Problem set 9: Topic in differentiation

Exercise 1

i. Find dy/dx by differentiating implicitly

ii. Solve the equation for y as function for x, and find dy/dx from that equation

iii. Confirm that the two results are consistent by expressing the derivative in part a as function of x alone

a) $x^3 + xy - 2x = 1$

i) By differentiating implicitly yields:

$$\frac{d}{dx}(x^3 + xy - 2x) = \frac{d}{dx}(1)$$

$$\Rightarrow 3x^2 + x\frac{dy}{dx} + y - 2 = 0$$

$$\Rightarrow x\frac{dy}{dx} = -3x^2 + 2 - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-3x^2 + 2 - y}{x}$$

ii) We'll write the equation as function of x

$$y = \frac{1 - x^3 + 2x}{x}$$

We'll find the derivative by using Quotient Rule:

$$\Rightarrow \frac{dy}{dx} = \frac{(-3x^2+2)x - (1-x^3+2x)}{x^2}$$

$$= \frac{-3x^3 + 2x + 1 + x^3 - 2x}{x^2}$$

$$= \frac{-2x^3 + 1}{x^2}$$

iii) From (i), we'll put $\frac{1-x^3+2x}{x}$ instead of y, we obtain:

$$\frac{dy}{dx} = \frac{-3x^2+2-y}{x} = \frac{-3x^2 + 2 - \left(\frac{1-x^3+2x}{x}\right)}{x}$$

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$$= \frac{-3x^3+2x-1+x^3-2x}{x^2} = \frac{-2x^3-1}{x^2}$$

We find the two results. are consistent

b) $xy = x-y$

i) By differentiating implicitly yields:

$$\frac{d}{dx}(xy) = \frac{d}{dx}(x-y)$$

$$\Rightarrow x\frac{dy}{dx}+y = 1+0$$

$$\Rightarrow x\frac{dy}{dx}+y = 1 \Rightarrow \frac{dy}{dx} = \frac{1-y}{x}$$

ii) We'll write the equation as function of x

$$y = -xy - x$$

$$\Rightarrow xy + y = -x$$

$$\Rightarrow y(x+1) = -x$$

$$\Rightarrow y = \frac{-x}{x+1}$$

we'll find derivative by using Quotient Rule:

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{x+1} \Leftrightarrow \frac{dy}{dx} = \frac{x'(x+1)-x(x+1)'}{(x+1)^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{-x+x+1}{(x+1)^2} = \frac{-1}{(x+1)^2}$$

iii) from (i) we'll put (ii) instead of ~~(x)~~ y we have

$$\frac{dy}{dx} = \frac{1-y}{x} = \frac{1-\left(-1/(x+1)^2\right)}{x}$$

$$= \frac{(x+1)^2+1}{x(x+1)^2}$$

Exercise 2 Find $\frac{dy}{dx}$ by implicit differentiation

a) $\frac{1}{y} + \frac{1}{x} = 1$

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$\Leftrightarrow \frac{dy}{dx}\left(\frac{1}{y}\right] + \frac{dy}{dx}\left(\frac{1}{x}\right) = \frac{dy}{dx}$ (1)

$\Leftrightarrow 0 \cancel{+} \frac{-1}{x^2} = 0$ $\qquad \frac{-\frac{d}{dx}(y)}{y^2} - \frac{1}{x^2} = 0$

$\Rightarrow \cancel{x = 0}$ $\qquad \frac{d}{dx}(y) = -\frac{y^2}{x^2}$

b) $x^3 - y^3 = 6xy$

$\Leftrightarrow \frac{dy}{dx}(x^3 - y^3) = \frac{dy}{dx} 6xy$

$\Rightarrow 3x^2 - 3y^2\frac{dy}{dx} = 6(y + x\frac{dy}{dx})$

$\Rightarrow$

$\Rightarrow \frac{dy}{dx} = \frac{-2y - x^2}{y^2 + 2x}$

c) $\sec(xy) = y$

$\Rightarrow \frac{dy}{dx} = \frac{-y\sec(xy)\tan(xy)}{x\sec(xy)\tan(xy) - 1}$

d) $x^2 = \frac{\cot y}{1 + \csc y}$

$\Leftrightarrow 2x = \frac{-\csc^2(y)\frac{d}{dx}(y)(1 + \csc(y)) + \cot^2(y)\csc(y)\frac{d}{dx}(y)}{(1 + \csc(y))^2}$

$\Leftrightarrow \frac{dy}{dx}y = \frac{2x + 4x\cdot\csc(y) + 2x\csc^2(y)}{\csc(y)(\cot^2(y) - \csc^2(y) - \csc(y))}$

$= \frac{2x + 4x\csc y + 2x\csc^2(y)}{\csc(y)(\cot^2 y - \csc^2(y) - \csc(y))}$

Exercise3 Find $\frac{d^2y}{dx^2}$ by implicit differentiation

a) $3x^2 - 4y^2 = 7$

$\Rightarrow 6x - 8y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{3x}{4y}$

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$\Rightarrow \frac{d^2y}{dx^2} = \cancel{\frac{3}{4y} x'} \; \frac{3 \cdot 4y - 4 \cdot 3x}{16y^2}$

$= \frac{12y - 12x}{16y^2} = \frac{12(y-x)}{16y^2} = \frac{3(y-x)}{4y^2}$

b) $2xy - y^2 = 3$

$\Leftrightarrow 2\left(y + x\frac{dy}{dx}\right) - 2y\frac{dy}{dx} = 0$

$\Leftrightarrow \frac{dy}{dx} = \frac{-y}{x-y}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1(x-y) - 0}{x-y} = -1$

Exercise 6 : Find coordinate of the point in the first quadrant at which the tangent line to the curve $x^3 - xy + y^3 = 0$ is parallel to the x-axis

Using implicit differentiation, we get

$\frac{d}{dx}(x^3 - xy + y^3) = 0$

$3x^2 - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}$

Since slope is parallel to x-axis

$\frac{dy}{dx} = 0 \Rightarrow y - 3x^2 = 0$

$\Rightarrow y = 3x^2$

Putting this on curve, we get

$x^3 - x(3x^2) + (3x^2)^3 = 0$

$\Rightarrow 27x^6 - 2x^3 = 0 \Rightarrow x = 0, \; x = \frac{\sqrt[3]{2}}{3}$

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$x = 0$ gives $y = 0$ which gives slope indeterminate.
Plugging $x = \frac{\sqrt[3]{2}}{3}$ in equation of curve, we get

$$\left(\frac{\sqrt[3]{2}}{3}\right)^3 - \left(\frac{\sqrt[3]{2}}{3}\right) y + y^3 = 4$$

numerically solving this, we get $y \approx 1.67$
$\Rightarrow$ Point is $(0.42, 1.67)$

Exercise 5: At what point(s) is the tangent line to the curve $y^2 = 2x^3$ perpendicular to the line $4x - 3y + 1 = 0$?

The slope of the line $4x - 3y + 1 = 0$ is $m_1 = 4/3$
($y = (4/3)x + 1/3$, compare it to $y = mx + c$)

Using implicit differentiation on $y^2 = 2x^3$, we get

$$2y \frac{dy}{dx} = 6x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{y}$$

let $m_2 = \frac{3x^2}{y}$. For two lines to be perpendicular,

$m_1 \cdot m_2 = -1$. This gives $\frac{3x^2}{y} \cdot \frac{4}{3} = -1$

$$\Rightarrow y = -4x^2$$

using this relation on $y^2 = 3x^3$, we get

$$16x^4 = 3x^3$$

This gives $x = 0$ and $x = 3/16$. We can calculate the value of y as

$$y^2 = 3(3/16)^3 = \pm 9/64$$

If we check the slope at $y = \pm 9/64$, it does not gives $3/4$ which is not perpendicular to the line.
Also at $x = 0$ the slope of the curve is indeterminate. Therefore, the point is $(3/16, -9/64)$

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Exercise 8 : Use implicit differentiation to show that the equation of the tangent line to the curve $y^2 = kx$ at (x_0, y_0) is $yy_0 = \frac{1}{2}k(x + x_0)$

Given curve $y = f(x)$, the equation of the tangent to $y = f(x)$ at (x_0, y_0) is given by

$$y = mx + n$$

where $m = f'(x_0)$ and $n = y_0 - mx_0$

for $y^2 = kx$, $2y\frac{dy}{dx} = k$

$$\Rightarrow f'(x) = \frac{dy}{dx} = \frac{k}{2y}$$

At (x_0, y_0), $m = f(x_0) = \frac{k}{2y_0}$

Now, $n = y_0 - mx_0 = y_0 - \frac{k}{2y_0}x_0$.

Thus the equation of the tangent line results

$$y = mx + n = \frac{k}{2y_0} + y_0 - \frac{kx_0}{2y_0}$$

$$yy_0 = \frac{k}{2}x + y_0^2 - \frac{kx_0}{2}$$

But $y_0^2 = kx_0$, then

$$yy_0 = \frac{k}{2}(x + 2x_0 - x_0)$$

$$\Rightarrow yy_0 = \frac{1}{2}k(x + x_0)$$

Exercise 9 : Find $\frac{dy}{dx}$ by first using algebraic properties of the nature logarithm function

a) let us find the derivative of y. First we use the quotient rule

$$\ln\left(\frac{a}{c}\right) = \ln(a) - \ln(c)$$

to obtain:

$$\ln \frac{(x+1)(x+2)^2}{(x+3)^3(x+4)^4} = \ln(x+1)(x+2)^2 - \ln\left((x+3)^3 (x+4)^4\right)$$

Product Rule) $\Rightarrow \ln(x+1) + \ln(x+2)^2 - \left(\ln(x+3)^3 + \ln(x+4)^4\right)$

$\Rightarrow \ln(x+1) + \ln(x+2)^2 - \ln(x+3)^3 - \ln(x+4)^4$

$\ln(a^r) = r\ln(a)$

$$\ln(x+1) + \ln(x+2)^2 - \ln(x+3)^3 - \ln(x+4)^4$$

$$= \ln(x+1) + 2\ln(x+2) - 3\ln(x+3) - 4\ln(x+4)$$

differentiating the above equation with respect to x, we get

$$\Rightarrow y' = \frac{1}{x+1} + \frac{2}{x+2} - \frac{3}{x+3} - \frac{4}{x+4}$$

b) $$y = \ln \frac{\sqrt{x}\sqrt[3]{x+1}}{\sin x \sec x}$$

$$= \ln\left(\sqrt{x}\sqrt[3]{x+1}\right) - \ln(\sin x \sec x)$$

$$= \ln\sqrt{x} + \ln\sqrt[3]{x+1} - \ln\sin x - \ln\sec x$$

$$= \ln x^{1/2} + \ln(x+1)^{1/3} - \ln\sin x - \ln\sec x$$

$$= \frac{1}{2}\ln x + \frac{1}{3}\ln(x+1) - \ln\sin x - \ln\sec x$$

$$y' = \frac{1}{2}\frac{d}{dx}\ln(x) + \frac{1}{3}\frac{d}{dx}\left(\ln(x+1)\right) - \frac{d}{dx}\ln\sin x - \frac{d}{dx}\ln\sec x$$

$$= \frac{1}{2}\frac{1}{x} + \frac{1}{3}\frac{1}{x+1} - \cot x - \tan x$$

$$= \frac{5x+3}{6x(x+1)} - \cot(x) - \tan(x)$$

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Exercise 10

a) $y = 2\ln 2x$

$\Leftrightarrow \dfrac{dy}{dx} = \dfrac{2}{x}$

b) $y = \sqrt[3]{\ln x + 1}$

$\Leftrightarrow \dfrac{dy}{dx} = \dfrac{1}{3x(\ln x + 1)^{\frac{2}{3}}}$

d) $y = \ln\left(x^{3/2}\sqrt{1+x^4}\right)$

$\Leftrightarrow \dfrac{dy}{dx} = \dfrac{3x^{1/2}\sqrt{1+x^4}}{2} + \dfrac{2x^{\frac{9}{2}}}{(1+x^4)^{1/2}}$

c) $y = \log(\ln x)$

$\Leftrightarrow y = \ln((\ln x)) \Leftrightarrow \dfrac{dy}{dx} = \dfrac{1}{\ln 10 \ln x} \cdot \ln x' \left(\dfrac{1}{x}\right)$

$\Leftrightarrow \cancel{\dfrac{dy}{dx} = \dfrac{1}{x\ln(x)}} \quad = \dfrac{1}{\ln 10\, x \ln x}$

e) $y = e^{\ln(x^2+1)} \quad \Leftrightarrow \cancel{\dfrac{dy}{dx} = \ln(x^2+1)' \cdot e^{\ln(x^2+1)}}$

$\Leftrightarrow \dfrac{dy}{dx} = (x^2+1)' = 2x$

f) $y = 2x e^{\sqrt{x}}$

$\Leftrightarrow \dfrac{dy}{dx} = 2e^{\sqrt{x}} + e^{\sqrt{x}}\sqrt{x}$

g) $y = x^{(e^x)}$

$\cancel{\Leftrightarrow \dfrac{dy}{dx} =}$

$\ln y = e^x \ln x$

$\Leftrightarrow \dfrac{dy}{dx} = e^x \ln x + e^x \dfrac{1}{x}$

$= y e^x\left(\ln x + \dfrac{1}{x}\right)$

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h) $y = (\ln x)^2$

$\Rightarrow \frac{dy}{dx} = 2\ln x \cdot (\ln x)' = \frac{2\ln x}{x}$

i) $y = \ln \sqrt[3]{x+1}$

$\Rightarrow y = \ln (x+1)^{1/3}$

$= \frac{1}{3} \ln(x+1)$

$\frac{dy}{dx} = \frac{1}{3(x+1)}$

(j) $y = \frac{1+\log x}{1-\log x}$

$\Rightarrow \frac{dy}{dx} = \frac{(1-\log x)(1+\log x)\frac{d}{dx} - (1+\log x)(1-\log x)\frac{d}{dx}}{(1-\log x)^2}$

$= \frac{(1-\log x)\left(\frac{1}{x\ln 10}\right) - (1+\log x)\left(-\frac{1}{x\ln 10}\right)}{}$

$= \frac{\frac{1-\log x}{x\ln 10} + \frac{1+\log x}{x\ln 10}}{(1-\log x)^2}$

$= \frac{\frac{2}{x\ln 10}}{(1-\log x)^2} = \frac{2}{x\ln 10\,(1-\log x)^2}$

(k) $y = \ln\left(\frac{\sqrt{x}\ \cos x}{1+x^2}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{\left(\frac{\cos(x)}{2\sqrt{x}} - \sqrt{x}\sin(x)\right)(1+x^2) - 2x\sqrt{x}\cos(x)}{(1+x^2)^2}$

(l) $y = \ln\left(\frac{1+e^x+e^{2x}}{1-e^{3x}}\right)$

$$= \ln(1+e^x+e^{2x}) - \ln(1-e^{3x})$$
$$= \ln(1+e^x+e^{2x}) - \ln(1-e^x)(1+e^x+e^{2x})$$
$$= -\ln(1-e^x)$$
$$\Rightarrow \frac{dy}{dx} = \frac{-e^x}{1-e^x}$$

n) $y = (1+x)^{\frac{1}{x}} \Rightarrow \ln y = \frac{1}{x}\ln(1+x)$

$$\frac{dy}{dx} = \frac{1}{x^2}\ln(1+x) + \frac{1}{x}\cdot\frac{1}{1+x}$$
$$\Rightarrow \frac{1}{(1+x)^{\frac{1}{x}}}\frac{dy}{dx} = \frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2}$$
$$\Rightarrow \frac{dy}{dx} = (1+x)^{\frac{1}{x}}\left[\frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2}\right]$$

Exercise 11 Find $\frac{dy}{dx}$ using logarithmic differentiation

a) $y = \frac{x^3}{\sqrt{x^2+1}}$

Take the logarithms of both sides of the given equat

$$\ln y = \ln\left(\frac{x^3}{\sqrt{x^2+1}}\right)$$
$$\ln\left(\frac{x^3}{\sqrt{x^2+1}}\right) = \ln x^3 - \ln(x^2+1)^{\frac{1}{2}}$$
$$\ln x^3 - \ln(x^2+1)^{\frac{1}{2}} = 3\ln x - \frac{1}{2}\ln(x^2+1)$$
$$\ln y = \ln\frac{x^3}{\sqrt{x^2+1}}$$
$$\ln y = 3\ln x - \frac{1}{2}\ln(x^2+1)$$

differentiationg both sides

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(3\ln x - \frac{1}{2}\ln(x^2+1)\right)$$

$$\frac{1}{y}\frac{dx}{dy} = 3\frac{d}{dx}\ln x - \frac{1}{2}\frac{d}{dx}\left(\ln(x^2+1)\right)$$

$$\frac{1}{y}y' = \frac{3}{x} - \frac{1}{2}\cdot\frac{2x}{x^2+1}$$

$$\Rightarrow y' = \frac{x^3}{\sqrt{x^2+1}}\left[\frac{3}{x} - \frac{x}{x^2+1}\right].$$

b) $y = \sqrt[3]{\dfrac{x^2-1}{x^2+1}}$

+ Find derivative of y

$$\ln y = \ln\sqrt[3]{\frac{x^2-1}{x^2+1}}$$

$$\ln y = \ln\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{3}} = \frac{1}{3}\ln\left(\frac{x^2-1}{x^2+1}\right)$$

$$\ln y = \frac{1}{3}\left(\ln(x^2-1) - \ln(x^2+1)\right)$$

differentiating both sides

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left[\frac{1}{3}\{\ln(x^2-1) - \ln(x^2+1)\}\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\frac{d}{dx}\ln(x^2-1) - \frac{1}{3}\frac{d}{dx}\left(\ln(x^2+1)\right)$$

$$\frac{1}{y}y' = \frac{4x}{3(x^2-1)(x^2+1)}$$

$$y' = y\left[\frac{4x}{3(x^2-1)(x^2+1)}\right]$$

Substitute the original function instead of y

$$y' = \sqrt[3]{\frac{x^2-1}{x^2+1}}\left(\frac{4x}{3(x^2-1)}\right)$$

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Exercise 12 Find a point on the graph of $y = e^{3x}$ at which the tangent line passes through the origin

The slope m of its tangent line $f(x) = mx + n$ is given by

$$m = \frac{dy}{dx} = \frac{d}{dx}\left(e^{3x}\right) = e^{3x}\frac{d}{dx}3x = 3e^{3x}$$

Now, the equation of the tangent line result at $x = x_0$

$$f(x) = m(x) + n = (3e^{3x_0})x + n$$

Since this line passes through the origin $(0,0)$

$$0 = (3e^{3x_0})(0) + n$$

$$\Rightarrow n = 0$$

Therefore we obtain the result

$$f(x) = (3e^{3x_0})x$$

Now, the point of tangency is given by

$$f(x_0) = y(x_0)$$

$$\Rightarrow 3x_0 e^{3x_0} = e^{3x_0} \Rightarrow 3x_0 = 1.$$

$$\Rightarrow x_0 = \frac{1}{3}$$

Thus

$$f = 3x\, e^{3(\frac{1}{3})} = 3e\, x$$

$$f(x) = (3e)x$$

Exercise 13

+ Find y', y'' → substitute into the original 2nd order

$$y'' - 2ay' + (a^2+b^2)y = 0$$

~~diff~~ • $\frac{d}{dx}(x^a) = a \cdot x^{a-1}$

• $(f \pm g)' = f' \pm g'$

$$\frac{d}{dx}(e^x) = e^x$$

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- $\frac{d}{dx}(f(u)) = \frac{df}{du} \cdot \frac{du}{dx}$
- $(f \cdot g)' = f' g + f g'$
- $\frac{d}{dx} \sin x = \cos x$
- $\frac{d}{dx} \cos x = -\sin x$

differentiating the given equation

$$y' = \frac{d}{dx}\left(e^{ax} \sin(bx)\right)$$

$$= (e^{ax})' \sin(bx) + \frac{d}{dx}(\sin(bx))\, e^{ax}$$

$$y' = \frac{d}{dx}(e^{ax}) \sin bx + \frac{d}{dx}(\sin(bx))\, e^{ax}$$

$$= e^{ax}\, a \sin(bx) + \cos(bx)\, b e^{ax}$$

differ

$$y'' = \frac{d}{dx}\left(e^{ax} \cdot a \cdot \sin(bx) + \cos(bx)\, b e^{ax}\right)$$

$$= a \frac{d}{dx}(e^{ax} \sin bx) + b \frac{d}{dx}(\cos(bx) e^{ax})$$

$$= a\left(e^{ax} \cdot a \cdot \sin(bx) + \cos(bx)\, b e^{ax}\right)$$

$$+ b\left(-b \cdot e^{ax} \sin(bx) + a e^{ax} \cos(bx)\right)$$

$$= a^2 e^{ax} \sin(bx) - b^2 \cdot e^{ax} \sin bx + 2ab e^{ax} \cos bx$$

+ substitute y', y''

$$(a^2 e^{ax} \sin(bx) - b^2 e^{ax} \sin bx + 2ab e^{ax} \cos(bx))$$

$$- 2a(e^{ax} \cdot a \cdot \sin(bx) + \cos(bx)\, b e^{ax}) + (a^2+b^2)(e^{ax} \sin(bx)) = 0$$

$\Rightarrow$ $0 = 0$

The function y satisfies $y'' - 2ay' + (a^2+b^2)y = 0$

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