Linear Algebra Notes

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1 Vector Spaces

1.1 Definition of Vector Space

Definition 1.1 (Vector Space). ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯
*A **vector space** (over a field $\mathbb{F}$) consists of a set V with two operations "$+$" and "$\cdot$" subject to the conditions that for all $\vec{v}, \vec{w}, \vec{u} \in V$ and scalars $r, s \in \mathbb{F}$:*

1. ***Closure under:***

    i *Vector addition: $\vec{v} + \vec{w} \in V$.*

    ii *Scalar multiplication: $r \cdot \vec{v} \in V$.*

2. ***Properties of vector addition:***

    iii *Commutativity: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.*

    iv *Associativity: $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$.*

3. ***Properties of scalar multiplication:***

    v *Associativity : $r \cdot (s \cdot \vec{v}) = (rs) \cdot \vec{v}$.*

    vi *Distributivity over scalar addition: $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$.*

    vii *Distributivity over vector addition: $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$.*

4. ***Inverse elements:***

    viii *Additive inverse: $\forall \vec{v} \in V, \exists -\vec{v} \in V : \vec{v} + (-\vec{v}) = \vec{0}$.*

5. ***Identity elements:***

    ix *Additive identity: $\exists \vec{0} \in V : \vec{0} + \vec{v} = \vec{v}, \quad \forall \vec{v} \in V$.*

    x *Multiplicative identity: $\exists 1 \in \mathbb{F} : 1 \cdot \vec{v} = \vec{v}, \quad \forall \vec{v} \in V$.*

For brevity, we will denote vectors as bold face letters instead of overhead arrows after this definition. For example, $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$.

Remark 1.1 ("Over a field"). *When we use the phrase "a vector space over a field $\mathbb{F}$", this means that the scalars that we use will be taken from the field $\mathbb{F}$. It does not mean that our vector space consists of $\mathbb{F}$-valued vectors. For example, the following vector space:*

$$L = \Big\{ (x, \alpha x) : x \in \mathbb{C}, \alpha \in \mathbb{R} \Big\}$$

is a vector space over $\mathbb{R}$ (scalar multiplications are done with real-valued scalars) even though the vectors are complex-valued.

Remark 1.2 (Trivial Space). *A vector space with one element is called a **trivial space**.*

Example 1.1 (A simple example). *The following is a vector space over $\mathbb{R}$:*

$$L = \Big\{ \begin{pmatrix} x & y \end{pmatrix}^{\top} : y = 3x \Big\}.$$

This is easy to verify. Let us go through each condition one by one. Let the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in L$ defined as follows:

$$\mathbf{u}_1 = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} x_2 \\ 3x_2 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} x_3 \\ 3x_3 \end{pmatrix}..$$

All the axioms of a vector space are satisfied. Let $\alpha, \beta \in \mathbb{R}$, we have:

1. Closure under vector addition: $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ 3x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 3(x_1 + x_2) \end{pmatrix} \in \mathrm{L}$.

2. Closure under scalar multiplication: $\alpha\mathbf{u}_1 = \alpha \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ 3\alpha x_1 \end{pmatrix} \in \mathrm{L}$.

3. Additive commutativity: $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} x_1 + x_2 \\ 3x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} x_2 + x_1 \\ 3x_2 + 3x_1 \end{pmatrix} = \mathbf{u}_2 + \mathbf{u}_1$.

4. Additive associativity: $(\mathbf{u}_1 + \mathbf{u}_2) + \mathbf{u}_3 = \begin{pmatrix} (x_1 + x_2) + x_3 \\ 3(x_1 + x_2) + 3x_3 \end{pmatrix} = \begin{pmatrix} x_1 + (x_2 + x_3) \\ 3x_1 + 3(x_2 + x_3) \end{pmatrix} = \mathbf{u}_1 + (\mathbf{u}_2 + \mathbf{u}_3)$.

5. ... (We can easily verify other axioms as well).

Example 1.2 (Polynomials of degree 3)**.** *Consider the following set of real-coefficients polynomials with degree of at most 3:*

$$\mathcal{P}_3 = \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \,\middle|\, a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

Then, $\mathcal{P}_3$ is a vector space over $\mathbb{R}$ under the following operations:

$$(a_0 + a_1 x + a_2 x^2 + a_3 x^3) + (b_0 + b_1 x + b_2 x^2 + b_3 x^3)$$
$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3,$$
$$\alpha \cdot (a_0 + a_1 x + a_2 x^2 + a_3 x^3) = (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + (\alpha a_3)x^3.$$

We can think of $\mathcal{P}_3$ as being "the same" as the vector space $\mathbb{R}^4$. For every set of real coefficients $a_0, \ldots, a_3$, we have the following correspondence:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ \text{corresponds to} \ \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Similarly, for any $n \geq 1$, $\mathcal{P}_n$ is also a vector space over $\mathbb{R}$.

Definition 1.2 (Vector Space as Abelian Group)**.** ──────────────
Let V be a vector space over a field $\mathbb{F}$ with two operations "$+$" and "$\cdot$". Then, $(V, +)$ is an (additive) Abelian group that satisfies the additional axioms on scalar multiplications:

    i $\lambda\mathbf{u} \in V$ for all $\lambda \in \mathbb{F}$ and $\mathbf{u} \in V$.   *(Closure under scalar multiplication)*

    ii $(\lambda\mu)\mathbf{u} = \lambda(\mu\mathbf{u})$ for all $\lambda, \mu \in \mathbb{F}$ and $\mathbf{v} \in V$.   *(Associativity)*

    iii $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$, $\mathbf{u}, \mathbf{v} \in V$ and $\lambda \in \mathbb{F}$.   *(Distributivity in V)*

    iv $(\lambda + \mu)\mathbf{u} = \lambda\mathbf{u} + \mu\mathbf{u}$, $\lambda, \mu \in \mathbb{F}$ and $\mathbf{u} \in V$.   *(Distributivity in $\mathbb{F}$)*

    v $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.   *(Multiplicative identity)*

Basically, we can think of vector spaces as Abelian groups with additional structures involving scalar multiplications. According to definition 1.1, the fact that V is an Abelian group makes it satisfy properties $(i), (iii), (iv), (vii)$ and $(viii)$. The additonal properties regarding scalar multiplication are listed above.

1.2   Subspace and Spanning Sets

In example 1.1, we saw a vector space over $\mathbb{R}$ that is a subset of $\mathbb{R}^2$, $\mathrm{L} = \{(x, 3x)^\top : x \in \mathbb{R}\}$, which is a line through the origin. In this section, we provide a formal definition for subspace of a vector space:

Definition 1.3 (Subspace). ───────────────────────────────────
For vector space V with two operations "$+$" and "$\cdot$". S is called a subspace of V if:

- $S \subseteq V$.

- S is a vector space under the inherited operations.

Proposition 1.1: Closure of Subspace under Linear Combination

For a non-empty subset S of a vector space V over a field $\mathbb{F}$. Under the inherited operations, the following statements are equivalent:

1. S is a subspace of V.

2. S is closed under linear combinations of any number of vectors: For all $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ and $\mathbf{u}_1, \ldots, \mathbf{u}_n \in S$, $\sum_{i=1}^{n} \lambda_i \mathbf{u}_i \in S$.

The above proposition gives us a convenient tool to work with whenever we want to prove that some subset of a vector space is a subspace. Instead of going through all the axioms in definition 1.1, we can just prove that the subset is closed under finite linear combinations. For example, we have the following proposition:

Proposition 1.2: Operations that Preserves Subspaces

Let V be a vector space over a field $\mathbb{F}$ and $U, W \subseteq V$ be subspaces. Then, the following subsets of V:

1. $U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U, \mathbf{w} \in W\}^a$,

2. $U \cap W = \{\mathbf{v} \in V : \mathbf{v} \in U \text{ and } \mathbf{v} \in W\}$

are all subspaces of V.

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    [a]Later we will learn that the direct sum $U \oplus W$ is also a subspace. Basically $U \oplus W = U + W$ where $U \cap W = \{0\}$.

Proof (Proposition 1.2). ───────────────────────────────────
We know that $U + W$ is definitely a subset of V because of its closure under vector addition. It is also trivial that $U \cap W \subseteq V$. Now we prove that any finite linear combination of each subset belongs to itself.

1. *$U + W$: Let $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ be a finite sequence of scalars and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in U + W$. Then, for any $1 \leq i \leq n$, we have $\mathbf{v}_i = \mathbf{u}_i + \mathbf{w}_i$ where $\mathbf{u}_i \in U$ and $\mathbf{w}_i \in W$. Therefore, we have:*

$$\sum_{i=1}^{n} \lambda_i \mathbf{v}_i = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i + \sum_{i=1}^{n} \lambda_i \mathbf{w}_i.$$

Since we have $\sum_{i=1}^{n} \lambda_i \mathbf{u}_i \in U$ and $\sum_{i=1}^{n} \lambda_i \mathbf{w}_i \in W$ due to closure under linear combinations of subspaces, we have $\sum_{i=1}^{n} \lambda_i \mathbf{v}_i \in U + W$. Therefore, $U + W$ is also closed under linear combinations and hence, a subspace of V.

2. $U \cap W$: Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in U \cap W$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$. It holds that $\sum_{i=1}^{n} \lambda_i \mathbf{v}_i \in U$ because U is a subspace. Since W is also a subspace $\sum_{i=1}^{n} \lambda_i \mathbf{v}_i \in W$. Therefore, $\sum_{i=1}^{n} \lambda_i \mathbf{v}_i \in U \cap W$.

$\square$.

Definition 1.4 (Span (Linear Closure)). ———————————————————————————
The span of a non empty subset S of a vector space over a field $\mathbb{F}$ is the set of all linear combinations of vectors from S.

$$\operatorname{span}(S) = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{v}_i : \mathbf{v}_i \in S, \lambda_i \in \mathbb{F} \right\}. \tag{1}$$

The span of an empty subset of a vector space is its trivial space ($\operatorname{span}(\emptyset) = \{0\}$*).*

1.3   Linear Independence

1.4   Basis and Dimension

A List of Definitions

B   Important Theorems

C   Important Corollaries

D   Important Propositions

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E References

References

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