Problem set 10

Exercise 1 : Evaluate the integral

a) $\int \left[\frac{1}{2x^3} + 4\sqrt{x}\right] dx = \int \left[\frac{1}{2}.x^{-3} + 4.x^{1/2}\right] dx$

$= \int \frac{1.\ x^{-2}}{-4} dx + 4\int \frac{x^{3/2}}{3/2} + C$

$= \frac{-1}{4x^2} + 4.\frac{x^{3/2}}{3/2} + c$

b) $\int \left[u^3 - 2u + 7\right] du = \frac{u^4}{u} - 2\frac{u^2}{2} + 7u + c$

$= \frac{u^4}{u} - u^2 + 7u + c$

d) $\int (4\sin x + 2\cos x).dx$

$= \int 4^{-\cos x} - 4\cos x + 2\sin x + C$

e) $\int \sec x(\tan x + \cos x)\, dx$

$= \int (\sec x \tan x)\, dx + \int (\sec x \cos x)\, dx$

$= \int \left(\frac{1}{\cos x} \frac{\sin x}{\cos x}\right) dx +$

$= \sec x + x + C = \frac{1}{\cos x} + x + C$

c) $\int \left[x^{-2/3} - 5e^x\right] dx = \int \cancel{x}\left(\frac{1}{x^{2/3}} - 5e^x\right) dx$

$= \int \frac{x^{1/3}}{1/3}\quad 3.x^{1/3} - 5e^x = \sqrt[3]{x} - 5e^x$

f) $\int\left(\frac{3}{4x} - \sec^2 x\right)dx = \frac{3}{4}\int \frac{1}{x}\,dx - \int \sec^2(x)\,dx$

$= \frac{3}{4}\ln x - \tan x + C$

g) $\int\left(\frac{1}{1+x^2} + \frac{2}{\sqrt{1-x^2}}\right)dx$

$= \int \arctan x + 2\arcsin x + C$

h) $\int\left(\frac{12}{x\sqrt{x^2-1}} + \frac{1-x^4}{1+x^2}\right)dx$

* Substitute $u = \sqrt{x^2-1}$

$\rightarrow \frac{du}{dx} = \frac{x}{\sqrt{x^2-1}}$

$\rightarrow dx = \frac{\sqrt{x^2-1}}{x}\,du$

$= \arctan\sqrt{x^2-1}$

* solve $\int \frac{1-x^4}{1+x^2}\,dx$

$= \int -\frac{x^4-1}{1+x^2}\,dx = -\int \frac{(x^2-1)(x^2+1)}{(x^2+1)}\,dx$

$= -\int (x^2-1)\,dx = -\left(\frac{x^3}{3} - x\right) + C$

* Plug in solved integrals

$= 12\arctan\sqrt{x^2-1} - \frac{x^3}{3} + x + C$

Exercise 2 : Solve the initial problem

a) $\frac{dy}{dx} = \frac{1-x}{\sqrt{x}} \rightarrow dy = \frac{1-x}{\sqrt{x}}\,dx$

integrate 2 side of the equation

$y(x) \quad \int dy = \int \frac{1-x}{\sqrt{x}}\,dx = \int \frac{1}{\sqrt{x}}\,dx - \int \frac{x}{\sqrt{x}}\,dx$

$$= \int \frac{1}{\sqrt{x}}\,dx - \int \sqrt{x}\,dx$$

$$= 2\sqrt{x} - \frac{2x^{\frac{3}{2}}}{3} + C$$

Substitute $y(1) = 0$

$$y(x) = 2\sqrt{x} - \frac{2}{3}x^{\frac{3}{2}} + C$$

$$2 - \frac{2}{3}1^{\frac{3}{2}} + C = 0$$

$$\Rightarrow 2 - \frac{2}{3} + C = 0$$

$$\Rightarrow C = \frac{-2}{3}$$

$$\Rightarrow y(x) = 2\sqrt{x} - \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{3}$$

b) $\frac{dy}{dx} = \cos x - 5e^{x}$, $y(0) = 0$

$$dy = (\cos x - 5e^{x})\,dx$$

$$\Rightarrow \int dy = \int (\cos x - 5e^{x})\,dx$$

$$= \sin x - 5e^{x} + C$$

Substitute $y(0) = 0$

$$\sin 0 - 5e^{0} + C = 0$$

$$\Rightarrow -5 \quad C = 5$$

c) $\frac{dy}{dx} = \sqrt[3]{x}$, $y(1) = 2$

$$\int \sqrt[3]{x} = \frac{3}{4}x^{\frac{4}{3}} + C$$

Substitute $y(1) = 2$

$$\frac{3}{4} \cdot 1 + C = 2 \Rightarrow C = \frac{-3}{4}$$

d) $\frac{dy}{dx} = x.e^{x^2}$, $y(0)=0$

$\int x.e^{x^2}$

Substitute $u = x^2 \rightarrow \frac{du}{dx} = 2x$

$\rightarrow dx = \frac{1}{2x} du$

* $\int x.e^{x^2} dx = \frac{1}{2}\int e^u du = \frac{\cancel{1}}{2} \frac{e^u}{2}$

$= \frac{e^{x^2}}{2} + C$

$\Rightarrow \frac{e^0}{2} + C = 0 \Rightarrow C = \frac{-1}{2}$

Exercise 3

a) * Consider $u = \sec x$

$\rightarrow du = \sec x \tan x \, dx$

$\int \sec x^2 x \tan x \, dx = \int u du$

$= \frac{u^2}{2} + C$

$= \frac{\sec^2 x}{2} + C$

\+ Consider $u = \tan x$

$\rightarrow du = \sec^2 x \, dx$

$\int \sec^2 x \tan x \, dx = \int u \, du$

$= \frac{u^2}{2} + C$

$= \frac{\tan^2 x}{2} + C$

$\Rightarrow$ different substitutes produce different solutions

b) $\int \sec^2 x \tan x \, dx = \frac{\sec^2 x}{2} + C$

$= \frac{1}{2}(1 + \tan^2 x) + C$

$$= \frac{\tan^2 x}{2} + \left(\frac{1}{2} + C\right)$$

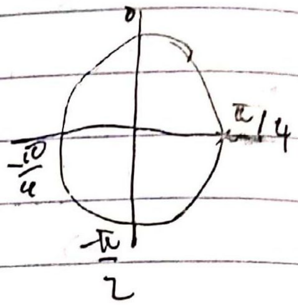
$$= \frac{\tan^2 x}{2} + C$$

Exercise 4 Tính tích phân không xác định

$+ \int_0^{\pi/4} \sec^2 x \tan x \, dx$

let $u = \tan x$

$\Rightarrow du = \sec^2 x \, dx$



$x = \frac{\pi}{4} \Rightarrow u = 1$

$x = 0 \Rightarrow u = 0$

Substitute

$$\int_0^1 u \, du$$

$$= \frac{u^2}{2}\Big|_0^1 = \frac{1^2 - 0^2}{2} = \frac{1}{2}$$

$\Rightarrow$

$+$ let $u = \sec x$

$\Rightarrow du = \sec x \tan x \, dx$

$x = \frac{\pi}{4} \Rightarrow u = 1$, $x = 0 \Rightarrow u = 0$

Substitute $\int_0^1 u \, du$

$$= \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$$

Exercise5 Tính nguyên hàm

$$\int \frac{x}{(x^2-1)\sqrt{x^4-2x^2}} dx$$

$=$ thay $u = x^2 - 1$

$u = x^2 - 1$

$\Rightarrow du = 2x\,dx$

$\Rightarrow x\,dx = \frac{du}{2}$

$\int \frac{x}{(x^2-1)\sqrt{x^4-2x^2}}dx = \int \frac{x}{(x^2-1)\sqrt{x^4-2x^2+1-1}}dx$

$= \int \frac{x}{x^2-1\sqrt{(x^2-1)^2-1}}dx = \int \frac{du}{2u\sqrt{(u^2-1)}}$

$= \frac{1}{2} \sec^{-1}|u| + C$

$= \frac{1}{2} \sec^{-1}|x^2-1| + C$

Exercise 6 : Tính nguyên hàm $\int \sqrt{1+x^{-2/3}}\,dx$

let $u = 1 + x^{2/3}$

$\Rightarrow du = \frac{2}{3}x^{-1/3}dx$

$\Rightarrow x^{-1/3}dx = \frac{3}{2}du$

$\int \sqrt{1+x^{-2/3}}\,dx = \int \sqrt{\left(1+\frac{1}{x^{2/3}}\right)}dx$

$= \int \sqrt{\left(\frac{x^{2/3}+1}{x^{2/3}}\right)}dx = \int \frac{\sqrt{x^{2/3}+1}}{\sqrt{x^{2/3}}}dx$

$= \int \frac{\sqrt{x^{2/3}+1}}{x^{1/3}}dx$ $\quad \left(x^{2/3 \cdot \frac{1}{2}}\right)$

$= \int \left(\sqrt{x^{2/3}+1} \cdot x^{-1/3}\right)dx$

$= \int \sqrt{u} \cdot \frac{3}{2}du = \int \frac{3}{2}u^{\frac{1}{2}}du$

$= \frac{3}{2}\frac{u^{3/2}}{3/2} + C = u^{3/2} + C$

$= (x^{2/3}+1)^{3/2} + C$

Exercise 11: Tính tích phân dùng lí thuyết cơ bản của CC và thuộc tính của tích phân vô định

a) $\int_{-3}^{0}(x^2-4x+7)\,dx = \frac{x^3}{3}-2x^2+7x \Big|_{-3}^{0}$

$= \left(\frac{0^3}{3}-2.0^2+7.0\right)-\left(\frac{(-3)^3}{3}-2.(-3)^2+7.(-3)\right)$

$= 0-(-48) = 48$

b) $\int_{-1}^{2} x(1+x^3)\,dx = \int_{-1}^{2}(x+x^4)\,dx = \left[\frac{x^2}{2}+\frac{x^5}{5}\right]\Big|_{-1}^{2}$

$= \left[\frac{2^2}{2}+\frac{2^5}{5}\right]-\left[\frac{(-1)^2}{2}+\frac{(-1)^5}{5}\right]$

$= \left(\frac{4}{2}+\frac{32}{5}\right)-\left(\frac{1}{2}-\frac{1}{5}\right) = 8.1$

c)

$\int_1^3 \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^3 = \frac{2}{3}$

d) $\int_1^8 (5x^{2/3}-4x^{-2})\,dx = \int_1^8 5x^{2/3}\,dx - \int_1^8 4x^{-2}\,dx$

$= 5\,\frac{x^{5/3}}{5/3}\Big|_1^8 - 4\,\frac{x^{-1}}{-1}\Big|_1^8$

$= 3.(8^{5/3}-1)+4(8^{-1}-1)$

$= 3(32-1)+4\left(\frac{-7}{8}\right)$

$= 93-\frac{7}{2} = \frac{179}{2}$

e) $\int_0^1 (x-\sec x\tan x)\,dx = \left(\frac{x^2}{2}-\sec x\right)\Big|_0^1$

$= \frac{1}{2}-\sec 1+1 = \frac{3}{2}-\sec 1$

f) $\int_1^4 \left(\frac{3}{\sqrt{t}} - 5.\sqrt{t} - t^{-3/2}\right) dt$

$= \int_1^4 \frac{3}{\sqrt{t}} - 5\int_1^4 \sqrt{t} - \int_1^4 t^{-3/2}$

$= 3\left(\frac{t^{1/2}}{\frac{1}{2}}\right)\Big|_1^4 - 5\frac{t^{3/2}}{\frac{3}{2}}\Big|_1^4 - \frac{t^{-1/2}}{\frac{-1}{2}}\Big|_1^4$

$= 6(2-1) - \frac{10}{3}(8-1) + 2\left(\frac{1}{2}-1\right)$

$= 6 - \frac{70}{3} - 1 = \frac{-55}{3}$

g) $\int_0^2 |2x-3|\,dx$

In the given interval at $x = \frac{3}{2}$ the function $|2x-3| = 0$ $\left(2x-3=0 \Rightarrow x = \frac{3}{2}\right.$

For $0 < x < \frac{3}{2} \Rightarrow |2x-3| < 0$

and $\frac{3}{2} < x < 2 \Rightarrow |2x-3| > 0$

$\Rightarrow \int_0^2 |2x-3|\,dx = \int_0^{3/2} |2x-3|\,dx + \int_{3/2}^{2} |2x-3|\,dx$

$= \left(3x - 2\frac{x^2}{2}\right)\Big|_0^{3/2} + \left(2\frac{x^2}{2} - 3x\right)\Big|_0^{3/2}$

$= \left(\frac{9}{2} - \frac{9}{4}\right) - 0 + \left[(4-6) - \left(\frac{9}{4} - \frac{9}{2}\right)\right]$

$= \frac{9}{2} - 2 = \frac{5}{2}$

h) $\int_0^{\pi/2} \left|\frac{1}{2} - \sin x\right| dx$

$\frac{1}{2} - \sin x = 0 \Leftrightarrow \sin x = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{6}$

$$\int_0^{\pi/2} \left|\frac{1}{2} - \sin x\right| dx = \int_0^{\pi/6} \left|\frac{1}{2} - \sin x\right| dx + \int_{\pi/6}^{\pi/2} \left|\frac{1}{2} - \sin x\right| dx$$

$$= \left(\frac{x}{2} + \cos x\right)\Big|_0^{\pi/6} + \left(\frac{x}{2} + \cos x\right)\Big|_{\pi/6}^{\pi/2}$$

$$= \left[\left(\frac{\pi}{12} + \frac{\sqrt{3}}{2}\right) - 1\right] - \left[\frac{\pi}{4} - \left(\frac{\pi}{12} + \frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{\pi}{6} + \sqrt{3} - 1 - \frac{\pi}{4}$$

$$= \frac{-\pi}{12} + \sqrt{3} - 1$$

<u>Exercise 12</u>: Tìm phần dưới đường cong $y = f(x)$ qua khoảng cho trước

a). $f(x) = \sqrt{x}$, $[1, 9]$

$$\int_1^9 f(x)\,dx = \int_1^9 \sqrt{x}\,dx = \int_1^9 x^{1/2}\,dx$$

$$= \frac{2}{3} x^{3/2} \Big|_1^9$$

$$= \frac{54}{3} - \frac{2}{3} = \frac{52}{3}$$

b) $f(x) = e^x$; $[1, 3]$

$$\int_1^3 e^x dx = e^x \Big|_1^3 = e^3 - e$$

<u>Exercise 13</u>: Tìm khu vực trên trục x dưới đường cong: vẽ

$y = (1-x)(x-2) = -x^2 + 3x - 2$

This is parable with vertex $v(o_x, x_y)$

$o_x = \frac{-b}{2a}$, $o_y = y(o_x) = y \cdot \frac{-b}{2a}$

$y = -x^2 + 3x - 2$

$\Rightarrow \quad v_x = \frac{-3}{2(-1)} = \frac{3}{2} > 0$

$v_y = y(v_x) = y\left(\frac{3}{2}\right) = \frac{1}{4} > 0$

We note $y(1) = y(2) = 0$ hence y is above the x axis on $[1,2]$. Therefore

$$A = \int_1^2 y(x)dx = \int_1^2 (-x^2 + 3x - 2)dx$$

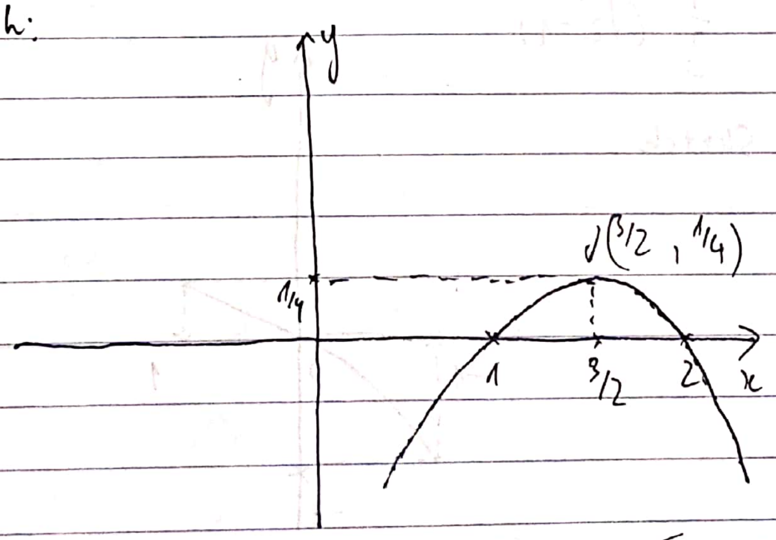
$$= \left(y' - \frac{x^3}{3} + 3\frac{x^2}{2} - 2x\right)\Big|_1^2$$

$$= -\frac{(2)^3}{3} + 3\frac{(2)^2}{3} - 2.(2) + \frac{1}{3} - 3\frac{1}{2} + 2.1$$

$$= \frac{-8}{3} + 6 - 4 + \frac{1}{3} - \frac{3}{2} + 2$$

$$= \frac{1}{6}$$

Sketch:



Exercise 14 : Vẽ và tìm tổng diện tích giữa đường cong và ~~trục~~ khoảng cho trước trên trục x (Ko lấy số âm)

a) $y = x^2 - 1$; $[0;3]$

$0 < x^2 - 1 < 3$ | $x^2 - 1 = 0$

$\Leftrightarrow 1 < x^2 < 2$ | $\Rightarrow x^2 = 1$

$\Leftrightarrow 1 < x < \pm\sqrt{2}$ | $\Rightarrow x = \pm 1$

$$\text{Area} = -\int_0^1 (x^2-1)\,dx + \int_1^3 (x^2-1)\,dx$$

$$= -\left(\frac{1}{3}x^3 - x\right)\Big|_0^1 + \left(\frac{1}{3}x^3 - x\right)\Big|_1^3$$

$$= -\left(\frac{1}{3}\cdot 1 - 1\right) + \left(\frac{1}{3}\cdot 0\right) + \left(\frac{1}{3}\cdot 3^3 - 3\right] - \left[\frac{1}{3}\cdot 1 \ldots\right.$$

$$= \frac{22}{3}$$

b) $y = \sqrt{x+1} - 1$; $[-1, 1]$

$x + 1 > 0$

$\Rightarrow x > -1$

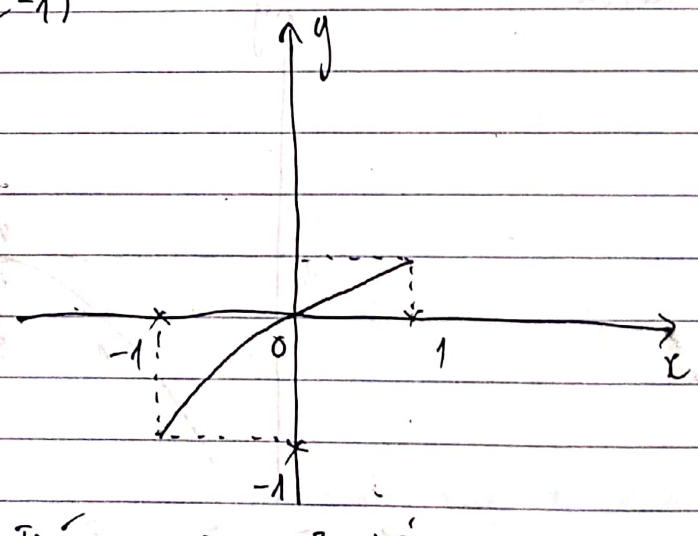
$$\text{Area} = -\int_{-1}^{0} (\sqrt{x+1} - 1)\,dx + \int_0^1 (\sqrt{x+1} - 1)\,dx$$

$$= -\left(\frac{2}{3}(x+1)^{3/2} - x\right)\Big|_{-1}^{0} + \left(\frac{2}{3}(x+1)^{3/2} - x\right)\Big|$$

$$= -\left(\frac{2}{3}(0+1)^{3/2} + 0\right) + \left(\frac{2}{3}(-1+1)^{3/2} - (-1)\right) + \ldots\ldots$$

$$= \frac{4}{3}(\sqrt{2} - 1)$$

Sketch



<u>Exercise 15</u> Tính ng hàm = p^2 thế

a) $\int_0^1 (2x+1)^4\,dx$

let $2x + 1 = u$

$\Rightarrow 2dx = du \Rightarrow dx = \frac{1}{2}du$

$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = 3$

$$\int_0^1 (2x+1)^4 dx = \frac{1}{2}\int_1^3 u^4 du$$

$$= \frac{1}{2}\left(\frac{u^5}{5}\right)\Big|_1^3$$

$$= \frac{1}{2}\left(\frac{3^5}{5} - \frac{1^5}{5}\right)$$

$$= \frac{121}{5}$$

b) $$\int_{-5}^{0} x\sqrt{4-x}\, dx$$

let $4 - x = u \Rightarrow x = 4 - u$

$\Rightarrow -dx = du$

$x = 0 \Rightarrow u = 4$

$x = -5 \Rightarrow u = 9$

Substitute

$$\int_{-5}^{0} x\sqrt{4-x}\, dx = \int_9^4 (4-u) . u^{1/2} . (-du)$$

$$= -\int_9^4 (4u^{1/2} - u^{3/2}) du = -4 \frac{1}{1/2+1} u^{1/2+1}\Big|_9^4$$

$$+ \frac{u^{3/2+1}}{3/2+1}\Big|_9^4$$

$$= -\frac{506}{15}$$

c) $$\int_0^1 \frac{dx}{\sqrt{3x+1}}$$

let $3x + 1 = u$

$\Rightarrow 3\, dx = du$

$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = 4$

Thứ...... Ngày......

Substitute $\int_1^4 u$ $\quad \int_0^1 \frac{dx}{\sqrt{3x+1}} = \int_0^1 \frac{1}{(3x+1)^{1/2}} dx$

$= \int_0^1 (3x+1)^{-1/2} dx$

$\Rightarrow \int_0^4 \int_1^4 u^{-1/2} \cdot \frac{1}{3} dx = \left(\frac{1}{3} \cdot \frac{u^{-1/2+1}}{-1/2+1}\right) \Big|_1^4$

$= \frac{2}{3} \cdot (4^{1/2} - 1^{1/2}) = \frac{2}{3}$

d) $\int_0^{\sqrt{\pi}} x \sin x^2 \, dx$

let $x^2 = u$

$\Rightarrow 2x\,dx = du$

$\Rightarrow x\,dx = \frac{du}{2}$

$x = 0 \Rightarrow u = 0$

$x = \sqrt{\pi} \Rightarrow u = \pi$

substitute in integration

$\frac{1}{2} \int_0^{\pi} \sin u \, du = \frac{1}{2} (-\cos x) \Big|_0^{\pi} = 1$

e) $\int_0^1 \sin^2(\pi x) \cos(\pi x) \, dx$

let $\sin \pi x = u$

$\Rightarrow \pi \cdot \cos \pi x \cdot dx = du$

$\Rightarrow \cos \pi x \cdot dx = \frac{du}{\pi}$

$x = 0 \Rightarrow u = 0$

$x = 1 \Rightarrow u = 0$

substitute on given integration

$$\frac{1}{\pi}\int_0^0 u^2\,du = 0$$

f) $$\int_e^{e^2} \frac{dx}{x\ln x}$$

let $\ln x = u$

$\Rightarrow \frac{dx}{x} = du$

$x = e \Rightarrow u = 1$

$x = e^2 \Rightarrow u = 2$

substitue on the given integration

$$\int_1^2 \frac{du}{u} = \ln u \Big|_1^2 = \ln 2 - \ln 1 = \ln 2$$

g) $$\int_0^1 \frac{dx}{\sqrt{e^x}} = \int_0^1 \frac{dx}{e^{x\cdot 1/2}} = \int_0^1 e^{-\frac{x}{2}}\,dx$$

let $u = \frac{-x}{2}$, $x = 0 \Rightarrow u = 0$

$x = 1 \Rightarrow u = -\frac{1}{2}$

$\Rightarrow \frac{du}{dx} = -\frac{1}{2}$ $\Rightarrow dx = -2du$

substitute onthe given integration

$$\int_0^{-1/2} e^u.(-2du) = -2e^u \Big|_0^{-1/2}$$

$$= -2(e^{-1/2} - e^0)$$

$$= -2e^{-1/2} + 2$$

h) $$\int_0^{2/\sqrt{3}} \frac{1}{4+9x^2}\,dx$$

$x = 0 \Rightarrow u = 0$

let $\frac{3x}{2} = u$, $x = \frac{2}{\sqrt{3}} \Rightarrow u = \frac{2\sqrt{3}}{2} = \sqrt{3}$

$\Rightarrow \frac{du}{dx} = \frac{3}{2} \Rightarrow dx = \frac{2}{3} du$

substitute

$$\int_0^{\sqrt{3}} \frac{2}{3} \cdot \frac{1}{\cancel{9x^2}\ 4u^2+4} = \int_0^{\sqrt{3}} \frac{1}{6} \cdot \frac{1}{u^2+1} \, du$$

$$= \frac{\arctan u}{6} \Big|_0^{\sqrt{3}}$$

$$= \frac{\arctan \sqrt{3}}{6} - \frac{\arctan \sqrt{0}}{6} = \frac{\arctan \sqrt{3}}{6}$$