A1.1. *Proof.* Given ϕ being a $\mathbb{C}$ automorphism, we have $1 = \phi(1) = \phi(i)\phi(-i)$. Since ϕ preserves $\mathbb{R}$, $\phi(i) = bi$ for some $b \in \mathbb{R}$. It follows that $\phi(-i) = -i/b$. Furthermore, we have $0 = \phi(0) = \phi(i) + \phi(-i) = bi - i/b$. Hence $b = \pm 1$: ϕ is identity when $b = 1$, and ϕ conjugation when $b = -1$. □

A1.2. (a) *Proof.* Let $p(x) \in \mathbb{R}[x]$ with $\deg p > 2$. If $p(x)$ has a real root, then we can reduce its degree by 1. If, on the other hand, $p(x)$ has no real roots, then by the Fundamental Theorem of Algebra, it has at least a complex root. We calim that for real polynomials, their complex roots come in pairs. To see this, assume $a + bi$ solves $p(x)$, that is,

$$0 = p(a + bi) = a_n(a + bi)^n + \cdots + a_1(a + bi) + a_0.$$

Then,

$$\begin{aligned} p(a - bi) &= a_n(a - bi)^n + \cdots + a_1(a - bi) + a_0 \\ &= a_n(\overline{a + bi})^n + \cdots + a_1(\overline{a + bi}) + a_0 \\ &= a_n\overline{(a + bi)^n} + \cdots + a_1\overline{(a + bi)} + a_0 \\ &= \overline{a_n(a + bi)^n + \cdots + a_1(a + bi) + a_0} \\ &= \overline{0} = 0. \end{aligned}$$

It follows that if $p(x)$ has a complex root, then we can reduce its degree by 2 at once. We can repeat the above process untile we reach degree of 2. In that case, assuming we found all its real roots, then the remaining polynomial is immediately irreducible in $\mathbb{R}[x]$, and its last two roots are complex conjugates. All in all, no irreducible polynomials in $\mathbb{R}[x]$ has degree greater than 2. □

(b) Let us consider the 8-th roots of unity for 8:

$$\begin{aligned} x^8 &= -8 \\ &= (8)^{-1/8}\,(-1)^{1/8}\,e^{ik\pi/8} \\ &= 2^{3/8}\exp\left(i\frac{(k+1)\pi}{8}\right), \end{aligned}$$

where we used the Euler identity in the last equality, and $k = 0, 1, \ldots, 7$. That is, in conjugate pair,

$$x = 2^{3/8}e^{i(\pm\pi/8)}, 2^{3/8}e^{i(\pm 3\pi/8)}, 2^{3/8}e^{i(\pm 5\pi/8)}, 2^{3/8}e^{i(\pm 7\pi/8)}.$$

Now grouping these roots together by conjugate pairs gives us all irreducible factors of $x^8 + 8$. For example,

$$\begin{aligned} &\left(x - 2^{3/8}e^{i(\pm\pi/8)}\right)\left(x - 2^{3/8}e^{i(\pm\pi/8)}\right) \\ &= x^2 - (2)\left(2^{3/8}\right)\cos(\pi/8)\,x + 2^{3/4} \\ &= x^2 - (2^{3/8})\left(\sqrt{2 + \sqrt{2}}\right)x + 2^{3/4}. \end{aligned}$$

Similarly, we can multiply out and simplify all other factors. All in all,

$$
\begin{aligned}
&x^8+8 \\
&= \left(x^2-(2^{3/8})\left(\sqrt{2+\sqrt{2}}\right)x+2^{3/4}\right)\left(x^2-(2^{3/8})\left(\sqrt{2-\sqrt{2}}\right)x+2^{3/4}\right) \\
&\left(x^2+(2^{3/8})\left(\sqrt{2-\sqrt{2}}\right)x+2^{3/4}\right)\left(x^2+(2^{3/8})\left(\sqrt{2+\sqrt{2}}\right)x+2^{3/4}\right).
\end{aligned}
$$

A1.3. We may consider the polar form in this problem. Then $P_0 = 0$, $P_1 = P_0 + \exp\left(i\frac{2\pi}{3}\right)$, $P_2 = P_1 + \exp\left(i2 \cdot \frac{2\pi}{3}\right)$, and so on. Inductively,

$$P_n = 1 + 2\omega + 3\omega^2 + \cdots + n\omega^{n-1},$$

where $\omega = \exp\left(\frac{2\pi}{3}\right)$. Observe that

$$
\begin{aligned}
P_n &= (1+\omega+\omega^2+\cdots+\omega^n)' \\
&= \left(\frac{1-\omega^{n+1}}{1-\omega}\right)' \\
&= \frac{1-(n+1)\omega^n+n\omega^{n+1}}{(1-\omega)^2} \\
&= \frac{1-(n+1)\exp\left(in\frac{2\pi}{3}\right)+n\exp\left(i(n+1)\frac{2\pi}{3}\right)}{1-\exp\left(i\frac{2\pi}{3}\right)} \\
&= \frac{1-(n+1)\exp\left(in\frac{2\pi}{3}\right)+(-1)^{2/3}n\exp\left(in\frac{2\pi}{3}\right)}{\left(1-(-1)^{2/3}\right)^2}.
\end{aligned}
$$

In the last equality, we used the fact that the cube root of -1 is $\exp\left(i\frac{\pi}{3}\right)$.

A1.4. *Proof.* Since $\mathbb{C} \cong_{\text{vec}} \mathbb{R}^2$, we only consider how f transforms 1 and i. Let $f(1) = a+bi$ and $f(i) = c+di$ where $a, b, c, d \in \mathbb{R}$. For any $z = x+yi \in \mathbb{C}$, we have

$$
\begin{aligned}
f(z) &= f(x+yi) \\
&= xf(1)+yf(i) \\
&= ax+bxi+cy+dyi \\
&= a\left(\frac{z+\bar{z}}{2}\right)+b\left(\frac{z+\bar{z}}{2}\right)i+c\left(\frac{z-\bar{z}}{2i}\right)+d\left(\frac{z-\bar{z}}{2i}\right)i \\
&= \frac{a+d}{2}z+\frac{a-d}{z}\bar{z}+\frac{b+c}{2}\bar{z}i+\frac{b-c}{2}zi \\
&= \left(\frac{a+d}{2}+\frac{b-c}{2}i\right)z+\left(\frac{a-d}{2}+\frac{b+c}{2}i\right)\bar{z}.
\end{aligned}
$$

$\square$