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Conjectures about Primes and Cyclic Numbers

Joel E. Cohen[1]
The Rockefeller University
1230 York Avenue, Box 20
New York, NY 10065
USA
cohen@rockefeller.edu

Abstract

A positive integer n is defined to be *cyclic* if and only if every group of size n is cyclic. Equivalently, the number n is cyclic if and only if n is relatively prime to the number of positive integers less than n that are relatively prime to n. Because every prime number is cyclic, it is natural to ask whether a (proved or conjectured) property of primes extends to cyclic numbers. I review proved or conjectured properties of primes (including some new conjectures about primes) and propose analogous conjectures about cyclic numbers. Using the 28,488,167 cyclic numbers less than 10^8, I test the conjectures about cyclic numbers and disprove the cyclic analog of the second conjecture about primes of Hardy and Littlewood. Proofs or disproofs of the remaining conjectures are invited.

1 Introduction

I propose some conjectures about cyclic numbers $\mathcal{C} := (c_1,\ c_2,\ \ldots)$ (sequence [37, A003277] in the *On-Line Encyclopedia of Integer Sequences* (OEIS)) based on analogous proved or conjectured properties of prime numbers $\mathcal{P} := (p_1,\ p_2,\ \ldots)$ (A000040). I test the conjectures about cyclic numbers (or, for brevity, cyclics) using the 28,488,167 cyclics less than 10^8. I also test some new conjectures about prime numbers (or, for brevity, primes) using the 50,847,534 primes less than 10^9. I invite proofs, disproofs, further numerical confirmations, counterexamples, and additional conjectures.

[1]Author's other affiliations: Departments of Statistics, Columbia University & University of Chicago.

A natural number (positive integer) $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ is *cyclic* if and only if there exists only one group of size n, up to isomorphism. If gcd is the greatest common divisor and $\varphi(n)$ is Euler's totient function (the number of positive integers less than n that are relatively prime to n, A000010), then, according to Szele [50], the number $n \in \mathbb{N}$ is cyclic if and only if $\gcd(n, \varphi(n)) = 1$.

I use Szele's condition to compute which $n \in \mathbb{N}$ are cyclic numbers. My first 10^4 computed cyclic numbers exactly match the first 10^4 cyclic numbers computed independently by T. D. Noe in A003277.

Michel Lagneau [37, A003277, November 18, 2012] asserted without proof that $n \in \mathbb{N}$ is cyclic if and only if $\varphi(n)^{\varphi(n)} \equiv 1 \bmod n$. Richard P. Stanley (personal communication, January 20 2025) gave an elegant short proof of Lagneau's condition. I quote it with his permission. First, assume that n is not cyclic, so $\gcd(n, \varphi(n)) = d > 1$. Then $\varphi(n)^{\varphi(n)}$ is divisible by d so cannot be congruent to 1 mod n. On the other hand, assume that n is cyclic. Euler's generalization of Fermat's little theorem implies that $k^{\varphi(n)} \equiv 1 \bmod n$ whenever $\gcd(k, n) = 1$. Putting $k = \varphi(n)$ completes the proof.

Alexei Kourbatov (personal communications, May 24 2025 and June 1 2025) pointed out that Lagneau's condition can be evaluated using the well-known algorithm [35, p. 71, algorithm 2.143] for modular exponentiation. Whether using Szele's criterion (as I do here) or Lagneau's criterion for a number to be cyclic, the most computationally expensive step is finding $\varphi(n)$.

The sequence $\mathcal{C}$ of cyclics begins $(1, 2, 3, 5, 7, 11, 13, 15, 17, 19, 23, 29, 31, 33, 35, 37, \ldots)$. Cyclics $\mathcal{C}$ are the union of primes $\mathcal{P}$ and the composite numbers $n \in \mathbb{N}$ such that n and $\varphi(n)$ are relatively prime or coprime (A050384, e.g., 1, 15, 33, 35, 51, 65, 69, 77, 85, 87, 91, 95, 115, 119, 123, 133, 141, 143, 145, 159, ...). The only cyclic that is a square is $c_1 = 1$. The only cyclic that is even is $c_2 = 2$. Consequently, the only cyclic of the form $n(n-1)$ for $n \in \mathbb{N}$ is $c_2 = 2$ with $n = 2$, and no cyclic is of the form $n(n+1)$ for $n \in \mathbb{N}$, because both $n(n-1)$ and $n(n+1)$ are even.

For an increasing integer sequence $a := (a(1),\ a(2),\ a(3),\ \ldots)$, the counting function of a evaluated at a positive real number x is the number of elements of a that are less than or equal to x. For example, the counting function $\pi(\cdot)$ of primes $\mathcal{P}$ satisfies $\pi(10) = 4$. The prime number theorem [21, 51] gives that $\pi(x) \sim x/\log x$ as $x \to \infty$. Let

$$C(x) := \sum_{\substack{m \le x \\ m \text{ cyclic}}} 1 \tag{1}$$

be the counting function of cyclic numbers, that is, the number of cyclic numbers that do not exceed positive real x (A061091). For $n = 1, \ldots, 20$, $C(n) = 1, 2, 3, 3, 4, 4, 5, 5, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10, 10$. Erdős [15] proved that

$$C(x) \sim \frac{x}{e^{\gamma} \log\log\log x} \text{ as } x \to \infty. \tag{2}$$

Here $\gamma \approx 0.5772156649\ldots$ is the Euler-Mascheroni constant and $e^{\gamma} \approx 1.78107241799\ldots$.

Pollack [40] gave an asymptotic series expansion

$$C(x) \sim \frac{x}{e^{\gamma} \log\log\log x} \left(1 - \frac{\gamma}{\log\log\log x} + \cdots\right) \text{ as } x \to \infty \tag{3}$$

with additional terms. I shall use just these first two terms.

Cyclics are much more abundant than primes asymptotically because $\lim_{x\to\infty} \pi(x)/C(x) = 0$. Hence, asymptotically, almost all cyclics are composite. John Campbell and I [4] observed that, since $C(c_n) = n$ by definition, (2) implies that

$$c_n \sim e^{\gamma} n \log\log\log n \text{ as } n \to \infty, \tag{4}$$

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 1,$$

$$\lim_{x\to\infty} \frac{x}{c_{C(x)}} = 1, \tag{5}$$

$$\lim_{x\to\infty} \frac{\log C(x)}{\log x} = 1.$$

The observation that cyclics are asymptotically much more abundant than primes motivates investigating which (proved or conjectured) properties of primes depend on their asymptotic scarcity relative to cyclics, and which properties of primes carry over (exactly or asymptotically) to the more abundant cyclics.

Another infinite increasing integer sequence that contains all primes is $\mathbb{N}$. But $\mathbb{N}$ does not share an elementary property that $\mathcal{P}$ and $\mathcal{C}$ share, namely, that the only even element of the sequence is 2. Similarly, while $\mathcal{P}$ contains no squares and $\mathcal{C}$ contains exactly one square, $\mathbb{N}$ includes infinitely many squares. Other infinite increasing integer sequences that share important properties with $\mathcal{P}$ and $\mathcal{C}$ remain to be investigated.

Campbell and I [4] proved two analogies between primes and cyclics. First, under the Riemann hypothesis, the nth prime gap satisfies $p_{n+1} - p_n = O(\sqrt{p_n} \log p_n)$ as $n \to \infty$ [10]. More precisely, under the Riemann hypothesis, for every $p_n > 3$, $p_{n+1} - p_n < \frac{22}{25}\sqrt{p_n} \log p_n$ [5]. We [4, Theorem 2] proved that, under the Riemann hypothesis, the first difference of consecutive cyclics satisfies $c_{n+1} - c_n = o\left(\sqrt{p_n} \log p_n\right)$. Second, if $m_n(\mathcal{P})$ is the mean and $v_n(\mathcal{P})$ is the variance of the first n primes, then asymptotically $v_n(\mathcal{P}) \sim (1/3)(m_n(\mathcal{P}))^2$ as $n \to \infty$ [6]. We [4, Theorem 1] proved, without the Riemann hypothesis, that the mean $m_n(\mathcal{C})$ and the variance $v_n(\mathcal{C})$ of the first n cyclics satisfy the same asymptotic relationship, $v_n(\mathcal{C}) \sim (1/3)(m_n(\mathcal{C}))^2$ as $n \to \infty$.

This project of generalizing from primes to cyclics is not guaranteed to succeed. After proposing in section 2 conjectures that numerical calculations have so far failed to reject, I give in section 3 a counterexample to show that the analog for cyclics of the second conjecture [22] of Hardy and Littlewood fails. This counterexample provides a further small, indirect hint in support of the belief of Hensley and Richards [25] that the second conjecture of Hardy and Littlewood for primes is false.

Because $\mathcal{P} \subset \mathcal{C}$, if infinitely many primes have property X, then infinitely many cyclics have property X. For example, Euler's proof that $\lim_{n\to\infty} \sum_{j=1}^{n} p_j^{-1} = \infty$ immediately implies that $\lim_{n\to\infty}\sum_{j=1}^{n} c_j^{-1} = \infty$. But if every prime has property X, it may be true or false, depending on property X, that every cyclic has property X. Conversely, if infinitely many cyclics have property X, it may be true or false, depending on property X, that infinitely many primes have property X. But if every cyclic has property X, then every prime has property X.

Consequently, when a conjecture about primes has been extensively verified numerically, if that conjecture immediately implies the corresponding conjecture about cyclics, there is no need, and I do not bother, to test numerically the analogous conjecture about cyclics. I test numerically only those conjectures about cyclics not immediately implied by properties of primes that are proved or conjectured and numerically supported.

A helpful referee pointed out that many additional questions could be asked about cyclics. For example, the referee asked, are the cyclics equidistributed over arithmetic progressions of a prescribed modulus? How does the sum of all cyclics less than or equal to positive real x behave as a function of x? The latter question leads to the first and only theorem of this paper, which reports, for a fixed positive integer k, the sum of the kth power of all cyclics less than or equal to a positive real x as a function of x.

Theorem 1. *Fix $k \in \mathbb{N}$. For $n \in \mathbb{N}$, if $c_1, \ldots, c_n \in \mathcal{C}$ are the first n cyclic numbers, then*

$$c_1^k + \cdots + c_n^k \sim \frac{nc_n^k}{k+1} \sim \frac{n^{k+1}e^{k\gamma}(\log\log\log n)^k}{k+1} \text{ as } x \to \infty. \tag{6}$$

In particular, $c_1 + \cdots + c_n \sim nc_n/2 \sim n^2 e^\gamma \log\log\log n/2$.

For positive real x, as $x \to \infty$, the sum of the kth power of all cyclics less than or equal to x is asymptotic to

$$\frac{C(x)c_{C(x)}^k}{k+1} \sim \frac{C(x)x^k}{k+1} \sim \frac{x^{k+1}}{(k+1)e^\gamma \log\log\log x}\left(1 - \frac{\gamma}{\log\log\log x}\right), \tag{7}$$

using (3). In particular, the sum of all cyclics less than or equal to positive real x is asymptotic to

$$\frac{C(x)x}{2} \sim \frac{x^2}{2e^\gamma \log\log\log x}\left(1 - \frac{\gamma}{\log\log\log x}\right). \tag{8}$$

Proof. Campbell and I [4, Theorem 1] showed that

$$n^{-1}(c_1^k + \cdots + c_n^k) \sim \frac{c_n^k}{k+1} \text{ as } x \to \infty.$$

Hence, using (4), $c_1^k + \cdots + c_n^k \sim nc_n^k/(k+1) \sim n(e^\gamma n \log\log\log n)^k/(k+1)$ $= n^{k+1}e^{k\gamma}(\log\log\log n)^k/(k+1)$, proving (6).

Replacing n in (6) by $C(x)$ and using (5) to approximate $c_{C(x)}$ yields (8). □

In general, $xC(x)/2 > nc_n/2$ because in general $x > c_n$ while $C(x) = n$. Figure 1 shows that $xC(x)/2$ and $nc_n/2$ closely approximate the exact sum of cyclics, and the asymptotic approximation on the right side of (8) consistently falls below the exact sum of cyclics and the other two approximations.

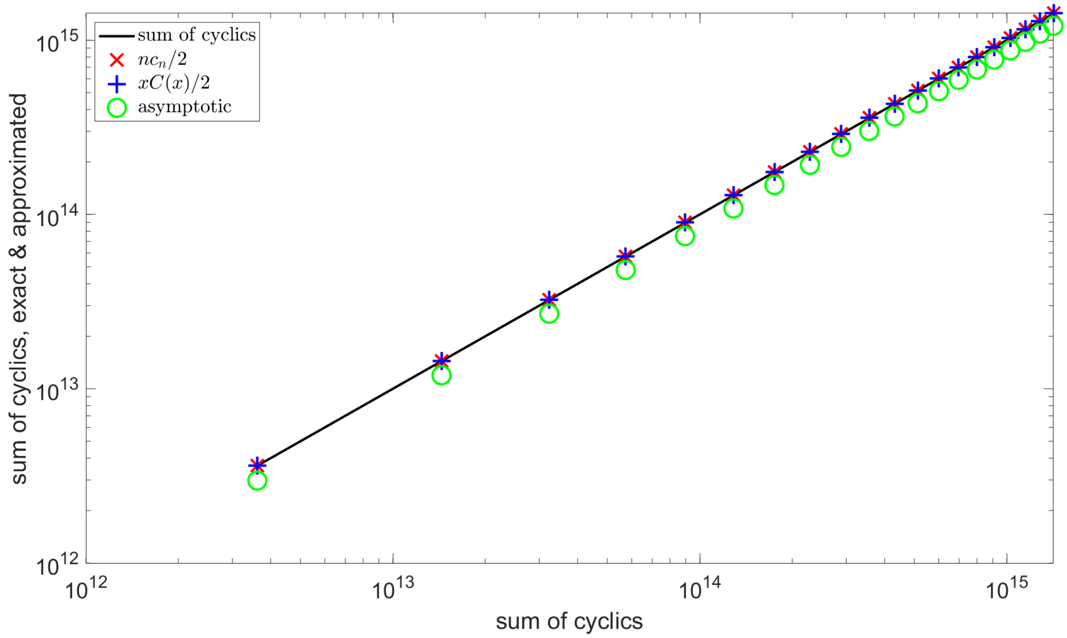

Figure 1: For $m = 1, 2, \ldots, 20$ and $x = m \times 5 \times 10^6$, the abscissa of each plotted point is the exact sum of cyclics less than or equal to x. The ordinate of each point compares the exact sum of cyclics less than or equal to x (solid black line, with abscissa = ordinate) with three approximations: $nc_n/2$ (red $\times$ marker); $xC(x)/2$ (blue $+$ marker); and the asymptotic approximation on the right side of (8) (open green circle).

2 Conjectures

2.1 Landau's list and Legendre's relatives

In 1912, Landau [31] presented four historic conjectures about primes. Extensively verified numerically, these conjectures are generally believed to be true but unproved as of 2025. Many unconfirmed and unrefuted claims to have proved one or several of Landau's conjectures have been published or posted but I am not aware that such a claimed proof has been independently confirmed.

If true, each of Landau's four conjectures about primes immediately implies the conjecture about cyclics that follows it below. But the conjecture about cyclics may be true even if the corresponding Landau conjecture about primes is false. Here are Landau's four conjectures about primes and analogous conjectures about cyclics. I include a few novel conjectures about cyclics suggested by these analogs of Landau's conjectures.

2.1.1 Landau's problem 1

First, Goldbach conjectured that every even $n \in \mathbb{N}$ greater than 2 is a sum of two primes.

Conjecture 2 (Goldbach analog for cyclics)**.** Every even $n \in \mathbb{N}$ greater than 2 is a sum of two cyclics.

On seeing Conjecture 2 in a prior draft of this paper, Carl Pomerance (personal communication, March 5 2025) [41] proved that every sufficiently large even n is a sum of two cyclics. I quote his result with his permission. Let $G(n)$ be the number of pairs of cyclics $c_1,\ c_2$ such that $c_1 + c_2 = n$. Then if n is even, Pomerance [41] proved,

$$G(n) \sim \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \cdot \frac{2n}{(e^\gamma \log\log\log n)^2} \cdot \prod_{\substack{p|n \\ 2<p<\log\log n}} \frac{p-1}{p-2} \text{ as } n \to \infty.$$

2.1.2 Landau's problem 2

Second in Landau's list, the twin prime conjecture states that there are infinitely many primes p such that $p + 2$ is also a prime. Here p and $p + 2$ are called twin primes.

Conjecture 3 (twin cyclics analog)**.** There are infinitely many cyclics $c \in \mathcal{C}$ such that $c+2$ is also cyclic.

On seeing Conjecture 3 in a prior draft of this paper, Carl Pomerance (personal communication, March 5 2025) [41] proved a much stronger result, which I quote with his permission. For positive real x, let $C_2(x)$ be the number of cyclics $c \leq x$ such that $c + 2$ is also cyclic. Then as $x \to \infty$,

$$C_2(x) \sim 2 \prod_{p>2} \left(1 - (p-1)^{-2}\right) x(e^\gamma \log\log\log x)^{-2}. \tag{9}$$

The right side of (9) approaches infinity as $x \to \infty$, proving Conjecture 3.

It is well known that the only prime p such that $p,\ p+2,\ p+4$ are all primes is $p = 3$, because if $p > 3$, one of $p,\ p+2,\ p+4$ must be divisible by 3. Cyclics are different. The composite cyclics (A050384) include multiple triplets, such as 141, 143, 145, and 213, 215, 217, and 319, 321, 323, and 391, 393, 395.

Conjecture 4 (cyclic triplets)**.** There are infinitely many composite cyclics $c \in \mathcal{C}$ such that $c,\ c+2,\ c+4$ are all composite cyclic.

Carl Pomerance (personal communication, March 5 2025) [41] proved a related result, which I quote with his permission: there are infinitely many cyclic triplets $c,\ c+2,\ c+4$ (not necessarily all composite cyclics, as in Conjecture 4), and their counting function is of order $x(\log\log\log x)^{-5/2}(\log x)^{-1/2}$ as $x \to \infty$. As Carl Pomerance pointed out (personal communication, March 24 2025), since the number of primes up to x is of order $x/\log x$, which is much smaller, asymptotically most of these triplets consist of three composites.

Further, the cyclics (whether prime or composite) include multiple quintuplets with successive gaps equal to 2, such as 11, 13, 15, 17, 19, and 29, 31, 33, 35, 37, and 65, 67, 69, 71, 73, and 83, 85, 87, 89, 91, and 137, 139, 141, 143, 145, and 209, 211, 213, 215, 217, and 263, 265, 267, 269, 271.

Conjecture 5 (cyclic quintuplets)**.** There are infinitely many cyclics $c \in \mathcal{C}$ such that $c,\ c+2,\ c+4,\ c+6,\ c+8$ are all cyclic.

Carl Pomerance (personal communication, March 5 2025) [41] proved a general result which implies, for example, that infinitely many all-cyclic 8-tuples have the form $n, n+2, n+4, n+6, n+8, n+10, n+12, n+14$ but there are no all-cyclic 9-tuples with the additional term $n+16$ because one of these nine numbers is divisible by 9, therefore not cyclic.

2.1.3 Landau's problem 3

Third in Landau's list, Legendre [33] conjectured that for every $n \in \mathbb{N}$, there exists a prime $p \in \mathcal{P}$ such that $n^2 < p < (n+1)^2$. Legendre's claimed proof was based on a prior claim that was false. Several claims to prove Legendre's conjecture have been published or posted but I am not aware that any has been independently confirmed.

Conjecture 6 (Legendre analog for cyclics)**.** For every $n \in \mathbb{N}$, there exists a cyclic $c \in \mathcal{C}$ such that $n^2 < c < (n+1)^2$.

A cyclic c such that $n^2 < c < (n+1)^2$ exists for all $1 \leq n \leq 9998$. The use of strict inequality in Conjecture 6 is justified because no cyclic other than 1 is a square.

Desboves [12, Theorem 2, p. 290] conjectured that for every $n \in \mathbb{N}$, there exist two primes p, p' such that $n^2 < p < p' < (n+1)^2$. Desboves asserted that if Legendre's conjecture is true, then his conjecture follows. The converse is obvious. I confirmed Desboves' conjecture numerically for the 50,847,534 primes less than 10^9. If true, Desboves' conjecture immediately implies Conjecture 7 about cyclics. But Conjecture 7 may be true even if Desboves' conjecture is false.

Conjecture 7 (Desboves analog for cyclics)**.** For every $n \in \mathbb{N}$, there exist two cyclics $c, c' \in \mathcal{C}$ such that $n^2 < c < c' < (n+1)^2$.

For every positive integer $n \leq 3161$, there exist $c, c' \in \mathcal{C}$ such that $n^2 < c < c' < (n+1)^2$.

The next conjectures generalize Legendre's and Desboves's conjectures to $k > 2$ primes and cyclics.

For $n \in \mathbb{N}$, let $N_{\mathcal{P}}(n) := \#\{p \in \mathcal{P} \mid p \in (n^2, (n+1)^2)\}$ be the number of primes in the interval $(n^2, (n+1)^2)$ (A014085 apart from an initial 0). For example, in the first 25 intervals $(n^2, (n+1)^2),\ n = 1, \ldots, 25$, the numbers of primes are, respectively, 2, 2, 2, 3, 2, 4, 3, 4, 3, 5, 4, 5, 5, 4, 6, 7, 5, 6, 6, 7, 7, 7, 6, 9, 8. For example, $N_{\mathcal{P}}(6) = 4$ because four primes, 37, 41, 43, 47, are between $6^2 = 36$ and $7^2 = 49$.

In a prior draft of this paper, I conjectured that $N_{\mathcal{P}}(n)$ is asymptotic (as $n \to \infty$) to a regularly varying function of n with positive index not exceeding 1. Recall that a regularly varying function [16, 47, 27] maps the positive half line $x > 0$ into the positive half line and takes the form $x \mapsto x^\rho \ell(x)$. The exponent ρ of x is a real number, commonly called the index of the regularly varying function, and $\ell(x)$ is a slowly varying function of x, that is, for every $\lambda > 0$, $\lim_{x\to\infty} \ell(\lambda x)/\ell(x) = 1$. A regularly varying function generalizes a power function $x \mapsto x^\rho$.

On seeing this conjecture, Pierre Deligne (personal communication, March 6 2025) refined my conjecture to a much more specific, heuristically plausible conjecture, which I quote with his permission. Deligne observed that the length of the interval $(n^2, (n+1)^2)$ is asymptotic to $2n$ and the probability that an integer in this interval is prime is asymptotic to $1/\log(n^2)$, so the number of primes in $(n^2, (n+1)^2)$ should be asymptotic to $2n/\log(n^2) = n/\log n$. Deligne commented, “Of course with no way to prove it.”

Conjecture 8 (Deligne’s conjecture: primes in intervals between successive squares)**.** As $n \to \infty$, the number $N_{\mathcal{P}}(n)$ of primes in the interval $(n^2, (n+1)^2)$ satisfies $N_{\mathcal{P}}(n) \sim n/\log n$.

Figure 2 (left) plots $N_{\mathcal{P}}(n)$ for $n = 1, \ldots, 31621$ (blue dots) and the asymptotic approximation $n/\log(n)$ (red line). The results support Deligne’s Conjecture 8 for primes.

Imitating Deligne’s heuristic argument for primes, the length of the interval $(n^2, (n+1)^2)$ is asymptotic to $2n$. The probability that an integer in this interval is cyclic should be proportional to $C((n+1)^2)/(n+1)^2$, which by (3) is asymptotic to

$$\frac{1}{e^\gamma \log\log\log((n+1)^2)} \left(1 - \frac{\gamma}{\log\log\log((n+1)^2)}\right).$$

Hence as $n \to \infty$, the number of cyclics in $(n^2, (n+1)^2)$ should be asymptotic to

$$\frac{2n}{e^\gamma \log\log\log((n+1)^2)} \left(1 - \frac{\gamma}{\log\log\log((n+1)^2)}\right) \sim \frac{2n}{e^\gamma \log\log\log(n)} \left(1 - \frac{\gamma}{\log\log\log(n)}\right). \tag{10}$$

This heuristic argument is not a proof of the following conjecture.

Conjecture 9 (cyclics in intervals between successive squares)**.** As $n \to \infty$, the number $N_c(n) := \#\{c \in \mathcal{C} \mid c \in (n^2, (n+1)^2)\}$ of cyclics in the interval $(n^2, (n+1)^2)$ is asymptotic to (10).

In my numerical calculations, the ratio of $N_c(n)$ to the corresponding function of n in (10) declines slowly toward 1 as n increases, despite the increase in Figure 2 (right) in the arithmetic difference between $N_c(n)$ and the corresponding function of n in (10).

Let L_p be the OEIS sequence A349997, defined as “Numbers k such that the number of primes in any [i.e., every] interval $[j^2, (j+1)^2], j > k$, is not less than the number of primes in the interval $[k^2, (k+1)^2]$.” Let $L_p(n)$ be the nth element of L_p in increasing order.

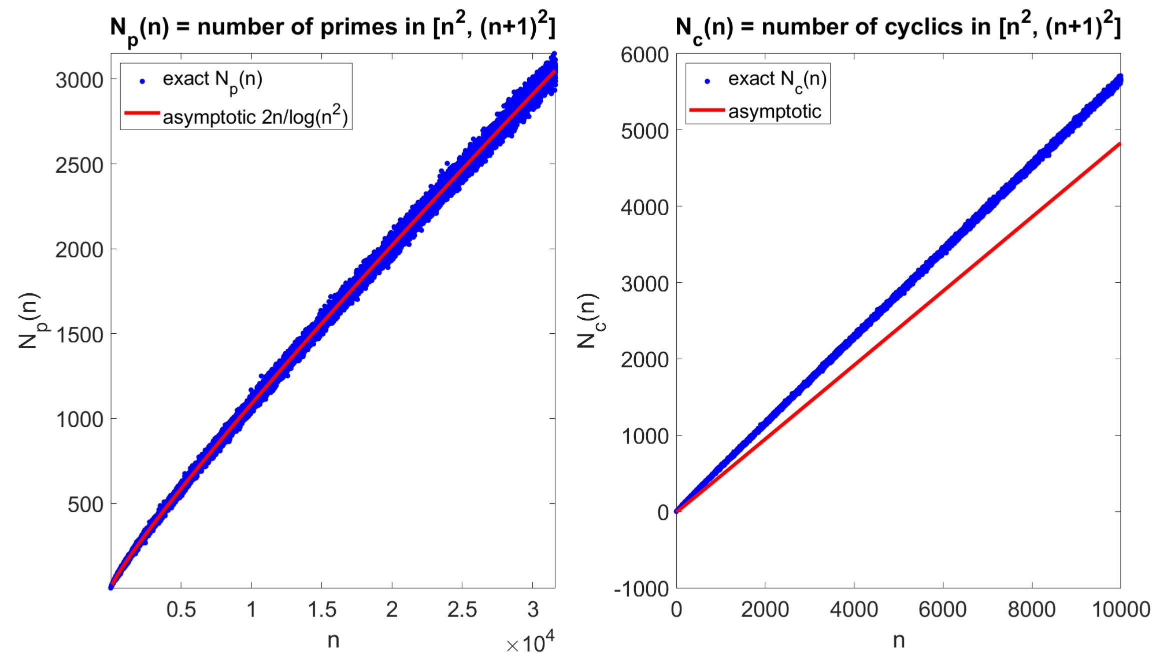

Figure 2: (left) The number $N_{\mathcal{P}}(n)$ of primes in the interval $(n^2, (n+1)^2)$ (blue dots) for $n = 1, \ldots, 31621$ and a conjectured asymptotic approximation $n/\log(n)$ (red curve). (right) The number $N_c(n)$ of cyclics in the interval $(n^2, (n+1)^2)$ (blue dots) for $n = 1, \ldots, 9998$, and a conjectured asymptotic approximation (10) (red line).

Conjecture 10 (k-fold Legendre for primes)**.** For primes, $L_p = \{1, 7, 11, 17, 18, 26, 27, 32, 46, 50, 56, 58, 85, 88, 92, 137, 143, 145, \ldots\}$.

For example, $L_p(2) = 7$ means that $N_p(7) = 3$ (i.e., three primes 53, 59, 61 lie between 7^2 and 8^2) and (conjecturally, based on available computations) for every $j > 7$, $N_p(j) \geq 3$.

Hugo Pfoertner tabulated 2414 (conjectural) values of L_p at OEIS A349997 without reporting the number of primes he considered. These numerical values are conjectural because they depend on an infinite sequence of primes not accessible to computation and not yet analyzed mathematically. Pfoertner's 2414 values appear (Figure 3 left) to be well approximated by an^b with $a \approx 0.257,\ b \approx 1.9475$.

Conjecture 11 (asymptotic k-fold Legendre for primes)**.** As $n \to \infty$, $L_p(n)$ is asymptotic to a regularly varying function with index b that satisfies $3/2 < b \leq 2$.

The counting function of the sequence $L_p(\cdot)$ is defined for each $m \in \mathbb{N}$ as $\sum_{L_p(n) \leq m} 1$. Because $L_p(\cdot)$ and its counting function are asymptotically inverses, a mathematical consequence of Conjecture 11 is that the counting function of $L_p(\cdot)$ is asymptotic to a regularly varying function with index $1/2 \leq 1/b < 2/3$ [47, pp. 21–27] [27, section 8, pp. 16–17]. Define

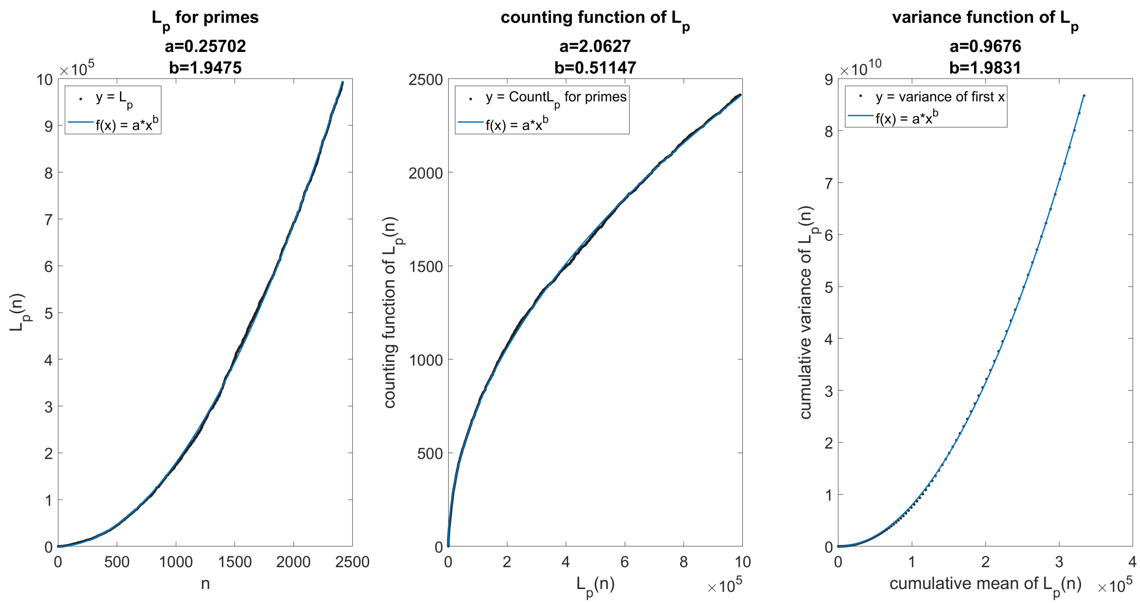

Figure 3: (left) The sequence L_p (A349997, black dots), and the power function (blue curve) fitted by least squares. (middle) The counting function of L_p (black dots) and the power function (blue curve) fitted by least squares. (right) Variance function of L_p (black dots) and a power function (blue curve) fitted by least squares.

$m_p(n)$ and $v_p(n)$ to be, respectively, the mean and the variance of $L_p(1), L_p(2), \ldots, L_p(n)$. Then applying [8, Theorem 1] to Conjecture 11 gives $v_p(n) \sim m_p(n)^2/\big((1/b)(1/b+2)\big) = b^2 m_p(n)^2/(1+2b)$.

The estimated exponent $b \approx 1.9475$ of the power law fitted to L_p (Figure 3 left) by least squares predicts that the counting function of L_p will be asymptotic to a power law with exponent $1/b \approx 0.5135$. The power law fitted by least squares to the counting function of L_p has exponent 0.5115 (Figure 3 middle), different by only 0.002.

The variance function of L_p is defined as the function $(0, \infty) \mapsto (0, \infty)$ from the mean of the first n elements of L_p to the variance of the first n elements of L_p, for all $n \in \mathbb{N}$. I estimated the variance function (Figure 3 right) using the first 25 elements of L_p, then the first 50, then the first 75, and so on in successively longer intervals, each embedded in the next, up to the first 2400 elements. The exponent $1/b$ of an asymptotic regularly varying counting function (Figure 3 middle) predicts that the asymptotic variance function (cumulative) will be a power function with exponent 2 and coefficient $1/\big((1/b)(1/b+2)\big) = b^2/(1+2b)$. The power law fitted by least squares to the variance function has exponent approximately 1.9831 (Figure 3 right), not greatly different from the predicted value 2. The estimated coefficient is approximately 0.9676, while $1/\big((1/b)(1/b+2)\big) \approx 0.7748$.

Since all primes are cyclics, if there are k or more primes in $(n^2, (n+1)^2)$, then there are

k or more cyclics in $(n^2, (n+1)^2)$.

Analogous to L_p, define L_c as numbers k such that the number of cyclics in every interval $[j^2, (j+1)^2], j > k$, is not less than the number of cyclics in the interval $[k^2, (k+1)^2]$. Using the identical algorithm used to calculate L_p for primes, with cyclics replacing primes as the input, I calculated 769 values of L_c based on the 28,488,167 cyclics less than 10^8. For example, as the first 25 cyclics are 1, 2, 3, 5, 7, 11, 13, 15, 17, 19, 23, 29, 31, 33, 35, 37, 41, 43, 47, 51, 53, 59, 61, 65, 67, the seven intervals $[1, 4]$, $[4, 9], \ldots$, $[49, 64]$ contain $N_c(n) =$ 2, 2, 3, 3, 4, 4, and 4 cyclics, and no later intervals *in these calculations* have fewer than 4 cyclics. Consequently, I conjecture that $L_c(1) = 1, L_c(2) = 3, L_c(3) = 5$. At greater length, I conjecture:

Conjecture 12 (k-fold Legendre for cyclics)**.** For cyclics, $L_c = (1, 3, 5, 8, 11, 14, 15, 16, 19, 21, 27, 29, 33, 38, 39, 46, 47, 51, 58, 61, 62, 66, 82, 86, 90, 104, 105, 108, 110, 118, 126, 127, 129, 131, 138, 141, 149, 152, 159, 161, 167, 170, 172, 174, 180, 182, 185, 187, \ldots)$.

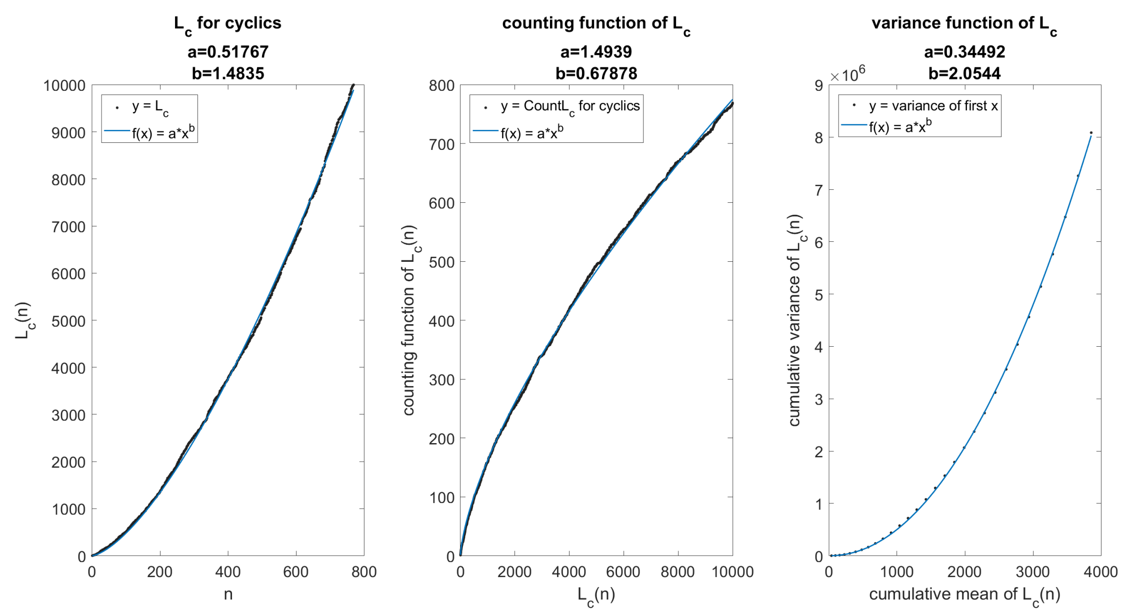

Figure 4: (left) The sequence L_c (black dots) and the power function (blue curve) fitted by least squares. (middle) The counting function of L_c (black dots) and the power function (blue curve) fitted by least squares. (right) The variance function (black dots) of L_c and a power function (blue curve) fitted by least squares.

Figure 4 (left) indicates that a power function approximates closely the calculated values of L_c for cyclics. The counting function of L_c (middle) and the variance function of L_c (right) are also well approximated by power functions.

Conjecture 13 (asymptotic k-fold Legendre for cyclics)**.** As $n \to \infty$, $L_c(n)$ is asymptotic to a regularly varying function with index that satisfies $1 \leq b \leq 2$.

2.1.4 Landau's problem 4

Fourth in Landau's list, the near-square conjecture states that infinitely many primes $p \in \mathcal{P}$ satisfy $p = n^2+1$ for some $n \in \mathbb{N}$. These primes (A002496) are one variety of the "near-square primes." Hardy and Littlewood [22] conjectured a still unproved asymptotic expression for the counting function of classes of primes including near-square primes. Western [55, p. 109, table] compared the conjectured asymptotic formula of Hardy and Littlewood with the numerically evaluated counting function of near-square primes using a list [11, pp. 238–239] by Cunningham of the values of $n < 15000$ such that $p = n^2 + 1$ is a near-square prime. I programmed the computation and found, unexpectedly, that every one of Western's 15 tabulated values of the counting function of near-square primes is one less than the corresponding value I obtained. The discrepancy is due to Cunningham's omission [11, p. 238] of $n = 1$ for the first near-square prime, $1^2 + 1 = 2$.

Golubew [20, pp. 10–12] tabulated the values of $n \in [1, 10000]$ such that $n^2 + 1 \in \mathcal{P}$ (including $n = 1$, unlike Cunningham) and conjectured [20, p. 13] that, for every $m \in \mathbb{N}$, the interval $(m^4, (m+1)^4)$ contains at least one near-square prime $p = n^2 + 1$ for some $n \in \mathbb{N}$. I confirmed Golubew's conjecture using the 50,847,534 prime numbers less than 10^9, which include 2379 near-square primes, the last being $999444997 = 31614^2+1$ (Figure 5, top right). Not every interval between successive cubes $(m^3, (m+1)^3)$ for $m \in \mathbb{N}$ contains at least one near-square prime (Figure 5, top left). For example, the interval $(9^3 = 729,\ 10^3 = 1000)$ contains no near-square prime.

The 28,488,167 cyclics less than 10^8 include 3,786 near-square cyclics equal to $n^2 + 1$ for some $n \in \mathbb{N}$, beginning with 2, 5, 17, 37, 65, 101, 145, 197, 257, 401, 485, 577, 677, 785, 901, 1157, 1297, 1601, 1765, 1937, 2117, 2501, 2917, 3137, 3365, 3601, 3845, 4097, 4357, 5477, 5777, 6085, 6401, 7057, 7397, 7745, 8101, 8465, 8837, 9217, 9605, 10001, 10817, 11237, 11665, 12101, 12545, 12997, 13457, 14401, 14885, 15377, 15877, 16385, 16901, 17957, 18497, 19045, 19601, 20165, 20737, 21317, 21905, 22501, 23717, 24337, 24965, 25601, 26897, 27557, 28901, 30977, 31685, 32401, 33857, 34597, 35345, 36101, 37637, 38417, 40001, 41617, 42437, 43265, 44101, 45797, 46657, 48401, 49285, 50177, 51077, 52901, 54757, 55697, 56645, 57601, 59537, 60517, 62501, 63505 and ending with $99880037 = 9994^2 + 1$ and $99,920,017 = 9996^2 + 1$.

Conjecture 14 (near-square analog for cyclics)**.** Infinitely many cyclics $c \in \mathcal{C}$ satisfy $c = n^2 + 1$ for some $n \in \mathbb{N}$.

After seeing Conjecture 14 in an earlier draft, Carl Pomerance (personal communication, June 2 2025) conjectured that $n^2 - 1$ is cyclic for infinitely many n. He suggested that the conjecture is plausible because all prime factors of $n^2 - 1$ are at most $n + 1$.

Conjecture 15 (near-square analog of Golubew for cyclics)**.** For every $m \in \mathbb{N}$, the interval $(m^3, (m+1)^3)$ contains at least one near-square cyclic $c = n^2 + 1$ for some $n \in \mathbb{N}$. For every

$m \in \mathbb{N}$, the interval $(m^4, (m+1)^4)$ contains at least two near-square cyclics $c = n^2 + 1$, $c' = n'^2 + 1$ for some $n < n' \in \mathbb{N}$.

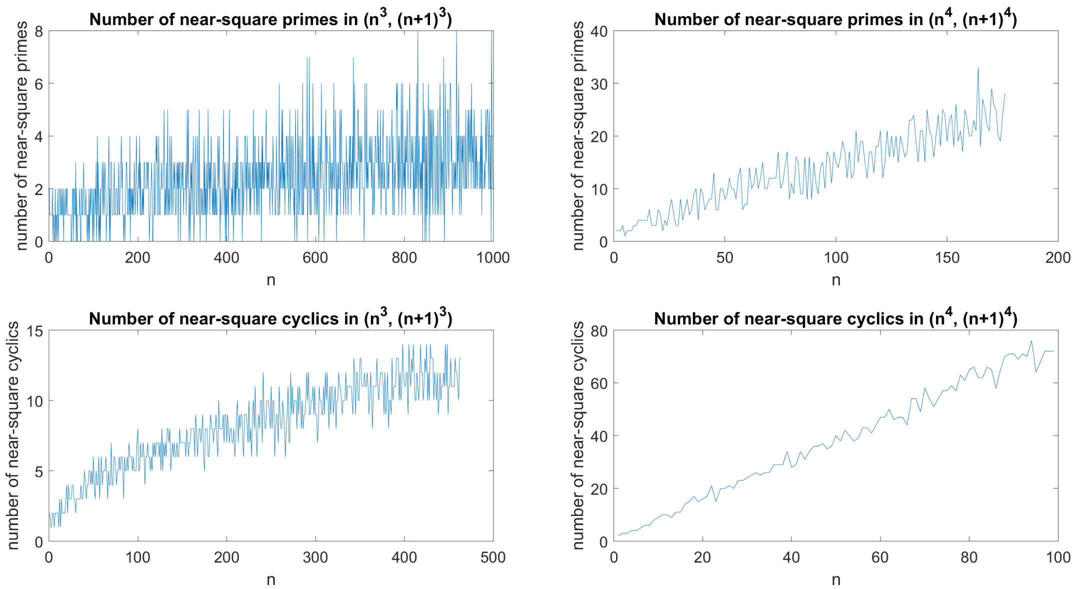

Figure 5: (top row) The number of near-square primes $p = m^2 + 1$ in each interval (left) $(n^3, (n+1)^3)$ and (right) $(n^4, (n+1)^4)$. (bottom row) The number of near-square cyclics $c = m^2 + 1$ in each interval (left) $(n^3, (n+1)^3)$ and (right) $(n^4, (n+1)^4)$.

2.2 Oppermann's conjecture

Oppermann [38, p. 174], in an unpublished lecture on March 9 1877, conjectured that for every $n > 1$, there exist two primes p, p' such that $n^2 - n < p < n^2 < p' < n^2 + n$. Several claims to prove Oppermann's conjecture have been published or posted but I am not aware that any has been independently confirmed.

The heuristic approach of Deligne's conjecture 8 suggests that, asymptotically as $n \to \infty$, each number in the interval $[n^2 - n, n^2]$ has probability $1/\log(n^2)$ of being prime, hence the number of primes in $[n^2 - n, n^2]$ should be asymptotic to $n/(2\log(n))$. The same argument and conjectured conclusion hold for the number of primes in $[n^2, n^2 + n]$. These suggestions make plausible the following conjecture:

Conjecture 16 (counting Oppermann primes)**.** As $n \to \infty$, the number of primes in $[n^2 - n, n^2]$ is asymptotic to $n/(2\log n)$ and the number of primes in $[n^2, n^2 + n]$ is also asymptotic to $n/(2\log n)$.

If true, Conjecture 16 would imply the following:

Conjecture 17 (k-fold Oppermann conjecture for primes)**.** For every $k \in \mathbb{N}$, there exists $N(k) \in \mathbb{N}$ such that for all $n > N(k)$, there exist at least k primes in $[n^2-n, n^2]$ and another at least k primes in $[n^2, n^2+n]$. In particular, $N(2) = 16$, $N(3) = 36$, $N(4) = 46$, $N(5) = 76$, $N(6) = N(7) = 79$, $N(8) = 85$, $N(9) = 118$, $N(10) = 136$, $N(11) = N(12) = 155$, $N(13) = 188$.

Oppermann's conjecture corresponds to $N(1) = 1$.

Remark 18. Oppermann's conjecture for primes implies Legendre's conjecture for primes. Specifically, if (a) for every $n \in \mathbb{N},\ n > 1$, there exists $p \in \mathcal{P}$ such that $n(n-1) < p < n^2$, or (b) for every $n \in \mathbb{N}$, there exists $p' \in \mathcal{P}$ such that $n^2 < p' < n(n+1)$, or (c) both (a) and (b) hold, then (d) for every $n \in \mathbb{N}$, there exists $p \in \mathcal{P}$ such that $n^2 < p < (n+1)^2$.

Proof. Since $(n-1)^2 < n(n-1)$ for all $n > 1$, (a) implies (d). Since $n(n+1) < (n+1)^2$ for all $n \in \mathbb{N}$, (b) implies (d). $\square$

Conjecture 19 (Oppermann analog for cyclics)**.** For every $n > 1$, there exist $c,\ c' \in \mathcal{C}$ such that $n^2 - n < c < n^2 < c' < n^2 + n$. The number of cyclics in $[n^2 - n, n^2]$ is asymptotic to

$$\frac{n}{e^\gamma \log\log\log(n^2)}\left(1 - \frac{\gamma}{\log\log\log(n^2)}\right) \sim \frac{n}{e^\gamma \log\log\log(n)}\left(1 - \frac{\gamma}{\log\log\log(n)}\right). \tag{11}$$

The number of cyclics in the interval $[n^2, n^2 + n]$ is asymptotic to

$$\frac{n}{e^\gamma \log\log\log(n(n+1))}\left(1 - \frac{\gamma}{\log\log\log(n(n+1))}\right) \sim \frac{n}{e^\gamma \log\log\log(n)}\left(1 - \frac{\gamma}{\log\log\log(n)}\right). \tag{12}$$

The sum of (11) and (12) is asymptotic to (10), and (11) is asymptotic to (12). For $n \in \mathbb{N}$ such that $n \geq 4$, one has (11) < (12).

For the primes less than 10^9, for each $n = 2, \ldots, 31622$, Figure 6 (left) plots $n/(2\log n)$ and the minimum of the numbers of primes in the two intervals $[n^2 - n, n^2]$ and $[n^2, n^2 + n]$. For the cyclics less than 10^8, Figure 6 (right) compares the asymptotic expression in (11) (red curve) with the minimum of the numbers of cyclics in the two intervals $[n^2 - n, n^2]$ and $[n^2, n^2 + n]$ (blue dot).

If true, Conjecture 19 would imply the following:

Conjecture 20 (k-fold Oppermann conjecture for cyclics)**.** For every $k \in \mathbb{N}$, there exists $N(k) \in \mathbb{N}$ such that for all $n > N(k)$, there exist at least k cyclic numbers in the interval $[n^2 - n, n^2]$ and another at least k prime numbers in the interval $[n^2, n^2 + n]$. In particular, $N(2) = 4$, $N(3) = 7$, $N(4) = 13$, $N(5) = 16$, $N(6) = 18$, $N(7) = 21$, $N(8) = 25$, $N(9) = 31$, $N(10) = N(11) = 32$, $N(12) = 40$, $N(13) = 44$.

Numerically, there exist $c, c' \in \mathcal{C}$ such that $n^2 - n < c < n^2 < c' < n^2 + n$ for each $1 < n \leq 9998$. For $n = 2, 3, \ldots, 25$, the lesser of the number of cyclics in $(n^2 - n, n^2)$ and the number of cyclics in $(n^2, n^2 + n)$ is 1, 1, 2, 1, 2, 2, 2, 3, 3, 3, 3, 4, 3, 5, 5, 4, 6, 5, 6, 7, 6, 7, 9, 7, and 7. All computed later elements in this sequence are 8 or larger.

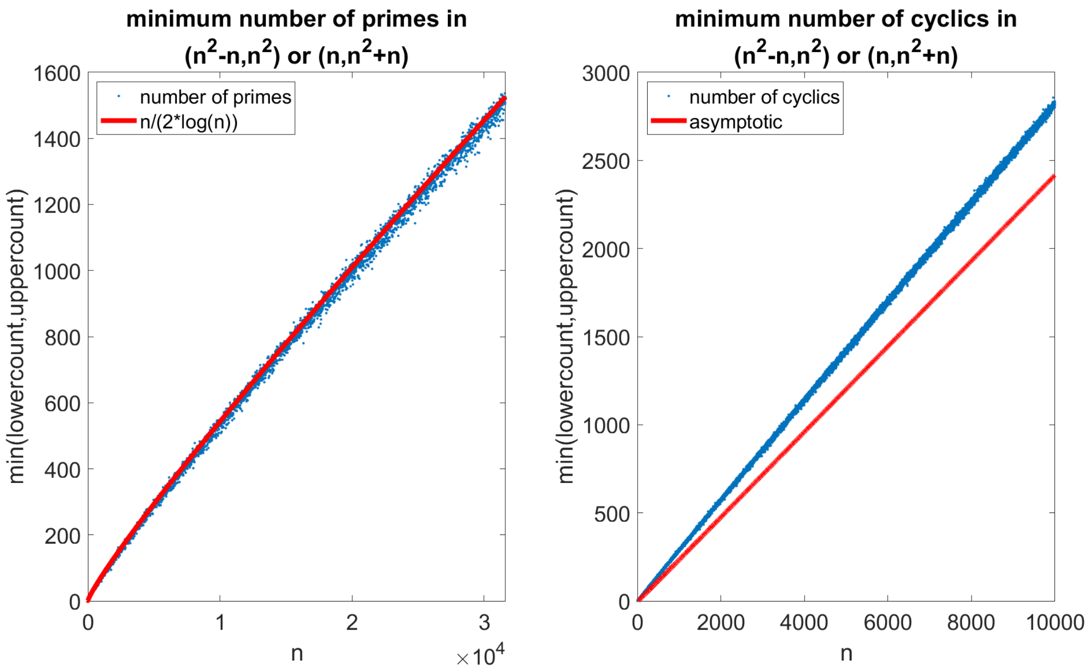

Figure 6: (left) For the primes less than 10^9, the minimum of the numbers of primes in the two intervals $[n^2 - n, n^2]$ and $[n^2, n^2 + n]$ is shown by a blue dot for every tenth value of n to avoid having the blue dots overwrite the red curve. The red curve plots $n/(2 \log n)$, $n = 2, \ldots, 31622$. (right) For the cyclics less than 10^8, for each $n = 20, 21, 22, \ldots, 9999$, a blue dot shows the minimum of the numbers of cyclics in the two intervals $[n^2 - n, n^2]$ and $[n^2, n^2 + n]$. The red line shows the asymptotic expression in (11).

2.3 Brocard's and Desboves' conjectures

Brocard [3] conjectured in 1904 that there are at least four primes between the squares of two successive primes, provided that the first prime be greater than 3. However, between the squares 9 and 25 of 3 and 5, respectively, there are more than four primes, namely, 11, 13, 17, 19, 23, so Brocard's proviso should be replaced by requiring that the first prime be greater than 2. Several claims to prove Brocard's conjecture have been published or posted but I am not aware that any has been independently confirmed.

Remark 21. Desboves' conjecture [12] that for every $n \in \mathbb{N}$, there exist $p, p' \in \mathcal{P}$ such that $n^2 < p < p' < (n+1)^2$ implies Brocard's (adjusted) conjecture [3] that there are at least four primes between the squares of two successive primes greater than 2. More generally, if p, p' are two primes both greater than 2, then Desboves' conjecture [12] implies that there are at least $2(|p' - p|)$ primes between p^2 and p'^2.

Proof. As 2 is the only even prime, every pair of consecutive primes except 2 and 3 is separated by at least one even number. If $p > 2$ and $p' = p + 2$ are twin primes and

$p < m < p'$, $m \in \mathbb{N}$, then, by Desboves' conjecture, there are at least two primes between p^2 and m^2. Again by Desboves' conjecture, there are at least two primes between m^2 and p'^2. So there are at least four primes between p^2 and p'^2. All other pairs $2 < p < p'$ of consecutive primes have more than one intervening even number and one or more intervening odd numbers between them, and each of those intervening numbers contributes two or more primes to the count of primes between p^2 and p'^2. $\square$

For $k \in \mathbb{N},\ k \geq 4$, define $B(k)$ to be the smallest $n \in \mathbb{N}$ such that, for all $m \in \mathbb{N}$ with $m \geq n$, there are always at least k primes between p_m^2 and p_{m+1}^2. For example, Brocard's conjecture asserts that $B(4) = 2$. (This 2 points to the second prime, $p_2 = 3$, not to the first prime 2.) T. D. Noe calculated the number of primes between p_n^2 and p_{n+1}^2 for $n \leq 10^4$; see A050216. Based on Noe's calculations, I conjecture the following.

Conjecture 22 (k-fold Brocard for primes)**.** Let $B(4) = 2, B(5) = 2, B(6) = 3, B(7) = B(8) = B(9) = 5, B(10) = B(11) = 7, B(12) = B(13) = B(14) = B(15) = B(16) = 10, B(17) = B(18) = B(19) = B(20) = 13$. Then for $k = 4, \dots, 20$ and for every $n \in \mathbb{N}$ such that $n \geq B(k)$, there exist at least k primes between p_n^2 and p_{n+1}^2. More generally, for every $k \in \mathbb{N}$, there exists $B(k) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq B(k)$, there exist at least k primes between n^2 and $(n+1)^2$.

Conjecture 23 (Brocard analog for cyclics)**.** For every $n > 2$, there exist at least six cyclics $c \in \mathcal{C}$ such that $c_n^2 < c < c_{n+1}^2$.

There are two cyclics $c_2 = 2,\ c_3 = 3$ in the open interval $(c_1^2 = 1, c_2^2 = 4)$. There are two cyclics $c_4 = 5,\ c_5 = 7$ in $(c_2^2 = 4, c_3^2 = 9)$. The six cyclics in $(c_3^2 = 9, c_4^2 = 25)$ are 11, 13, 15, 17, 19, and 23. For $n = 1, \dots, 25$, the number of cyclics $c \in \mathcal{C}$ such that $c_n^2 < c < c_{n+1}^2$ is 2, 2, 6, 8, 25, 16, 17, 21, 22, 56, 102, 36, 36, 45, 49, 96, 52, 113, 125, 65, 206, 80, 152, 83, 84. For $3 \leq n \leq 1009$ (with $c_{1009} = 3157$), there are at least six cyclics $c \in \mathcal{C}$ such that $c_n^2 < c < c_{n+1}^2$.

If the k-fold Brocard conjecture for primes is true, then it is immediate that for $k = 4, \dots, 20$ and for every $n \in \mathbb{N}$ such that $n \geq B(k)$, there exist at least k cyclics between p_n^2 and p_{n+1}^2, since all primes are cyclics. The following analog for cyclic numbers is not an obvious consequence of the k-fold Brocard conjecture for primes, and may hold true even if Conjecture 22 is false.

Conjecture 24 (k-fold Brocard analog for cyclics)**.** For every $k \in \mathbb{N}$, there exists $C(k) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq C(k)$, there exist at least k cyclic numbers c such that $c_n^2 < c < c_{n+1}^2$.

Numerically, $C(2) = 1$, i.e., for every $n \geq 1$ computed here, there exist at least 2 cyclic numbers c such that $c_n^2 < c < c_{n+1}^2$.

2.4 Schinzel's conjectures

Schinzel [49, p. 155, Conjecture P_1] conjectured in 1961 that, for real numbers $x \geq 117$, there is at least one prime between x and $x + \sqrt{x}$. When x is evaluated only at primes, the exceptional cases (among the primes less than 10^9, in my computations) where there is *no* prime between p_n and $p_n + \sqrt{p_n}$ are $p_2 = 3, p_4 = 7, p_6 = 13, p_9 = 23, p_{11} = 31$, and $p_{30} = 113$. A stronger conjecture by Schinzel [49, p. 156] is that, for real number $x \geq 8$, there is at least one prime between x and $x + (\log x)^2$. When x is evaluated only at primes, the exceptional cases (among the primes less than 10^9) where there is *no* prime between p_n and $p_n + (\log p_n)^2$ are $p_1 = 2, p_2 = 3$, and $p_4 = 7$. With their different lower bounds, both conjectures have been confirmed numerically for $x \leq 4.44 \times 10^{12}$. Conjecture P_1 implies Conjecture P of Sierpiński [49, p. 153]. Both of Schinzel's conjectures have obvious analogs for cyclics and one analog not so obvious. I verified these conjectures for the cyclics less than 10^8.

Conjecture 25 (Schinzel Conjecture P_1 analog for cyclics)**.** For every $n \in \mathbb{N}$,

$$c_{n+1} \leq c_n + \sqrt{c_n}$$

except for $c_3 = 3, c_5 = 7$, and $c_{11} = 23$.

Conjecture 26 (Schinzel conjecture for $\log^2$ analog for cyclics)**.** For every $n \in \mathbb{N}$,

$$c_{n+1} \leq c_n + (\log c_n)^2$$

except for $c_1 = 1, c_2 = 2, c_3 = 3$, and $c_5 = 7$.

Conjecture 27 (Schinzel-type conjecture for $2 \times \log$ analog for cyclics)**.** For every $n \in \mathbb{N}$,

$$c_{n+1} \leq c_n + 2 \log c_n$$

except for $c_1 = 1$ and $c_5 = 7$.

2.5 Golubew's conjectures

Noting Legendre's conjecture for primes, Golubew [19, p. 85] conjectured in 1957 that for $n \in \mathbb{N}$, there is at least one pair of twin primes $p,\ p+2$ between n^3 and $(n+1)^3$. Further, he conjectured that, for $n \in \mathbb{N}$, there is at least one quartet of primes $p,\ p+2,\ p+6,\ p+8$ between n^5 and $(n+1)^5$. For $n = 1, \ldots, 25$, I find that the number of pairs of twin primes between n^3 and $(n+1)^3$ is 2, 2, 3, 3, 5, 5, 4, 6, 5, 11, 9, 12, 11, 12, 17, 17, 16, 19, 16, 18, 24, 22, 17, 22, and 26. For $n = 1, \ldots, 10^3 - 1$ such that $(n+1)^3 \leq 10^9$, I find numerically that the number of pairs of twin primes between n^3 and $(n+1)^3$ is never less than two. As examples, if $n = 1$, then $(3, 5)$ and $(5, 7)$ are two pairs of twin primes between 1 and $(n+1)^3 = 8$; and if $n = 2$, then $(11, 13)$ and $(17, 19)$ are two pairs of twin primes between 8 and 27. Moreover, I find 3 or more pairs of twin primes between n^3 and $(n+1)^3$ for all $3 \leq n \leq 999$, 4 or more pairs of twin primes between n^3 and $(n+1)^3$ for all $5 \leq n \leq 999$, 5 or more pairs of twin primes between n^3 and $(n+1)^3$ for all $8 \leq n \leq 999$, 6 or more pairs of twin primes between n^3 and $(n+1)^3$ for all $10 \leq n \leq 999$, and so on.

Golubew [19, p. 84, Table 2] tabulated the number of pairs of twin primes between n^3 and $(n+1)^3$ for $n = 1, \ldots, 80$. I confirmed his counts with five exceptions, which I believe to be his errors. I list the five cases in which I challenge his results with three numbers: n, his count of twin primes between n^3 and $(n+1)^3$, and my count of twin primes between n^3 and $(n+1)^3$. These five cases are: 25, 27, 26; 26, 31, 32; 70, 109, 119; 74, 130, 131; 80, 160, 161.

Conjecture 28 (number of twin primes between consecutive cubes)**.** For every $n \in \mathbb{N}$, the number of pairs of twin primes between n^3 and $(n+1)^3$ is never less than two. More generally, for every $k \in \mathbb{N}$, there exists $N(k) \in \mathbb{N}$ such that for all $n \geq N(k)$ there are at least k pairs of twin primes between n^3 and $(n+1)^3$. Specifically, $N(1) = N(2) = 1$, $N(3) = 3$, $N(4) = 5$, $N(5) = 8$, $N(6) = N(7) = N(8) = N(9) = 10$, $N(10) = 11$, $N(11) = 13$, $N(12) = N(13) = N(14) = N(15) = N(16) = 15$, and $N(17) = 20$.

Two consecutive primes p, p' are defined [56, p. 336] to be cousin primes if $|p - p'| = 4$ and defined to be sexy primes if $|p - p'| = 6$. By these definitions, 3 and 7 are not cousin primes and 11 and 17 are not sexy primes because they are not consecutive. Every other pair of primes p, p' with $|p-p'| = 4$ is consecutive, hence cousin. Many other pairs of primes p, p' with $|p - p'| = 6$ are not consecutive, hence not sexy. For $n = 1, \ldots, 25$, I find that the number of pairs of cousin primes between n^3 and $(n+1)^3$ is 0, 2, 2, 5, 3, 5, 8, 3, 11, 7, 12, 7, 15, 14, 13, 10, 19, 13, 20, 21, 22, 23, 24, 28, and 31. For $n = 1, \ldots, 25$, I find that the number of pairs of sexy primes between n^3 and $(n+1)^3$ is 0, 0, 3, 2, 5, 6, 7, 11, 7, 15, 11, 12, 19, 15, 20, 21, 30, 27, 29, 33, 30, 37, 43, 36, and 52. These numbers and all the following counts of cousin primes between successive cubes suggest the following conjectures:

Conjecture 29 (number of cousin primes between consecutive cubes)**.** For $n \in \mathbb{N}$ with $n > 1$, the number of pairs of cousin primes between n^3 and $(n+1)^3$ is never less than two. More generally, for every $k \in \mathbb{N}$, there exists $N(k) \in \mathbb{N}$ such that, for all $n \geq N(k)$, there are at least k pairs of cousin primes between n^3 and $(n+1)^3$. Specifically, $N(1) = N(2) = 2$, $N(3) = 8$, $N(4) = N(5) = N(6) = N(7) = 9$, $N(8) = N(9) = N(10) = 12$.

Conjecture 30 (number of sexy primes between consecutive cubes)**.** For $n \in \mathbb{N}$ with $n > 2$, the number of pairs of sexy primes between n^3 and $(n+1)^3$ is never less than two. (While 3 pairs of sexy primes occur between 3^3 and 4^3, only 2 pairs of sexy primes occur between 4^3 and 5^3.) More generally, for every $k \in \mathbb{N}$, there exists $N(k) \in \mathbb{N}$ such that, for all $n \geq N(k)$, there are at least k pairs of sexy primes between n^3 and $(n+1)^3$. Specifically, $N(1) = N(2) = 3$, $N(3) = N(4) = N(5) = 5$, $N(6) = 6$, $N(7) = 7$, $N(8) = N(9) = N(10) = N(11) = 11$.

Conjecture 31 (asymptotic k-fold primes between consecutive cubes)**.** As $n \to \infty$, the numbers of pairs of twin primes, cousin primes, and sexy primes between consecutive cubes n^3 and $(n+1)^3$ are asymptotic to regularly varying functions of n with indices between 3/2 and 2, and the indices for twin primes and cousin primes are identical.

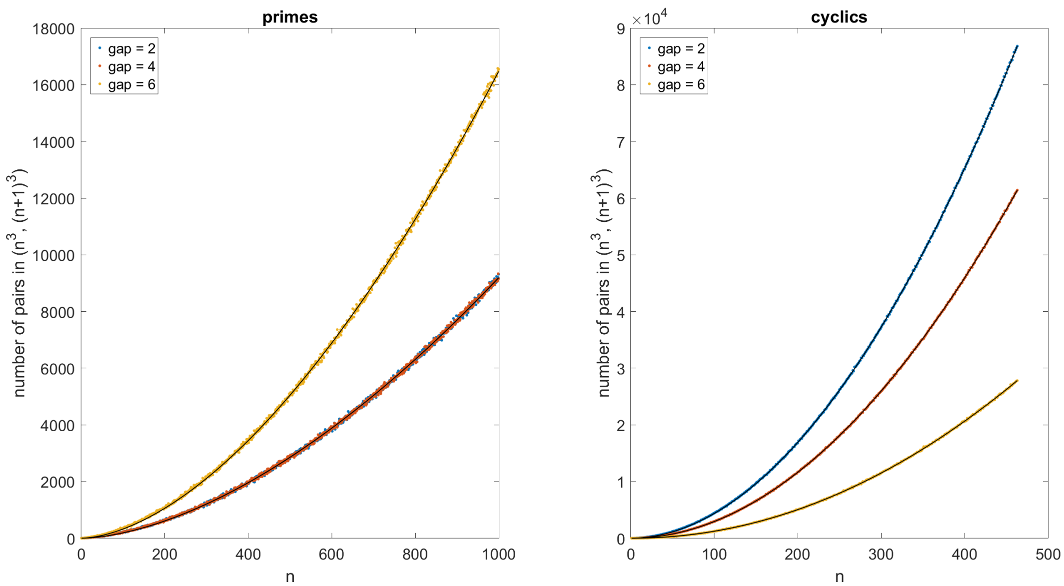

Figure 7: (left) Each dot represents, on the vertical axis, the number of pairs of consecutive primes p_m, p_{m+1} in the interval between n^3 and $(n+1)^3$ for the value of n on the horizontal axis. Gaps $p_{m+1} - p_m = 2, 4, 6$ correspond to twin (blue dots), cousin (red dots), and sexy (yellow dots) primes. Black curves show power functions an^b fitted by least squares. For twin primes (gap = 2), $a = 0.079, b = 1.689$. For cousin primes (gap = 4), $a = 0.0804, b = 1.686$. For sexy primes (gap = 6), $a = 0.124, b = 1.708$. (right) Each dot represents, on the vertical axis, the number of pairs of consecutive cyclics c_m, c_{m+1} in the interval between n^3 and $(n+1)^3$ for the value of n on the horizontal axis. Gaps $c_{m+1} - c_m = 2, 4, 6$ correspond to twin (blue dots), cousin (red dots), and sexy (yellow dots) cyclics. Black curves show power functions an^b fitted by least squares. For twin cyclics (gap = 2), $a = 0.5493, b = 1.950$. For cousin cyclics (gap = 4), $a = 0.3187, b = 1.983$. For sexy cyclics (gap = 6), $a = 0.1051, b = 2.034$.

Figure 7 (left) supports Conjecture 31 numerically.

Turning from primes to cyclics: twin cyclics, cousin cyclics, and sexy cyclics are pairs of consecutive cyclics with first difference 2, 4, and 6, respectively. For $n = 1, \ldots, 25$, I find that the numbers of pairs of twin cyclics between n^3 and $(n+1)^3$ are 2, 4, 7, 13, 17, 22, 32, 41, 44, 57, 70, 80, 99, 107, 122, 132, 142, 171, 189, 220, 221, 239, 271, 292, and 310. For $n = 1, \ldots, 25$, I find that the numbers of pairs of cousin cyclics between n^3 and $(n+1)^3$ are 0, 1, 3, 8, 8, 14, 15, 22, 29, 37, 36, 51, 50, 69, 71, 95, 92, 97, 120, 129, 142, 149, 177, 175, and 194. For $n = 1, \ldots, 25$, I find that the numbers of pairs of sexy cyclics between n^3 and $(n+1)^3$ are 0, 0, 1, 0, 2, 4, 7, 7, 9, 8, 13, 13, 19, 17, 16, 23, 38, 44, 36, 42, 46, 58, 54, 67, and 70. These numbers and all the following counts of cyclics between successive cubes $(n^3, (n+1)^3)$ suggest the following:

Conjecture 32 (number of twin cyclics between consecutive cubes)**.** For every $n \in \mathbb{N}$, the number of pairs of twin cyclics between n^3 and $(n+1)^3$ is never less than two. For every $k \in \mathbb{N}$, there exists $N(k) \in \mathbb{N}$ such that for all $n \geq N(k)$ there are at least k pairs of twin cyclics between n^3 and $(n+1)^3$. Specifically, $N(1) = N(2) = 1$, $N(3) = N(4) = 2$, $N(4) = N(5) = N(6) = N(7) = 3$, $N(8) = N(9) = N(10) = N(11) = N(12) = N(13) = 4$.

Conjecture 33 (number of cousin cyclics between consecutive cubes)**.** For $n \in \mathbb{N}$ with $n > 1$, the number of pairs of cousin cyclics between n^3 and $(n+1)^3$ is never less than one. For every $k \in \mathbb{N}$, there exists $N(k) \in \mathbb{N}$ such that for all $n \geq N(k)$ there are at least k pairs of cousin cyclics between n^3 and $(n+1)^3$. Specifically, $N(1) = 2$, $N(2) = N(3) = 3$, $N(4) = N(5) = N(6) = N(7) = N(8) = 4$, $N(9) = N(10) = N(11) = N(12) = N(13) = N(14) = 6$.

Conjecture 34 (number of sexy cyclics between consecutive cubes)**.** For $n \in \mathbb{N}$ with $n \geq 5$, the number of pairs of sexy cyclics between n^3 and $(n+1)^3$ is never less than two. For every $k \in \mathbb{N}$, there exists $N(k) \in \mathbb{N}$ such that for all $n \geq N(k)$ there are at least k pairs of sexy cyclics between n^3 and $(n+1)^3$. Specifically, $N(1) = N(2) = 5$, $N(3) = N(4) = 6$, $N(5) = N(6) = N(7) = 7$, $N(8) = 10$, $N(9) = N(10) = N(11) = N(12) = N(13) = 11$.

Conjecture 35 (asymptotic k-fold cyclics between consecutive cubes)**.** As $n \to \infty$, the numbers of twin cyclics, cousin cyclics, and sexy cyclics between consecutive cubes n^3 and $(n+1)^3$ are asymptotic to regularly varying functions of n with indices between 1 and 5/2.

2.6 Sophie Germain primes and cyclics

In approaching a proof of Fermat's Last Theorem, Sophie Germain considered (as a very special case of much more general hypotheses) pairs of primes $(p, 2p+1)$ such as $(3, 7)$ and $(5, 11)$. A prime p such that $2p+1$ is prime is called a Sophie Germain prime (or an SG prime), and the prime $2p + 1$ is called a safe or auxiliary prime. Identifying the first appearance historically of SG primes is challenging because much of Germain's work was never published, appeared in correspondence, is mentioned in the work of other mathematicians, or was published anonymously or pseudonymously [32].

It is conjectured but unproved that there are infinitely many SG primes. For positive real x, let $\pi_{SG}(x)$ be the counting function of SG primes, i.e., the number of SG primes less than or equal to x. It is conjectured but not proved [48, p. 123] that, as $x \to \infty$, we have

$$\pi_{SG}(x) \sim \left(2 \prod_{\{p \in \mathcal{P} | p > 2\}} \frac{p(p-2)}{(p-1)^2} \right) \frac{x}{(\log x)^2}.$$

If there are infinitely many SG primes, then Conjecture 36 is true, but Conjecture 36 may be true even if there are only finitely many SG primes.

Define a cyclic $c \in \mathcal{C}$ to be a Sophie Germain cyclic (or an SG cyclic) if $2c + 1 \in \mathcal{C}$.

Conjecture 36 (infinitely many SG cyclics)**.** There are infinitely many Sophie Germain cyclics.

After seeing Conjecture 36 in a draft of this paper, Carl Pomerance (personal communication, May 24 2025) announced that he can prove that the number of Sophie Germain cyclics less than x is asymptotic to $cx(e^\gamma \log\log\log x)^{-2}$ for an appropriate $c > 0$, and hence that Conjecture 36 is true.

Let σ_n be the nth SG cyclic. The first 25 SG cyclics are 1, 2, 3, 5, 7, 11, 15, 17, 23, 29, 33, 35, 41, 43, 47, 51, 53, 59, 61, 65, 69, 71, 79, 83, 89. For example, $\sigma_7 = 15$ is an SG cyclic because $2 \times 15 + 1 = 31 \in \mathcal{P}$ is cyclic, although 31 itself is not an SG cyclic, as the following list shows. The first 25 cyclics that are *not* SG cyclics are 13, 19, 31, 37, 67, 73, 77, 85, 87, 91, 97, 101, 103, 109, 115, 137, 139, 145, 157, 163, 177, 181, 187, 193, 199. For example, $2 \times 13 + 1 = 27$.

It is well known that every SG prime except SG prime 3 is congruent to 2 mod 3, for if an SG prime p were congruent to 1 mod 3, then $2p + 1$ would be congruent to 3 mod 3, i.e., composite, contradicting the assumption that p is an SG prime. The SG cyclics are different.

Conjecture 37 (SG cyclics mod 3)**.** As the number of SG cyclics grows without limit, the number of SG cyclics congruent to j mod 3, $j = 1, 2, 3$, grows without limit, and the limiting fraction of SG cyclics congruent to 1 mod 3 equals the limiting fraction of SG cyclics congruent to 3 mod 3.

After seeing an earlier version of Conjecture 37, Carl Pomerance (personal communication, May 24 2025) announced that he can show that the fractions of SG cyclics congruent to j mod 3 approach the limits $w_1 = w_3 = 0$ and $w_2 = 1$.

Based on the first 3,441,316 SG cyclics, the proportions of SG cyclics congruent to 1, 2, and 3 mod 3 are approximately 0.1360, 0.7252, 0.1388. Based on the first 6,882,632 SG cyclics (twice as many), the proportions of SG cyclics congruent to 1, 2, and 3 mod 3 are approximately 0.1342, 0.7290, 0.1368.

Conjecture 38 (Desboves analog for SG cyclics)**.** For every $n \in \mathbb{N}$, there exists at least two SG cyclics in $(n^2, (n + 1)^2)$.

For $n = 1, \ldots, 7070$, every interval $(n^2, (n + 1)^2)$ contains at least two SG cyclics. For example, for $n = 1, \ldots, 25$, the number of SG cyclics in each interval $(n^2, (n + 1)^2)$ is 2, 2, 2, 2, 3, 3, 4, 4, 3, 3, 6, 5, 4, 7, 6, 5, 8, 9, 6, 10, 7, 8, 7, 8, 9.

2.7 Firoozbakht's and related conjectures

Firoozbakht conjectured in 1982 that, if p_n is the nth prime starting from $p_1 = 2$, then $(p_n)^{1/n}$ decreases strictly as $n \in \mathbb{N}$ increases [44]. Firoozbakht's conjecture has been verified numerically for all primes less than $2^{64} \approx 1.844 \times 10^{19}$ [52].

Ribenboim [43, p. 185] dated this conjecture, communicated to him by the author, "from about 1992," but Kourbatov [29] gives what appears to be the correct date, 1982 [44].

Campbell and I [4] first stated Conjectures 39–41 of Firoozbakht type concerning cyclic numbers and tested them using only the cyclics not exceeding 4×10^6. Here I test Conjectures 39–41 plus two new Conjectures 42 and 43 using the cyclics less than 10^8.

Conjecture 39 (Firoozbakht analog for cyclics 1)**.** For every positive integer n excluding $n = 1, 2, 3$ and 5, we have

$$c_n^{1/n} > c_{n+1}^{1/(n+1)}.$$

The four exceptions are $1 < 2^{1/2}$, $2^{1/2} < 3^{1/3}$, $3^{1/3} < 5^{1/4}$, and $7^{1/5} < 11^{1/6}$.

Conjecture 40 (Firoozbakht analog for cyclics 2)**.** For every positive integer $n > 1$, we have

$$c_n^{1/(n-1)} > c_{n+1}^{1/n}.$$

If $1^{1/0} = 1^{\infty} := 1$, then the only exception is $c_1 = 1 < c_2^1 = 2$.

Conjecture 41 (Firoozbakht analog for cyclics 3)**.** For every $k \in \mathbb{N}$, there exists a least $m \in \mathbb{N}$, call it $N(k)$, such that, for all $n > N(k)$, we have

$$c_n^{1/(n+k)} > c_{n+1}^{1/(n+k+1)}.$$

In particular, if $k = 1$ or $k = 2$, then $N(k) = 5$; and if $k = 3$ or $k = 4$, then $N(k) = 11$.

For the primes among the cyclic numbers, Conjecture 39 is stronger (gives tighter inequalities) than Firoozbakht's conjecture [4]. For example, Conjecture 39 gives $c_6^{1/6} = 11^{1/6} \approx 1.4913 > c_7^{1/7} = 13^{1/7} \approx 1.4426$ whereas Firoozbakht's conjecture gives $p_5^{1/5} = 11^{1/5} \approx 1.6154 > p_6^{1/6} = 13^{1/6} \approx 1.5334$.

Conjecture 42 (Firoozbakht analog for cyclics 4)**.** For every $k \in \{0\} \cup \mathbb{N}$, define

$$\bar{c}(k) := \max_{n \in \mathbb{N}} c_n^{1/(n+k)}.$$

Then

$$\bar{c}(0) \approx 1.4953 > \bar{c}(1) \approx 1.4085, > \bar{c}(2) \approx 1.3495, > \bar{c}(3) \approx 1.3053, > \bar{c}(4) \approx 1.2710 > \dots .$$

Conjecture 43 (Firoozbakht analog for SG cyclics)**.** For every $n \in \mathbb{N}$ excluding $n = 1, 2, 3$ and 5, we have

$$\sigma_n^{1/n} > \sigma_{n+1}^{1/(n+1)}.$$

The four exceptions are $1 < 2^{1/2}$, $2^{1/2} < 3^{1/3}$, $3^{1/3} < 5^{1/4}$, and $7^{1/5} < 11^{1/6}$, exactly as in Conjecture 39.

I confirmed Conjecture 43 for the first 6,882,632 SG cyclics.

2.8 Andrica and related conjectures

Andrica [1] conjectured that, for all $n \in \mathbb{N}$, we have $\Delta\sqrt{p_n} := \sqrt{p_{n+1}} - \sqrt{p_n} < 1$. Visser [52] verified Visser's stronger version (quoted below) of Andrica's conjecture for all primes less than $2^{64} \approx 1.84 \times 10^{19}$. Several claims to prove Andrica's conjecture have been published or posted but I am not aware that any has been independently confirmed.

Conjecture 44 (Andrica analog for cyclics)**.** For all $n \in \mathbb{N}$, $\Delta\sqrt{c_n} := (c_{n+1})^{1/2} - (c_n)^{1/2} < 1$.

Ribenboim [43, p. 191] observed that the conjecture

$$\lim_{n\to\infty}(\sqrt{p_{n+1}} - \sqrt{p_n}) = 0 \tag{13}$$

would imply Andrica's conjecture. An editor [18, p. 61] remarked that "it is a difficult and as yet unsolved problem whether" (13) is true. No originator of conjecture (13) is given in [18, 43]. Wolf [57] gives an impressive heuristic argument for the truth of (13).

Figure 8 provides numerical support for conjecture (13) and for Conjecture 46 below with $t = 1/2$, which is the analogous conjecture for cyclics. I also verified Andrica's conjecture for primes less than 10^9 and its analog Conjecture 46 (with $t = 1/2$) for cyclics less than 10^8.

Conjecture 45 deals with $p_{n+1}^t - p_n^t$ for real $t \in (0, 1/2]$. Because $d^2\big((p+g)^t - p^t\big)/dg\,dt = (g+p)^t/(g+p) + t(g+p)^{t-1}\log(g+p) > 0$ for $g > 0$, $t > 0$, and $p > 1$, the difference $p_{n+1}^t - p_n^t$ is an increasing function of t and of $p_{n+1} - p_n$.

Conjecture 45 (generalized analog for primes)**.** For real $t \in (0, 1/2]$, we have

$$\lim_{n\to\infty}(p_{n+1}^t - p_n^t) = 0. \tag{14}$$

Conjecture 45 obviously implies (13).

Conjecture 46 (generalized analog for cyclics)**.** For real $t \in (0, 1/2]$, we have

$$\lim_{n\to\infty}(c_{n+1}^t - c_n^t) = 0. \tag{15}$$

Visser [53, p. 182] conjectured that, except for $n \in \{2, 4, 6, 9, 11, 30\}$ corresponding to $p_n \in \{3, 7, 13, 23, 31, 113\}$,

$$\Delta\sqrt{p_n} := \sqrt{p_{n+1}} - \sqrt{p_n} < \frac{1}{2}.$$

A generalization of the obvious analog for cyclics is as follows:

Conjecture 47 (Visser analog for cyclics)**.** For a fraction $\epsilon \in (0, 1/2)$, there exists $N(\epsilon) \in \mathbb{N}$ such that, for all $n > N(\epsilon)$, we have

$$\Delta\sqrt{c_n} := \sqrt{c_{n+1}} - \sqrt{c_n} < \epsilon. \tag{16}$$

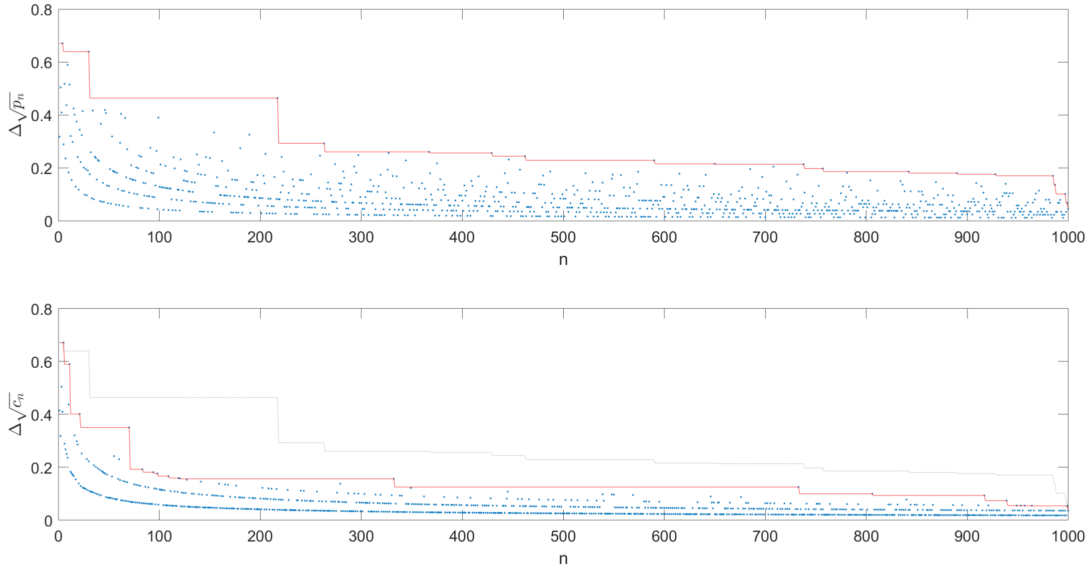

Figure 8: (above) The values (blue dots) of $\Delta\sqrt{p_n} := \sqrt{p_{n+1}} - \sqrt{p_n}$ for $n = 1, \ldots, 1000$ and the reverse cumulative maximum (red line), that is, the cumulative maximum starting from $\Delta\sqrt{p_{1000}}$ and working back to $\Delta\sqrt{p_1}$. The decrease of the reverse cumulative maximum as n increases displays the approach of $\Delta\sqrt{p_n}$ in the direction of 0 as n increases. (below) The values (blue dots) of $\Delta\sqrt{c_n} := \sqrt{c_{n+1}} - \sqrt{c_n}$ for $n = 1, \ldots, 1000$ and the reverse cumulative maximum of $\Delta\sqrt{c_n}$ for cyclics (red line). The grey line reproduces the reverse cumulative maximum for primes. The grey line for primes is never less than the red line for cyclics in these examples.

In particular, except for $n \in \{3, 5, 11\}$ corresponding to $c_n \in \{3, 7, 23\}$,

$$\Delta\sqrt{c_n} := \sqrt{c_{n+1}} - \sqrt{c_n} < \frac{1}{2} \tag{17}$$

and except for $n \in \{1, 3, 4, 5, 10, 11, 21, 70\}$ corresponding to $c_n \in \{1, 3, 5, 7, 19, 23, 53, 199\}$,

$$\Delta\sqrt{c_n} := \sqrt{c_{n+1}} - \sqrt{c_n} < \frac{1}{3}. \tag{18}$$

Kosyak, Moree, Sofos, and Zhang (hereafter KMSZ) conjectured [28, p. 216, Eq. (2)] (see also [36]) that if $n \geq 31$ (so $p_n \geq 127$), then

$$p_{n+1} - p_n < \sqrt{p_n} + 1.$$

A generalization of this conjecture consistent with the primes less than 10^9 is as follows:

Conjecture 48 (generalized KMSZ for primes)**.** For finite positive or negative integer k, there exists $N(k) \in \mathbb{N}$ such that, for all $n > N(k)$ (strict inequality), we have

$$p_{n+1} - p_n < \sqrt{p_n} + k. \tag{19}$$

In particular, for $k \in [-20, -17]$, $N(k) = 263$; for $k \in [-16, -3]$, $N(k) = 217$; for $k = -2$, $N(k) = 34$; for $k \in [-1, +3]$, $N(k) = 30$ ($k = 1$ is the KMSZ conjecture); and for $k \geq 4$, $N(k) = 0$, i.e., (19) holds for all $n \in \mathbb{N}$.

An analogous conjecture is consistent with the cyclics less than 10^8.

Conjecture 49 (generalized KMSZ for cyclics)**.** For finite positive or negative integer k, there exists $N(k) \in \mathbb{N}$ such that, for all $n > N(k)$ (strict inequality), we have

$$c_{n+1} - c_n < \sqrt{c_n} + k. \tag{20}$$

In particular, for $k = -20, -19, -18, \ldots, -1, 0, +1$, the corresponding 22 values of $N(k)$ are $N(k) = 216, 208, 176, 176, 159, 141, 127, 120, 109, 98, 83, 70, 70, 70, 70, 70, 23, 21, 21, 11, 11, 11$, and for $k \geq 2$, $N(k) = 0$, i.e., (20) holds for all $n \in \mathbb{N}$.

As mentioned in the Introduction, Carneiro et al. [5] proved that, under the Riemann hypothesis, for every $p_n > 3$, $p_{n+1} - p_n < \frac{22}{25}\sqrt{p_n} \log p_n$.

Conjecture 50 (Carneiro analog for cyclics)**.** For every $c_n > 3$, we have

$$c_{n+1} - c_n < \frac{22}{25}\sqrt{c_n} \log c_n. \tag{21}$$

The conjectured upper bounds on prime gaps in Conjecture 48 and on cyclic gaps in Conjecture 49 and Conjecture 50 are very likely far from the best possible upper bounds. For example, for the primes less than 10^9, $\max_n(p_{n+1} - p_n) = 282$ (the gap following prime 436273009) while, for the next to last of these primes, $\sqrt{p_n} + 4 \approx 31626.7755$. Similarly, for the cyclics less than 10^8, $\max_n(c_{n+1} - c_n) = 24$ while, for the next to last of these cyclics, $\sqrt{c_n} + 2 \approx 10001.9998$. It seems worth exploring further generalizations of (19) and (20) obtained by replacing the square root on the right sides by an exponent $\epsilon \in (0, 1/2)$.

Conjecture 51 (Carneiro analog for SG cyclics)**.** For every SG cyclic $\sigma_n > 3$, we have

$$\sigma_{n+1} - \sigma_n < \frac{22}{25}\sqrt{\sigma_n} \log \sigma_n. \tag{22}$$

I confirmed Conjecture 51 for the first 6,882,632 SG cyclics.

2.9 Rosser, Dusart and related conjectures

The prime number theorem [21, 51] is equivalent to $p_n \sim n \log n$. Rosser [45] proved that, for all $n \in \mathbb{N}$, $p_n > n \log n$. Dusart [14] proved that, for all $n > 1$, $p_n > n(\log n + \log\log n - 1)$. By analogy with Rosser's and Dusart's inequalities for primes, I conjecture, using (4) for cyclics, the following:

Conjecture 52 (Rosser analog for cyclics)**.** For all $n > 1$,

$$c_n > e^\gamma n \log\log\log n; \tag{23}$$

Conjecture 53 (Dusart analog for cyclics)**.** For all $n > 1$,

$$c_n > e^\gamma n(\log\log\log n + \log\log\log\log n). \tag{24}$$

Conjectures 44, 46, 47, 49, 50, 52, and 53 are consistent with all numerically evaluated $c_n \in (1, 10^8)$.

After seeing Conjectures 52 and 53 in a draft of this paper, Carl Pomerance (personal communication, June 3 2025) computed that

$$c_n = e^\gamma n(\log\log\log n + \gamma + o(1))$$

(which refines (4)). So Conjecture 52 holds for n sufficiently large and Conjecture 53 fails for n sufficiently large. The numerical results in this paper fail to distinguish between Conjecture 52 and Conjecture 53 because $\log\log\log\log 10^8 \approx 0.0670 < \gamma \approx 0.5772$.

2.10 Additive and multiplicative inequalities

Extending the Bertrand-Chebyshev theorem that $2p_n > p_{n+1}$ for all $n \in \mathbb{N}$, Ishikawa [26, Theorem 1], Gallot et al. [17, Lemma 10], and I [9, Theorem 2] proved independently, and with different proofs, that if $n > 1$, then $p_n + p_{n+1} > p_{n+2}$. When $n = 1$, equality holds: $p_1 + p_2 = p_3 = 5$.

Conjecture 54 (Ishikawa analog for cyclics)**.** For all $n > 2$,

$$c_n + c_{n+1} > c_{n+2}. \tag{25}$$

When $n = 1$ or $n = 2$, equality holds: $c_1 + c_2 = c_3 = 3$ and $c_2 + c_3 = c_4 = 5$.

Conjecture 55 (Ishikawa analog for SG cyclics)**.** For all $n > 2$,

$$\sigma_n + \sigma_{n+1} > \sigma_{n+2}. \tag{26}$$

I verified this conjecture for the first 6,882,632 SG cyclics. When $n = 1$ or $n = 2$, equality holds: $\sigma_1 + \sigma_2 = \sigma_3 = 3$ and $\sigma_2 + \sigma_3 = \sigma_4 = 5$.

More generally [9, Theorem 1], if $b_1, \ldots, b_g$ are $g > 1$ nonnegative integers (not necessarily distinct), and $d_1, \ldots, d_h$ are h positive integers (not necessarily distinct), with $1 \leq h < g$, then there exists a positive integer N such that, for all $n \geq N$, we have

$$p_{n-b_1} + p_{n-b_2} + \cdots + p_{n-b_g} > p_{n+d_1} + \cdots + p_{n+d_h}.$$

A concrete example [9] is $p_n + p_{n+1} + p_{n+2} > p_{n+3} + p_{n+4}$, proved for all $n \geq N = 8$. Analogously, for cyclic numbers, I make the following conjecture:

Conjecture 56 (sum-3-versus-sum-2 analog for cyclics)**.** For all $n > 9$,

$$LHS(n) := c_n + c_{n+1} + c_{n+2} > RHS(n) := c_{n+3} + c_{n+4}. \tag{27}$$

For $n = 1, 2, 3, 4, 5, 8, 9$, I find numerically that $LHS(n) < RHS(n)$. For $n = 6, 7$ and all computed values of $n > 9$, I find numerically that $LHS(n) > RHS(n)$.

According to Ribenboim [43, p. 185], Dusart [13] proved Mandl's conjecture that $(p_1 + p_2 + \cdots + p_n)/n \leq p_n/2$ for all $n > 8$. The inequality also holds for $n = 7$. In all the examples I know, the inequality is strict wherever the weak inequality holds.

Conjecture 57 (Dusart-Mandl analog for cyclics)**.** For all $n > 5$,

$$\frac{c_1 + c_2 + \cdots + c_n}{n} < \frac{c_n}{2}.$$

The opposite strict inequality holds for $n = 1, 2, 3, 4, 5$.

Conjecture 58 (Dusart-Mandl analog for SG cyclics)**.** For all $n > 5$,

$$\frac{\sigma_1 + \sigma_2 + \cdots + \sigma_n}{n} < \frac{\sigma_n}{2}.$$

I verified this conjecture for the first 6,882,632 SG cyclics. The opposite strict inequality holds for $n = 1, 2, 3, 4, 5$.

Panaitopol [39] and I [7] proved independently a multiplicative inequality for primes: if $m \in \mathbb{N}$, $n \in \mathbb{N}$, and $m \leq n$, then $p_{m \cdot n} < p_m p_n$ unless $(m, n) = (3, 4)$ or $(m, n) = (4, 4)$, in which cases the reverse strict inequality holds.

Conjecture 59 (Panaitopol analog for cyclics)**.** If $m \in \mathbb{N}$, $n \in \mathbb{N}$, and $3 \leq m \leq n$, then $c_{m \cdot n} < c_m c_n$ unless $(m, n) = (3, 3)$ or $(m, n) = (5, h)$ for $h = 5, 6, 7, 8, 9, 10$, in which cases the reverse strict inequality holds.

Vrba [54] conjectured (in 2010, according to Kourbatov [30]) that

$$\lim_{n \to \infty} \frac{p_n}{\left(\prod_{j=1}^{n} p_j\right)^{1/n}} = e. \tag{28}$$

Sándor and Verroken [46] in 2011 and independently Kourbatov [30] in 2016 proved (28). Hassani [23, 24] and Kourbatov [30] bounded the approach of $p_n/(\prod_{j=1}^{n} p_j)^{1/n}$ to e. Kourbatov [30] gave a short proof and calculated higher-order terms in a series expansion.

Conjecture 60 (Vrba analog for cyclics)**.**

$$\lim_{n\to\infty} \frac{c_n}{\left(\prod_{j=1}^{n} c_j\right)^{1/n}} = e.$$

For $n = 28488167$, I calculate $c_n = 99999997$ and $c_n/(\prod_{j=1}^{n} c_j)^{1/n} \approx 2.7362$, not a bad approximation to $e \approx 2.7183$. To circumvent numerical overflow of $\prod_{j=1}^{n} c_j$ in this calculation, I computed $c_n/(\prod_{j=1}^{n} c_j)^{1/n}$ by means of the equivalent $\exp\{\log c_n - (\sum_{j=1}^{n} \log c_j)/n\}$.

Hassani [23, Theorem 1.1] also proved that

$$\lim_{n\to\infty} \frac{(p_1 + \cdots + p_n)/n}{(p_1 \times \cdots \times p_n)^{1/n}} = \frac{e}{2}.$$

Conjecture 61 (Hassani analog for cyclics)**.**

$$\lim_{n\to\infty} \frac{(c_1 + \cdots + c_n)/n}{(c_1 \times \cdots \times c_n)^{1/n}} = \frac{e}{2}.$$

For $n = 28488167$, I calculate

$$\frac{(c_1 + \cdots + c_n)/n}{(c_1 \times \cdots \times c_n)^{1/n}} \approx 1.3638, \qquad \frac{e}{2} \approx 1.3591.$$

If Conjectures 60 and 61 are true, then dividing the former equality by the latter equality yields an equality Campbell and I [4, Theorem 1] proved:

$$\lim_{n\to\infty} \frac{c_n}{(c_1 + \cdots + c_n)/n} = 2.$$

Thus the difference between $(c_1 + c_2 + \cdots + c_n)/n$ and $c_n/2$, which appear in an inequality in Conjecture 57, is proved to vanish asymptotically as $n \to \infty$.

Analogs of Conjectures 60 and 61 for SG cyclics, replacing c_n by σ_n, are obvious and are numerically plausible.

2.11 Sequence of absolute difference sequences: Proth, Gilbreath

Define $\{a(n) \mid n \in \{0\} \cup \mathbb{N}\}$ to be a *sequence of absolute difference sequences* (hereafter, SADS) if, for each $n \in \{0\} \cup \mathbb{N}$, $a(n) = (a_1(n), a_2(n), a_3(n), \ldots)$ is an infinite sequence of real numbers such that, for all $n \in \mathbb{N}$ and all $m \in \mathbb{N}$, we have $a_m(n) = |a_m(n-1) - a_{m+1}(n-1)|$. In more detail, starting from an arbitrary initial real sequence $a(0) := (a_1(0), a_2(0), a_3(0), \ldots, a_m(0), a_{m+1}(0), \ldots)$, the next sequence is $a(1) := (|a_1(0) - a_2(0)|, |a_2(0) - a_3(0)|, \ldots, |a_m(0) - a_{m+1}(0)|, \ldots)$, and the elements of each successive sequence $a(n)$ give the absolute values of the first differences of elements of the preceding sequence.

Proth [42] in 1878 computed numerically the behavior of a SADS starting from the first seven primes (he included 1 as the first prime, which I ignore) and observed that every successor sequence begins with 1. For example, $a_1(1) = |3-2| = 1$ and $a_1(2) = ||3-2| - |5-3|| = |1-2| = 1$, and so on. He suggested that $a_1(n) = 1$ for all $n \in \mathbb{N}$. The journal editor, Eugène Catalan, in a gentle concluding footnote, asked (my translation): "Are not the *theorems* of M. Proth that one has just read rather postulates?" [≪Est-ce que les *théorèmes* de M. Proth, qu'on vient de lire, ne sont pas, plutôt, des postulata?≫] I take Proth's suggestion as a conjecture, generally known as N. L. Gilbreath's conjecture [43, pp. 191-192]. The Proth-Gilbreath conjecture has been verified for the primes less than 10^{13}.

Conjecture 62 (Proth-Gilbreath analog for cyclics)**.** A SADS starting from the cyclic numbers $\mathcal{C}$ after omitting $c_1 = 1$ has 1 as the first element of every successor sequence.

I verified this conjecture for 1 million successor sequences of the cyclic numbers following but not including $c_1 = 1$.

Conjecture 63 (Proth-Gilbreath analog for SG cyclics)**.** A SADS starting from the SG cyclics after omitting $\sigma_1 = 1$ has 1 as the first element of every successor sequence.

I verified this conjecture for 1 million successor sequences of the SG cyclics following but not including $\sigma_1 = 1$.

3 Second Hardy and Littlewood conjecture: cyclic analog is false

The second conjecture of Hardy and Littlewood [22] states that $\pi(m+n) \leq \pi(m) + \pi(n)$ for all integers $2 \leq m \leq n$.

Conjecture 64 (Hardy and Littlewood analog for SG primes)**.** The counting function of SG primes obeys $\pi_{SG}(m+n) \leq \pi_{SG}(m) + \pi_{SG}(n)$ for all $2 \leq m \leq n$.

I verified Conjecture 64 for all $2 \leq m \leq 910664$, $m \leq n \leq 999997$.

Using the counting function $C(\cdot)$ (1) of cyclic numbers, an analog for cyclics of the second conjecture of Hardy and Littlewood, starting from $1 \leq m \leq n$, is as follows:

Conjecture 65 (Hardy and Littlewood analog for cyclics)**.** For all integers $1 \leq m \leq n$, $C(m+n) \leq C(m) + C(n)$.

This conjecture is false. Let $m = 209 = c_{71}$, so $C(209) = 71$. Let $n = 389 = c_{128}$, so $C(389) = 128$. Then $C(m+n) = C(598) = 200$ because $c_{200} = 595 < 598 < c_{201} = 599$. Thus $C(m+n) = C(598) = 200 > C(m) + C(n) = 71 + 128 = 199$. Counterexamples like this one are abundant.

To the extent that analogies between primes and cyclics are valid, this counterexample gives a further small hint to support the belief of Hensley and Richards [25] that the second conjecture of Hardy and Littlewood [22] about primes is false.

Conjecture 66 (Hardy and Littlewood analog for Sophie Germain cyclics)**.** For all integers $1 \le m \le n$, $C_\sigma(m+n) \le C_\sigma(m) + C_\sigma(n)$.

The counterexample to Conjecture 65 for cyclics, $m = 209 = \sigma_{46}$, $n = 389 = \sigma_{83}$ is not a counterexample to Conjecture 66 for SG cyclics because $\sigma_{120} = 593 < 598 < \sigma_{121} = 599$ and therefore $C_\sigma(209 + 389) = C_\sigma(598) = 120 < C_\sigma(209) + C_\sigma(389) = 46 + 83 = 129$.

For $m = 1, \ldots, 10^6$ and $n = m, \ldots, 10^6$, I found no counterexamples to Conjecture 66 for SG cyclics. It remains open.

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References

[1] D. Andrica, Note on a conjecture in prime number theory, *Stud. Univ. Babeş–Bolyai Math.* **31** (4) (1986), 44–48,

[2] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Encyclopedia of Mathematics and its Applications, Vol. 27, Cambridge University Press, 1987.

[3] H. Brocard, Response to Problem 2181. *L'intermédiaire des mathématiques* **11** (1904), 149.

[4] J. Campbell and J. E. Cohen, Cyclic numbers and gaps, *Integers* **25** (2025), Paper A23.

[5] E. Carneiro, M. B. Milinovich and K. Soundararajan, Fourier optimization and prime gaps, *Comment. Math. Helv.* **94** (2019), 533–568.

[6] J. E. Cohen, Statistics of primes (and probably twin primes) satisfy Taylor's law from ecology, *Amer. Statist.* **70** (2016), 399–404.

[7] J. E. Cohen, Multiplicative inequalities for primes and the prime counting function, *Integers* **22** (2022), Paper A106.

[8] J. E. Cohen, Integer sequences with regularly varying counting functions have power-law variance functions, *Integers* **23** (2023), Paper A87.

[9] J. E. Cohen, Generalizations of Bertrand's postulate to sums of any number of primes, *Math. Mag.* **96** (2023), 428–432.

[10] H. Cramér, Some theorems concerning prime numbers, *Ark. Mat. Astron. Fys.* **15** (1921), 1–33.

[11] Alan J. C. Cunningham, *Binomial Factorisations, Giving Extensive Congruence-Tables and Factorisation-Tables*, Vol. 1. Francis Hodgson, 1923.

[12] H. A. Desboves, Sur un théorème de Legendre et son application à la recherche de limites qui comprennent entre elles des nombres premiers. *Nouv. Ann. Math.* **14** (1855), 281–295.

[13] P. Dusart, Autour de la fonction qui compte le nombre de nombres premiers, Ph.D. thesis, Université de Limoges, 1998.

[14] P. Dusart, The kth prime is greater than $k(\log k + \log\log k - 1)$ for $k \geq 2$, *Math. Comp.* **68** (1999), 411–415.

[15] P. Erdős, Some asymptotic formulas in number theory, *J. Indian Math. Soc. (N.S.)* **12** (1948), 75–78.

[16] W. Feller, *An Introduction to Probability Theory and its Applications. Vol. II*, John Wiley & Sons, 1971.

[17] Y. Gallot, P. Moree, and H. Hommersom, Value distribution of cyclotomic polynomial coefficients, *Unif. Distrib. Theory* **6** (2011), 177–206.

[18] S. W. Golomb et al., Limits of differences of square roots, Problem E2506, *Amer. Math. Monthly* **83** (1976), 60–61.

[19] W. A. Golubew, Abzahlung von "Vierlingen" und "Fünflingen" bis zu 5000000 und von "Sechslingen" von 0 bis 14000000, *Anz. Osterr. Akad. Wiss. Math. Nat. Kl.* **94** (1957), 82–87.

[20] W. A. Golubew, Primzahlen der Form x^2+1, *Anz. Osterr. Akad. Wiss. Math. Nat. Kl.* **95** (1958), 9–13.

[21] J. Hadamard, Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques, *Bull. Soc. Math. France* **24** (1896), 199–220.

[22] G. H. Hardy and J. E. Littlewood, Some problems of 'Partitio Numerorum.' III. On the expression of a number as a sum of primes, *Acta Math.* **44** (1923), 1–70.

[23] M. Hassani, On the ratio of the arithmetic and geometric means of the prime numbers and the number e, *Int. J. Number Theory* **9** (2013), 1593–1603.

[24] M. Hassani, On the arithmetic-geometric means of positive integers and the number e, *Appl. Math. E-Notes*, **14** (2014), 250–255.

[25] D. Hensley and I. Richards, Primes in intervals, *Acta Arith.*, **25** (1974), 375–391.

[26] H. Ishikawa, Über die Verteilung der Primzahlen, *Scientific Reports Tokyo Bunrika Daigaku, A* **2** (1934), 27–40.

[27] P. Kevei, Regularly varying functions, April 2019. Available at https://math.u-szeged.hu/~kevei/tanitas/1819regvar/RegVar_notes.pdf.

[28] Alexandre Kosyak, Pieter Moree, Efthymios Sofos, and Bin Zhang, Cyclotomic polynomials with prescribed height and prime number theory, *Mathematika* **67** (2021), 214–234.

[29] A. Kourbatov, Upper bounds for prime gaps related to Firoozbakht's conjecture, *J. Integer Sequences* **18** (2015), Article 15.11.2. Corrections in Arxiv preprint arXiv:1506.03042v4 [math.NT], March 12 2019, available at https://arxiv.org/abs/1506.03042v4.

[30] A. Kourbatov, On the geometric mean of the first n primes, Arxiv preprint arXiv:1603.00855 [math.NT], March 2 2016, available at https://arxiv.org/abs/1603.00855.

[31] Edmund Landau, Gelöste und ungelöste Probleme aus der Theorie der Primzahlverteilung und der Riemannschen Zetafunktion. *Jahresber. Deutsch. Math.-Ver.* **21** (1912), 208–228.

[32] R. Laubenbacher and D. Pengelley, *Voici ce que j'ai trouvé*: Sophie Germain's grand plan to prove Fermat's last theorem. *Historia Math.* **37** (2010), 641–692.

[33] A.-M. Legendre, *Essai sur la théorie des nombres*, 2nd ed., Courcier, 1808.

[34] MATLAB, Version 24.2.0.2740171 (R2024b) Update 1, MathWorks Inc., https://www.mathworks.com.

[35] A. J. Menezes, P. C. van Oorschot, and S. A. Vanstone, *Handbook of Applied Cryptography*, Taylor and Francis , 1996.

[36] P. Moree and E. Sofos, Hasse pairs and Andrica's conjecture. Appendix A, pp. 19–22, of: E. Agathocleous, A. Joux, and D. Taufer, Elliptic curves over Hasse pairs, Max-Planck-Institut für Mathematik, Preprint Series 2024 (14), June 27 2024. Available at https://archive.mpim-bonn.mpg.de/id/eprint/5072/1/mpim-preprint_2024-14.pdf.

[37] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, 2025, https://oeis.org.

[38] L. Oppermann, Om vor Kundskab om Primtallenes Maengde mellem givne Graendser, *Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandlinger og dets Medlemmers Arbejder* (1882), 169–179.

[39] L. Panaitopol, On the inequality $p_a p_b > p_{ab}$. *Bull. Math. Soc. Sci. Math. Roumanie N.S.* **41** (1998), 135–139.

[40] P. Pollack, Numbers which are orders only of cyclic groups, *Proc. Amer. Math. Soc.* **150** (2022), 515–524.

[41] C. Pomerance, Patterns for cyclic numbers, Unpublished manuscript submitted for publication, March 12 2025, https://math.dartmouth.edu/~carlp/cyclic.pdf. Revision June 3 2025, https://math.dartmouth.edu/~carlp/cyclicrev.pdf. Paper 247, https://math.dartmouth.edu/~carlp/.

[42] F. Proth, Sur la série des nombres premiers, *Nouv. Corresp. Math.* **4** (1878), 236–240.

[43] P. Ribenboim, *The Little Book of Bigger Primes*, 2d ed., Springer, New York, 2004.

[44] C. Rivera, Conjecture 30. The Firoozbakht Conjecture, https://www.primepuzzles.net/conjectures/conj_030.htm, in: *The prime puzzles and problems connection.* https://www.primepuzzles.net/ppp.htm. The Puzzlers: Farideh Firoozbakht (1962–2019), https://www.primepuzzles.net/thepuzzlers/Firoozbakht.htm.

[45] J. B. Rosser, The n-th prime is greater than $n \log n$, *Proc. London Math. Soc.* **45** (1939), 21–44.

[46] J. Sándor and A. Verroken, On a limit involving the product of prime numbers, *Notes Number Theory Discrete Math.* **17** (2) (2011), 1–3.

[47] E. Seneta, *Regularly Varying Functions*, Springer-Verlag, 1976.

[48] V. Shoup, *A Computational Introduction to Number Theory and Algebra*, 2d ed., Cambridge University Press, 2009.

[49] Wacław Sierpiński and Andrzej Schinzel, *Elementary Theory of Numbers*, Mathematical Library, Vol. 31. 2d English ed., revised and enlarged by Andrzej Schinzel. Elsevier North-Holland, Amsterdam, and PWN Polish Scientific Publishers, 1988.

[50] T. Szele, Über die endlichen Ordnungszahlen, zu denen nur eine Gruppe gehört, *Comment. Math. Helv.* **20** (1947), 265–267.

[51] Ch.-J. de la Vallée Poussin, Sur la fonction $\zeta(s)$ de Riemann et le nombre des nombres premiers inférieurs à une limite donnée, *Mém. couronnés et autres Mém. publ. l'Acad. roy. de Belgique* **59** (1899), 1–74.

[52] Matt Visser, Verifying the Firoozbakht, Nicholson, and Farhadian conjectures up to the 81st maximal prime gap, *Mathematics* **7** (2019), 691.

[53] Matt Visser, Strong version of Andrica's conjecture, *Int. Math. Forum* **14** (4) (2019), 181–188.

[54] Anton Vrba, Conjecture 67. Primes & e, in *Problems & Puzzles: Conjectures*, https://www.primepuzzles.net/conjectures/conj_067.htm.

[55] A. E. Western, Note on the number of primes of the form $n^2 + 1$, *Proc. Cambridge Philos. Soc.* **21** (1922), 108–109.

[56] M. Wolf, Random walk on the prime numbers, *Physica A* **250** (1998), 335–344.

[57] M. Wolf, A note on the Andrica conjecture. Arxiv preprint arXiv:1010.3945 [math.NT], October 19 2010. Available at https://arxiv.org/abs/1010.3945.

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