Linear Algebra Notes

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Contents

1 Vector Spaces

1.1 Definition of Vector Space

Definition 1.1 (Vector Space)**.**
*A **vector space** (over a field $\mathbb{F}$) consists of a set V with two operations "$+$" and "$\cdot$" subject to the conditions that for all $\vec{v}, \vec{w}, \vec{u} \in V$ and scalars $r, s \in \mathbb{F}$:*

1. ***Closure under:***
   - *i* *Vector addition:* $\vec{v} + \vec{w} \in V$.
   - *ii* *Scalar multiplication:* $r \cdot \vec{v} \in V$.
2. ***Properties of vector addition:***
   - *iii* *Commutativity:* $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.
   - *iv* *Associativity:* $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$.
3. ***Properties of scalar multiplication:***
   - *v* *Associativity :* $r \cdot (s \cdot \vec{v}) = (rs) \cdot \vec{v}$.
   - *vi* *Distributivity over scalar addition:* $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$.
   - *vii* *Distributivity over vector addition:* $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$.
4. ***Inverse elements:***
   - *viii* *Additive inverse:* $\forall \vec{v} \in V, \exists -\vec{v} \in V : \vec{v} + (-\vec{v}) = \vec{0}$.
5. ***Identity elements:***
   - *ix* *Additive identity:* $\exists \vec{0} \in V : \vec{0} + \vec{v} = \vec{v}, \quad \forall \vec{v} \in V$.
   - *x* *Multiplicative identity:* $\exists 1 \in \mathbb{F} : 1 \cdot \vec{v} = \vec{v}, \quad \forall \vec{v} \in V$.

For brevity, we will denote vectors as bold face letters instead of overhead arrows after this definition. For example, $\mathbf{u}, \mathbf{v}$ *and* $\mathbf{w}$.

Remark 1.1 ("Over a field")**.** *When we use the phrase "a vector space over a field $\mathbb{F}$", this means that the scalars that we use will be taken from the field $\mathbb{F}$. It does not mean that our vector space consists of $\mathbb{F}$-valued vectors. For example, the following vector space:*

$$\mathrm{L} = \Big\{ (x, \alpha x) : x \in \mathbb{C}, \alpha \in \mathbb{R} \Big\}$$

is a vector space over $\mathbb{R}$ (scalar multiplications are done with real-valued scalars) even though the vectors are complex-valued.

Remark 1.2 (Trivial Space)**.** *A vector space with one element is called a **trivial space**.*

Example 1.1 (A simple example)**.** *The following is a vector space over $\mathbb{R}$:*

$$\mathrm{L} = \Big\{ \begin{pmatrix} x & y \end{pmatrix}^{\top} : y = 3x \Big\}.$$

This is easy to verify. Let us go through each condition one by one. Let the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathrm{L}$ defined as follows:

$$\mathbf{u}_1 = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} x_2 \\ 3x_2 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} x_3 \\ 3x_3 \end{pmatrix}..$$

All the axioms of a vector space are satisfied. Let $\alpha, \beta \in \mathbb{R}$, we have:

1. Closure under vector addition*:* $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ 3x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 3(x_1 + x_2) \end{pmatrix} \in \mathrm{L}$.

2. Closure under scalar multiplication*:* $\alpha \mathbf{u}_1 = \alpha \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ 3\alpha x_1 \end{pmatrix} \in \mathrm{L}$.

3. Additive commutativity*:* $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} x_1 + x_2 \\ 3x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} x_2 + x_1 \\ 3x_2 + 3x_1 \end{pmatrix} = \mathbf{u}_2 + \mathbf{u}_1$.

4. Additive associativity*:* $(\mathbf{u}_1 + \mathbf{u}_2) + \mathbf{u}_3 = \begin{pmatrix} (x_1 + x_2) + x_3 \\ 3(x_1 + x_2) + 3x_3 \end{pmatrix} = \begin{pmatrix} x_1 + (x_2 + x_3) \\ 3x_1 + 3(x_2 + x_3) \end{pmatrix} = \mathbf{u}_1 + (\mathbf{u}_2 + \mathbf{u}_3)$.

5. ... (We can easily verify other axioms as well).

Example 1.2 (Polynomials of degree 3)**.** *Consider the following set of real-coefficients polynomials with degree of at most 3:*

$$\mathcal{P}_3 = \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \middle| a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

Then, $\mathcal{P}_3$ is a vector space over $\mathbb{R}$ under the following operations:

$$\begin{aligned} &(a_0 + a_1 x + a_2 x^2 + a_3 x^3) + (b_0 + b_1 x + b_2 x^2 + b_3 x^3) \\ &\qquad = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3, \\ &\alpha \cdot (a_0 + a_1 x + a_2 x^2 + a_3 x^3) = (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + (\alpha a_3)x^3. \end{aligned}$$

We can think of $\mathcal{P}_3$ as being "the same" as the vector space $\mathbb{R}^4$. For every set of real coefficients $a_0, \ldots, a_3$, we have the following correspondence:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 \text{ corresponds to } \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Similarly, for any $n \geq 1$, $\mathcal{P}_n$ is also a vector space over $\mathbb{R}$.

Definition 1.2 (Vector Space as Abelian Group)**.**
Let V be a vector space over a field $\mathbb{F}$ with two operations "$+$" and "$\cdot$". Then, $(V, +)$ is an (additive) Abelian group that satisfies the additional axioms on scalar multiplications:

- *i* $\lambda \mathbf{u} \in V$ *for all* $\lambda \in \mathbb{F}$ *and* $\mathbf{u} \in V$. *(Closure under scalar multiplication)*
- *ii* $(\lambda\mu)\mathbf{u} = \lambda(\mu\mathbf{u})$ *for all* $\lambda, \mu \in \mathbb{F}$ *and* $\mathbf{v} \in V$. *(Associativity)*
- *iii* $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$, $\mathbf{u}, \mathbf{v} \in V$ *and* $\lambda \in \mathbb{F}$. *(Distributivity in V)*
- *iv* $(\lambda + \mu)\mathbf{u} = \lambda\mathbf{u} + \mu\mathbf{u}$, $\lambda, \mu \in \mathbb{F}$ *and* $\mathbf{u} \in V$. *(Distributivity in $\mathbb{F}$)*
- *v* $1\mathbf{u} = \mathbf{u}$ *for all* $\mathbf{u} \in V$. *(Multiplicative identity)*

Basically, we can think of vector spaces as Abelian groups with additional structures involving scalar multiplications. According to definition 1.1, the fact that V is an Abelian group makes it satisfy properties $(i), (iii), (iv), (vii)$ and $(viii)$. The additonal properties regarding scalar multiplication are listed above.

1.2 Subspace and Spanning Sets

In example 1.1, we saw a vector space over $\mathbb{R}$ that is a subset of $\mathbb{R}^2$, $\mathrm{L} = \{(x, 3x)^\top : x \in \mathbb{R}\}$, which is a line through the origin. In this section, we provide a formal definition for subspace of a vector space:

Definition 1.3 (Subspace).

For vector space V with two operations "$+$" and "$\cdot$". S is called a subspace of V if:

- $S \subseteq V$.
- *S is a vector space under the inherited operations.*

Proposition 1.1: Closure of Subspace under Linear Combination

1.3 Linear Independence

1.4 Basis and Dimension

A List of Definitions

B Important Theorems

C Important Corollaries

D Important Propositions

E References

References

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