

Linear Algebra Notes

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1 Vector Spaces

1.1 Definition of Vector Space

Definition 1.1 (Vector Space).

A **vector space** (over a field \mathbb{F}) consists of a set V with two operations “+” and “ \cdot ” subject to the conditions that for all $\vec{v}, \vec{w}, \vec{u} \in V$ and scalars $r, s \in \mathbb{F}$:

1. **Closure under:**

- Vector addition: $\vec{v} + \vec{w} \in V$.
- Scalar multiplication: $r \cdot \vec{v} \in V$.

2. **Properties of vector addition:**

- Commutativity: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.
- Associativity: $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$.

3. **Properties of scalar multiplication:**

- Distributivity over scalar addition: $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$.
- Distributivity over vector addition: $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$.

4. **Inverse elements:**

- Additive inverse: $\forall \vec{v} \in V, \exists -\vec{v} \in V : \vec{v} + (-\vec{v}) = \vec{0}$.

5. **Identity elements:**

- Additive identity: $\exists \vec{0} \in V : \vec{0} + \vec{v} = \vec{v}, \quad \forall \vec{v} \in V$.
- Multiplicative identity: $\exists 1 \in \mathbb{F} : 1 \cdot \vec{v} = \vec{v}, \quad \forall \vec{v} \in V$.

For brevity, we will denote vectors as bold face letters instead of overhead arrows after this definition. For example, \mathbf{u}, \mathbf{v} and \mathbf{w} .

Remark 1.1 (“Over a field”). When we use the phrase “a vector space over a field \mathbb{F} ”, this means that the scalars that we use will be taken from the field \mathbb{F} . It does not mean that our vector space consists of \mathbb{F} -valued vectors. For example, the following vector space:

$$L = \left\{ (x, \alpha x) : x \in \mathbb{C}, \alpha \in \mathbb{R} \right\}$$

is a vector space over \mathbb{R} (scalar multiplications are done with real-valued scalars) even though the vectors are complex-valued.

Remark 1.2 (Trivial Space). A vector space with one element is called a **trivial space**.

Example 1.1 (A simple example). The following is a vector space over \mathbb{R} :

$$L = \left\{ \begin{pmatrix} x & y \end{pmatrix}^\top : y = 3x \right\}.$$

This is easy to verify. Let us go through each condition one by one. Let the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in L$ defined as follows:

$$\mathbf{u}_1 = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} x_2 \\ 3x_2 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} x_3 \\ 3x_3 \end{pmatrix}.$$

All the axioms of a vector space are satisfied. Let $\alpha, \beta \in \mathbb{R}$, we have:

1. Closure under vector addition: $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ 3x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 3(x_1 + x_2) \end{pmatrix} \in L.$
2. Closure under scalar multiplication: $\alpha \mathbf{u}_1 = \alpha \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ 3\alpha x_1 \end{pmatrix} \in L.$
3. Additive commutativity: $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} x_1 + x_2 \\ 3x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} x_2 + x_1 \\ 3x_2 + 3x_1 \end{pmatrix} = \mathbf{u}_2 + \mathbf{u}_1.$
4. Additive associativity: $(\mathbf{u}_1 + \mathbf{u}_2) + \mathbf{u}_3 = \begin{pmatrix} (x_1 + x_2) + x_3 \\ 3(x_1 + x_2) + 3x_3 \end{pmatrix} = \begin{pmatrix} x_1 + (x_2 + x_3) \\ 3x_1 + 3(x_2 + x_3) \end{pmatrix} = \mathbf{u}_1 + (\mathbf{u}_2 + \mathbf{u}_3).$
5. ... (We can easily verify other axioms as well).

Example 1.2 (Polynomials of degree 3). Consider the following set of real-coefficients polynomials with degree of at most 3:

$$\mathcal{P}_3 = \left\{ a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

Then, \mathcal{P}_3 is a vector space over \mathbb{R} under the following operations:

$$\begin{aligned} (a_0 + a_1x + a_2x^2 + a_3x^3) + (b_0 + b_1x + b_2x^2 + b_3x^3) \\ = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3, \\ \alpha \cdot (a_0 + a_1x + a_2x^2 + a_3x^3) = (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + (\alpha a_3)x^3. \end{aligned}$$

We can think of \mathcal{P}_3 as being “the same” as the vector space \mathbb{R}^4 . For every set of real coefficients a_0, \dots, a_3 , we have the following correspondence:

$$a_0 + a_1x + a_2x^2 + a_3x^3 \text{ corresponds to } \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

1.2 Linear Independence

1.3 Basis and Dimension

A List of Definitions

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B Important Theorems

C Important Corollaries

D Important Propositions

E References

References

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