2.1

(1) $f(z)=\frac{1}{z}$

解：因为 $\lim\limits_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=\lim\limits_{\Delta z\to 0}\frac{\frac{1}{z+\Delta z}-\frac{1}{z}}{\Delta z}$

$$=\lim_{\Delta z\to 0}\frac{z-z-\Delta z}{\Delta z(z+\Delta z)z}=-\frac{1}{z^2}\ (z\neq 0)$$

所以 $f'(z)=\left(\frac{1}{z}\right)'=-\frac{1}{z^2}\ (z\neq 0)$

2.2

(1) $f(z)=\bar{z}^2\cdot z^2$

解：$f(z)=\bar{z}\cdot z^2=|z|^2\cdot z=(x+y)^2(x+iy)$

$$=x(x+y^2)+iy(x^2+y^2)$$

由 $u(x,y)=x(x^2+y^2)$，~~that~~ $v(x,y)=y(x^2+y^2)$

$u_x=x^2+y^2+2x^2$，$u_y=2xy$

$v_y=x^2+y^2+2y^2$，$v_x=2xy$

要 $u_x=v_y$，$u_y=-v_x$，当且仅当 $x=y=0$，而 u_x, u_y, v_x, v_y 均连续．故 $f(z)=\bar{z}\cdot z^2$ 仅在 z 处可导．

(3) $f(z)=x^3-3xy^2+i(3x^2y-y^3)$

解：$u(x,y)=x^3-3xy^2$，$v(x,y)=3x^2y-y^3$，$u_x=3x^2-3y^2$，

$u_y=-6xy$，$v_x=6xy$，$v_y=3x^2-3y^2$．

四个偏导数连续且 $u_x = v_y$，$u_y = -v_x$ 都成立.
因此，$f(z)$ 在整个复平面上处处可导. 也处处解析.

2.3

(1) $\frac{1}{z^2-1}$

解：$f(z)=\frac{1}{z^2-1}$ 是有理函数. 但要除去 $z^2-1=0$ 的点，其它处处解析. 故全平面除去 $z=1$ 及 $z=-1$ 的区域为 $f(z)$ 的解析区域. 奇点为 $z=\pm1$. $f(z)$ 导数为 $f'(z)=\left(\frac{1}{z^2-1}\right)'=$

$$\frac{-2z}{(z^2-1)^2}$$

2.4

(2) $v=u^2$

解：$\because f(z)$ 在D中解析且有 $f(z)=u+iu^2$,

由C-R方程：
$$\begin{cases} \frac{\partial u}{\partial x} = 2u\frac{\partial u}{\partial y}, \\ \frac{\partial u}{\partial y} = -2u\frac{\partial u}{\partial x} \end{cases}$$

则可推出 $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=0$，即 $u=C$，故：

$f(z)$ 必为D中常数！

2.7 证明：$\because \frac{\partial u}{\partial x}=2x$. $\frac{\partial^2 u}{\partial x^2}=2$，$\frac{\partial u}{\partial y}=-2y$，

$$\frac{\partial u}{\partial y^2}=-2，\text{则} \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=2+(-2)=0$$

故 $u=x^2-y^2$ 是调和函数. 又 $\frac{\partial u}{\partial x}=\frac{-2xy}{(x^2+y^2)^2}$，

$$\frac{\partial^2 u}{\partial x^2}=\frac{-2y^3+6x^2y}{(x^2+y^2)^2}，\frac{\partial u}{\partial y}=\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{2y^3-6x^2y}{(x^2+y^2)^2}$$

则 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$. 故 $v=\frac{y}{x^2+y^2}$ 是调和函数.

但 $\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}$，$\frac{\partial u}{\partial y}\neq-\frac{\partial u}{\partial x}$，故 $u+iv$ 不是解析函数

2.9 (1) $u=(x-y)(x^2+4xy+y^2)$

因 $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=3x^2+6xy-3y^2$，所以：

$$v=\int(3x^2+6xy-3y^2)\,dy=3x^2y+3xy^2-y^3+\varphi(x)$$

又 $\frac{\partial u}{\partial x}=6xy+3y^2+\varphi'(x)$. 而 $\frac{\partial u}{\partial x}=3x^2-6xy-3y^2$.

所以 $\varphi'(x)=-3x^2$，则 $\varphi(x)=-x^3+C$（常数）

$\therefore f(z) = u + iv$

$$= (x-y)(x^2+4xy+y^2) + i(3x^2y+3xy^2-y^3-x^3+C)$$

$$= (1-i)x^2(x+iy) - y^2(1-i)(x+iy) + 2x^2y(1+i) - 2xy^2(1-i) + Ci$$

$$= z(1-i)(x^2-y^2) + 2xyi \cdot z(1-i) + Ci$$

$$= (1-i)z(x^2-y^2+2xyi) + Ci$$

$$= (1-i)z^3 + Ci$$

(3) $u = 2(x-1)y,\ f(2) = -i$

解: $\because \dfrac{\partial u}{\partial x} = 2y,\ \dfrac{\partial u}{\partial y} = 2(x-1)$ 由于 $f(z)$ 有解析性.

故 $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = -2(x-1)$

$v = \int -2(x-1)\,dx = -(x-1)^2 + \psi(y)$

又 $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial x} = 2y$, 而 $\dfrac{\partial v}{\partial y} = \psi'(y)$.

所以. $\psi'(y) = 2y,\ \psi(y) = y^2 + C$

则 $v = -(x-1)^2 + y^2 + C$

故 $f(z) = 2(x-1)y + i[-(x-1)^2 + y^2 + C]$

由 $f(2) = -i$ 得 $f(2) = i(-1+C) = -i$, 推出 $C = 0$

即 $f(z) = 2(x-1)y + i(y^3 - x^2 + 2x - 1) = i(-z^2+2z-1)$

$= -i(z-1)^2$

2.13 试解下列方程.

(1) $e^z = 1+\sqrt{3}i$

解：$e^z = 1+\sqrt{3}i = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 2e^{i(\frac{\pi}{3}+2k\pi)}$

$= e^{\ln 2 + i(2k\pi + \frac{\pi}{3})}$, $k=0, \pm1, \pm2, \cdots$

故 $z = \ln 2 + i(2k\pi + \frac{\pi}{3})$, $k=0, \pm1, \pm2\cdots$

(3) $\sin z = i\sinh 1$;

解：$\sin z = i\sinh 1 = i(-i)\sin i = \sin i$

所以 $z = 2k\pi + i$ 或 $z = (2k-1)\pi - i$,

k 为整数

2.14

(1) $\cos i$

解：$\cos i = \dfrac{e^{i\cdot i} + e^{-i\cdot i}}{2} = \dfrac{e^{-1}+e^{1}}{2}$

(2) $\operatorname{Ln}(-3+4i)$

解：$\operatorname{Ln}(-3+4i) = \ln 5 + i\operatorname{Arg}(-3+4i)$

$= \ln 5 + i(2k\pi + \pi - \arctan\frac{4}{3})$

$k=0, \pm1, \pm2\cdots$

2.15 证明:

(1) $\sin z = \sin x \cosh y + i\cos x \sinh y$

证明: $\sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$$= \sin x \frac{e^{i\cdot iy}+e^{-i\cdot iy}}{2} + \cos x \frac{e^{i\cdot iy}-e^{-i\cdot iy}}{2i}$$

$$= \sin x \frac{e^{-y}+e^{y}}{2} - i\cos x \frac{e^{-y}-e^{y}}{2}$$

$$= \sin x \cosh y + i\cos x \sinh y$$

2.16

(1) $\cosh^2 z - \sinh^2 z = 1$

证: 因 $\sinh^2 z = \left(\frac{e^{z}-e^{-z}}{2}\right) = \frac{e^{2z}+e^{-2z}-2}{4}$

$$\cosh^2 z = \left(\frac{e^{z}+e^{-z}}{2}\right) = \frac{e^{2z}+e^{-2z}+2}{4}$$

故 $\cosh^2 z - \sinh^2 z = \frac{e^{2z}+e^{-2z}+2}{4} - \frac{e^{2z}+e^{-2z}-2}{4} = 1$

2.17 证明: $\cosh z$ 的反函数是 $\mathrm{Arcosh}\, z = \ln(z+\sqrt{z^2-1})$

设 $z = \cosh w$, 且 $w = \mathrm{Arcosh}\, z$, 由

$$z = \cosh w = \frac{1}{2}(e^{w}+e^{-w})$$

知 $2z = e^{w}+e^{-w}$

即 $e^{2w}-2ze^{w}+1=0$

解得 $e^{w}=z\pm\sqrt{z^2-1})$

故 $w=\ln(z+\sqrt{z^2-1})$