

[Enter your subject] Notes

Nong Minh Hieu¹

¹ School of Physical and Mathematical Sciences, Nanyang Technological University (NTU - Singapore)

Contents

1	Overview of measure theory	2
1.1	Algebra and σ -algebra	2
1.2	Lebesgue measure	2
1.3	Measurable spaces	5
1.4	Measurable functions	6
2	Dynkin's π-λ Theorem	7
2.1	π -system and λ -system	7
2.2	Theorem and proof	9
3	Caratheodory Extension Theorem	10
3.1	Semi-ring of sets	10
3.2	σ -finiteness of measure	10
3.3	Theorem and proof	10
4	Manual	14
4.1	Theorems, Propositions, Lemmas, Corollaries	14
4.2	Definitions	14
4.3	Proofs	14
A	List of Definitions	16
B	Important Theorems	16
C	Important Corollaries	16
D	Important Propositions	16
E	References	17

1 Overview of measure theory

1.1 Algebra and σ -algebra

Definition 1.1 (Algebra).

Let X be a set and \mathcal{A} be a collection of subsets of X . Then, we say that \mathcal{A} is an algebra if it satisfies:

- **Closure under complement** : If $E \in \mathcal{A} \implies E^c \in \mathcal{A}$.
- **Closure under finite union** : For all finite collection $\{E_n\}_{n=1}^N \subset \mathcal{A} \implies \bigcup_{n=1}^N E_n \in \mathcal{A}$.

Definition 1.2 (σ -algebra).

Let X be a set and \mathcal{A} be a collection of subsets of X . Then, we say that \mathcal{A} is a σ -algebra if it is:

- **Closure under complement** : If $E \in \mathcal{A} \implies E^c \in \mathcal{A}$.
- **Closure under countable union** : For all countable collection $\{E_n\}_{n=1}^\infty \subset \mathcal{A} \implies \bigcup_{n=1}^\infty E_n \in \mathcal{A}$.

Definition 1.3 (Borel- σ -algebra).

Let Σ be the set of all the σ -algebras generated by open intervals in \mathbb{R} . Then, the Borel- σ -algebra is the smallest σ -algebra containing the open intervals:

$$\mathcal{B} = \bigcap_{\mathcal{A} \in \Sigma} \mathcal{A}$$

Proposition 1.1: Disjoint union in algebra

Let \mathcal{A} be an algebra and let $\{E_n\}_{n=1}^\infty$ be a countable collection of subsets in \mathcal{A} . Then, there exists a countable disjoint subsets $\{F_n\}_{n=1}^\infty$ such that:

$$\bigcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty F_n$$

Proof (Proposition 1.1).

Let $G_m = \bigcup_{n=1}^m E_n$, we have $G_1 \subset G_2 \subset G_3 \subset \dots \subset G_N$. It is easy to see that $\bigcup_{n=1}^N G_n = \bigcup_{n=1}^N E_n$. Now, define the collection $\{F_n\}_{n=1}^\infty$ as followed:

$$F_n = \begin{cases} G_1 & \text{When } n = 1 \\ G_n \setminus G_{n-1} & \text{When } n \geq 2 \end{cases}$$

Hence, we have $\bigcup_{n=1}^N F_n = \bigcup_{n=1}^N G_n \implies \bigcup_{n=1}^N F_n = \bigcup_{n=1}^N E_n$. □.

1.2 Lebesgue measure

Overview : The definition of Lebesgue measure stems from the need to construct a more general notion of integral (the Lebesgue integral) because the simple notion of Riemann integral is

incomplete. For example, $L_R^1([0, 1])$ (space of absolutely Riemann-integrable functions) is not a Banach space.

The construction of the Lebesgue integral over \mathbb{R} requires a notion of "measure" on subsets of \mathbb{R} , which, ideally satisfies the following conditions:

- $\mu : \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty)$ where $\mathcal{P}(\mathbb{R})$ denotes the power set of \mathbb{R} .
- μ **extends the measure of interval length** l . Meaning, if $I \subset \mathbb{R}$ is an interval, $\mu(I) = l(I)$.
- **Countable additivity** : Let $\{E_n\}_{n=1}^\infty \subset X$ be a collection of disjoint subsets of \mathbb{R} , then $\mu(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \mu(E_n)$.
- **Translation invariance** : For $E \subset \mathbb{R}, x \in \mathbb{R}$, we have $\mu(E + x) = \mu(E)$.

However, it is widely known that the construction of such measure is not possible because of the existence of non-measurable sets (Vitali sets [3]).

Definition 1.4 (Lebesgue outer measure).

Let $E \subset \mathbb{R}$. The Lebesgue outer measure (or simply "outer measure") is a mapping from the power set of \mathbb{R} to $[0, \infty)$ such that:

$$\mu^*(E) = \inf \left\{ \sum_{n=1}^\infty l(I_n) : I_n \text{ are open intervals; } E \subseteq \bigcup_{n=1}^\infty I_n \right\}$$

Where l denotes interval length. Without proving, we will just acknowledge the fact that the Lebesgue outer measure satisfies the second and the fourth conditions. However, **the outer measure is countably sub-additive rather than countably additive**. To account for this, we look at the definition of the Caratheodory criterion below.

Definition 1.5 (Caratheodory criterion - Lebesgue measurable sets).

Let $E \subseteq \mathbb{R}$. The set E is called **Lebesgue measurable** if for all $A \subseteq \mathbb{R}$, we have:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

The above condition is called the **Caratheodory criterion**. We denote the set of Lebesgue measurable subsets as \mathcal{M} :

$$\mathcal{M} = \left\{ E \subseteq \mathbb{R} : \forall A \subseteq \mathbb{R}, \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \right\}$$

Remark : Note that by countable sub-additivity, we will always have $\mu^*(A) \geq \mu^*(A \cap E)$

Definition 1.6 (Lebesgue measure).

The Lebesgue measure (denoted μ) is simply the Lebesgue outer measure μ^* restricted to the set of Lebesgue measurable sets \mathcal{M} :

$$\mu : \mathcal{M} \rightarrow [0, \infty); \quad \mu := \mu^*|_{\mathcal{M}}$$

Proposition 1.2: Measure of intersection with measurable collection

Let $A \subseteq \mathbb{R}$ and let $\{E_n\}_{n=1}^N$ be a finite disjoint collection of Lebesgue measurable sets. Then, we have:

$$\mu^* \left(A \cap \left[\bigcup_{n=1}^N E_n \right] \right) = \sum_{n=1}^N \mu^*(A \cap E_n)$$

Proof (Proposition 1.2).

We will prove this by induction. For $N = 1$, both sides are identical. For the inductive step, suppose that the above proposition is true for $N = m$. We have to prove that it is true for $N = m + 1$.

Since E_{m+1} is measurable, using the Caratheodory criterion, we have:

$$\begin{aligned} \mu^* \left(A \cap \left[\bigcup_{n=1}^{m+1} E_n \right] \right) &= \mu^* \left(A \cap \left[\bigcup_{n=1}^{m+1} E_n \right] \cap E_{m+1} \right) + \mu^* \left(A \cap \left[\bigcup_{n=1}^{m+1} E_n \right] \cap E_{m+1}^c \right) \\ &= \mu^*(A \cap E_{m+1}) + \mu^* \left(A \cap \left[\bigcup_{n=1}^{m+1} E_n \right] \cap E_{m+1}^c \right) \end{aligned}$$

Since E_n is disjoint for all $1 \leq n \leq m + 1$. We have:

$$\bigcup_{n=1}^m E_n \subset E_{m+1}^c \implies \left[\bigcup_{n=1}^{m+1} E_n \right] \cap E_{m+1}^c = \bigcup_{n=1}^m E_n$$

Finally, we have

$$\begin{aligned} \mu^* \left(A \cap \left[\bigcup_{n=1}^{m+1} E_n \right] \right) &= \mu^*(A \cap E_{m+1}) + \mu^* \left(A \cap \left[\bigcup_{n=1}^m E_n \right] \right) \\ &= \mu^*(A \cap E_{m+1}) + \sum_{n=1}^m \mu^*(A \cap E_n) \\ &= \sum_{n=1}^{m+1} \mu^*(A \cap E_n) \end{aligned}$$

□.

Proposition 1.3: \mathcal{M} is σ -algebra

The set of Lebesgue measurable subsets \mathcal{M} is a σ -algebra.

Proof (Proposition 1.3).

We first prove that \mathcal{M} is an algebra. Then, for all countable collection of Lebesgue measurable sets $\{E_n\}_{n=1}^\infty$ such that $E = \bigcup_{n=1}^\infty E_n$, $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$.

Claim 1 : \mathcal{M} is an algebra

We have to prove that \mathcal{M} is both closed under complement and finite union.

- **Closure under complement** : Trivial due to the symmetry of the Caratheodory criterion.
- **Closure under finite union** : Let E_1, E_2 be two Lebesgue measurable sets. We have:

$$\begin{aligned}
\mu^*(A \cap (E_1 \cup E_2)) &= \mu^*((A \cap E_1) \cup (A \cap E_2)) \\
&= \mu^*((A \cap E_1) \cup (A \cap E_2 \cap E_1^c)) \\
&\leq \mu^*(A \cap E_1) + \mu^*(A \cap E_2 \cap E_1^c) \quad (\text{Countable sub-additivity}) \\
&= \mu^*(A) - \mu^*(A \cap E_1^c) + \mu^*(A \cap E_2 \cap E_1^c) \\
&= \mu^*(A) - [\mu^*(A \cap E_1^c) - \mu^*([A \cap E_1^c] \cap E_2)] \\
&= \mu^*(A) - \mu^*(A \cap E_1^c \cap E_2^c) = \mu^*(A) - \mu^*(A \cap [E_1 \cup E_2]^c) \\
\implies \mu^*(A) &\geq \mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap [E_1 \cup E_2]^c) \\
&\implies E_1 \cap E_2 \in \mathcal{M}
\end{aligned}$$

Claim 2 : \mathcal{M} is a σ -algebra

Given $\{E_n\}_{n=1}^\infty$ be a countable collection of Lebesgue measurable sets and let $E = \bigcup_{n=1}^\infty E_n$. By proposition 1.1, there exists another countable **disjoint** collection of Lebesgue measurable sets $\{F_n\}_{n=1}^\infty$ such that $\bigcup_{n=1}^\infty F_n = \bigcup_{n=1}^\infty E_n = E$.

For any integer $N \geq 1$, we have $\bigcup_{n=1}^N F_n$ is Lebesgue measurable because \mathcal{M} is an algebra. Hence, we have:

$$\begin{aligned}
\mu^*(A) &= \mu^*\left(A \cap \left[\bigcup_{n=1}^N F_n\right]\right) + \mu^*\left(A \cap \left[\bigcup_{n=1}^N F_n\right]^c\right) \\
&\geq \mu^*\left(A \cap \left[\bigcup_{n=1}^N F_n\right]\right) + \mu^*\left(A \cap \left[\bigcup_{n=1}^\infty F_n\right]^c\right) = \mu^*\left(A \cap \left[\bigcup_{n=1}^N F_n\right]\right) + \mu^*(A \cap E^c)
\end{aligned}$$

By proposition 1.2, we have:

$$\mu^*(A) \geq \sum_{n=1}^N \mu^*(A \cap F_n) + \mu^*(A \cap E^c)$$

Taking $N \rightarrow \infty$, we have:

$$\begin{aligned}
\mu^*(A) &\geq \sum_{n=1}^\infty \mu^*(A \cap F_n) + \mu^*(A \cap E^c) \\
&= \mu^*\left(A \cap \left[\bigcup_{n=1}^\infty F_n\right]\right) + \mu^*(A \cap E^c) = \mu^*(A \cap E) + \mu^*(A \cap E^c)
\end{aligned}$$

Hence, \mathcal{M} is closed under countable union and is a σ -algebra. □.

1.3 Measurable spaces

Definition 1.7 (Measurable space).

Let E be a set and \mathcal{E} be a σ -algebra over E . Then, the pair (E, \mathcal{E}) is called a **measurable space**. The elements in \mathcal{E} are called **measurable sets**. When E is a topological space and \mathcal{E} is the Borel- σ -algebra on E , then the elements in \mathcal{E} are also called **Borel sets**.

Definition 1.8 (Product of measurable spaces).

Let (E, \mathcal{E}) and (F, \mathcal{F}) be measurable spaces. For $A \subset E, B \subset F$, we denote the product of A, B , denoted $A \times B$, as the set of all pairs (x, y) such that $x \in A, y \in B$. The set $A \times B$ is then called a **measurable rectangle**. The measurable space $(E \times F, \mathcal{E} \otimes \mathcal{F})$ is called the product of measurable spaces (E, \mathcal{E}) and (F, \mathcal{F}) where $\mathcal{E} \otimes \mathcal{F}$ is a σ -algebra over $E \times F$:

$$\mathcal{E} \otimes \mathcal{F} = \left\{ A \times B : A \in \mathcal{E}, B \in \mathcal{F} \right\}$$

Definition 1.9 (Measure & Measure space).

Let (E, \mathcal{E}) be a measurable space. A measure is a mapping $\mu : \mathcal{E} \rightarrow [0, \infty]$ (Including infinity) such that:

- **Empty set has zero measure** : $\mu(\emptyset) = 0$.
- **Countable additivity** : For a collection of disjoint measurable sets $\{E_n\}_{n=1}^{\infty}$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

Remark : Note that **monotonicity** and **translation invariance** are properties specific to Lebesgue measure only. A general measure need not to have those properties.

1.4 Measurable functions

Definition 1.10 (Function (Mapping)).

A **mapping** or a **function** $f : E \rightarrow F$ from E to F is a rule that assigns every element $f(x) \in F$ to an element $x \in E$. Given a subset $B \subset F$, the pre-image of B under a function $f : E \rightarrow F$ is given by:

$$f^{-1}(B) = \left\{ x \in E : f(x) \in B \right\}$$

Proposition 1.4: Properties of functions

Let $f : E \rightarrow F$ be a function, the following properties hold:

- $f^{-1}(\emptyset) = \emptyset$.
- $f^{-1}(F) = E$.
- $f^{-1}(B \setminus C) = f^{-1}(B) \setminus f^{-1}(C)$.
- $f^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(B_n)$.
- $f^{-1}\left(\bigcap_{n=1}^{\infty} B_n\right) = \bigcap_{n=1}^{\infty} f^{-1}(B_n)$.

Definition 1.11 (Measurable function).

Let (E, \mathcal{E}) and (F, \mathcal{F}) be measurable spaces, $f : E \rightarrow F$ be a function. Then, f is called **measurable relative to \mathcal{E} and \mathcal{F}** (or $(\mathcal{E}, \mathcal{F})$ -measurable) if:

$$\forall B \in \mathcal{F} : f^{-1}(B) \in \mathcal{E}$$

2 Dynkin's π - λ Theorem

Before diving into the theorem, we should familiarise ourselves with the relevant definitions. Specifically, what is a π -system and what is a λ -system.

2.1 π -system and λ -system

Definition 2.1 (π -system).

Given a set X . A collection \mathcal{P} of subsets of X is called a π -system if it is **closed under intersection**.

The simplest example of a π -system is the set of any single elements of X or the set that contains only the empty set. However, we are more interested in some of the more non-trivial examples of π -system:

- The set of half-open intervals (from the left) : $\{(-\infty, a] : a \in \mathbb{R}\}$.
- The set of half-open intervals (from the right) : $\{[a, \infty) : a \in \mathbb{R}\}$.
- The set of closed intervals are also a π -system if the empty set is included : $\{[a, b] : a, b \in \mathbb{R}; a \leq b\} \cup \{\emptyset\}$.
- If $\mathcal{P}_1, \mathcal{P}_2$ are π -systems over X_1, X_2 then the Cartesian products $\mathcal{P}_1 \times \mathcal{P}_2 = \{A_1 \times A_2 : A_1 \in \mathcal{P}_1, A_2 \in \mathcal{P}_2\}$ is also a π -system over $X_1 \times X_2$.
- Any σ -algebra is a π -system.

Definition 2.2 (λ -system).

Given a set X . A collection of \mathcal{D} of subsets of X is called a λ -system if it satisfies the following conditions:

- $X \in \mathcal{D}$
- **Closure under relative complement** : If $A, B \in \mathcal{D}$ and $A \subseteq B \implies B \setminus A \in \mathcal{D}$.
- **Closure under countable disjoint union** : If there exists a countable collection of disjoint sets $\{A_n\}_{n=1}^{\infty} \subset \mathcal{D}$. Then, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$.

Now that we see that λ -system is actually a slightly more complicated algebraic structure than a π -system. However, one thing we can notice is that any σ -algebra is also a λ -system. More generally, we have the following proposition.

Proposition 2.1: σ -algebra = π -system + λ -system

Every σ -algebra is both a π -system and a λ -system.

Proof (Proposition 2.1).

Given a set X and let \mathcal{A} be a σ -algebra on X .

Claim 1 : \mathcal{A} is a π -system over X

We have to prove that \mathcal{A} is closed under (finite) intersection. We know that \mathcal{A} is closed under countable intersection. Hence, for all countable collection of sets $\{A_n\}_{n=1}^{\infty}$ in \mathcal{A} , we have:

$$A = \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$$

For all $N \geq 1$, we have:

$$\begin{aligned} \bigcap_{n=1}^N A_n &= A \setminus \bigcap_{m=N+1}^{\infty} A_m \\ &= A \cap \left(\bigcup_{m=N+1}^{\infty} A_m^c \right) = \bigcup_{m=N+1}^{\infty} (A \cap A_m^c) \end{aligned}$$

For all $m \geq N+1$, $A \cap A_m^c$ is a countable intersection of sets in \mathcal{A} . Hence, $\bigcup_{m=N+1}^{\infty} (A \cap A_m^c)$ is a countable union of sets in \mathcal{A} . Hence, $\bigcap_{n=1}^N A_n \in \mathcal{A}$.

Therefore, \mathcal{A} is closed under finite intersection and is a π -system.

Claim 2 : \mathcal{A} is a λ -system over X

We have to prove that:

- $X \in \mathcal{A}$: Trivial.
- \mathcal{A} is closed under relative complement : For $A, B \in \mathcal{A}$ and $A \subseteq B$, we have $B \setminus A = B \cap A^c \in \mathcal{A}$ (By closure under intersection).
- \mathcal{A} is closed under countable disjoint union : Trivial since \mathcal{A} is already closed under countable union.

Hence, \mathcal{A} is a λ -system over X .

□.

2.2 Theorem and proof

Theorem 2.1: Dynkin's π - λ Theorem

If \mathcal{D} is a λ -system containing the π -system \mathcal{P} . Then, it also contains the σ -algebra generated by \mathcal{P} .

$$\mathcal{P} \subseteq \mathcal{D} \implies \sigma(\mathcal{P}) \subseteq \mathcal{D}$$

In other words, if a λ -system contains a π -system, it also contains the smallest σ -algebra containing the π -system.

Proof (Theorem 2.1).

We will prove the theorem by using the smallest λ -system generated from \mathcal{P} . We will prove that:

- (i) $\lambda(\mathcal{P})$ is a σ -algebra.
- (ii) $\sigma(\mathcal{P}) \subseteq \lambda(\mathcal{P})$.
- (iii) Since $\lambda(\mathcal{P}) \subseteq \mathcal{D} \implies \sigma(\mathcal{P}) \subseteq \lambda(\mathcal{P}) \subseteq \mathcal{D}$.

Obviously, we know that $\lambda(\mathcal{P})$ is a λ -system, we have to prove that it is also a π -system. Meaning, $\forall A, B \in \lambda(\mathcal{P}) \implies A \cap B \in \lambda(\mathcal{P})$. Let $A \in \lambda(\mathcal{P})$ be an arbitrary set and define:

$$\mathcal{L}_A = \{E : A \cap E \in \lambda(\mathcal{P})\}$$

Claim 1 : \mathcal{L}_A is a λ -system

We have:

- $X \in \mathcal{L}_A$ because $A \cap X = A \in \lambda(\mathcal{P})$.
- $\forall P, Q \in \mathcal{L}_A, P \subseteq Q \implies Q - P \in \mathcal{L}_A$ because we have:
 - $A \cap (Q - P) = (A \cap Q) - (A \cap P)$.
 - $(A \cap Q), (A \cap P) \in \lambda(\mathcal{P})$ and we have $A \cap P \subseteq A \cap Q$. Hence, $(A \cap Q) - (A \cap P) \in \lambda(\mathcal{P})$.
- $\forall \{E_n\}_{n=1}^{\infty} \subseteq \mathcal{L}_A$ be a disjoint collection, we have $\bigcup_{n=1}^{\infty} E_n \in \mathcal{L}_A$ because:
 - $A \cap \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \{A \cap E_n\}$.
 - For all $E_n, A \cap E_n \in \lambda(\mathcal{P})$ and disjoint, hence $\bigcup_{n=1}^{\infty} \{A \cap E_n\} \in \mathcal{L}_A$.

Claim 2 : $A \in \mathcal{P} \implies \lambda(\mathcal{P}) \subseteq \mathcal{L}_A$

We have:

- $\forall C \in \mathcal{P} : A \cap C \in \mathcal{P}$ because both $A, C \in \mathcal{P}$.
 - $\implies A \cap C \in \lambda(\mathcal{P})$.
 - $\implies \forall C \in \mathcal{P} : C \in \mathcal{L}_A$.
 - $\implies \mathcal{P} \subseteq \mathcal{L}_A$ (Meaning \mathcal{L}_A is a λ -system generated by \mathcal{P}).
- But, we already stated that $\lambda(\mathcal{P})$ is the smallest λ -system generated by \mathcal{P} . Hence, we have $\lambda(\mathcal{P}) \subseteq \mathcal{L}_A$.

Claim 3 : $B \in \lambda(\mathcal{P}) \implies \lambda(\mathcal{P}) \subseteq \mathcal{L}_B$

- We already proved that $\forall A \in \mathcal{P} : \lambda(\mathcal{P}) \subseteq \mathcal{L}_A$.
- Hence, for an arbitrary $B \in \lambda(\mathcal{P}) \implies B \in \mathcal{L}_A$.
- In other words : $\forall A \in \mathcal{P} : A \cap B \in \lambda(\mathcal{P}) \implies \forall A \in \mathcal{P} : A \in \mathcal{L}_B$.
 - $\implies \mathcal{P} \subseteq \mathcal{L}_B$.

$\implies \lambda(\mathcal{P}) \subseteq \mathcal{L}_B$ (Again, $\lambda(\mathcal{P})$ is the smallest λ -system containing \mathcal{P}).

From **Claim 3**, we can conclude that for two arbitrary sets $A, B \in \lambda(\mathcal{P}) \implies A \in \mathcal{L}_B$ (and vice versa). Therefore, $A \cap B \in \lambda(\mathcal{P})$ and $\lambda(\mathcal{P})$ is also a π -system. Finally, we conclude that $\lambda(\mathcal{P})$ is a σ -algebra \square .

(ii) We have proven that $\lambda(\mathcal{P})$ is a σ -algebra. We also have that $\sigma(\mathcal{P})$ is the smallest σ -algebra generated by \mathcal{P} . Hence, $\sigma(\mathcal{P}) \subseteq \lambda(\mathcal{P})$ \square .

(iii) Since $\lambda(\mathcal{P})$ is the smallest λ -system containing \mathcal{P} . We have $\lambda(\mathcal{P}) \subseteq \mathcal{D}$. Finally, we have $\sigma(\mathcal{P}) \subseteq \lambda(\mathcal{P}) \subseteq \mathcal{D}$. \square .

3 Caratheodory Extension Theorem

3.1 Semi-ring of sets

Definition 3.1 (Semi-ring of sets).

Given a set X , a collection of subsets of X - \mathcal{A} is called a semi-ring of subsets in X if it satisfies the following conditions:

- $\emptyset \in \mathcal{A}$.
- **Closure under intersection** : $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$.
- If $A, B \in \mathcal{A}$, there exists a **finite disjoint** collection of subsets $\{I_n\}_{n=1}^N \subset \mathcal{A}$ such that:

$$A \setminus B = \bigcup_{n=1}^N I_n$$

Remark : Notice that any semi-ring of sets over a set X is also a π -system over X .

3.2 σ -finiteness of measure

Definition 3.2 (σ -finite measure).

Let (X, \mathcal{A}) be a measurable space and $\mu : \mathcal{A} \rightarrow [0, \infty]$ be a measure (or pre-measure) defined on it. Then μ is called σ -finite if X can be **covered by countably many measurable sets with finite measure**. In other words, There exists $\{S_n\}_{n=1}^\infty \subset \mathcal{A}$, $\mu(S_n) < \infty$ such that $X = \bigcup_{n=1}^\infty S_n$.

3.3 Theorem and proof

The Caratheodory Extension Theorem involves the uniqueness and existence of extension of pre-measures. Before diving into the theorem, we look at the following lemma ??, which will help us prove the uniqueness.

Proposition 3.1: Uniqueness of measure

Suppose that μ_1, μ_2 are measures on (X, \mathcal{E}) such that $\mu_1(X) = \mu_2(X) < \infty$. Let $\mathcal{A} \subset \mathcal{E}$ be a π -system over X such that $\mathcal{E} = \sigma(\mathcal{A})$. Then,

$$\mu_1 \Big|_{\mathcal{A}} := \mu_2 \Big|_{\mathcal{A}} \implies \mu_1 := \mu_2$$

In other words, if two measures agree on a π -system, they also agree on the σ -algebra generated by that π -system.

Proof (Proposition 3.1). _____

Let \mathcal{D} be the set where μ_1, μ_2 agrees:

$$\mathcal{D} = \left\{ A \in \mathcal{E} : \mu_1(A) = \mu_2(A) \right\}$$

Hence, we have $\mathcal{A} \subseteq \mathcal{D}$.

Claim : \mathcal{D} is a λ -system

- $E \in \mathcal{D}$ by assumption.
- If $A, B \in \mathcal{D}$ and $A \subseteq B$. Since $B = (B \setminus A) \cup A$, we have:

$$\begin{aligned} \mu_1(B) &= \mu_1(A) + \mu_1(B \setminus A) \\ \mu_2(B) &= \mu_2(A) + \mu_2(B \setminus A) \end{aligned}$$

But we have $\mu_1(A) = \mu_2(A)$ and $\mu_1(B) = \mu_2(B)$. Hence, $\mu_1(B \setminus A) = \mu_2(B \setminus A)$ and $B \setminus A \in \mathcal{D}$.

- Let $\{A_n\}_{n=1}^{\infty}$ be a countable disjoint sets in \mathcal{D} . Since $\mu_1(A_n) = \mu_2(A_n)$ for all $n \geq 1$. Hence:

$$\sum_{n=1}^{\infty} \mu_1(A_n) = \sum_{n=1}^{\infty} \mu_2(A_n) \implies \mu_1\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu_2\left(\bigcup_{n=1}^{\infty} A_n\right)$$

Therefore, we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$.

Now that we have proved that \mathcal{D} is a λ -system that contains a π -system, by Dynkin's π - λ theorem 2.1, we have $\mathcal{A} \subseteq \sigma(\mathcal{A}) \subseteq \mathcal{D}$. Therefore:

$$\mu_1 \Big|_{\sigma(\mathcal{A})} := \mu_2 \Big|_{\sigma(\mathcal{A})} \text{ or } \mu_1 \Big|_{\mathcal{E}} := \mu_2 \Big|_{\mathcal{E}}$$

□.

Theorem 3.1: Caratheodory Extension Theorem

Let X be a set and \mathcal{A} be a semi-ring of sets over X . Let $\mu : \mathcal{A} \rightarrow [0, \infty]$ be a pre-measure defined on the semi-ring of sets. Then,

- There exists an extension of $\mu, \tilde{\mu} : \sigma(\mathcal{A}) \rightarrow [0, \infty]$, which is a measure on the σ -algebra generated by the semi-ring.
- If μ is σ -finite, then $\tilde{\mu}$ is unique.

Proof (Theorem 3.1).

We have to prove both the existence and uniqueness of $\tilde{\mu}$.

(i) **Existence of $\tilde{\mu}$**

We start by defining the outer measure $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ of the pre-measure as followed:

$$\mu^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \mu(E_n) : E_n \in \mathcal{A}, E \subseteq \bigcup_{n=1}^{\infty} E_n \right\}$$

Now we restrict μ^* to the set of Caratheodory-measurable subsets only:

$$\mathcal{M} = \left\{ E \subseteq X : \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c), \forall A \subseteq X \right\}$$

The strategy is to prove the following:

- \mathcal{M} is a σ -algebra $\implies \mu^*$ is a measure on \mathcal{M} .
- $\mathcal{A} \subset \mathcal{M} \implies \sigma(\mathcal{A}) \subset \mathcal{M}$.
- Finally, conclude that $\mu^* : \sigma(\mathcal{A}) \rightarrow [0, \infty]$ is an extension of μ to $\sigma(\mathcal{A})$.

Claim 1 : \mathcal{M} is a σ -algebra (In other words, μ^* is indeed a measure on \mathcal{M})

It is trivial to prove closure under complement because of the symmetry in Caratheodory criterion. Hence, we will focus on proving closure under countable union.

Let $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$, we have to prove that $E = \bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$. To do this, we use the same technique that we used to prove proposition 1.3 for Lebesgue measurable subsets. We make use of the following lemmas for the proof:

- **Proposition 1.1** : In an algebra \mathcal{M} , for any countable collection $\{E_n\}_{n=1}^{\infty}$, there exists a countable disjoint collection $\{F_n\}_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$.
- **Proposition 1.2** : For any finite disjoint collection of measurable sets $\{E_n\}_{n=1}^N$:

$$\mu^* \left(A \cap \bigcup_{n=1}^N E_n \right) = \sum_{n=1}^N \mu^*(A \cap E_n)$$

Claim 2 : $\mathcal{A} \subset \mathcal{M}$

For any $E \in \mathcal{A}$, we have to show that for all $A \subseteq X$, we have:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

By the definition of the outer measure μ^* , for all $\epsilon > 0$, we can always find a countable collection $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ such that:

$$A \subset \bigcup_{n=1}^{\infty} A_n \text{ and } \mu^*(A) + \epsilon \geq \sum_{n=1}^{\infty} \mu^*(A_n)$$

Then, we also have:

$$\begin{aligned} A \cap E &\subseteq \bigcup_{n=1}^{\infty} (A_n \cap E) \\ A \cap E^c &\subseteq \bigcup_{n=1}^{\infty} (A_n \cap E^c) \end{aligned}$$

Hence, we have:

$$\begin{aligned}\mu^*(A \cap E) + \mu^*(A \cap E^c) &\leq \mu^*\left(\bigcup_{n=1}^{\infty} (A_n \cap E)\right) + \mu^*\left(\bigcup_{n=1}^{\infty} (A_n \cap E^c)\right) \\ &= \mu^*\left(\bigcup_{n=1}^{\infty} (A_n \cap E)\right) + \mu^*\left(\bigcup_{n=1}^{\infty} (A_n \setminus E)\right)\end{aligned}$$

Since we have $A_n \cap E \in \mathcal{A}$ for all $n \geq 1$, we have:

$$\mu^*\left(\bigcup_{n=1}^{\infty} (A_n \cap E)\right) = \sum_{n=1}^{\infty} \mu^*(A_n \cap E)$$

Furthermore, we can always write $A_n \setminus E$ as a finite disjoint union of elements in \mathcal{A} , we also have:

$$\mu^*\left(\bigcup_{n=1}^{\infty} (A_n \setminus E)\right) = \sum_{n=1}^{\infty} \mu^*(A_n \setminus E)$$

Therefore, we can rewrite the above inequality as:

$$\begin{aligned}\mu^*(A \cap E) + \mu^*(A \cap E^c) &\leq \sum_{n=1}^{\infty} \mu^*(A_n \cap E) + \sum_{n=1}^{\infty} \mu^*(A_n \setminus E) \\ &= \sum_{n=1}^{\infty} \left(\mu^*(A_n \cap E) + \mu^*(A_n \cap E^c) \right) \\ &= \sum_{n=1}^{\infty} \mu^*(A_n) \\ &\leq \mu^*(A) + \epsilon\end{aligned}$$

Taking $\epsilon \rightarrow 0$, we have $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$. Hence, E is measurable and we conclude that $\mathcal{A} \subset \mathcal{M}$.

Claim 3 : $\sigma(\mathcal{A}) \subset \mathcal{M}$

This is a direct consequence of **Claim 2** due to the Dynkin's π - λ theorem 2.1.

With **Claim 1**, **Claim 2** and **Claim 3**, we define $\tilde{\mu}$ as a measure such that $\tilde{\mu}|_{\sigma(\mathcal{A})} := \mu^*|_{\sigma(\mathcal{A})}$ and conclude the proof for existence of an extension measure.

(ii) Uniqueness of $\tilde{\mu}$

Suppose that μ_1 and μ_2 are two measures defined on $(X, \sigma(\mathcal{A}))$ such that $\mu_1|_{\mathcal{A}} := \mu_2|_{\mathcal{A}}$. By proposition 3.1, since μ_1, μ_2 agrees on a π -system and $\mu_1(X) = \mu_2(X) < \infty$, μ_1 and μ_2 are identical and thus conclude the proof for uniqueness.

□.

4 Manual

4.1 Theorems, Propositions, Lemmas, Corollaries

Definition 4.1 (Manual - Theorems).

To define a new theorem/proposition/lemma/corollary, use the following common format:

```
\begin{theorem}{<theorem_name>}{<theorem_refid>}
    %% Content of theorem
\end{theorem}

%% ----- %%
% Replace theorem with
% - corollary.
% - proposition.
% - lemma.
%% ----- %%
```

To refer to the theorem, we can use the following format:

```
\ref{thm:<theorem_refid>}

%% ----- %%
% Replace the "thm" prefix with the following prefixes
% - coro: for Corollaries.
% - prop: for Propositions.
% - lem: for Lemmas.
%% ----- %%
```

4.2 Definitions

Definition 4.2 (Manual - Definitions).

Create a new definition using the following format:

```
\begin{definition}[<definition_name>]
    \label{def:<definition_refid>}
    %% Contents of definition
\end{definition}
```

4.3 Proofs

Definition 4.3 (Manual - Proofs).

When creating a proof for a theorem/proposition/lemma/corollary, use the following format:

```
\begin{proof*}[\ref{thm:<theorem_refid>}]
    %% You can also add sub-proofs, which will be indented
    \begin{subproof}{subproof_title}
        %% Contents of sub-proof
    \end{subproof}
\end{proof*}
```

Remember to always associate the proof with the corresponding theorem/proposition/lemma/corollary using the reference index.

A List of Definitions

1.1	Definition (Algebra)	2
1.2	Definition (σ -algebra)	2
1.3	Definition (Borel- σ -algebra)	2
1.4	Definition (Lebesgue outer measure)	3
1.5	Definition (Caratheodory criterion - Lebesgue measurable sets)	3
1.6	Definition (Lebesgue measure)	3
1.7	Definition (Measurable space)	5
1.8	Definition (Product of measurable spaces)	6
1.9	Definition (Measure & Measure space)	6
1.10	Definition (Function (Mapping))	6
1.11	Definition (Measurable function)	7
2.1	Definition (π -system)	7
2.2	Definition (λ -system)	8
3.1	Definition (Semi-ring of sets)	10
3.2	Definition (σ -finite measure)	10
4.1	Definition (Manual - Theorems)	14
4.2	Definition (Manual - Definitions)	14
4.3	Definition (Manual - Proofs)	14

B Important Theorems

2.1	Dynkin's π - λ Theorem	9
3.1	Caratheodory Extension Theorem	11

C Important Corollaries

D Important Propositions

1.1	Disjoint union in algebra	2
1.2	Measure of intersection with measurable collection	4
1.3	\mathcal{M} is σ -algebra	4
1.4	Properties of functions	7
2.1	σ -algebra = π -system + λ -system	8
3.1	Uniqueness of measure	11

E References

References

- [1] Rick Durrett. *Probability: Theory and Examples*. 4th. USA: Cambridge University Press, 2010. ISBN: 0521765390.
- [2] Erhan undefinedinar. *Probability and Stochastics*. Springer New York, 2011. ISBN: 9780387878591. DOI: [10.1007/978-0-387-87859-1](https://doi.org/10.1007/978-0-387-87859-1). URL: <http://dx.doi.org/10.1007/978-0-387-87859-1>.
- [3] Wikipedia. *Vitali set* — *Wikipedia, The Free Encyclopedia*. <http://en.wikipedia.org/w/index.php?title=Vitali%20set&oldid=1187241923>. [Online; accessed 24-December-2023]. 2023.