Large scale Finite Element solvers for the large eddy simulation of incompressible turbulent flows

author:
ORIOL COLOMÉS GENÉ

supervisor:
SANTIAGO BADIA

Departament d'Enginyeria Civil i Ambiental

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Escola de Camins
Escola Tècnica Superior d'Enginyeria de Camins, Canals i Ports
UPC BARCELONATECH

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2. Residual-based VMS
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5. Segregated VMS
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Motivation

Thesis motivation

Highly scalable Finite Element (FE) framework for Large Eddy Simulations (LES) of incompressible turbulent flows

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How to get there?

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1. Variational MultiScale (VMS) methods as LES models.
2. Time integration schemes with velocity-pressure segregation.

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Thesis motivation
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How to get there?

1. Variational MultiScale (VMS) methods as LES models.
2. Time integration schemes with velocity-pressure segregation.
3. Highly scalable algorithms based on Domain Decomposition (DD) and block preconditioners.

Motivation

Step by step...

- Residual-based VMS as LES models.

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- Adaptive time stepping techniques.
- Velocity-pressure segregation.
- Scalable solvers.

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- Application.

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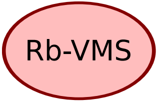
Rb-VMS

Implicit LES

ILES: only numerical dissipation (for stabilization) acts as turbulent model

- Not based on filtering of the Navier-Stokes equations
- Rely on numerical artifacts, no modification at the continuous level

Incomp. Navier Stokes equations

Find $\mathbf{u}$ and p defined in Ω

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} &= \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

with appropriate boundary conditions on Γ.

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with appropriate boundary conditions on Γ.

The weak problem is: $\forall \mathbf{v} \in \mathcal{V}_0$ and $\forall q \in \mathcal{Q}_0$, find $\mathbf{u} \in \mathcal{V}$ and $p \in \mathcal{Q}$ such that

$$\begin{aligned} (\mathbf{v}, \partial_t \mathbf{u})_\Omega + (\nabla\mathbf{v}, \nu\nabla\mathbf{u})_\Omega + b\left(\mathbf{u},\mathbf{u},\mathbf{v}\right) - (\nabla\cdot\mathbf{v}, p)_\Omega &= \langle \mathbf{v}, \mathbf{f}\rangle_\Omega \\ (q, \nabla\cdot\mathbf{u})_\Omega &= 0 \end{aligned}$$

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where

$$b(\mathbf{a}, \mathbf{u}, \mathbf{v}) = \quad \langle \mathbf{v}, \mathbf{a} \cdot \nabla \mathbf{u} \rangle_\Omega$$

Incomp. Navier Stokes equations

Find $\mathbf{u}$ and p defined in Ω

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where

$$b(\mathbf{a}, \mathbf{u}, \mathbf{v}) = \frac{1}{2} \langle \mathbf{v}, \mathbf{a} \cdot \nabla \mathbf{u} \rangle_\Omega - \frac{1}{2} \langle \mathbf{a} \cdot \nabla \mathbf{v}, \mathbf{u} \rangle_\Omega + \frac{1}{2} \langle \mathbf{v}, \mathbf{n} \cdot \mathbf{a} \mathbf{u} \rangle_\Gamma$$

VMS decomposition (Hughes 1995)

A decomposition of spaces $\mathcal{V}$ and $\mathcal{Q}$ given by

$$\mathcal{V} = \mathcal{V}_h \oplus \widetilde{\mathcal{V}}, \quad \mathcal{Q} = \mathcal{Q}_h \oplus \widetilde{\mathcal{Q}}$$

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$$\mathcal{V} = \mathcal{V}_h \oplus \widetilde{\mathcal{V}}, \quad \mathcal{Q} = \mathcal{Q}_h \oplus \widetilde{\mathcal{Q}}$$

is applied to the function and test spaces

$$\mathbf{u} = \mathbf{u}_h + \widetilde{\mathbf{u}}, \quad p = p_h + \widetilde{p}$$
$$\mathbf{v} = \mathbf{v}_h + \widetilde{\mathbf{v}}, \quad q = q_h + \widetilde{q}$$

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We keep all the (eight) contributions from the splitting of the convective term

$$\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{u}_h \cdot \nabla \mathbf{u}_h + \widetilde{\mathbf{u}} \cdot \nabla \mathbf{u}_h + \mathbf{u}_h \cdot \nabla \widetilde{\mathbf{u}} + \widetilde{\mathbf{u}} \cdot \nabla \widetilde{\mathbf{u}}$$

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and all the (four) contributions from the temporal term

$$\partial_t \mathbf{u} = \partial_t \mathbf{u}_h + \partial_t \widetilde{\mathbf{u}}$$

Semidiscrete problem

FEM equations

$$B\left((\mathbf{u}_h, p_h);(\widetilde{\mathbf{u}}, \widetilde{p});(\mathbf{v}_h, q_h)\right) = L\left(\mathbf{v}_h, q_h\right)$$

SGS equations

$$B\left((\widetilde{\mathbf{u}}, \widetilde{p});(\mathbf{u}_h, p_h);(\widetilde{\mathbf{v}}, \widetilde{q})\right) = L\left(\widetilde{\mathbf{v}}, \widetilde{q}\right)$$

Semidiscrete problem

FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

$$+ (\mathbf{v}_h, \partial_t \widetilde{\mathbf{u}})_\Omega + (\mathcal{L}^* \mathbf{v}_h, \widetilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \widetilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

SGS equations

$$\partial_t \widetilde{\mathbf{u}} + \tau_m^{-1} \widetilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$$\tau_m = \left(\frac{c_1 \nu}{h^2} + \frac{c_2 |\mathbf{a}|}{h} \right)^{-1}, \quad \tau_c = \frac{h^2}{c_1 \tau_m}$$

$$\mathbf{a} = \mathbf{u}_h + \widetilde{\mathbf{u}}$$

Semidiscrete problem

FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

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$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

SGS equations

$$\partial_t \widetilde{\mathbf{u}} + \tau_m^{-1} \widetilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$$\mathbf{R}_m := \mathbf{f} - \partial_t \mathbf{u}_h - \mathcal{L} \mathbf{u}_h - \nabla p_h, \quad R_c := -\nabla \cdot \mathbf{u}_h$$

Semidiscrete problem

FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$
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$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

SGS equations

$$\partial_t \widetilde{\mathbf{u}} + \tau_m^{-1} \widetilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = I \quad \text{(ASGS)}, \qquad \mathcal{P} = P_h^\perp = I - P_h \quad \text{(OSS)}$$

Semidiscrete problem

FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

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$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

SGS equations

$$\partial_t \widetilde{\mathbf{u}} + \tau_m^{-1} \widetilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = I \quad \text{(ASGS)}, \qquad \mathcal{P} = \mathrm{P}_\mathrm{h}^\perp = \mathrm{I} - \mathrm{P}_\mathrm{h} \quad \text{(OSS)}$$

$$\mathbf{a} = \mathbf{u}_h + \widetilde{\mathbf{u}}$$

Summary

	Sgs space	Sgs dynamics	Advection
1	ASGS	Static	Linear
2	ASGS	Dynamic	Linear
3	ASGS	Dynamic	Nonlinear
4	OSS	Static	Linear
5	OSS	Dynamic	Linear
6	OSS	Dynamic	Nonlinear

1 It is the most standard method (SUPG for linear elements) up to the choice of the stabilization parameters. Unknown stability properties.

4 Strictly pointwise positive for linear elements (no backscatter).

5 Convergent to weak solutions of NS equations (Badia & Gutierrez 2012).

Energy statements

FE counterpart:

$$B\left((\mathbf{u}_h, p_h); (\widetilde{\mathbf{u}}, \widetilde{p}); (\mathbf{u}_h, p_h)\right) = L\left(\mathbf{u}_h, p_h\right)$$

SGS counterpart:

$$B\left((\widetilde{\mathbf{u}}, \widetilde{p}); (\mathbf{u}_h, p_h); (\widetilde{\mathbf{u}}, \widetilde{p})\right) = L\left(\widetilde{\mathbf{u}}, \widetilde{p}\right)$$

TOTAL:

$$\begin{aligned} & B\left((\mathbf{u}_h, p_h); (\widetilde{\mathbf{u}}, \widetilde{p}); (\mathbf{u}_h, p_h)\right) \\ + & B\left((\widetilde{\mathbf{u}}, \widetilde{p}); (\mathbf{u}_h, p_h); (\widetilde{\mathbf{u}}, \widetilde{p})\right) = L\left(\mathbf{u}_h, p_h\right) + L\left(\widetilde{\mathbf{u}}, \widetilde{p}\right) \end{aligned}$$

Energy statements

FE counterpart:

$$\frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) + (\partial_t \widetilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \widetilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \widetilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle$$

SGS counterpart:

$$\frac{1}{2} d_t \|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\widetilde{p}\|^2 + (\mathcal{P}(\partial_t \mathbf{u}_h), \widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \widetilde{p}) = \langle \mathcal{P}(\mathbf{f}), \widetilde{\mathbf{u}} \rangle$$

TOTAL:

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h) + (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2 + (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \nu\|\nabla\mathbf{u}_h\|^2$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\tilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\tilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\tilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\tilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\tilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\tilde{p}) = \langle\mathcal{P}(\mathbf{f}),\tilde{\mathbf{u}}\rangle$$

TOTAL:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \frac{1}{2}d_t\|\tilde{\mathbf{u}}\|^2 + \nu\|\nabla\mathbf{u}_h\|^2$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
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TOTAL:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}})$$
$$- (\nabla\cdot\mathbf{u}_h,\widetilde{p}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p})$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h) + (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2 + (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2 + (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle + \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

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$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL: Static subscales

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}})$$
$$- (\nabla\cdot\mathbf{u}_h,\widetilde{p}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle + \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h)$$
$$+ (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$+ (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL: **Dynamic subscales - ASGS**

$$\frac{1}{2}d_t\|\mathbf{u}_h+\widetilde{\mathbf{u}}\|^2 \qquad + \nu\|\nabla\mathbf{u}_h\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2$$
$$-2\,(\nu\Delta\mathbf{u}_h,\widetilde{\mathbf{u}})$$
$$= \langle\mathbf{f},\mathbf{u}_h\rangle + \langle\mathbf{f},\widetilde{\mathbf{u}}\rangle$$

Energy statements

FE counterpart:

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + b(\mathbf{a},\mathbf{u}_h,\mathbf{u}_h) + (\partial_t\widetilde{\mathbf{u}},\mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h,p_h),\widetilde{\mathbf{u}}) - (\nabla\cdot\mathbf{u}_h,\widetilde{p}) = \langle\mathbf{f},\mathbf{u}_h\rangle$$

SGS counterpart:

$$\frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2 + (\mathcal{P}(\partial_t\mathbf{u}_h),\widetilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h,p_h)),\widetilde{\mathbf{u}}) - (\mathcal{P}(\nabla\cdot\mathbf{u}_h),\widetilde{p}) = \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

TOTAL: Dynamic subscales - OSS

$$\frac{1}{2}d_t\|\mathbf{u}_h\|^2 + \frac{1}{2}d_t\|\widetilde{\mathbf{u}}\|^2 + \nu\|\nabla\mathbf{u}_h\|^2 + \tau_m^{-1}\|\widetilde{\mathbf{u}}\|^2 + \tau_c^{-1}\|\widetilde{p}\|^2 - 2\,(\nu\Delta\mathbf{u}_h,\widetilde{\mathbf{u}}) = \langle\mathbf{f},\mathbf{u}_h\rangle + \langle\mathcal{P}(\mathbf{f}),\widetilde{\mathbf{u}}\rangle$$

Numerical experiments

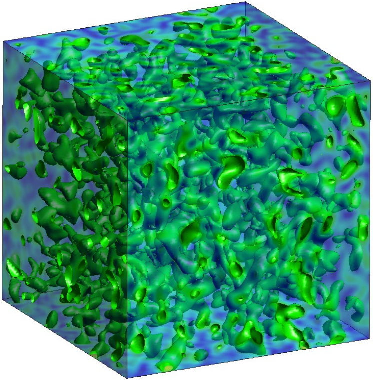
Three different turbulent benchmarks:

- Decaying of Homogeneous Isotropic Turbulence (DHIT).
- Taylor-Green Vortex (TGV) flow.
- Turbulent Channel Flow (TCF).

DHIT Decay of Homogeneous Isotropic Turbulence

Problem setting:

- Prescribed initial energy spectra corresponding to $Re_\lambda = 952$.
- Setting defined in AGARD database (Mansour & Wray 1993).
- A (very simple) time step adaptation technique is used.
- Different mesh discretizations (Q_1/Q_1 and Q_2/Q_2).



DHIT Decay of Homogeneous Isotropic Turbulence

Energy espectra (models):

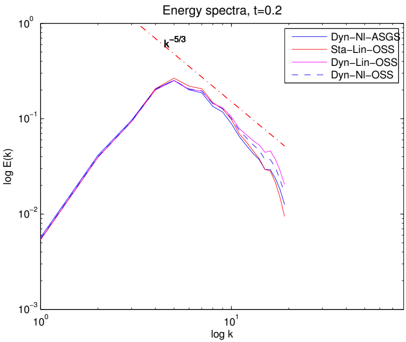


Figure : $32^3 - Q1,\ t = 0.2s$

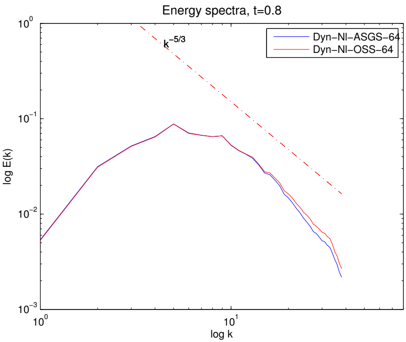
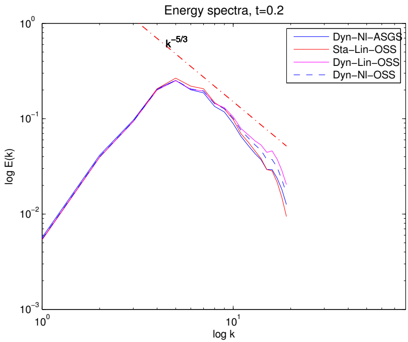


Figure : $64^3 - Q1,\ t = 0.8s$

DHIT Decay of Homogeneous Isotropic Turbulence

Energy espectra (models):



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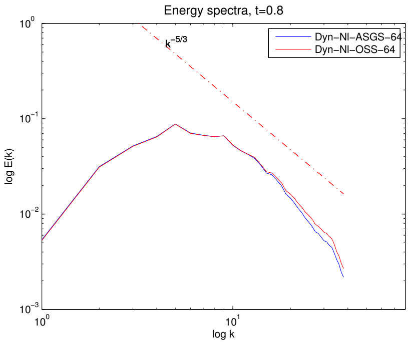
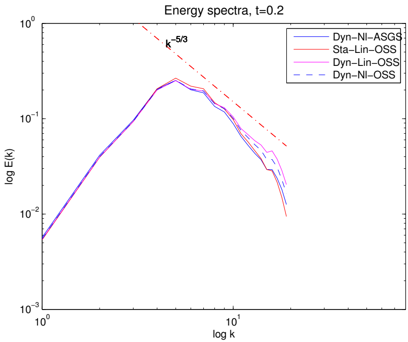


Figure : $64^3 - Q1,\ t = 0.8s$

- Small differences between methods (physical sense).

DHIT Decay of Homogeneous Isotropic Turbulence

Energy espectra (models):



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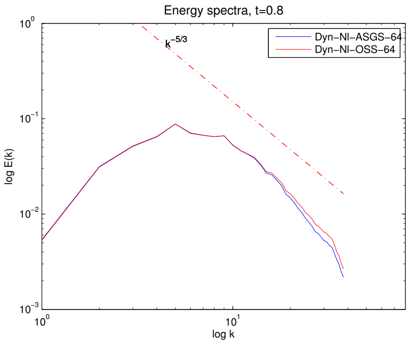


Figure : $64^3 - Q1,\ t = 0.8s$

- Small differences between methods (physical sense).
- Even more similar when we refine the mesh.

DHIT Decay of Homogeneous Isotropic Turbulence

Computational cost (models):

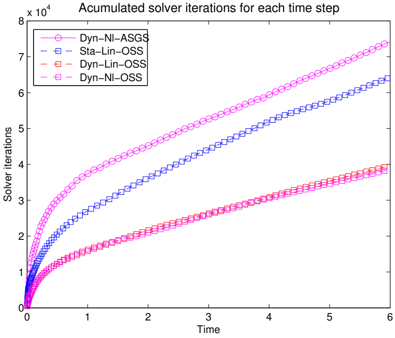
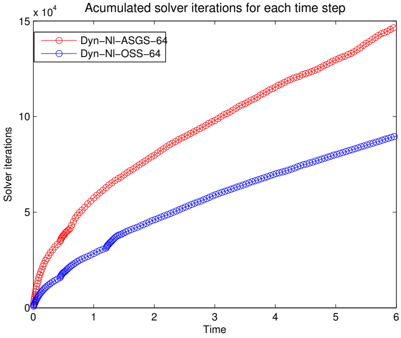


Figure : $32^3 - Q1$



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DHIT Decay of Homogeneous Isotropic Turbulence

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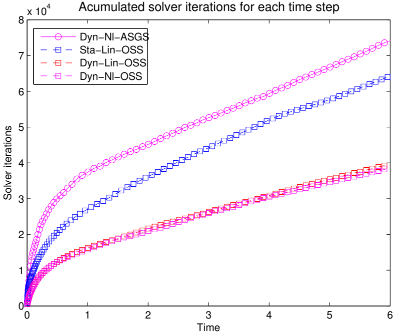

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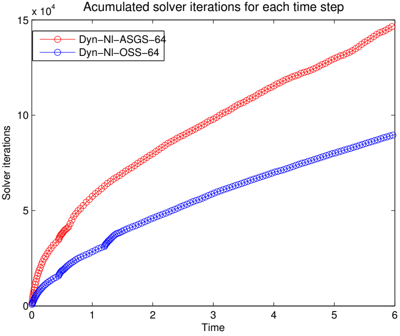

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DHIT Decay of Homogeneous Isotropic Turbulence

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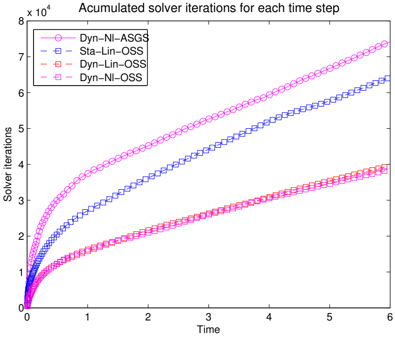

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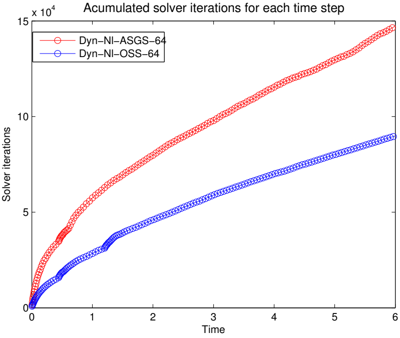
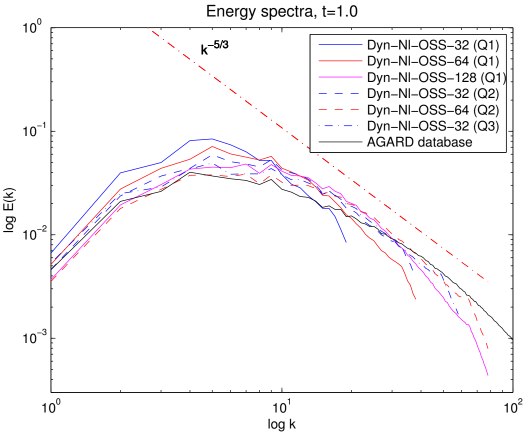


Figure : $64^3 - Q1$

- Big differences between methods (computational sense).
- Dynamic versions of OSS method are the most efficient.

DHIT Decay of Homogeneous Isotropic Turbulence

Energy espectra (refinement):



- Results become closer to the DNS when we refine the mesh.

TGV Taylor-Green Vortex flow

Problem setting:

- Prescribed initial condition.
- $Re = 1600$.
- Different mesh discretizations (Q_1/Q_1 and Q_2/Q_2).

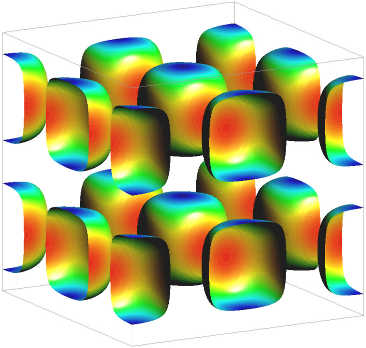


Figure : Initial vorticity isosurface $|\omega| = 1$

TGV Taylor-Green Vortex flow

Problem setting:

- Prescribed initial condition.
- $Re = 1600$.
- Different mesh discretizations (Q_1/Q_1 and Q_2/Q_2).

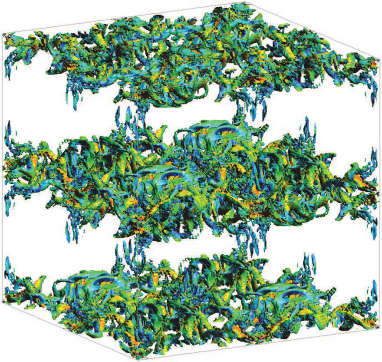


Figure : Vorticity isosurfaces $|\omega| = 9.0$

TGV Taylor-Green Vortex flow

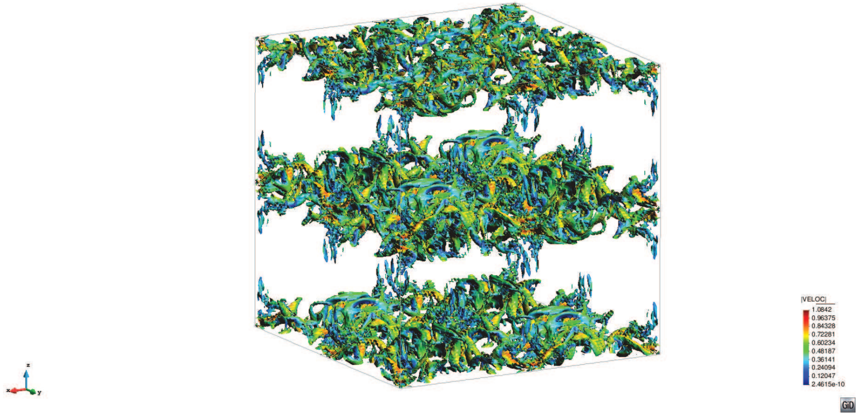


Figure : Velocity isosurface

TGV Taylor-Green Vortex flow

Energy dissipation rate (refinement):

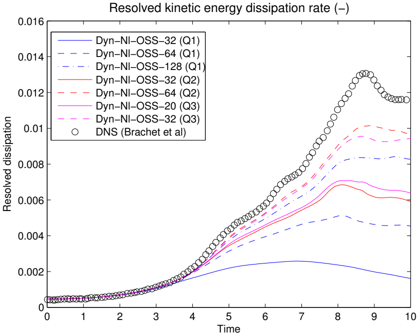


Figure : Resolved scales

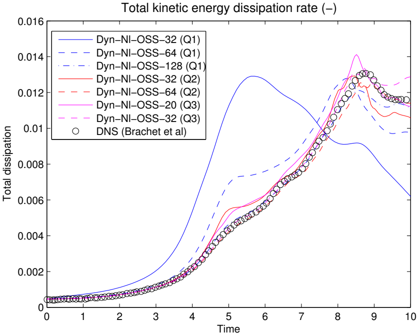


Figure : Total

TGV Taylor-Green Vortex flow

Energy dissipation rate (refinement):

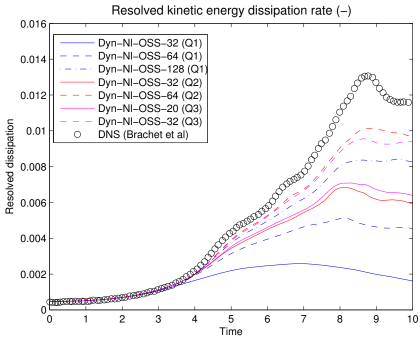


Figure : Resolved scales

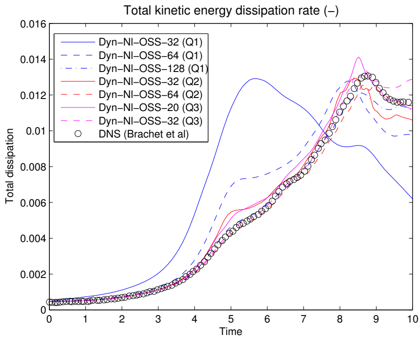


Figure : Total

- Good agreement with the DNS taking account the subscales.

TGV Taylor-Green Vortex flow

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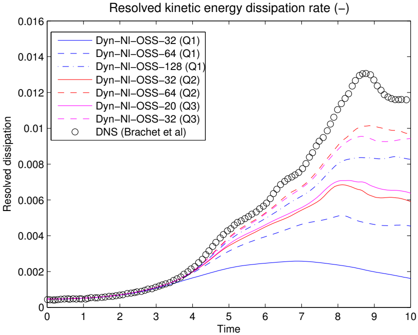

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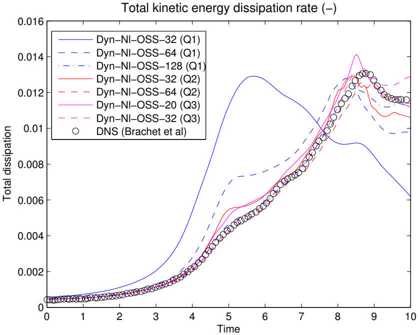


Figure : Total

- Good agreement with the DNS taking account the subscales.
- More accurate results increasing the order of approximation.

TGV Taylor-Green Vortex flow

- All results until now are compared against **DNS**.
- Are our methods comparable with **LES** models?

TGV Taylor-Green Vortex flow

Energy dissipation rate (against LES model):

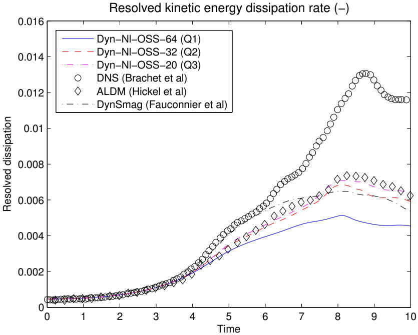
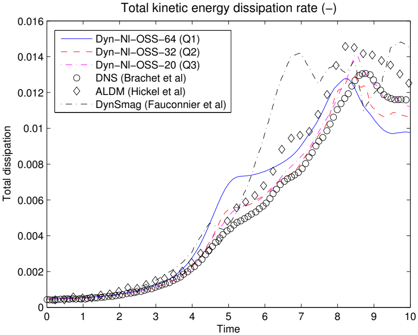


Figure : Resolved scales



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TGV Taylor-Green Vortex flow

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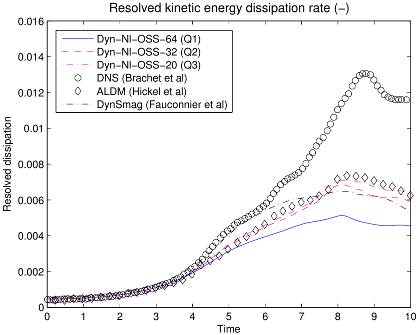

Figure : Resolved scales

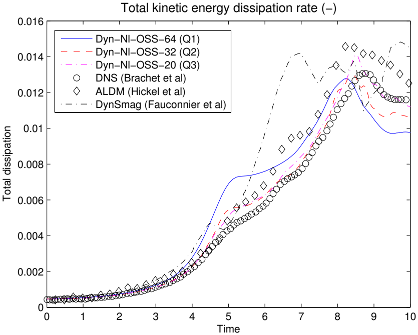

Figure : Total

- Good agreement with the LES models on resolved scales.

TGV Taylor-Green Vortex flow

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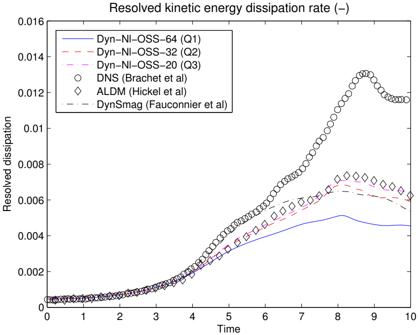


Figure : Resolved scales

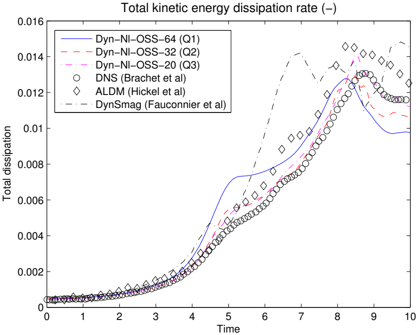


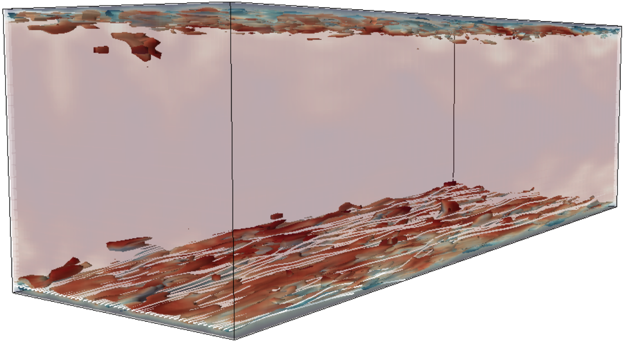
Figure : Total

- Good agreement with the LES models on resolved scales.
- Better results than LES models adding subscales counterpart.

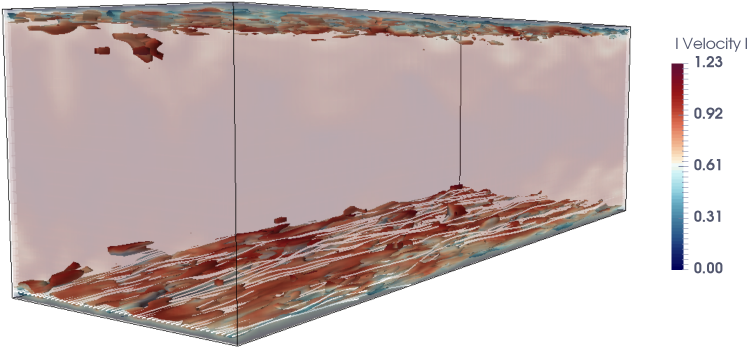
TCF Turbulent Channel Flow

Problem setting:

- Wall bounded flow.
- $Re_\tau = 180$ and $Re_\tau = 395$.
- Mesh resolution: $32^3 - Q1$.

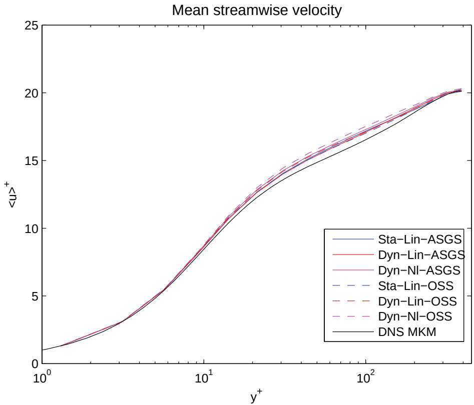


TCF Turbulent Channel Flow



TCF Turbulent Channel Flow

Mean streamwise velocity (models):



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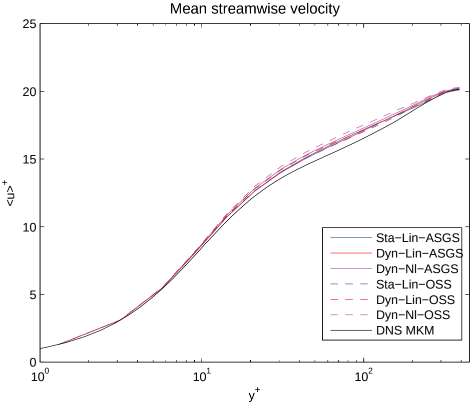

- Small differences between methods.

TCF Turbulent Channel Flow

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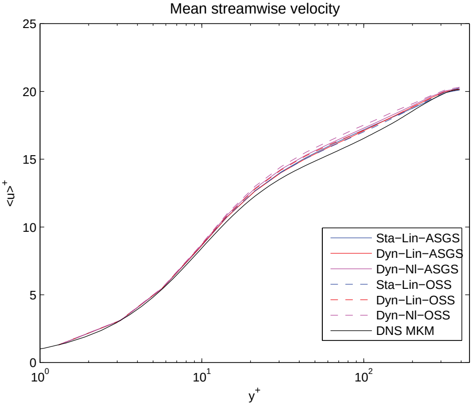

- Small differences between methods.
- Very accurate results compared against the DNS.

TCF Turbulent Channel Flow

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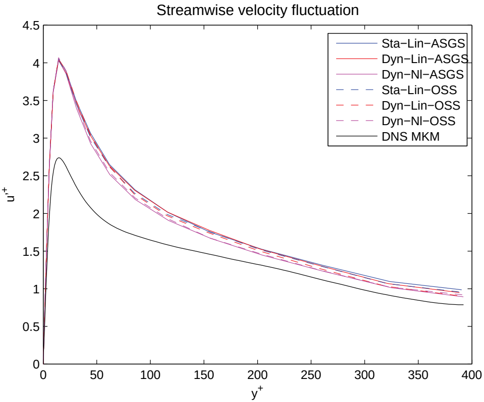

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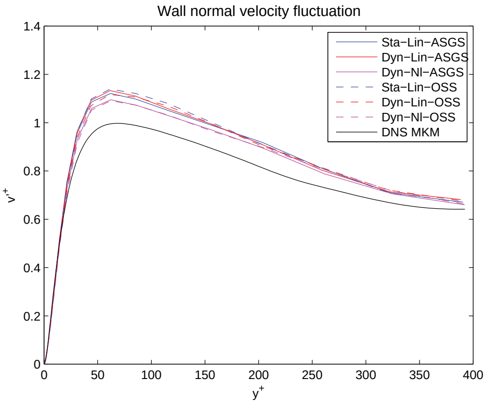

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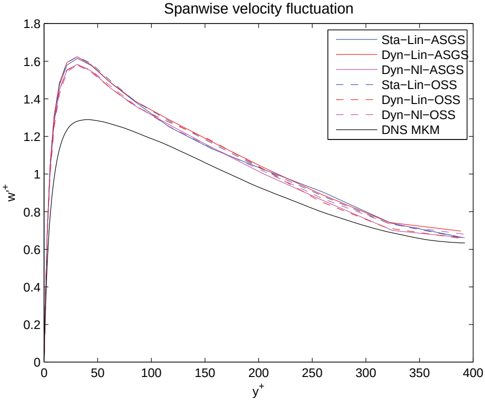

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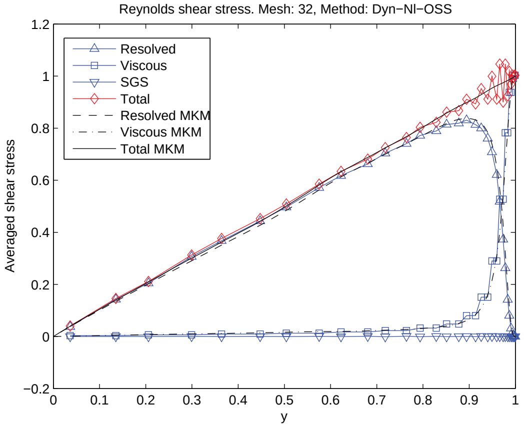
TCF Turbulent Channel Flow

Reynolds shear stress (models):



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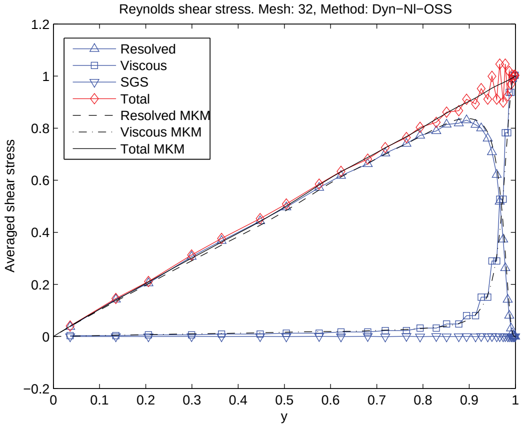

- Almost identical to the DNS.

TCF Turbulent Channel Flow

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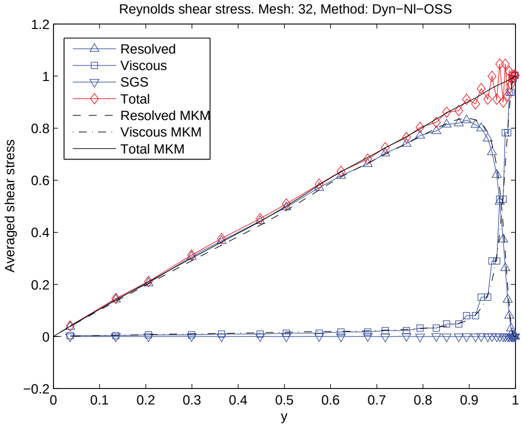

- Almost identical to the DNS.
- SGS counterpart does not contribute to the Reynolds shear stress.

RB-VMS Conclusions

- VMS formulations of NS equations can be used for the numerical simulation of turbulent flows.

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- Our particular VMS modelling ingredients
  - Dynamic subscales
  - Nonlinear subscales
  - Orthgonal subscales

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- Among them dynamic and orthogonal subscales (linear or nonlinear) are the most effective.

RB-VMS Conclusions

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  seem to be important to simulate turbulent flows.
- Among them dynamic and orthogonal subscales (linear or nonlinear) are the most effective.
- The skewsymmetric formulation is important to keep stability.

RB-VMS Limitations

- **ASGS:**

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  - Poorly matrix conditioning $\Rightarrow$ High number of solver iterations.

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RB-VMS Limitations

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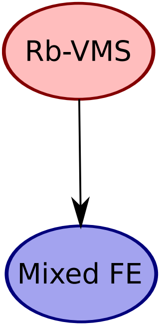
  - Poorly matrix conditioning ⇒ High number of solver iterations.
  - No explicit projections ⇒ Low number of nonlinear iterations.
- **OSS:**
  - Better matrix conditioning ⇒ Low number of solver iterations.
  - Explicit projection treatment ⇒ High number of nonlinear iterations.
- **Desired:**
  - OSS with implicit projections.

1. Motivation
2. Residual-based VMS
3. Mixed FE VMS
   - Formulation
   - Block-preconditioning
   - Numerical experiments
   - Conclusions
4. Segregated Runge-Kutta
5. Segregated VMS
6. Conclusions



Motivation

Step by step...

- Residual-based VMS as LES models.
- Mixed FE formulations LES.
- High-order FE methods.
- High-order time integration schemes.
- Adaptive time stepping techniques.
- Velocity-pressure segregation.
- Scalable solvers.
- Application.

Semidiscrete problem

FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$
$$+ (\mathbf{v}_h, \partial_t \widetilde{\mathbf{u}})_\Omega + (\mathcal{L}^* \mathbf{v}_h, \widetilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \widetilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$
$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

SGS equations

$$\partial_t \widetilde{\mathbf{u}} + \tau_m^{-1} \widetilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$
$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = I \quad \text{(ASGS)}, \qquad \mathcal{P} = \mathrm{P}_\mathrm{h}^\perp = \mathrm{I} - \mathrm{P}_\mathrm{h} \quad \text{(OSS)}$$

$$\mathbf{a} = \mathbf{u}_h + \widetilde{\mathbf{u}}$$

Semidiscrete problem

FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b\left(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h\right) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

$$+ (\mathcal{L}^* \mathbf{v}_h, \widetilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \widetilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

SGS equations

$$\tau_m^{-1} \widetilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \widetilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = \mathrm{P}_\mathrm{h}^\perp = \mathrm{I} - \mathrm{P}_\mathrm{h} \quad \text{(OSS)}$$

$$\mathbf{a} = \mathbf{u}_h$$

Term-by-term OSS

FEM equations

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$
$$+ (\mathcal{L}^* \mathbf{v}_h, \widetilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \widetilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \widetilde{\mathbf{u}})_{\Omega^h} = 0$$

Term-by-term OSS

Term-by-term OSS (Codina 2008)

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

$$+ \left(\tau_m \mathbf{a} \cdot \nabla \mathbf{v}_h, \mathcal{P}_h^\perp(\mathbf{a} \cdot \nabla \mathbf{u}_h)\right)_{\Omega^h}$$

$$+ \left(\tau_c \nabla \cdot \mathbf{v}_h, \mathcal{P}_h^\perp(\nabla \cdot \mathbf{u}_h)\right)_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega + \left(\tau_m \nabla q_h, \mathcal{P}_h^\perp(\nabla p_h)\right)_{\Omega^h} = 0$$

Term-by-term OSS

Term-by-term OSS (Codina 2008)

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$
$$+ (\tau_m \mathbf{a} \cdot \nabla \mathbf{v}_h, \mathbf{a} \cdot \nabla \mathbf{u}_h)_{\Omega^h} - (\tau_m \mathbf{a} \cdot \nabla \mathbf{v}_h, \boldsymbol{\eta}_h)_{\Omega^h}$$
$$+ (\tau_c \nabla \cdot \mathbf{v}_h, \nabla \cdot \mathbf{u}_h)_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega + (\tau_m \nabla q_h, \nabla p_h)_{\Omega^h} - (\tau_m \nabla q_h, \boldsymbol{\xi}_h)_{\Omega^h} = 0$$

$$\boldsymbol{\eta}_h := \mathcal{P}_h(\mathbf{a} \cdot \nabla \mathbf{u}_h)$$
$$\boldsymbol{\xi}_h := \mathcal{P}_h(\nabla p_h)$$
$$\mathcal{P}_h(\nabla \cdot \mathbf{u}_h) \approx 0$$

Matricial form

- ASGS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G \\ D & L_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \end{bmatrix},$$

Matricial form

- ASGS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G \\ D & L_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \end{bmatrix},$$

- Term-by-term OSS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

Matricial form

- ASGS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G \\ D & L_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \end{bmatrix},$$

- Term-by-term OSS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Term-by-term OSS with Inf-sup stable elements (mixed FE):

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_\tau \\ D & 0 & 0 \\ -B_\tau^T & 0 & M_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

Matricial form

- ASGS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G \\ D & L_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \end{bmatrix},$$

- Term-by-term OSS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Term-by-term OSS with Inf-sup stable elements (mixed FE):

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_\tau \\ D & 0 & 0 \\ -B_\tau^T & 0 & M_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

Index-2 DAE!!!

Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

Block-recursive preconditioning

General case:

$$\left[\begin{array}{c} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{array}\right] + \left[\begin{array}{cc:cc} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hdashline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \left[\begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \hdashline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_u \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{array}\right],$$

Block-recursive preconditioning

General case:

$$\left[\begin{array}{c} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right] + \left[\begin{array}{cc|cc} K+C+A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \left[\begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \hline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_u \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right],$$

- Outer block matrix:

$$\tilde{A} = \left[\begin{array}{cc|cc} M_{\eta,\tau} & 0 & -B_{\eta,\tau}^T & 0 \\ 0 & M_{\xi,\tau} & 0 & -B_{\xi,\tau}^T \\ \hline B_{\eta,\tau} & 0 & K+C+A_\tau & G \\ 0 & B_{\xi,\tau} & D & L_\tau \end{array}\right]$$

Block-recursive preconditioning

General case:

$$\left[\begin{array}{c} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right] + \left[\begin{array}{cc|cc} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \left[\begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \hline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_u \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right],$$

- Outer block matrix:

Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \left[\begin{array}{cc|cc} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \hline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Outer block matrix:

$$\tilde{A} = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ \tilde{B}_\tau & \tilde{K}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}_\tau \tilde{M}_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix},$$

Block-recursive preconditioning

General case:

$$\left[\begin{array}{c} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right] + \left[\begin{array}{cc|cc} K+C+A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \left[\begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \hline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_u \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right],$$

- Outer block matrix:

$$\tilde{A} = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ \tilde{B}_\tau & \tilde{K}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}_\tau \tilde{M}_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix},$$

$$\tilde{S} = \tilde{K}_\tau + \tilde{B}_\tau \tilde{M}_\tau^{-1} \tilde{B}_\tau^T,$$

Block-recursive preconditioning

General case:

$$\left[\begin{array}{c} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right] + \left[\begin{array}{cc|cc} K+C+A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \left[\begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \hline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_u \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right],$$

- Outer block matrix:

$$\tilde{A} = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ \tilde{B}_\tau & \tilde{K}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}_\tau \tilde{M}_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix},$$

$$\tilde{S} = \tilde{K}_\tau + \tilde{B}_\tau \tilde{M}_\tau^{-1} \tilde{B}_\tau^T,$$

$$P_U(\tilde{A}) = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{M}_\tau^{-1} & \tilde{M}_\tau^{-1} \tilde{B}_\tau^T \tilde{S}^{-1} \\ 0 & \tilde{S}^{-1} \end{bmatrix},$$

Block-recursive preconditioning

General case:

$$\left[\begin{array}{c} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right] + \left[\begin{array}{cc|cc} K+C+A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \left[\begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \hline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_u \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right],$$

- Outer block matrix:

$$\tilde{A} = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ \tilde{B}_\tau & \tilde{K}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}_\tau \tilde{M}_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix},$$

$$\tilde{S} = \tilde{K}_\tau + \tilde{B}_\tau \tilde{M}_\tau^{-1} \tilde{B}_\tau^T,$$

$$P_U(\tilde{A}) = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{M}_\tau^{-1} & \tilde{M}_\tau^{-1} \tilde{B}_\tau^T \tilde{S}^{-1} \\ 0 & \tilde{S}^{-1} \end{bmatrix},$$

$$\tilde{S}^{-1} \approx \tilde{K}_\tau^{-1}.$$

Block-recursive preconditioning

General case:

$$\left[\begin{array}{c} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{array}\right] + \left[\begin{array}{cc:cc} K+C+A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hdashline -B^T_{\eta,\tau} & 0 & M_{\eta,\tau} & 0 \\ 0 & -B^T_{\xi,\tau} & 0 & M_{\xi,\tau} \end{array}\right] \left[\begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \hdashline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_u \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{array}\right],$$

- Inner block matrix:

$$\tilde{K}_\tau = \begin{bmatrix} K_\tau & G \\ D & L_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ DK_\tau{}^{-1} & I \end{bmatrix} \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix},$$

Block-recursive preconditioning

General case:

$$
\left[\begin{array}{c} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{array}\right]
+
\left[\begin{array}{c|c:cc} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ \hline D & L_\tau & 0 & B_{\xi,\tau} \\ \hdashline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right]
\left[\begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \hdashline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{array}\right]
=
\left[\begin{array}{c} \mathbf{F}_u \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{array}\right],
$$

- Inner block matrix:

$$
\tilde{K}_\tau = \begin{bmatrix} K_\tau & G \\ D & L_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ DK_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix},
$$

$$
S = L_\tau - DK_\tau G,
$$

Block-recursive preconditioning

General case:

$$\left[\begin{array}{c} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{array}\right] + \left[\begin{array}{cc:cc} K+C+A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hdashline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \left[\begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \hdashline \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_u \\ \mathbf{0} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{array}\right],$$

- Inner block matrix:

$$\tilde{K}_\tau = \begin{bmatrix} K_\tau & G \\ D & L_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ D{K_\tau}^{-1} & I \end{bmatrix} \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix},$$

$$S = L_\tau - DK_\tau G,$$

$$P_U(\tilde{K}_\tau) = \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix}^{-1} = \begin{bmatrix} K_\tau^{-1} & -K_\tau^{-1}GS^{-1} \\ 0 & S^{-1} \end{bmatrix},$$

Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \left[\begin{array}{cc|cc} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ \hline -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{array}\right] \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Inner block matrix:

$$\tilde{K}_\tau = \begin{bmatrix} K_\tau & G \\ D & L_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ DK_\tau{}^{-1} & I \end{bmatrix} \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix},$$

$$S = L_\tau - DK_\tau G,$$

$$P_U(\tilde{K}_\tau) = \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix}^{-1} = \begin{bmatrix} K_\tau^{-1} & -K_\tau^{-1} G S^{-1} \\ 0 & S^{-1} \end{bmatrix},$$

$$\tilde{S}^{-1} \approx L_p^{-1} \quad (\delta t L_p^{-1}).$$

Numerical experiments

Manufactured analytical solution:

- Colliding flow.

Two different turbulent benchmarks:

- Taylor-Green Vortex (TGV) flow.
- Turbulent Channel Flow (TCF).

Colliding flow

Problem setting:

- Analytical solution.
- $Re = 25$.
- Mesh refinement: 4^3 to 64^3 $Q1/Q1$ elements (ASGS and OSS) or 2^3 to $32^3 Q2/Q1$ elements (OSS-ISS)

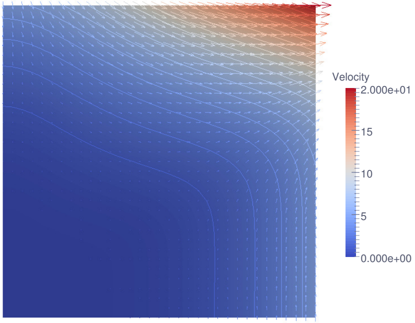
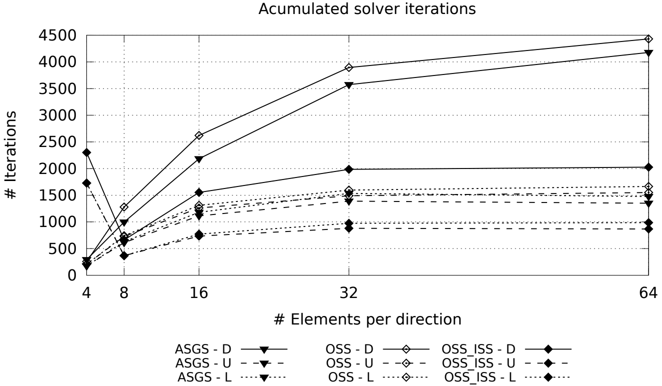


Figure : Velocity field

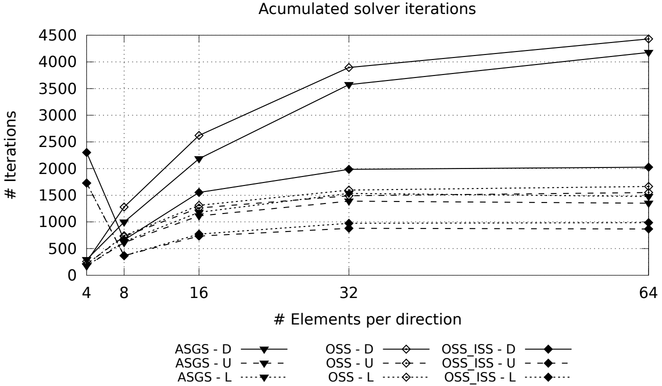
Colliding flow

Acumulated solver iterations: (using $P_U(\widetilde{A})$)



Colliding flow

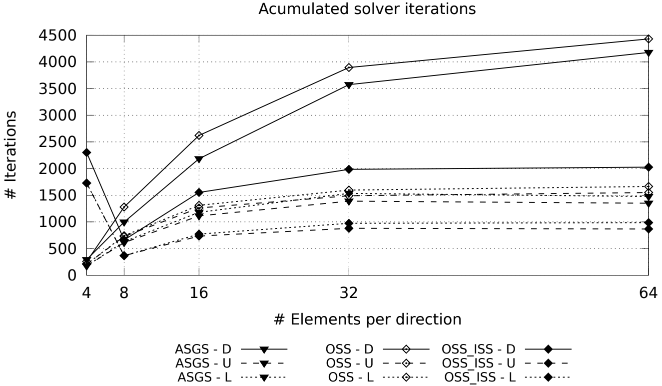
Acumulated solver iterations: (using $P_U(\widetilde{A})$)



- $P_U(\widetilde{K}_\tau)$ and $P_L(\widetilde{K}_\tau)$ scalable block-preconditioners for all methods.

Colliding flow

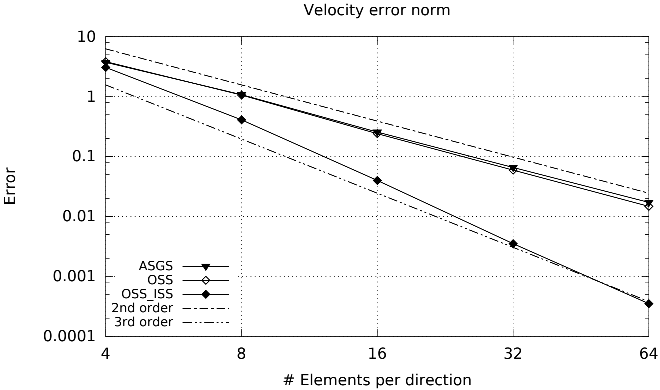
Acumulated solver iterations: (using $P_U(\widetilde{A})$)



- $P_U(\widetilde{K}_\tau)$ and $P_L(\widetilde{K}_\tau)$ scalable block-preconditioners for all methods.
- Less solver iterations for the OSS-ISS method with the same velocity DOFs.

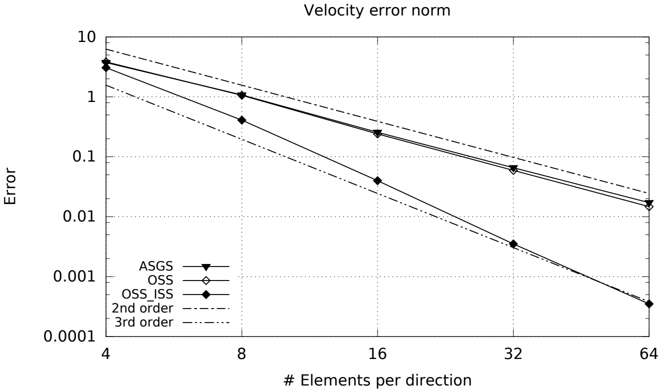
Colliding flow

Accuracy: Velocity error



Colliding flow

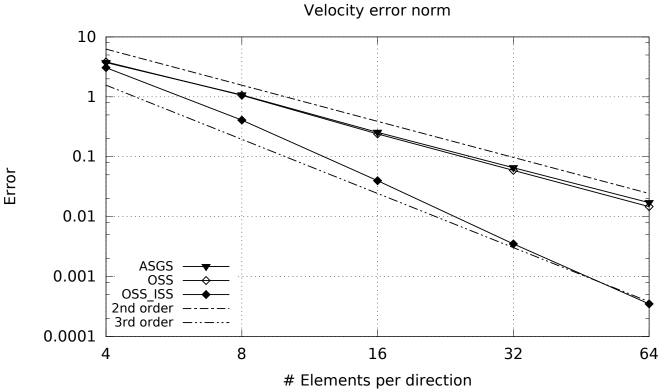
Accuracy: Velocity error



- 2nd order convergence rate for ASGS and OSS methods.

Colliding flow

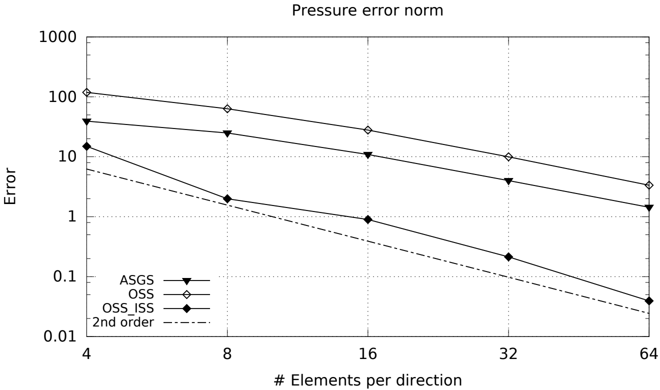
Accuracy: Pressure error



- 2nd order convergence rate for ASGS and OSS methods.
- 3rd order convergence rate for OSS-ISS method.

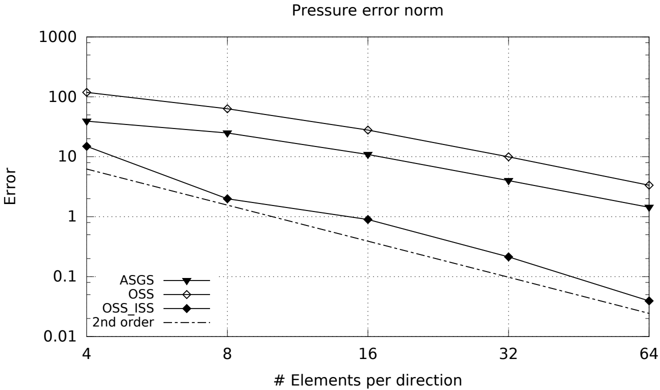
Colliding flow

Accuracy: Pressure error



Colliding flow

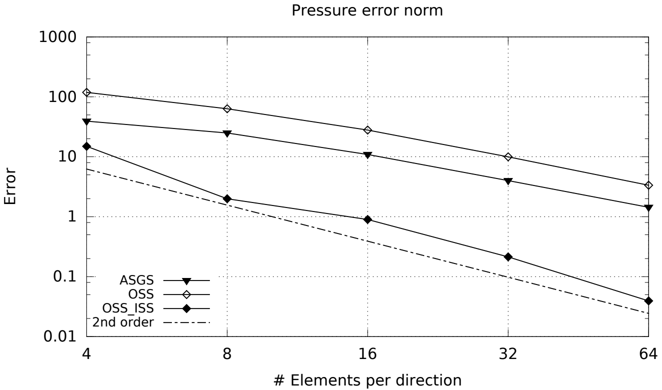
Accuracy:



- 2nd order convergence rate for all methods.

Colliding flow

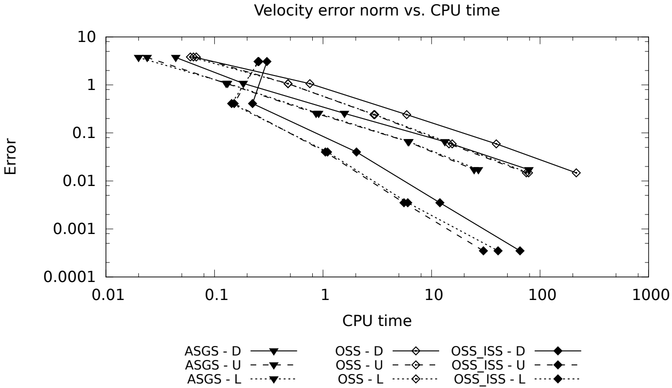
Accuracy:



- 2nd order convergence rate for all methods.
- Best accuracy for OSS-ISS method.

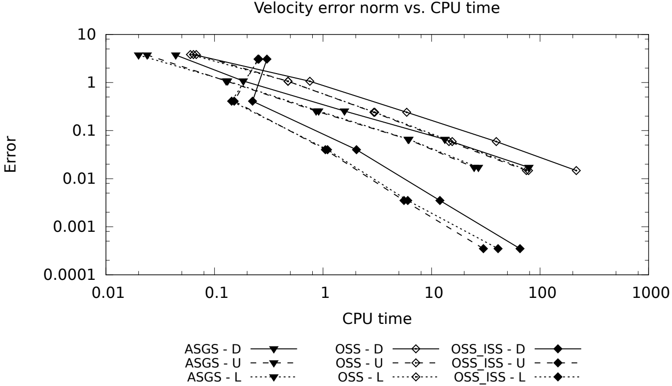
Colliding flow

Efficiency: Velocity



Colliding flow

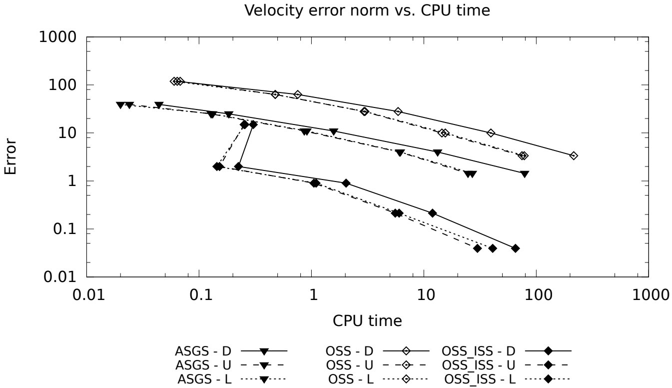
Efficiency: Velocity



- OSS-ISS the most efficient method.

Colliding flow

Efficiency: Pressure



- Also for pressures.

TGV Taylor-Green Vortex flow

Energy dissipation rate (different methods):

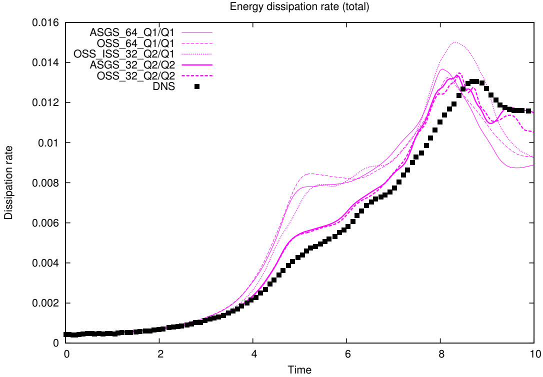


Figure : Total energy dissipation rate

TGV Taylor-Green Vortex flow

Energy dissipation rate (different methods):

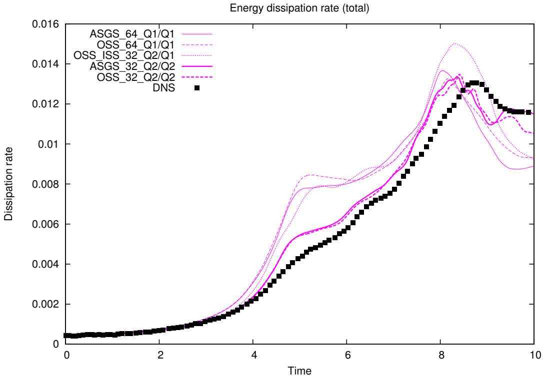


Figure : Total energy dissipation rate

- Good agreement with the DNS (coarse mesh).

TGV Taylor-Green Vortex flow

Energy dissipation rate (different methods):

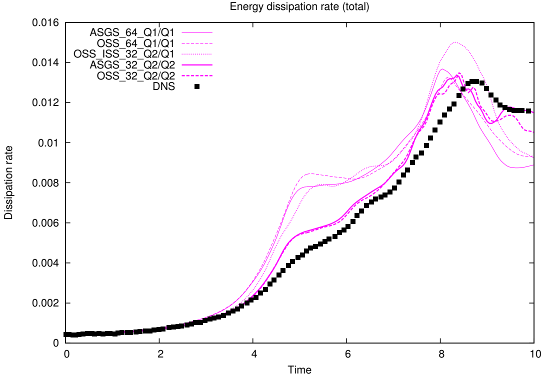


Figure : Total energy dissipation rate

- Good agreement with the DNS (coarse mesh).
- More accurate results with equal-order elements.

TGV Taylor-Green Vortex flow

Energy dissipation rate (refinement OSS-ISS):

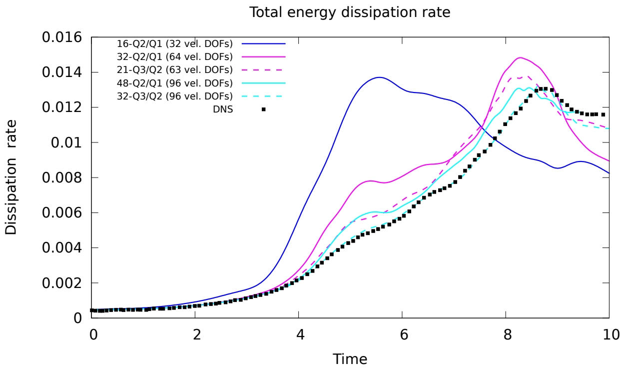


Figure : Total energy dissipation rate

TGV Taylor-Green Vortex flow

Energy dissipation rate (refinement OSS-ISS):

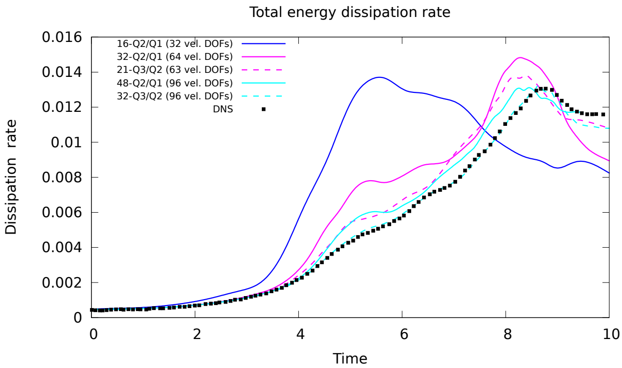


Figure : Total energy dissipation rate

- Good agreement with the DNS.

TGV Taylor-Green Vortex flow

Energy dissipation rate (refinement OSS-ISS):

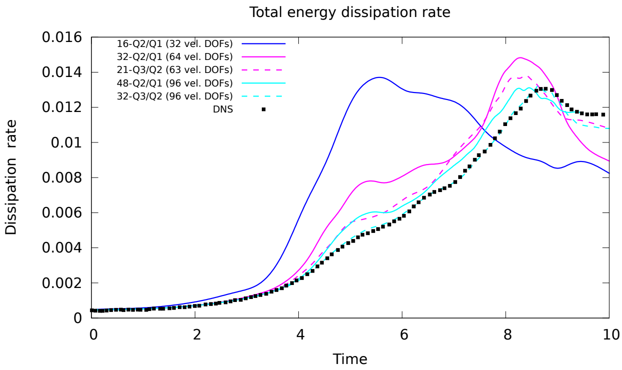


Figure : Total energy dissipation rate

- Good agreement with the DNS.
- 32^3 $Q3/Q2$ elements mesh on top of DNS.

TGV Taylor-Green Vortex flow

Effect of the grad-div term $(\tau_c \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})$: (coarse mesh)

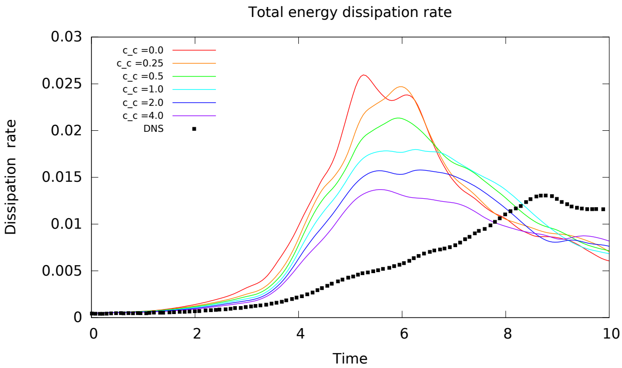


Figure : Total energy dissipation rate.

TGV Taylor-Green Vortex flow

Effect of the grad-div term $(\tau_c \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})$**:** (coarse mesh)

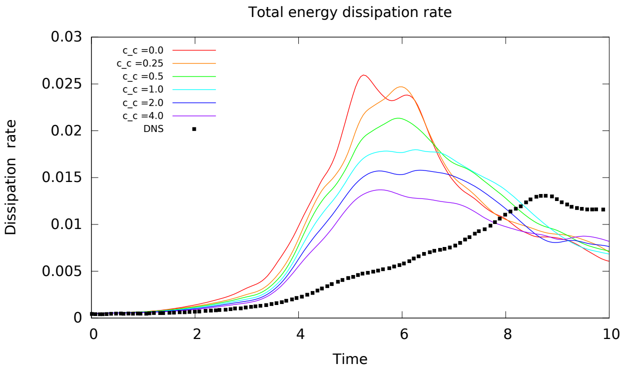


Figure : Total energy dissipation rate.

- Bad results when $c_c \to 0$.

TGV Taylor-Green Vortex flow

Effect of the grad-div term $(\tau_c \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})$: (coarse mesh)

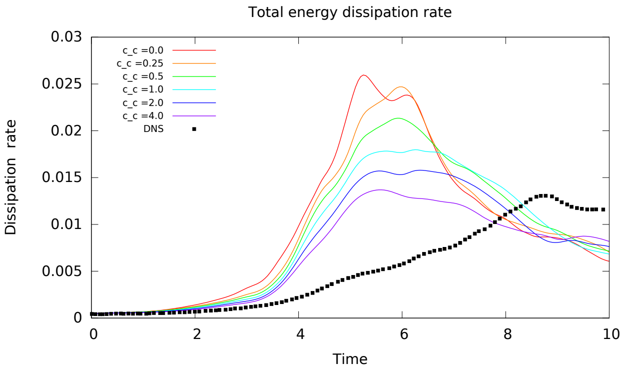


Figure : Total energy dissipation rate.

- Bad results when $c_c \to 0$.
- Best option $c_c = 4.0$.

TGV Taylor-Green Vortex flow

Effect of the grad-div term $(\tau_c \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})$: (coarse mesh)

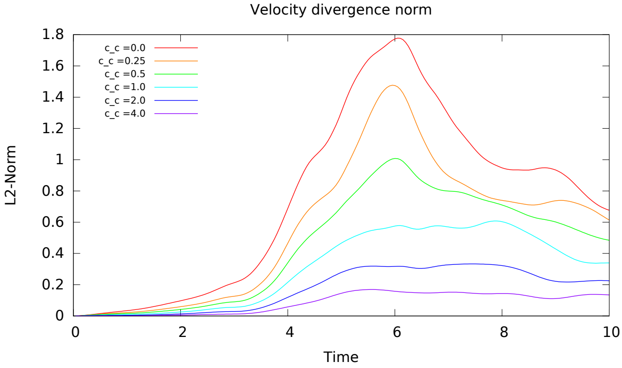


Figure : $\|\nabla \cdot \mathbf{u}\|$.

- Bad results when $c_c \to 0$.

TGV Taylor-Green Vortex flow

Effect of the grad-div term $(\tau_c \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})$: (coarse mesh)

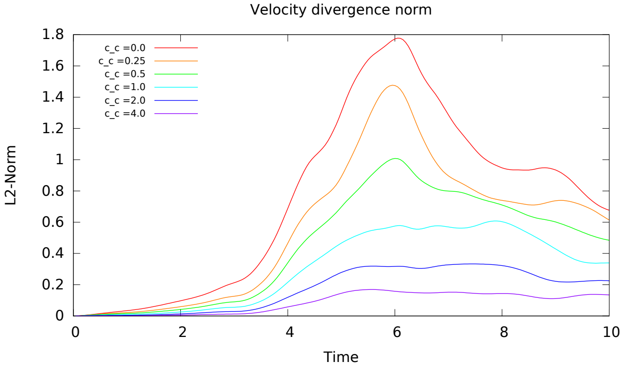


Figure : $\|\nabla \cdot \mathbf{u}\|$.

- Bad results when $c_c \to 0$.
- Incompressibility constraint not satisfied.

TGV Taylor-Green Vortex flow

Effect of the grad-div term $(\tau_c \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})$: (coarse mesh)

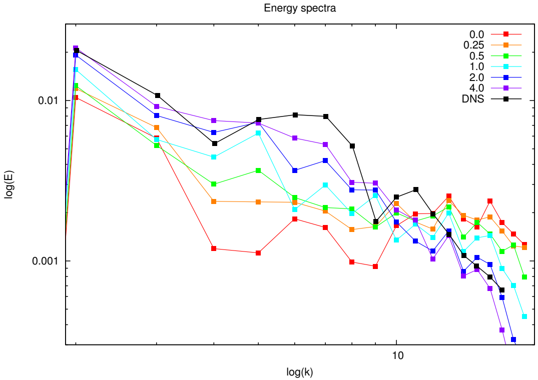


Figure : Energy spectra.

- Bad results when $c_c \to 0$.

TGV Taylor-Green Vortex flow

Effect of the grad-div term $(\tau_c \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})$**:** (coarse mesh)

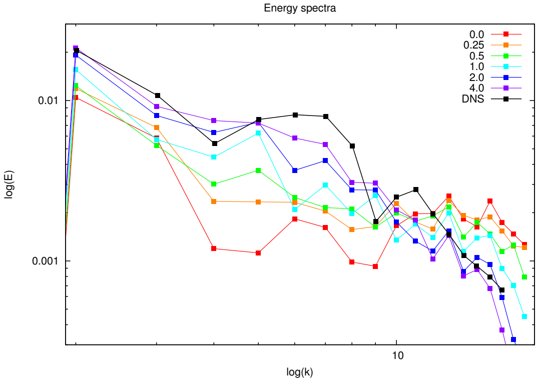


Figure : Energy spectra.

- Bad results when $c_c \to 0$.
- Overdissipation on the large scales when $c_c \to 0$.

TGV Taylor-Green Vortex flow

Effect of the grad-div term $(\tau_c \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})$**:** (coarse mesh)

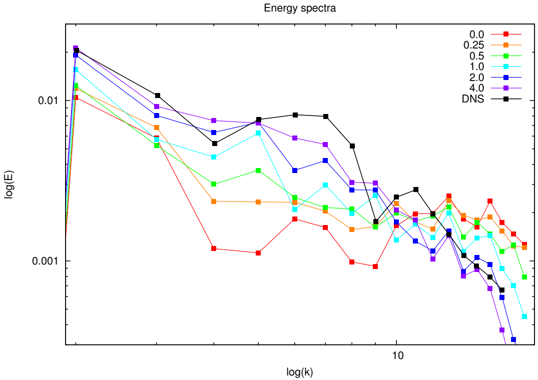
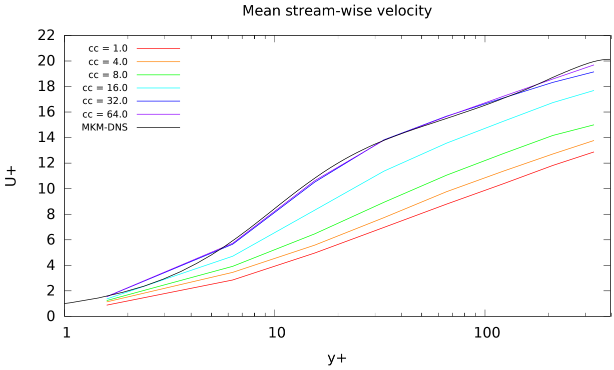


Figure : Energy spectra.

- Bad results when $c_c \to 0$.
- Overdissipation on the large scales when $c_c \to 0$.

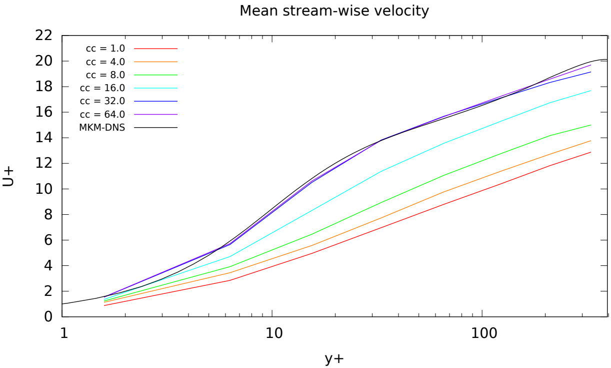
TCF Turbulent Channel Flow at $Re_\tau = 395$

Mean streamwise velocity (models):



TCF Turbulent Channel Flow at $Re_\tau = 395$

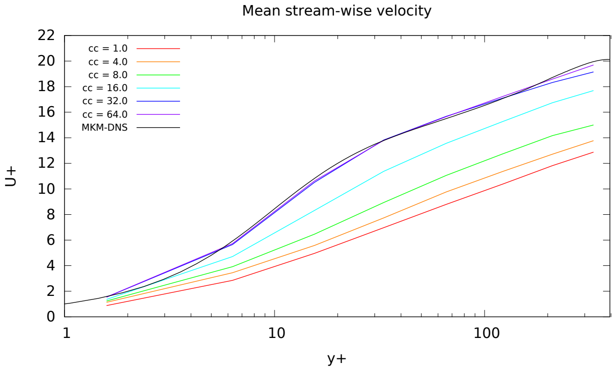
Mean streamwise velocity (models):



- Same behaviour observed for TGV test.

TCF Turbulent Channel Flow at $Re_\tau = 395$

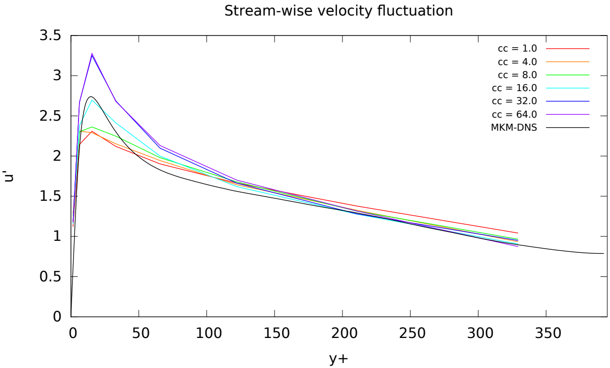
Mean streamwise velocity (models):



- Same behaviour observed for TGV test.
- Best option $c_c = 32.0$.

TCF Turbulent Channel Flow at $Re_\tau = 395$

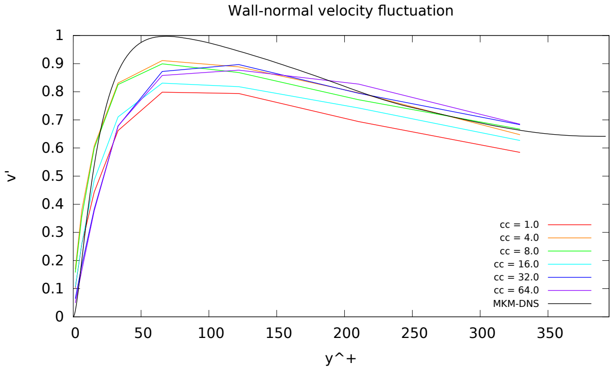
Mean streamwise velocity (models):



- Same behaviour observed for TGV test.
- Best option $c_c = 32.0$.

TCF Turbulent Channel Flow at $Re_\tau = 395$

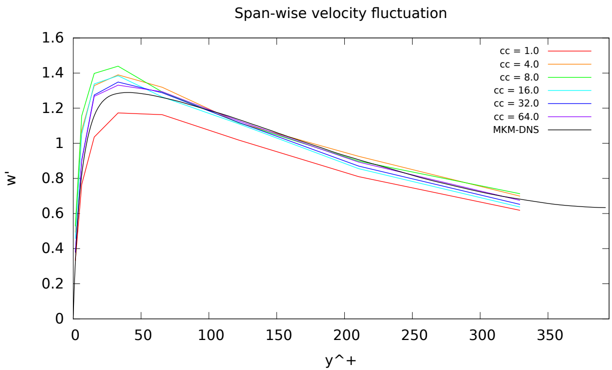
Mean streamwise velocity (models):



- Same behaviour observed for TGV test.
- Best option $c_c = 32.0$.

TCF Turbulent Channel Flow at $Re_\tau = 395$

Mean streamwise velocity (models):



- Same behaviour observed for TGV test.
- Best option $c_c = 32.0$.

Mixed FE VMS Conclusions

- Scalable recursive block-preconditioners.

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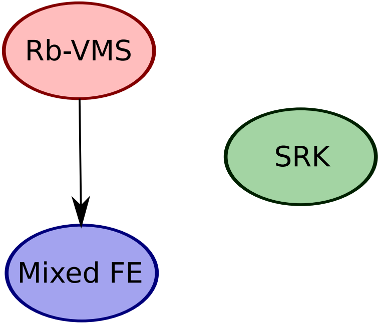
- Mixed FE OSS the most efficient method.

Mixed FE VMS Conclusions

- Scalable recursive block-preconditioners.
- Sligthly better results for equal-order interpolations.
- Mixed FE OSS the most efficient method.
- Mixed FE OSS keeps the index-2 DAE nature in time of the problem.

Mixed FE VMS Limitations

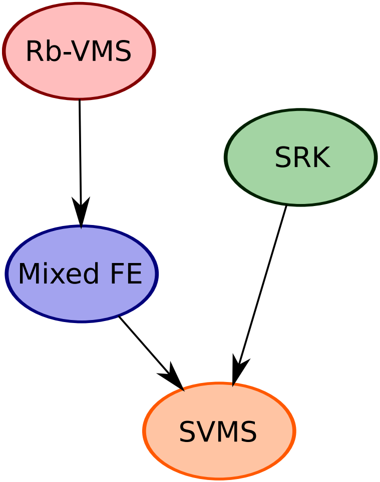
- Strong dependency on the grad-div stabilization term.


Rb-VMS
SRK
Mixed FE

Motivation

Step by step...

- Residual-based VMS as LES models.
- Mixed FE formulations LES.
- High-order FE methods.
- High-order time integration schemes.
- Adaptive time stepping techniques.
- Velocity-pressure segregation.
- Scalable solvers.
- Application.


Rb-VMS
SRK
Mixed FE
SVMS

Outline

- Line 1.

Outline

- Line 1.
- Line 2.
  Less formal

Outline

- Line 1.
- Line 2.
  Less formal
- Line 3.
  Less formal, different color.

Blocks

Standard Block

This is a standard block.

Example Block

This is an example block.

Alert Block

This is an alert block.

Questions?