

Projection matrices

The vector \mathbf{v}' is a result of multiplying the vector \mathbf{v} with transformation matrix \mathbf{M} (“postmultiplying”):

$$\mathbf{v}' = \mathbf{M}\mathbf{v}$$

For a three-dimensional vector $\mathbf{v} = \begin{pmatrix} x & y & z \end{pmatrix}^T$ we use the four-dimensional homogenous coordinates $\mathbf{v} = \begin{pmatrix} x & y & z & w \end{pmatrix}^T$.

Now

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

where

$$x' = m_{11}x + m_{12}y + m_{13}z + m_{14}w$$

$$y' = m_{21}x + m_{22}y + m_{23}z + m_{24}w$$

$$z' = m_{31}x + m_{32}y + m_{33}z + m_{34}w$$

$$w' = m_{41}x + m_{42}y + m_{43}z + m_{44}w$$

...

Our goal is the perspective projection matrix \mathbf{M}_{proj} defined as:

$$\mathbf{M}_{\text{proj}} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{l+r}{l-r} & 0 \\ 0 & \frac{2n}{t-b} & \frac{b-t}{b+t} & 0 \\ 0 & 0 & \frac{f+n}{n-f} & \frac{2fn}{f-n} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Here n is the *near* clipping plane, f the *far* clipping plane, r the *right* clipping plane, l the *left* clipping plane, t the *top* clipping plane, and b the *bottom* clipping plane: $left \leq x \leq right$, $bottom \leq y \leq top$, $far \leq z \leq near$.

The complete projection matrix is

$$\mathbf{M}_{\text{proj}} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{l-r} & 0 \\ 0 & \frac{2n}{t-b} & \frac{b-t}{f+n} & 0 \\ 0 & 0 & \frac{n-f}{n-f} & \frac{2fn}{f-n} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

If the viewing volume is symmetric, which is $r = -l$ and $t = -b$, then

$$\begin{aligned} r + l &= 0 & t + b &= 0 \\ r - l &= 2r & t - b &= 2t \end{aligned}$$

so the matrix can be simplified to

$$\mathbf{M}'_{\text{proj}} = \begin{pmatrix} \frac{n}{r} & 0 & 0 & 0 \\ 0 & \frac{n}{t} & 0 & 0 \\ 0 & 0 & \frac{f+n}{n-f} & \frac{2fn}{f-n} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

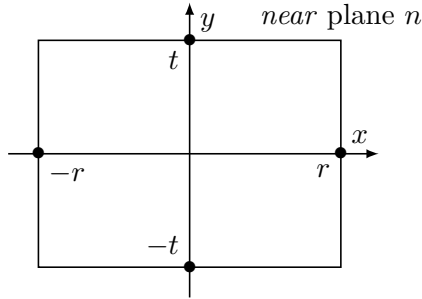


Figure 1. The *near* plane when the viewing volume is symmetric.