1. (a) We know that the expectation value of the square of the separation distance between the two particles is given by

$$\left\langle (x_1 - x_2)^2 \right\rangle = \left\langle x_1^2 \right\rangle + \left\langle x_2^2 \right\rangle - 2\left\langle x_1 x_2 \right\rangle .$$

In particular, for a system made of two distinguishable particles with $\Psi(x_1, x_2) = \psi_m(x_1)\psi_n(x_2)$, we have

$$\left\langle (x_1 - x_2)^2 \right\rangle_{\text{disting.}} = \left\langle x^2 \right\rangle_m + \left\langle x^2 \right\rangle_n - 2\left\langle x \right\rangle_m \left\langle x \right\rangle_n ;$$

on the other hand, for a system made of two identical particles with

$$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_m(x_1)\psi_n(x_2) \pm \psi_n(x_1)\psi_m(x_2)\right],$$

we have

$$\left\langle (x_1 - x_2)^2 \right\rangle_{\pm} = \left\langle (x_1 - x_2)^2 \right\rangle_{\text{disting.}} \mp 2\left|\left\langle x \right\rangle_{mn}\right|^2 ,$$

where

$$\left\langle x \right\rangle_{mn} \equiv \left\langle m \right| x \left| n \right\rangle .$$

Now recall that with quantum harmonic oscillators, we can work with ladder operators. In particular, the position operator in terms of ladder operators is given by

$$x = \sqrt{\frac{\hbar}{2m_0\omega}}\left(a + a^\dagger\right)$$

so that

$$\begin{aligned} x^2 &= \frac{\hbar}{2m_0\omega}\left(aa + a^\dagger a^\dagger + aa^\dagger + a^\dagger a\right) \\ &= \frac{2}{\hbar\omega}\hat{H}. \end{aligned}$$

Lastly let us recall the fact that for harmonic oscillators, $\langle x \rangle = 0$.

(i) Distinguishable particles:

$$\begin{aligned} \left\langle (x_1 - x_2)^2 \right\rangle_{\text{disting.}} &= \left\langle x^2 \right\rangle_m + \left\langle x^2 \right\rangle_n - \cancelto{0}{2\left\langle x \right\rangle_m \left\langle x \right\rangle_n} \\ &= \frac{\hbar}{2m_0\omega}\left\{\left\langle m \right| x^2 \left| m \right\rangle + \left\langle n \right| x^2 \left| n \right\rangle\right\} \\ &= \frac{\hbar}{2m_0\omega}\left\{\left\langle m \right|\left(aa^\dagger + a^\dagger a\right)\left| m \right\rangle + \left\langle n \right|\left(aa^\dagger + a^\dagger a\right)\left| n \right\rangle\right\} \\ &= \frac{\hbar}{2m_0\omega}\frac{2}{\hbar\omega}\left\{\left\langle m \right| \hat{H} \left| m \right\rangle + \left\langle n \right| \hat{H} \left| n \right\rangle\right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\hbar}{m_0\omega^2}\left(E_m + E_n\right)\\
&= \frac{\hbar}{m_0\omega^2}\hbar\omega\left(m + \frac{1}{2} + n + \frac{1}{2}\right)\\
&= \boxed{\frac{\hbar}{m_0\omega}(m+n+1).}
\end{aligned}
$$

(ii) Bosons:

$$
\begin{aligned}
\langle x\rangle_{mn} &= \langle m|\, x\, |n\rangle\\
&= \sqrt{\frac{\hbar}{2m_0\omega}}\,\langle m|\, a^\dagger + a\, |n\rangle\\
&= \sqrt{\frac{\hbar}{2m_0\omega}}\left\{\langle m|\, a^\dagger\, |n\rangle + \langle m|\, a\, |n\rangle\right\}\\
&= \sqrt{\frac{\hbar}{2m_0\omega}}\left\{\sqrt{n+1}\,\langle m|n+1\rangle + \sqrt{n}\,\langle m|n-1\rangle\right\}\\
&= \sqrt{\frac{\hbar}{2m_0\omega}}\left(\delta_m^{n+1}\sqrt{n+1} + \delta_m^{n-1}\sqrt{n}\right).
\end{aligned}
$$

Similarly, we found

$$
\langle x\rangle_{nm} = \sqrt{\frac{\hbar}{2m_0\omega}}\left(\delta_m^{n+1}\sqrt{m} + \delta_m^{n-1}\sqrt{m+1}\right).
$$

Therefore,

$$
\begin{aligned}
|\,\langle x\rangle_{mn}\,|^2 &= \langle x\rangle_{mn}\,\langle x\rangle_{nm}\\
&= \frac{\hbar}{2m_0\omega}\underbrace{\left(\delta_m^{n+1}\sqrt{m}\sqrt{n+1} + \delta_m^{n-1}\sqrt{m+1}\sqrt{n}\right)}_{>0}.
\end{aligned}
$$

All in all,

$$
\begin{aligned}
\left\langle (x_1-x_2)^2\right\rangle_{\text{bosons}} &= \left\langle (x_1-x_2)^2\right\rangle_{\text{disting.}} - 2\,|\langle x\rangle_{mn}|^2\\
&= \boxed{\frac{\hbar}{m_0\omega}\left(m+n+1 - \delta_m^{n+1}\sqrt{m}\sqrt{n+1} - \delta_m^{n-1}\sqrt{m+1}\sqrt{n}\right).}
\end{aligned}
$$

(iii) Fermions:
Analogously,

$$
\begin{aligned}
\left\langle (x_1-x_2)^2\right\rangle_{\text{fermions}} &= \left\langle (x_1-x_2)^2\right\rangle_{\text{disting.}} + 2\,|\langle x\rangle_{mn}|^2\\
&= \boxed{\frac{\hbar}{m_0\omega}\left(m+n+1 + \delta_m^{n+1}\sqrt{m}\sqrt{n+1} + \delta_m^{n-1}\sqrt{m+1}\sqrt{n}\right).}
\end{aligned}
$$

(b) With first two Hermite polynomials $H_0(\xi) = 1$ and $H_1(\xi) = 2\xi$, we can write down

$$\begin{aligned}
\psi_0(x) &= \left(\frac{m_0\omega}{\pi\hbar}\right)^{1/4} H_0(\xi)e^{-\xi^2/2} = \left(\frac{m_0\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2};\\
\psi_1(x) &= \frac{1}{\sqrt{2}}\left(\frac{m_0\omega}{\pi\hbar}\right)^{1/4} H_1(\xi)e^{-\xi^2/2} = \frac{1}{\sqrt{2}}\left(\frac{m_0\omega}{\pi\hbar}\right)^{1/4} (2\xi)\, e^{-\xi^2/2}.
\end{aligned}$$

Then for distinguishable particless:

$$\begin{aligned}
\Psi_{\text{disting.}}(x_1, x_2) &= \psi_0(x_1)\psi_1(x_2)\\
&= \boxed{\sqrt{2}\left(\frac{m_0\omega}{\pi\hbar}\right)^{1/2} \xi_2 \exp\left(-\frac{\xi_1^2}{2} - \frac{\xi_2^2}{2}\right)}
\end{aligned}$$

so that

$$\boxed{|\Psi_{\text{disting.}}(x_1, x_2)|^2 = \frac{2}{\pi x_0^2}\xi_2^2 \exp\bigl(-\xi_1^2 - \xi_2^2\bigr).}$$

Now for identical particles:

$$\begin{aligned}
\Psi_{\pm}(x_1, x_2) &= \frac{1}{\sqrt{2}}\left[\psi_0(x_1)\psi_1(x_2) \pm \psi_1(x_1)\psi_0(x_2)\right]\\
&= \frac{1}{\sqrt{2}}\left(\Psi_{\text{disting.}}(x_1, x_2) \pm \Psi_{\text{disting.}}(x_2, x_1)\right)\\
&= \boxed{\left(\frac{m_0\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{\xi_1^2}{2} - \frac{\xi_2^2}{2}\right)(\xi_2 \pm \xi_1)}
\end{aligned}$$

so that

$$\boxed{|\Psi_{\pm}(x_1, x_2)|^2 = \frac{1}{\pi x_0^2}\exp\bigl(-\xi_1^2 - \xi_2^2\bigr)(\xi_2 \pm \xi_1)^2.}$$

All in all,

$$\boxed{\pi x_0^2|\Psi(x_1, x_2)|^2 = \begin{cases} 2\xi_2^2\exp\bigl(-\xi_1^2 - \xi_2^2\bigr) & \text{distinguishable,}\\ \exp\bigl(-\xi_1^2 - \xi_2^2\bigr)(\xi_2 + \xi_1)^2 & \text{bosons,}\\ \exp\bigl(-\xi_1^2 - \xi_2^2\bigr)(\xi_2 - \xi_1)^2 & \text{fermions.}\end{cases}}$$

Now if $x_1 = x_2 = x$, then

$$\boxed{\pi x_0^2|\Psi(x, x)|^2 = \begin{cases} 2\xi^2\exp\bigl(-2\xi^2\bigr) & \text{distinguishable,}\\ 4\xi^2\exp\bigl(-2\xi^2\bigr) & \text{bosons,}\\ 0 & \text{fermions.}\end{cases}}$$

```
atSamePosition = Plot[{2 \[Xi]^2 Exp[-2 \[Xi]^2], 4 \[Xi]^2 Exp
    [-2 \[Xi]^2], 0}, {\[Xi], -2, 2}, PlotLegends -> Placed[{"
    distinguishable", "bosons", "fermions"}, {0.89, 0.75}]]
```

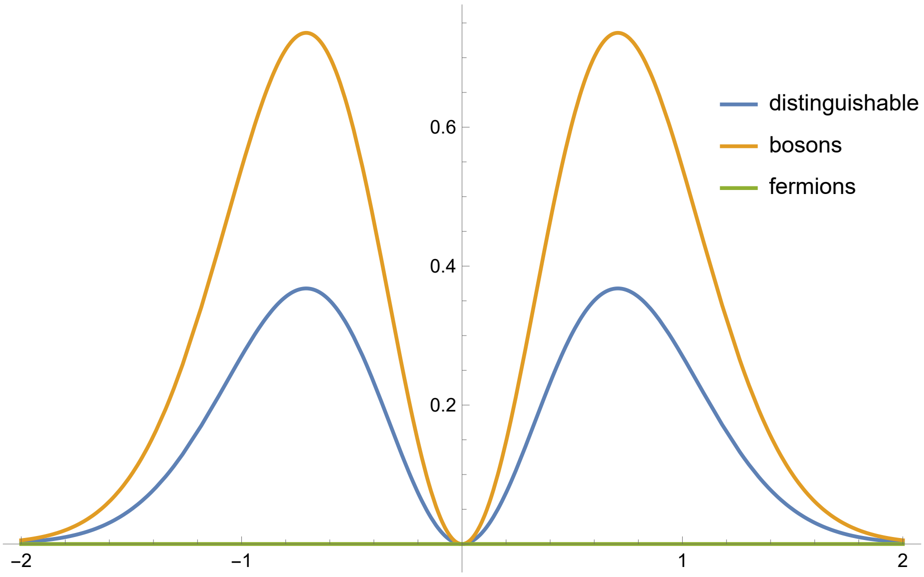
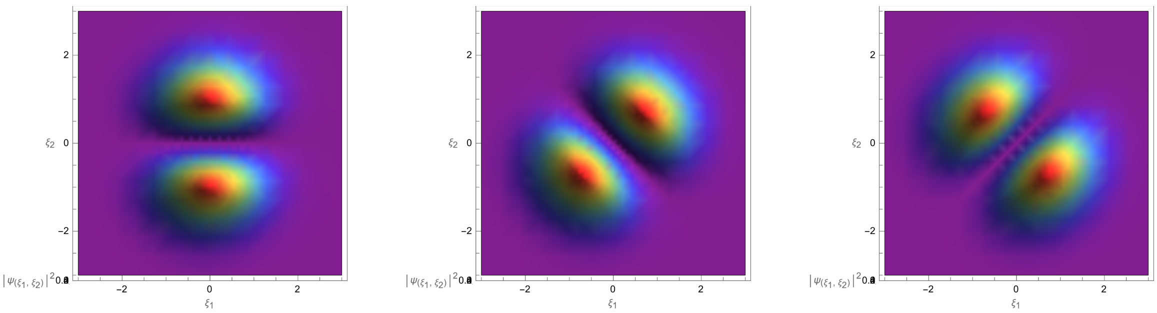


Figure 1: A two-particles system at $x_1 = x_2 = x$. By the Pauli exclusion principle, two fermions cannot be found at the same position. Due to exchange "forces" arising from symmetrization, two bosons are more likely to be close to each other.

```
distinguishable = Plot3D[2xi2^2 Exp[-xi1^2- xi2^2], {xi1, -3,
    3}, {xi2, -3, 3}, AxesLabel -> {Subscript[\[Xi], 1],
    Subscript[\[Xi], 2], Superscript[Abs[Subscript[\[Psi], Row
    [{"(", Subscript[\[Xi], 1], ", ", Subscript[\[Xi], 2], ")
    "}]]], 2]}, Mesh -> None, PlotRange -> All, ColorFunction ->
     Function[{x, y, z}, ColorData["Rainbow"][z]],
    ColorFunctionScaling -> True, Mesh -> None, ViewPoint -> {0,
     0, Infinity}]
```

```
bosons = Plot3D[Exp[-xi1^2 - xi2^2] (xi1 + xi2)^2, {xi1, -3, 3},
     {xi2, -3, 3}, AxesLabel -> {Subscript[\[Xi], 1], Subscript
    [\[Xi], 2], Superscript[Abs[Subscript[\[Psi], Row[{"(",
    Subscript[\[Xi], 1], ", ", Subscript[\[Xi], 2], ")"}]]],
    2]}, Mesh -> None, PlotRange -> All, ColorFunction ->
    Function[{x, y, z}, ColorData["Rainbow"][z]],
    ColorFunctionScaling -> True, Mesh -> None, ViewPoint -> {0,
     0, Infinity}]
```

```
fermions = Plot3D[Exp[-xi1^2 - xi2^2] (xi1 - xi2)^2, {xi1, -3,
    3}, {xi2, -3, 3}, AxesLabel -> {Subscript[\[Xi], 1],
    Subscript[\[Xi], 2], Superscript[Abs[Subscript[\[Psi], Row
    [{"(", Subscript[\[Xi], 1], ", ", Subscript[\[Xi], 2], ")
    "}]]], 2]}, Mesh -> None, PlotRange -> All, ColorFunction ->
     Function[{x, y, z}, ColorData["Rainbow"][z]],
    ColorFunctionScaling -> True, Mesh -> None, ViewPoint -> {0,
     0, Infinity}]
```



(a) Distinguishable particles (b) Bosons (c) Fermions

Figure 2: Top view of $\pi x_0^2 |\Psi(x_1, x_2)|^2$ vs. ξ_1 and ξ_2. Note that looking at the section of the $\xi_1 = \xi_2$ plane in this 3d plot, we obtain the result of Figure 1.