Linear Algebra Notes

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Contents

1 Vector Spaces

1.1 Definition of Vector Space

Definition 1.1 (Vector Space)**.**
A **vector space** (over a field $\mathbb{F}$) consists of a set V with two operations "$+$" and "$\cdot$" subject to the conditions that for all $\vec{v}, \vec{w}, \vec{u} \in V$ and scalars $r, s \in \mathbb{F}$:

1. **Closure under**:
   - i Vector addition: $\vec{v} + \vec{w} \in V$.
   - ii Scalar multiplication: $r \cdot \vec{v} \in V$.
2. **Properties of vector addition**:
   - iii Commutativity: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.
   - iv Associativity: $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$.
3. **Properties of scalar multiplication**:
   - v Associativity : $r \cdot (s \cdot \vec{v}) = (rs) \cdot \vec{v}$.
   - vi Distributivity over scalar addition: $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$.
   - vii Distributivity over vector addition: $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$.
4. **Inverse elements**:
   - viii Additive inverse: $\forall \vec{v} \in V, \exists - \vec{v} \in V : \vec{v} + (-\vec{v}) = \vec{0}$.
5. **Identity elements**:
   - ix Additive identity: $\exists \vec{0} \in V : \vec{0} + \vec{v} = \vec{v}, \quad \forall \vec{v} \in V$.
   - x Multiplicative identity: $\exists \mathbf{1} \in \mathbb{F} : \mathbf{1} \cdot \vec{v} = \vec{v}, \quad \forall \vec{v} \in V$.

For brevity, we will denote vectors as bold face letters instead of overhead arrows after this definition. For example, $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$.

Remark 1.1 ("Over a field")**.** *When we use the phrase "a vector space over a field $\mathbb{F}$", this means that the scalars that we use will be taken from the field $\mathbb{F}$. It does not mean that our vector space consists of $\mathbb{F}$-valued vectors. For example, the following vector space:*

$$\mathrm{L} = \Big\{ (x, \alpha x) : x \in \mathbb{C}, \alpha \in \mathbb{R} \Big\}$$

is a vector space over $\mathbb{R}$ (scalar multiplications are done with real-valued scalars) even though the vectors are complex-valued.

Remark 1.2 (Trivial Space)**.** *A vector space with one element is called a **trivial space**.*

Example 1.1 (A simple example)**.** *The following is a vector space over $\mathbb{R}$:*

$$\mathrm{L} = \Big\{ \begin{pmatrix} x & y \end{pmatrix}^\top : y = 3x \Big\}.$$

This is easy to verify. Let us go through each condition one by one. Let the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathrm{L}$ defined as follows:

$$\mathbf{u}_1 = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} x_2 \\ 3x_2 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} x_3 \\ 3x_3 \end{pmatrix}..$$

All the axioms of a vector space are satisfied. Let $\alpha, \beta \in \mathbb{R}$, we have:

1. Closure under vector addition: $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ 3x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 3(x_1 + x_2) \end{pmatrix} \in \mathrm{L}$.

2. Closure under scalar multiplication: $\alpha\mathbf{u}_1 = \alpha \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ 3\alpha x_1 \end{pmatrix} \in \mathrm{L}$.

3. Additive commutativity: $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} x_1 + x_2 \\ 3x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} x_2 + x_1 \\ 3x_2 + 3x_1 \end{pmatrix} = \mathbf{u}_2 + \mathbf{u}_1$.

4. Additive associativity: $(\mathbf{u}_1 + \mathbf{u}_2) + \mathbf{u}_3 = \begin{pmatrix} (x_1 + x_2) + x_3 \\ 3(x_1 + x_2) + 3x_3 \end{pmatrix} = \begin{pmatrix} x_1 + (x_2 + x_3) \\ 3x_1 + 3(x_2 + x_3) \end{pmatrix} = \mathbf{u}_1 + (\mathbf{u}_2 + \mathbf{u}_3)$.

5. . . . *(We can easily verify other axioms as well).*

Example 1.2 (Polynomials of degree 3)**.** *Consider the following set of real-coefficients polynomials with degree of at most 3:*

$$\mathcal{P}_3 = \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \middle| a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

Then, $\mathcal{P}_3$ is a vector space over $\mathbb{R}$ under the following operations:

$$\begin{aligned} (a_0 + a_1 x + a_2 x^2 + a_3 x^3) &+ (b_0 + b_1 x + b_2 x^2 + b_3 x^3) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3, \\ \alpha \cdot (a_0 + a_1 x + a_2 x^2 + a_3 x^3) &= (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + (\alpha a_3)x^3. \end{aligned}$$

We can think of $\mathcal{P}_3$ as being "the same" as the vector space $\mathbb{R}^4$. For every set of real coefficients $a_0, \ldots, a_3$, we have the following correspondence:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 \text{ corresponds to } \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Similarly, for any $n \geq 1$, $\mathcal{P}_n$ is also a vector space over $\mathbb{R}$.

Definition 1.2 (Vector Space as Abelian Group)**.**
Let V be a vector space over a field $\mathbb{F}$ with two operations "$+$" and "$\cdot$". Then, $(V, +)$ is an (additive) Abelian group that satisfies the additional axioms on scalar multiplications:

- i $\lambda\mathbf{u} \in V$ for all $\lambda \in \mathbb{F}$ and $\mathbf{u} \in V$. (Closure under scalar multiplication).
- ii $(\lambda\mu)\mathbf{u} = \lambda(\mu\mathbf{u})$ for all $\lambda, \mu \in \mathbb{F}$ and $\mathbf{v} \in V$. (Associativity)
- iii $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$, $\mathbf{u}, \mathbf{v} \in V$ and $\lambda \in \mathbb{F}$. (Distributivity in V)
- iv $(\lambda + \mu)\mathbf{u} = \lambda\mathbf{u} + \mu\mathbf{u}$, $\lambda, \mu \in \mathbb{F}$ and $\mathbf{u} \in V$. (Distributivity in $\mathbb{F}$)
- v $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$. (Multiplicative identity)

Basically, we can think of vector spaces as Abelian groups with additional structures involving scalar multiplications. According to definition 1.1, the fact that V is an Abelian group makes it satisfy properties $(i), (iii), (iv), (vii)$ and $(viii)$. The additonal properties regarding scalar multiplication are listed above.

1.2 Subspace and Spanning Sets

In example 1.1, we saw a vector space over $\mathbb{R}$ that is a subset of $\mathbb{R}^2$, $\mathrm{L} = \{(x, 3x)^\top : x \in \mathbb{R}\}$, which is a line through the origin. In this section, we provide a formal definition for subspace of a vector space:

Definition 1.3 (Subspace).
For vector space V with two operations "$+$" and "$\cdot$". S is called a subspace of V if:

- $S \subseteq V$.
- S is a vector space under the inherited operations.

Proposition 1.1: Closure of Subspace under Linear Combination

For a non-empty subset S of a vector space V over a field $\mathbb{F}$. Under the inherited operations, the following statements are equivalent:

1. S is a subspace of V.
2. S is closed under linear combinations of any number of vectors: For all $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ and $\mathbf{u}_1, \ldots, \mathbf{u}_n \in S$, $\sum_{i=1}^n \lambda_i \mathbf{u}_i \in S$.

The above proposition gives us a convenient tool to work with whenever we want to prove that some subset of a vector space is a subspace. Instead of going through all the axioms in definition 1.1, we can just prove that the subset is closed under finite linear combinations. For example, we have the following proposition:

Proposition 1.2: Operations that Preserves Subspaces

Let V be a vector space over a field $\mathbb{F}$ and $U, W \subseteq V$ be subspaces. Then, the following subsets of V:

1. $U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U, \mathbf{w} \in W\}$[a],
2. $U \cap W = \{\mathbf{v} \in V : \mathbf{v} \in U \text{ and } \mathbf{v} \in W\}$

are all subspaces of V.

[a] Later we will learn that the direct sum $U \oplus W$ is also a subspace. Basically $U \oplus W = U + W$ where $U \cap W = \{0\}$.

Proof (Proposition 1.2).
We know that $U + W$ is definitely a subset of V because of its closure under vector addition. It is also trivial that $U \cap W \subseteq V$. Now we prove that any finite linear combination of each subset belongs to itself.

1. $U + W$: Let $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ be a finite sequence of scalars and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in U + W$. Then, for any $1 \leq i \leq n$, we have $\mathbf{v}_i = \mathbf{u}_i + \mathbf{w}_i$ where $\mathbf{u}_i \in U$ and $\mathbf{w}_i \in W$. Therefore, we have:

$$\sum_{i=1}^n \lambda_i \mathbf{v}_i = \sum_{i=1}^n \lambda_i \mathbf{u}_i + \sum_{i=1}^n \lambda_i \mathbf{w}_i.$$

   Since we have $\sum_{i=1}^n \lambda_i \mathbf{u}_i \in U$ and $\sum_{i=1}^n \lambda_i \mathbf{w}_i \in W$ due to closure under linear combinations of subspaces, we have $\sum_{i=1}^n \lambda_i \mathbf{v}_i \in U + W$. Therefore, $U + W$ is also closed under linear combinations and hence, a subspace of V.

2. $U \cap W$: Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in U \cap W$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$. It holds that $\sum_{i=1}^{n} \lambda_i \mathbf{v}_i \in U$ because U is a subspace. Since W is also a subspace $\sum_{i=1}^{n} \lambda_i \mathbf{v}_i \in W$. Therefore, $\sum_{i=1}^{n} \lambda_i \mathbf{v}_i \in U \cap W$.

$\square$.

Definition 1.4 (Span (Linear Closure)).

The span of a non empty subset S of a vector space over a field $\mathbb{F}$ is the set of all linear combinations of vectors from S.

$$\operatorname{span}(S) = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{v}_i : \mathbf{v}_i \in S, \lambda_i \in \mathbb{F} \right\}. \tag{1}$$

The span of an empty subset of a vector space is its trivial space ($\operatorname{span}(\emptyset) = \{0\}$).

Proposition 1.3: Span of Subsets in Vector Spaces

In a vector space, the span of any subset is a subspace.

Proof (Proposition 1.3).

Let V be a vector space over $\mathbb{F}$ and $S \subseteq V$. Denote that $\tilde{S} = \operatorname{span}(S)$. Obviously $\tilde{S} \subseteq V$ due to closure under linear combinations of vector spaces. We have to prove that $\tilde{S}$ is also closed under linear combinations. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \tilde{S}$. For each $\mathbf{v}_i$, we have:

$$\mathbf{v}_i = \sum_{j=1}^{n_i} \alpha_{i,j} \mathbf{u}_{i,j} \text{ where } \alpha_{i,1}, \ldots, \alpha_{i,n_i} \in \mathbb{F}, \mathbf{u}_{i,1}, \ldots, \mathbf{u}_{i,n_i} \in S.$$

Therefore, we have:

$$\sum_{i=1}^{n} \lambda_i \mathbf{v}_i = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \alpha_{i,j} \mathbf{u}_{i,j} \in \tilde{S}.$$

Therefore, $\tilde{S}$ is a subspace of V. $\square$.

Proposition 1.4: Adding Vectors to Span

Let V be a vector space over a field $\mathbb{F}$ and $S \subseteq V$. Let $\mathbf{v} \in V$, then

$$\operatorname{span}(S \cup \{\mathbf{v}\}) = \operatorname{span}(S) \iff \mathbf{v} \in \operatorname{span}(S). \tag{2}$$

Proof (Proposition 1.4).

We immediately have $\operatorname{span}(S \cup \{\mathbf{v}\}) = \operatorname{span}(S) \implies \mathbf{v} \in \operatorname{span}(S)$ because if $\mathbf{v} \notin \operatorname{span}(S)$, then the two sets cannot be equal because $\mathbf{v} \in \operatorname{span}(S \cup \{\mathbf{v}\})$.

To prove the opposite side, suppose that $\mathbf{v} \in \operatorname{span}(S)$. Hence, we can write $\mathbf{v}$ as a linear combination of vectors in S:

$$\mathbf{v} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i, \quad \lambda_i \in \mathbb{F}, \mathbf{u}_i \in S.$$

We prove that $\operatorname{span}(S \cup \{\mathbf{v}\}) = \operatorname{span}(S)$ by showing that:

1. $\operatorname{span}(S) \subseteq \operatorname{span}(S \cup \{\mathbf{v}\})$.

2. $\text{span}(S \cup \{\mathbf{v}\}) \subseteq \text{span}(S)$.

The first containment is clear. To prove the second point, we prove that for any $\mathbf{w} \in \text{span}(S \cup \{\mathbf{v}\})$, $\mathbf{w} \in \text{span}(S)$. Let $\alpha_1, \ldots, \alpha_{m+1} \in \mathbb{F}$ and $\mathbf{s}_1, \ldots, \mathbf{s}_m \in S$, we can write $\mathbf{w}$ as follows:

$$\begin{aligned}\mathbf{w} &= \sum_{j=1}^{m} \alpha_j \mathbf{s}_j + \alpha_{m+1}\mathbf{v} \\ &= \sum_{j=1}^{m} \alpha_j \mathbf{s}_j + \alpha_{m+1} \sum_{i=1}^{n} \lambda_i \mathbf{u}_i \\ &= \sum_{j=1}^{m} \alpha_j \mathbf{s}_j + \sum_{i=1}^{n} \alpha_{m+1}\lambda_i \mathbf{u}_i,\end{aligned}$$

which is a linear combination of vectors in S. Hence, $\mathbf{w} \in \text{span}(S)$. Therefore, we conclude that $\text{span}(S \cup \{\mathbf{v}\}) \subseteq \text{span}(S)$. $\square$.

Corollary 1.1: Adding Indepdent Vector to Span

Let V be a vector space over a field $\mathbb{F}$ and $S \subseteq V$. Let $\mathbf{v} \in V$, then

$$\text{span}(S) \subset \text{span}(S \cup \{\mathbf{v}\}) \iff \mathbf{v} \notin \text{span}(S). \tag{3}$$

Proof (Corollary 1.1).
We prove from both sides:

1. $\text{span}(S) \subset \text{span}(S \cup \{\mathbf{v}\}) \implies \mathbf{v} \notin \text{span}(S)$: Suppose the opposite that $\mathbf{v} \in \text{span}(S)$. Then, by proposition 1.4, we have $\text{span}(S) = \text{span}(S \cup \{\mathbf{v}\})$, which contradicts the initial assumption.

2. $\mathbf{v} \notin \text{span}(S) \implies \text{span}(S) \subset \text{span}(S \cup \{\mathbf{v}\})$: Similar to the above, we prove by contradiction using proposition 1.4.

$\square$.

Proposition 1.5: Removing Vectors from Span

Let $\mathbf{v} \in S$. Then, removing $\mathbf{v}$ from S does not shrink the span if and only if it is a linear combination of other vectors in the set.

$$\text{span}(S \setminus \{\mathbf{v}\}) = \text{span}(S) \iff \mathbf{v} \in \text{span}(S \setminus \{\mathbf{v}\}). \tag{4}$$

Proof (Proposition 1.5).
This is a direct consequence of proposition 1.4 (By applying the proposition with $\tilde{S} = S \setminus \{\mathbf{v}\}$). From proposition 1.5, we have the following corollary. $\square$.

Corollary 1.2: Remove Vector from Linearly Independent Sets

A set S is linearly independent if and only if removing any vector from S shrinks the span. In other words:

$$\forall \mathbf{v} \in S : \text{span}(S \setminus \{\mathbf{v}\}) \subset \text{span}(S). \tag{5}$$

Proof (Corollary 1.2).
We prove by contradiction. Suppose that we have $\mathbf{v} \in S$ and $\text{span}(S \setminus \{\mathbf{v}\}) = \text{span}(S)$. Then, by

proposition 1.5, we have $\mathbf{v} \in \text{span}(S \setminus \{\mathbf{v}\})$. Hence, $\mathbf{v}$ is a linear combination of the other vectors in S, which is a contradiction since we assumed S to be independent. $\square$.

1.3 Linear Independence

Definition 1.5 (Linear Independence).
We have the following definitions of linear independence:

1. In any vector space, a finite set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is said to be linearly indepedent if **<u>NONE of its elements</u> is a linear combination of the others from the set**. Otherwise, the set is linearly dependent.

2. A finite set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is linearly independent if and only if:
$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n = 0 \implies \alpha_i = 0, \quad \forall 1 \le i \le n.$$

3. A finite set of vectors $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent if and only if:
$$\forall 1 \le k \le n : \mathbf{v}_k \notin \text{span}(S \setminus \{\mathbf{v}_k\}),$$
In other words, every vector in S is not spanned by the remaining vectors of S.

4. An <u>infinite</u> set of vectors $V = \{\mathbf{v}_k\}_{k=1}^{\infty}$ is linearly indepedent if every finite subset of V is linearly independent.

Definition 1.6 (Relation between Linear Independence and Span).
From propositions 1.4, 1.5 and corollaries 1.1, 1.2, we have the following properties that relate linear independence to spanning sets:

	$\text{span}(S) = \text{span}(\tilde{S})$	$\text{span}(S) \neq \text{span}(\tilde{S})$
Add $\mathbf{v}$ to S	$\mathbf{v} \in \text{span}(S)$	$\mathbf{v} \notin \text{span}(S)$
Remove $\mathbf{v}$ from S	$\mathbf{v} \in \text{span}(\tilde{S})$	$\mathbf{v} \notin \text{span}(\tilde{S})$

Table 1: Conditions for $\text{span}(\tilde{S}) = \text{span}(S)$ where $\tilde{S}$ is resulted from adding $\mathbf{v}$ to or removing $\mathbf{v}$ from S. Note that for a vector $\mathbf{u}$ and a (finite) set V, when we write $\mathbf{u} \notin \text{span}(V)$, it is the same as saying "$\mathbf{u}$ is independent of all vectors in V".

Definition 1.7 (Properties of Linear Independence).
Let V be a vector space (over a field $\mathbb{F}$) and $S \subseteq V$ be a finite subset of V. Then, we have:

1. S is linearly dependent if and only if there are distinct $\mathbf{v}_0, \dots, \mathbf{v}_n \in S$ and $\lambda_1, \dots, \lambda_n \in S$ (not all zeros) such that $\mathbf{v}_0 = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i$.

2. Let $S_1 \subseteq S \subseteq S_2$, then:
   - S is linearly independent $\implies$ S_2 is linearly independent.
   - S is linearly dependent $\implies$ S_1 is linearly dependent.

3. There always exists $\tilde{S} \subseteq S$ such that $\tilde{S}$ is linearly independent and $\text{span}(S) = \text{span}(\tilde{S})$.

1.4 Basis and Dimension

A List of Definitions

B Important Theorems

C Important Corollaries

D Important Propositions

E References

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