Large scale Finite Element solvers for the large eddy simulation of incompressible turbulent flows

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1. Motivation
2. Residual-based VMS
3. Mixed FE VMS
4. Segregated Runge-Kutta
5. Segregated VMS
6. Conclusions

Motivation

Thesis motivation

Highly scalable Finite Element (FE) framework for Large Eddy Simulations (LES) of incompressible turbulent flows

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How to get there?

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1. Variational MultiScale (VMS) methods as LES models.

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1. Variational MultiScale (VMS) methods as LES models.
2. Time integration schemes with velocity-pressure segregation.

Motivation

Thesis motivation

Highly scalable Finite Element (FE) framework for Large Eddy Simulations (LES) of incompressible turbulent flows

How to get there?

1. Variational MultiScale (VMS) methods as LES models.
2. Time integration schemes with velocity-pressure segregation.
3. Highly scalable algorithms based on Domain Decomposition (DD) and block preconditioners.

Motivation

Step by step...

- Residual-based VMS as LES models.

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- Mixed FE formulations LES.

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- High-order time integration schemes.
- Adaptive time stepping techniques.
- Velocity-pressure segregation.
- Scalable solvers.

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- Application.

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- Scalable solvers.
- Application.

1. Motivation

2. Residual-based VMS
   - Formulation
   - Energy statements
   - Numerical experiments
   - Conclusions

Rb-VMS

3. Mixed FE VMS

4. Segregated Runge-Kutta

5. Segregated VMS

6. Conclusions

Implicit LES

ILES: only numerical dissipation (for stabilization) acts as turbulent model

- Not based on filtering of the Navier-Stokes equations
- Rely on numerical artifacts, no modification at the continuous level

Incomp. Navier Stokes equations

Find $\mathbf{u}$ and p defined in Ω

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = \mathbf{f}$$
$$\nabla \cdot \mathbf{u} = 0$$

with appropriate boundary conditions on Γ.

Incomp. Navier Stokes equations

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with appropriate boundary conditions on Γ.

The weak problem is: $\forall \mathbf{v} \in \mathcal{V}_0$ and $\forall q \in \mathcal{Q}_0$, find $\mathbf{u} \in \mathcal{V}$ and $p \in \mathcal{Q}$ such that

$$\begin{aligned} (\mathbf{v}, \partial_t \mathbf{u})_\Omega + (\nabla \mathbf{v}, \nu \nabla \mathbf{u})_\Omega + b\,(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p)_\Omega &= \langle \mathbf{v}, \mathbf{f} \rangle_\Omega \\ (q, \nabla \cdot \mathbf{u})_\Omega &= 0 \end{aligned}$$

Incomp. Navier Stokes equations

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where

$$b(\mathbf{a}, \mathbf{u}, \mathbf{v}) = \quad \langle \mathbf{v}, \mathbf{a} \cdot \nabla \mathbf{u} \rangle_\Omega$$

Incomp. Navier Stokes equations

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where

$$b(\mathbf{a}, \mathbf{u}, \mathbf{v}) = \frac{1}{2} \langle \mathbf{v}, \mathbf{a} \cdot \nabla \mathbf{u} \rangle_\Omega - \frac{1}{2} \langle \mathbf{a} \cdot \nabla \mathbf{v}, \mathbf{u} \rangle_\Omega + \frac{1}{2} \langle \mathbf{v}, \mathbf{n} \cdot \mathbf{a} \mathbf{u} \rangle_\Gamma$$

VMS decomposition (Hughes 1995)

A decomposition of spaces $\mathcal{V}$ and $\mathcal{Q}$ given by

$$\mathcal{V} = \mathcal{V}_h \oplus \widetilde{\mathcal{V}}, \quad \mathcal{Q} = \mathcal{Q}_h \oplus \widetilde{\mathcal{Q}}$$

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$$\mathcal{V} = \mathcal{V}_h \oplus \widetilde{\mathcal{V}}, \quad \mathcal{Q} = \mathcal{Q}_h \oplus \widetilde{\mathcal{Q}}$$

is applied to the function and test spaces

$$\mathbf{u} = \mathbf{u}_h + \widetilde{\mathbf{u}}, \quad p = p_h + \widetilde{p}$$
$$\mathbf{v} = \mathbf{v}_h + \widetilde{\mathbf{v}}, \quad q = q_h + \widetilde{q}$$

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is applied to the function and test spaces

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_h + \widetilde{\mathbf{u}}, \quad & p &= p_h + \widetilde{p} \\ \mathbf{v} &= \mathbf{v}_h + \widetilde{\mathbf{v}}, \quad & q &= q_h + \widetilde{q} \end{aligned}$$

We keep all the (eight) contributions from the splitting of the convective term

$$\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{u}_h \cdot \nabla \mathbf{u}_h + \widetilde{\mathbf{u}} \cdot \nabla \mathbf{u}_h + \mathbf{u}_h \cdot \nabla \widetilde{\mathbf{u}} + \widetilde{\mathbf{u}} \cdot \nabla \widetilde{\mathbf{u}}$$

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$$\mathbf{u} = \mathbf{u}_h + \widetilde{\mathbf{u}}, \quad p = p_h + \widetilde{p}$$
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We keep all the (eight) contributions from the splitting of the convective term

$$\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{u}_h \cdot \nabla \mathbf{u}_h + \widetilde{\mathbf{u}} \cdot \nabla \mathbf{u}_h + \mathbf{u}_h \cdot \nabla \widetilde{\mathbf{u}} + \widetilde{\mathbf{u}} \cdot \nabla \widetilde{\mathbf{u}}$$

and all the (four) contributions from the temporal term

$$\partial_t \mathbf{u} = \partial_t \mathbf{u}_h + \partial_t \widetilde{\mathbf{u}}$$

Semidiscrete problem

FEM equations

$$B\left((\mathbf{u}_h, p_h);(\widetilde{\mathbf{u}}, \widetilde{p});(\mathbf{v}_h, q_h)\right) = L\left(\mathbf{v}_h, q_h\right)$$

SGS equations

$$B\left((\widetilde{\mathbf{u}}, \widetilde{p});(\mathbf{u}_h, p_h);(\widetilde{\mathbf{v}}, \widetilde{q})\right) = L\left(\widetilde{\mathbf{v}}, \widetilde{q}\right)$$

Semidiscrete problem

FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$
$$+ (\mathbf{v}_h, \partial_t \widetilde{\mathbf{u}})_\Omega + (\mathcal{L}^* \mathbf{v}_h, \widetilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \widetilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

SGS equations

$$\partial_t \widetilde{\mathbf{u}} + \tau_m^{-1} \widetilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$$\tau_m = \left(\frac{c_1 \nu}{h^2} + \frac{c_2 |\mathbf{a}|}{h} \right)^{-1}, \quad \tau_c = \frac{h^2}{c_1 \tau_m}$$

$$\mathbf{a} = \mathbf{u}_h + \widetilde{\mathbf{u}}$$

Semidiscrete problem

FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

$$+ (\mathbf{v}_h, \partial_t \widetilde{\mathbf{u}})_\Omega + (\mathcal{L}^* \mathbf{v}_h, \widetilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \widetilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

SGS equations

$$\partial_t \widetilde{\mathbf{u}} + \tau_m^{-1} \widetilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$$\mathbf{R}_m := \mathbf{f} - \partial_t \mathbf{u}_h - \mathcal{L} \mathbf{u}_h - \nabla p_h, \quad R_c := -\nabla \cdot \mathbf{u}_h$$

Semidiscrete problem

FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b\left(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h\right) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

$$+ (\mathbf{v}_h, \partial_t \widetilde{\mathbf{u}})_\Omega + (\mathcal{L}^* \mathbf{v}_h, \widetilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \widetilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

SGS equations

$$\partial_t \widetilde{\mathbf{u}} + \tau_m^{-1} \widetilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = I \quad \text{(ASGS)}, \qquad \mathcal{P} = P_h^\perp = I - P_h \quad \text{(OSS)}$$

Semidiscrete problem

FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

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$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

SGS equations

$$\partial_t \widetilde{\mathbf{u}} + \tau_m^{-1} \widetilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$\mathcal{P} = I$ (ASGS), $\mathcal{P} = \mathrm{P}_\mathrm{h}^\perp = \mathrm{I} - \mathrm{P}_\mathrm{h}$ (OSS)

$$\mathbf{a} = \mathbf{u}_h + \widetilde{\mathbf{u}}$$

Summary

	Sgs space	Sgs dynamics	Advection
1	ASGS	Static	Linear
2	ASGS	Dynamic	Linear
3	ASGS	Dynamic	Nonlinear
4	OSS	Static	Linear
5	OSS	Dynamic	Linear
6	OSS	Dynamic	Nonlinear

1 It is the most standard method (SUPG for linear elements) up to the choice of the stabilization parameters. Unknown stability properties.

4 Strictly pointwise positive for linear elements (no backscatter).

5 Convergent to weak solutions of NS equations (Badia & Gutierrez 2012).

Energy statements

FE counterpart:

$$B\left(\left(\mathbf{u}_h, p_h\right);\left(\widetilde{\mathbf{u}}, \widetilde{p}\right);\left(\mathbf{u}_h, p_h\right)\right) = L\left(\mathbf{u}_h, p_h\right)$$

SGS counterpart:

$$B\left(\left(\widetilde{\mathbf{u}}, \widetilde{p}\right);\left(\mathbf{u}_h, p_h\right);\left(\widetilde{\mathbf{u}}, \widetilde{p}\right)\right) = L\left(\widetilde{\mathbf{u}}, \widetilde{p}\right)$$

TOTAL:

$$\begin{aligned} & B\left(\left(\mathbf{u}_h, p_h\right);\left(\widetilde{\mathbf{u}}, \widetilde{p}\right);\left(\mathbf{u}_h, p_h\right)\right) \\ +& B\left(\left(\widetilde{\mathbf{u}}, \widetilde{p}\right);\left(\mathbf{u}_h, p_h\right);\left(\widetilde{\mathbf{u}}, \widetilde{p}\right)\right) = L\left(\mathbf{u}_h, p_h\right) + L\left(\widetilde{\mathbf{u}}, \widetilde{p}\right) \end{aligned}$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL:

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \nu\|\nabla\mathbf{u}_h\|^2$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \nu\|\nabla\mathbf{u}_h\|^2$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h) + (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2 + (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}})$$
$$- (\nabla\cdot\mathbf{u}_h,\widetilde{p}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p})$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}})$$
$$- (\nabla\cdot\mathbf{u}_h,\widetilde{p}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle + \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

Energy statements

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$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
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$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL: Static subscales

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}})$$
$$- (\nabla\cdot\mathbf{u}_h,\widetilde{p}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle + \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL: **Dynamic subscales - ASGS**

$$\frac{1}{2}d_t\|\mathbf{u}_h+\widetilde{\mathbf{u}}\|^2 \qquad + \nu\|\nabla\mathbf{u}_h\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$-2(\nu\Delta\mathbf{u}_h,\widetilde{\mathbf{u}})$$
$$= \langle\mathbf{f},\mathbf{u}_h\rangle + \langle\mathbf{f},\widetilde{\mathbf{u}}\rangle$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

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$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL: **Dynamic subscales - OSS**

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$- 2\,(\nu\Delta\mathbf{u}_h,\widetilde{\mathbf{u}})$$
$$= \langle\mathbf{f},\mathbf{u}_h\rangle + \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

Numerical experiments

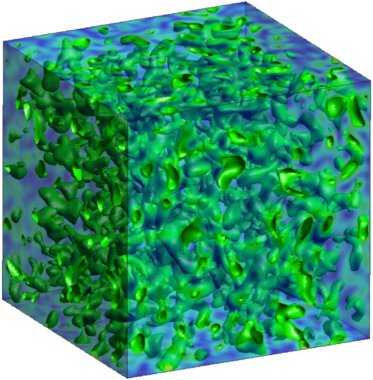
Three different turbulent benchmarks:

- Decaying of Homogeneous Isotropic Turbulence (DHIT).
- Taylor-Green Vortex (TGV) flow.
- Turbulent Channel Flow (TCF).

DHIT Decay of Homogeneous Isotropic Turbulence

Problem setting:

- Prescribed initial energy spectra corresponding to $Re_\lambda = 952$.
- Setting defined in AGARD database (Mansour & Wray 1993).
- A (very simple) time step adaptation technique is used.
- Different mesh discretizations (Q_1/Q_1 and Q_2/Q_2).



DHIT Decay of Homogeneous Isotropic Turbulence

Energy espectra (models):

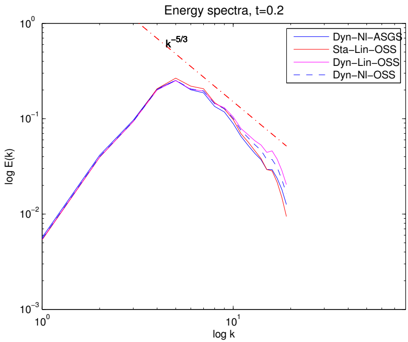


Figure : $32^3 - Q1, \; t = 0.2s$

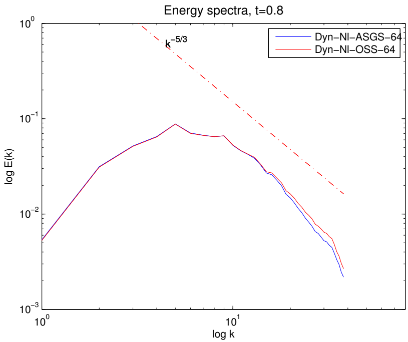
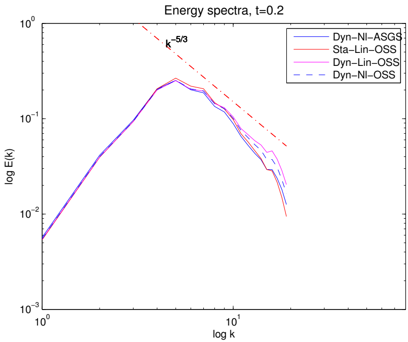


Figure : $64^3 - Q1, \; t = 0.8s$

DHIT Decay of Homogeneous Isotropic Turbulence

Energy espectra (models):



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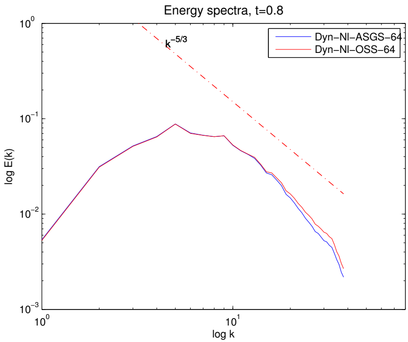

Figure : $64^3 - Q1,\ t = 0.8s$

- Small differences between methods (physical sense).

DHIT Decay of Homogeneous Isotropic Turbulence

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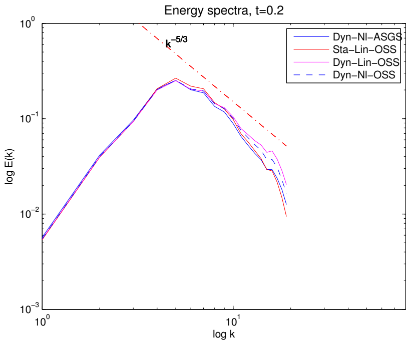

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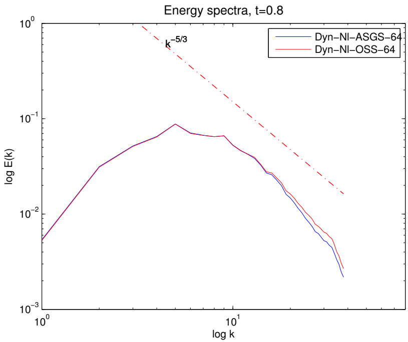
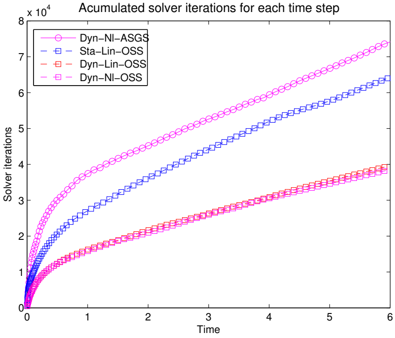


Figure : $64^3 - Q1,\ t = 0.8s$

- Small differences between methods (physical sense).
- Even more similar when we refine the mesh.

DHIT Decay of Homogeneous Isotropic Turbulence

Computational cost (models):



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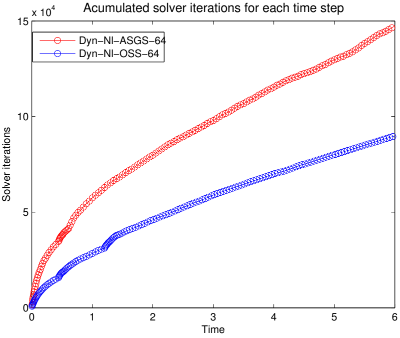

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DHIT Decay of Homogeneous Isotropic Turbulence

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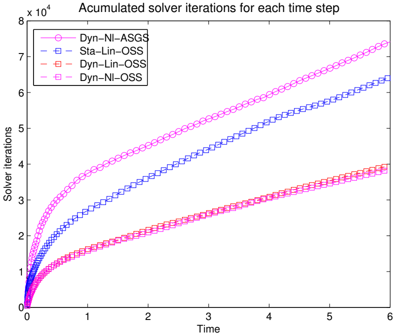

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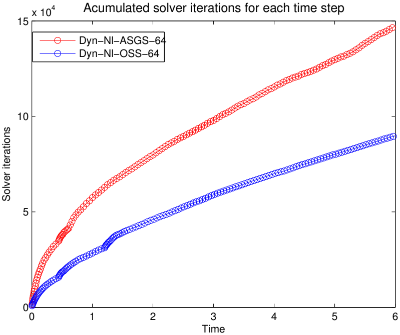

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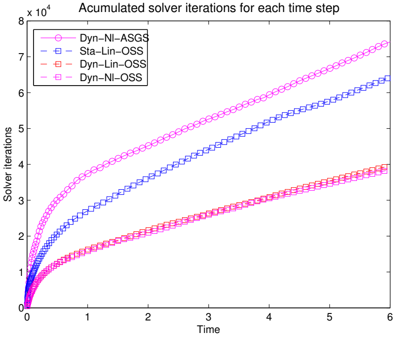

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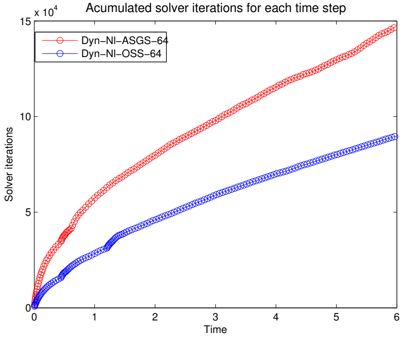
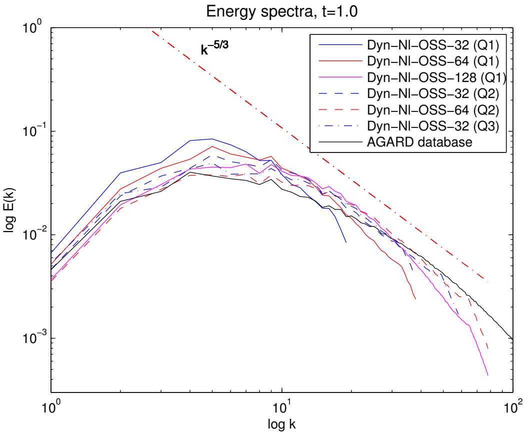


Figure : $64^3 - Q1$

- Big differences between methods (computational sense).
- Dynamic versions of OSS method are the most efficient.

DHIT Decay of Homogeneous Isotropic Turbulence

Energy espectra (refinement):



- Results become closer to the DNS when we refine the mesh.

TGV Taylor-Green Vortex flow

Problem setting:

- Prescribed initial condition.
- $Re = 1600$.
- Different mesh discretizations (Q_1/Q_1 and Q_2/Q_2).

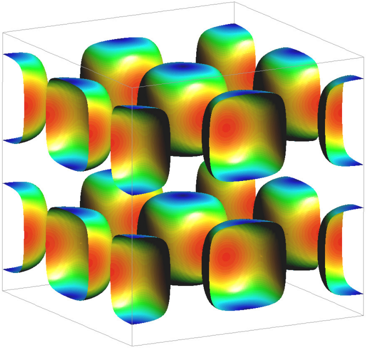


Figure : Initial vorticity isosurface $|\omega| = 1$

TGV Taylor-Green Vortex flow

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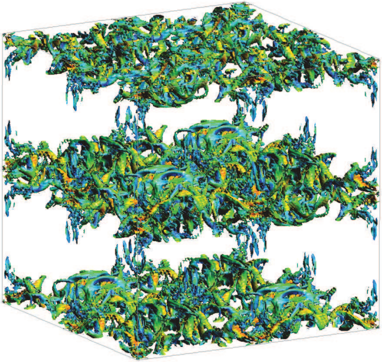

Figure : Vorticity isosurfaces $|\omega| = 9.0$

TGV Taylor-Green Vortex flow

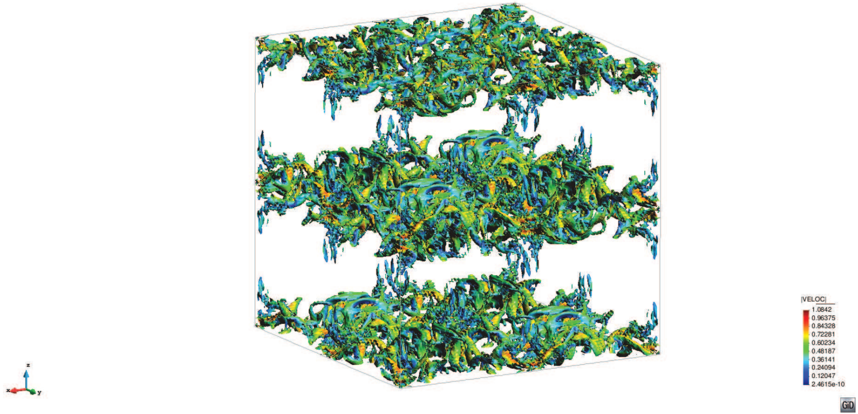

Figure : Velocity isosurface

TGV Taylor-Green Vortex flow

Energy dissipation rate (refinement):

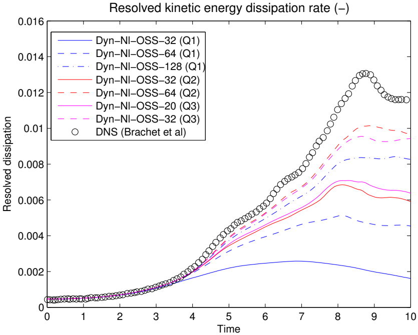

Figure : Resolved scales

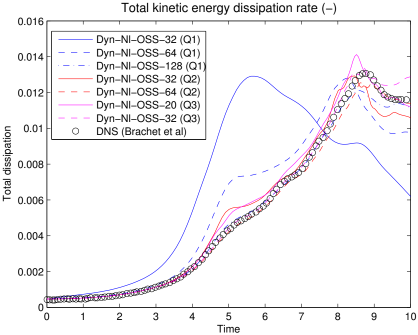

Figure : Total

TGV Taylor-Green Vortex flow

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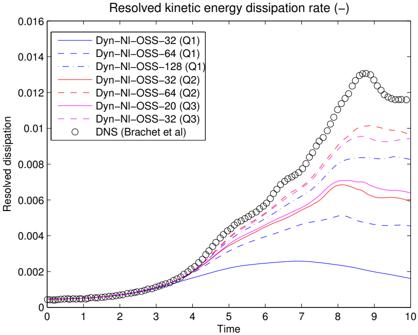

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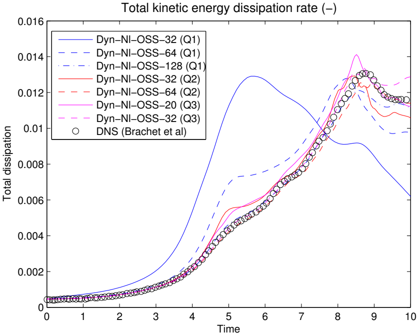

Figure : Total

- Good agreement with the DNS taking account the subscales.

TGV Taylor-Green Vortex flow

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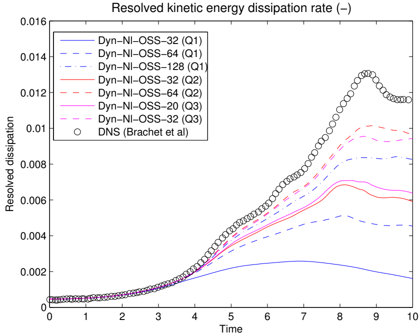

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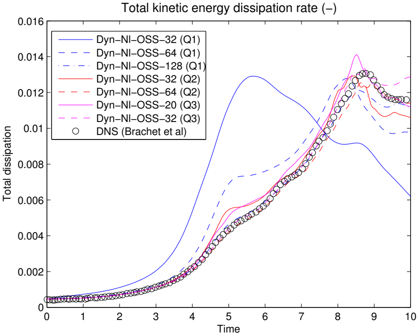

Figure : Total

- Good agreement with the DNS taking account the subscales.
- More accurate results increasing the order of approximation.

TGV Taylor-Green Vortex flow

- All results until now are compared against **DNS**.
- Are our methods comparable with **LES** models?

TGV Taylor-Green Vortex flow

Energy dissipation rate (against LES model):

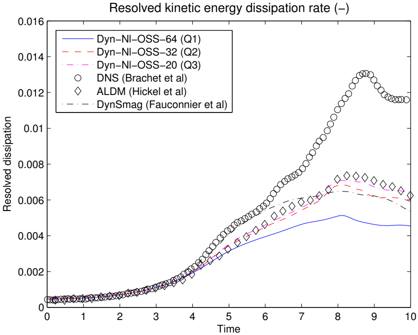

Figure : Resolved scales

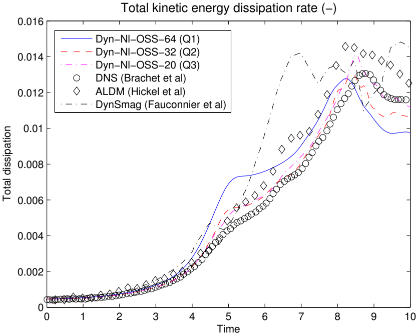

Figure : Total

TGV Taylor-Green Vortex flow

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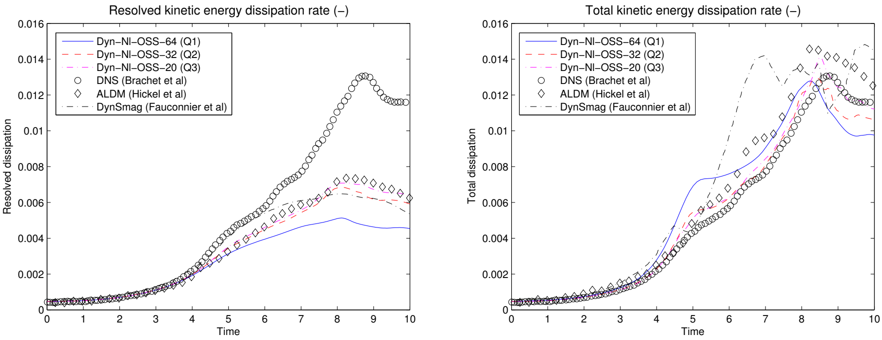

Figure : Resolved scales

Figure : Total

- Good agreement with the LES models on resolved scales.

TGV Taylor-Green Vortex flow

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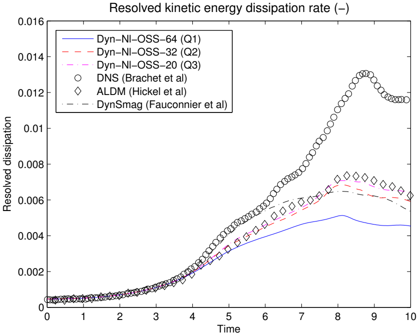


Figure : Resolved scales

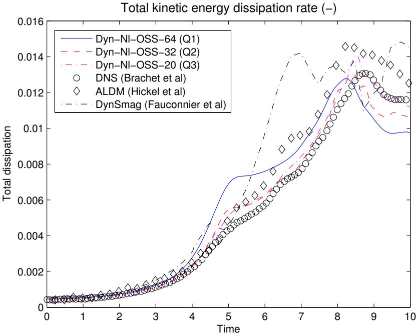


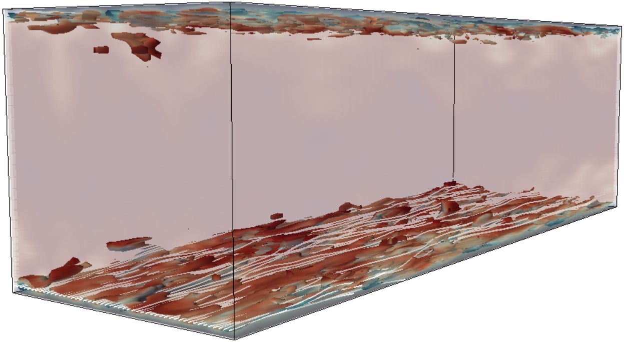
Figure : Total

- Good agreement with the LES models on resolved scales.
- Better results than LES models adding subscales counterpart.

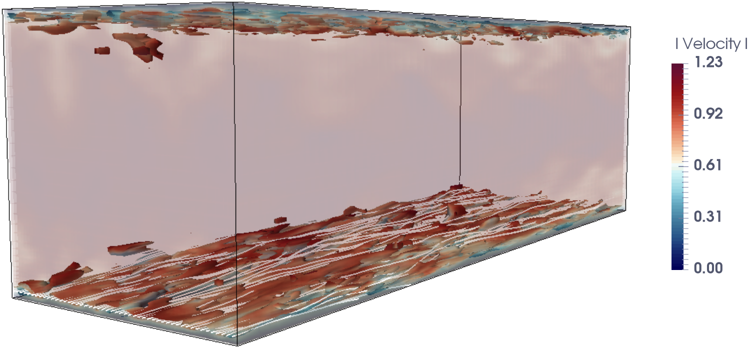
TCF Turbulent Channel Flow

Problem setting:

- Wall bounded flow.
- $Re_\tau = 180$ and $Re_\tau = 395$.
- Mesh resolution: $32^3 - Q1$.

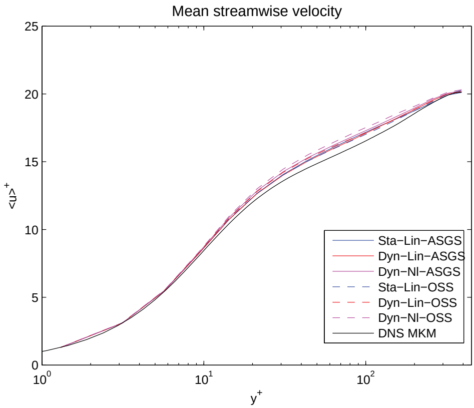


TCF Turbulent Channel Flow



TCF Turbulent Channel Flow

Mean streamwise velocity (models):



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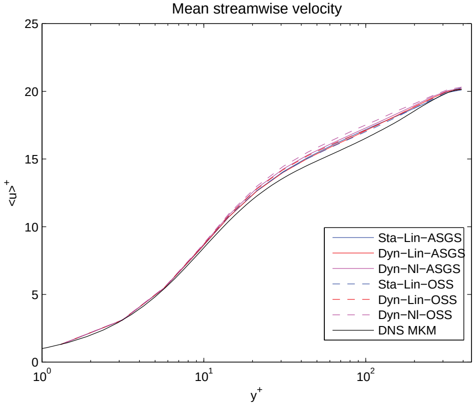

- Small differences between methods.

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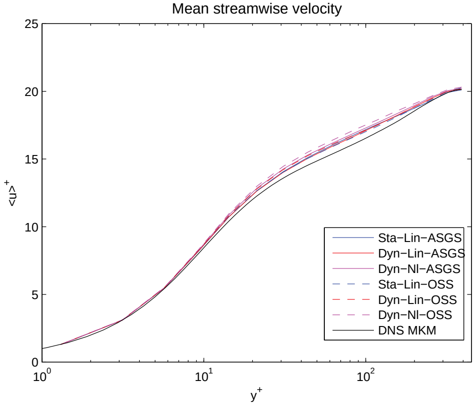

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- Very accurate results compared against the DNS.

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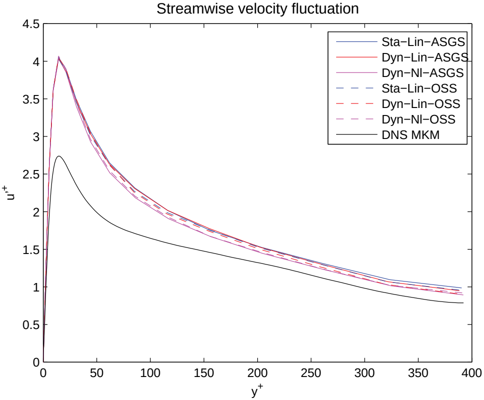

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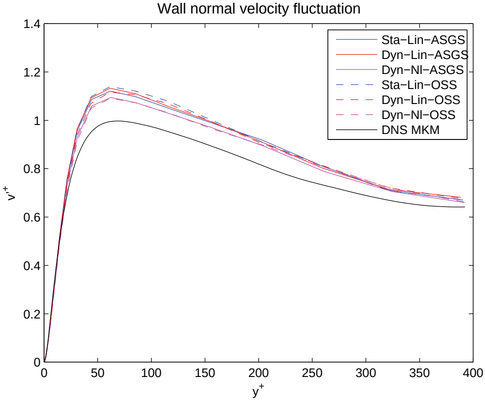

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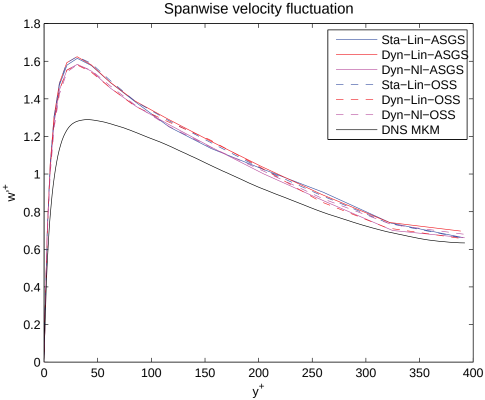

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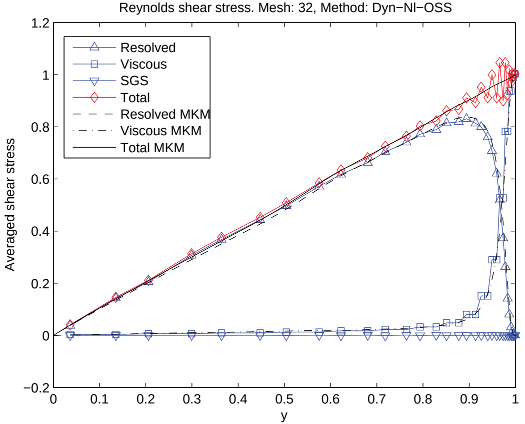
TCF Turbulent Channel Flow

Reynolds shear stress (models):



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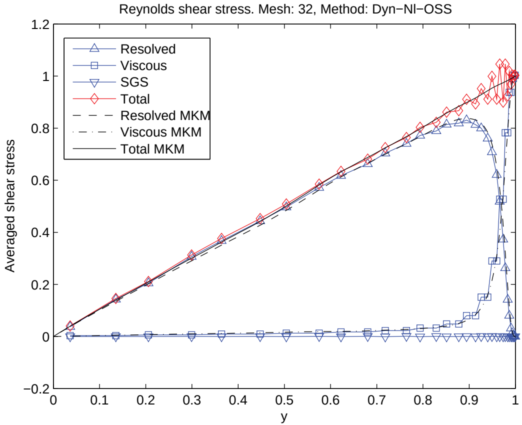

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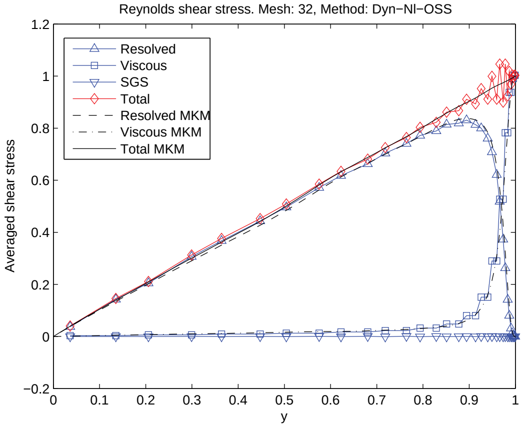

- Almost identical to the DNS.
- SGS counterpart does not contribute to the Reynolds shear stress.

RB-VMS Conclusions

- VMS formulations of NS equations can be used for the numerical simulation of turbulent flows.

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- Our particular VMS modelling ingredients
  - Dynamic subscales
  - Nonlinear subscales
  - Orthgonal subscales

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- Among them dynamic and orthogonal subscales (linear or nonlinear) are the most effective.
- The skewsymmetric formulation is important to keep stability.

RB-VMS Limitations

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RB-VMS Limitations

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  - Poorly matrix conditioning ⇒ High number of solver iterations.
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- **OSS:**
  - Better matrix conditioning ⇒ Low number of solver iterations.
  - Explicit projection treatment ⇒ High number of nonlinear iterations.
- **Desired:**
  - OSS with implicit projections.

1. Motivation

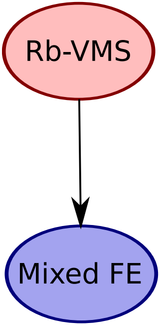
2. Residual-based VMS

3. Mixed FE VMS
   - Formulation
   - Block-preconditioning
   - Numerical experiments
   - Conclusions

4. Segregated Runge-Kutta

5. Segregated VMS

6. Conclusions



Motivation

Step by step...

- Residual-based VMS as LES models.
- Mixed FE formulations LES.
- High-order FE methods.
- High-order time integration schemes.
- Adaptive time stepping techniques.
- Velocity-pressure segregation.
- Scalable solvers.
- Application.

Semidiscrete problem

FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$
$$+ (\mathbf{v}_h, \partial_t \widetilde{\mathbf{u}})_\Omega + (\mathcal{L}^* \mathbf{v}_h, \widetilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \widetilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$
$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

SGS equations

$$\partial_t \widetilde{\mathbf{u}} + \tau_m^{-1} \widetilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$
$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = I \quad \text{(ASGS)}, \qquad \mathcal{P} = \mathrm{P}_\mathrm{h}^\perp = \mathrm{I} - \mathrm{P}_\mathrm{h} \quad \text{(OSS)}$$

$$\mathbf{a} = \mathbf{u}_h + \widetilde{\mathbf{u}}$$

Semidiscrete problem

FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$
$$+ (\mathcal{L}^* \mathbf{v}_h, \widetilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \widetilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

SGS equations

$$\tau_m^{-1} \widetilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$
$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = \mathrm{P}_\mathrm{h}^\perp = \mathrm{I} - \mathrm{P}_\mathrm{h} \quad (\mathrm{OSS})$$

$$\mathbf{a} = \mathbf{u}_h$$

Term-by-term OSS

FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

$$+ (\mathcal{L}^* \mathbf{v}_h, \widetilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \widetilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

Term-by-term OSS

Term-by-term OSS (Codina 2008)

$$
\begin{aligned}
&(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b\left(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h\right) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega \\
&+ \left(\tau_m \mathbf{a} \cdot \nabla \mathbf{v}_h, \mathcal{P}_h^\perp(\mathbf{a} \cdot \nabla \mathbf{u}_h)\right)_{\Omega^h} \\
&+ \left(\tau_c \nabla \cdot \mathbf{v}_h, \mathcal{P}_h^\perp(\nabla \cdot \mathbf{u}_h)\right)_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega
\end{aligned}
$$

$$
(q_h, \nabla \cdot \mathbf{u}_h)_\Omega + \left(\tau_m \nabla q_h, \mathcal{P}_h^\perp(\nabla p_h)\right)_{\Omega^h} = 0
$$

Term-by-term OSS

Term-by-term OSS (Codina 2008)

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b\left(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h\right) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

$$+ (\tau_m \mathbf{a} \cdot \nabla \mathbf{v}_h, \mathbf{a} \cdot \nabla \mathbf{u}_h)_{\Omega^h} - (\tau_m \mathbf{a} \cdot \nabla \mathbf{v}_h, \boldsymbol{\eta}_h)_{\Omega^h}$$

$$+ (\tau_c \nabla \cdot \mathbf{v}_h, \nabla \cdot \mathbf{u}_h)_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega + (\tau_m \nabla q_h, \nabla p_h)_{\Omega^h} - (\tau_m \nabla q_h, \boldsymbol{\xi}_h)_{\Omega^h} = 0$$

$$\boldsymbol{\eta}_h := \mathcal{P}_h(\mathbf{a} \cdot \nabla \mathbf{u}_h)$$

$$\boldsymbol{\xi}_h := \mathcal{P}_h(\nabla p_h)$$

$$\mathcal{P}_h(\nabla \cdot \mathbf{u}_h) \approx 0$$

Matricial form

- ASGS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G \\ D & L_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \end{bmatrix},$$

Matricial form

- ASGS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G \\ D & L_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \end{bmatrix},$$

- Term-by-term OSS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

Matricial form

- ASGS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G \\ D & L_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \end{bmatrix},$$

- Term-by-term OSS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Term-by-term OSS with Inf-sup stable elements (mixed FE):

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_\tau \\ D & 0 & 0 \\ -B_\tau^T & 0 & M_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

Matricial form

- ASGS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K+C+A_\tau & G \\ D & L_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \end{bmatrix},$$

- Term-by-term OSS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K+C+A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B^T_{\eta,\tau} & 0 & M_{\eta,\tau} & 0 \\ 0 & -B^T_{\xi,\tau} & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Term-by-term OSS with Inf-sup stable elements (mixed FE):

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K+C+A_\tau & G & B_\tau \\ D & 0 & 0 \\ -B^T_\tau & 0 & M_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

Index-2 DAE!!!

Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

Block-recursive preconditioning

General case:

$$\left[\begin{array}{c} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{array}\right] + \left[\begin{array}{cc:cc} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hdashline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \left[\begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \hdashline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_u \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{array}\right],$$

Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \left[\begin{array}{cc:cc} K+C+A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hdashline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \hdashline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Outer block matrix:

$$\tilde{A} = \left[\begin{array}{cc:cc} M_{\eta,\tau} & 0 & -B_{\eta,\tau}^T & 0 \\ 0 & M_{\xi,\tau} & 0 & -B_{\xi,\tau}^T \\ \hdashline B_{\eta,\tau} & 0 & K+C+A_\tau & G \\ 0 & B_{\xi,\tau} & D & L_\tau \end{array}\right]$$

Block-recursive preconditioning

General case:

$$\left[\begin{array}{c} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{array}\right] + \left[\begin{array}{cc:cc} K+C+A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hdashline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \left[\begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \hdashline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_u \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{array}\right],$$

- Outer block matrix:

Block-recursive preconditioning

General case:

$$
\left[\begin{array}{c} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right] + \left[\begin{array}{cc|cc} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \left[\begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \hline \mathbf{\Upsilon} \\ \mathbf{\Xi} \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_u \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right],
$$

- Outer block matrix:

$$
\tilde{A} = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ \tilde{B}_\tau & \tilde{K}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}_\tau \tilde{M}_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix},
$$

Block-recursive preconditioning

General case:

$$\left[\begin{array}{c} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right] + \left[\begin{array}{cc|cc} K+C+A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \left[\begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \hline \mathbf{\Upsilon} \\ \mathbf{\Xi} \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_u \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right],$$

- Outer block matrix:

$$\tilde{A} = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ \tilde{B}_\tau & \tilde{K}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}_\tau \tilde{M}_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix},$$

$$\tilde{S} = \tilde{K}_\tau + \tilde{B}_\tau \tilde{M}_\tau^{-1} \tilde{B}_\tau^T,$$

Block-recursive preconditioning

General case:

$$\left[\begin{array}{c} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right] + \left[\begin{array}{cc|cc} K+C+A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \left[\begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \hline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_u \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right],$$

- Outer block matrix:

$$\tilde{A} = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ \tilde{B}_\tau & \tilde{K}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}_\tau \tilde{M}_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix},$$

$$\tilde{S} = \tilde{K}_\tau + \tilde{B}_\tau \tilde{M}_\tau^{-1} \tilde{B}_\tau^T,$$

$$P_U(\tilde{A}) = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{M}_\tau^{-1} & \tilde{M}_\tau^{-1} \tilde{B}_\tau^T \tilde{S}^{-1} \\ 0 & \tilde{S}^{-1} \end{bmatrix},$$

Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \left[\begin{array}{cc|cc} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \hline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Outer block matrix:

$$\tilde{A} = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ \tilde{B}_\tau & \tilde{K}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}_\tau \tilde{M}_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix},$$

$$\tilde{S} = \tilde{K}_\tau + \tilde{B}_\tau \tilde{M}_\tau^{-1} \tilde{B}_\tau^T,$$

$$P_U(\tilde{A}) = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{M}_\tau^{-1} & \tilde{M}_\tau^{-1} \tilde{B}_\tau^T \tilde{S}^{-1} \\ 0 & \tilde{S}^{-1} \end{bmatrix},$$

$$\tilde{S}^{-1} \approx \tilde{K}_\tau^{-1}.$$

Block-recursive preconditioning

General case:

$$
\left[\begin{array}{c} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{array}\right] + \left[\begin{array}{cc:cc} K+C+A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hdashline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \left[\begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \hdashline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_u \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{array}\right],
$$

- Inner block matrix:

$$
\tilde{K}_\tau = \begin{bmatrix} K_\tau & G \\ D & L_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ DK_\tau{}^{-1} & I \end{bmatrix} \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix},
$$

Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \left[\begin{array}{cc:cc} K+C+A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hdashline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \hdashline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Inner block matrix:

$$\tilde{K}_\tau = \begin{bmatrix} K_\tau & G \\ D & L_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ DK_\tau{}^{-1} & I \end{bmatrix} \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix},$$

$$S = L_\tau - DK_\tau G,$$

Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \left[\begin{array}{cc:cc} K+C+A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hdashline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Inner block matrix:

$$\tilde{K}_\tau = \begin{bmatrix} K_\tau & G \\ D & L_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ DK_\tau{}^{-1} & I \end{bmatrix} \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix},$$

$$S = L_\tau - DK_\tau G,$$

$$P_U(\tilde{K}_\tau) = \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix}^{-1} = \begin{bmatrix} K_\tau^{-1} & -K_\tau^{-1}GS^{-1} \\ 0 & S^{-1} \end{bmatrix},$$

Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \left[\begin{array}{cc:cc} K+C+A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hdashline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \hdashline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Inner block matrix:

$$\tilde{K}_\tau = \begin{bmatrix} K_\tau & G \\ D & L_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ DK_\tau{}^{-1} & I \end{bmatrix} \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix},$$

$$S = L_\tau - DK_\tau G,$$

$$P_U(\tilde{K}_\tau) = \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix}^{-1} = \begin{bmatrix} K_\tau^{-1} & -K_\tau^{-1}GS^{-1} \\ 0 & S^{-1} \end{bmatrix},$$

$$\tilde{S}^{-1} \approx L_p^{-1} \quad (\delta t L_p^{-1}).$$

Numerical experiments

Manufactured analytical solution:

- Colliding flow.

Two different turbulent benchmarks:

- Taylor-Green Vortex (TGV) flow.
- Turbulent Channel Flow (TCF).

Colliding flow

Problem setting:

- Analytical solution.
- $Re = 25$.
- Mesh refinement: 4^3 to 64^3 $Q1/Q1$ elements (ASGS and OSS) or 2^3 to $32^3 Q2/Q1$ elements (OSS-ISS)

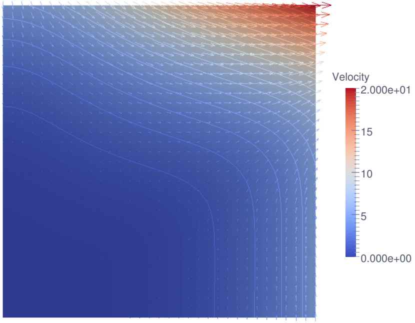
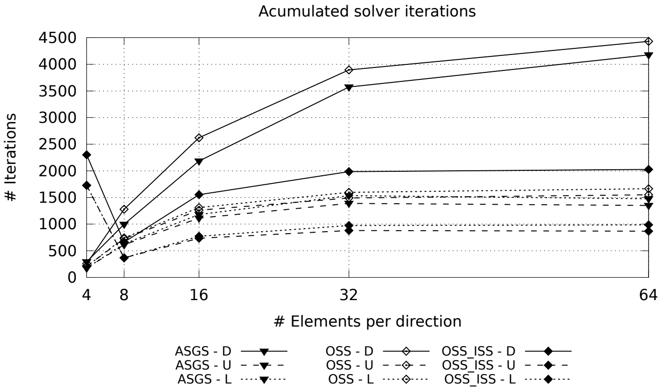


Figure : Velocity field

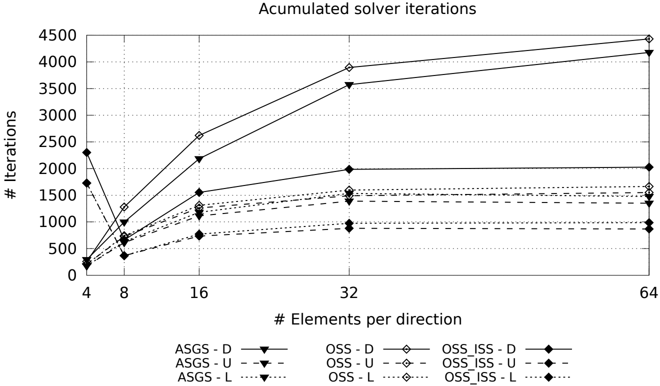
Colliding flow

Acumulated solver iterations: (using $P_U(\widetilde{A})$)



Colliding flow

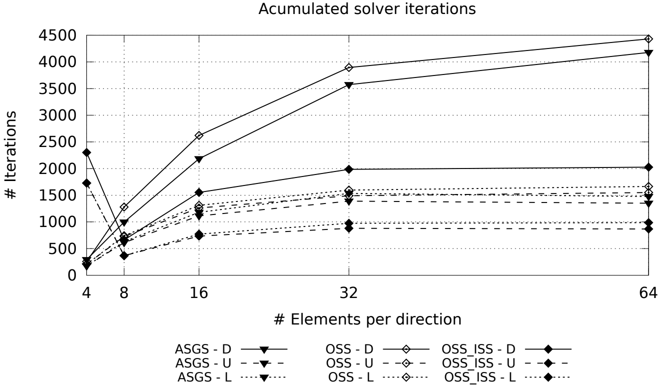
Acumulated solver iterations: (using $P_U(\widetilde{A})$)



- $P_U(\widetilde{K}_\tau)$ and $P_L(\widetilde{K}_\tau)$ scalable block-preconditioners for all methods.

Colliding flow

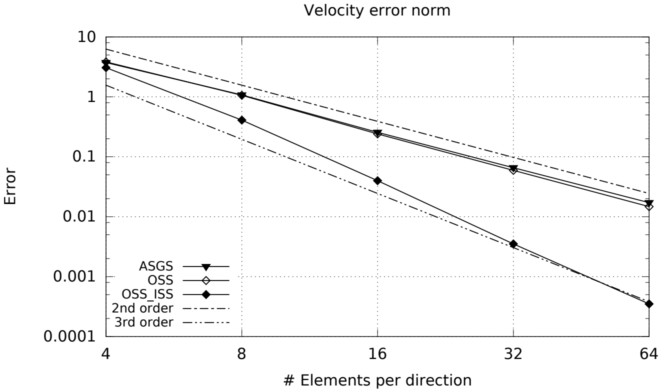
Acumulated solver iterations: (using $P_U(\widetilde{A})$)



- $P_U(\widetilde{K}_\tau)$ and $P_L(\widetilde{K}_\tau)$ scalable block-preconditioners for all methods.
- Less solver iterations for the OSS-ISS method with the same velocity DOFs.

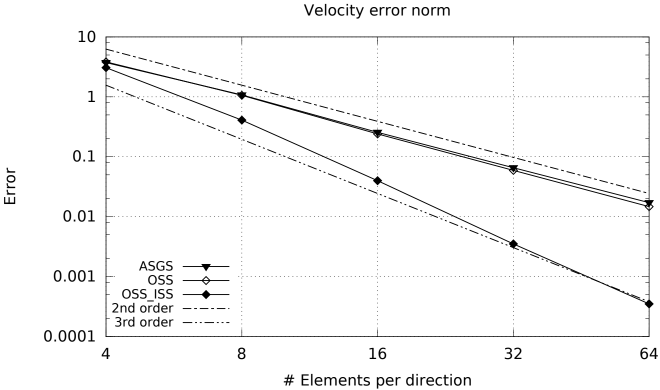
Colliding flow

Accuracy: Velocity error



Colliding flow

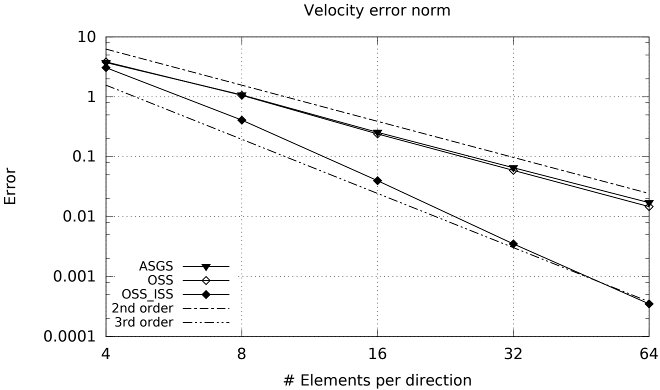
Accuracy: Velocity error



- 2nd order convergence rate for ASGS and OSS methods.

Colliding flow

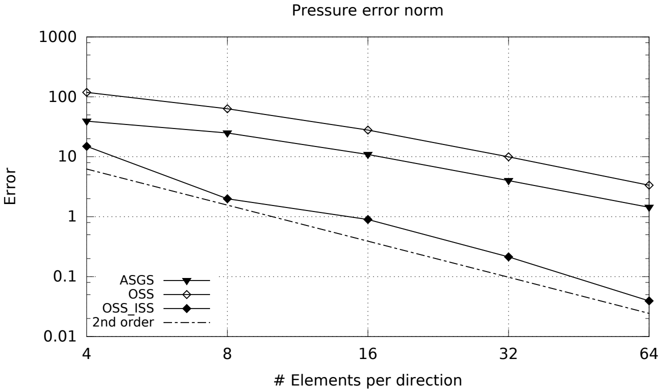
Accuracy: Pressure error



- 2nd order convergence rate for ASGS and OSS methods.
- 3rd order convergence rate for OSS-ISS method.

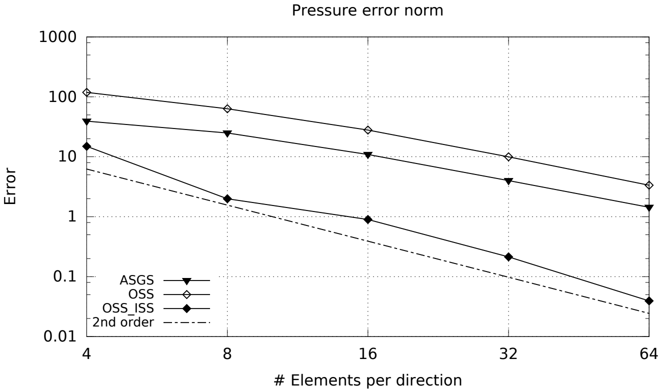
Colliding flow

Accuracy: Pressure error



Colliding flow

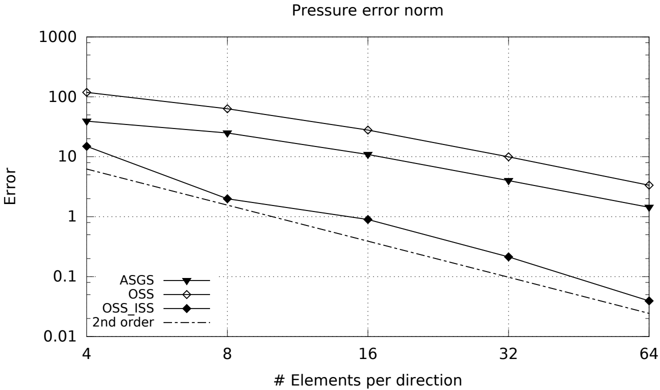
Accuracy:



- 2nd order convergence rate for all methods.

Colliding flow

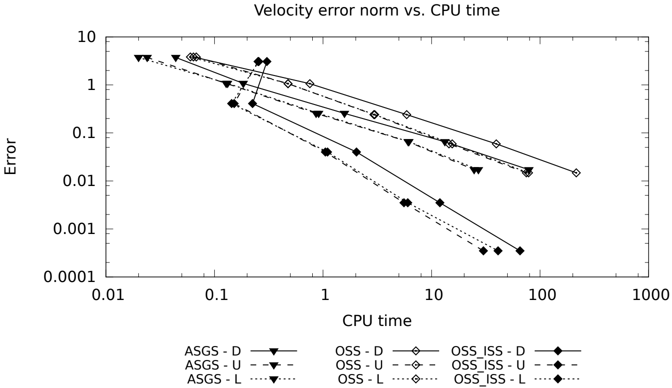
Accuracy:



- 2nd order convergence rate for all methods.
- Best accuracy for OSS-ISS method.

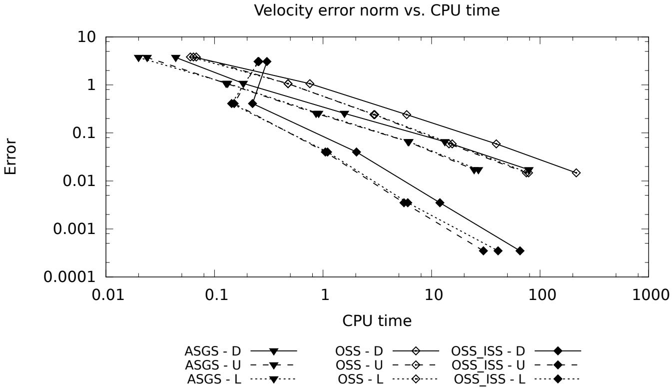
Colliding flow

Efficiency: Velocity



Colliding flow

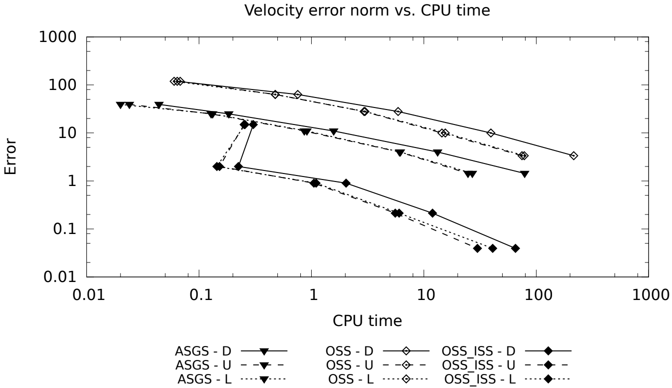
Efficiency: Velocity



- OSS-ISS the most efficient method.

Colliding flow

Efficiency: Pressure



- Also for pressures.

TGV Taylor-Green Vortex flow

Energy dissipation rate (different methods):

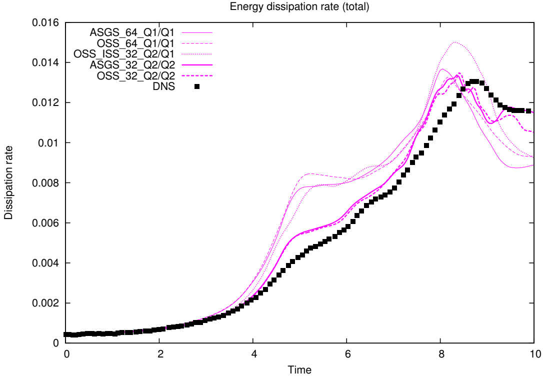


Figure : Total energy dissipation rate

TGV Taylor-Green Vortex flow

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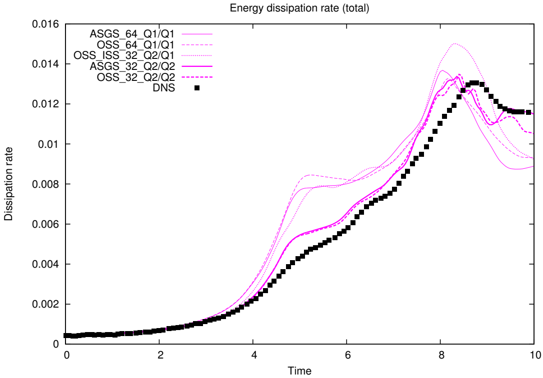

Figure : Total energy dissipation rate

- Good agreement with the DNS (coarse mesh).

TGV Taylor-Green Vortex flow

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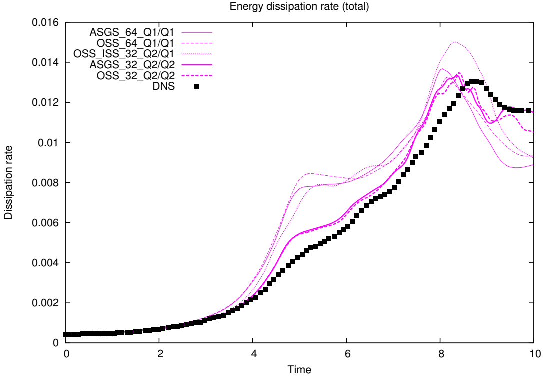

Figure : Total energy dissipation rate

- Good agreement with the DNS (coarse mesh).
- More accurate results with equal-order elements.

TGV Taylor-Green Vortex flow

Energy dissipation rate (refinement OSS-ISS):

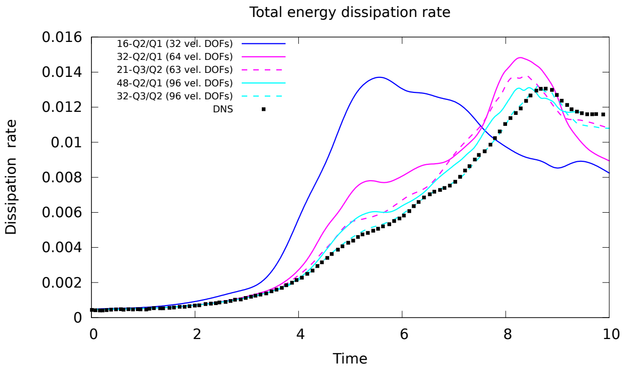

Figure : Total energy dissipation rate

TGV Taylor-Green Vortex flow

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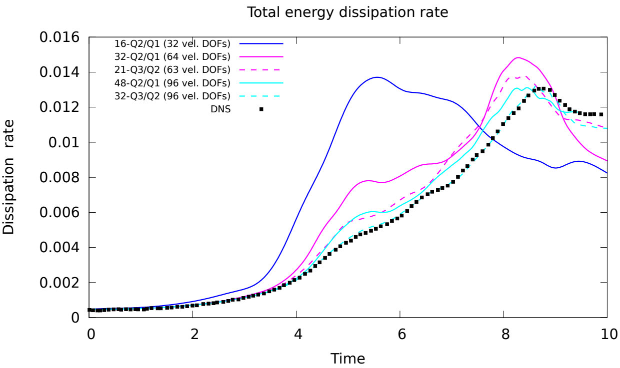

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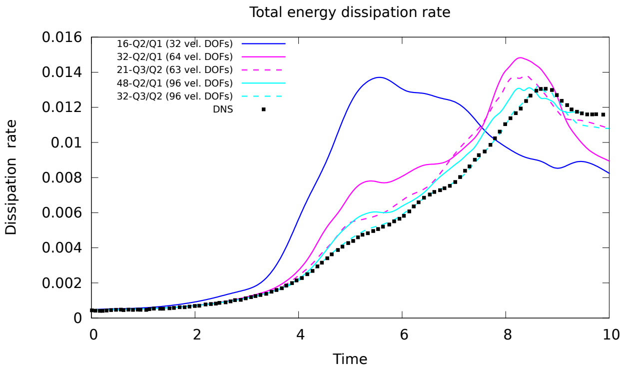
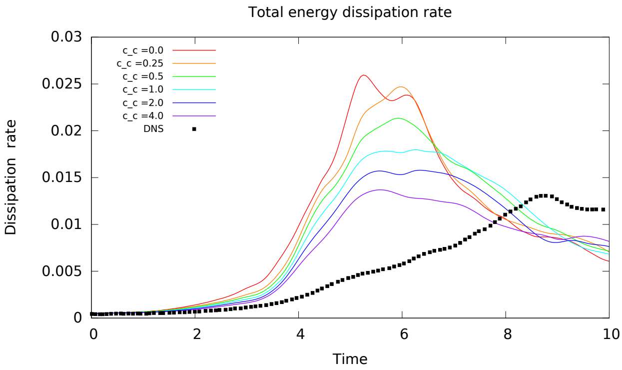


Figure : Total energy dissipation rate

- Good agreement with the DNS.
- 32^3 $Q3/Q2$ elements mesh on top of DNS.

TGV Taylor-Green Vortex flow

Effect of the grad-div term $(\tau_c \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})$**:** (coarse mesh)



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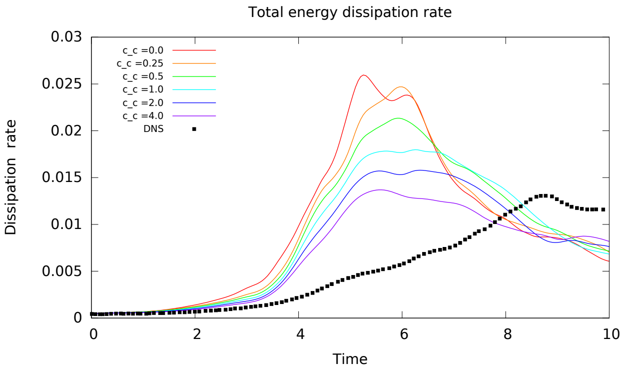

Figure : Total energy dissipation rate.

- Bad results when $c_c \to 0$.

TGV Taylor-Green Vortex flow

Effect of the grad-div term $(\tau_c \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})$: (coarse mesh)

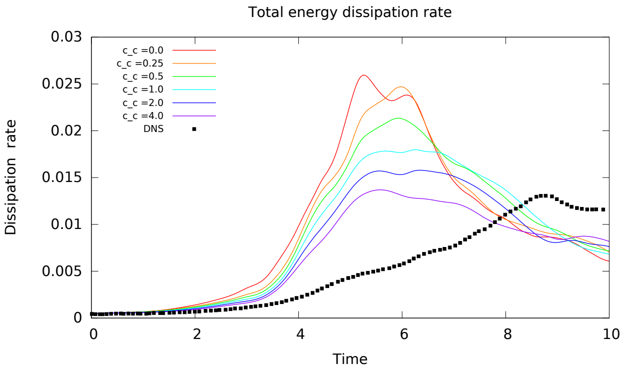

Figure : Total energy dissipation rate.

- Bad results when $c_c \to 0$.
- Best option $c_c = 4.0$.

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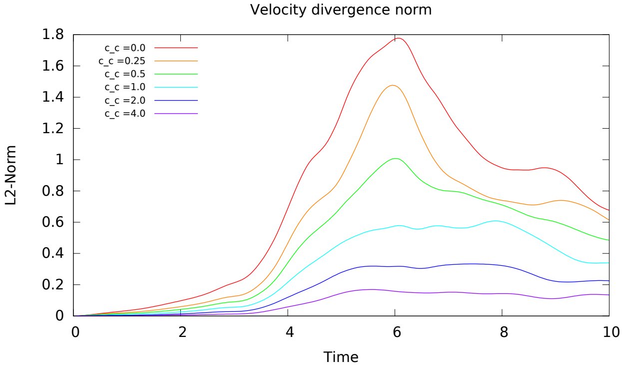

Figure : $\|\nabla \cdot \mathbf{u}\|$.

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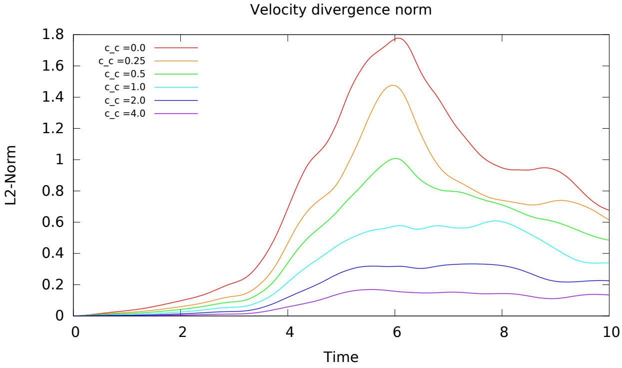

Figure : $\|\nabla \cdot \mathbf{u}\|$.

- Bad results when $c_c \to 0$.
- Incompressibility constraint not satisfied.

TGV Taylor-Green Vortex flow

Effect of the grad-div term $(\tau_c \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})$: (coarse mesh)

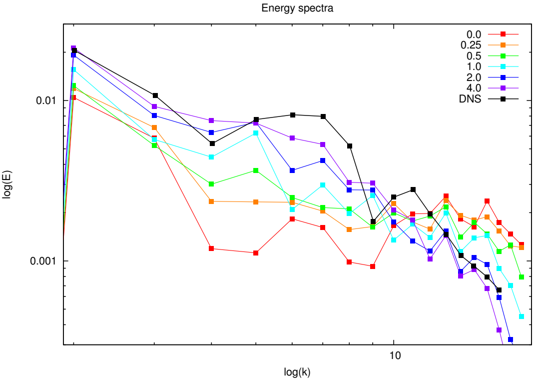

Figure : Energy spectra.

- Bad results when $c_c \to 0$.

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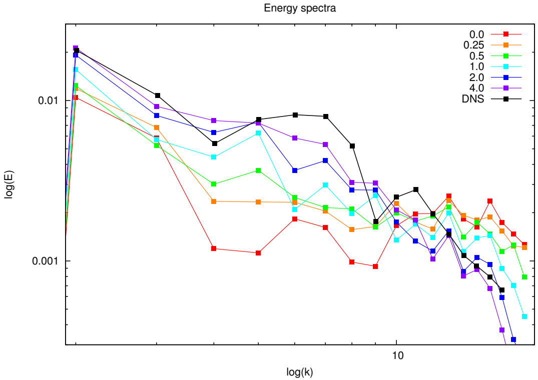

Figure : Energy spectra.

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- Overdissipation on the large scales when $c_c \to 0$.

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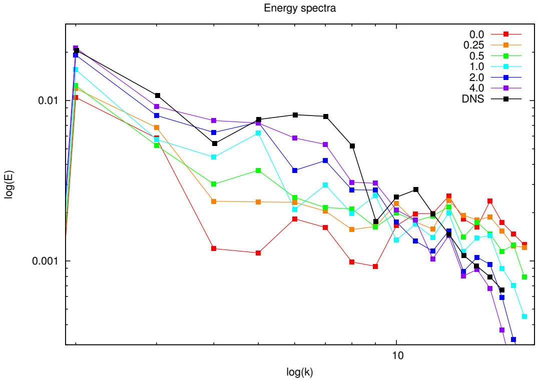

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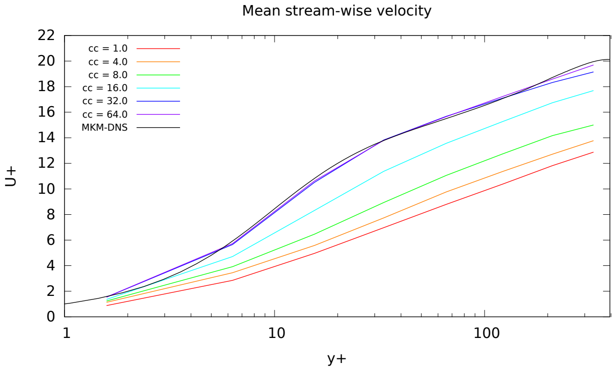
TCF Turbulent Channel Flow at $Re_\tau = 395$

Mean streamwise velocity (models):



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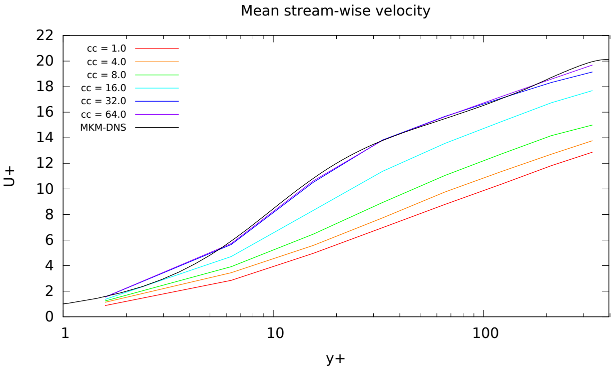

- Same behaviour observed for TGV test.

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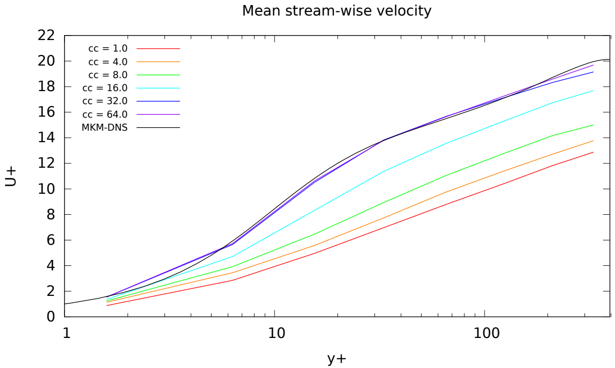

- Same behaviour observed for TGV test.
- Best option $c_c = 32.0$.

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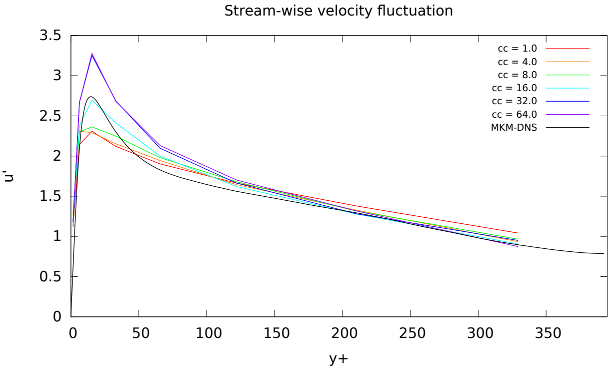

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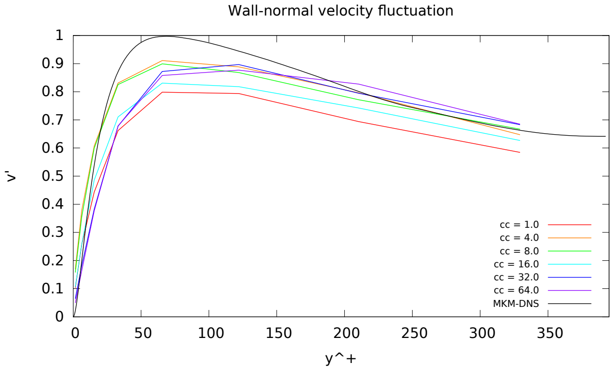

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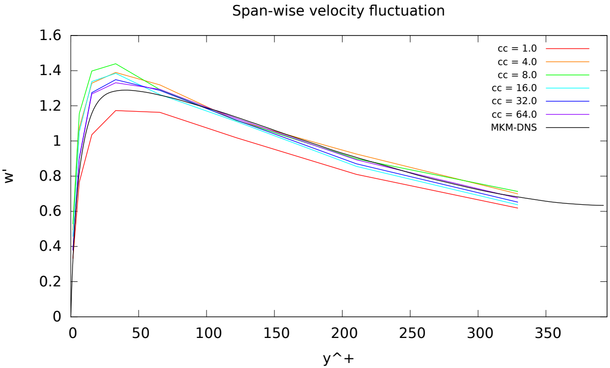

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Mixed FE VMS Conclusions

- Scalable recursive block-preconditioners.

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Mixed FE VMS Conclusions

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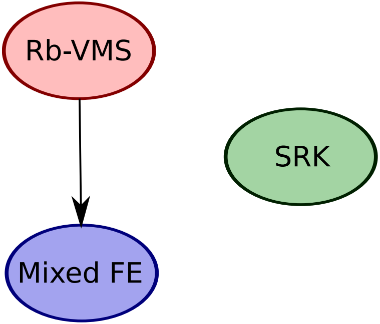
- Sligthly better results for equal-order interpolations.
- Mixed FE OSS the most efficient method.
- Mixed FE OSS keeps the index-2 DAE nature in time of the problem.

Mixed FE VMS Limitations

- Strong dependency on the grad-div stabilization term.



Rb-VMS
SRK
Mixed FE

Motivation

Step by step...

- Residual-based VMS as LES models.
- Mixed FE formulations LES.
- High-order FE methods.
- High-order time integration schemes.
- Adaptive time stepping techniques.
- Velocity-pressure segregation.
- Scalable solvers.
- Application.

Incomp. Navier Stokes equations

Find $\mathbf{u}$ and p defined in Ω

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} &= \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

with appropriate boundary conditions on Γ.

Incomp. Navier Stokes equations

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- Index-2 DAE system (time integration with care)

Incomp. Navier Stokes equations

Find $\mathbf{U}$ and $\mathbf{P}$

$$\begin{aligned} \dot{\mathbf{U}} + M^{-1}(K + C(\mathbf{U}))\mathbf{U} + M^{-1}G\mathbf{P} &= \mathbf{F}, \\ D\mathbf{U} &= \mathbb{G}, \end{aligned}$$

with appropriate boundary conditions on Γ.

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Incomp. Navier Stokes equations

Find **U** and **P**

$$
\begin{aligned}
\dot{\mathbf{U}} + M^{-1}(K + C(\mathbf{U}))\mathbf{U} + M^{-1}G\mathbf{P} &= \mathbf{F}, \\
-DM^{-1}G\mathbf{P} &= DM^{-1}(K + C(\mathbf{U})\mathbf{U} - \mathbf{F}) + \dot{\mathbb{G}},
\end{aligned}
$$

with appropriate boundary conditions on Γ.

- Laminar case (no convection stabilization required).
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Velocity-pressure splitting

Existing methods: Fractional step / pressure segregation methods

$$\dot{\mathbf{U}} + M^{-1}(K + C(\mathbf{U}))\mathbf{U} + M^{-1}G\mathbf{P}^{*} = \mathbf{F}, \qquad \text{in } (t^n, t^{n+1}],$$

$$-DM^{-1}G\mathbf{P} = DM^{-1}(K + C(\mathbf{U})\mathbf{U} - \mathbf{F}) + \mathbb{G}, \qquad \text{in } t^{n+1},$$

Some properties:

- P^{*} extrapolation from previous time steps
- Time-marching scheme for the momentum eq'on (ODE system!)
- Velocity and pressure computations are segregated
- A must for computational efficiency (coercive blocks)
- 2nd order splitting error (>2 unstable)

Velocity-pressure splitting

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High-order time integration:

- Fractional step techniques + (e.g.) Runge-Kutta for momentum equation: [Nikitin'06; Knikker'09; Kampanis et al'06...]
- Introduce segregation error (2nd order)

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High-order time integration:

- Fractional step techniques + (e.g.) Runge-Kutta for momentum equation: [Nikitin'06; Knikker'09; Kampanis et al'06...]
- Introduce segregation error (2nd order)

Drawbacks:

- Segregation error can jeopardize high order time integrators
- Effect of segregation on pressure error unclear (not reported in literature)

Velocity-pressure splitting

Existing methods: Half-explicit RK methods (HERK)

$$\frac{1}{\delta t}\mathbf{U}_i = \frac{1}{\delta t}\mathbf{U}_n + \sum_{j=1}^{i-1} \widehat{a}_{ij} M^{-1}(KU_j + C(U_j)U_j + GP_j),$$

$$D\mathbf{U}_i = \mathbb{G}(t_i),$$

Velocity-pressure splitting

Existing methods: Half-explicit RK methods (HERK)

$$\frac{1}{\delta t}\mathbf{U}_i = \frac{1}{\delta t}\mathbf{U}_n - \sum_{j=1}^{i-1} \widehat{a}_{ij} M^{-1}(KU_j + C(U_j)U_j + GP_j),$$

$$D\mathbf{U}_n - \delta t \sum_{j=1}^{i-1} \widehat{a}_{ij} DM^{-1}(KU_j + C(U_j)U_j + GP_j) = \mathbb{G}(t_i).$$

High-order time integration:

- Recently applied to the Navier-Stokes equations [Sanderse & Koren'12]
- Velocity-pressure segregation by the time integrator
- No additional splitting needed (high-order feasible, not spoiled)

Velocity-pressure splitting

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Drawbacks:

- All terms must be treated explicitly (viscous + convective terms)
- Implicit treatment of interest when δt_{CFL} smaller than finest time scales of interest [Verstappen & Veldman'03; Vreman & Kuerten'14]
- A straightforward application with implicit/IMEX integrators leads to, e..g, $D(M + \delta t K_V)^{-1} G$ matrix for the pressure (not affordable)

Segregated RK methods

- **Observation:** No implicit / IMEX high-order integrators with implicit velocity-pressure segregation

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- **Target:** Develop such algorithms (of RK type)
- **Result:** Segregated RK methods [Colomés & SB'15]

Segregated RK methods

The idea:

1. Consider the projected momentum eq'on on the discretely divergence free space

$$
\begin{aligned}
&M\dot{\mathbf{U}} + (K + C(\mathbf{U}))\mathbf{U} + G\mathbf{P} = \mathbf{F},\\
&D\mathbf{U} = \mathbb{G}
\end{aligned}
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$$M\cancel{\mathbf{U}} + (K + C(\mathbf{U}))\mathbf{U} + G\mathbf{P} = \mathbf{F},$$
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Segregated RK methods

The idea:

1. Consider the projected momentum eq'on on the discretely divergence free space

$$M\cancel{\mathbf{U}} + P(K + C(\mathbf{U}))\mathbf{U} = P\mathbf{F} + G(DM^{-1}G)^{-1}\cancel{\mathbb{G}},$$

$$\text{with} \quad P := (I - G(DM^{-1}G)^{-1}DM^{-1})$$

Segregated RK methods

The idea:

1. Consider the projected momentum eq'on on the discretely divergence free space
2. Integrate the resulting ODE system w/ preferred IMEX RK method (diagonally implicit)...

$$M\dot{\mathbf{U}} = \mathcal{F}(\mathbf{U}) + \mathcal{G}(\mathbf{U}), \qquad \mathcal{F}: \text{ implicit terms}, \quad \mathcal{G}: \text{ explicit ones}$$

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The idea:

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3. $\mathcal{G}$ must include $G(DM^{-1}G)^{-1}\left(DM^{-1}(K + C(\mathbf{U})\mathbf{U} - \mathbf{F}) + \mathbb{G}\right)$ for velocity-pressure segregation

$$M\mathbf{Ʉ} = \mathcal{F}(\mathbf{U}) + \mathcal{G}(\mathbf{U}), \qquad \mathcal{F}: \text{ implicit terms}, \quad \mathcal{G}: \text{ explicit ones}$$

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3. $\mathcal{G}$ must include $G(DM^{-1}G)^{-1}\left(DM^{-1}(K + C(\mathbf{U})\mathbf{U} - \mathbf{F}) + \mathbb{G}\right)$ for velocity-pressure segregation
4. The rest of terms can be treated implicitly / explicitly

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Segregated RK methods

It leads to the time-discrete algorithm:

$$\frac{1}{\delta t} M\mathbf{U}_i = \frac{1}{\delta t} M\mathbf{U}_n + \sum_{j=1}^{i} a_{ij}\mathcal{F}(\mathbf{U}_j) + \sum_{j=1}^{i-1} \widehat{a}_{ij}\mathcal{G}(\mathbf{U}_j)$$

After re-ordering (pressure again):

$$\frac{1}{\delta t} M\mathbf{U}_i = \frac{1}{\delta t} M\mathbf{U}_n + \sum_{j=1}^{i} a_{ij}\mathcal{F}(\mathbf{U}_j) + \sum_{j=1}^{i-1} \widehat{a}_{ij}\mathcal{G}(\mathbf{U}_j, \mathbf{P}_j),$$

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- Viscous / convective can be treated implicitly / explicitly

$$\mathcal{F}(\mathbf{U}) := -K\mathbf{U} + \mathbf{F} - C(\mathbf{U})\mathbf{U}, \qquad \mathcal{G}(\cancel{\mathbf{U}}, P) = -GP \qquad \text{or}$$

$$\mathcal{F}(\mathbf{U}) := -K\mathbf{U} + \mathbf{F}, \qquad \mathcal{G}(\mathbf{U}, P) = -C(\mathbf{U})\mathbf{U} - GP$$

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$$-DM^{-1}G(\mathbf{P}_i) = DM^{-1}((K + C(\mathbf{U}_i))\mathbf{U}_i - \mathbf{F}(t_i)) + \mathbb{G}(t_i)$$

Features:

1. Segregation at the time integration level (no additional splitting)
2. High order achievable (the one of the ODE RK integration)
3. Implicit LES turbulent model (stabilization terms)

Some properties

Some proved results [Colomés & SB'15]:

- HERK and SRK methods are *equivalent* for full explicit treatment
- However, in SRK methods we are not enforcing implicitly the constraint...
- We have proved that $DU(t) = G$ as soon as strongly imposed velocity traces are polynomials or order p and $DU(0) = G(0)$ (p: order of IMEX RK time integrator)
- Equivalent when weakly enforced boundary conditions (dG)
- Equal-order & optimal velocity-pressure methods (of interest in FSI, etc.)

Numerical experiments

Manufactured analytical solutions:

- Simple $sin(t) \cdot \exp(t)$ function.

Laminar benchmark:

- 2D Laminar flow around a cylinder.

Manufactured analytical solution

Problem setting:

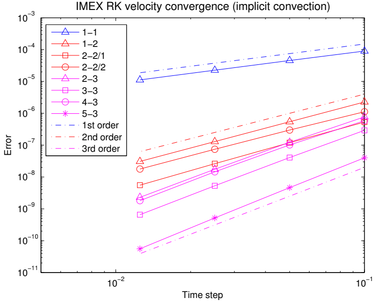
- Analytical solution:

$$\mathbf{u}(x, y, t) = \begin{bmatrix} x \\ -y \end{bmatrix} \sin(\frac{\pi}{10} t) \exp(\frac{t}{25}),$$
$$p(x, y) = x + y.$$

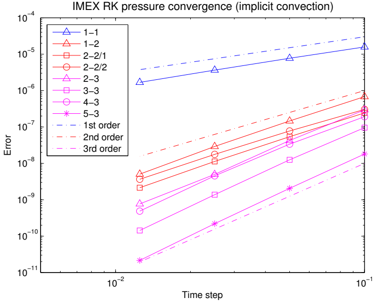
- $Re = 1/10/100$.
- Different IMEX Butcher tableaus with 1st, 2nd and 3rd order.

Manufactured analytical solution

Implicit convection:



(a) Velocity convergence, $\nu = 0.01$

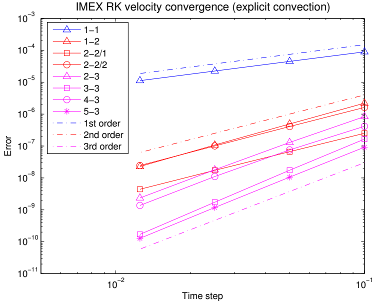


(b) Pressure convergence, $\nu = 0.01$

Figure : Fully implicit SRK.

Manufactured analytical solution

Explicit convection:



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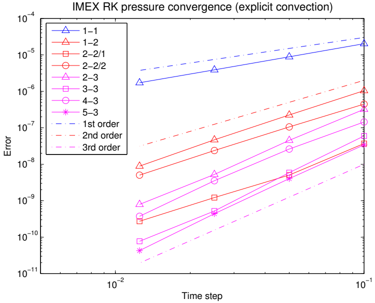


(b) Pressure convergence, $\nu = 0.01$

Figure : SRK convergence with convection integrated explicitly and diffusion integrated implicitly.

2D Laminar flow around a cylinder

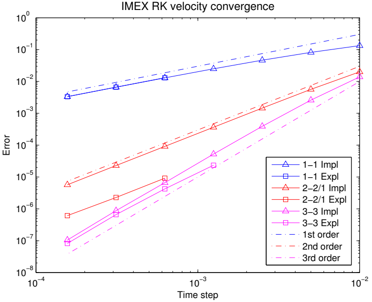
Problem setting:

- Widely used benchmark.
- $Re = 100$.
- Time convergence.
- Drag and Lift coefficients.

Figure : Vorticity field at $t = 8.0$.

2D Laminar flow around a cylinder

Implicit convection:



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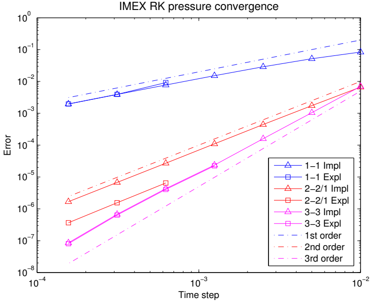
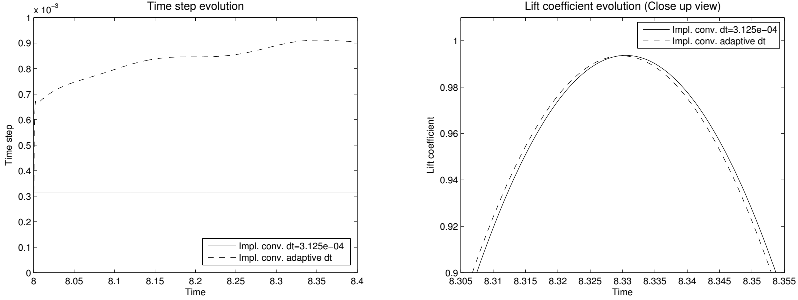


(b) Pressure convergence

Figure : Fully implicit and IMEX-SRK convergence rate comparison.

2D Laminar flow around a cylinder

Adaptive time stepping:



(a) Time step evolution

(b) Lift coefficient (zoom)

Figure : Adaptive time stepping.

Segregated Runge-Kutta Conclusions

- Velocity-pressure segregated by the time integrator (IMEX RK)

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- No additional splitting needed
- Equal order in time (pressure error not spoiled)

Segregated Runge-Kutta Limitations

- Only for index-2 DAE.

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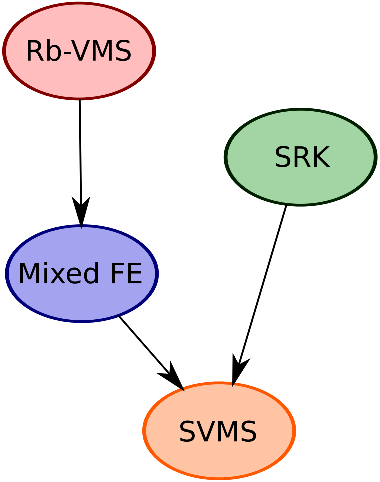
- Only for index-2 DAE.
- Not applicable for equal-order stabilization methods.


Rb-VMS
SRK
Mixed FE
SVMS

Motivation

Step by step...

- Residual-based VMS as LES models.
- Mixed FE formulations LES.
- High-order FE methods.
- High-order time integration schemes.
- Adaptive time stepping techniques.
- Velocity-pressure segregation.
- Scalable solvers.
- Application.

Looking backward...

VMS as LES models: Residual-based VMS methods

- Velocity + pressure stabilization, applied to equal-order FEs.
- Intensively tested methods (several works in the literature).
- Nature of the problem changes, not DAE-2 type (!).
- It prevents us to use segregated RK methods.

Looking backward...

VMS as LES models: Mixed FE VMS methods

- Only the term that we need, i.e., convection stabilization.
- To keep accuracy, we use orthogonal projections ($\mathcal{P}_h^{\perp}$).
- Many choices for the $\mathcal{P}_h$ projector: local, global (OSS).
- Discrete problem still index-2 DAE.
- Segregated RK schemes can be used.

Looking backward...

VMS as LES models: Mixed FE VMS methods

- Only the term that we need, i.e., convection stabilization.
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- Many choices for the $\mathcal{P}_h$ projector: local, global (OSS).
- Discrete problem still index-2 DAE.
- Segregated RK schemes can be used.

Now, we can use Segregated RK methods for ILES of turbulent flows

Formulation

OSS-ISS: Semi-discrete form

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C(\mathbf{U}) + A_\tau & G & B_\tau \\ D & 0 & 0 \\ -B_\tau^T & 0 & M_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

Formulation

OSS-ISS: Semi-discrete form

$$
\begin{aligned}
M\dot{\mathbf{U}} + (K + C(\mathbf{U}) + A_\tau)\mathbf{U} + G\mathbf{P} + B_\tau \boldsymbol{\Upsilon} &= \mathbf{F}_u, \\
M_\tau \boldsymbol{\Upsilon} - B_\tau^T \mathbf{U} &= \mathbf{0}, \\
D\mathbf{U} &= \mathbf{0}.
\end{aligned}
$$

Formulation

OSS-ISS:

1. Consider the projected momentum eq'on on the discretely divergence free space

$$
\begin{aligned}
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& M_\tau \boldsymbol{\Upsilon} - B_\tau^T \mathbf{U} = \mathbf{0}, \\
& -DM^{-1}G\mathbf{P} = DM^{-1}(K + C(\mathbf{U})\mathbf{U} + B_\tau \boldsymbol{\Upsilon} - \mathbf{F}_u).
\end{aligned}
$$

Formulation

OSS-ISS:

1. Consider the projected momentum eq'on on the discretely divergence free space
2. Integrate the resulting ODE system w/ preferred IMEX RK method (diagonally implicit)...

$$M\dot{\mathbf{U}} = \mathcal{F}(\mathbf{U}, \boldsymbol{\Upsilon}) + \mathcal{G}(\mathbf{U}, \mathbf{P}), \qquad \mathcal{F}: \text{ implicit terms}, \quad \mathcal{G}: \text{ explicit ones}$$

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4. We treat the projection term implicit or explicit, according the convective term.

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Final discrete problem

At each stage of the Runge-Kutta scheme: (explicit convection version)

$$\frac{1}{\delta t}M\mathbf{U}_i = \frac{1}{\delta t}M\mathbf{U}_n + \sum_{j=1}^{i} a_{ij}\mathcal{F}(\mathbf{U}_j) + \sum_{j=1}^{i-1}\widehat{a}_{ij}\mathcal{G}(\mathbf{U}_j, \boldsymbol{\Upsilon}_j, \mathbf{P}_j),$$

$$M_\tau\boldsymbol{\Upsilon}_i - B_\tau^T\mathbf{U}_i = \mathbf{0},$$

$$-DM^{-1}G(\mathbf{P}_i) = DM^{-1}((K + C(\mathbf{U}_i) + A_\tau)\mathbf{U}_i + B_\tau\boldsymbol{\Upsilon}_i - \mathbf{F}_u(t_i)).$$

Final discrete problem

At each stage of the Runge-Kutta scheme: (explicit convection version)

$$\left(\frac{1}{\delta t}M + a_{ii}K\right)\mathbf{U}_i = \frac{1}{\delta t}M\mathbf{U}_n + \sum_{j=1}^{i-1}\left[a_{ij}\mathcal{F}(\mathbf{U}_j) + \widehat{a}_{ij}\mathcal{G}(\mathbf{U}_j, \boldsymbol{\Upsilon}_j, \mathbf{P}_j)\right],$$

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- Elasticity-type matrix for the momentum equation.

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- Elasticity-type matrix for the momentum equation.
- Mass-type matrix for the projection equation.

Final discrete problem

At each stage of the Runge-Kutta scheme: (explicit convection version)

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$$M_\tau\boldsymbol{\Upsilon}_i - B_\tau^T\mathbf{U}_i = \mathbf{0},$$

$$-DM^{-1}G(\mathbf{P}_i) = DM^{-1}((K + C(\mathbf{U}_i) + A_\tau)\mathbf{U}_i + B_\tau\boldsymbol{\Upsilon}_i - \mathbf{F}_u(t_i)).$$

- Elasticity-type matrix for the momentum equation.
- Mass-type matrix for the projection equation.
- Darcy-type system for the pressure equation.

Large scale simulations

Momentum system matrix: $\frac{1}{\delta t}M + a_{ii}K$

- Implicit viscous: Balancing domain decomposition (BDDC) for $M + \delta t K$
- Coercive operator, algorithmically scalable algorithm, freezed as mesh fixed

Large scale simulations

Momentum system matrix: $\frac{1}{\delta t}M + a_{ii}K$

- Implicit viscous: Balancing domain decomposition (BDDC) for $M + \delta t K$
- Coercive operator, algorithmically scalable algorithm, freezed as mesh fixed

Pressure system matrix: $DM^{-1}G$

- Spectrally equivalent to standard Laplacian L
- Written as

$$\begin{bmatrix} M & G \\ D & 0 \end{bmatrix} \quad \text{preconditioned w/} \quad \begin{bmatrix} M & 0 \\ 0 & L_{\text{BDDC}} \end{bmatrix}$$

- Optimal + scalable method (BDDC)

Outline

- Line 1.

Outline

- Line 1.
- Line 2.
  Less formal

Outline

- Line 1.
- Line 2.
  Less formal
- Line 3.
  Less formal, different color.

Blocks

Standard Block

This is a standard block.

Example Block

This is an example block.

Alert Block

This is an alert block.

Questions?