# Large scale Finite Element solvers for the large eddy simulation of incompressible turbulent flows

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# Motivation

**Thesis motivation**

Highly scalable Finite Element (FE) framework for Large Eddy Simulations (LES) of incompressible turbulent flows

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1. Variational MultiScale (VMS) methods as LES models.
2. Time integration schemes with velocity-pressure segregation.

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Highly scalable Finite Element (FE) framework for Large Eddy Simulations (LES) of incompressible turbulent flows

**How to get there?**

1. Variational MultiScale (VMS) methods as LES models.
2. Time integration schemes with velocity-pressure segregation.
3. Highly scalable algorithms based on Domain Decomposition (DD) and block preconditioners.

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**Step by step...**

- Residual-based VMS as LES models.

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- Application.

Rb-VMS

# Implicit LES

**ILES**: only numerical dissipation (for stabilization) acts as turbulent model

- Not based on filtering of the Navier-Stokes equations
- Rely on numerical artifacts, no modification at the continuous level

# Incomp. Navier Stokes equations

Find $\mathbf{u}$ and $p$ defined in $\Omega$

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} &= \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

with appropriate boundary conditions on $\Gamma$.

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The weak problem is: $\forall \mathbf{v} \in \mathcal{V}_0$ and $\forall q \in \mathcal{Q}_0$, find $\mathbf{u} \in \mathcal{V}$ and $p \in \mathcal{Q}$ such that

$$\begin{aligned} (\mathbf{v}, \partial_t \mathbf{u})_\Omega + (\nabla \mathbf{v}, \nu \nabla \mathbf{u})_\Omega + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p)_\Omega &= \langle \mathbf{v}, \mathbf{f} \rangle_\Omega \\ (q, \nabla \cdot \mathbf{u})_\Omega &= 0 \end{aligned}$$

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where

$$b(\mathbf{a}, \mathbf{u}, \mathbf{v}) = \quad \langle \mathbf{v}, \mathbf{a} \cdot \nabla \mathbf{u} \rangle_\Omega$$

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where

$$b\,(\mathbf{a}, \mathbf{u}, \mathbf{v}) = \frac{1}{2} \langle \mathbf{v}, \mathbf{a} \cdot \nabla \mathbf{u} \rangle_\Omega - \frac{1}{2} \langle \mathbf{a} \cdot \nabla \mathbf{v}, \mathbf{u} \rangle_\Omega + \frac{1}{2} \langle \mathbf{v}, \mathbf{n} \cdot \mathbf{a} \mathbf{u} \rangle_\Gamma$$

# VMS decomposition (Hughes 1995)

A decomposition of spaces $\mathcal{V}$ and $\mathcal{Q}$ given by

$$\mathcal{V} = \mathcal{V}_h \oplus \widetilde{\mathcal{V}}, \quad \mathcal{Q} = \mathcal{Q}_h \oplus \widetilde{\mathcal{Q}}$$

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is applied to the function and test spaces

$$\mathbf{u} = \mathbf{u}_h + \widetilde{\mathbf{u}}, \quad p = p_h + \widetilde{p}$$
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We keep all the (eight) contributions from the splitting of the convective term

$$\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{u}_h \cdot \nabla \mathbf{u}_h + \widetilde{\mathbf{u}} \cdot \nabla \mathbf{u}_h + \mathbf{u}_h \cdot \nabla \widetilde{\mathbf{u}} + \widetilde{\mathbf{u}} \cdot \nabla \widetilde{\mathbf{u}}$$

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$$\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{u}_h \cdot \nabla \mathbf{u}_h + \widetilde{\mathbf{u}} \cdot \nabla \mathbf{u}_h + \mathbf{u}_h \cdot \nabla \widetilde{\mathbf{u}} + \widetilde{\mathbf{u}} \cdot \nabla \widetilde{\mathbf{u}}$$

and all the (four) contributions from the temporal term

$$\partial_t \mathbf{u} = \partial_t \mathbf{u}_h + \partial_t \widetilde{\mathbf{u}}$$

# Semidiscrete problem

## FEM equations

$$B\left((\mathbf{u}_h, p_h); (\widetilde{\mathbf{u}}, \widetilde{p}); (\mathbf{v}_h, q_h)\right) = L\left(\mathbf{v}_h, q_h\right)$$

## SGS equations

$$B\left((\widetilde{\mathbf{u}}, \widetilde{p}); (\mathbf{u}_h, p_h); (\widetilde{\mathbf{v}}, \widetilde{q})\right) = L\left(\widetilde{\mathbf{v}}, \widetilde{q}\right)$$

# Semidiscrete problem

## FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$
$$+ (\mathbf{v}_h, \partial_t \widetilde{\mathbf{u}})_\Omega + (\mathcal{L}^* \mathbf{v}_h, \widetilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \widetilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

## SGS equations

$$\partial_t \widetilde{\mathbf{u}} + \tau_m^{-1} \widetilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$$\tau_m = \left( \frac{c_1 \nu}{h^2} + \frac{c_2 |\mathbf{a}|}{h} \right)^{-1}, \quad \tau_c = \frac{h^2}{c_1 \tau_m}$$

$$\mathbf{a} = \mathbf{u}_h + \widetilde{\mathbf{u}}$$

# Semidiscrete problem

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$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

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$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$$\mathbf{R}_m := \mathbf{f} - \partial_t \mathbf{u}_h - \mathcal{L} \mathbf{u}_h - \nabla p_h, \quad R_c := -\nabla \cdot \mathbf{u}_h$$

# Semidiscrete problem

## FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$
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$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

## SGS equations

$$\partial_t \widetilde{\mathbf{u}} + \tau_m^{-1} \widetilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$
$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = I \quad \text{(ASGS)}, \qquad \mathcal{P} = P_h^\perp = I - P_h \quad \text{(OSS)}$$

# Semidiscrete problem

## FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

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$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

## SGS equations

$$\partial_t \widetilde{\mathbf{u}} + \tau_m^{-1} \widetilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = I \quad \text{(ASGS)}, \qquad \mathcal{P} = \mathrm{P}_\mathrm{h}^\perp = \mathrm{I} - \mathrm{P}_\mathrm{h} \quad \text{(OSS)}$$

$$\mathbf{a} = \mathbf{u}_h + \widetilde{\mathbf{u}}$$

# Summary

| | Sgs space | Sgs dynamics | Advection |
|---|---|---|---|
| 1 | ASGS | Static | Linear |
| 2 | ASGS | Dynamic | Linear |
| 3 | ASGS | Dynamic | Nonlinear |
| 4 | OSS | Static | Linear |
| 5 | OSS | Dynamic | Linear |
| 6 | OSS | Dynamic | Nonlinear |

1 It is the most standard method (SUPG for linear elements) up to the choice of the stabilization parameters. Unknown stability properties.

4 Strictly pointwise positive for linear elements (no backscatter).

5 Convergent to weak solutions of NS equations (Badia & Gutierrez 2012).

# Energy statements

**FE counterpart:**

$$B\left((\mathbf{u}_h, p_h)\,;(\widetilde{\mathbf{u}}, \widetilde{p})\,;(\mathbf{u}_h, p_h)\right) = L\left(\mathbf{u}_h, p_h\right)$$

**SGS counterpart:**

$$B\left((\widetilde{\mathbf{u}}, \widetilde{p})\,;(\mathbf{u}_h, p_h)\,;(\widetilde{\mathbf{u}}, \widetilde{p})\right) = L\left(\widetilde{\mathbf{u}}, \widetilde{p}\right)$$

**TOTAL:**

$$\begin{aligned} &B\left((\mathbf{u}_h, p_h)\,;(\widetilde{\mathbf{u}}, \widetilde{p})\,;(\mathbf{u}_h, p_h)\right) \\ +&B\left((\widetilde{\mathbf{u}}, \widetilde{p})\,;(\mathbf{u}_h, p_h)\,;(\widetilde{\mathbf{u}}, \widetilde{p})\right) = L\left(\mathbf{u}_h, p_h\right) + L\left(\widetilde{\mathbf{u}}, \widetilde{p}\right) \end{aligned}$$

# Energy statements

**FE counterpart:**

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

**SGS counterpart:**

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

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$$- (\nabla\cdot\mathbf{u}_h,\widetilde{p}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p})$$

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**SGS counterpart:**

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$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2 + (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

**TOTAL: Static subscales**

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2 + (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle + \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

# Energy statements

**FE counterpart:**

$$\frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) + (\partial_t \widetilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \widetilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \widetilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle$$

**SGS counterpart:**

$$\frac{1}{2} d_t \|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\widetilde{p}\|^2 + (\mathcal{P}(\partial_t \mathbf{u}_h), \widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \widetilde{p}) = \langle \mathcal{P}(\mathbf{f}), \widetilde{\mathbf{u}} \rangle$$

**TOTAL:** **Dynamic subscales - ASGS**

$$\frac{1}{2} d_t \|\mathbf{u}_h + \widetilde{\mathbf{u}}\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + \tau_m^{-1} \|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\widetilde{p}\|^2 - 2\,(\nu \Delta \mathbf{u}_h, \widetilde{\mathbf{u}}) = \langle \mathbf{f}, \mathbf{u}_h \rangle + \langle \mathbf{f}, \widetilde{\mathbf{u}} \rangle$$

# Energy statements

**FE counterpart:**

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\tilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\tilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\tilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

**SGS counterpart:**

$$\frac{1}{2}d_t\|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\tilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\tilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\tilde{p}) = \langle\mathcal{P}(\mathbf{f}),\tilde{\mathbf{u}}\rangle$$

**TOTAL: Dynamic subscales - OSS**

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \frac{1}{2}d_t\|\tilde{\mathbf{u}}\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + \tau_m^{-1}\|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\tilde{p}\|^2$$
$$-2\,(\nu\Delta\mathbf{u}_h,\tilde{\mathbf{u}})$$
$$= \langle\mathbf{f},\mathbf{u}_h\rangle + \langle\mathcal{P}(\mathbf{f}),\tilde{\mathbf{u}}\rangle$$

1. Motivation

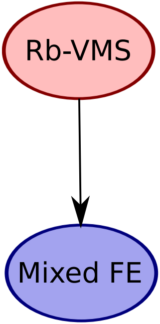
2. Residual-based VMS

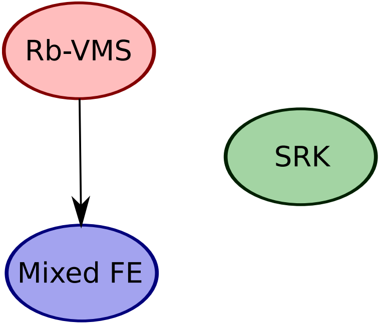
3. Mixed FE VMS
   - Formulation
   - Block-preconditioning
   - Numerical experiments
   - Conclusions

4. Segregated Runge-Kutta

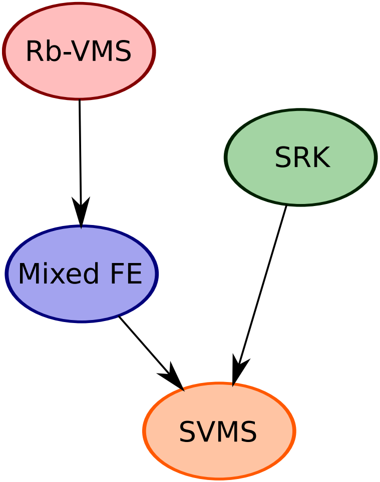
5. Segregated VMS

6. Conclusions





Rb-VMS
SRK
Mixed FE


Rb-VMS
SRK
Mixed FE
SVMS

# Outline

- Line 1.

# Outline

- Line 1.
- Line 2.
  Less formal

# Outline

- Line 1.
- Line 2.
  Less formal
- Line 3.
  Less formal, different color.

# Blocks

## Standard Block

This is a standard block.

## Example Block

This is an example block.

## Alert Block

This is an alert block.

# Questions?