Projection matrices

The vector $\mathbf{v}'$ is a result of multiplying the vector $\mathbf{v}$ with transformation matrix $\mathbf{M}$ ("postmultiplying"):

$$\mathbf{v}' = \mathbf{M}\mathbf{v}$$

For a three-dimensional vector $\mathbf{v} = \begin{pmatrix} x & y & z \end{pmatrix}^{\mathbf{T}}$ we use the four-dimensional homogenous coordinates $\mathbf{v} = \begin{pmatrix} x & y & z & w \end{pmatrix}^{\mathbf{T}}$.

Now

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

where

$$\begin{aligned} x' &= m_{11}x + m_{12}y + m_{13}z + m_{14}w \\ y' &= m_{21}x + m_{22}y + m_{23}z + m_{24}w \\ z' &= m_{31}x + m_{32}y + m_{33}z + m_{34}w \\ w' &= m_{41}x + m_{42}y + m_{43}z + m_{44}w \end{aligned}$$

...

Our goal is the perspective projection matrix $\mathbf{M}_{\text{proj}}$ defined as:

$$\mathbf{M}_{\text{proj}} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{l+r}{l-r} & 0 \\ 0 & \frac{2n}{t-b} & \frac{b+t}{b-t} & 0 \\ 0 & 0 & \frac{f+n}{n-f} & \frac{2fn}{f-n} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Here n is the *near* clipping plane, f the *far* clipping plane, r the *right* clipping plane, l the *left* clipping plane, t the *top* clipping plane, and b the *bottom* clipping plane: *left* $\leq x \leq$ *right*, *bottom* $\leq y \leq$ *top*, *far* $\leq z \leq$ *near*.

The complete projection matrix is

$$\mathbf{M}_{\text{proj}} = \begin{pmatrix} \dfrac{2n}{r-l} & 0 & \dfrac{r+l}{l-r} & 0 \\ 0 & \dfrac{2n}{t-b} & \dfrac{t+b}{b-t} & 0 \\ 0 & 0 & \dfrac{f+n}{n-f} & \dfrac{2fn}{f-n} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

If the viewing volume is symmetric, which is $r = -l$ and $t = -b$, then

$$\begin{aligned} r + l &= 0 & t + b &= 0 \\ r - l &= 2r & t - b &= 2t \end{aligned}$$

so the matrix can be simplified to

$$\mathbf{M}'_{\text{proj}} = \begin{pmatrix} \dfrac{n}{r} & 0 & 0 & 0 \\ 0 & \dfrac{n}{t} & 0 & 0 \\ 0 & 0 & \dfrac{f+n}{n-f} & \dfrac{2fn}{f-n} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

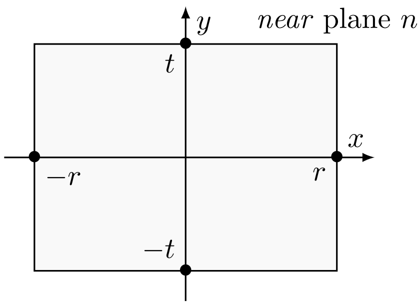

Figure 1. The *near* plane when the viewing volume is symmetric.

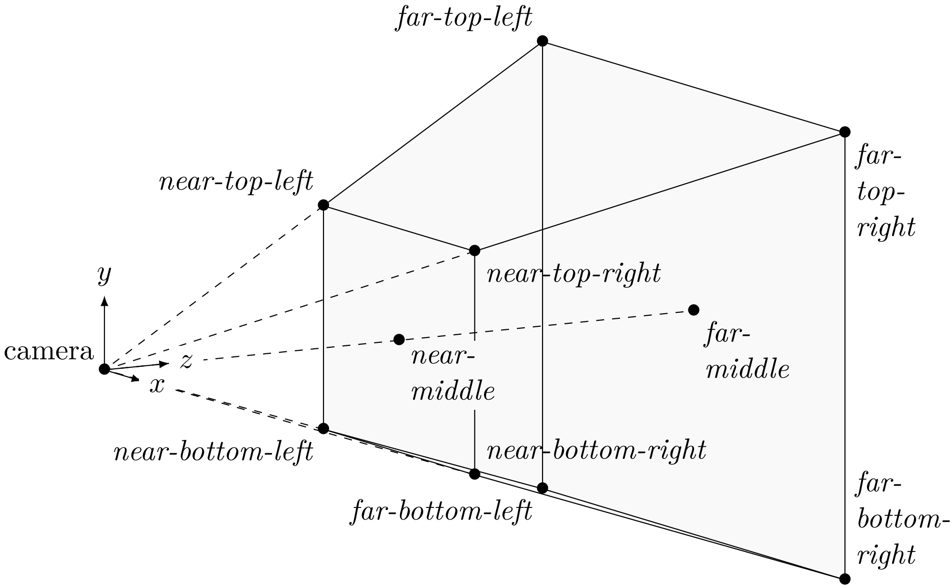

Figure 2. The view frustum with a left-handed coordinate system. The z-axis to opposite direction would give us a right-handed coordinate system. Image is projected to *near* plane.

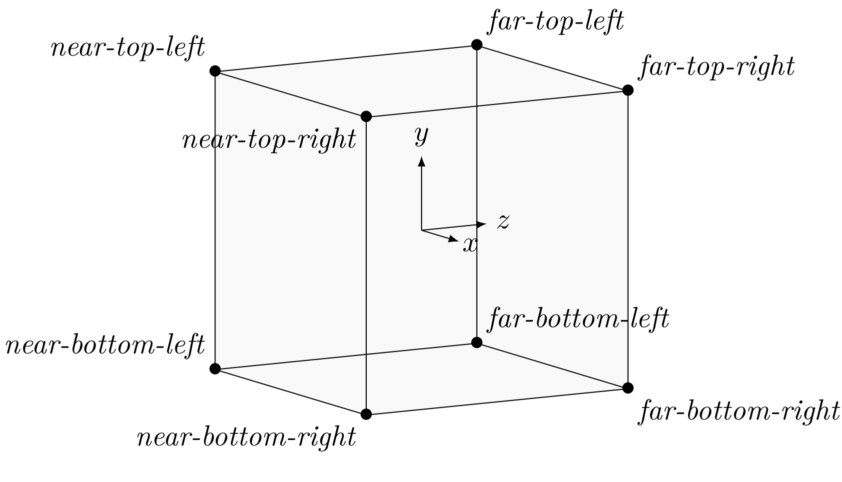

Figure 3. Normalized device coordinates (NDC).

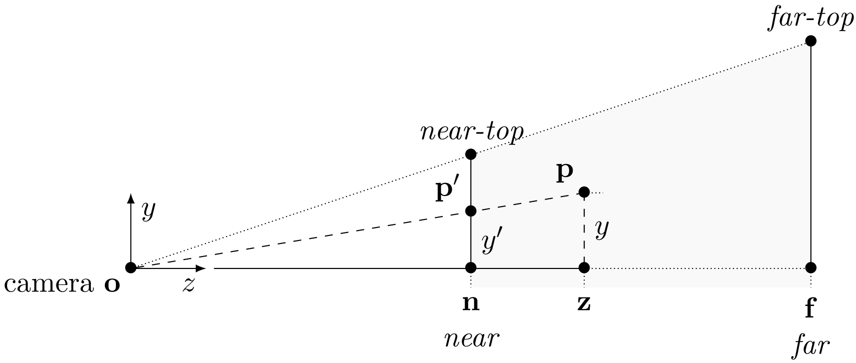

Figure 4. Projection $\mathbf{p}'$ of point $\mathbf{p}$ to *near* plane (side view).

Here the triangles $\Delta\mathbf{ozp}$ and $\Delta\mathbf{onp}'$ are similar. Therefore

$$\frac{|\mathbf{np}'|}{|\mathbf{zp}|} = \frac{|\mathbf{on}|}{|\mathbf{oz}|} \quad \Rightarrow \quad \frac{y'}{y} = \frac{n}{z}$$

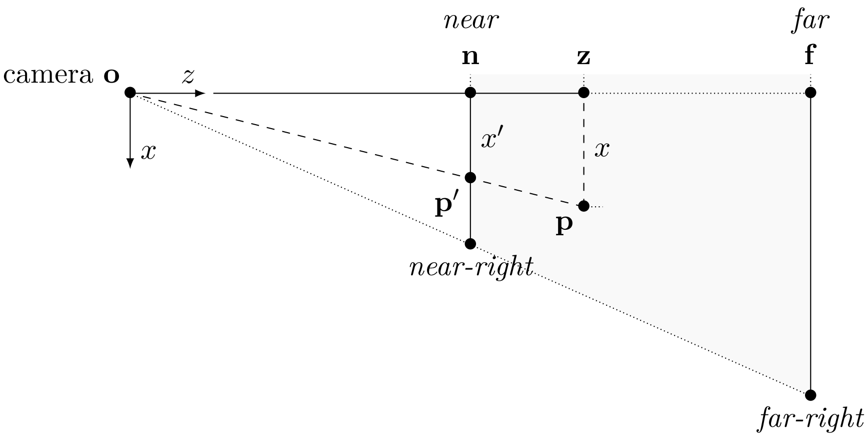

Figure 5. Projection $\mathbf{p}'$ of point $\mathbf{p}$ to *near* plane (top view).

Here the triangles $\Delta\mathbf{ozp}$ and $\Delta\mathbf{onp}'$ are similar. Therefore

$$\frac{|\mathbf{np}'|}{|\mathbf{zp}|} = \frac{|\mathbf{on}|}{|\mathbf{oz}|} \quad \Rightarrow \quad \frac{x'}{x} = \frac{n}{z}$$