45.616	47.269	49.244	51.805	55.758	59.342	60.436	63.691
45	46.761	50.985	52.729	54.810	57.505	61.656	65.410	66.555	69.957
50	51.892	56.334	58.164	60.346	63.167	67.505	71.420	72.613	76.154
60	62.135	66.981	68.972	71.341	74.397	79.082	83.298	84.580	88.380
70	72.358	77.577	79.715	82.255	85.527	90.530	95.020	96.390	100.430
80	82.566	88.130	90.410	93.110	96.580	101.880	106.630	108.070	112.330
90	92.760	98.650	101.050	103.900	107.570	113.150	118.140	119.650	124.120
100	102.950	109.140	111.670	114.660	118.500	124.340	129.560	131.140	135.810

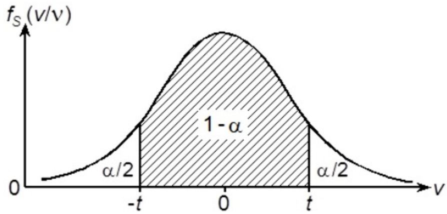
Student's t-Distribution – Two-sided, Symmetric Confidence Intervals

The corresponding values of t for the parameters v and $(1-a)$ in form of two-sided, symmetric confidence intervals are given.

The following is applicable for t:

$W(-t < T \leq t) = 1 - \alpha$

with $T =$ random variable



v	$1-\alpha$								
	0.100	**0.150**	**0.200**	**0.250**	**0.300**	**0.350**	**0.400**	**0.450**	**0.500**
1	0.158	0.240	0.325	0.414	0.510	0.613	0.727	0.854	1.000
2	0.142	0.215	0.289	0.365	0.445	0.528	0.617	0.713	0.816
3	0.137	0.206	0.277	0.349	0.424	0.502	0.584	0.671	0.765
4	0.134	0.202	0.271	0.341	0.414	0.490	0.569	0.652	0.741
5	0.132	0.199	0.267	0.337	0.408	0.482	0.559	0.641	0.727
6	0.131	0.197	0.265	0.334	0.404	0.477	0.553	0.633	0.718
7	0.130	0.196	0.263	0.331	0.402	0.474	0.549	0.628	0.711
8	0.130	0.195	0.262	0.330	0.399	0.471	0.546	0.624	0.706
9	0.129	0.195	0.261	0.329	0.398	0.469	0.543	0.621	0.703
10	0.129	0.194	0.260	0.328	0.397	0.468	0.542	0.619	0.700
11	0.129	0.194	0.260	0.327	0.396	0.466	0.540	0.617	0.697
12	0.128	0.193	0.259	0.326	0.395	0.465	0.539	0.615	0.695
13	0.128	0.193	0.259	0.325	0.394	0.464	0.538	0.614	0.694
14	0.128	0.193	0.258	0.325	0.393	0.464	0.537	0.613	0.692
15	0.128	0.192	0.258	0.325	0.393	0.463	0.536	0.612	0.691
16	0.128	0.192	0.258	0.324	0.392	0.462	0.535	0.611	0.690
17	0.128	0.192	0.257	0.324	0.392	0.462	0.534	0.610	0.689
18	0.127	0.192	0.257	0.324	0.392	0.461	0.534	0.609	0.688
19	0.127	0.192	0.257	0.323	0.391	0.461	0.533	0.609	0.688
20	0.127	0.192	0.257	0.323	0.391	0.461	0.533	0.608	0.687
21	0.127	0.191	0.257	0.323	0.391	0.460	0.532	0.608	0.686
22	0.127	0.191	0.256	0.323	0.390	0.460	0.532	0.607	0.686
23	0.127	0.191	0.256	0.322	0.390	0.460	0.532	0.607	0.685
24	0.127	0.191	0.256	0.322	0.390	0.460	0.531	0.606	0.685
25	0.127	0.191	0.256	0.322	0.390	0.459	0.531	0.606	0.684

Student's t-Distribution – Two-sided, Symmetric Confidence Intervals

ν	$1-\alpha$								
	0.100	**0.150**	**0.200**	**0.250**	**0.300**	**0.350**	**0.400**	**0.450**	**0.500**
26	0.127	0.191	0.256	0.322	0.390	0.459	0.531	0.606	0.684
27	0.127	0.191	0.256	0.322	0.389	0.459	0.531	0.605	0.684
28	0.127	0.191	0.256	0.322	0.389	0.459	0.530	0.605	0.683
29	0.127	0.191	0.256	0.322	0.389	0.459	0.530	0.605	0.683
30	0.127	0.191	0.256	0.322	0.389	0.458	0.530	0.605	0.683
40	0.126	0.190	0.255	0.321	0.388	0.457	0.529	0.603	0.681
50	0.126	0.190	0.255	0.320	0.388	0.457	0.528	0.602	0.679
100	0.126	0.190	0.254	0.320	0.386	0.455	0.526	0.600	0.677
150	0.126	0.189	0.254	0.319	0.386	0.455	0.526	0.599	0.676
∞	0.126	0.189	0.253	0.319	0.385	0.454	0.524	0.598	0.675

Student's t-Distribution – Two-sided, Symmetric Confidence Intervals

ν	$1-\alpha$								
	0.600	**0.700**	**0.800**	**0.850**	**0.900**	**0.950**	**0.975**	**0.990**	**0.995**
1	1.376	1.963	3.078	4.165	6.314	12.706	25.452	63.657	127.321
2	1.061	1.386	1.886	2.282	2.920	4.303	6.205	9.925	14.089
3	0.978	1.250	1.638	1.924	2.353	3.182	4.177	5.841	7.453
4	0.941	1.190	1.533	1.778	2.132	2.776	3.495	4.604	5.598
5	0.920	1.156	1.476	1.699	2.015	2.571	3.163	4.032	4.773
6	0.906	1.134	1.440	1.650	1.943	2.447	2.969	3.707	4.317
7	0.896	1.119	1.415	1.617	1.895	2.365	2.841	3.499	4.029
8	0.889	1.108	1.397	1.592	1.860	2.306	2.752	3.355	3.833
9	0.883	1.100	1.383	1.574	1.833	2.262	2.685	3.250	3.690
10	0.879	1.093	1.372	1.559	1.812	2.228	2.634	3.169	3.581
11	0.876	1.088	1.363	1.548	1.796	2.201	2.593	3.106	3.497
12	0.873	1.083	1.356	1.538	1.782	2.179	2.560	3.055	3.428
13	0.870	1.079	1.350	1.530	1.771	2.160	2.533	3.012	3.372
14	0.868	1.076	1.345	1.523	1.761	2.145	2.510	2.977	3.326
15	0.866	1.074	1.341	1.517	1.753	2.131	2.490	2.947	3.286
16	0.865	1.071	1.337	1.512	1.746	2.120	2.473	2.921	3.252
17	0.863	1.069	1.333	1.508	1.740	2.110	2.458	2.898	3.222
18	0.862	1.067	1.330	1.504	1.734	2.101	2.445	2.878	3.197
19	0.861	1.066	1.328	1.500	1.729	2.093	2.433	2.861	3.174
20	0.860	1.064	1.325	1.497	1.725	2.086	2.423	2.845	3.153
21	0.859	1.063	1.323	1.494	1.721	2.080	2.414	2.831	3.135
22	0.858	1.061	1.321	1.492	1.717	2.074	2.405	2.819	3.119
23	0.858	1.060	1.319	1.489	1.714	2.069	2.398	2.807	3.104
24	0.857	1.059	1.318	1.487	1.711	2.064	2.391	2.797	3.091
25	0.856	1.058	1.316	1.485	1.708	2.060	2.385	2.787	3.078
26	0.856	1.058	1.315	1.483	1.706	2.056	2.379	2.779	3.067
27	0.855	1.057	1.314	1.482	1.703	2.052	2.373	2.771	3.057
28	0.855	1.056	1.313	1.480	1.701	2.048	2.368	2.763	3.047
29	0.854	1.055	1.311	1.479	1.699	2.045	2.364	2.756	3.038
30	0.854	1.055	1.310	1.477	1.697	2.042	2.360	2.750	3.030
40	0.851	1.050	1.303	1.468	1.684	2.021	2.329	2.704	2.971
50	0.849	1.047	1.299	1.462	1.676	2.009	2.311	2.678	2.937
100	0.845	1.042	1.290	1.451	1.660	1.984	2.276	2.626	2.871
150	0.844	1.040	1.287	1.447	1.655	1.976	2.264	2.609	2.849
∞	0.842	1.036	1.282	1.440	1.645	1.960	2.242	2.576	2.808

Student's t-Distribution – Distribution Function

The corresponding values of t for the parameters v and $(1-a)$ in form of a distribution function are given.

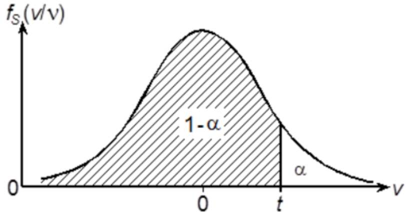
The following is applicable for t:

$$W(-\infty < T \leq t) = F_S\left(\frac{t}{v}\right) = 1-\alpha$$

with T = random variable

The following applies:

$$F_S\left(\frac{-t}{v}\right) = 1 - F_S\left(\frac{t}{v}\right)$$



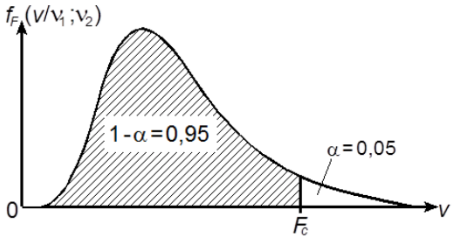
v	$1-\alpha$								
	0.600	**0.700**	**0.800**	**0.900**	**0.950**	**0.975**	**0.990**	**0.995**	**0.999**
1	0.325	0.727	1.376	3.078	6.314	12.706	31.821	63.657	318.309
2	0.289	0.617	1.061	1.886	2.920	4.303	6.965	9.925	22.327
3	0.277	0.584	0.978	1.638	2.353	3.182	4.541	5.841	10.215
4	0.271	0.569	0.941	1.533	2.132	2.776	3.747	4.604	7.173
5	0.267	0.559	0.920	1.476	2.015	2.571	3.365	4.032	5.893
6	0.265	0.553	0.906	1.440	1.943	2.447	3.143	3.707	5.208
7	0.263	0.549	0.896	1.415	1.895	2.365	2.998	3.499	4.785
8	0.262	0.546	0.889	1.397	1.860	2.306	2.896	3.355	4.501
9	0.261	0.543	0.883	1.383	1.833	2.262	2.821	3.250	4.297
10	0.260	0.542	0.879	1.372	1.812	2.228	2.764	3.169	4.144
11	0.260	0.540	0.876	1.363	1.796	2.201	2.718	3.106	4.025
12	0.259	0.539	0.873	1.356	1.782	2.179	2.681	3.055	3.930
13	0.259	0.538	0.870	1.350	1.771	2.160	2.650	3.012	3.852
14	0.258	0.537	0.868	1.345	1.761	2.145	2.624	2.977	3.787
15	0.258	0.536	0.866	1.341	1.753	2.131	2.602	2.947	3.733
16	0.258	0.535	0.865	1.337	1.746	2.120	2.583	2.921	3.686
17	0.257	0.534	0.863	1.333	1.740	2.110	2.567	2.898	3,646
18	0.257	0.534	0.862	1.330	1.734	2.101	2.552	2.878	3.610
19	0.257	0.533	0.861	1.328	1.729	2.093	2.539	2.861	3.579
20	0.257	0.533	0.860	1.325	1.725	2.086	2.528	2.845	3.552
21	0.257	0.532	0.859	1.323	1.721	2.080	2.518	2.831	3.527
22	0.256	0.532	0.858	1.321	1.717	2.074	2.508	2.819	3.505
23	0.256	0.532	0.858	1.319	1.714	2.069	2.500	2.807	3.485
24	0.256	0.531	0.857	1.318	1.711	2.064	2.492	2.797	3.467
25	0.256	0.531	0.856	1.316	1.708	2.060	2.485	2.787	3.450
30	0.256	0.530	0.854	1.310	1.697	2.042	2.457	2.750	3.385
40	0.255	0.529	0.851	1.303	1.684	2.021	2.423	2.704	3.307
50	0.255	0.528	0.849	1.299	1.676	2.009	2.403	2.678	3.261
100	0.254	0.526	0.845	1.290	1.660	1.984	2.364	2.626	3.174
150	0.254	0.526	0.844	1.287	1.655	1.976	2.351	2.609	3.145
∞	0.253	0.524	0.842	1.282	1.645	1.960	2.326	2.576	3.090

F-Distribution – Distribution Function with $\alpha = 0.05$

The corresponding values of F_c for the parameters ν_1 and ν_2 in form of a F-distribution function with $(1-a) = 0.95$ are given.

The following is applicable for F_c:

$$W(0 < F \leq F_c) = F\left(\frac{F_c}{\nu_1;\ \nu_2}\right)$$
$$= 1-\alpha = 0.95$$



with F = random variable

and $F_{\alpha;\,\nu_1;\,\nu_2} = \dfrac{1}{F_{1-\alpha;\,\nu_1;\,\nu_2}}$

with $\nu_1 = n_1 - 1$
and $\nu_2 = n_2 - 1$

ν_2	ν_1										
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	242.98
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.40
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.70
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.41
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.28
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.24
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.22
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.20

F-Distribution – Distribution Function with $\alpha = 0.05$

v_2	v_1										
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.17
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.14
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.04
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.99
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.95
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02	1.97	1.93
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.91
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94	1.90
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.89
150	3.90	3.06	2.66	2.43	2.27	2.16	2.07	2.00	1.94	1.89	1.85
200	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.84
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.79

F-Distribution – Distribution Function with $\alpha = 0.05$

ν_2	ν_1										
	12	**13**	**14**	**15**	**20**	**30**	**40**	**50**	**100**	**200**	∞
1	243.91	244.69	245.36	245.95	248.01	250.1	251.14	251.77	253.04	253.68	254.31
2	19.41	19.42	19.42	19.43	19.45	19.46	19.47	19.48	19.49	19.49	19.50
3	8.74	8.73	8.71	8.70	8.66	8.62	8.59	8.58	8.55	8.54	8.53
4	5.91	5.89	5.87	5.86	5.80	5.75	5.72	5.70	5.66	5.65	5.63
5	4.68	4.66	4.64	4.62	4.56	4.50	4.46	4.44	4.41	4.39	4.37
6	4.00	3.98	3.96	3.94	3.87	3.81	3.77	3.75	3.71	3.69	3.67
7	3.57	3.55	3.53	3.51	3.44	3.38	3.34	3.32	3.27	3.25	3.23
8	3.28	3.26	3.24	3.22	3.15	3.08	3.04	3.02	2.97	2.95	2.93
9	3.07	3.05	3.03	3.01	2.94	2.86	2.83	2.80	2.76	2.73	2.71
10	2.91	2.89	2.86	2.85	2.77	2.70	2.66	2.64	2.59	2.56	2.54
11	2.79	2.76	2.74	2.72	2.65	2.57	2.53	2.51	2.46	2.43	2.40
12	2.69	2.66	2.64	2.62	2.54	2.47	2.43	2.40	2.35	2.32	2.30
13	2.60	2.58	2.55	2.53	2.46	2.38	2.34	2.31	2.26	2.23	2.21
14	2.53	2.51	2.48	2.46	2.39	2.31	2.27	2.24	2.19	2.16	2.13
15	2.48	2.45	2.42	2.40	2.33	2.25	2.20	2.18	2.12	2.10	2.07
16	2.42	2.40	2.37	2.35	2.28	2.19	2.15	2.12	2.07	2.04	2.01
17	2.38	2.35	2.33	2.31	2.23	2.15	2.10	2.08	2.02	1.99	1.96
18	2.34	2.31	2.29	2.27	2.19	2.11	2.06	2.04	1.98	1.95	1.92
19	2.31	2.28	2.26	2.23	2.16	2.07	2.03	2.00	1.94	1.91	1.88
20	2.28	2.25	2.22	2.20	2.12	2.04	1.99	1.97	1.91	1.88	1.84
21	2.25	2.22	2.20	2.18	2.10	2.01	1.96	1.94	1.88	1.84	1.81
22	2.23	2.20	2.17	2.15	2.07	1.98	1.94	1.91	1.85	1.82	1.78
23	2.20	2.18	2.15	2.13	2.05	1.96	1.91	1.88	1.82	1.79	1.76
24	2.18	2.15	2.13	2.11	2.03	1.94	1.89	1.86	1.80	1.77	1.73
25	2.16	2.14	2.11	2.09	2.01	1.92	1.87	1.84	1.78	1.75	1.71
26	2.15	2.12	2.09	2.07	1.99	1.90	1.85	1.82	1.76	1.73	1.69
27	2.13	2.10	2.08	2.06	1.97	1.88	1.84	1.81	1.74	1.71	1.67
28	2.12	2.09	2.06	2.04	1.96	1.87	1.82	1.79	1.73	1.69	1.65
29	2.10	2.08	2.05	2.03	1.94	1.85	1.81	1.77	1.71	1.67	1.64
30	2.09	2.06	2.04	2.01	1.93	1.84	1.79	1.76	1.70	1.66	1.62
40	2.00	1.97	1.95	1.92	1.84	1.74	1.69	1.66	1.59	1.55	1.51
50	1.95	1.92	1.89	1.87	1.78	1.69	1.63	1.60	1.52	1.48	1.44
60	1.92	1.89	1.86	1.84	1.75	1.65	1.59	1.56	1.48	1.44	1.39
70	1.89	1.86	1.84	1.81	1.72	1.62	1.57	1.53	1.45	1.40	1.35
80	1.88	1.84	1.82	1.79	1.70	1.60	1.54	1.51	1.43	1.38	1.32
90	1.86	1.83	1.80	1.78	1.69	1.59	1.53	1.49	1.41	1.36	1.30
100	1.85	1.82	1.79	1.77	1.68	1.57	1.52	1.48	1.39	1.34	1.28
150	1.82	1.79	1.76	1.73	1.64	1.54	1.48	1.44	1.34	1.29	1.22
200	1.80	1.77	1.74	1.72	1.62	1.52	1.46	1.41	1.32	1.26	1.19
∞	1.75	1.72	1.69	1.67	1.57	1.46	1.39	1.35	1.24	1.17	1.01

F-Distribution – Distribution Function with $\alpha = 0.01$

The corresponding values of F_c for the parameters v_1 and v_2 in form of a F-distribution function with $(1-a) = 0.99$ are given.

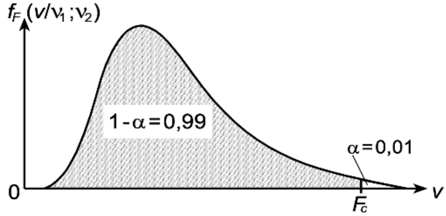
The following is applicable for F_c:

$$W(0 < F \leq F_c) = F\left(\frac{F_c}{v_1;\ v_2}\right) = 1 - \alpha = 0.99$$

with F = random variable

and $F_{\alpha;\,v_1;\,v_2} = \dfrac{1}{F_{1-\alpha;\,v_1;\,v_2}}$

with $v_1 = n_1 - 1$
and $v_2 = n_2 - 1$



v_2	v_1 = 1	2	3	4	5	6	7	8	9	10	11
1	4052	4999	5403	5625	5764	5859	5928	5981	6022	6056	6083
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.41
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.13
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.45
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.96
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.79
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.54
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.73
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.18
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.77
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.46
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.22
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	4.02
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.86
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.73
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.62
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.52
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.43
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.36
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.29
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.24
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.18
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.14
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.09
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	3.06

F-Distribution – Distribution Function with $\alpha = 0.01$

v_2	v_1										
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	3.02
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.99
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.96
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.93
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.91
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.73
50	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.78	2.70	2.63
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.56
70	7.01	4.92	4.07	3.60	3.29	3.07	2.91	2.78	2.67	2.59	2.51
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.64	2.55	2.48
90	6.93	4.85	4.01	3.53	3.23	3.01	2.84	2.72	2.61	2.52	2.45
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59	2.50	2.43
150	6.81	4.75	3.91	3.45	3.14	2.92	2.76	2.63	2.53	2.44	2.37
200	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50	2.41	2.34
∞	6.64	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.25

F-Distribution – Distribution Function with $\alpha = 0.01$

ν_2	ν_1										
	12	**13**	**14**	**15**	**20**	**30**	**40**	**50**	**100**	**200**	∞
1	6106	6126	6143	6157	6209	6261	6287	6303	6334	6350	6366
2	99.42	99.42	99.43	99.43	99.45	99.47	99.47	99.48	99.49	99.49	99.50
3	27.05	26.98	26.92	26.87	26.69	26.50	26.41	26.35	26.24	26.18	26.13
4	14.37	14.31	14.25	14.20	14.02	13.84	13.75	13.69	13.58	13.52	13.46
5	9.89	9.82	9.77	9.72	9.55	9.38	9.29	9.24	9.13	9.08	9.02
6	7.72	7.66	7.60	7.56	7.40	7.23	7.14	7.09	6.99	6.93	6.88
7	6.47	6.41	6.36	6.31	6.16	5.99	5.91	5.86	5.75	5.70	5.65
8	5.67	5.61	5.56	5.52	5.36	5.20	5.12	5.07	4.96	4.91	4.86
9	5.11	5.05	5.01	4.96	4.81	4.65	4.57	4.52	4.41	4.36	4.31
10	4.71	4.65	4.60	4.56	4.41	4.25	4.17	4.12	4.01	3.96	3.91
11	4.40	4.34	4.29	4.25	4.10	3.94	3.86	3.81	3.71	3.66	3.60
12	4.16	4.10	4.05	4.01	3.86	3.70	3.62	3.57	3.47	3.41	3.36
13	3.96	3.91	3.86	3.82	3.66	3.51	3.43	3.38	3.27	3.22	3.17
14	3.80	3.75	3.70	3.66	3.51	3.35	3.27	3.22	3.11	3.06	3.00
15	3.67	3.61	3.56	3.52	3.37	3.21	3.13	3.08	2.98	2.92	2.87
16	3.55	3.50	3.45	3.41	3.26	3.10	3.02	2.97	2.86	2.81	2.75
17	3.46	3.40	3.35	3.31	3.16	3.00	2.92	2.87	2.76	2.71	2.65
18	3.37	3.32	3.27	3.23	3.08	2.92	2.84	2.78	2.68	2.62	2.57
19	3.30	3.24	3.19	3.15	3.00	2.84	2.76	2.71	2.60	2.55	2.49
20	3.23	3.18	3.13	3.09	2.94	2.78	2.69	2.64	2.54	2.48	2.42
21	3.17	3.12	3.07	3.03	2.88	2.72	2.64	2.58	2.48	2.42	2.36
22	3.12	3.07	3.02	2.98	2.83	2.67	2.58	2.53	2.42	2.36	2.31
23	3.07	3.02	2.97	2.93	2.78	2.62	2.54	2.48	2.37	2.32	2.26
24	3.03	2.98	2.93	2.89	2.74	2.58	2.49	2.44	2.33	2.27	2.21
25	2.99	2.94	2.89	2.85	2.70	2.54	2.45	2.40	2.29	2.23	2.17
26	2.96	2.90	2.86	2.81	2.66	2.50	2.42	2.36	2.25	2.19	2.13
27	2.93	2.87	2.82	2.78	2.63	2.47	2.38	2.33	2.22	2.16	2.10
28	2.90	2.84	2.79	2.75	2.60	2.44	2.35	2.30	2.19	2.13	2.06
29	2.87	2.81	2.77	2.73	2.57	2.41	2.33	2.27	2.16	2.10	2.03
30	2.84	2.79	2.74	2.70	2.55	2.39	2.30	2.25	2.13	2.07	2.01
40	2.66	2.61	2.56	2.52	2.37	2.20	2.11	2.06	1.94	1.87	1.80
50	2.56	2.51	2.46	2.42	2.27	2.10	2.01	1.95	1.82	1.76	1.68
60	2.50	2.44	2.39	2.35	2.20	2.03	1.94	1.88	1.75	1.68	1.60
70	2.45	2.40	2.35	2.31	2.15	1.98	1.89	1.83	1.70	1.62	1.54
80	2.42	2.36	2.31	2.27	2.12	1.94	1.85	1.79	1.65	1.58	1.49
90	2.39	2.33	2.29	2.24	2.09	1.92	1.82	1.76	1.62	1.55	1.46
100	2.37	2.31	2.27	2.22	2.07	1.89	1.80	1.74	1.60	1.52	1.43
150	2.31	2.25	2.20	2.16	2.00	1.83	1.73	1.66	1.52	1.43	1.33
200	2.27	2.22	2.17	2.13	1.97	1.79	1.69	1.63	1.48	1.39	1.28
∞	2.18	2.13	2.08	2.04	1.88	1.70	1.59	1.52	1.36	1.25	1.01

Appendix B
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