

Linear Algebra Notes

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1 Vector Spaces

1.1 Definition of Vector Space

Definition 1.1 (Vector Space).

A **vector space** (over a field \mathbb{F}) consists of a set V with two operations “+” and “ \cdot ” subject to the conditions that for all $\vec{v}, \vec{w}, \vec{u} \in V$ and scalars $r, s \in \mathbb{F}$:

1. **Closure under:**

- i Vector addition: $\vec{v} + \vec{w} \in V$.
- ii Scalar multiplication: $r \cdot \vec{v} \in V$.

2. **Properties of vector addition:**

- iii Commutativity: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.
- iv Associativity: $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$.

3. **Properties of scalar multiplication:**

- v Associativity : $r \cdot (s \cdot \vec{v}) = (rs) \cdot \vec{v}$.
- vi Distributivity over scalar addition: $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$.
- vii Distributivity over vector addition: $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$.

4. **Inverse elements:**

- viii Additive inverse: $\forall \vec{v} \in V, \exists -\vec{v} \in V : \vec{v} + (-\vec{v}) = \vec{0}$.

5. **Identity elements:**

- ix Additive identity: $\exists \vec{0} \in V : \vec{0} + \vec{v} = \vec{v}, \quad \forall \vec{v} \in V$.
- x Multiplicative identity: $\exists \mathbf{1} \in \mathbb{F} : \mathbf{1} \cdot \vec{v} = \vec{v}, \quad \forall \vec{v} \in V$.

For brevity, we will denote vectors as bold face letters instead of overhead arrows after this definition. For example, \mathbf{u}, \mathbf{v} and \mathbf{w} .

Remark 1.1 (“Over a field”). When we use the phrase “a vector space over a field \mathbb{F} ”, this means that the scalars that we use will be taken from the field \mathbb{F} . It does not mean that our vector space consists of \mathbb{F} -valued vectors. For example, the following vector space:

$$L = \left\{ (x, \alpha x) : x \in \mathbb{C}, \alpha \in \mathbb{R} \right\}$$

is a vector space over \mathbb{R} (scalar multiplications are done with real-valued scalars) even though the vectors are complex-valued.

Remark 1.2 (Trivial Space). A vector space with one element is called a **trivial space**.

Example 1.1 (A simple example). The following is a vector space over \mathbb{R} :

$$L = \left\{ \begin{pmatrix} x & y \end{pmatrix}^\top : y = 3x \right\}.$$

This is easy to verify. Let us go through each condition one by one. Let the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in L$ defined as follows:

$$\mathbf{u}_1 = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} x_2 \\ 3x_2 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} x_3 \\ 3x_3 \end{pmatrix} ..$$

All the axioms of a vector space are satisfied. Let $\alpha, \beta \in \mathbb{R}$, we have:

1. Closure under vector addition: $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ 3x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 3(x_1 + x_2) \end{pmatrix} \in L$.
2. Closure under scalar multiplication: $\alpha \mathbf{u}_1 = \alpha \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ 3\alpha x_1 \end{pmatrix} \in L$.
3. Additive commutativity: $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} x_1 + x_2 \\ 3x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} x_2 + x_1 \\ 3x_2 + 3x_1 \end{pmatrix} = \mathbf{u}_2 + \mathbf{u}_1$.
4. Additive associativity: $(\mathbf{u}_1 + \mathbf{u}_2) + \mathbf{u}_3 = \begin{pmatrix} (x_1 + x_2) + x_3 \\ 3(x_1 + x_2) + 3x_3 \end{pmatrix} = \begin{pmatrix} x_1 + (x_2 + x_3) \\ 3x_1 + 3(x_2 + x_3) \end{pmatrix} = \mathbf{u}_1 + (\mathbf{u}_2 + \mathbf{u}_3)$.
5. ... (We can easily verify other axioms as well).

Example 1.2 (Polynomials of degree 3). Consider the following set of real-coefficients polynomials with degree of at most 3:

$$\mathcal{P}_3 = \left\{ a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

Then, \mathcal{P}_3 is a vector space over \mathbb{R} under the following operations:

$$\begin{aligned} (a_0 + a_1x + a_2x^2 + a_3x^3) + (b_0 + b_1x + b_2x^2 + b_3x^3) \\ = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3, \\ \alpha \cdot (a_0 + a_1x + a_2x^2 + a_3x^3) = (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + (\alpha a_3)x^3. \end{aligned}$$

We can think of \mathcal{P}_3 as being “the same” as the vector space \mathbb{R}^4 . For every set of real coefficients a_0, \dots, a_3 , we have the following correspondence:

$$a_0 + a_1x + a_2x^2 + a_3x^3 \text{ corresponds to } \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Similarly, for any $n \geq 1$, \mathcal{P}_n is also a vector space over \mathbb{R} .

Definition 1.2 (Vector Space as Abelian Group). _____

Let V be a vector space over a field \mathbb{F} with two operations “+” and “·”. Then, $(V, +)$ is an (additive) Abelian group that satisfies the additional axioms on scalar multiplications:

- i $\lambda \mathbf{u} \in V$ for all $\lambda \in \mathbb{F}$ and $\mathbf{u} \in V$. (Closure under scalar multiplication).
- ii $(\lambda\mu)\mathbf{u} = \lambda(\mu\mathbf{u})$ for all $\lambda, \mu \in \mathbb{F}$ and $\mathbf{u} \in V$. (Associativity)
- iii $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$, $\mathbf{u}, \mathbf{v} \in V$ and $\lambda \in \mathbb{F}$. (Distributivity in V)
- iv $(\lambda + \mu)\mathbf{u} = \lambda\mathbf{u} + \mu\mathbf{u}$, $\lambda, \mu \in \mathbb{F}$ and $\mathbf{u} \in V$. (Distributivity in \mathbb{F})
- v $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$. (Multiplicative identity)

Basically, we can think of vector spaces as Abelian groups with additional structures involving scalar multiplications. According to definition 1.1, the fact that V is an Abelian group makes it satisfy properties (i), (iii), (iv), (vii) and (viii). The additional properties regarding scalar multiplication are listed above.

1.2 Subspace and Spanning Sets

In example 1.1, we saw a vector space over \mathbb{R} that is a subset of \mathbb{R}^2 , $L = \{(x, 3x)^\top : x \in \mathbb{R}\}$, which is a line through the origin. In this section, we provide a formal definition for subspace of a vector space:

Definition 1.3 (Subspace).

For vector space V with two operations “+” and “ \cdot ”. S is called a subspace of V if:

- $S \subseteq V$.
- S is a vector space under the inherited operations.

Proposition 1.1: Closure of Subspace under Linear Combination

For a non-empty subset S of a vector space V over a field \mathbb{F} . Under the inherited operations, the following statements are equivalent:

1. S is a subspace of V .
2. S is closed under linear combinations of any number of vectors: For all $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ and $\mathbf{u}_1, \dots, \mathbf{u}_n \in S$, $\sum_{i=1}^n \lambda_i \mathbf{u}_i \in S$.

The above proposition gives us a convenient tool to work with whenever we want to prove that some subset of a vector space is a subspace. Instead of going through all the axioms in definition 1.1, we can just prove that the subset is closed under finite linear combinations. For example, we have the following proposition:

Proposition 1.2: Operations that Preserves Subspaces

Let V be a vector space over a field \mathbb{F} and $U, W \subseteq V$ be subspaces. Then, the following subsets of V :

1. $U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U, \mathbf{w} \in W\}$ ^a,
2. $U \cap W = \{\mathbf{v} \in V : \mathbf{v} \in U \text{ and } \mathbf{v} \in W\}$

are all subspaces of V .

^aLater we will learn that the direct sum $U \oplus W$ is also a subspace. Basically $U \oplus W = U + W$ where $U \cap W = \{0\}$.

Proof (Proposition 1.2).

We know that $U + W$ is definitely a subset of V because of its closure under vector addition. It is also trivial that $U \cap W \subseteq V$. Now we prove that any finite linear combination of each subset belongs to itself.

1. $U + W$: Let $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ be a finite sequence of scalars and $\mathbf{v}_1, \dots, \mathbf{v}_n \in U + W$. Then, for any $1 \leq i \leq n$, we have $\mathbf{v}_i = \mathbf{u}_i + \mathbf{w}_i$ where $\mathbf{u}_i \in U$ and $\mathbf{w}_i \in W$. Therefore, we have:

$$\sum_{i=1}^n \lambda_i \mathbf{v}_i = \sum_{i=1}^n \lambda_i \mathbf{u}_i + \sum_{i=1}^n \lambda_i \mathbf{w}_i.$$

Since we have $\sum_{i=1}^n \lambda_i \mathbf{u}_i \in U$ and $\sum_{i=1}^n \lambda_i \mathbf{w}_i \in W$ due to closure under linear combinations of subspaces, we have $\sum_{i=1}^n \lambda_i \mathbf{v}_i \in U + W$. Therefore, $U + W$ is also closed under linear combinations and hence, a subspace of V .

2. $U \cap W$: Let $\mathbf{v}_1, \dots, \mathbf{v}_n \in U \cap W$ and $\lambda_1, \dots, \lambda_n \in \mathbb{F}$. It holds that $\sum_{i=1}^n \lambda_i \mathbf{v}_i \in U$ because U is a subspace. Since W is also a subspace $\sum_{i=1}^n \lambda_i \mathbf{v}_i \in W$. Therefore, $\sum_{i=1}^n \lambda_i \mathbf{v}_i \in U \cap W$.

□.

Definition 1.4 (Span (Linear Closure)).

The span of a non empty subset S of a vector space over a field \mathbb{F} is the set of all linear combinations of vectors from S .

$$\text{span}(S) = \left\{ \sum_{i=1}^n \lambda_i \mathbf{v}_i : \mathbf{v}_i \in S, \lambda_i \in \mathbb{F} \right\}. \quad (1)$$

The span of an empty subset of a vector space is its trivial space ($\text{span}(\emptyset) = \{0\}$).

Proposition 1.3: Span of Subsets in Vector Spaces

In a vector space, the span of any subset is a subspace.

Proof (Proposition 1.3).

Let V be a vector space over \mathbb{F} and $S \subseteq V$. Denote that $\tilde{S} = \text{span}(S)$. Obviously $\tilde{S} \subseteq V$ due to closure under linear combinations of vector spaces. We have to prove that \tilde{S} is also closed under linear combinations. Let $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ and $\mathbf{v}_1, \dots, \mathbf{v}_n \in \tilde{S}$. For each \mathbf{v}_i , we have:

$$\mathbf{v}_i = \sum_{j=1}^{n_i} \alpha_{i,j} \mathbf{u}_{i,j} \text{ where } \alpha_{i,1}, \dots, \alpha_{i,n_i} \in \mathbb{F}, \mathbf{u}_{i,1}, \dots, \mathbf{u}_{i,n_i} \in S.$$

Therefore, we have:

$$\sum_{i=1}^n \lambda_i \mathbf{v}_i = \sum_{i=1}^n \sum_{j=1}^{n_i} \alpha_{i,j} \lambda_i \mathbf{u}_{i,j} \in \tilde{S}.$$

Therefore, \tilde{S} is a subspace of V .

□.

Proposition 1.4: Adding Vectors to Span

Let V be a vector space over a field \mathbb{F} and $S \subseteq V$. Let $\mathbf{v} \in V$, then

$$\text{span}(S \cup \{\mathbf{v}\}) = \text{span}(S) \iff \mathbf{v} \in \text{span}(S). \quad (2)$$

Proof (Proposition 1.4).

We immediately have $\text{span}(S \cup \{\mathbf{v}\}) = \text{span}(S) \implies \mathbf{v} \in \text{span}(S)$ because if $\mathbf{v} \notin \text{span}(S)$, then the two sets cannot be equal because $\mathbf{v} \in \text{span}(S \cup \{\mathbf{v}\})$.

To prove the opposite side, suppose that $\mathbf{v} \in \text{span}(S)$. Hence, we can write \mathbf{v} as a linear combination of vectors in S :

$$\mathbf{v} = \sum_{i=1}^n \lambda_i \mathbf{u}_i, \quad \lambda_i \in \mathbb{F}, \mathbf{u}_i \in S.$$

We prove that $\text{span}(S \cup \{\mathbf{v}\}) = \text{span}(S)$ by showing that:

1. $\text{span}(S) \subseteq \text{span}(S \cup \{\mathbf{v}\})$.

2. $\text{span}(S \cup \{\mathbf{v}\}) \subseteq \text{span}(S)$.

The first containment is clear. To prove the second point, we prove that for any $\mathbf{w} \in \text{span}(S \cup \{\mathbf{v}\})$, $\mathbf{w} \in \text{span}(S)$. Let $\alpha_1, \dots, \alpha_{m+1} \in \mathbb{F}$ and $\mathbf{s}_1, \dots, \mathbf{s}_m \in S$, we can write \mathbf{w} as follows:

$$\begin{aligned}\mathbf{w} &= \sum_{j=1}^m \alpha_j \mathbf{s}_j + \alpha_{m+1} \mathbf{v} \\ &= \sum_{j=1}^m \alpha_j \mathbf{s}_j + \alpha_{m+1} \sum_{i=1}^n \lambda_i \mathbf{u}_i \\ &= \sum_{j=1}^m \alpha_j \mathbf{s}_j + \sum_{i=1}^n \alpha_{m+1} \lambda_i \mathbf{u}_i,\end{aligned}$$

which is a linear combination of vectors in S . Hence, $\mathbf{w} \in \text{span}(S)$. Therefore, we conclude that $\text{span}(S \cup \{\mathbf{v}\}) \subseteq \text{span}(S)$. \square

Corollary 1.1: Adding Independent Vector to Span

Let V be a vector space over a field \mathbb{F} and $S \subseteq V$. Let $\mathbf{v} \in V$, then

$$\text{span}(S) \subset \text{span}(S \cup \{\mathbf{v}\}) \iff \mathbf{v} \notin \text{span}(S). \quad (3)$$

Proof (Corollary 1.1). _____

We prove from both sides:

1. $\text{span}(S) \subset \text{span}(S \cup \{\mathbf{v}\}) \implies \mathbf{v} \notin \text{span}(S)$: Suppose the opposite that $\mathbf{v} \in \text{span}(S)$. Then, by proposition 1.4, we have $\text{span}(S) = \text{span}(S \cup \{\mathbf{v}\})$, which contradicts the initial assumption.
2. $\mathbf{v} \notin \text{span}(S) \implies \text{span}(S) \subset \text{span}(S \cup \{\mathbf{v}\})$: Similar to the above, we prove by contradiction using proposition 1.4.

\square .

Proposition 1.5: Removing Vectors from Span

Let $\mathbf{v} \in S$. Then, removing \mathbf{v} from S does not shrink the span if and only if it is a linear combination of other vectors in the set.

$$\text{span}(S \setminus \{\mathbf{v}\}) = \text{span}(S) \iff \mathbf{v} \in \text{span}(S \setminus \{\mathbf{v}\}). \quad (4)$$

Proof (Proposition 1.5). _____

This is a direct consequence of proposition 1.4 (By applying the proposition with $\tilde{S} = S \setminus \{\mathbf{v}\}$). From proposition 1.5, we have the following corollary. \square

Corollary 1.2: Remove Vector from Linearly Independent Sets

A set S is linearly independent if and only if removing any vector from S shrinks the span. In other words:

$$\forall \mathbf{v} \in S : \text{span}(S \setminus \{\mathbf{v}\}) \subset \text{span}(S). \quad (5)$$

Proof (Corollary 1.2). _____

We prove by contradiction. Suppose that we have $\mathbf{v} \in S$ and $\text{span}(S \setminus \{\mathbf{v}\}) = \text{span}(S)$. Then, by

proposition 1.5, we have $\mathbf{v} \in \text{span}(S \setminus \{\mathbf{v}\})$. Hence, \mathbf{v} is a linear combination of the other vectors in S , which is a contradiction since we assumed S to be independent. \square .

1.3 Linear Independence

Definition 1.5 (Linear Independence). _____

We have the following definitions of linear independence:

1. In any vector space, a finite set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is said to be linearly independent if **NONE of its elements is a linear combination of the others from the set**. Otherwise, the set is linearly dependent.
2. A finite set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is linearly independent if and only if:

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = 0 \implies \alpha_i = 0, \quad \forall 1 \leq i \leq n.$$

3. A finite set of vectors $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent if and only if:

$$\forall 1 \leq k \leq n : \mathbf{v}_k \notin \text{span}(S \setminus \{\mathbf{v}_k\}),$$

In other words, every vector in S is not spanned by the remaining vectors of S .

4. An infinite set of vectors $V = \{\mathbf{v}_k\}_{k=1}^{\infty}$ is linearly independent if every finite subset of V is linearly independent.

Definition 1.6 (Relation between Linear Independence and Span). _____

From propositions 1.4, 1.5 and corollaries 1.1, 1.2, we have the following properties that relate linear independence to spanning sets:

	$\text{span}(S) = \text{span}(\tilde{S})$	$\text{span}(S) \neq \text{span}(\tilde{S})$
Add \mathbf{v} to S	$\mathbf{v} \in \text{span}(S)$	$\mathbf{v} \notin \text{span}(S)$
Remove \mathbf{v} from S	$\mathbf{v} \in \text{span}(\tilde{S})$	$\mathbf{v} \notin \text{span}(\tilde{S})$

Table 1: Conditions for $\text{span}(\tilde{S}) = \text{span}(S)$ where \tilde{S} is resulted from adding \mathbf{v} to or removing \mathbf{v} from S . Note that for a vector \mathbf{u} and a (finite) set V , when we write $\mathbf{u} \notin \text{span}(V)$, it is the same as saying “ \mathbf{u} is independent of all vectors in V ”.

Definition 1.7 (Properties of Linear Independence). _____

Let V be a vector space (over a field \mathbb{F}) and $S \subseteq V$ be a finite subset of V . Then, we have:

1. S is linearly dependent if and only if there are distinct $\mathbf{v}_0, \dots, \mathbf{v}_n \in S$ and $\lambda_1, \dots, \lambda_n \in S$ (not all zeros) such that $\mathbf{v}_0 = \sum_{i=1}^n \lambda_i \mathbf{v}_i$.
2. Let $S_1 \subseteq S \subseteq S_2$, then:
 - S is linearly independent $\implies S_1$ is linearly independent.
 - S is linearly dependent $\implies S_2$ is linearly dependent.

3. There always exists $\tilde{S} \subseteq S$ such that \tilde{S} is linearly independent and $\text{span}(S) = \text{span}(\tilde{S})$.

Example 1.3 (Finding the Minimum Spanning Set). *Suppose that we have a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where each $\mathbf{v}_i \in \mathbb{R}^3$. Then, we can find the subset $S_1 \subseteq S$ by:*

- (i) *Write S as a matrix A where the i^{th} column is \mathbf{v}_i .*
- (ii) *Write A in reduced row echelon form: $A \rightarrow A_{\text{rref}}$.*

1.4 Basis and Dimension

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E References

References

- [1] Rick Durrett. *Probability: Theory and Examples*. 4th. USA: Cambridge University Press, 2010. ISBN: 0521765390.
- [2] Erhan undefinedinar. *Probability and Stochastics*. Springer New York, 2011. ISBN: 9780387878591. DOI: [10.1007/978-0-387-87859-1](https://doi.org/10.1007/978-0-387-87859-1). URL: <http://dx.doi.org/10.1007/978-0-387-87859-1>.
- [3] Wikipedia. *Vitali set* — *Wikipedia, The Free Encyclopedia*. <http://en.wikipedia.org/w/index.php?title=Vitali%20set&oldid=1187241923>. [Online; accessed 24-December-2023]. 2023.