

Large scale Finite Element solvers for the large eddy simulation of incompressible turbulent flows

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March 31 2016



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1. Motivation

2. Residual-based VMS

3. Mixed FE VMS

4. Segregated Runge-Kutta

5. Segregated VMS

6. Conclusions

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(High-performance Computing (HPC))

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 - Many industrial applications
 - Understanding the phenomena (physics)
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- Take advantage of increasing computational capacity
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- In a Finite Element (FE) framework

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Thesis goal

Highly scalable Finite Element (FE) framework for Large Eddy Simulations (LES) of incompressible turbulent flows

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1. Variational MultiScale (VMS) methods as LES models
2. Time integration schemes with **velocity-pressure segregation**

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Thesis goal

Highly scalable Finite Element (FE) framework for Large Eddy Simulations (LES) of incompressible turbulent flows

How to get there?

1. Variational MultiScale (VMS) methods as LES models
2. Time integration schemes with velocity-pressure segregation
3. Highly scalable algorithms based on **Domain Decomposition (DD)** and **block preconditioners**

Objective

Step by step...

- Residual-based VMS as LES models

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- Scalable solvers

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1. Motivation

Rb-VMS

2. Residual-based VMS

Formulation

Numerical experiments

Conclusions

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4. Segregated Runge-Kutta

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Implicit LES

ILES: only numerical dissipation (for stabilization) acts as turbulent model

- Not based on filtering of the Navier-Stokes equations
- Rely on numerical artifacts, no modification at the continuous level

Incomp. Navier Stokes equations

Find \mathbf{u} and p defined in Ω

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

with appropriate boundary conditions on Γ

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The weak problem is: $\forall \mathbf{v} \in \mathcal{V}_0$ and $\forall q \in \mathcal{Q}_0$, find $\mathbf{u} \in \mathcal{V}$ and $p \in \mathcal{Q}$ such that

$$\begin{aligned} (\mathbf{v}, \partial_t \mathbf{u})_\Omega + (\nabla \mathbf{v}, \nu \nabla \mathbf{u})_\Omega + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p)_\Omega &= \langle \mathbf{v}, \mathbf{f} \rangle_\Omega \\ (q, \nabla \cdot \mathbf{u})_\Omega &= 0 \end{aligned}$$

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where

$$b(\mathbf{a}, \mathbf{u}, \mathbf{v}) = \langle \mathbf{v}, \mathbf{a} \cdot \nabla \mathbf{u} \rangle_\Omega$$

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where

$$b(\mathbf{a}, \mathbf{u}, \mathbf{v}) = \frac{1}{2} \langle \mathbf{v}, \mathbf{a} \cdot \nabla \mathbf{u} \rangle_\Omega - \frac{1}{2} \langle \mathbf{a} \cdot \nabla \mathbf{v}, \mathbf{u} \rangle_\Omega + \frac{1}{2} \langle \mathbf{v}, \mathbf{n} \cdot \mathbf{a} \mathbf{u} \rangle_\Gamma$$

VMS decomposition (Hughes 1995)

A decomposition of spaces \mathcal{V} and \mathcal{Q} given by

$$\mathcal{V} = \mathcal{V}_h \oplus \tilde{\mathcal{V}}, \quad \mathcal{Q} = \mathcal{Q}_h \oplus \tilde{\mathcal{Q}}$$

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is applied to the function and test spaces

$$\mathbf{u} = \mathbf{u}_h + \tilde{\mathbf{u}}, \quad p = p_h + \tilde{p}$$

$$\mathbf{v} = \mathbf{v}_h + \tilde{\mathbf{v}}, \quad q = q_h + \tilde{q}$$

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We keep all the (eight) contributions from the splitting of the convective term

$$\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{u}_h \cdot \nabla \mathbf{u}_h + \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_h + \mathbf{u}_h \cdot \nabla \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}$$

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and all the (four) contributions from the temporal term

$$\partial_t \mathbf{u} = \partial_t \mathbf{u}_h + \partial_t \tilde{\mathbf{u}}$$

Semidiscrete problem

FEM equations

$$B((\mathbf{u}_h, p_h); (\tilde{\mathbf{u}}, \tilde{p}) ; (\mathbf{v}_h, q_h)) = L(\mathbf{v}_h, q_h)$$

SGS equations

$$B((\tilde{\mathbf{u}}, \tilde{p}) ; (\mathbf{u}_h, p_h) ; (\tilde{\mathbf{v}}, \tilde{q})) = L(\tilde{\mathbf{v}}, \tilde{q})$$

Semidiscrete problem

FEM equations

$$\begin{aligned}
 & (\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega \\
 & + (\mathbf{v}_h, \partial_t \tilde{\mathbf{u}})_\Omega + (\mathcal{L}^* \mathbf{v}_h, \tilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \tilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega \\
 & (q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \tilde{\mathbf{u}})_{\Omega^h} = 0
 \end{aligned}$$

SGS equations

$$\partial_t \tilde{\mathbf{u}} + \tau_m^{-1} \tilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \tilde{p} = \mathcal{P} R_c$$

$$\tau_m = \left(\frac{c_1 \nu}{h^2} + \frac{c_2 |\mathbf{a}|}{h} \right)^{-1}, \quad \tau_c = \frac{h^2}{c_1 \tau_m}$$

$$\mathbf{a} = \mathbf{u}_h + \tilde{\mathbf{u}}$$

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$$\tau_c^{-1} \tilde{p} = \mathcal{P} \mathbf{R}_c$$

$$\mathbf{R}_m := \mathbf{f} - \partial_t \mathbf{u}_h - \mathcal{L} \mathbf{u}_h - \nabla p_h, \quad \mathbf{R}_c := -\nabla \cdot \mathbf{u}_h$$

Semidiscrete problem

FEM equations

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$$\tau_c^{-1} \tilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = I \quad (\text{ASGS}), \quad \mathcal{P} = P_h^\perp = I - P_h \quad (\text{OSS})$$

Semidiscrete problem

FEM equations

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 & (q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \tilde{\mathbf{u}})_{\Omega^h} = 0
 \end{aligned}$$

SGS equations

$$\partial_t \tilde{\mathbf{u}} + \tau_m^{-1} \tilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m \quad (\text{static/dynamic})$$

$$\tau_c^{-1} \tilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = I \quad (\text{ASGS}), \quad \mathcal{P} = P_h^\perp = I - P_h \quad (\text{OSS})$$

$$\mathbf{a} = \mathbf{u}_h + \tilde{\mathbf{u}} \quad (\text{linear/nonlinear})$$

Numerical experiments

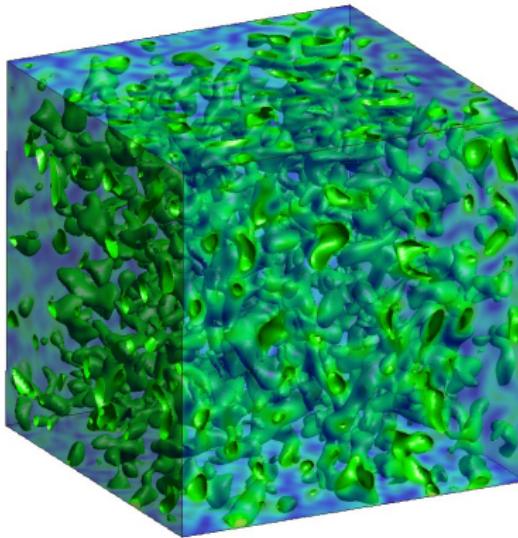
Three different turbulent benchmarks:

- Decaying of Homogeneous Isotropic Turbulence (DHIT)
- Taylor-Green Vortex (TGV) flow
- Turbulent Channel Flow (TCF)

DHIT Decay of Homogeneous Isotropic Turbulence

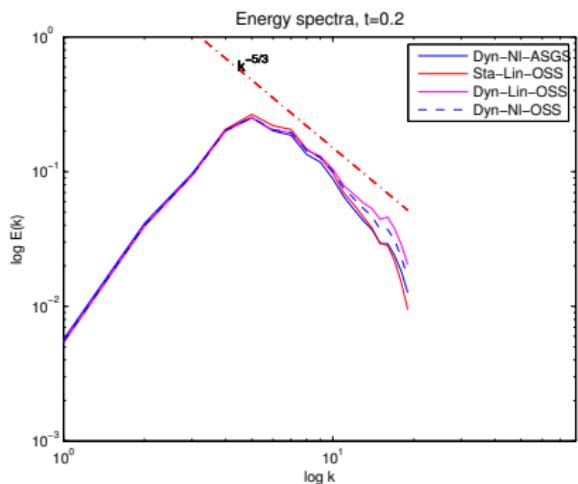
Problem setting:

- Prescribed initial energy spectra corresponding to $Re_\lambda = 952$
- Setting defined in AGARD database [Mansour & Wray, 1993]
- Different mesh discretizations (Q_1/Q_1 and Q_2/Q_2)

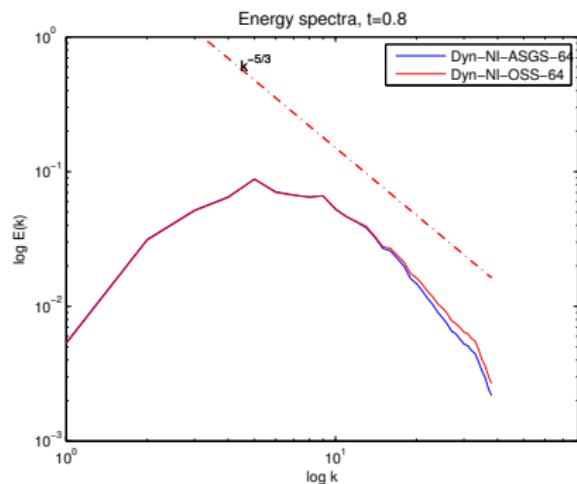


DHIT Decay of Homogeneous Isotropic Turbulence

Energy spectra (models):



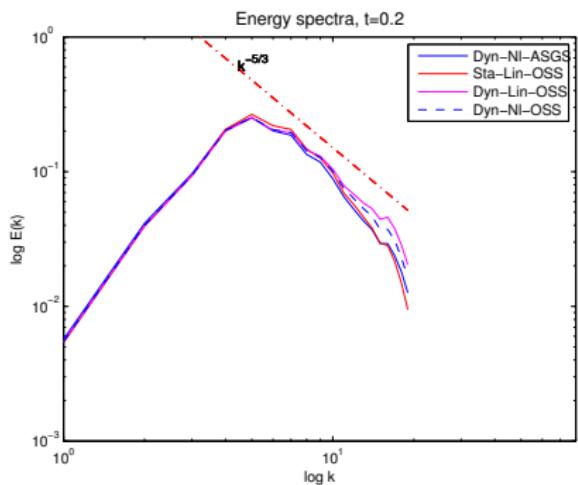
$32^3 - Q1, t = 0.2s$



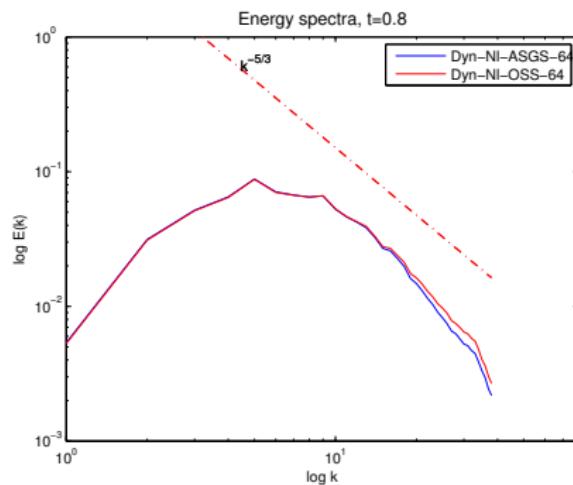
$64^3 - Q1, t = 0.8s$

DHIT Decay of Homogeneous Isotropic Turbulence

Energy spectra (models):



$32^3 - Q1, t = 0.2s$

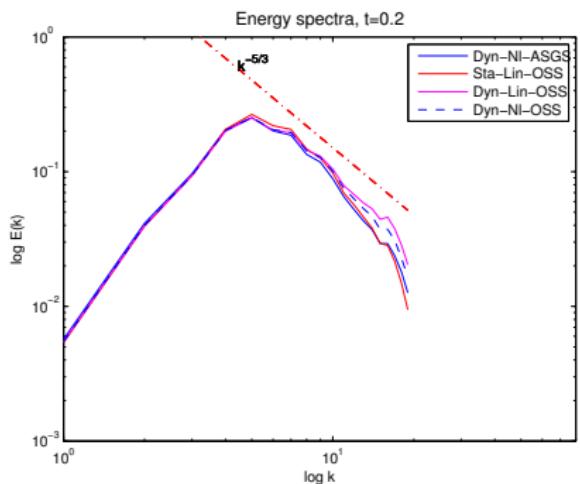


$64^3 - Q1, t = 0.8s$

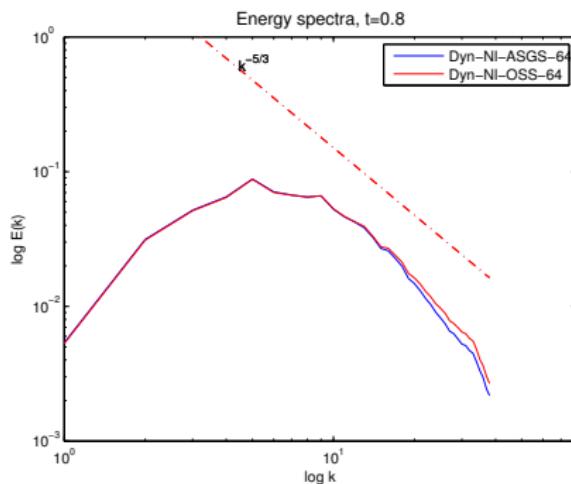
- Small differences between methods (physical sense)

DHIT Decay of Homogeneous Isotropic Turbulence

Energy spectra (models):



$32^3 - Q1, t = 0.2s$

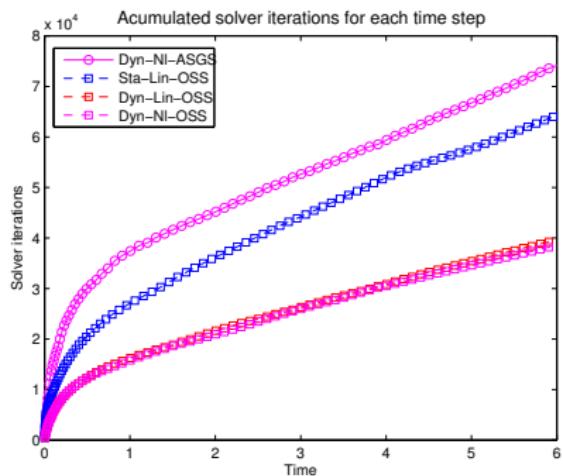


$64^3 - Q1, t = 0.8s$

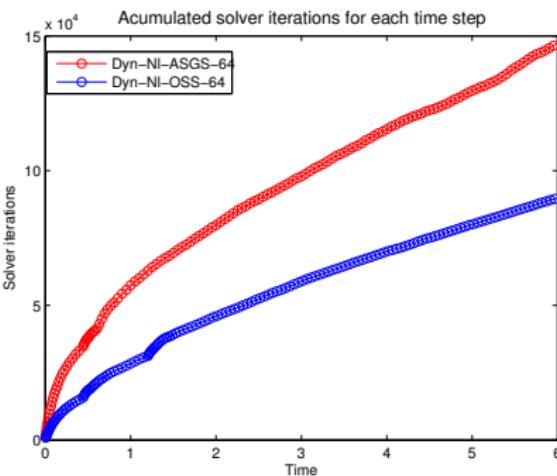
- Small differences between methods (physical sense)
- Even more similar when we refine the mesh

DHIT Decay of Homogeneous Isotropic Turbulence

Computational cost (models):



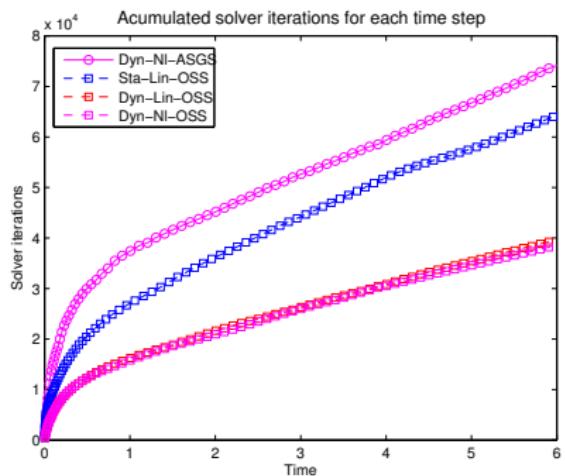
$32^3 - Q1$



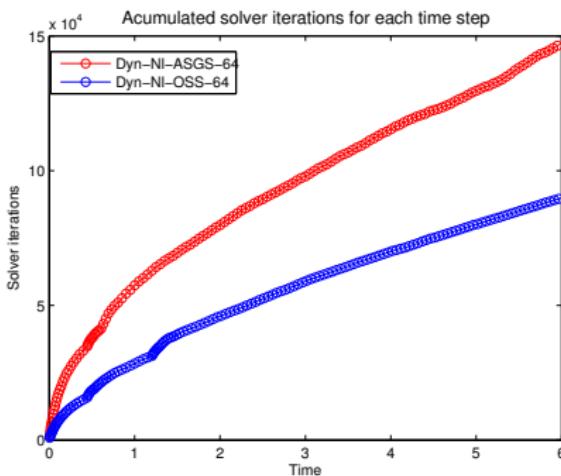
$64^3 - Q1$

DHIT Decay of Homogeneous Isotropic Turbulence

Computational cost (models):



$32^3 - Q1$

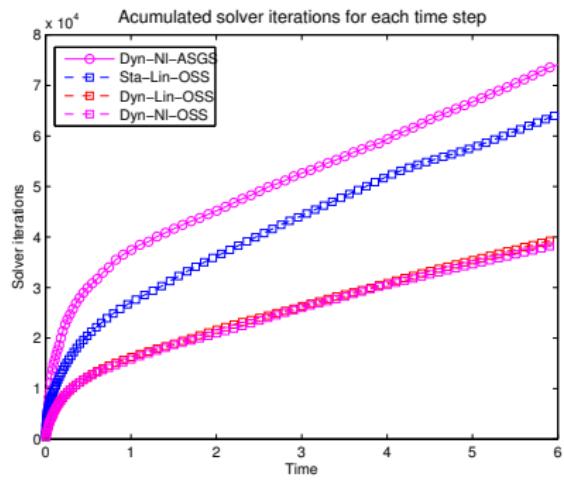


$64^3 - Q1$

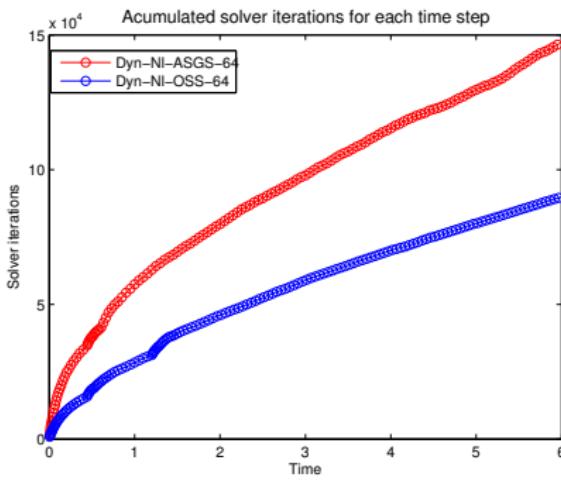
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DHIT Decay of Homogeneous Isotropic Turbulence

Computational cost (models):



$32^3 - Q1$

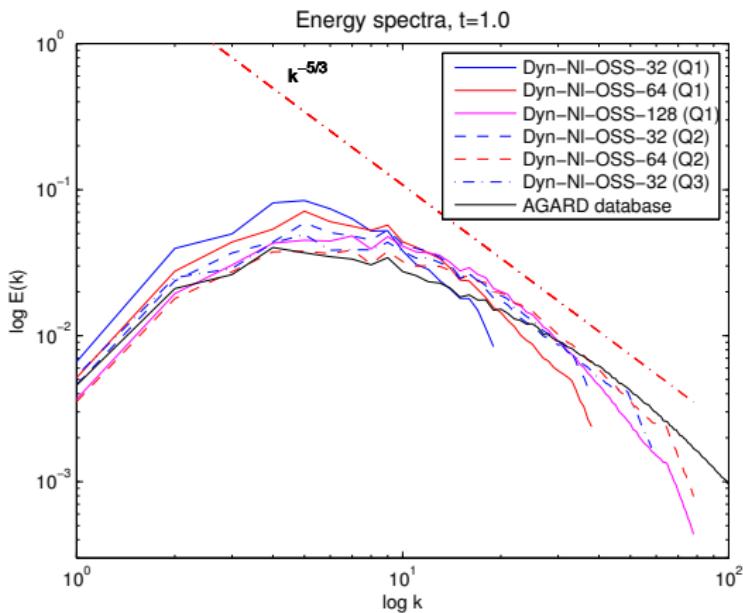


$64^3 - Q1$

- Big differences between methods (computational sense)
- **Dynamic versions of OSS method are the most efficient**

DHIT Decay of Homogeneous Isotropic Turbulence

Energy spectra (refinement):

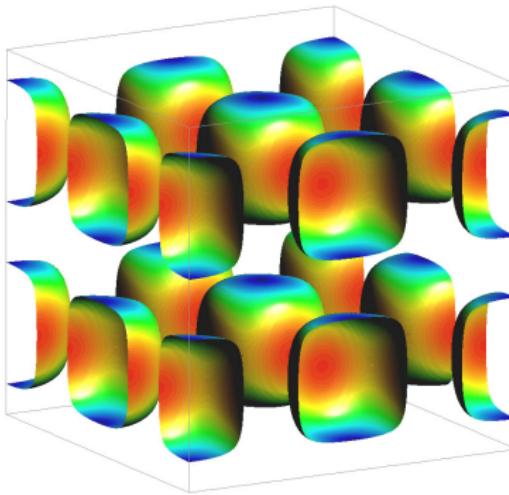


- Results become closer to the DNS when we refine the mesh

TGV Taylor-Green Vortex flow

Problem setting:

- Prescribed initial condition
- $Re = 1600$
- Different mesh discretizations (Q_1/Q_1 and Q_2/Q_2)

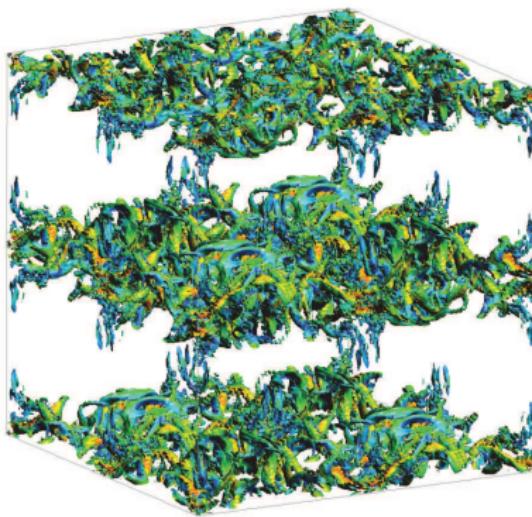


Initial vorticity isosurface $|\omega| = 1$

TGV Taylor-Green Vortex flow

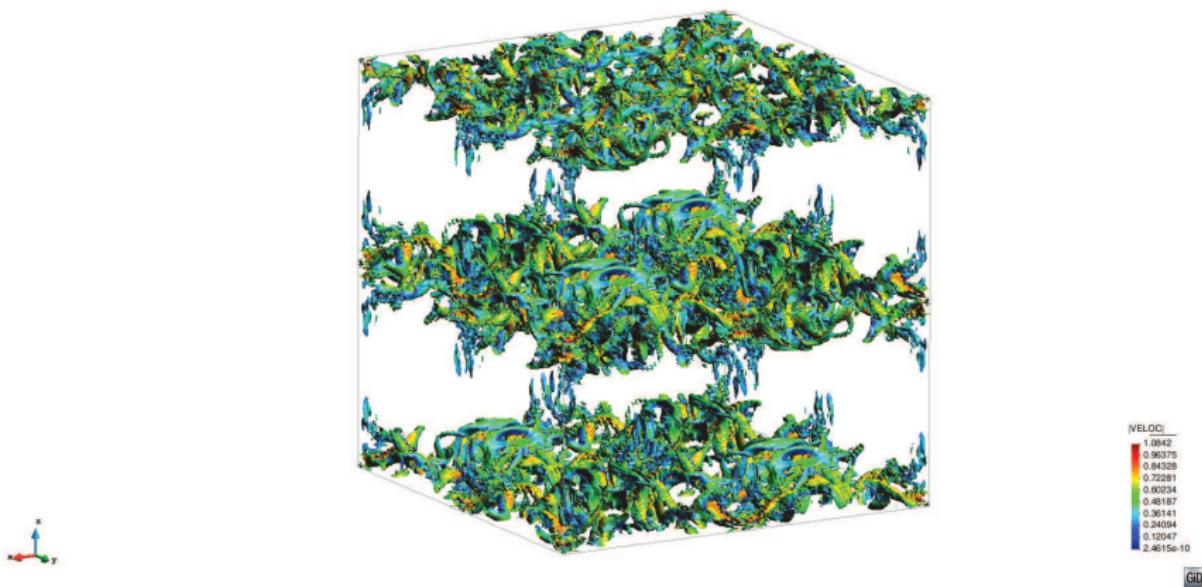
Problem setting:

- Prescribed initial condition
- $Re = 1600$
- Different mesh discretizations (Q_1/Q_1 and Q_2/Q_2)



Vorticity isosurfaces $|\omega| = 9.0$

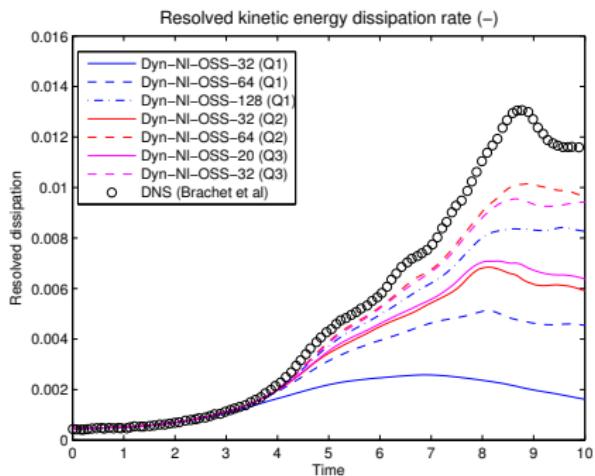
TGV Taylor-Green Vortex flow



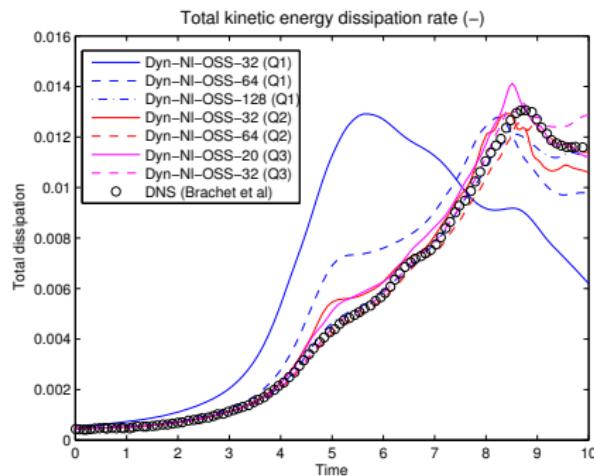
Velocity isosurface

TGV Taylor-Green Vortex flow

Energy dissipation rate (refinement):



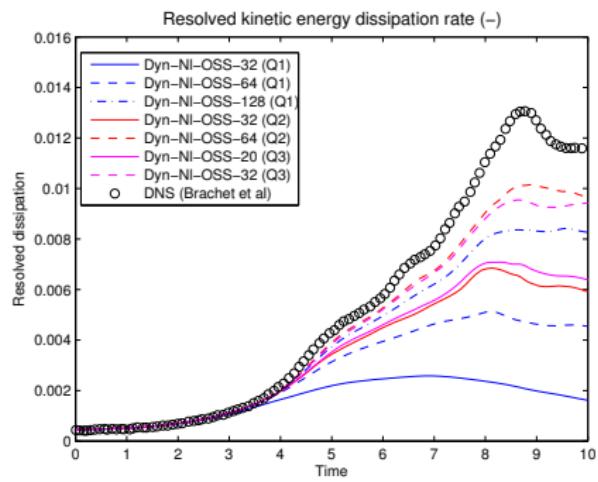
Resolved scales



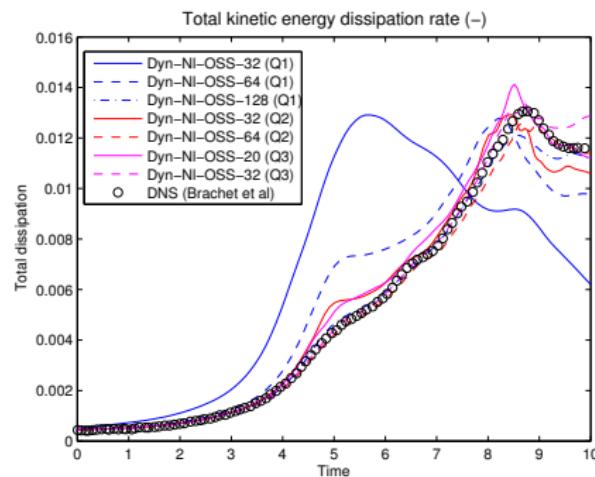
Total

TGV Taylor-Green Vortex flow

Energy dissipation rate (refinement):



Resolved scales

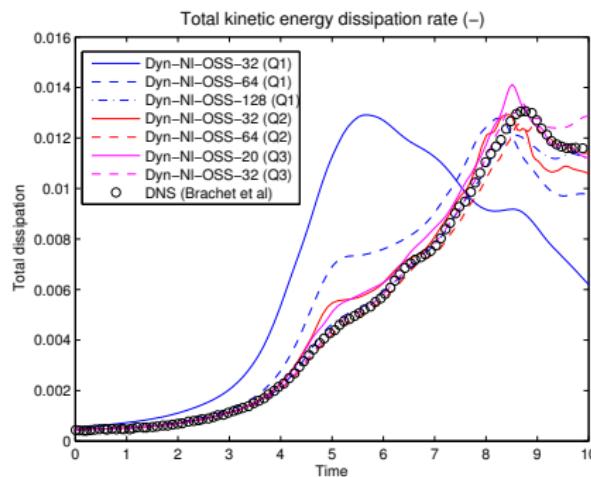
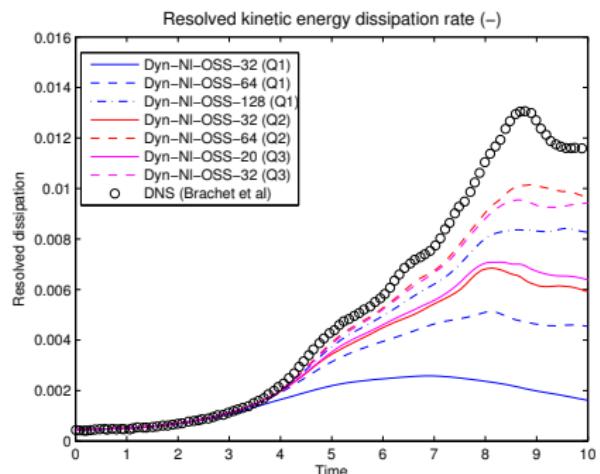


Total

- Good agreement with the DNS taking account the subscales

TGV Taylor-Green Vortex flow

Energy dissipation rate (refinement):



Resolved scales

Total

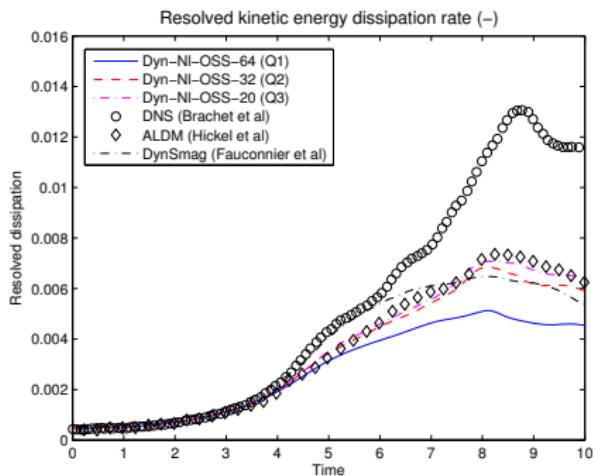
- Good agreement with the DNS taking account the subscales
- More accurate results increasing the order of approximation

TGV Taylor-Green Vortex flow

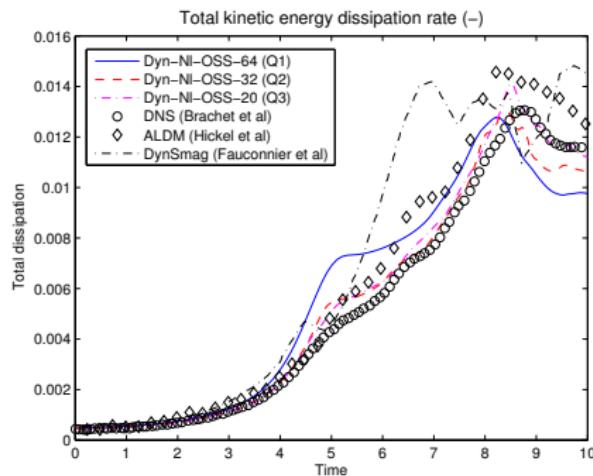
- All results until now are compared against **DNS**
- Are our methods comparable with **LES** models?

TGV Taylor-Green Vortex flow

Energy dissipation rate (against LES model):



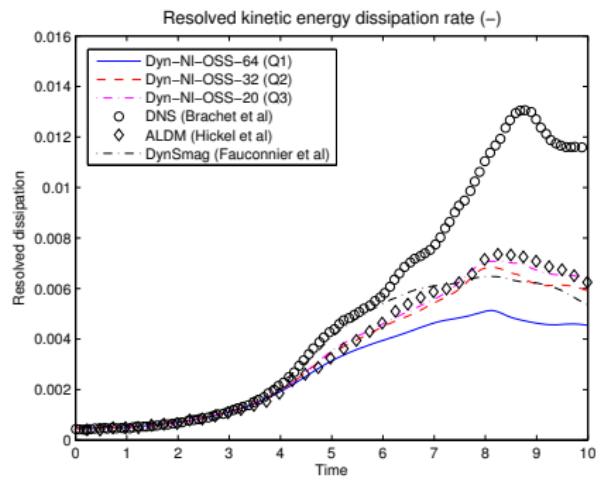
Resolved scales



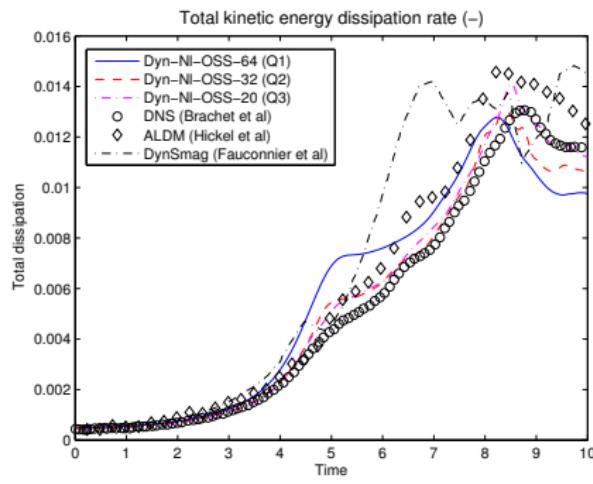
Total

TGV Taylor-Green Vortex flow

Energy dissipation rate (against LES model):



Resolved scales

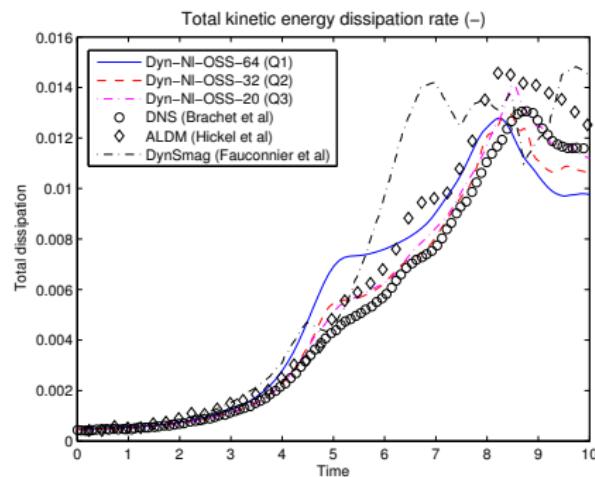
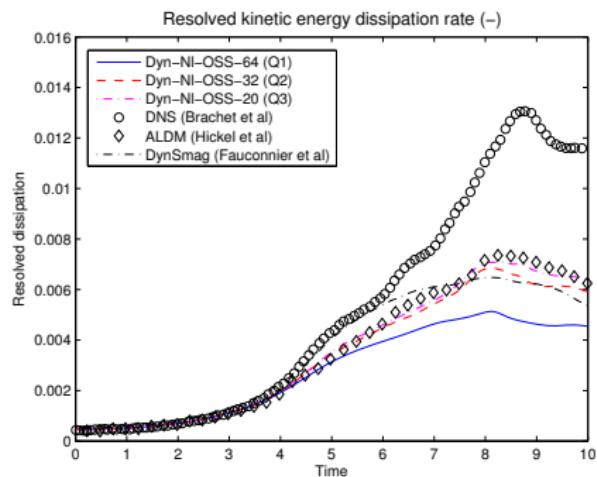


Total

- Good agreement with the LES models on resolved scales

TGV Taylor-Green Vortex flow

Energy dissipation rate (against LES model):



Resolved scales

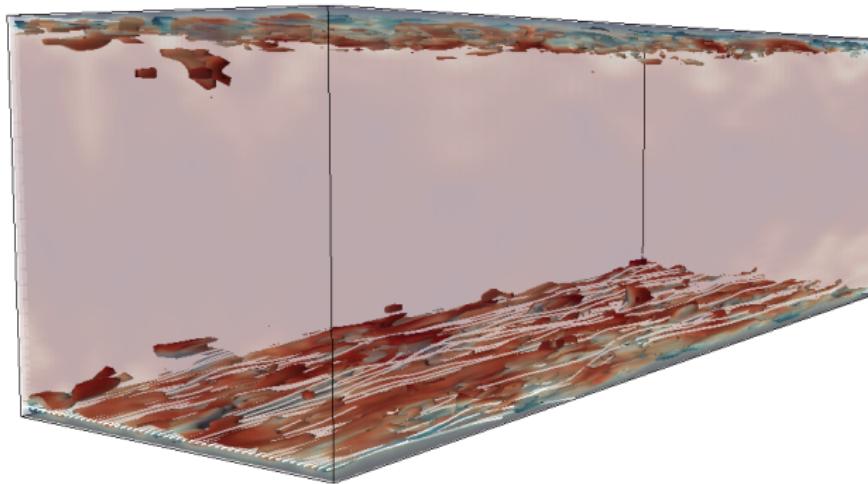
Total

- Good agreement with the LES models on resolved scales
- Better results than LES models adding subscales counterpart

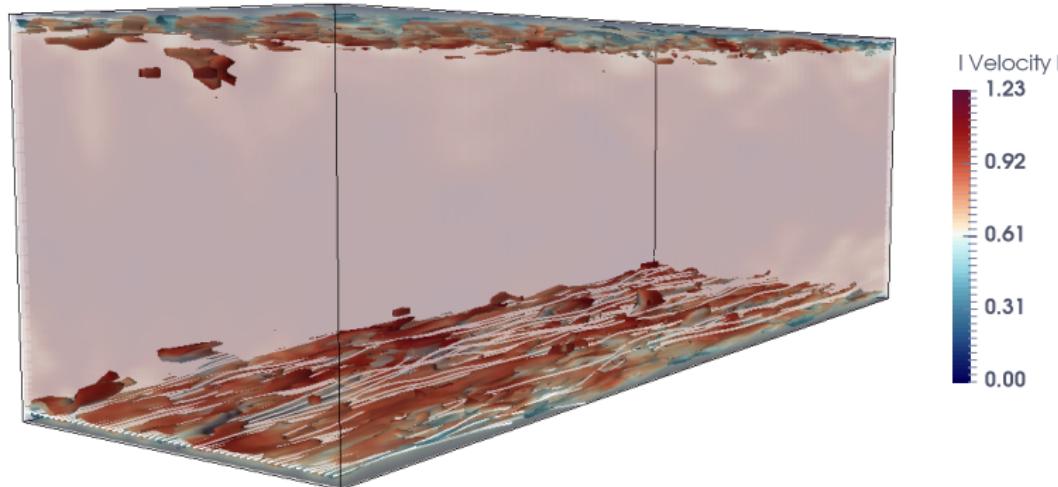
TCF Turbulent Channel Flow

Problem setting:

- Wall bounded flow
- $Re_\tau = 180$ and $Re_\tau = 395$
- Mesh resolution: $32^3 - Q1$ (stretched elements near the wall)

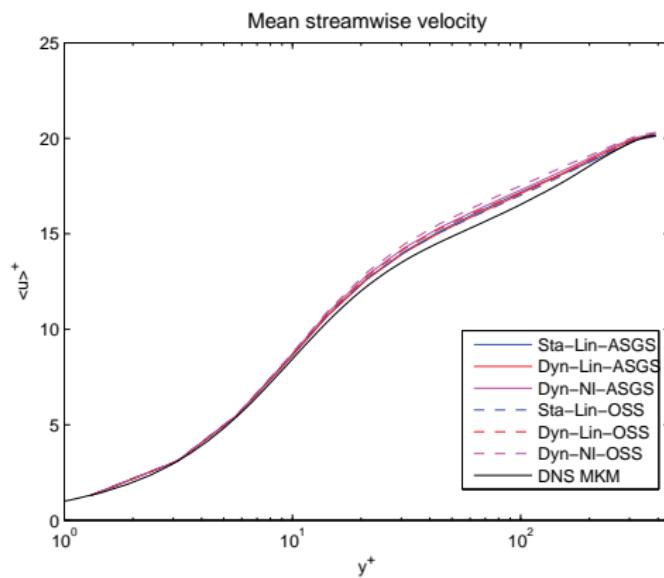


TCF Turbulent Channel Flow



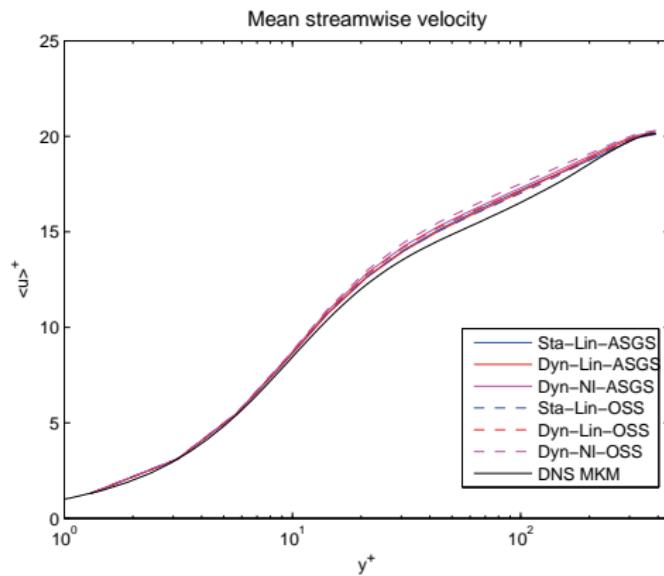
TCF Turbulent Channel Flow

Mean streamwise velocity (models):



TCF Turbulent Channel Flow

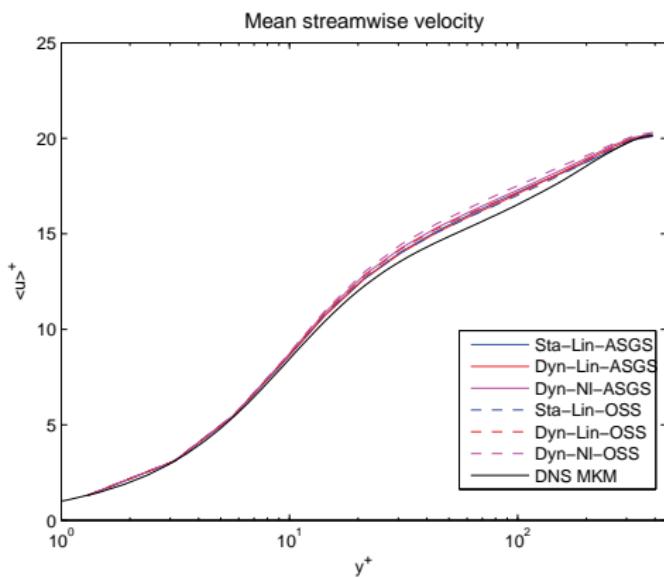
Mean streamwise velocity (models):



- Small differences between methods

TCF Turbulent Channel Flow

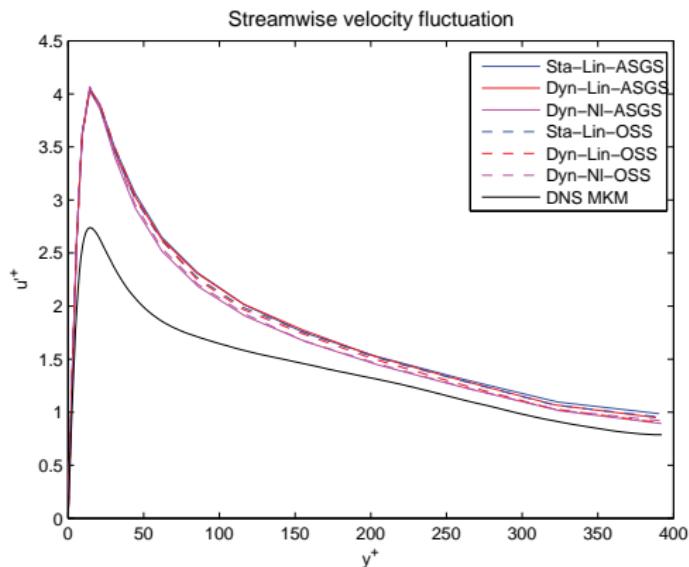
Mean streamwise velocity (models):



- Small differences between methods
- Very accurate results compared against the DNS

TCF Turbulent Channel Flow

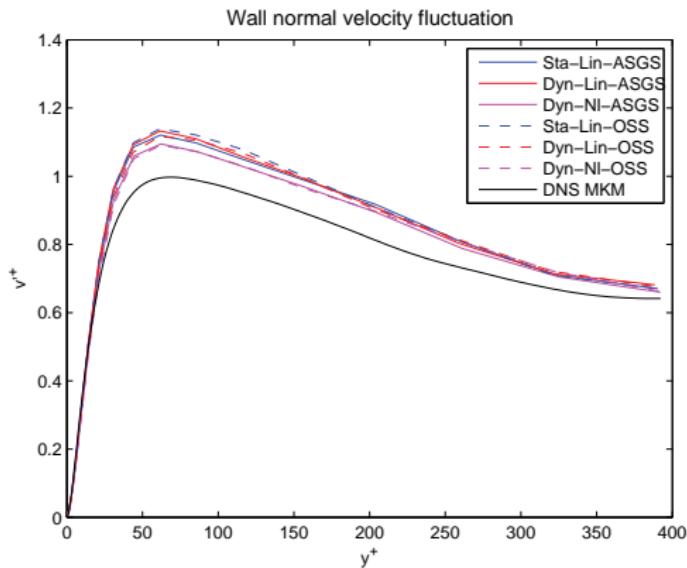
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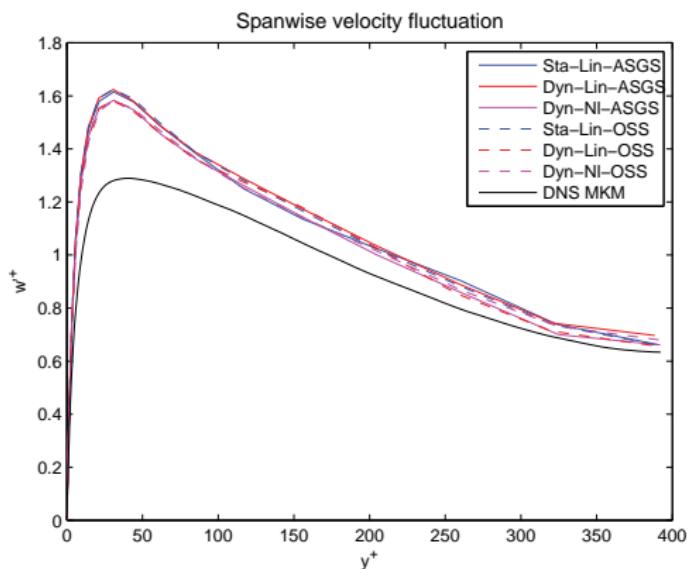
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RB-VMS Conclusions

- VMS formulations of NS equations can be used for the numerical simulation of turbulent flows

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 - Nonlinear subscales
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Contribution:

-  OC, Santiago Badia, Ramon Codina and Javier Principe
Assessment of variational multiscale models for the large eddy simulation of turbulent incompressible flows

Computer Methods in Applied Mechanics and Engineering, 2015

RB-VMS Limitations

- ASGS:

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- **Desired:**

- OSS with implicit projections

1. Motivation

2. Residual-based VMS

3. Mixed FE VMS

Formulation

Block-preconditioning

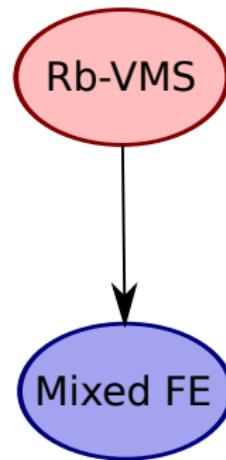
Numerical experiments

Conclusions

4. Segregated Runge-Kutta

5. Segregated VMS

6. Conclusions



Motivation

Step by step...

- Residual-based VMS as LES models
- Mixed FE formulations LES
- High-order FE methods
- High-order time integration schemes
- Adaptive time stepping techniques
- Velocity-pressure segregation
- Scalable solvers
- Application

Semidiscrete problem

FEM equations

$$\begin{aligned}
 & (\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega \\
 & + (\mathbf{v}_h, \partial_t \tilde{\mathbf{u}})_\Omega + (\mathcal{L}^* \mathbf{v}_h, \tilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \tilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega \\
 & (q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \tilde{\mathbf{u}})_{\Omega^h} = 0
 \end{aligned}$$

SGS equations

$$\partial_t \tilde{\mathbf{u}} + \tau_m^{-1} \tilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \tilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = I \quad (\text{ASGS}), \quad \mathcal{P} = \mathbf{P}_h^\perp = \mathbf{I} - \mathbf{P}_h \quad (\text{OSS})$$

$$\mathbf{a} = \mathbf{u}_h + \tilde{\mathbf{u}}$$

Semidiscrete problem

FEM equations

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SGS equations

$$\tau_m^{-1} \tilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \tilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = \mathbf{P}_h^\perp = \mathbf{I} - \mathbf{P}_h \quad (\text{OSS})$$

$$\mathbf{a} = \mathbf{u}_h$$

Term-by-term OSS

FEM equations

$$\begin{aligned}
 & (\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega \\
 & + (\mathcal{L}^* \mathbf{v}_h, \tilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \tilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega \\
 & (q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \tilde{\mathbf{u}})_{\Omega^h} = 0
 \end{aligned}$$

Term-by-term OSS

Term-by-term OSS (Codina 2008)

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

$$+ \left(\tau_m \mathbf{a} \cdot \nabla \mathbf{v}_h, \mathcal{P}_h^\perp(\mathbf{a} \cdot \nabla \mathbf{u}_h) \right)_{\Omega^h}$$

$$+ (\tau_c \nabla \cdot \mathbf{v}_h, \nabla \cdot \mathbf{u}_h)_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega + \left(\tau_m \nabla q_h, \mathcal{P}_h^\perp(\nabla p_h) \right)_{\Omega^h} = 0$$

Term-by-term OSS

Term-by-term OSS (Codina 2008)

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

$$+ (\tau_m \mathbf{a} \cdot \nabla \mathbf{v}_h, \mathbf{a} \cdot \nabla \mathbf{u}_h)_{\Omega^h} - (\tau_m \mathbf{a} \cdot \nabla \mathbf{v}_h, \boldsymbol{\eta}_h)_{\Omega^h}$$

$$+ (\tau_c \nabla \cdot \mathbf{v}_h, \nabla \cdot \mathbf{u}_h)_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega + (\tau_m \nabla q_h, \nabla p_h)_{\Omega^h} - (\tau_m \nabla q_h, \boldsymbol{\xi}_h)_{\Omega^h} = 0$$

$$\boldsymbol{\eta}_h := \mathcal{P}_h(\mathbf{a} \cdot \nabla \mathbf{u}_h)$$

$$\boldsymbol{\xi}_h := \mathcal{P}_h(\nabla p_h)$$

Matrix form

- ASGS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G + G_\tau \\ D + D_\tau & L_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{G} \end{bmatrix},$$

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- Term-by-term OSS with Inf-sup stable elements (mixed FE):

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_\tau \\ D & 0 & 0 \\ -B_\tau^T & 0 & M_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{G} \\ \mathbf{0} \end{bmatrix},$$

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Index-2 DAE!!!

Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{G} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

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- Outer block matrix:

$$\tilde{A} = \begin{bmatrix} M_{\eta,\tau} & 0 & | & -B_{\eta,\tau}^T & 0 \\ 0 & M_{\xi,\tau} & | & 0 & -B_{\xi,\tau}^T \\ B_{\eta,\tau} & 0 & | & K + C + A_\tau & G \\ 0 & B_{\xi,\tau} & | & D & L_\tau \end{bmatrix}$$

Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & | & B_{\eta,\tau} & 0 \\ D & L_\tau & | & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & | & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & | & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{G} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

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$$\tilde{A} = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ \tilde{B}_\tau & \tilde{K}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}_\tau \tilde{M}_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix},$$

Block-recursive preconditioning

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$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & | & B_{\eta,\tau} & 0 \\ D & L_\tau & | & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & | & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & | & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{G} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Outer block matrix:

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ \tilde{B}_\tau & \tilde{K}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}_\tau \tilde{M}_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix}, \\ \tilde{S} &= \tilde{K}_\tau + \tilde{B}_\tau \tilde{M}_\tau^{-1} \tilde{B}_\tau^T, \end{aligned}$$

Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & | & B_{\eta,\tau} & 0 \\ D & L_\tau & | & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & | & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & | & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \Xi \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{G} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Outer block matrix:

$$\tilde{A} = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ \tilde{B}_\tau & \tilde{K}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}_\tau \tilde{M}_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix},$$

$$\tilde{S} = \tilde{K}_\tau + \tilde{B}_\tau \tilde{M}_\tau^{-1} \tilde{B}_\tau^T,$$

$$P_U(\tilde{A}) = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{M}_\tau^{-1} & \tilde{M}_\tau^{-1} \tilde{B}_\tau^T \tilde{S}^{-1} \\ 0 & \tilde{S}^{-1} \end{bmatrix},$$

Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{G} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

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$$\tilde{S} = \tilde{K}_\tau + \tilde{B}_\tau \tilde{M}_\tau^{-1} \tilde{B}_\tau^T,$$

$$P_U(\tilde{A}) = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{M}_\tau^{-1} & \tilde{M}_\tau^{-1} \tilde{B}_\tau^T \tilde{S}^{-1} \\ 0 & \tilde{S}^{-1} \end{bmatrix},$$

$$\tilde{S}^{-1} \approx \tilde{K}_\tau^{-1}.$$

Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{G} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Inner block matrix:

$$\tilde{K}_\tau = \begin{bmatrix} K_\tau & G \\ D & L_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ DK_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix},$$

Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{G} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Inner block matrix:

$$\begin{aligned} \tilde{K}_\tau &= \begin{bmatrix} K_\tau & G \\ D & L_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ DK_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix}, \\ S &= L_\tau - DK_\tau^{-1}G, \end{aligned}$$

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$$\tilde{S}^{-1} \approx L_p^{-1}.$$

Numerical experiments

Manufactured analytical solution:

- Colliding flow

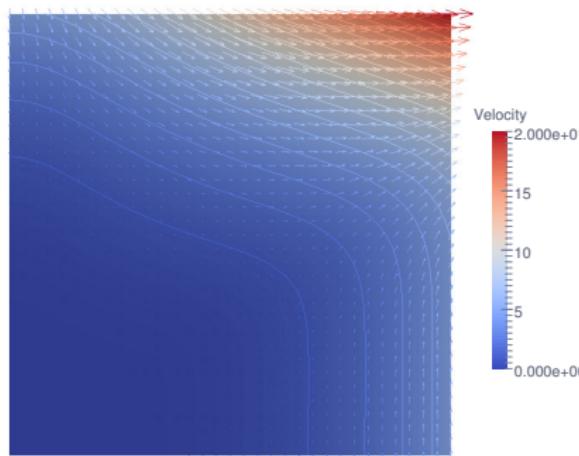
Two different turbulent benchmarks:

- Taylor-Green Vortex (TGV) flow
- Turbulent Channel Flow (TCF)

Colliding flow

Problem setting:

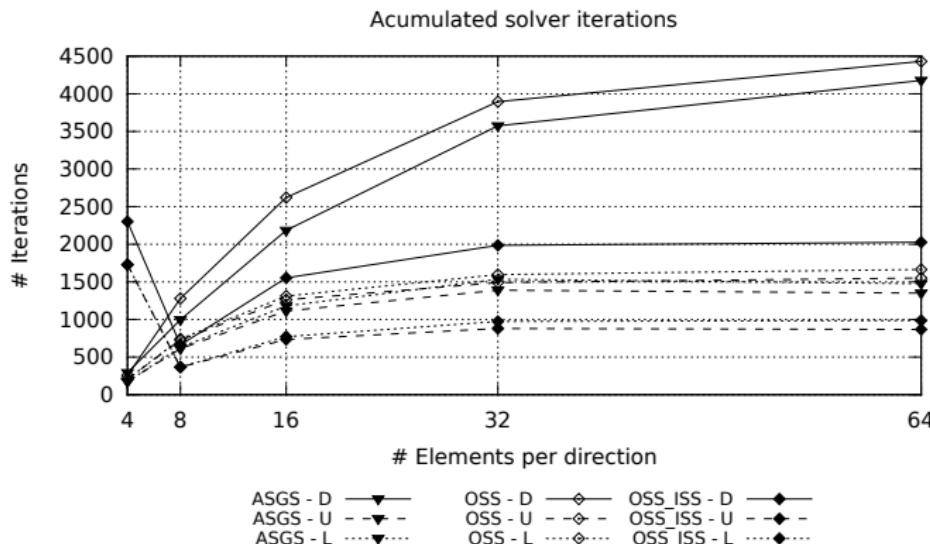
- Analytical solution
- $Re = 25$
- Mesh refinement: 4^3 to 64^3 Q1/Q1 elements (ASGS and OSS) or 2^3 to 32^3 Q2/Q1 elements (OSS-ISS)



Velocity field

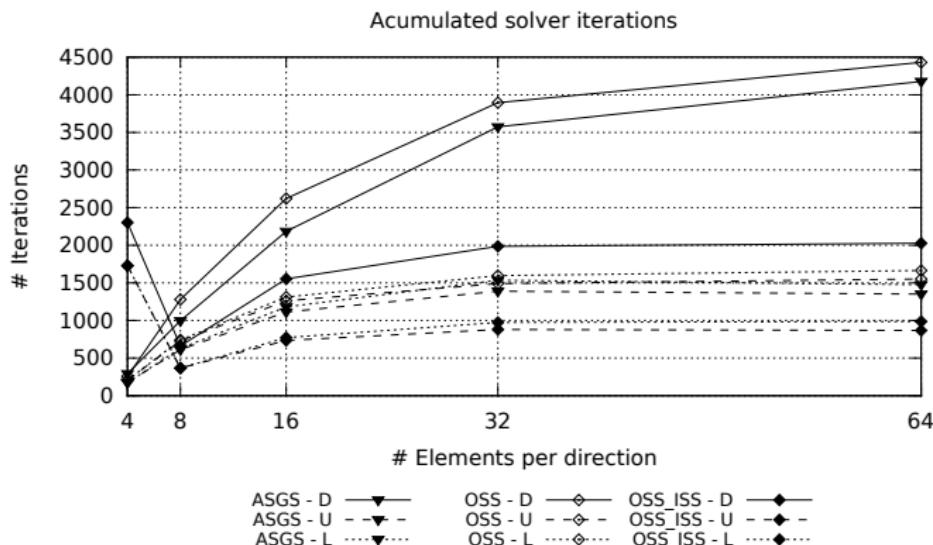
Colliding flow

Accumulated solver iterations: (using $P_U(\tilde{A})$)



Colliding flow

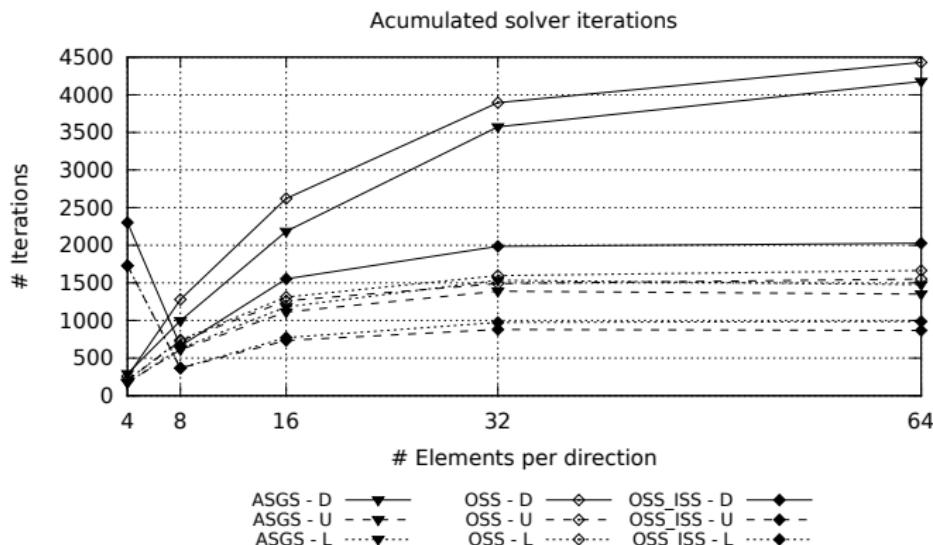
Accumulated solver iterations: (using $P_U(\tilde{A})$)



- $P_U(\tilde{K}_\tau)$ and $P_L(\tilde{K}_\tau)$ optimal block-preconditioners for all methods

Colliding flow

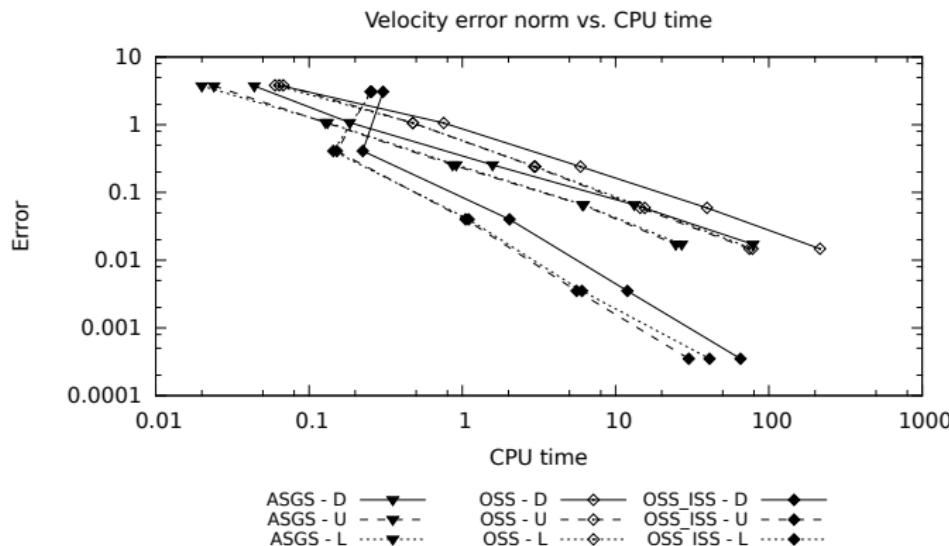
Accumulated solver iterations: (using $P_U(\tilde{A})$)



- $P_U(\tilde{K}_\tau)$ and $P_L(\tilde{K}_\tau)$ optimal block-preconditioners for all methods
- Less solver iterations for the OSS-ISS method with the same velocity DOFs

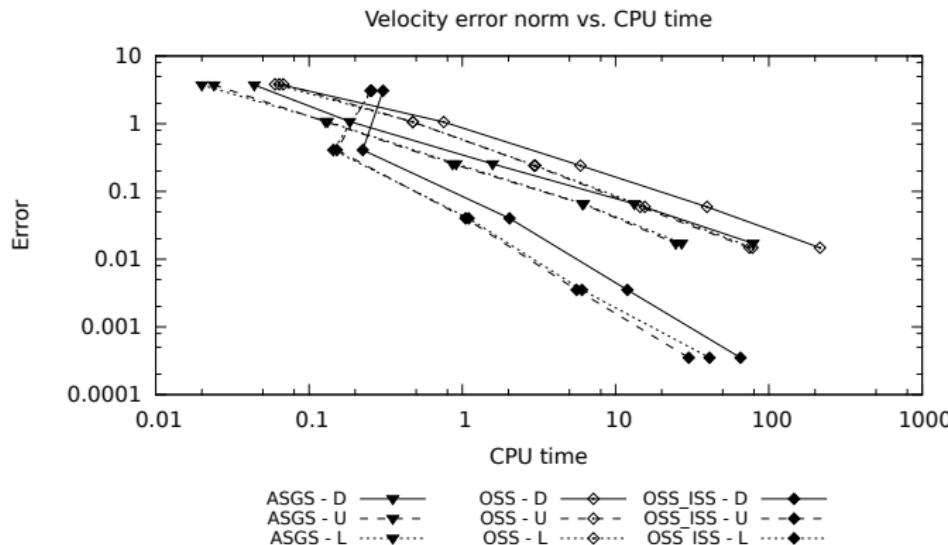
Colliding flow

Efficiency: Velocity



Colliding flow

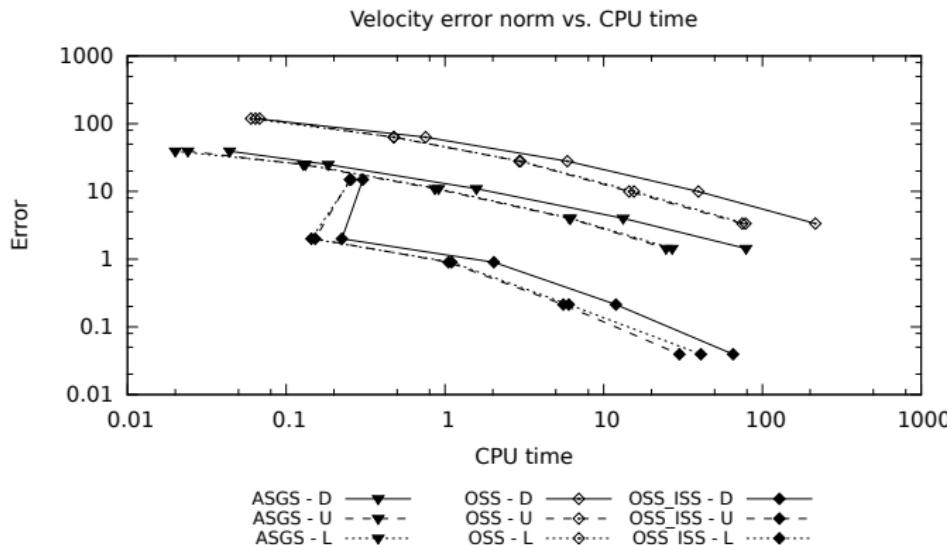
Efficiency: Velocity



- OSS-ISS the most efficient method

Colliding flow

Efficiency: Pressure

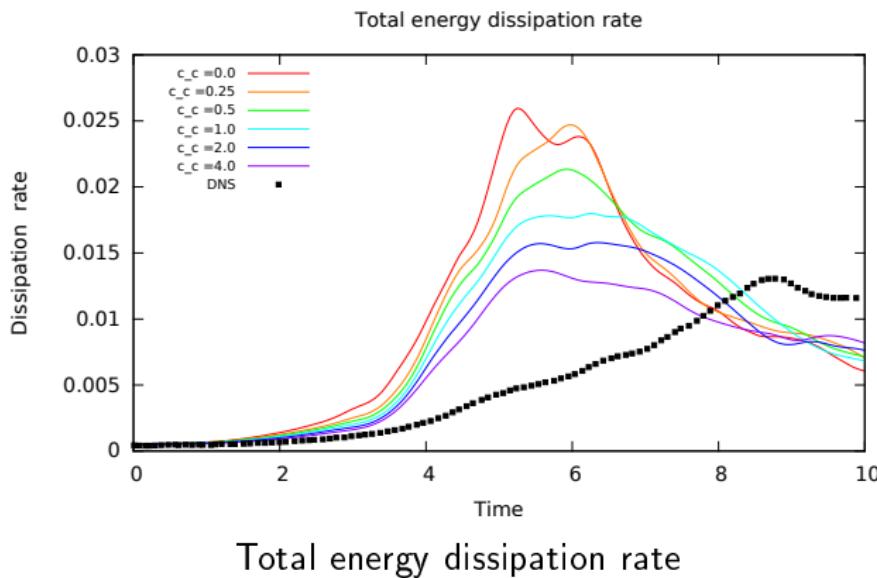


- Also for pressures

The method works properly for the laminar case, but what happens for turbulent problems?

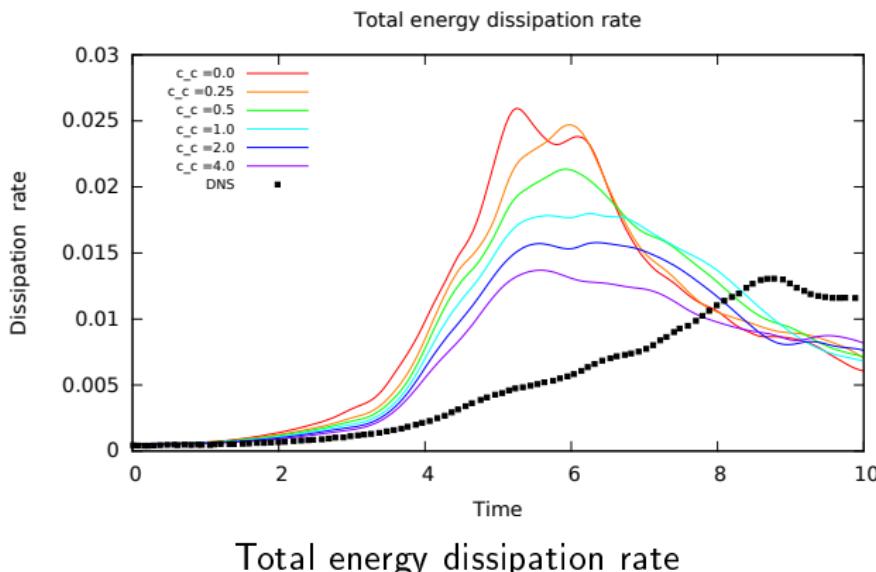
TGV Taylor-Green Vortex flow

Effect of the grad-div term ($\tau_c \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}$): (coarse mesh)



TGV Taylor-Green Vortex flow

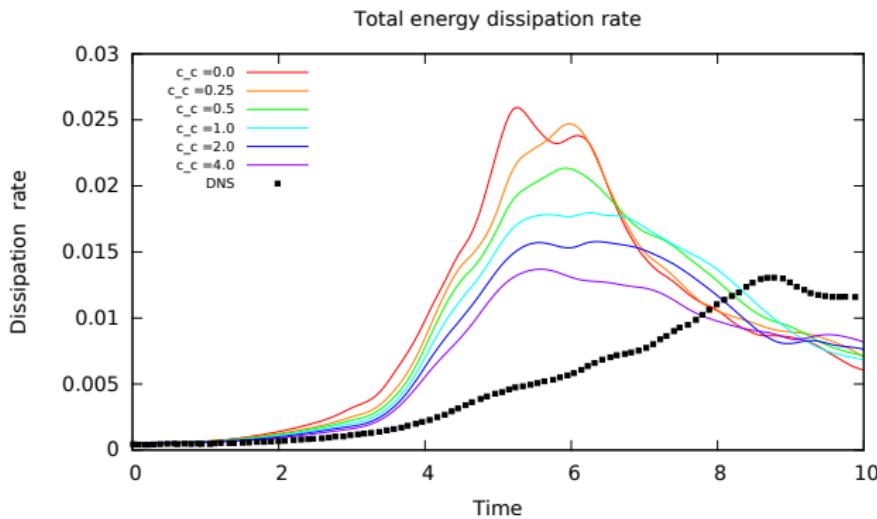
Effect of the grad-div term ($\tau_c \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}$): (coarse mesh)



- Bad results when $c_c \rightarrow 0$

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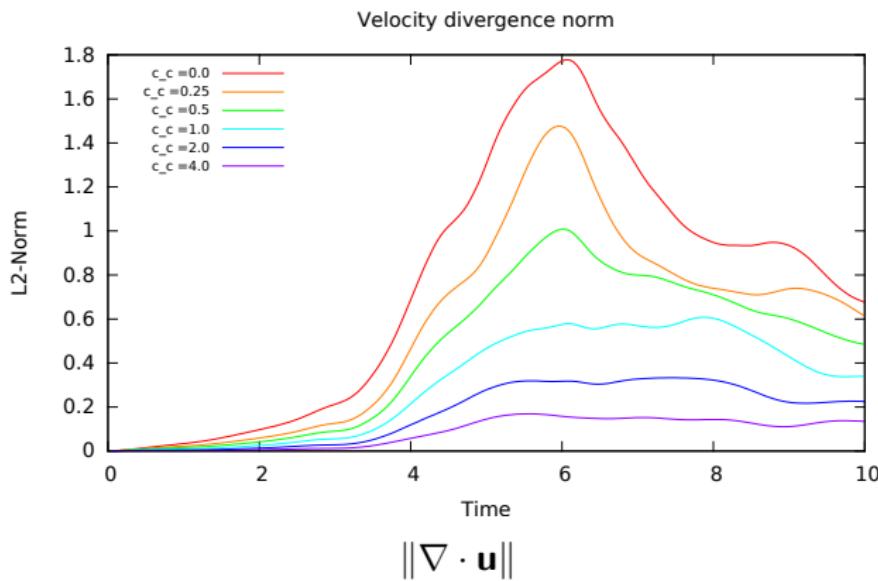


Total energy dissipation rate

- Bad results when $c_c \rightarrow 0$
- Best option $c_c = 4.0$

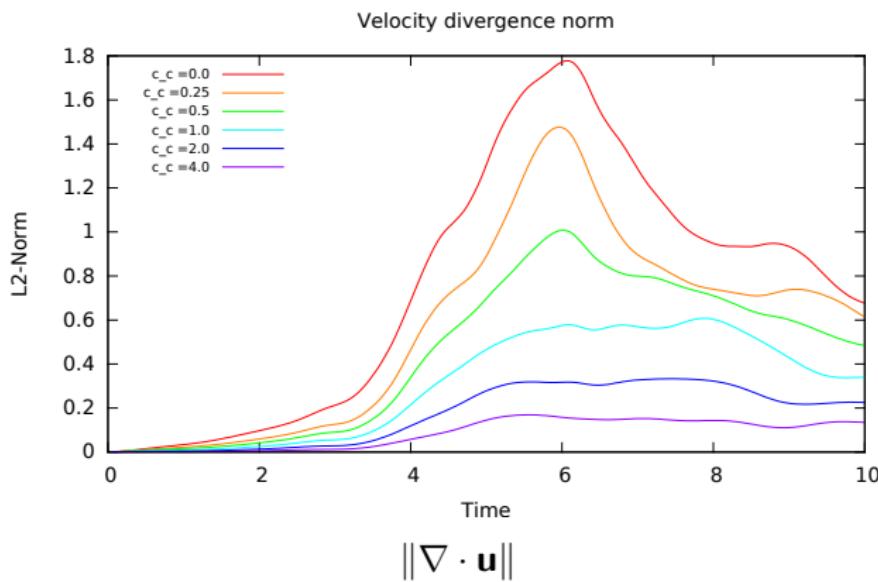
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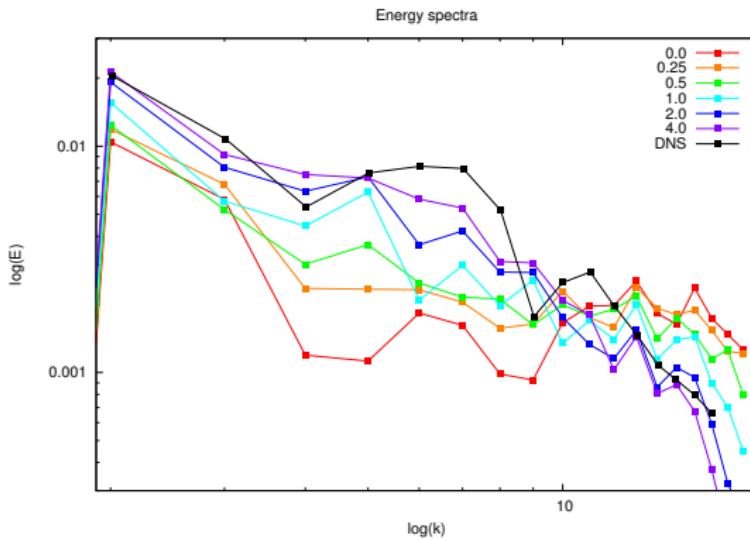
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- Incompressibility constraint not satisfied

TGV Taylor-Green Vortex flow

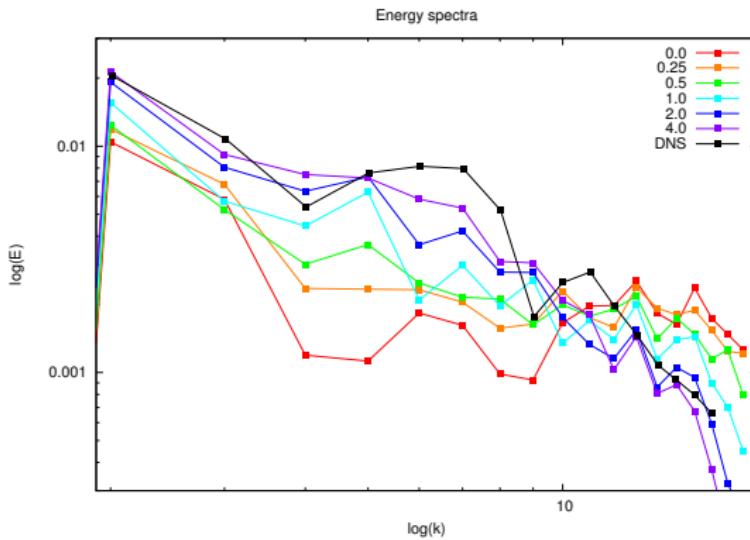
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Energy spectra

TGV Taylor-Green Vortex flow

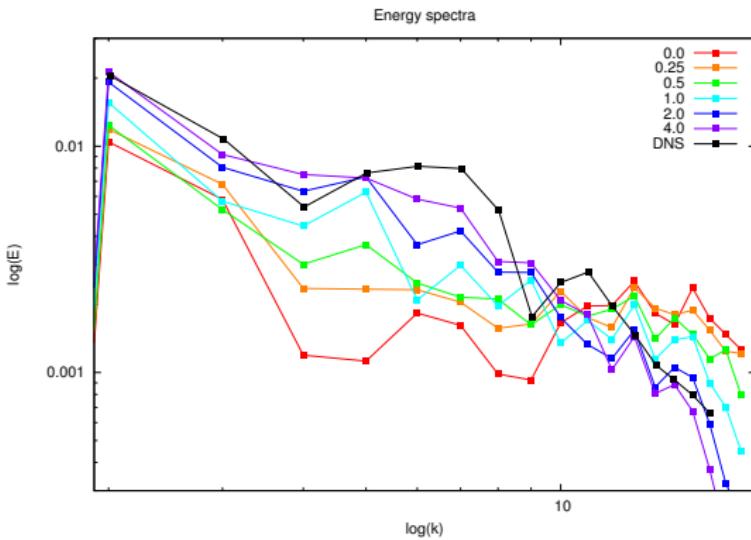
Effect of the grad-div term ($\tau_c \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}$): (coarse mesh)



- Overdissipation on the large scales when $c_c \rightarrow 0$

TGV Taylor-Green Vortex flow

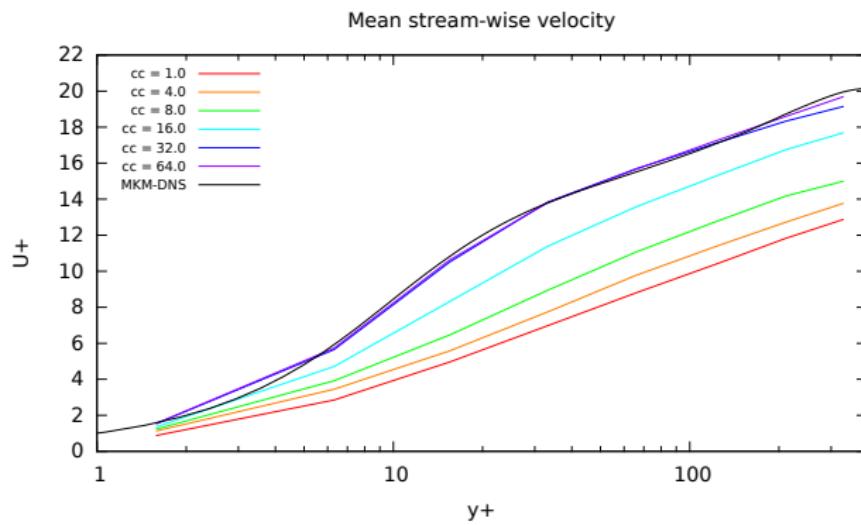
Effect of the grad-div term ($\tau_c \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}$): (coarse mesh)



- Overdissipation on the large scales when $c_c \rightarrow 0$
- Infradissipation on the **small scales** when $c_c \rightarrow 0$

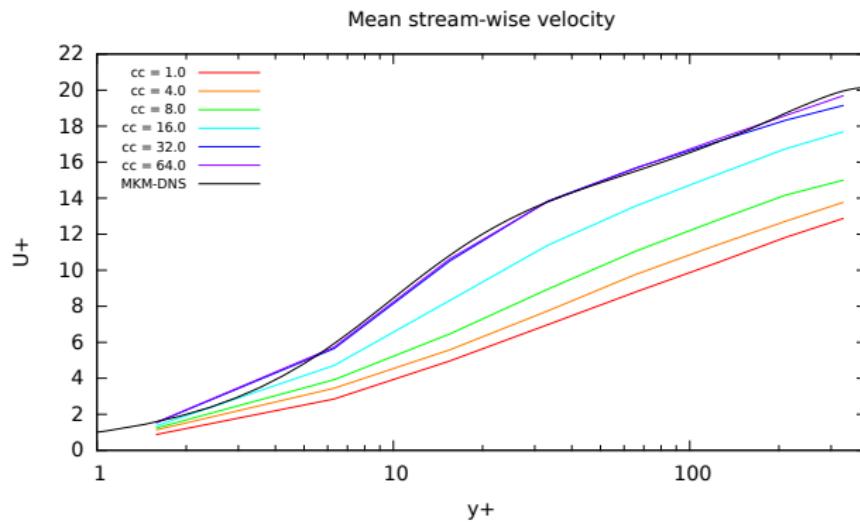
TCF Turbulent Channel Flow at $Re_\tau = 395$

Mean streamwise velocity (models):



TCF Turbulent Channel Flow at $Re_\tau = 395$

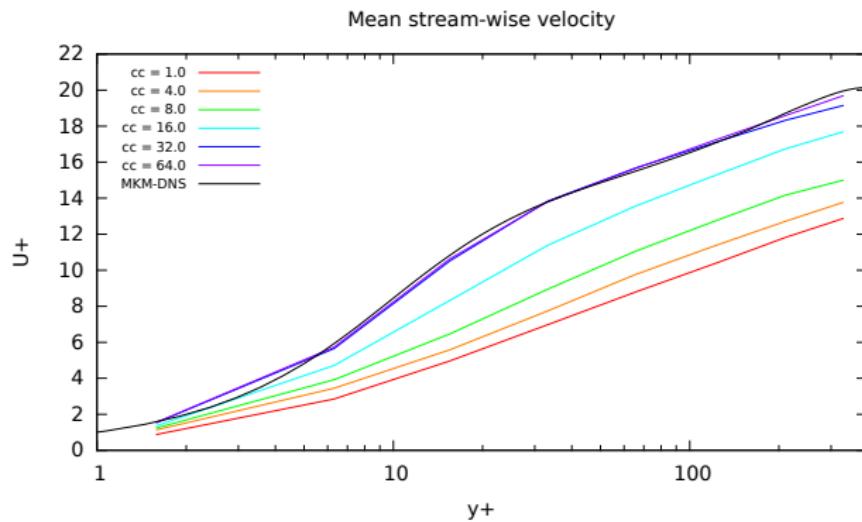
Mean streamwise velocity (models):



- Same behaviour observed for TCF test

TCF Turbulent Channel Flow at $Re_\tau = 395$

Mean streamwise velocity (models):



- Same behaviour observed for TCF test
- Best option $c_c = 32.0$

Mixed FE VMS Conclusions

- Optimal recursive block-preconditioners

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Contribution:

 OC, Santiago Badia and Javier Principe
Mixed finite element methods with convection stabilization for the large eddy simulation of incompressible turbulent flows
Computer Methods in Applied Mechanics and Engineering, 2016

Mixed FE VMS Limitations

- Strong dependency on the grad-div stabilization term

Mixed FE VMS Limitations

- Strong dependency on the grad-div stabilization term
- Optimal choice of c_c is problem dependent

1. Motivation

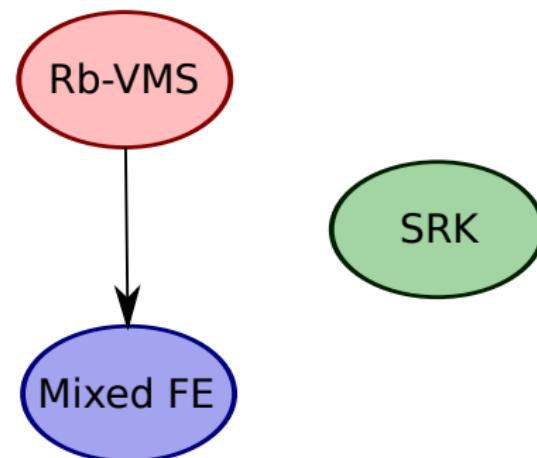
2. Residual-based VMS

3. Mixed FE VMS

4. Segregated Runge-Kutta Formulation Numerical experiments Conclusions

5. Segregated VMS

6. Conclusions



Motivation

Step by step...

- Residual-based VMS as LES models
- Mixed FE formulations LES
- High-order FE methods
- High-order time integration schemes
- Adaptive time stepping techniques
- Velocity-pressure segregation
- Scalable solvers
- Application

Incomp. Navier Stokes equations

Find \mathbf{u} and p defined in Ω

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

with appropriate boundary conditions on Γ

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- **Laminar** case (no convection stabilization required)

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- Saddle-point structure (stability via **mixed FE**)

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- Index-2 DAE system (time integration with care)

Incomp. Navier Stokes equations

Find \mathbf{U} and \mathbf{P}

$$\begin{aligned} M\dot{\mathbf{U}} + (K + C(\mathbf{U}))\mathbf{U} + G\mathbf{P} &= \mathbf{F}, \\ D\mathbf{U} &= \mathbf{G} \end{aligned}$$

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We want to integrate in time with:

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- Velocity-pressure segregation

Velocity-pressure splitting

Existing methods: Fractional step / pressure segregation methods

$$\begin{aligned} \dot{\mathbf{U}} + M^{-1}(K + C(\mathbf{U}))\mathbf{U} + M^{-1}G\mathbf{P}^* &= \mathbf{F}, & \text{in } (t^n, t^{n+1}], \\ -DM^{-1}G\mathbf{P} &= DM^{-1}(K + C(\mathbf{U})\mathbf{U} - \mathbf{F}) + \dot{\mathbf{G}}, & \text{in } t^{n+1}, \end{aligned}$$

with P^* extrapolation from previous time steps

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with P^* extrapolation from previous time steps

High-order time integration:

- Fractional step techniques + (e.g.) Runge-Kutta for momentum equation: [Nikitin'06; Knikker'09; Kampanis et al'06...]
- Fractional step techniques + (e.g.) BDF3 for momentum equation: [Castillo & Codina '15]
- Introduce segregation error (2nd order)

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with P^* extrapolation from previous time steps

Drawbacks:

- Segregation error can jeopardize high order time integrators
- Effect of segregation on pressure error unclear (not reported in literature)

Velocity-pressure splitting

Existing methods: Half-explicit RK methods (HERK)

$$\frac{1}{\delta t} \mathbf{U}_i = \frac{1}{\delta t} \mathbf{U}_n + \sum_{j=1}^{i-1} \hat{a}_{ij} M^{-1} (K \mathbf{U}_j + C(\mathbf{U}_j) \mathbf{U}_j + G \mathbf{P}_j),$$

$$D \mathbf{U}_i = \mathbf{G}(t_i),$$

Velocity-pressure splitting

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$$D\mathbf{U}_n - \delta t \sum_{j=1}^{i-1} \hat{a}_{ij} DM^{-1} (KU_j + C(U_j)U_j + GP_j) = \mathbf{G}(t_i).$$

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High-order time integration:

- Recently applied to the Navier-Stokes equations [Sanderse & Koren '12]
- Velocity-pressure segregation by the time integrator
- No additional splitting needed (high-order feasible, not spoiled)

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Drawbacks:

- All terms must be treated explicitly (viscous + convective terms)
- A straightforward application with implicit/IMEX integrators leads to, e.g, $D(M + \delta t K_V)^{-1} G$ matrix for the pressure (not affordable)

Segregated RK methods

- **Observation:** No implicit / IMEX high-order integrators with implicit velocity-pressure segregation

Segregated RK methods

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Segregated RK methods

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- **Target:** Develop such algorithms (of RK type)
- **Result:** Segregated RK methods [OC & Badia, 2015]

Segregated RK methods

The idea:

1. Consider the projected momentum eq'on on the discretely divergence free space

$$\begin{aligned} M\dot{\mathbf{U}} + (K + C(\mathbf{U}))\mathbf{U} + G\mathbf{P} &= \mathbf{F}, \\ D\mathbf{U} &= \mathbf{G} \end{aligned}$$

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Segregated RK methods

The idea:

1. Consider the projected momentum eq'on on the discretely divergence free space

$$\dot{M\bar{\mathbf{U}}} + P(K + C(\mathbf{U}))\mathbf{U} = P\mathbf{F} + G(DM^{-1}G)^{-1}\dot{\mathbf{G}},$$

with $P := (I - G(DM^{-1}G)^{-1}DM^{-1})$

Segregated RK methods

The idea:

1. Consider the projected momentum eq'on on the discretely divergence free space
2. Integrate the resulting ODE system w/ preferred IMEX RK method (diagonally implicit)...

$$M\dot{\bar{U}} = \mathcal{F}(\bar{U}) + \mathcal{G}(\bar{U}), \quad \mathcal{F} : \text{implicit terms}, \quad \mathcal{G} : \text{explicit ones}$$

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3. \mathcal{G} must include $G(DM^{-1}G)^{-1} \left(DM^{-1}(K + C(\mathbf{U})\mathbf{U} - \mathbf{F}) + \dot{\mathbf{G}} \right)$ for velocity-pressure segregation

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4. The rest of terms can be treated implicitly / explicitly

$$M\dot{\mathbf{U}} = \mathcal{F}(\mathbf{U}) + \mathcal{G}(\mathbf{U}), \quad \mathcal{F} : \text{implicit terms}, \quad \mathcal{G} : \text{explicit ones}$$

Segregated RK methods

At each stage i , the time-discrete algorithm reads:

$$\frac{1}{\delta t} M \mathbf{U}_i = \frac{1}{\delta t} M \mathbf{U}_n + \sum_{j=1}^i a_{ij} \mathcal{F}(\mathbf{U}_j) + \sum_{j=1}^{i-1} \hat{a}_{ij} \mathcal{G}(\mathbf{U}_j)$$

Segregated RK methods

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After re-ordering (pressure again):

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$$-DM^{-1} G(\mathbf{P}_i) = DM^{-1}((K + C(\mathbf{U}_i))\mathbf{U}_i - \mathbf{F}(t_i)) + \dot{\mathbf{G}}(t_i)$$

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$$-DM^{-1} G(\mathbf{P}_i) = DM^{-1}((K + C(\mathbf{U}_i))\mathbf{U}_i - \mathbf{F}(t_i)) + \dot{\mathbf{G}}(t_i)$$

- Viscous / convective can be treated implicitly / explicitly

$$\mathcal{F}(\mathbf{U}) := -K\mathbf{U} + \mathbf{F} - C(\mathbf{U})\mathbf{U}, \quad \tilde{\mathcal{G}}(\mathbf{U}, P) = -GP \quad \text{or}$$

$$\mathcal{F}(\mathbf{U}) := -K\mathbf{U} + \mathbf{F}, \quad \tilde{\mathcal{G}}(\mathbf{U}, P) = -C(\mathbf{U})\mathbf{U} - GP$$

SRK Some properties

- Segregation at the time integration level (no additional splitting)
- High order achievable (the one of the ODE RK integration)
- Implicit LES turbulent model (stabilization terms)
- Equal-order & optimal velocity-pressure methods (of interest in FSI, etc.)
- HERK and SRK methods are *equivalent* for full explicit treatment

Numerical experiments

Manufactured analytical solutions:

- Simple $\sin(t) \cdot \exp(t)$ function

Laminar benchmark:

- 2D Laminar flow around a cylinder

Manufactured analytical solution

Problem setting:

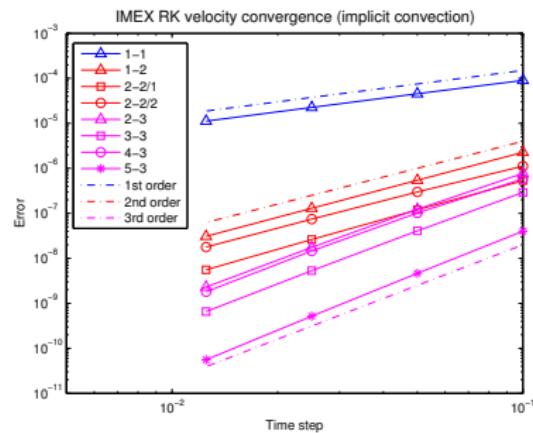
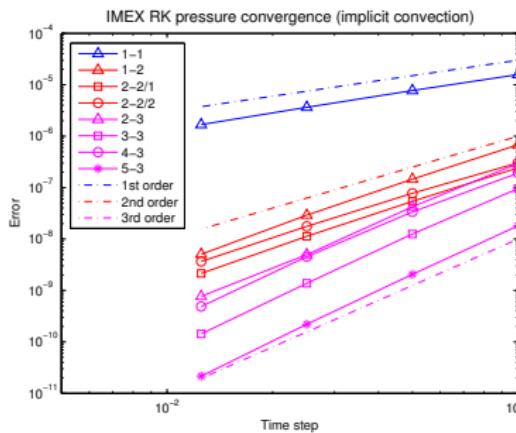
- Analytical solution:

$$\mathbf{u}(x, y, t) = \begin{bmatrix} x \\ -y \end{bmatrix} \sin\left(\frac{\pi}{10}t\right) \exp\left(\frac{t}{25}\right),$$
$$p(x, y) = x + y$$

- $Re = 1/10/100$
- Different IMEX Butcher tableaus with 1st, 2nd and 3rd order

Manufactured analytical solution

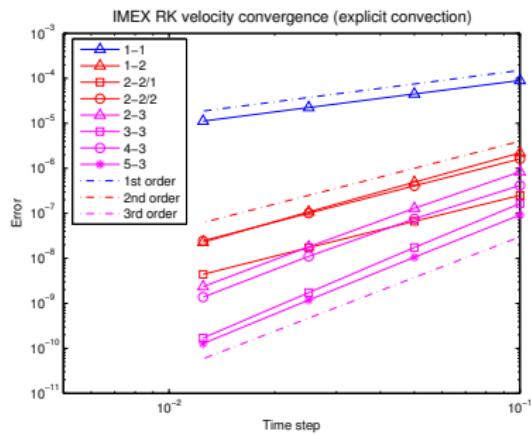
Implicit convection:

(a) Velocity convergence, $\nu = 0.01$ (b) Pressure convergence, $\nu = 0.01$

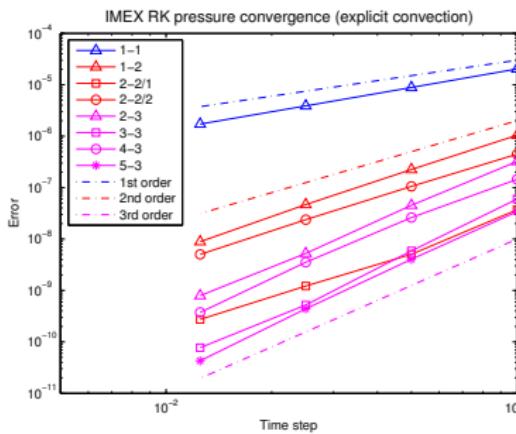
Fully implicit SRK

Manufactured analytical solution

Explicit convection:



(a) Velocity convergence, $\nu = 0.01$



(b) Pressure convergence, $\nu = 0.01$

SRK convergence with convection integrated explicitly and diffusion integrated implicitly

2D Laminar flow around a cylinder

Problem setting:

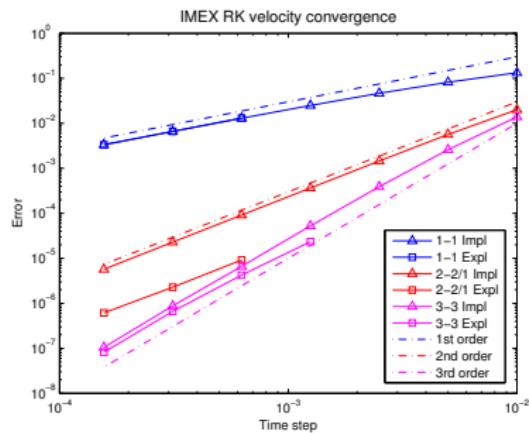
- Widely used benchmark.
- $Re = 100$.
- Time convergence.
- Drag and Lift coefficients.



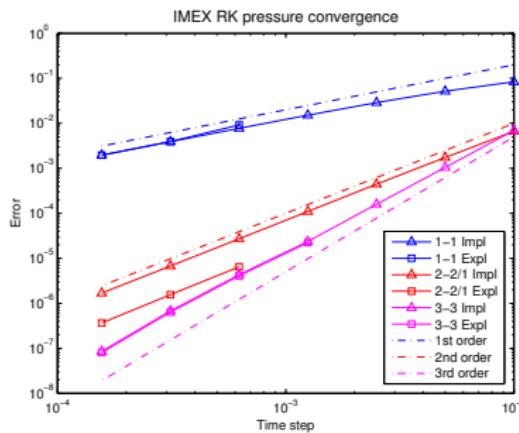
Vorticity field at $t = 8.0$.

2D Laminar flow around a cylinder

Implicit & explicit convection:



(a) Velocity convergence



(b) Pressure convergence

Fully implicit and IMEX-SRK convergence rate comparison.

Segregated Runge-Kutta Conclusions

- Velocity-pressure segregated by the time integrator (IMEX RK)

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Contribution:



OC and Santiago Badia

Segregated Runge-Kutta schemes for the incompressible Navier-Stokes equations

International Journal for Numerical Methods in Engineering, 2016

Segregated Runge-Kutta Limitations

- Only for index-2 DAE

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- Only for index-2 DAE
- Not applicable for equal-order stabilization methods

1. Motivation

2. Residual-based VMS

3. Mixed FE VMS

4. Segregated Runge-Kutta

5. Segregated VMS

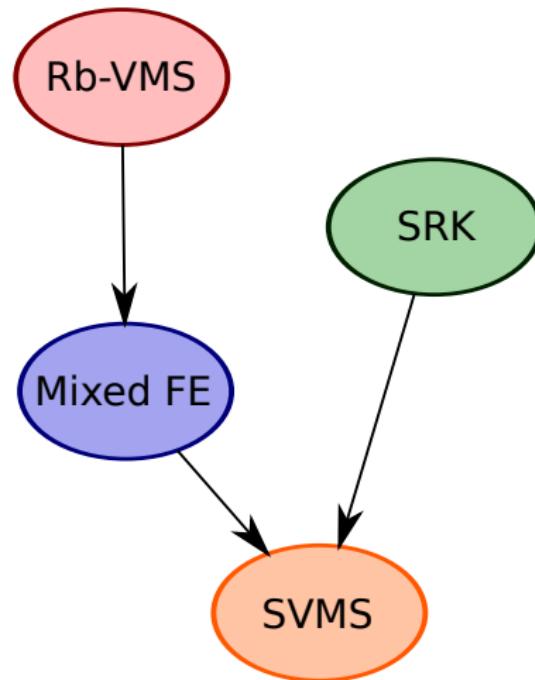
Formulation

Large-scale solvers

Numerical experiments

Conclusions

6. Conclusions



Motivation

Step by step...

- Residual-based VMS as LES models
- Mixed FE formulations LES
- High-order FE methods
- High-order time integration schemes
- Adaptive time stepping techniques
- Velocity-pressure segregation
- Scalable solvers
- Application

Looking backward...

VMS as LES models: Residual-based VMS methods

- Velocity + pressure stabilization, applied to equal-order FEs
- Intensively tested methods (several works in the literature)
- Nature of the problem changes, not DAE-2 type (!)
- It prevents us to use segregated RK methods

Looking backward...

VMS as LES models: Mixed FE VMS methods

- Only the term that we need, i.e., convection stabilization
- To keep accuracy, we use orthogonal projections (\mathcal{P}_h^\perp)
- Many choices for the \mathcal{P}_h projector: local, global (OSS)
- Discrete problem still index-2 DAE
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Now, we can use Segregated RK methods for ILES of turbulent flows

Formulation

OSS-ISS: Semi-discrete form

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C(\mathbf{U}) + A_\tau & G & B_\tau \\ D & 0 & 0 \\ -B_\tau^T & 0 & M_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{G} \\ \mathbf{0} \end{bmatrix},$$

Formulation

OSS-ISS: Semi-discrete form

$$\begin{aligned} M\dot{\mathbf{U}} + (K + C(\mathbf{U}) + A_\tau)\mathbf{U} + G\mathbf{P} + B_\tau\boldsymbol{\Upsilon} &= \mathbf{F}_u, \\ M_\tau\boldsymbol{\Upsilon} - B_\tau^T\mathbf{U} &= \mathbf{0}, \\ D\mathbf{U} &= \mathbf{G}. \end{aligned}$$

Formulation

OSS-ISS:

1. Consider the projected momentum eq'on on the discretely divergence free space

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1. Consider the projected momentum eq'on on the discretely divergence free space
2. Integrate the resulting ODE system w/ preferred IMEX RK method (diagonally implicit)...

$$\dot{M\mathbf{U}} = \mathcal{F}(\mathbf{U}, \boldsymbol{\Upsilon}) + \mathcal{G}(\mathbf{U}, \mathbf{P}), \quad \mathcal{F} : \text{implicit terms, } \mathcal{G} : \text{explicit ones}$$

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$$G(DM^{-1}G)^{-1} \left(DM^{-1}(K + C(\mathbf{U}) + A_\tau)\mathbf{U} + B_\tau \boldsymbol{\Upsilon} - \mathbf{F}_u \right) + \dot{\mathbf{G}} \quad \text{for velocity-pressure segregation}$$

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Final discrete problem

At each stage of the Runge-Kutta scheme: (explicit convective term and projection version)

$$\frac{1}{\delta t} M \mathbf{U}_i = \frac{1}{\delta t} M \mathbf{U}_n + \sum_{j=1}^i a_{ij} \mathcal{F}(\mathbf{U}_j) + \sum_{j=1}^{i-1} \hat{a}_{ij} \mathcal{G}(\mathbf{U}_j, \boldsymbol{\Upsilon}_j, \mathbf{P}_j),$$

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- **Mass-type** matrix for the projection equation

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- Elasticity-type matrix for the momentum equation
- Mass-type matrix for the projection equation
- **Darcy-type** system for the pressure equation

Large scale simulations

Momentum system matrix: $\frac{1}{\delta t} M + a_{ii} K$

- Multilevel/overlapped Balancing domain decomposition (BDDC) for $M + \delta t K$
- Excellent weak scalability results (up to $\sim 460,000$ cores) [Badia *et al* '15]

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Pressure system matrix: $DM^{-1}G$

- Spectrally equivalent to standard Laplacian L_p
- Written as

$$\begin{bmatrix} M & G \\ D & 0 \end{bmatrix} \quad \text{preconditioned w/} \quad \begin{bmatrix} M & 0 \\ 0 & L_p \end{bmatrix}$$

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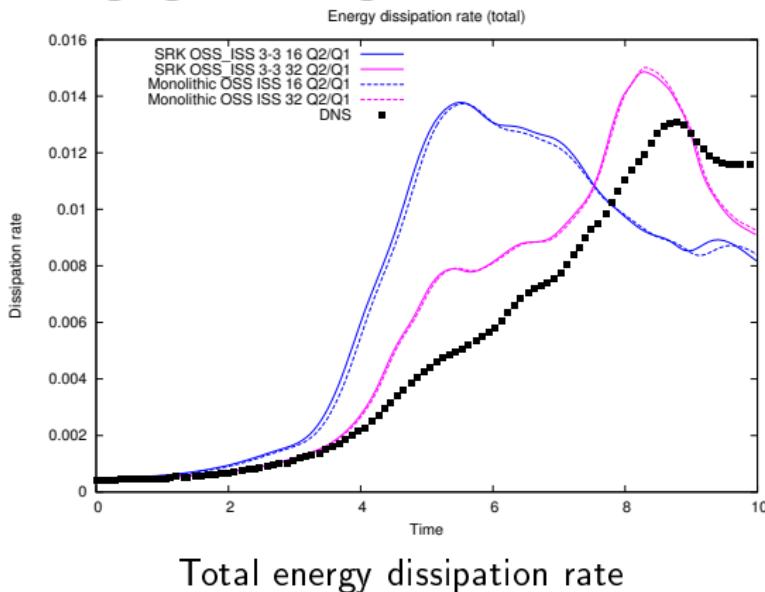
Numerical experiments

Three different turbulent benchmarks:

- Taylor-Green Vortex (TGV) flow
- Turbulent Channel Flow (TCF)
- Flow around a NACA profile

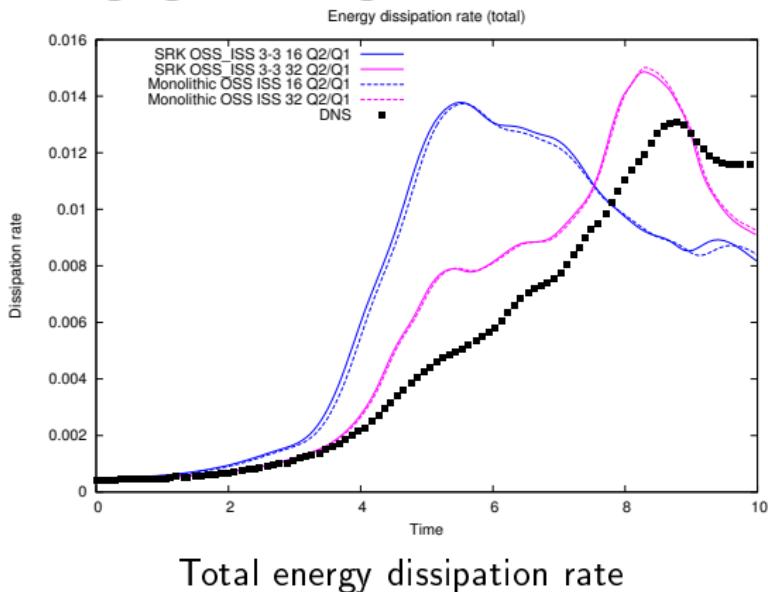
TGV Taylor-Green Vortex flow

Monolithic vs Segregated Runge-Kutta:



TGV Taylor-Green Vortex flow

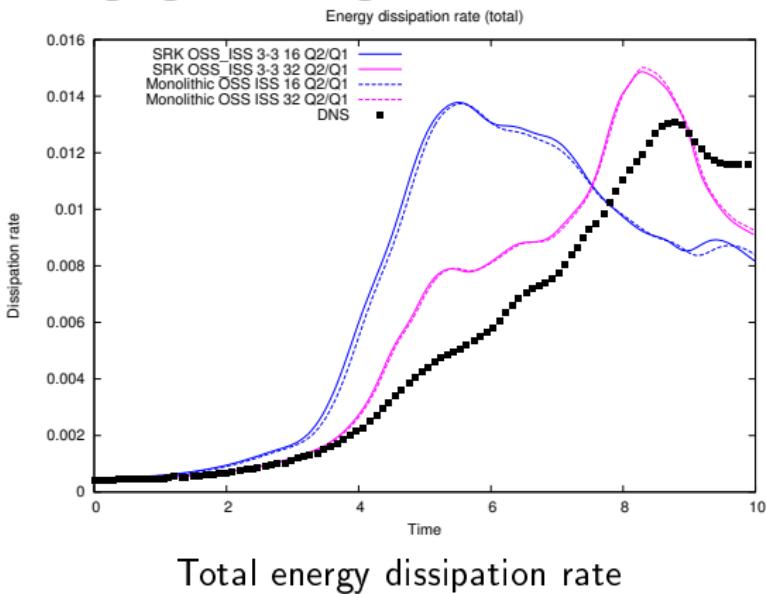
Monolithic vs Segregated Runge-Kutta:



- Almost **Identical results** obtained with Crank-Nicolson

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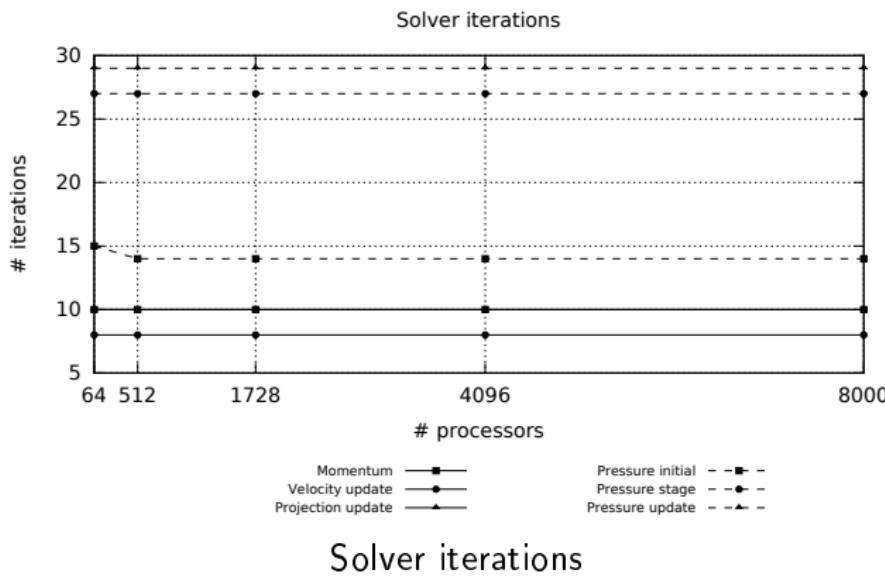
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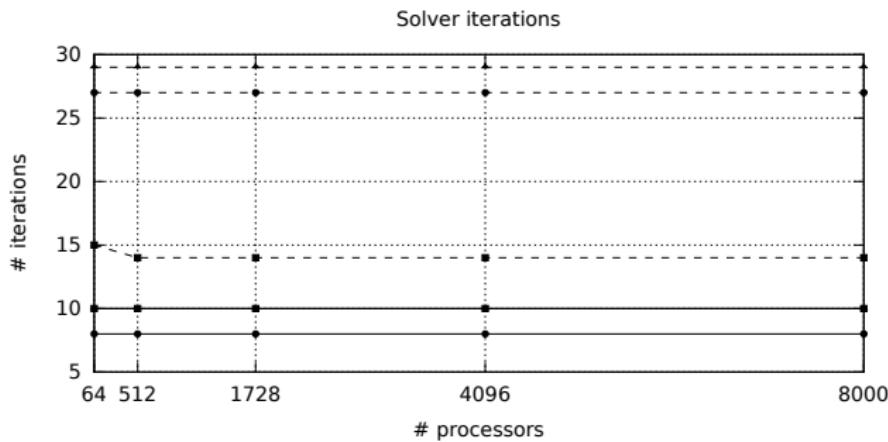
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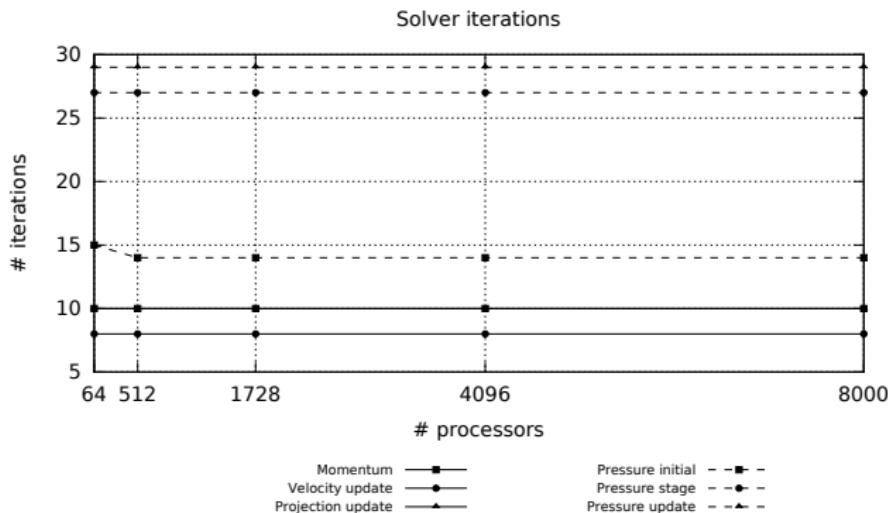


Solver iterations

- Perfect scalability in terms of solver iterations

TGV Taylor-Green Vortex flow

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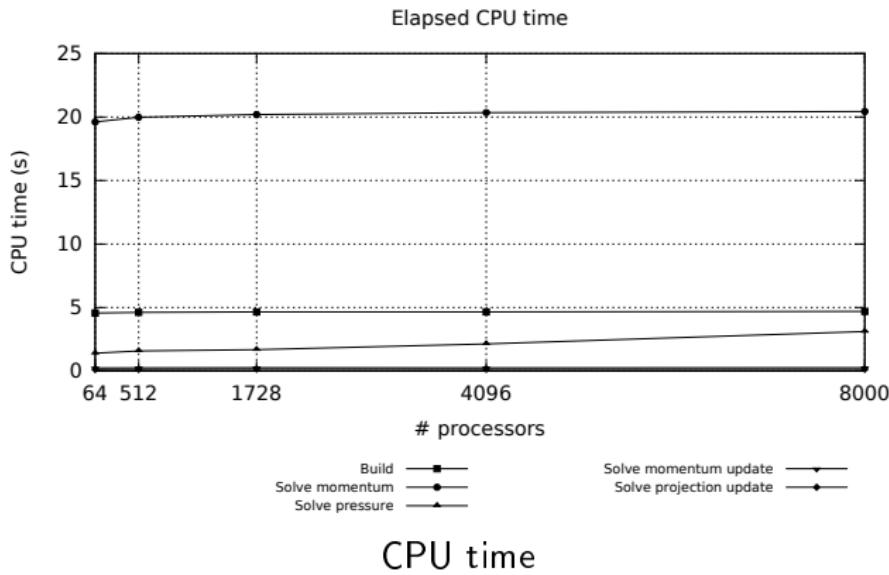


Solver iterations

- Perfect scalability in terms of solver iterations
- Darcy-type problem requires **more iterations**

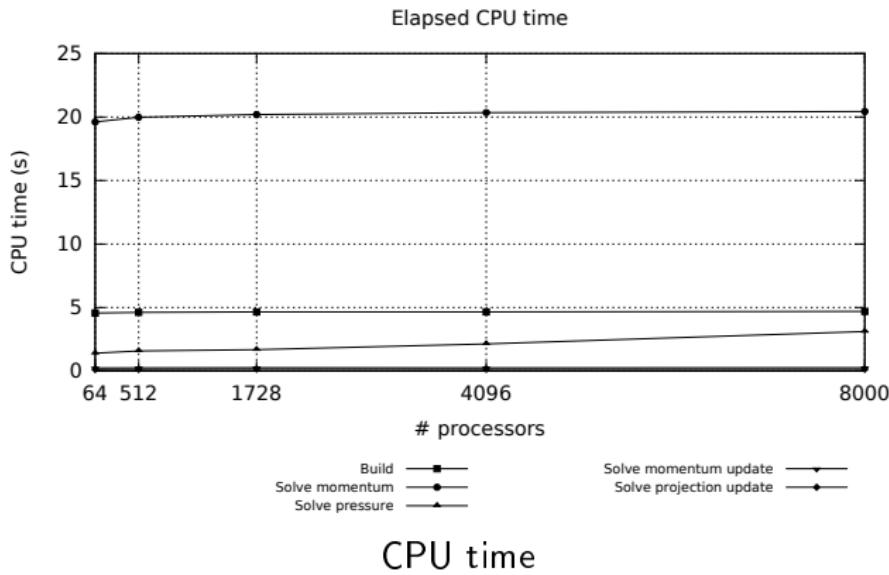
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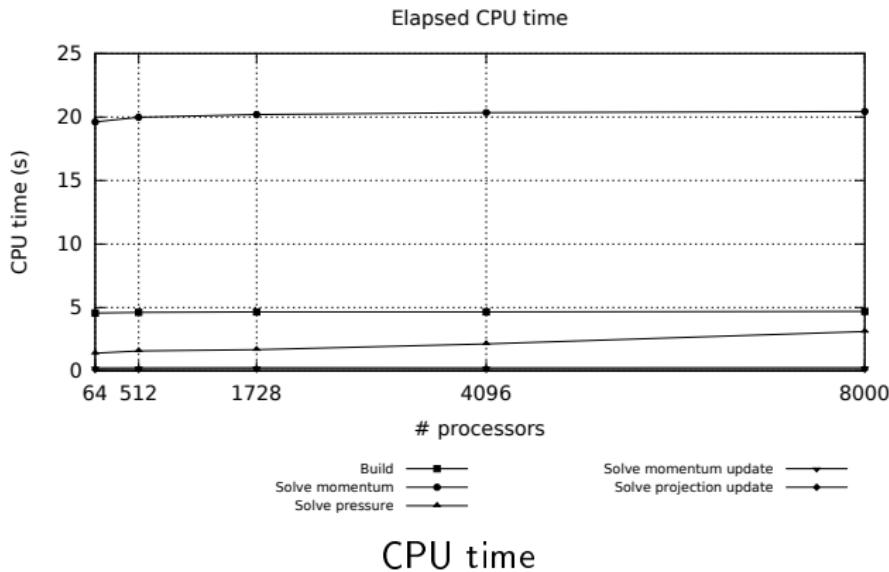
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- Perfect scalability in CPU time consumed

TGV Taylor-Green Vortex flow

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- Perfect scalability in CPU time consumed
- Momentum problem requires more time

TCF Turbulent Channel Flow at $Re_\tau = 395$

LES for wall bounded flows?

Problem:

- LES models cannot accurately simulate boundary layers without stretched meshes

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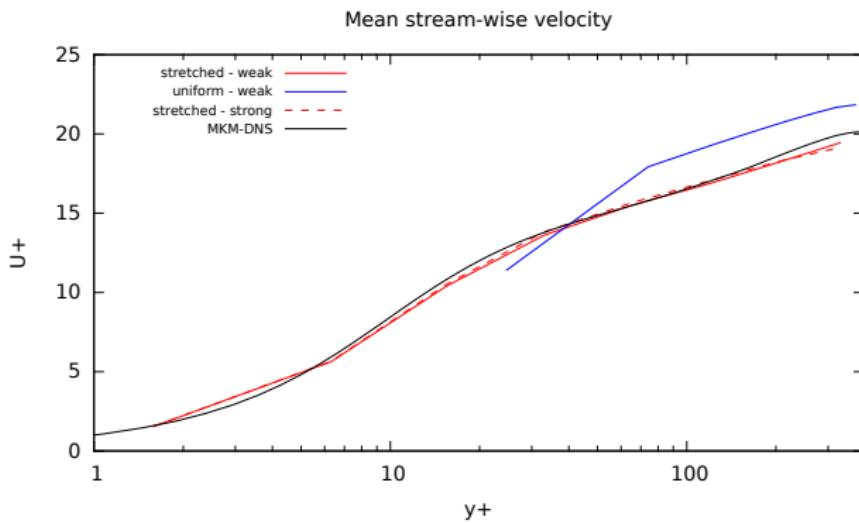
- LES models cannot accurately simulate boundary layers without stretched meshes

Solution:

- Use of weak boundary conditions (Nitche's method)
- Impose the tangential traction on the wall from a wall law model [Bazilevs et al '07]
- We also impose weakly the wall-normal component

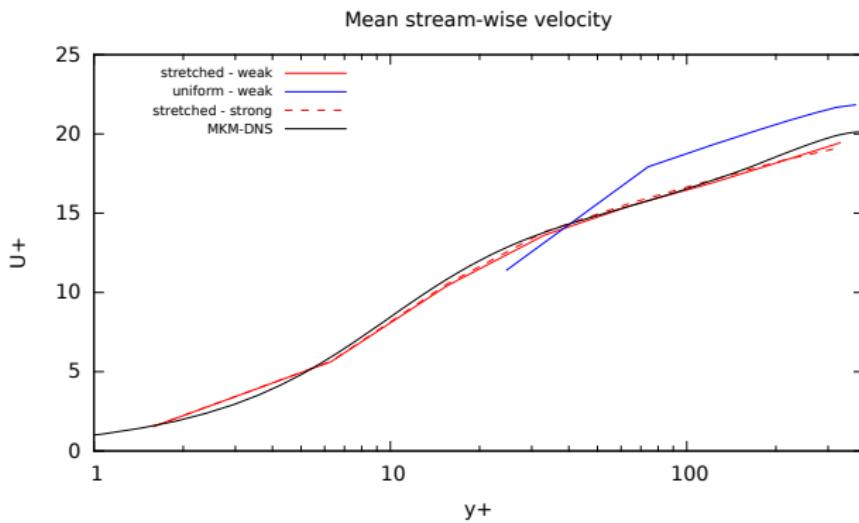
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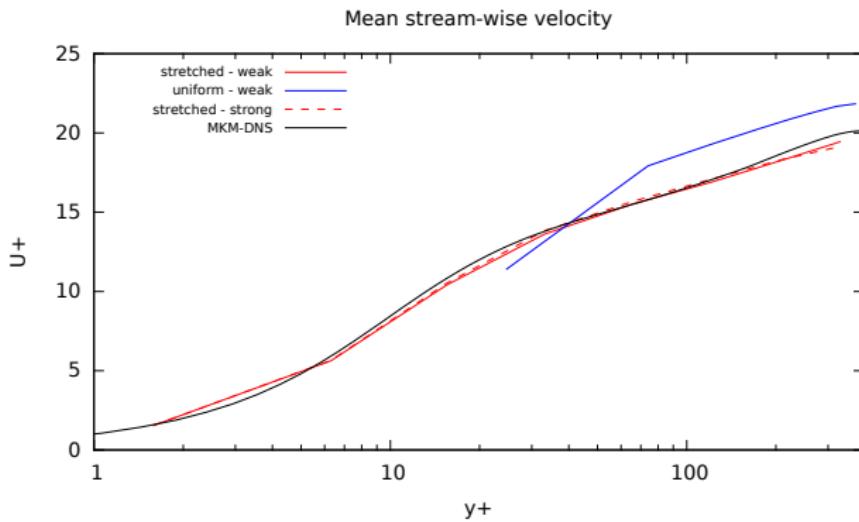
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- We recover the strong BC results when stretched mesh is used

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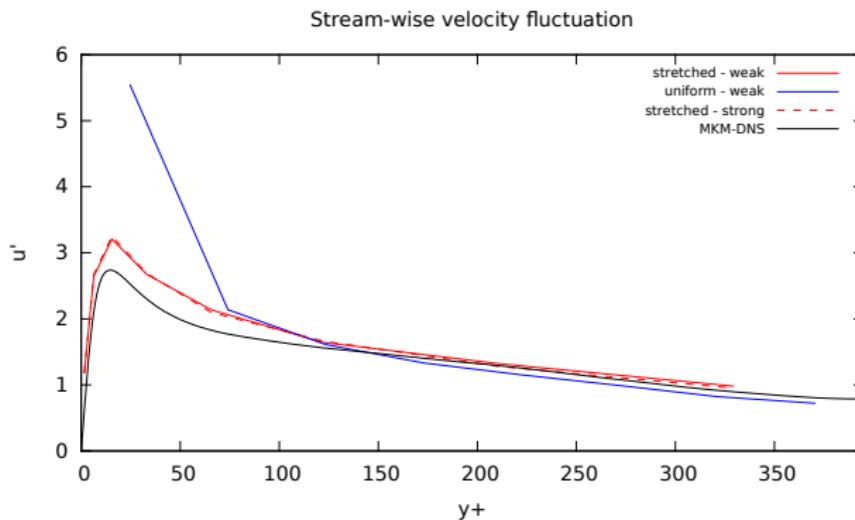
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- **Acceptable results for uniform meshes**

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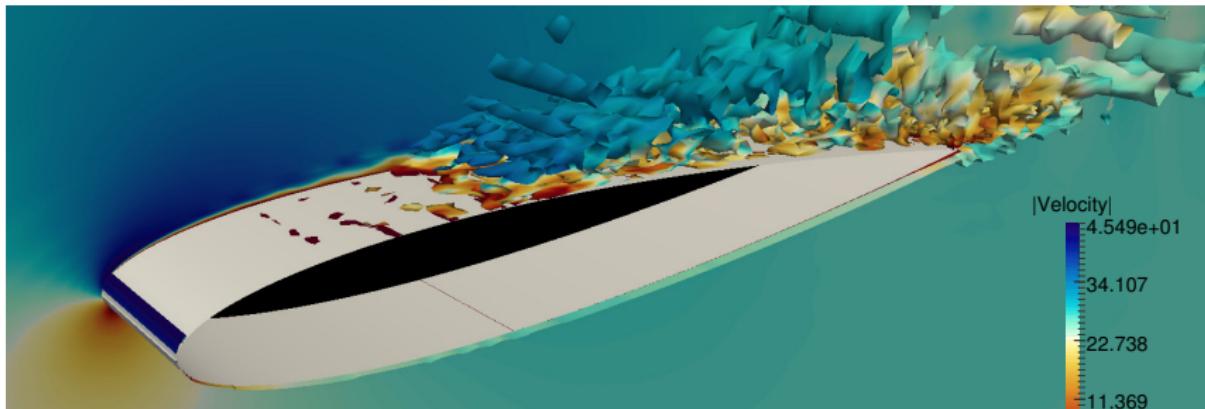
Weak boundary conditions:



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- Acceptable results for uniform meshes

Turbulent flow around a NACA profile

- Low Reynolds number, $Re = 23000$
- Weak boundary conditions **also on the wall-normal component**
- Coarse mesh



Q-criterion isosurfaces for a fully developed flow

Segregated VMS Conclusions

- Implicit LES for mixed FEs + convection stabilization
 - Similar behavior as VMS-type solvers
 - Not affecting nature of the system (index-2 DAE)
- Large scale computations
 - Segregation leads to coercive blocks to be solved
 - Highly scalable multilevel/overlapped implementation of BDDC
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Contribution:



OC and Santiago Badia

Segregated Runge-Kutta time integration of convection-stabilized mixed finite element schemes for wall-unresolved LES of incompressible flows

Submitted, 2016

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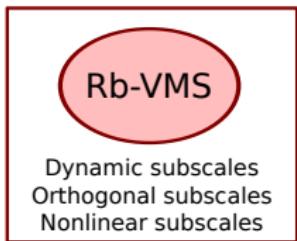
3. Mixed FE VMS

4. Segregated Runge-Kutta

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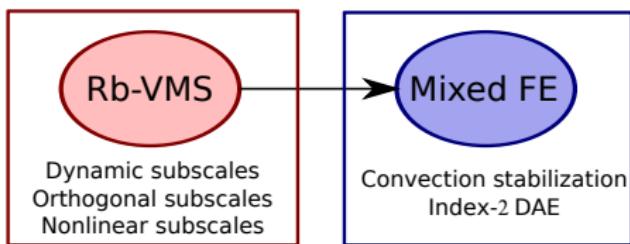
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Final Conclusions



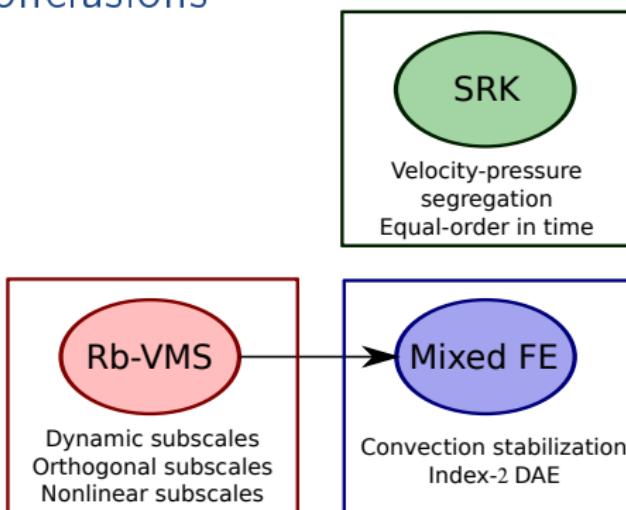
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Final Conclusions



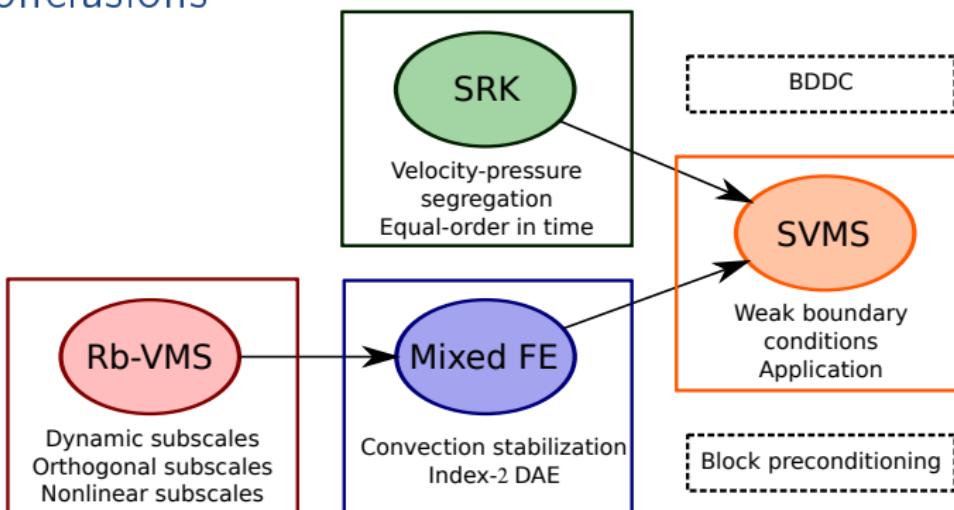
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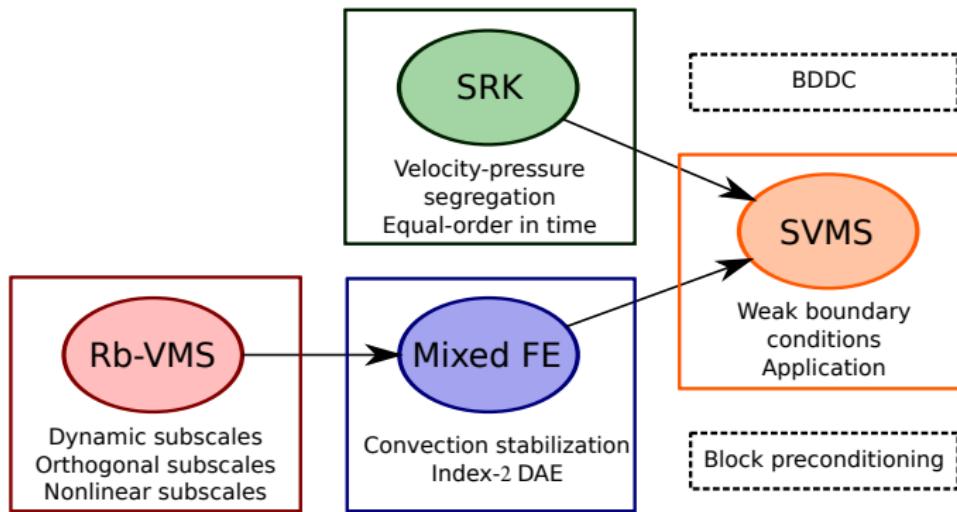
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- Defined Mixed FE methods with convection stabilization
- Developed novel Segregated Runge-Kutta methods
- **Highly-scalable FE solvers** for the simulation of turbulent incompressible flows

Final Conclusions



Thesis goal

Highly scalable Finite Element (FE) framework for Large Eddy Simulations (LES) of incompressible turbulent flows

Thank you!

-  OC, Santiago Badia, Ramon Codina and Javier Principe
Assessment of variational multiscale models for the large eddy simulation of turbulent incompressible flows
Computer Methods in Applied Mechanics and Engineering, 2015
-  OC and Santiago Badia
Segregated Runge-Kutta schemes for the incompressible Navier-Stokes equations
International Journal for Numerical Methods in Engineering, 2016
-  OC, Santiago Badia and Javier Principe
Mixed finite element methods with convection stabilization for the large eddy simulation of incompressible turbulent flows
Computer Methods in Applied Mechanics and Engineering, 2016
-  OC and Santiago Badia
Segregated Runge-Kutta time integration of convection-stabilized mixed finite element schemes for wall-unresolved LES of incompressible flows
Submitted, 2016

Rb-VMS Summary

	Sgs space	Sgs dynamics	Advection
1	ASGS	Static	Linear
2	ASGS	Dynamic	Linear
3	ASGS	Dynamic	Nonlinear
4	OSS	Static	Linear
5	OSS	Dynamic	Linear
6	OSS	Dynamic	Nonlinear

- 1 It is the most standard method (SUPG for linear elements) up to the choice of the stabilization parameters. Unknown stability properties.
- 4 Strictly pointwise positive for linear elements (no backscatter).
- 5 Convergent to weak solutions of NS equations (Badia & Gutierrez 2012).

Energy statements

FE counterpart:

$$B((\mathbf{u}_h, p_h); (\tilde{\mathbf{u}}, \tilde{p}); (\mathbf{u}_h, p_h)) = L(\mathbf{u}_h, p_h)$$

SGS counterpart:

$$B((\tilde{\mathbf{u}}, \tilde{p}); (\mathbf{u}_h, p_h); (\tilde{\mathbf{u}}, \tilde{p})) = L(\tilde{\mathbf{u}}, \tilde{p})$$

TOTAL:

$$\begin{aligned} & B((\mathbf{u}_h, p_h); (\tilde{\mathbf{u}}, \tilde{p}); (\mathbf{u}_h, p_h)) \\ & + B((\tilde{\mathbf{u}}, \tilde{p}); (\mathbf{u}_h, p_h); (\tilde{\mathbf{u}}, \tilde{p})) = L(\mathbf{u}_h, p_h) + L(\tilde{\mathbf{u}}, \tilde{p}) \end{aligned}$$

Energy statements

FE counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle \end{aligned}$$

SGS counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

TOTAL:

Energy statements

FE counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle \end{aligned}$$

SGS counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

TOTAL:

$$\frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2$$

Energy statements

FE counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle \end{aligned}$$

SGS counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

TOTAL:

$$\frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \nu \|\nabla \mathbf{u}_h\|^2$$

Energy statements

FE counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle \end{aligned}$$

SGS counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

TOTAL:

$$\frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \nu \|\nabla \mathbf{u}_h\|^2$$

Energy statements

FE counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle \end{aligned}$$

SGS counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

TOTAL:

$$\frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2$$

Energy statements

FE counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle \end{aligned}$$

SGS counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

TOTAL:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) \\ & - (\nabla \cdot \mathbf{u}_h, \tilde{p}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) \end{aligned}$$

Energy statements

FE counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle \end{aligned}$$

SGS counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

TOTAL:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) \\ & - (\nabla \cdot \mathbf{u}_h, \tilde{p}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle + \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

Energy statements

FE counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle \end{aligned}$$

SGS counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

TOTAL: Static subscales

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) \\ & - (\nabla \cdot \mathbf{u}_h, \tilde{p}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle + \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

Energy statements

FE counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle \end{aligned}$$

SGS counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

TOTAL: Dynamic subscales - ASGS

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h + \tilde{\mathbf{u}}\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & - 2(\nu \Delta \mathbf{u}_h, \tilde{\mathbf{u}}) = \langle \mathbf{f}, \mathbf{u}_h \rangle + \langle \mathbf{f}, \tilde{\mathbf{u}} \rangle \end{aligned}$$

Energy statements

FE counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle \end{aligned}$$

SGS counterpart:

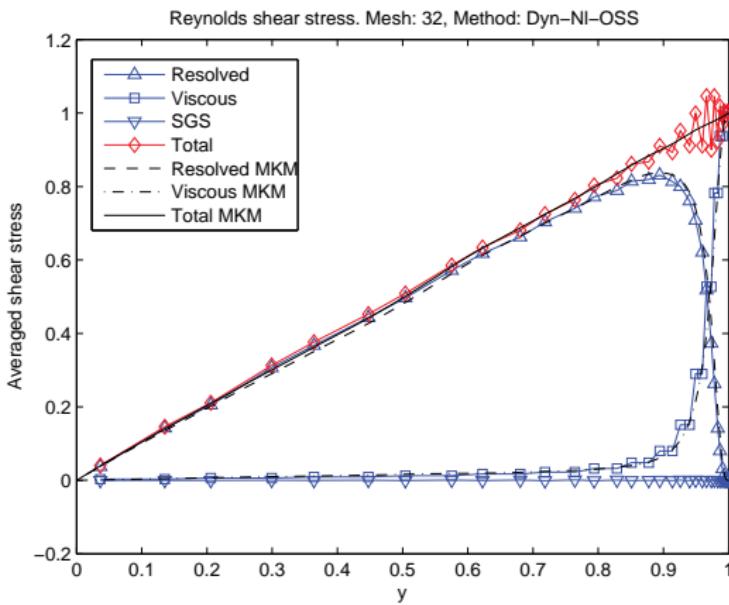
$$\begin{aligned} & \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

TOTAL: Dynamic subscales - OSS

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & - 2(\nu \Delta \mathbf{u}_h, \tilde{\mathbf{u}}) \\ & = \langle \mathbf{f}, \mathbf{u}_h \rangle + \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

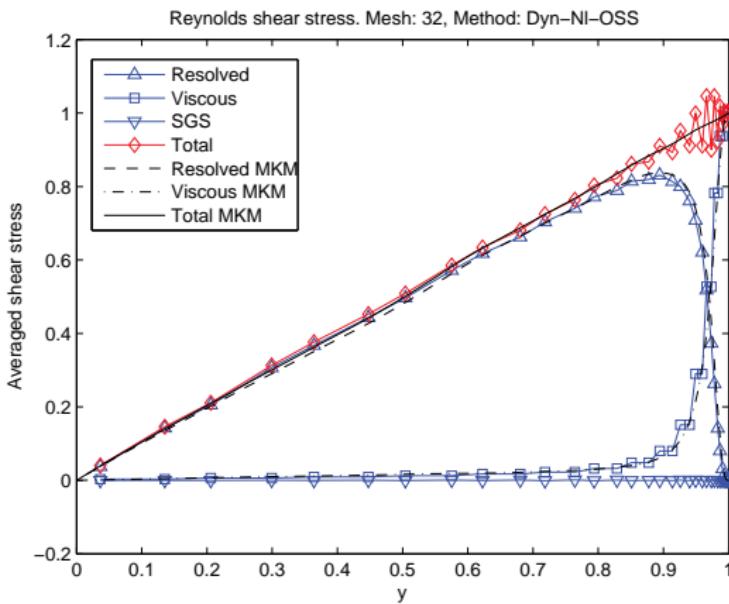
Rb-VMS TCF Turbulent Channel Flow

Reynolds shear stress (models):



Rb-VMS TCF Turbulent Channel Flow

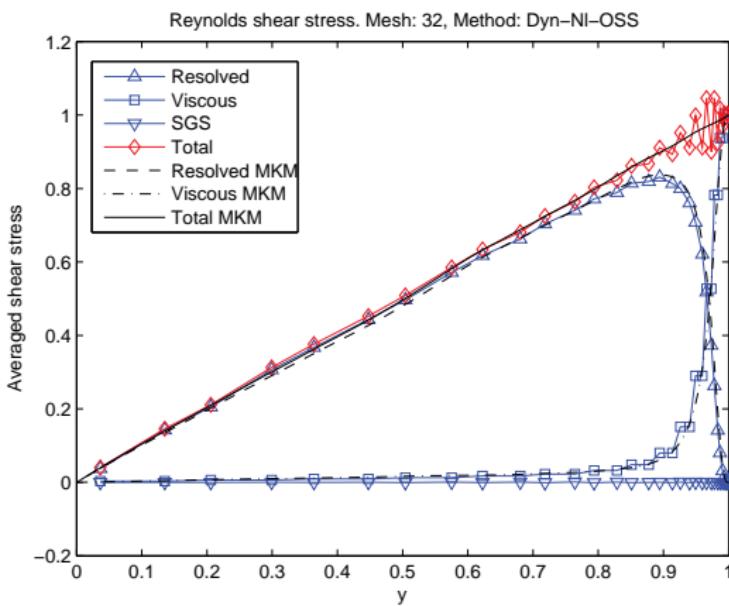
Reynolds shear stress (models):



- Almost identical to the DNS.

Rb-VMS TCF Turbulent Channel Flow

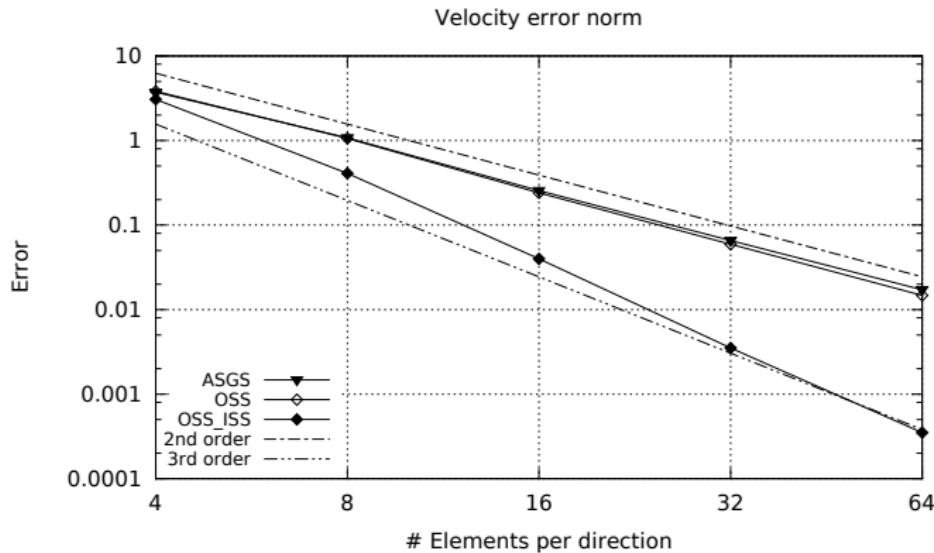
Reynolds shear stress (models):



- Almost identical to the DNS.
- SGS counterpart does not contribute to the Reynolds shear stress.

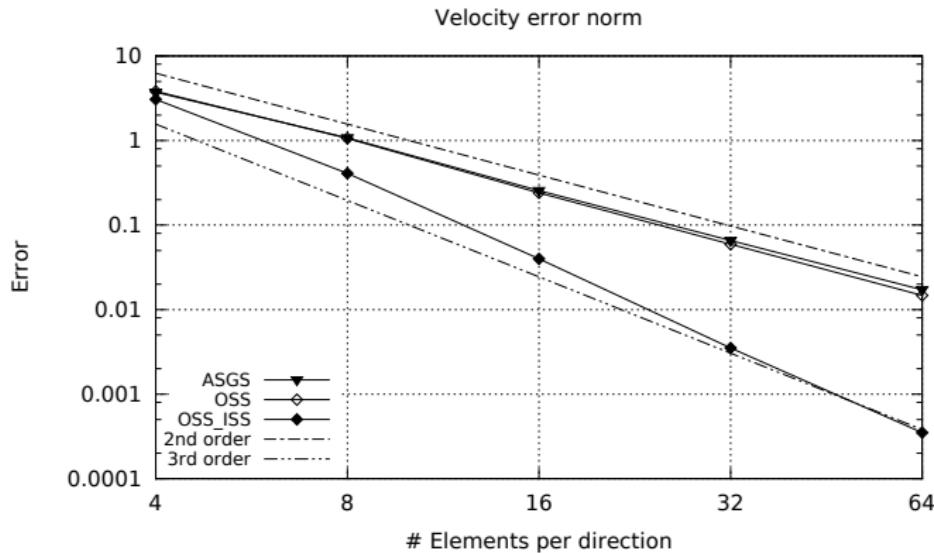
Mixed FE Colliding flow

Accuracy: Velocity error



Mixed FE Colliding flow

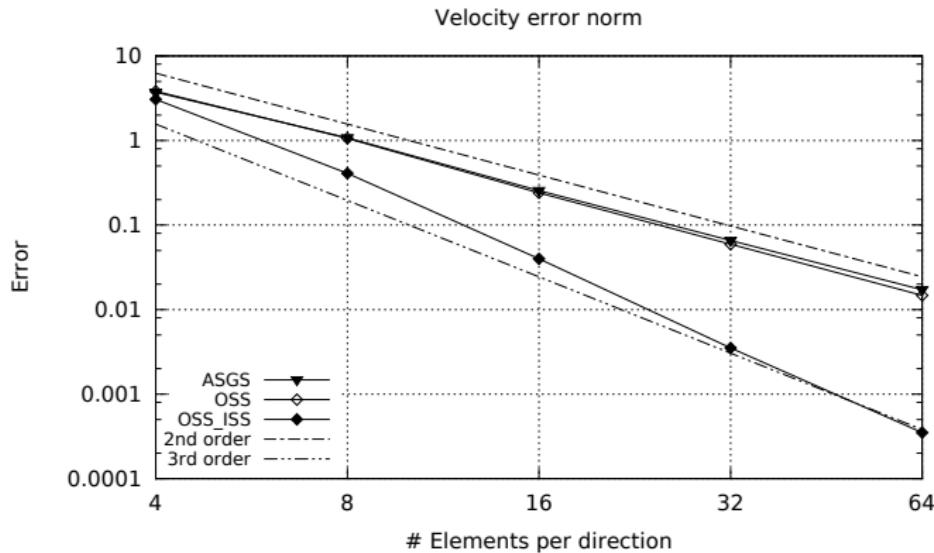
Accuracy: Velocity error



- 2nd order convergence rate for ASGS and OSS methods.

Mixed FE Colliding flow

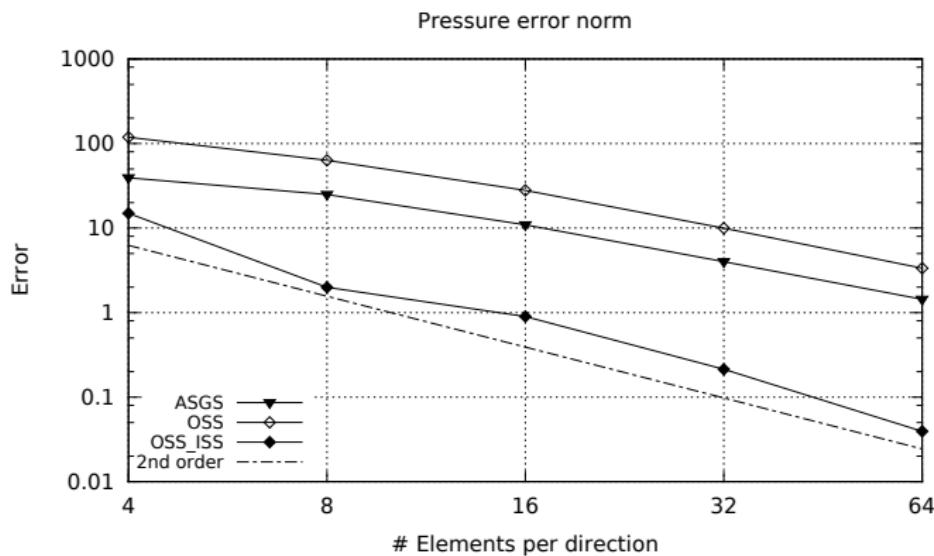
Accuracy: Pressure error



- 2nd order convergence rate for ASGS and OSS methods.
- **3rd order** convergence rate for OSS-ISS method.

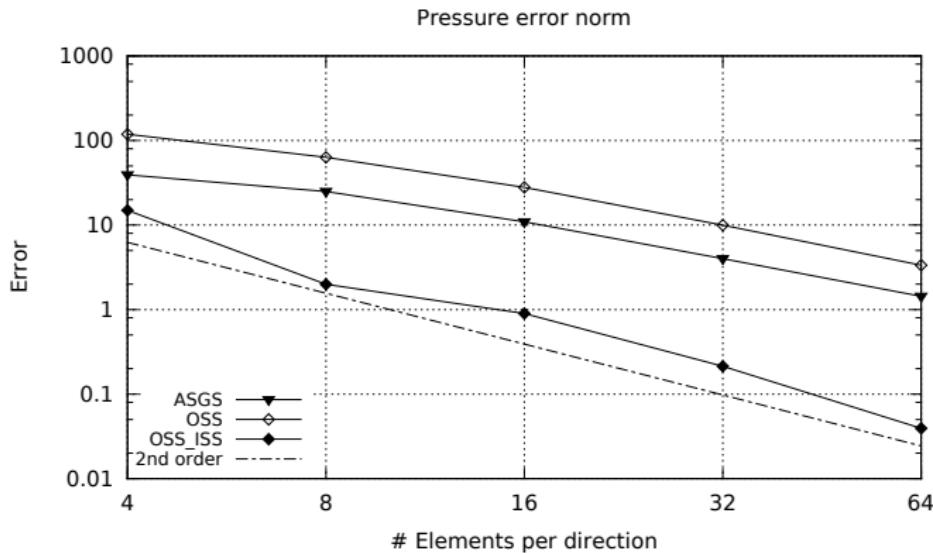
Mixed FE Colliding flow

Accuracy: Pressure error



Mixed FE Colliding flow

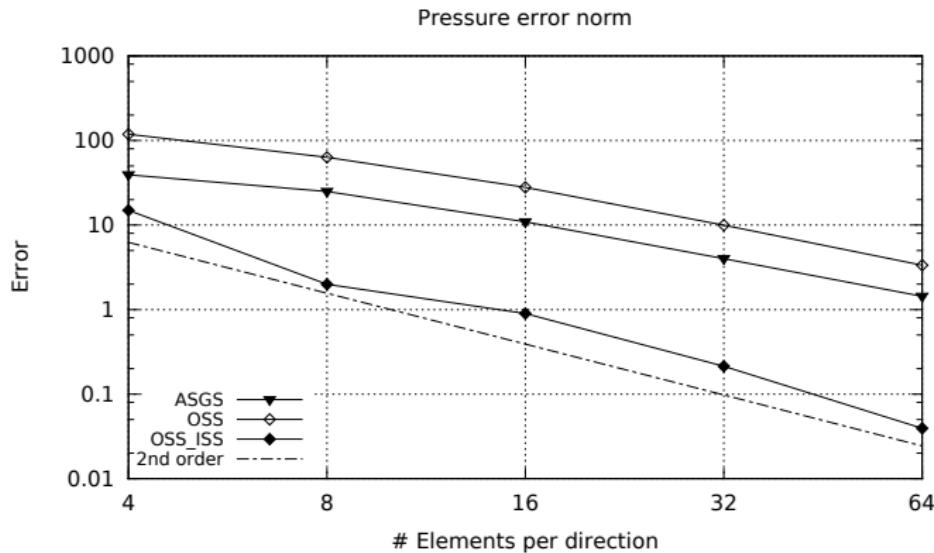
Accuracy:



- **2nd order** convergence rate for all methods.

Mixed FE Colliding flow

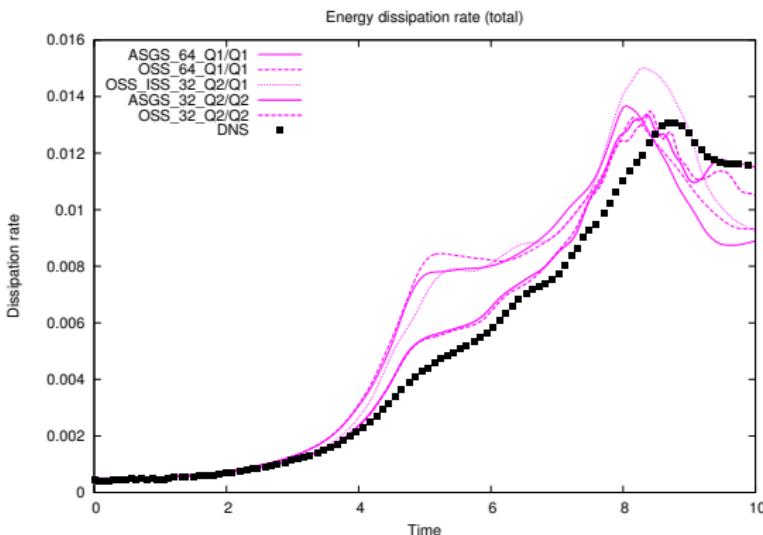
Accuracy:



- 2nd order convergence rate for all methods.
- Best accuracy for OSS-ISS method.

Mixed FE TGV Taylor-Green Vortex flow

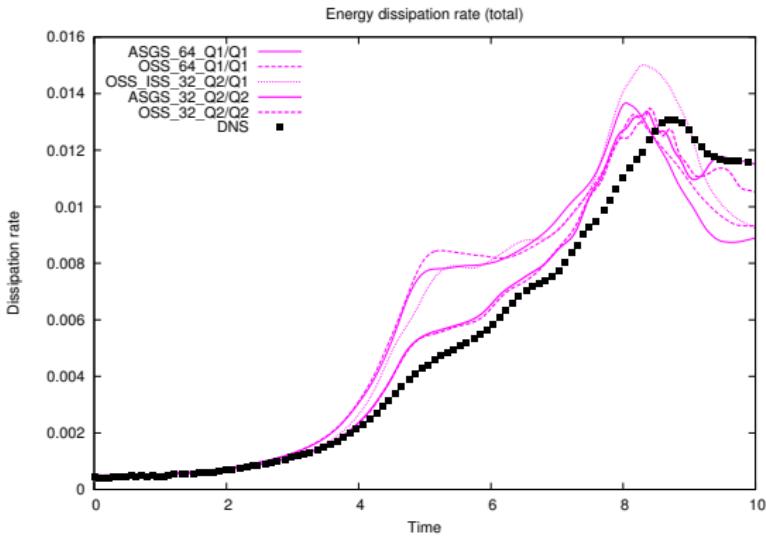
Energy dissipation rate (different methods):



Total energy dissipation rate

Mixed FE TGV Taylor-Green Vortex flow

Energy dissipation rate (different methods):

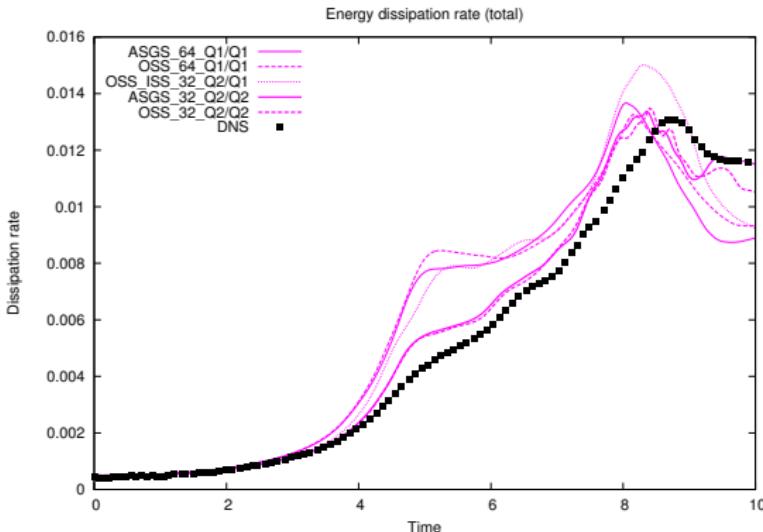


Total energy dissipation rate

- Good agreement with the DNS (coarse mesh).

Mixed FE TGV Taylor-Green Vortex flow

Energy dissipation rate (different methods):

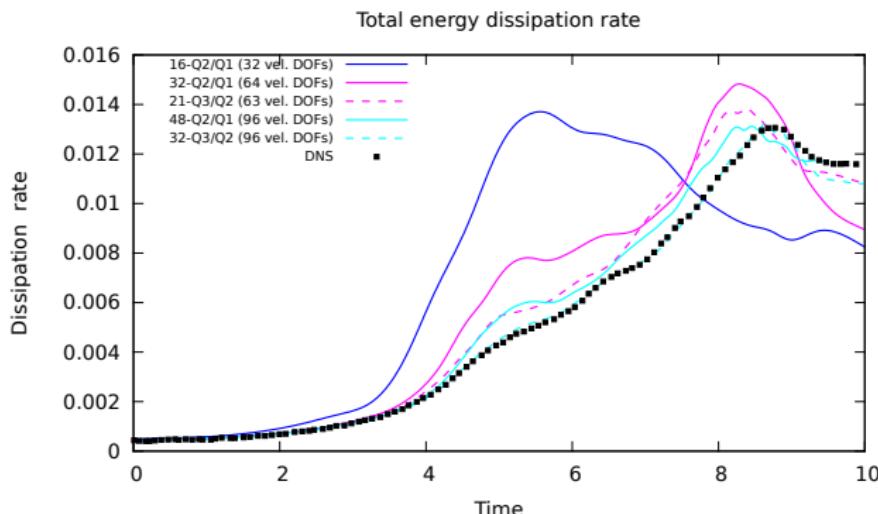


Total energy dissipation rate

- Good agreement with the DNS (coarse mesh).
- More accurate results with equal-order elements.

Mixed FE TGV Taylor-Green Vortex flow

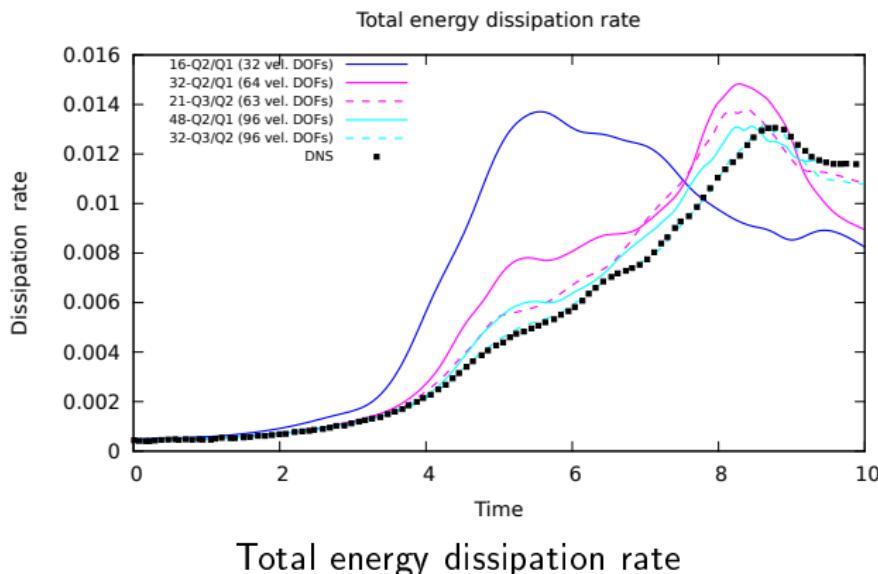
Energy dissipation rate (refinement OSS-ISS):



Total energy dissipation rate

Mixed FE TGV Taylor-Green Vortex flow

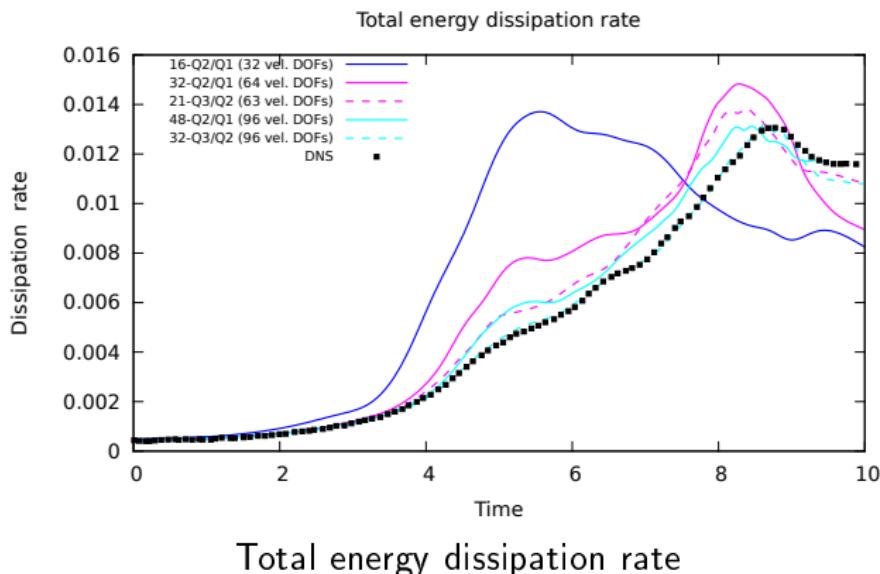
Energy dissipation rate (refinement OSS-ISS):



- Good agreement with the DNS.

Mixed FE TGV Taylor-Green Vortex flow

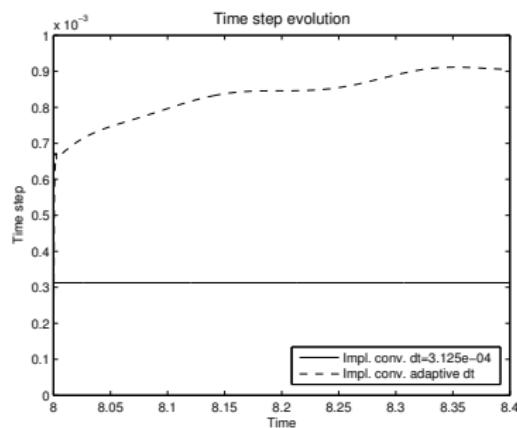
Energy dissipation rate (refinement OSS-ISS):



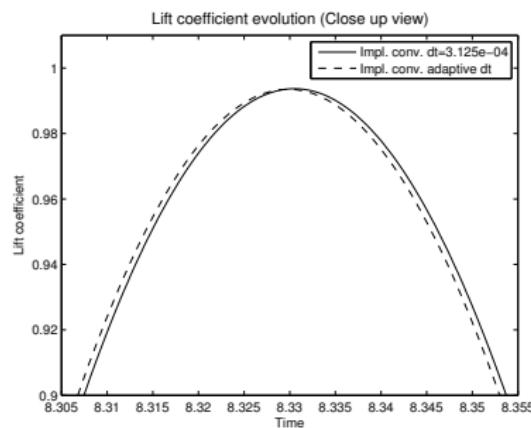
- Good agreement with the DNS.
- 32^3 Q3/Q2 elements mesh on top of DNS.

SRK 2D Laminar flow around a cylinder

Adaptive time stepping:



(a) Time step evolution



(b) Lift coefficient (zoom)

Adaptive time stepping.