

# Large scale Finite Element solvers for the large eddy simulation of incompressible turbulent flows

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## 1. Motivation

## 2. Residual-based VMS

## 3. Mixed FE VMS

## 4. Segregated Runge-Kutta

## 5. Segregated VMS

## 6. Conclusions

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## Thesis motivation

Highly scalable Finite Element (FE) framework for Large Eddy Simulations (LES) of incompressible turbulent flows

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Highly scalable Finite Element (FE) framework for Large Eddy Simulations (LES) of incompressible turbulent flows

## How to get there?

1. Variational MultiScale (VMS) methods as LES models.
2. Time integration schemes with velocity-pressure segregation.
3. Highly scalable algorithms based on **Domain Decomposition (DD)** and **block preconditioners**.

# Motivation

## Step by step...

- Residual-based VMS as LES models.

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## 1. Motivation

Rb-VMS

## 2. Residual-based VMS

Formulation

Energy statements

Numerical experiments

Conclusions

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## 4. Segregated Runge-Kutta

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## 6. Conclusions

# Implicit LES

**ILES:** only numerical dissipation (for stabilization) acts as turbulent model

- Not based on filtering of the Navier-Stokes equations
- Rely on numerical artifacts, no modification at the continuous level

# Incomp. Navier Stokes equations

Find  $\mathbf{u}$  and  $p$  defined in  $\Omega$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

with appropriate boundary conditions on  $\Gamma$ .

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with appropriate boundary conditions on  $\Gamma$ .

The weak problem is:  $\forall \mathbf{v} \in \mathcal{V}_0$  and  $\forall q \in \mathcal{Q}_0$ , find  $\mathbf{u} \in \mathcal{V}$  and  $p \in \mathcal{Q}$  such that

$$\begin{aligned} (\mathbf{v}, \partial_t \mathbf{u})_\Omega + (\nabla \mathbf{v}, \nu \nabla \mathbf{u})_\Omega + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p)_\Omega &= \langle \mathbf{v}, \mathbf{f} \rangle_\Omega \\ (q, \nabla \cdot \mathbf{u})_\Omega &= 0 \end{aligned}$$

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where

$$b(\mathbf{a}, \mathbf{u}, \mathbf{v}) = \langle \mathbf{v}, \mathbf{a} \cdot \nabla \mathbf{u} \rangle_\Omega$$

# Incomp. Navier Stokes equations

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where

$$b(\mathbf{a}, \mathbf{u}, \mathbf{v}) = \frac{1}{2} \langle \mathbf{v}, \mathbf{a} \cdot \nabla \mathbf{u} \rangle_\Omega - \frac{1}{2} \langle \mathbf{a} \cdot \nabla \mathbf{v}, \mathbf{u} \rangle_\Omega + \frac{1}{2} \langle \mathbf{v}, \mathbf{n} \cdot \mathbf{a} \mathbf{u} \rangle_\Gamma$$

# VMS decomposition (Hughes 1995)

A decomposition of spaces  $\mathcal{V}$  and  $\mathcal{Q}$  given by

$$\mathcal{V} = \mathcal{V}_h \oplus \tilde{\mathcal{V}}, \quad \mathcal{Q} = \mathcal{Q}_h \oplus \tilde{\mathcal{Q}}$$

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$$\mathcal{V} = \mathcal{V}_h \oplus \tilde{\mathcal{V}}, \quad \mathcal{Q} = \mathcal{Q}_h \oplus \tilde{\mathcal{Q}}$$

is applied to the function and test spaces

$$\mathbf{u} = \mathbf{u}_h + \tilde{\mathbf{u}}, \quad p = p_h + \tilde{p}$$

$$\mathbf{v} = \mathbf{v}_h + \tilde{\mathbf{v}}, \quad q = q_h + \tilde{q}$$

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We keep all the (eight) contributions from the splitting of the convective term

$$\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{u}_h \cdot \nabla \mathbf{u}_h + \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_h + \mathbf{u}_h \cdot \nabla \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}$$

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is applied to the function and test spaces

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We keep all the (eight) contributions from the splitting of the convective term

$$\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{u}_h \cdot \nabla \mathbf{u}_h + \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_h + \mathbf{u}_h \cdot \nabla \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}$$

and all the (four) contributions from the temporal term

$$\partial_t \mathbf{u} = \partial_t \mathbf{u}_h + \partial_t \tilde{\mathbf{u}}$$

# Semidiscrete problem

## FEM equations

$$B((\mathbf{u}_h, p_h); (\tilde{\mathbf{u}}, \tilde{p}) ; (\mathbf{v}_h, q_h)) = L(\mathbf{v}_h, q_h)$$

## SGS equations

$$B((\tilde{\mathbf{u}}, \tilde{p}) ; (\mathbf{u}_h, p_h) ; (\tilde{\mathbf{v}}, \tilde{q})) = L(\tilde{\mathbf{v}}, \tilde{q})$$

# Semidiscrete problem

## FEM equations

$$\begin{aligned}
 & (\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega \\
 & + (\mathbf{v}_h, \partial_t \tilde{\mathbf{u}})_\Omega + (\mathcal{L}^* \mathbf{v}_h, \tilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \tilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega \\
 & (q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \tilde{\mathbf{u}})_{\Omega^h} = 0
 \end{aligned}$$

## SGS equations

$$\partial_t \tilde{\mathbf{u}} + \tau_m^{-1} \tilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \tilde{p} = \mathcal{P} R_c$$

$$\tau_m = \left( \frac{c_1 \nu}{h^2} + \frac{c_2 |\mathbf{a}|}{h} \right)^{-1}, \quad \tau_c = \frac{h^2}{c_1 \tau_m}$$

$$\mathbf{a} = \mathbf{u}_h + \tilde{\mathbf{u}}$$

# Semidiscrete problem

## FEM equations

$$\begin{aligned}
 & (\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega \\
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 & (q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \tilde{\mathbf{u}})_{\Omega^h} = 0
 \end{aligned}$$

## SGS equations

$$\partial_t \tilde{\mathbf{u}} + \tau_m^{-1} \tilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \tilde{p} = \mathcal{P} \mathbf{R}_c$$

$$\mathbf{R}_m := \mathbf{f} - \partial_t \mathbf{u}_h - \mathcal{L} \mathbf{u}_h - \nabla p_h, \quad \mathbf{R}_c := -\nabla \cdot \mathbf{u}_h$$

# Semidiscrete problem

## FEM equations

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 & (\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega \\
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 & (q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \tilde{\mathbf{u}})_{\Omega^h} = 0
 \end{aligned}$$

## SGS equations

$$\partial_t \tilde{\mathbf{u}} + \tau_m^{-1} \tilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \tilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = I \quad (\text{ASGS}), \quad \mathcal{P} = P_h^\perp = I - P_h \quad (\text{OSS})$$

# Semidiscrete problem

## FEM equations

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 & (q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \tilde{\mathbf{u}})_{\Omega^h} = 0
 \end{aligned}$$

## SGS equations

$$\partial_t \tilde{\mathbf{u}} + \tau_m^{-1} \tilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

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$$\mathbf{a} = \mathbf{u}_h + \tilde{\mathbf{u}}$$

# Summary

	Sgs space	Sgs dynamics	Advection
1	ASGS	Static	Linear
2	ASGS	Dynamic	Linear
3	ASGS	Dynamic	Nonlinear
4	OSS	Static	Linear
5	OSS	Dynamic	Linear
6	OSS	Dynamic	Nonlinear

- 1 It is the most standard method (SUPG for linear elements) up to the choice of the stabilization parameters. Unknown stability properties.
- 4 Strictly pointwise positive for linear elements (no backscatter).
- 5 Convergent to weak solutions of NS equations (Badia & Gutierrez 2012).

# Energy statements

**FE counterpart:**

$$B((\mathbf{u}_h, p_h); (\tilde{\mathbf{u}}, \tilde{p}); (\mathbf{u}_h, p_h)) = L(\mathbf{u}_h, p_h)$$

**SGS counterpart:**

$$B((\tilde{\mathbf{u}}, \tilde{p}); (\mathbf{u}_h, p_h); (\tilde{\mathbf{u}}, \tilde{p})) = L(\tilde{\mathbf{u}}, \tilde{p})$$

**TOTAL:**

$$\begin{aligned} & B((\mathbf{u}_h, p_h); (\tilde{\mathbf{u}}, \tilde{p}); (\mathbf{u}_h, p_h)) \\ & + B((\tilde{\mathbf{u}}, \tilde{p}); (\mathbf{u}_h, p_h); (\tilde{\mathbf{u}}, \tilde{p})) = L(\mathbf{u}_h, p_h) + L(\tilde{\mathbf{u}}, \tilde{p}) \end{aligned}$$

# Energy statements

## FE counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle \end{aligned}$$

## SGS counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

## TOTAL:

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**TOTAL:**

$$\frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2$$

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# Energy statements

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## TOTAL:

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$$\begin{aligned} & \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

## TOTAL:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) \\ & - (\nabla \cdot \mathbf{u}_h, \tilde{p}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle + \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

# Energy statements

## FE counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle \end{aligned}$$

## SGS counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

## TOTAL: Static subscales

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) \\ & - (\nabla \cdot \mathbf{u}_h, \tilde{p}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle + \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

# Energy statements

**FE counterpart:**

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle \end{aligned}$$

**SGS counterpart:**

$$\begin{aligned} & \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

**TOTAL: Dynamic subscales - ASGS**

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h + \tilde{\mathbf{u}}\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & - 2(\nu \Delta \mathbf{u}_h, \tilde{\mathbf{u}}) = \langle \mathbf{f}, \mathbf{u}_h \rangle + \langle \mathbf{f}, \tilde{\mathbf{u}} \rangle \end{aligned}$$

# Energy statements

## FE counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + b(\mathbf{a}, \mathbf{u}_h, \mathbf{u}_h) \\ & + (\partial_t \tilde{\mathbf{u}}, \mathbf{u}_h) + (\mathcal{L}_a^*(\mathbf{u}_h, p_h), \tilde{\mathbf{u}}) - (\nabla \cdot \mathbf{u}_h, \tilde{p}) = \langle \mathbf{f}, \mathbf{u}_h \rangle \end{aligned}$$

## SGS counterpart:

$$\begin{aligned} & \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & + (\mathcal{P}(\partial_t \mathbf{u}_h), \tilde{\mathbf{u}}) + (\mathcal{P}(\mathcal{L}_a^*(\mathbf{u}_h, p_h)), \tilde{\mathbf{u}}) - (\mathcal{P}(\nabla \cdot \mathbf{u}_h), \tilde{p}) = \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

## TOTAL: Dynamic subscales - OSS

$$\begin{aligned} & \frac{1}{2} d_t \|\mathbf{u}_h\|^2 + \frac{1}{2} d_t \|\tilde{\mathbf{u}}\|^2 + \nu \|\nabla \mathbf{u}_h\|^2 + \tau_m^{-1} \|\tilde{\mathbf{u}}\|^2 + \tau_c^{-1} \|\tilde{p}\|^2 \\ & - 2(\nu \Delta \mathbf{u}_h, \tilde{\mathbf{u}}) \\ & = \langle \mathbf{f}, \mathbf{u}_h \rangle + \langle \mathcal{P}(\mathbf{f}), \tilde{\mathbf{u}} \rangle \end{aligned}$$

# Numerical experiments

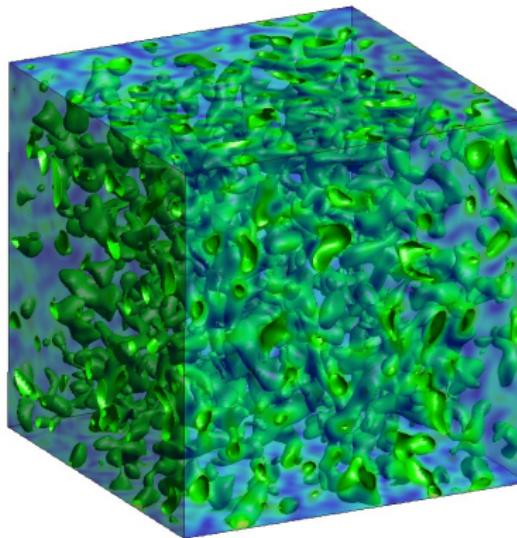
Three different turbulent benchmarks:

- Decaying of Homogeneous Isotropic Turbulence (DHIT).
- Taylor-Green Vortex (TGV) flow.
- Turbulent Channel Flow (TCF).

# DHIT Decay of Homogeneous Isotropic Turbulence

## Problem setting:

- Prescribed initial energy spectra corresponding to  $Re_\lambda = 952$ .
- Setting defined in AGARD database (Mansour & Wray 1993).
- A (very simple) time step adaptation technique is used.
- Different mesh discretizations ( $Q_1/Q_1$  and  $Q_2/Q_2$ ).



# DHIT Decay of Homogeneous Isotropic Turbulence

## Energy spectra (models):

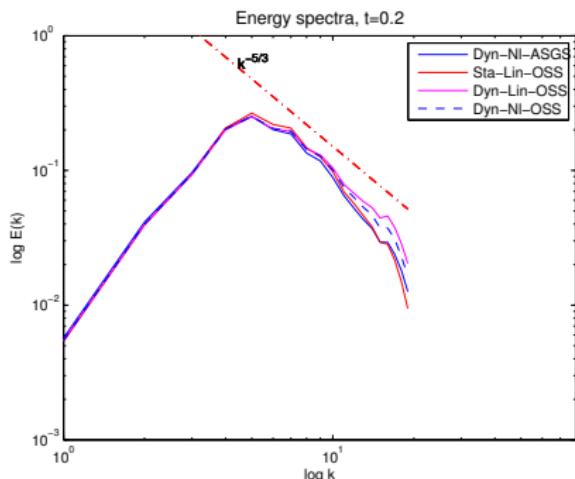


Figure :  $32^3 - Q1$ ,  $t = 0.2s$

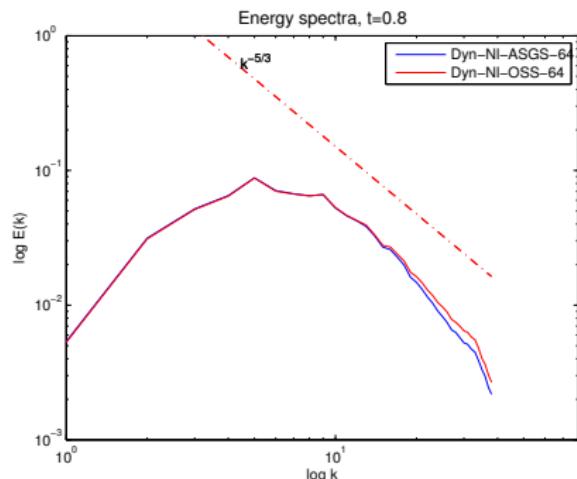


Figure :  $64^3 - Q1$ ,  $t = 0.8s$

# DHIT Decay of Homogeneous Isotropic Turbulence

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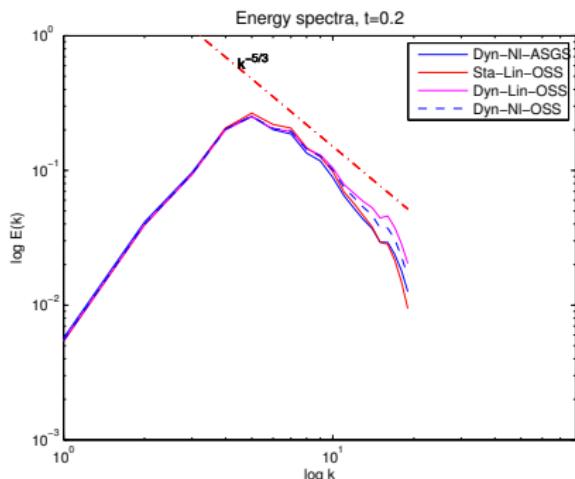


Figure :  $32^3 - Q1$ ,  $t = 0.2s$

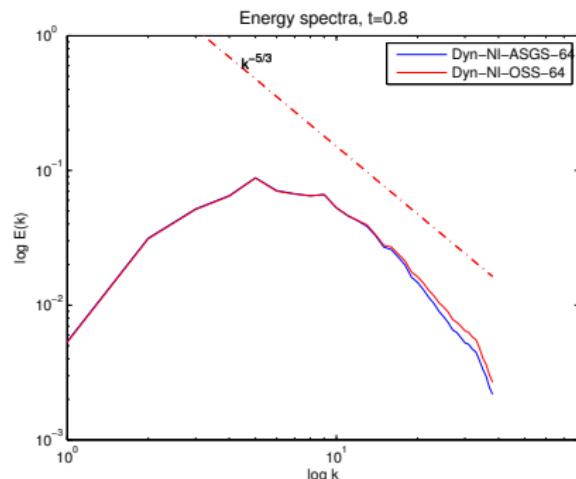


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- Small differences between methods (physical sense).

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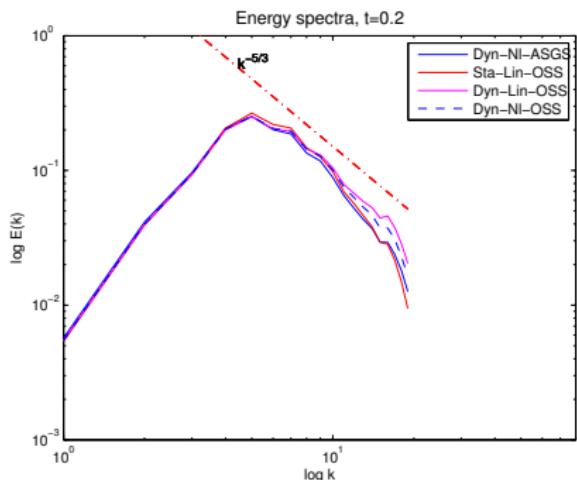


Figure :  $32^3 - Q1$ ,  $t = 0.2s$

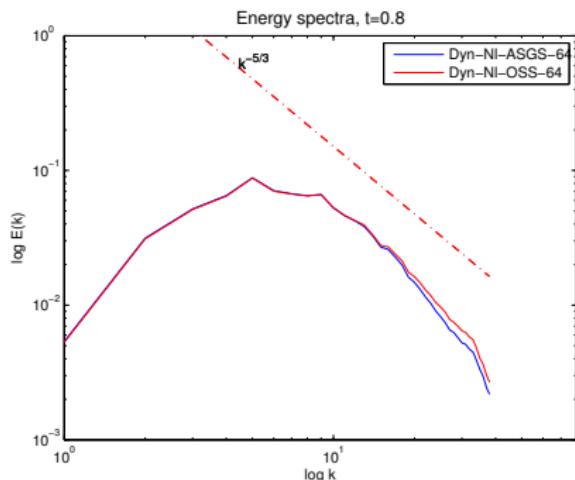


Figure :  $64^3 - Q1$ ,  $t = 0.8s$

- Small differences between methods (physical sense).
- Even more similar when we refine the mesh.

# DHIT Decay of Homogeneous Isotropic Turbulence

## Computational cost (models):

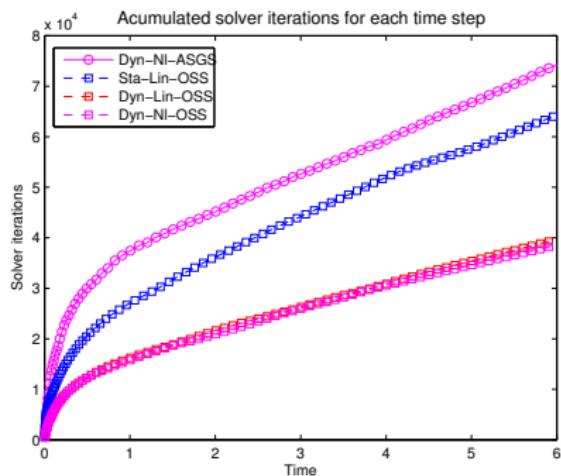


Figure :  $32^3$  – Q1

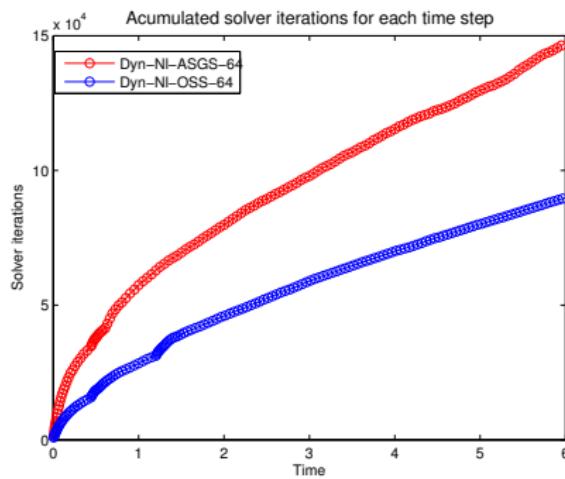


Figure :  $64^3$  – Q1

# DHIT Decay of Homogeneous Isotropic Turbulence

## Computational cost (models):

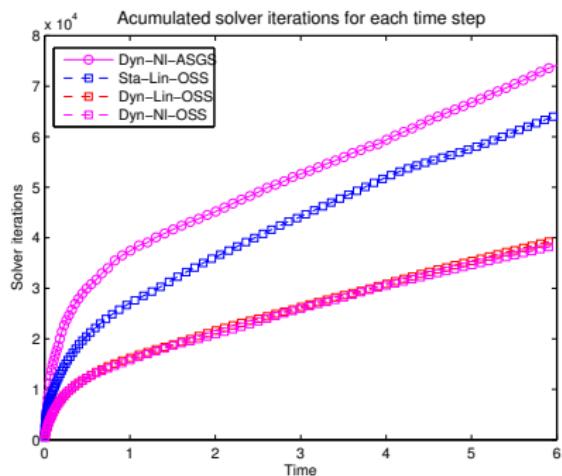


Figure :  $32^3$  – Q1

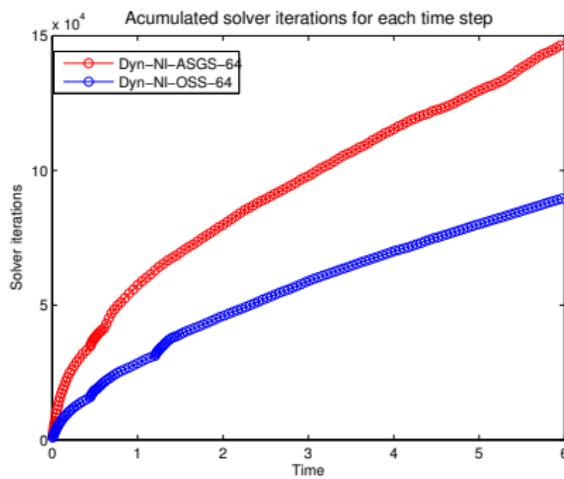


Figure :  $64^3$  – Q1

- Big differences between methods (computational sense).

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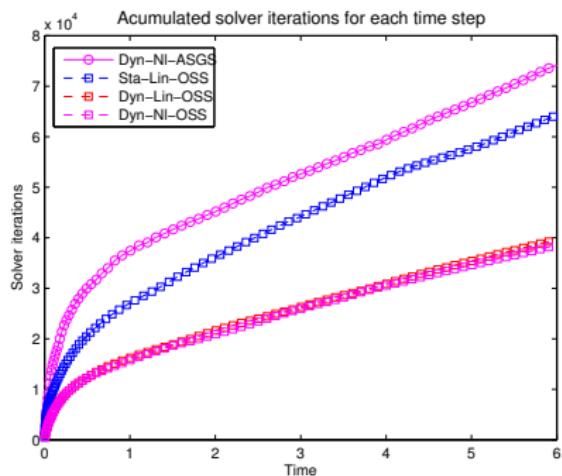


Figure :  $32^3$  – Q1

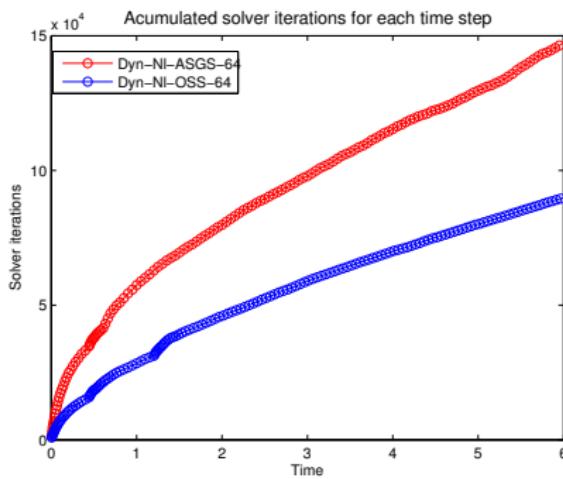
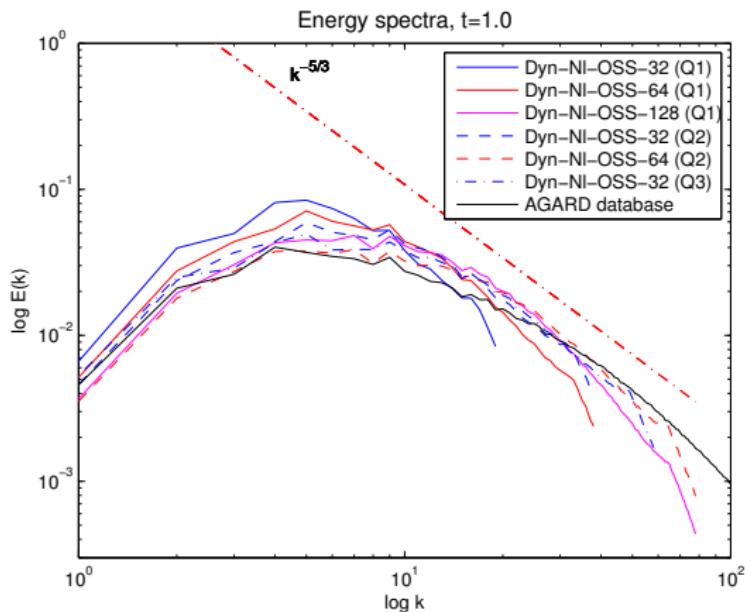


Figure :  $64^3$  – Q1

- Big differences between methods (computational sense).
- **Dynamic** versions of **OSS** method are **the most efficient**.

# DHIT Decay of Homogeneous Isotropic Turbulence

## Energy spectra (refinement):



- Results become closer to the DNS when we refine the mesh.

# TGV Taylor-Green Vortex flow

## Problem setting:

- Prescribed initial condition.
- $Re = 1600$ .
- Different mesh discretizations ( $Q_1/Q_1$  and  $Q_2/Q_2$ ).

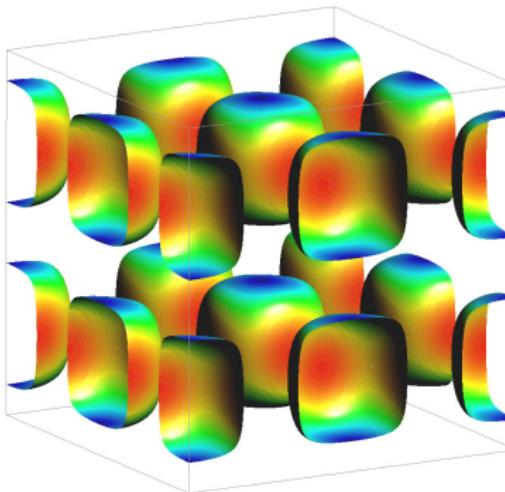


Figure : Initial vorticity isosurface  $|\omega| = 1$

# TGV Taylor-Green Vortex flow

## Problem setting:

- Prescribed initial condition.
- $Re = 1600$ .
- Different mesh discretizations ( $Q_1/Q_1$  and  $Q_2/Q_2$ ).

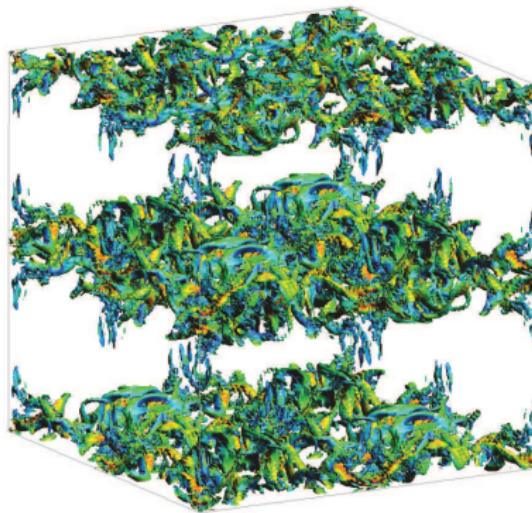


Figure : Vorticity isosurfaces  $|\omega| = 9.0$

# TGV Taylor-Green Vortex flow

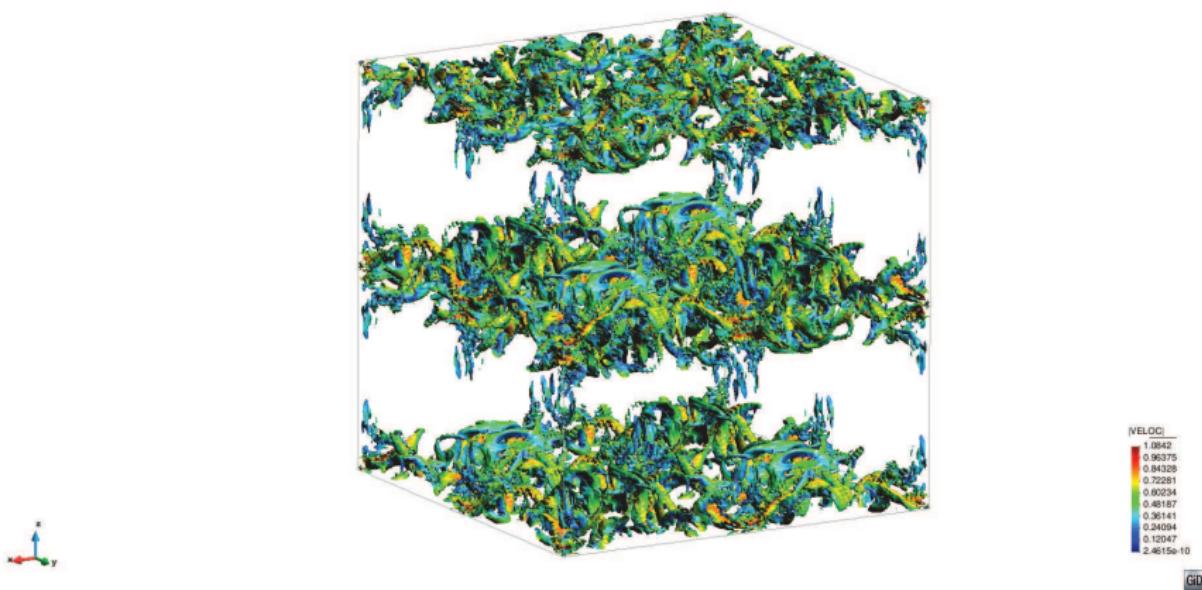


Figure : Velocity isosurface

# TGV Taylor-Green Vortex flow

## Energy dissipation rate (refinement):

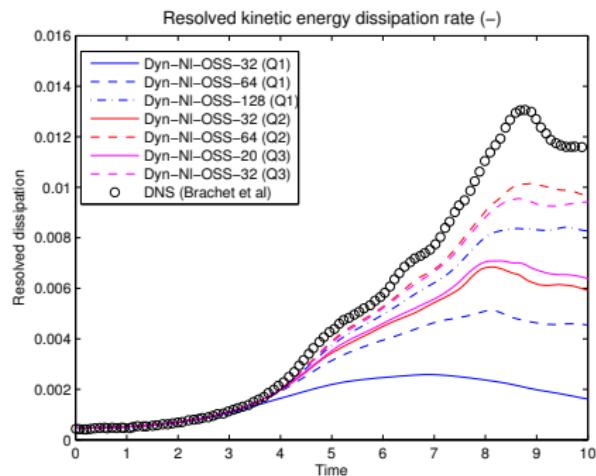


Figure : Resolved scales

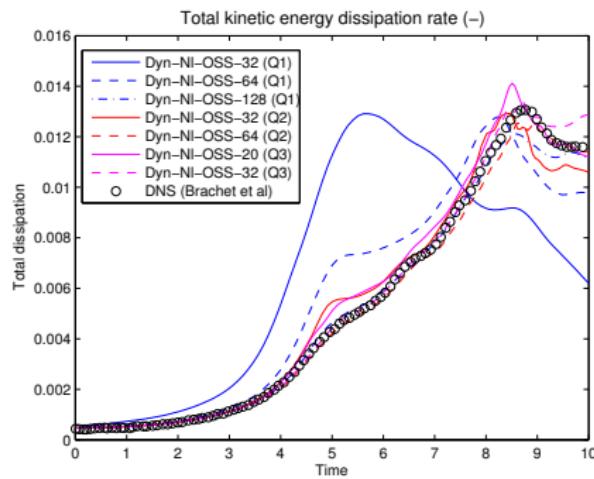


Figure : Total

# TGV Taylor-Green Vortex flow

## Energy dissipation rate (refinement):

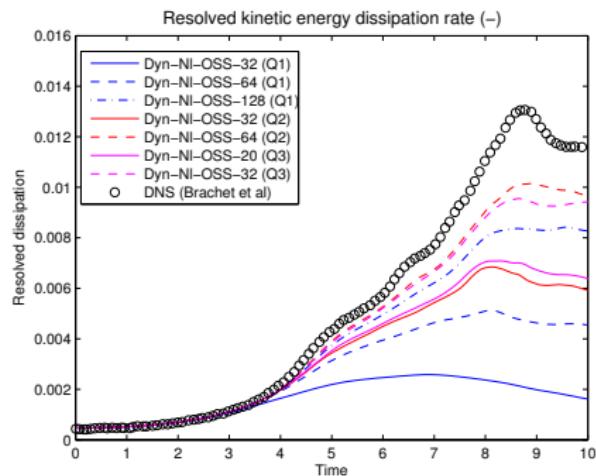


Figure : Resolved scales

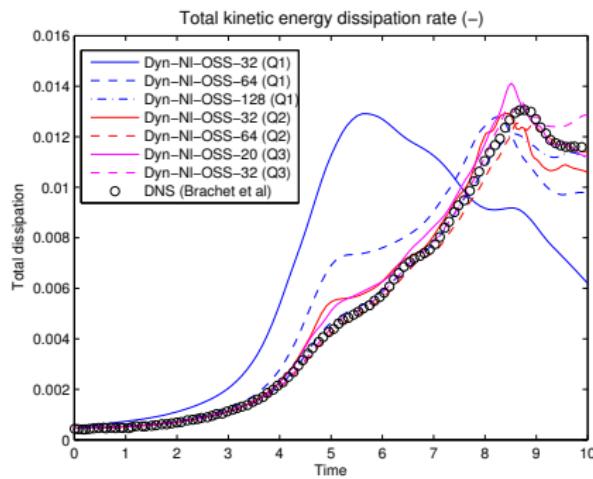


Figure : Total

- Good agreement with the DNS taking account the subscales.

# TGV Taylor-Green Vortex flow

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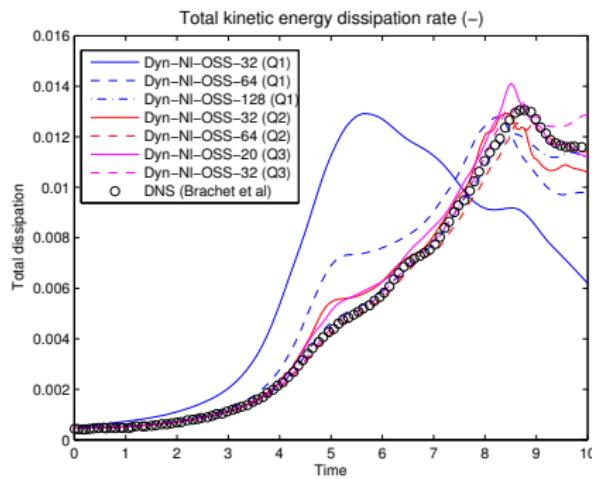
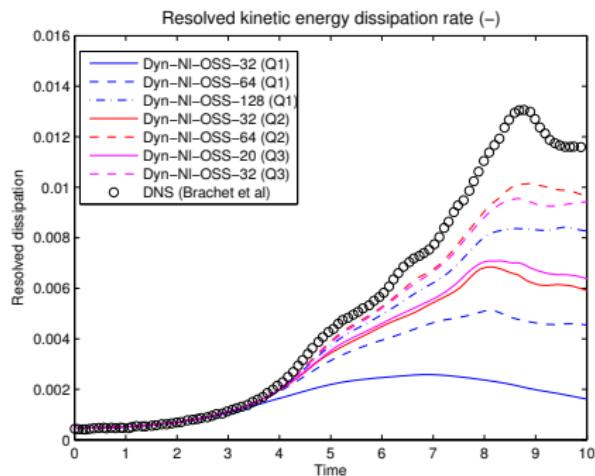


Figure : Resolved scales

Figure : Total

- Good agreement with the DNS taking account the subscales.
- More accurate results increasing the order of approximation.

# TGV Taylor-Green Vortex flow

- All results until now are compared against **DNS**.
- Are our methods comparable with **LES** models?

# TGV Taylor-Green Vortex flow

## Energy dissipation rate (against LES model):

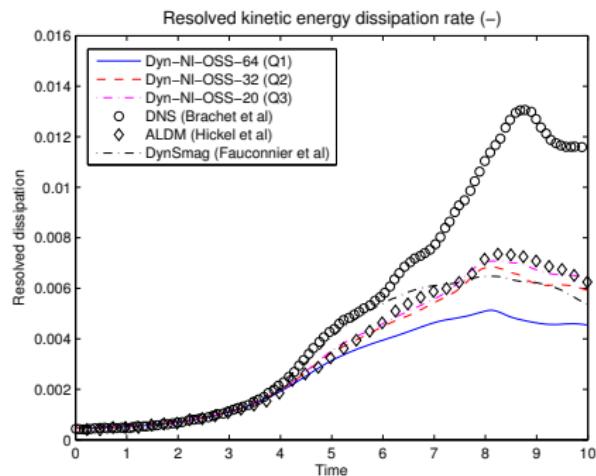


Figure : Resolved scales

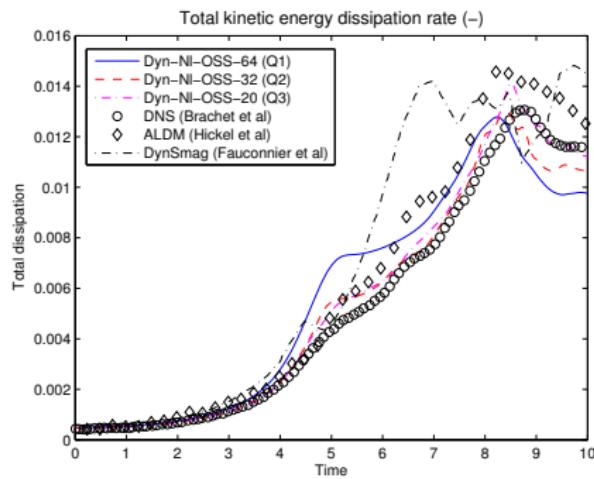


Figure : Total

# TGV Taylor-Green Vortex flow

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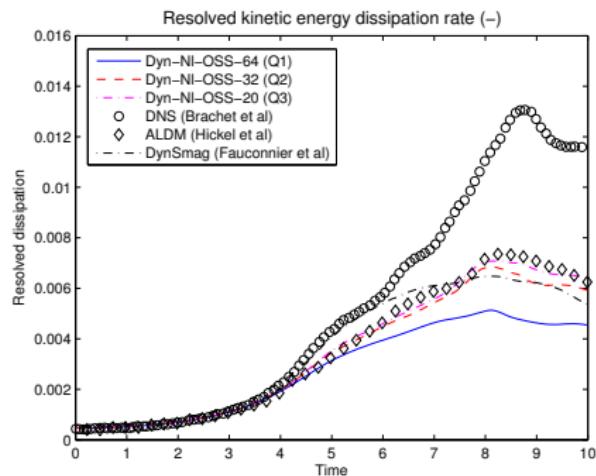


Figure : Resolved scales

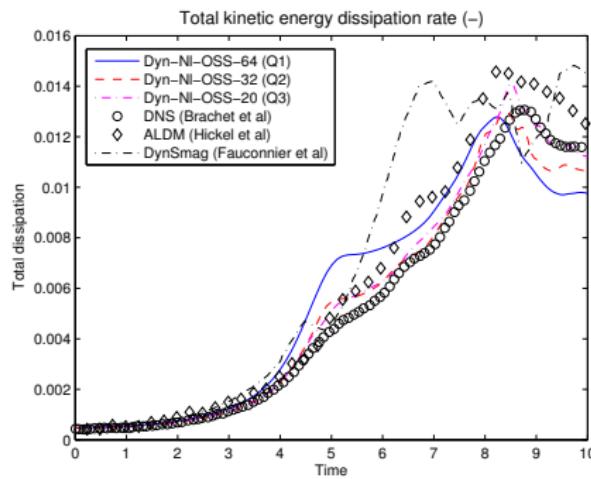


Figure : Total

- Good agreement with the LES models on resolved scales.

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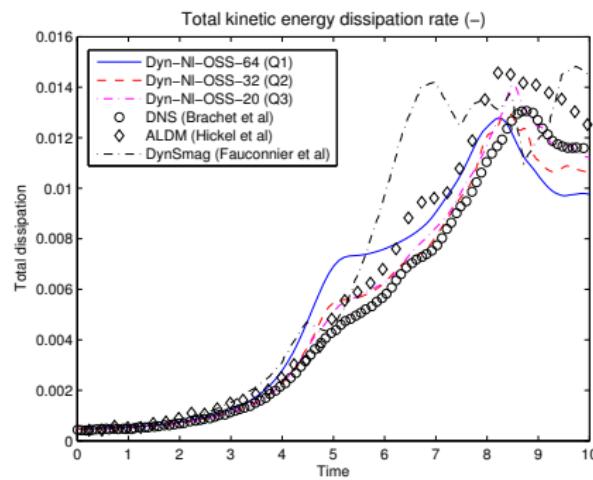
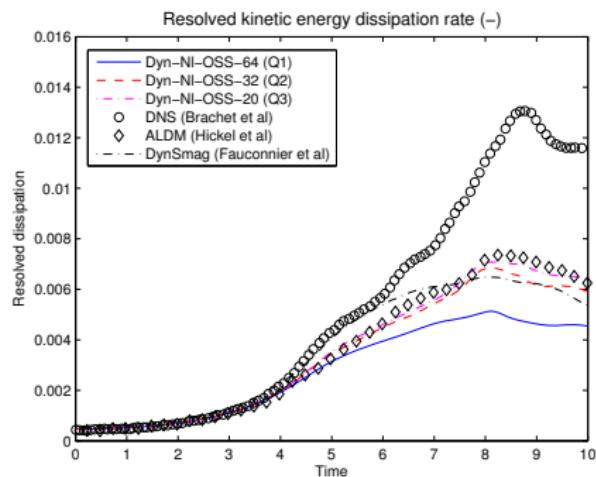


Figure : Resolved scales

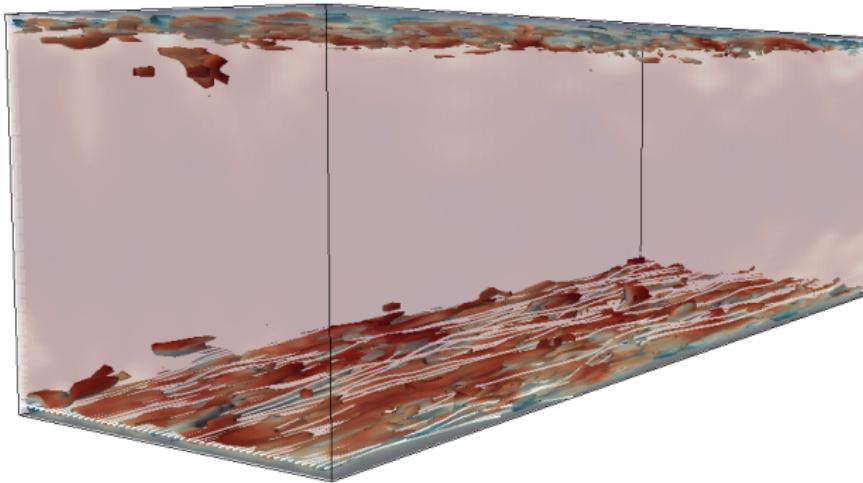
Figure : Total

- Good agreement with the LES models on resolved scales.
- Better results than LES models adding subscales counterpart.

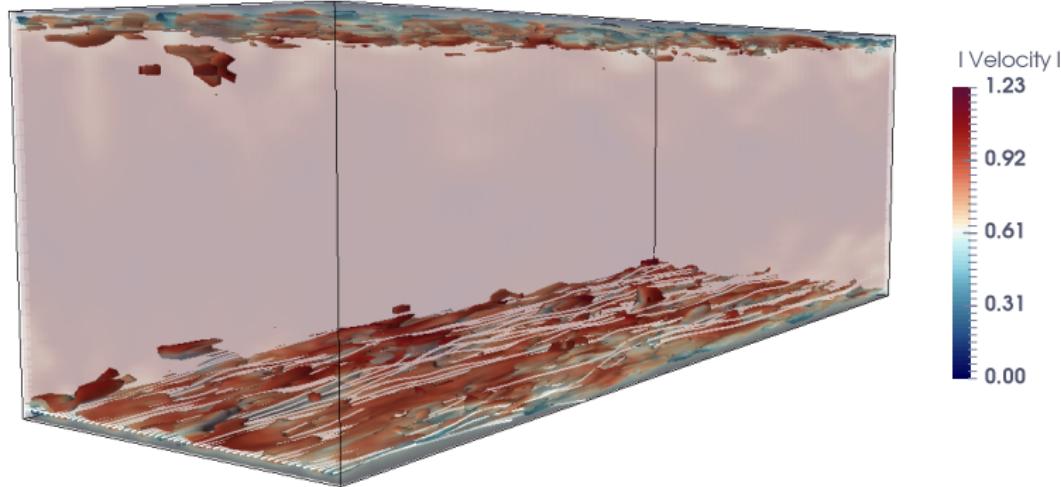
# TCF Turbulent Channel Flow

## Problem setting:

- Wall bounded flow.
- $Re_\tau = 180$  and  $Re_\tau = 395$ .
- Mesh resolution:  $32^3 - Q1$ .

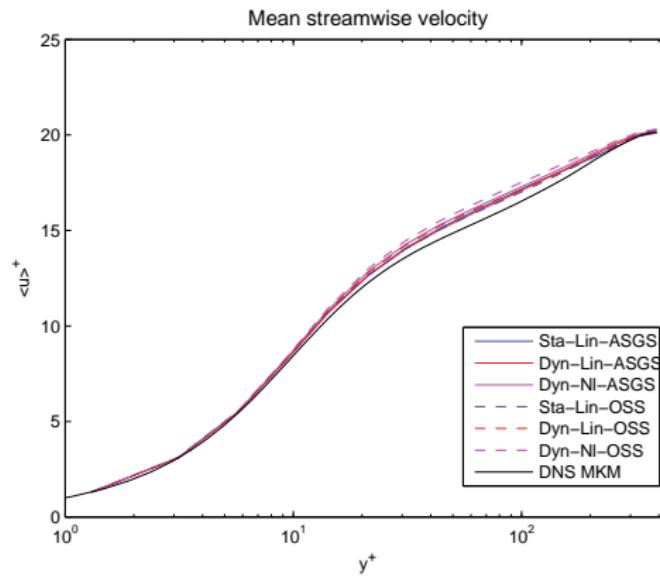


# TCF Turbulent Channel Flow



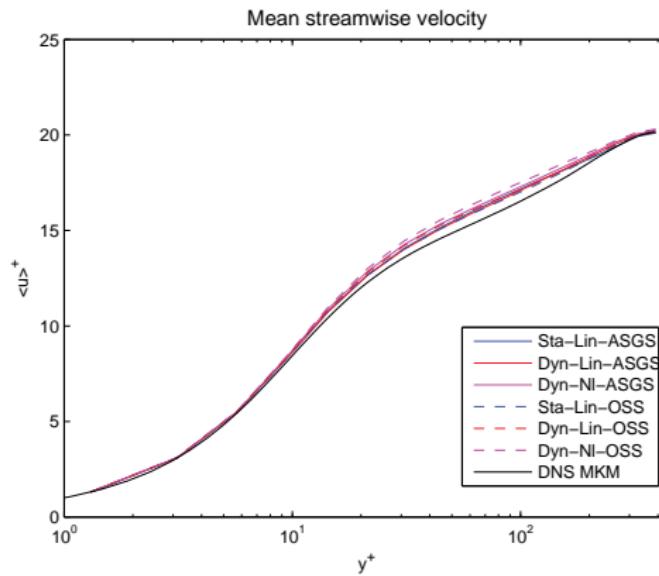
# TCF Turbulent Channel Flow

## Mean streamwise velocity (models):



# TCF Turbulent Channel Flow

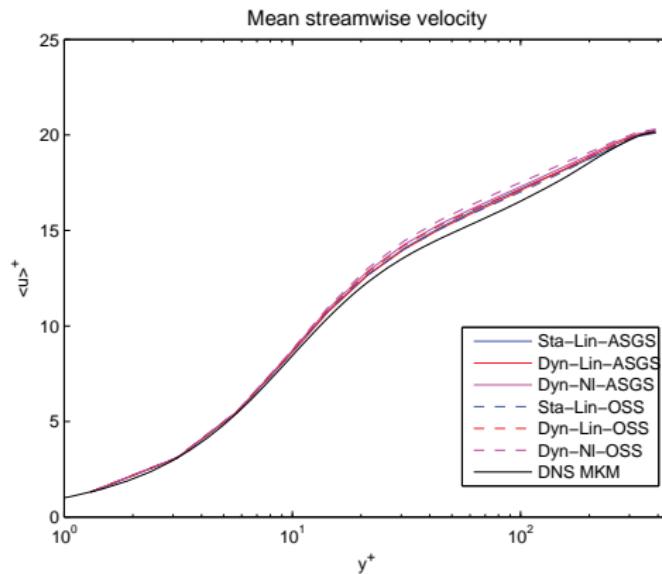
## Mean streamwise velocity (models):



- Small differences between methods.

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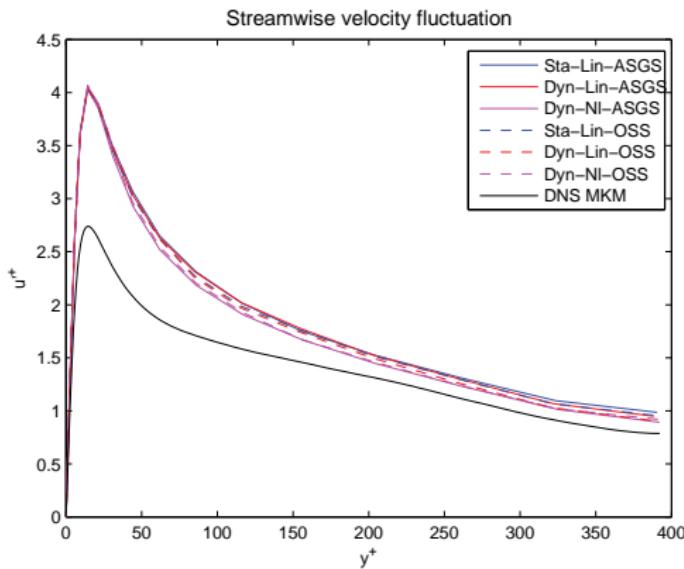
## Mean streamwise velocity (models):



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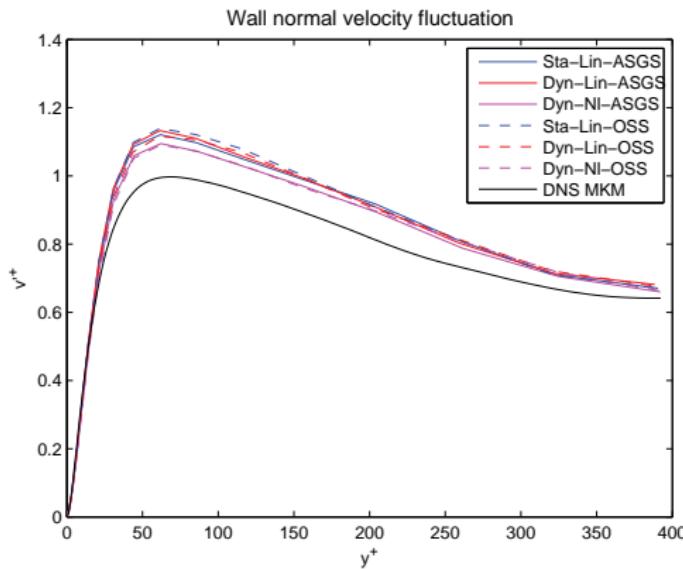
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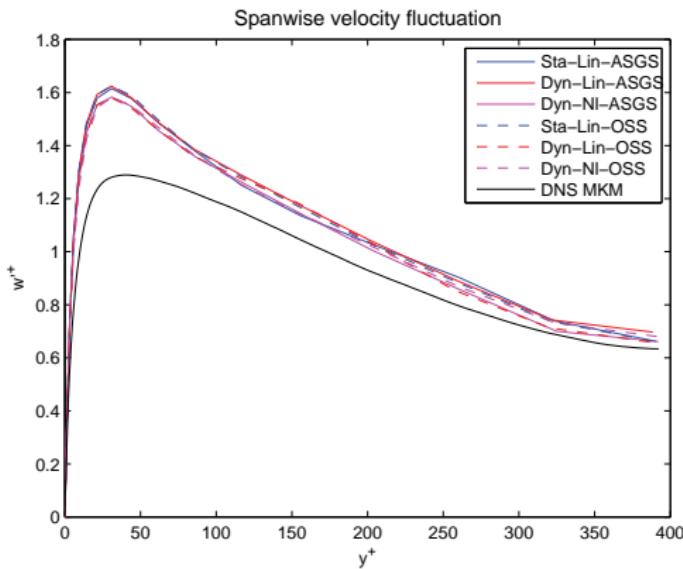
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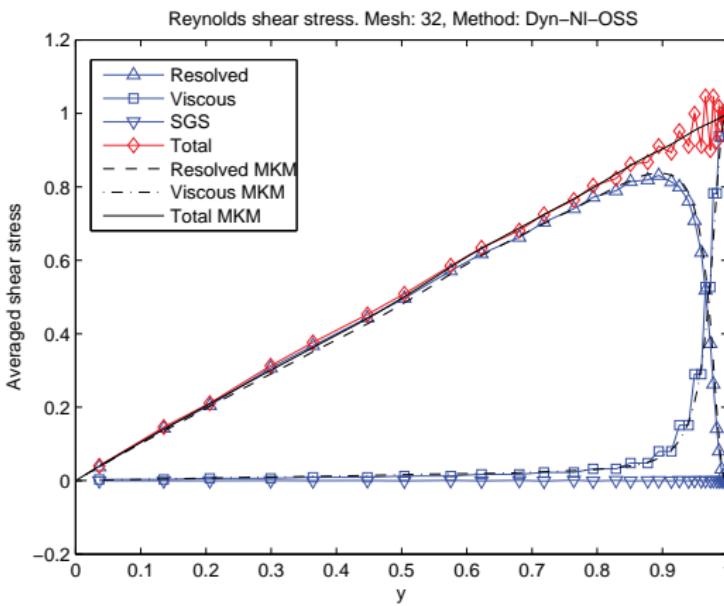
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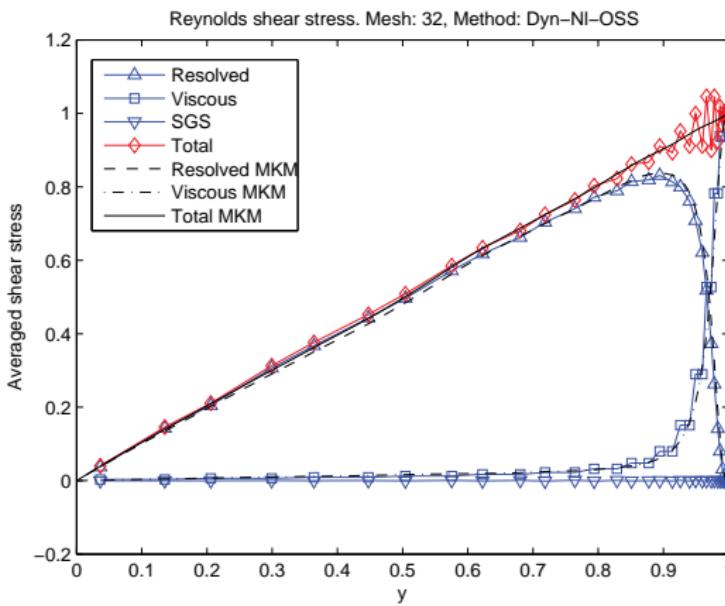
# TCF Turbulent Channel Flow

## Reynolds shear stress (models):



# TCF Turbulent Channel Flow

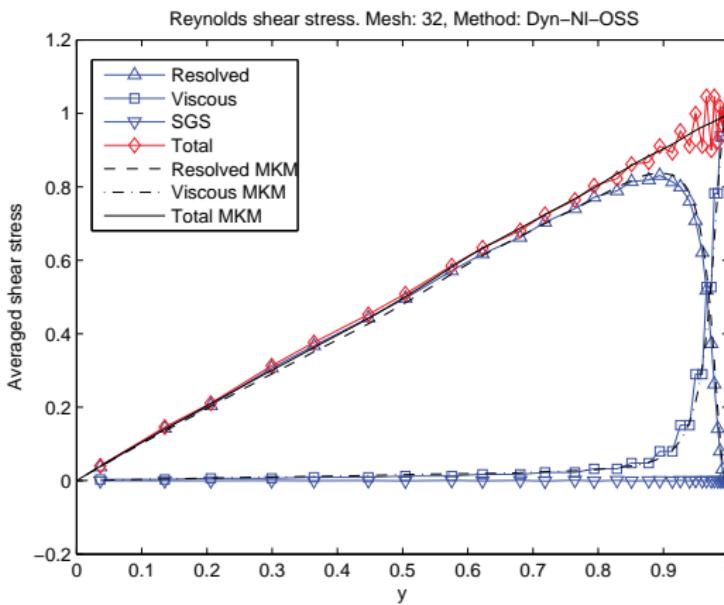
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- Almost identical to the DNS.

# TCF Turbulent Channel Flow

## Reynolds shear stress (models):



- Almost identical to the DNS.
- SGS counterpart does not contribute to the Reynolds shear stress.

# RB-VMS Conclusions

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- Among them dynamic and orthogonal subscales (linear or nonlinear) are the most effective.
- The skewsymmetric formulation is important to keep stability.

# RB-VMS Limitations

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- **Desired:**

- OSS with implicit projections.

## 1. Motivation

## 2. Residual-based VMS

## 3. Mixed FE VMS

Formulation

Block-preconditioning

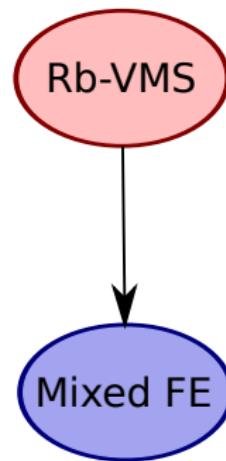
Numerical experiments

Conclusions

## 4. Segregated Runge-Kutta

## 5. Segregated VMS

## 6. Conclusions



# Semidiscrete problem

## FEM equations

$$\begin{aligned}
 & (\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega \\
 & + (\mathbf{v}_h, \partial_t \tilde{\mathbf{u}})_\Omega + (\mathcal{L}^* \mathbf{v}_h, \tilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \tilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega \\
 & (q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \tilde{\mathbf{u}})_{\Omega^h} = 0
 \end{aligned}$$

## SGS equations

$$\partial_t \tilde{\mathbf{u}} + \tau_m^{-1} \tilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \tilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = I \quad (\text{ASGS}), \quad \mathcal{P} = \mathbf{P}_h^\perp = \mathbf{I} - \mathbf{P}_h \quad (\text{OSS})$$

$$\mathbf{a} = \mathbf{u}_h + \tilde{\mathbf{u}}$$

# Semidiscrete problem

## FEM equations

$$\begin{aligned}
 & (\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega \\
 & + (\mathcal{L}^* \mathbf{v}_h, \tilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \tilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega \\
 & (q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \tilde{\mathbf{u}})_{\Omega^h} = 0
 \end{aligned}$$

## SGS equations

$$\tau_m^{-1} \tilde{\mathbf{u}} = \mathcal{P} \mathbf{R}_m$$

$$\tau_c^{-1} \tilde{p} = \mathcal{P} R_c$$

$$\mathcal{P} = \mathbf{P}_h^\perp = \mathbf{I} - \mathbf{P}_h \quad (\text{OSS})$$

$$\mathbf{a} = \mathbf{u}_h$$

# Term-by-term OSS

## FEM equations

$$\begin{aligned} & (\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega \\ & + (\mathcal{L}^* \mathbf{v}_h, \tilde{\mathbf{u}})_{\Omega^h} - (\nabla \cdot \mathbf{v}_h, \tilde{p})_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega \\ & (q_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\nabla q_h, \tilde{\mathbf{u}})_{\Omega^h} = 0 \end{aligned}$$

# Term-by-term OSS

## Term-by-term OSS (Codina 2008)

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

$$+ \left( \tau_m \mathbf{a} \cdot \nabla \mathbf{v}_h, \mathcal{P}_h^\perp(\mathbf{a} \cdot \nabla \mathbf{u}_h) \right)_{\Omega^h}$$

$$+ \left( \tau_c \nabla \cdot \mathbf{v}_h, \mathcal{P}_h^\perp(\nabla \cdot \mathbf{u}_h) \right)_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega + \left( \tau_m \nabla q_h, \mathcal{P}_h^\perp(\nabla p_h) \right)_{\Omega^h} = 0$$

# Term-by-term OSS

## Term-by-term OSS (Codina 2008)

$$(\mathbf{v}_h, \partial_t \mathbf{u}_h)_\Omega + b(\mathbf{a}, \mathbf{u}_h, \mathbf{v}_h) + (\nabla \mathbf{v}_h, \nu \nabla \mathbf{u}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$$

$$+ (\tau_m \mathbf{a} \cdot \nabla \mathbf{v}_h, \mathbf{a} \cdot \nabla \mathbf{u}_h)_{\Omega^h} - (\tau_m \mathbf{a} \cdot \nabla \mathbf{v}_h, \boldsymbol{\eta}_h)_{\Omega^h}$$

$$+ (\tau_c \nabla \cdot \mathbf{v}_h, \nabla \cdot \mathbf{u}_h)_{\Omega^h} = \langle \mathbf{v}_h, \mathbf{f} \rangle_\Omega$$

$$(q_h, \nabla \cdot \mathbf{u}_h)_\Omega + (\tau_m \nabla q_h, \nabla p_h)_{\Omega^h} - (\tau_m \nabla q_h, \boldsymbol{\xi}_h)_{\Omega^h} = 0$$

$$\boldsymbol{\eta}_h := \mathcal{P}_h(\mathbf{a} \cdot \nabla \mathbf{u}_h)$$

$$\boldsymbol{\xi}_h := \mathcal{P}_h(\nabla p_h)$$

$$\mathcal{P}_h(\nabla \cdot \mathbf{u}_h) \approx 0$$

# Matricial form

- ASGS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G \\ D & L_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \end{bmatrix},$$

# Matricial form

- ASGS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G \\ D & L_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \end{bmatrix},$$

- Term-by-term OSS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

# Matricial form

- ASGS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G \\ D & L_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \end{bmatrix},$$

- Term-by-term OSS:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Term-by-term OSS with Inf-sup stable elements (mixed FE):

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_\tau \\ D & 0 & 0 \\ -B_\tau^T & 0 & M_\tau \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

# Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \boldsymbol{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

# Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & | & B_{\eta,\tau} & 0 \\ -D & L_\tau & | & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & | & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & | & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \mathbf{T} \\ \Xi \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

# Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\bar{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & | & B_{\eta,\tau} & 0 \\ D & L_\tau & | & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & | & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & | & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{U}} \\ \bar{\mathbf{P}} \\ \bar{\boldsymbol{\tau}} \\ \Xi \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Outer block matrix:

$$\tilde{A} = \begin{bmatrix} M_{\eta,\tau} & 0 & | & -B_{\eta,\tau}^T & 0 \\ 0 & M_{\xi,\tau} & | & 0 & -B_{\xi,\tau}^T \\ B_{\eta,\tau} & 0 & | & K + C + A_\tau & G \\ 0 & B_{\xi,\tau} & | & D & L_\tau \end{bmatrix}$$

# Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\bar{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{U}} \\ \bar{\mathbf{P}} \\ \bar{\boldsymbol{\Upsilon}} \\ \Xi \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Outer block matrix:

# Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\bar{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{U}} \\ \bar{\mathbf{P}} \\ \bar{\boldsymbol{\Upsilon}} \\ \Xi \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Outer block matrix:

$$\tilde{A} = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ \tilde{B}_\tau & \tilde{K}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}_\tau \tilde{M}_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix},$$

# Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\mathbf{U} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Upsilon} \\ \Xi \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Outer block matrix:

$$\tilde{A} = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ \tilde{B}_\tau & \tilde{K}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}_\tau \tilde{M}_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix},$$

$$\tilde{S} = \tilde{K}_\tau + \tilde{B}_\tau \tilde{M}_\tau^{-1} \tilde{B}_\tau^T,$$

# Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\bar{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{U}} \\ \bar{\mathbf{P}} \\ \bar{\boldsymbol{\Upsilon}} \\ \Xi \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Outer block matrix:

$$\tilde{A} = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ \tilde{B}_\tau & \tilde{K}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}_\tau \tilde{M}_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix},$$

$$\tilde{S} = \tilde{K}_\tau + \tilde{B}_\tau \tilde{M}_\tau^{-1} \tilde{B}_\tau^T,$$

$$P_U(\tilde{A}) = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{M}_\tau^{-1} & \tilde{M}_\tau^{-1} \tilde{B}_\tau^T \tilde{S}^{-1} \\ 0 & \tilde{S}^{-1} \end{bmatrix},$$

# Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\bar{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{U}} \\ \bar{\mathbf{P}} \\ \bar{\boldsymbol{\Upsilon}} \\ \Xi \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Outer block matrix:

$$\tilde{A} = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ \tilde{B}_\tau & \tilde{K}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}_\tau \tilde{M}_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix},$$

$$\tilde{S} = \tilde{K}_\tau + \tilde{B}_\tau \tilde{M}_\tau^{-1} \tilde{B}_\tau^T,$$

$$P_U(\tilde{A}) = \begin{bmatrix} \tilde{M}_\tau & -\tilde{B}_\tau^T \\ 0 & \tilde{S} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{M}_\tau^{-1} & \tilde{M}_\tau^{-1} \tilde{B}_\tau^T \tilde{S}^{-1} \\ 0 & \tilde{S}^{-1} \end{bmatrix},$$

$$\tilde{S}^{-1} \approx \tilde{K}_\tau^{-1}.$$

# Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \bar{\boldsymbol{\tau}} \\ \Xi \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Inner block matrix:

$$\tilde{K}_\tau = \begin{bmatrix} K_\tau & G \\ D & L_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ DK_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix},$$

# Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \bar{\boldsymbol{\tau}} \\ \Xi \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Inner block matrix:

$$\tilde{K}_\tau = \begin{bmatrix} K_\tau & G \\ D & L_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ DK_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix},$$

$$S = L_\tau - DK_\tau G,$$

# Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \bar{\boldsymbol{\tau}} \\ \Xi \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Inner block matrix:

$$\tilde{K}_\tau = \begin{bmatrix} K_\tau & G \\ D & L_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ DK_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix},$$

$$S = L_\tau - DK_\tau G,$$

$$P_U(\tilde{K}_\tau) = \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix}^{-1} = \begin{bmatrix} K_\tau^{-1} & -K_\tau^{-1}GS^{-1} \\ 0 & S^{-1} \end{bmatrix},$$

# Block-recursive preconditioning

General case:

$$\begin{bmatrix} M\dot{\mathbf{U}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} K + C + A_\tau & G & B_{\eta,\tau} & 0 \\ D & L_\tau & 0 & B_{\xi,\tau} \\ -B_{\eta,\tau}^T & 0 & M_{\eta,\tau} & 0 \\ 0 & -B_{\xi,\tau}^T & 0 & M_{\xi,\tau} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \bar{\boldsymbol{\tau}} \\ \Xi \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

- Inner block matrix:

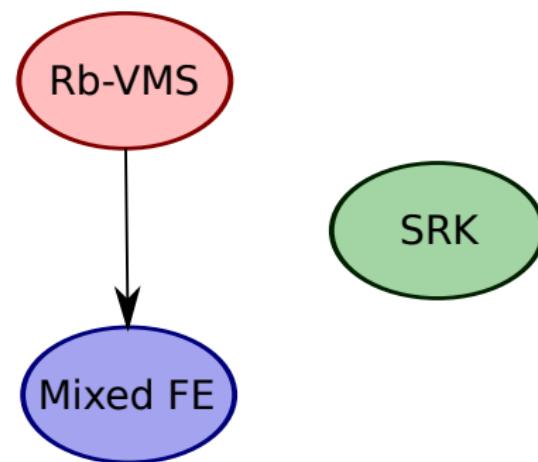
$$\tilde{K}_\tau = \begin{bmatrix} K_\tau & G \\ D & L_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ DK_\tau^{-1} & I \end{bmatrix} \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix},$$

$$S = L_\tau - DK_\tau G,$$

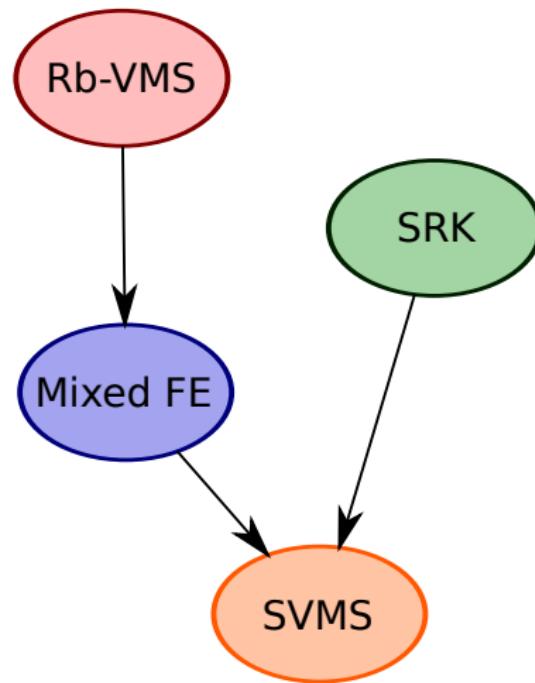
$$P_U(\tilde{K}_\tau) = \begin{bmatrix} K_\tau & G \\ 0 & S \end{bmatrix}^{-1} = \begin{bmatrix} K_\tau^{-1} & -K_\tau^{-1}GS^{-1} \\ 0 & S^{-1} \end{bmatrix},$$

$$\tilde{S}^{-1} \approx L_p^{-1} (\delta t L_p^{-1}).$$

1. Motivation
2. Residual-based VMS
3. Mixed FE VMS
4. Segregated Runge-Kutta  
Formulation  
Numerical experiments  
Conclusions
5. Segregated VMS
6. Conclusions



1. Motivation
2. Residual-based VMS
3. Mixed FE VMS
4. Segregated Runge-Kutta
5. Segregated VMS
  - Formulation
  - Block-preconditioning
  - Numerical experiments
  - Conclusions
6. Conclusions



## 1. Motivation

## 2. Residual-based VMS

## 3. Mixed FE VMS

## 4. Segregated Runge-Kutta

## 5. Segregated VMS

## 6. Conclusions

# Outline

- Line 1.

# Outline

- Line 1.
- Line 2.

Less formal

# Outline

- Line 1.
- Line 2.  
*Less formal*
- Line 3.  
*Less formal, different color.*

# Blocks

## Standard Block

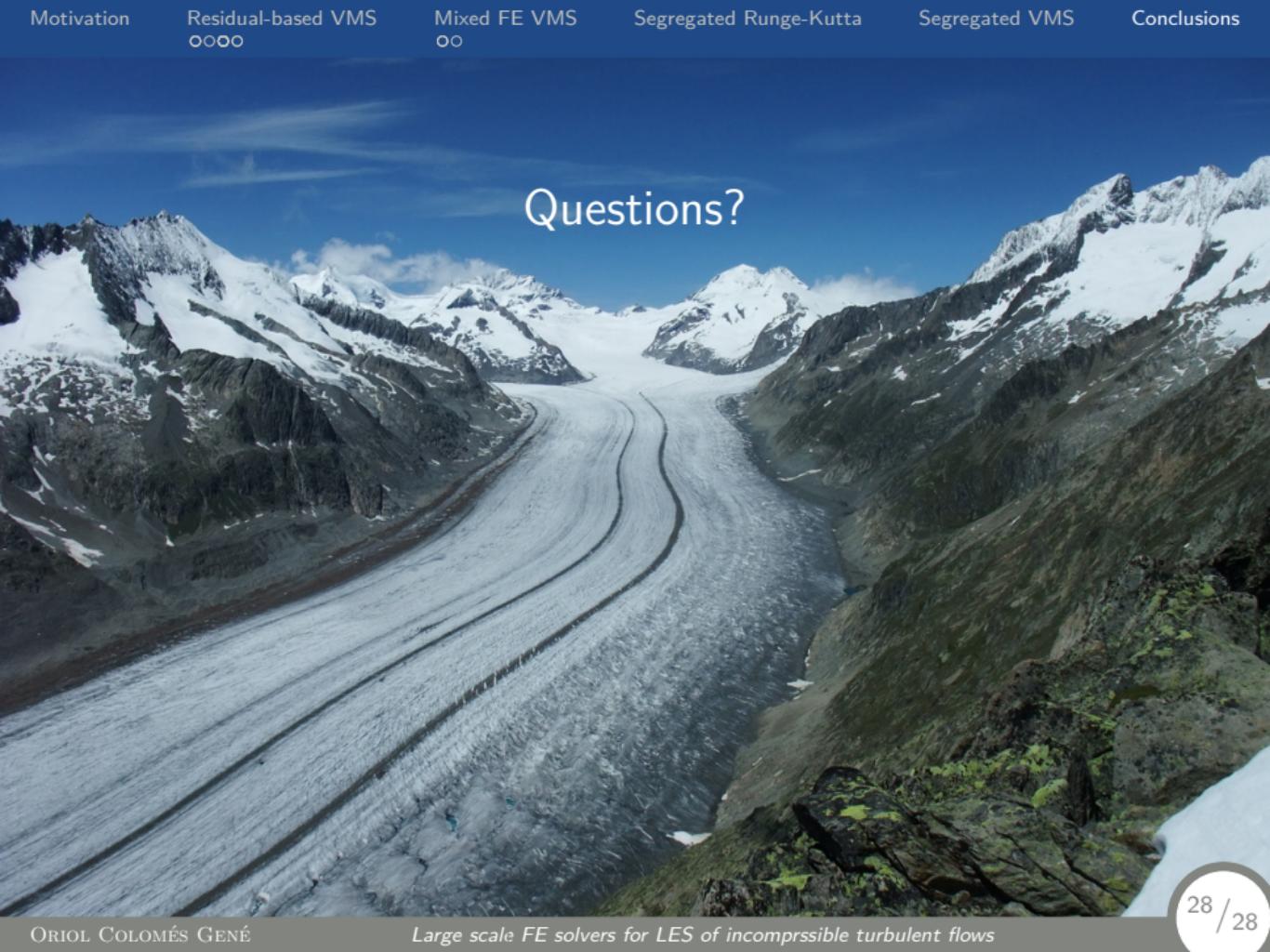
This is a standard block.

## Example Block

This is an example block.

## Alert Block

This is an alert block.

A wide-angle photograph of a massive glacier, likely the Aletsch Glacier in the Swiss Alps, winding its way through a deep valley between towering, snow-capped mountains. The glacier's surface is textured and shows signs of significant movement. The surrounding mountains are rugged, with patches of snow and exposed rock. The sky above is a clear, vibrant blue with a few wispy clouds.

Questions?