Probability Theory Notes

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1 Chapter 1 - Measurable spaces & Measures

1.1 Overview of measure theory

1.1.1 Algebra and σ-algebra

Definition 1.1 (Algebra).

Let X be a set and $\mathcal{A}$ be a collection of subsets of X. Then, we say that $\mathcal{A}$ is an algebra if it satisfies:

- ***Closure under complement*** *: If $E \in \mathcal{A} \implies E^c \in \mathcal{A}$.*
- ***Closure under finite union*** *: For all finite collection $\{E_n\}_{n=1}^N \subset \mathcal{A} \implies \bigcup_{n=1}^N E_n \in \mathcal{A}$.*

Definition 1.2 (σ-algebra).

Let X be a set and $\mathcal{A}$ be a collection of subsets of X. Then, we say that $\mathcal{A}$ is a σ-algebra if it is:

- ***Closure under complement*** *: If $E \in \mathcal{A} \implies E^c \in \mathcal{A}$.*
- ***Closure under countable union*** *: For all countable collection $\{E_n\}_{n=1}^\infty \subset \mathcal{A} \implies \bigcup_{n=1}^\infty E_n \in \mathcal{A}$.*

Definition 1.3 (Borel-σ-algebra).

Let Σ be the set of all the σ-algebras generated by open intervals in $\mathbb{R}$. Then, the Borel-σ-algebra is the smallest σ-algebra containing the open intervals:

$$\mathcal{B} = \bigcap_{\mathcal{A} \in \Sigma} \mathcal{A}$$

Proposition 1.1: Disjoint union in algebra

Let $\mathcal{A}$ be an algebra and let $\{E_n\}_{n=1}^\infty$ be a countable collection of subsets in $\mathcal{A}$. Then, there exists a countable disjoint subsets $\{F_n\}_{n=1}^\infty$ such that:

$$\bigcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty F_n$$

Proof (Proposition 1.1).

Let $G_m = \bigcup_{n=1}^m E_n$, we have $G_1 \subset G_2 \subset G_3 \subset \cdots \subset G_N$. It is easy to see that $\bigcup_{n=1}^N G_n = \bigcup_{n=1}^N E_n$. Now, define the collection $\{F_n\}_{n=1}^\infty$ as followed:

$$F_n = \begin{cases} G_1 & \text{When } n = 1 \\ \\ G_n \setminus G_{n-1} & \text{When } n \geq 2 \end{cases}$$

Hence, we have $\bigcup_{n=1}^N F_n = \bigcup_{n=1}^N G_n \implies \bigcup_{n=1}^N F_n = \bigcup_{n=1}^N E_n$. $\square$.

1.1.2 Measurable spaces

Definition 1.4 (Measurable space).
*Let E be a set and $\mathcal{E}$ be a σ-algebra over E. Then, the pair $(E, \mathcal{E})$ is called a **measurable space**. The elements in $\mathcal{E}$ are called **measurable sets**. When E is a topological space and $\mathcal{E}$ is the Borel-σ-algebra on E, then the elements in $\mathcal{E}$ are also called **Borel sets**.*

Definition 1.5 (Product of measurable spaces).
*Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces. For $A \subset E, B \subset F$, we denote the product of A, B, denoted $A \times B$, as the set of all pairs (x, y) such that $x \in A, y \in B$. The set $A \times B$ is then called a **measurable rectangle**. The measurable space $(E \times F, \mathcal{E} \otimes \mathcal{F})$ is called the product of measurable spaces $(E, \mathcal{E})$ and $(F, \mathcal{F})$ where $\mathcal{E} \otimes \mathcal{F}$ is a σ-algebra over $E \times F$:*

$$\mathcal{E} \otimes \mathcal{F} = \Big\{ A \times B : A \in \mathcal{E}, B \in \mathcal{F} \Big\}$$

1.1.3 Measures & Measure space

Definition 1.6 (Measure & Measure space).
Let $(E, \mathcal{E})$ be a measurable space. A measure is a mapping $\mu : \mathcal{E} \to [0, \infty]$ (Including infinity) such that:

- ***Empty set has zero measure** : $\mu(\emptyset) = 0$.*
- ***Countable (disjoint) additivity** : For a collection of disjoint measurable sets $\{E_n\}_{n=1}^{\infty}$, we have*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

*The triplet $(E, \mathcal{E}, \mu)$ is called the **Measure space** and μ is called a measure on the measurable space $(E, \mathcal{E})$.*

Remark : Note that **translation invariance** is not included because this property is specific to Lebesgue measure only. A general measure need not to have translation invariance.

Examples : Here are some of the most common examples of measures

- **Dirac measures** δ_x : Let $(E, \mathcal{E})$ be a measurable space and let $x \in E$ be a fixed point. For all $A \in \mathcal{E}$, defined:

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ \\ 0 & \text{if } x \notin A \end{cases}$$

  Then, δ_x is a measure on $(E, \mathcal{E})$ and it is called the **Dirac measure sitting at** x.

- **Counting measures** : Let $(E, \mathcal{E})$ be a measurable space and $D \subset E$ be countable. For each $A \in \mathcal{E}$, $\nu_D(A)$ is the number of points in $A \cap D$:

$$\nu_D(A) = \sum_{x \in D} \delta_x(A), \quad A \in \mathcal{A}$$

- **Discrete measures** : Let $(E, \mathcal{E})$ be a measurable space and $D \subset E$ be countable. For each $x \in D$, define $m : D \to (0, \infty)$ be a function that assigns a positive number to x. Define:

$$\nu_D^m(A) = \sum_{x \in D} m(x)\delta_x(A), \quad A \in \mathcal{A}$$

  Then, ν_D^m is called a **discrete measure** on $(E, \mathcal{E})$. We can understand $m(x)$ as a mass attached to each point $x \in D$.

Proposition 1.2: Properties of measures

Let μ be a measure on a measurable space $(X, \mathcal{A})$. Then, the following properties hold for all measurable sets A, B and a countable collection (not necessarily disjoint) of measurable sets $\{E_n\}_{n=1}^{\infty}$.

- **Monotonicity** : $A \subseteq B \implies \mu(A) \leq \mu(B)$.
- **Countable sub-additivity** : $\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leq \sum_{n=1}^{\infty} \mu(E_n)$.
- **Continuity**:
  - **Continuity from below** : $E_n \uparrow E \implies \mu(E_n) \uparrow \mu(E)$.
  - **Continuity from above** : $E_n \downarrow E \implies \mu(E_n) \downarrow \mu(E)$.

Proof (Proposition 1.2).

We prove each property one by one:

Monotonicity

If $A \subseteq B$, we have:

$$\begin{aligned} \mu(B) &= \mu((B \setminus A) \cup A) \\ &= \mu(B \setminus A) + \mu(A) \quad \textit{(Countable (disjoint) additivity)} \\ &\geq \mu(A) \end{aligned}$$

Countable sub-additivity

For two measurable sets A, B, we have:

$$\begin{aligned} \mu(A \cup B) &= \mu((A \cup B) \setminus A) + \mu(A) \\ &= \mu(B \setminus A) + \mu(A) \\ &\leq \mu(B) + \mu(A) \end{aligned}$$

Hence, extend the argument inductively, for a countable collection $\{E_n\}_{n=1}^{\infty}$, we have:

$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

Continuity

(**i**) ***Continuity from below*** *: Let $\{E_n\}_{n=1}^{\infty}$ be an increasing collection of measurable sets such* *that $E_1 \subseteq E_2 \subseteq \dots$ and $\bigcup_{n=1}^{\infty} E_n = E$. Construct a countable collection of disjoint measurable sets $\{F_n\}_{n=1}^{\infty}$ such that:*

$$\begin{cases} F_1 &= E_1 \\ \\ F_n &= E_n \setminus E_{n-1},\ n \geq 2 \end{cases}$$

Apparently $\{F_n\}_{n=1}^{\infty}$ is a disjoint collection and we have $E_n = \bigcup_{k=1}^{n} F_k$. Hence, we have:

$$\begin{aligned} \lim_{n\to\infty} \mu(E_n) &= \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} F_k\right) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(F_k) = \sum_{k=1}^{\infty} \mu(F_k) \\ &= \mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu(E) \end{aligned}$$

(ii) ***Continuity from above*** *: Let $\{E_n\}_{n=1}^{\infty}$ be an decreasing collection of measurable sets such that $E_1 \supseteq E_2 \supseteq \dots$ and $\bigcap_{n=1}^{\infty} E_n = E$. We have:*

$$\begin{aligned} \mu(E) &= \mu\left(\bigcap_{n=1}^{\infty} E_n\right) \\ &= \mu\left(\left[\bigcup_{n=1}^{\infty} E_n^c\right]^c\right) = \mu\left(X \setminus \left[\bigcup_{n=1}^{\infty} E_n^c\right]\right) \\ &= \mu(X) - \mu\left(\bigcup_{n=1}^{\infty} E_n^c\right) \quad \textit{(By monotonicity)} \\ &= \mu(X) - \lim_{n\to\infty} \mu(E_n^c) \quad \textit{(As proven in } \textbf{(i)}\textit{)} \\ &= \lim_{n\to\infty} \mu(X \setminus E_n^c) = \lim_{n\to\infty} \mu(E_n) \end{aligned}$$

$\square$.

1.1.4 Lebesgue Measure

Overview : The definition of Lebesgue measure stems from the need to construct a more general notion of integral (the Lebesgue integral) because the simple notion of Riemann integral is incomplete. For example, $L_R^1([0,1])$ (space of absolutely Riemann-integrable functions) is not a Banach space.

The construction of the Lebesgue integral over $\mathbb{R}$ requires a notion of "measure" on subsets of $\mathbb{R}$, which, ideally satisfies the following conditions:

- $\mu : \mathcal{P}(\mathbb{R}) \to [0,\infty)$ where $\mathcal{P}(\mathbb{R})$ denotes the power set of $\mathbb{R}$.
- μ **extends the measure of interval length** l. Meaning, if $I \subset \mathbb{R}$ is an interval, $\mu(I) = l(I)$.
- **Countable additivity** : Let $\{E_n\}_{n=1}^{\infty} \subset X$ be a collection of disjoint subsets of $\mathbb{R}$, then $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$.
- **Translation invariance** : For $E \subset \mathbb{R}, x \in \mathbb{R}$, we have $\mu(E + x) = \mu(E)$.

However, it is widely known that the construction of such measure is not possible because of the existence of non-measurable sets (Vitali sets [4]).

Definition 1.7 (Lebesgue outer measure).
Let $E \subset \mathbb{R}$. The Lebesgue outer measure (or simply "outer measure") is a mapping from the power set of $\mathbb{R}$ to $[0, \infty)$ such that:

$$\mu^*(E) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : I_n \text{ are open intervals}; E \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$$

Where l denotes interval length. Without proving, we will just acknowledge the fact that the Lebesgue outer measure satisfies the second and the fourth conditions. However, ***the outer measure is countably sub-additive rather than countably additive****. To account for this, we look at the definition of the Caratheodory criterion below.*

Definition 1.8 (Caratheodory criterion - Lebesgue measurable sets).
Let $E \subseteq \mathbb{R}$. The set E is called ***Lebgesgue measurable*** *if for all $A \subseteq \mathbb{R}$, we have:*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

The above condition is called the ***Caratheodory criterion****. We denote the set of Lebesgue measurable subsets as $\mathcal{M}$:*

$$\mathcal{M} = \left\{ E \subseteq \mathbb{R} : \forall A \subseteq \mathbb{R}, \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \right\}$$

Remark : Note that by countable sub-additivity, we will always have $\mu^*(A) \geq \mu^*(A \cap E)$

Definition 1.9 (Lebesgue measure).
The Lebesgue measure (denoted μ) is simply the Lebesgue outer measure μ^* restricted to the set of Lebesgue measurable sets $\mathcal{M}$:

$$\mu : \mathcal{M} \to [0, \infty); \quad \mu := \mu^* \Big|_{\mathcal{M}}$$

Proposition 1.3: Measure of intersection with measurable collection

Let $A \subseteq \mathbb{R}$ and let $\{E_n\}_{n=1}^N$ be a finite disjoint collection of Lebesgue measurable sets. Then, we have:

$$\mu^*\left(A \cap \left[\bigcup_{n=1}^{N} E_n\right]\right) = \sum_{n=1}^{N} \mu^*(A \cap E_n)$$

Proof (Proposition 1.3).
We will prove this by induction. For $N = 1$, both sides are identical. For the inductive step, suppose that the above proposition is true for $N = m$. We have to prove that it is true for $N = m + 1$.

Since E_{m+1} is measurable, using the Caratheodory criterion, we have:

$$\begin{aligned}\mu^*\left(A\cap\left[\bigcup_{n=1}^{m+1}E_n\right]\right)&=\mu^*\left(A\cap\left[\bigcup_{n=1}^{m+1}E_n\right]\cap E_{m+1}\right)+\mu^*\left(A\cap\left[\bigcup_{n=1}^{m+1}E_n\right]\cap E_{m+1}^c\right)\\&=\mu^*(A\cap E_{m+1})+\mu^*\left(A\cap\left[\bigcup_{n=1}^{m+1}E_n\right]\cap E_{m+1}^c\right)\end{aligned}$$

Since E_n is disjoint for all $1\le n\le m+1$. We have:

$$\bigcup_{n=1}^{m}E_n\subset E_{m+1}^c\implies\left[\bigcup_{n=1}^{m+1}E_n\right]\cap E_{m+1}^c=\bigcup_{n=1}^{m}E_n$$

Finally, we have

$$\begin{aligned}\mu^*\left(A\cap\left[\bigcup_{n=1}^{m+1}E_n\right]\right)&=\mu^*(A\cap E_{m+1})+\mu^*\left(A\cap\left[\bigcup_{n=1}^{m}E_n\right]\right)\\&=\mu^*(A\cap E_{m+1})+\sum_{n=1}^{m}\mu^*(A\cap E_n)\\&=\sum_{n=1}^{m+1}\mu^*(A\cap E_n)\end{aligned}$$

$\square$.

Proposition 1.4: $\mathcal{M}$ is σ-algebra

The set of Lebesgue measurable subsets $\mathcal{M}$ is a σ-algebra.

Proof (Proposition 1.4).

We first prove that $\mathcal{M}$ is an algebra. Then, for all countable collection of Lebesgue measurable sets $\{E_n\}_{n=1}^{\infty}$ such that $E=\bigcup_{n=1}^{\infty}E_n$, $\mu^*(A)\ge\mu^*(A\cap E)+\mu^(A\cap E^c)$.

Claim 1 : $\mathcal{M}$ is an algebra

We have to prove that $\mathcal{M}$ is both closed under complement and finite union.

- ***Closure under complement** : Trivial due to the symmetry of the Caratheodory criterion.*
- ***Closure under finite union** : Let E_1, E_2 be two Lebesgue measurable sets. We have:*

$$\begin{aligned}\mu^*(A\cap(E_1\cup E_2))&=\mu^*((A\cap E_1)\cup(A\cap E_2))\\&=\mu^*((A\cap E_1)\cup(A\cap E_2\cap E_1^c))\\&\le\mu^*(A\cap E_1)+\mu^*(A\cap E_2\cap E_1^c)\quad\textit{(Countable sub-additivity)}\\&=\mu^*(A)-\mu^*(A\cap E_1^c)+\mu^*(A\cap E_2\cap E_1^c)\\&=\mu^*(A)-\left[\mu^*(A\cap E_1^c)-\mu^*([A\cap E_1^c]\cap E_2)\right]\\&=\mu^*(A)-\mu^*(A\cap E_1^c\cap E_2^c)=\mu^*(A)-\mu^*\left(A\cap[E_1\cup E_2]^c\right)\end{aligned}$$

$$\implies\mu^*(A)\ge\mu^*\Big(A\cap(E_1\cup E_2)\Big)+\mu^*\left(A\cap[E_1\cup E_2]^c\right)$$

$$\implies E_1\cap E_2\in\mathcal{M}$$

Claim 2 : $\mathcal{M}$ is a σ-algebra
Given $\{E_n\}_{n=1}^{\infty}$ be a countable collection of Lebesgue measurable sets and let $E = \bigcup_{n=1}^{\infty} E_n$. By proposition 1.1, there exists another countable ***disjoint*** *collection of Lebesgue measurable sets $\{F_n\}_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n = E$.*

For any integer $N \geq 1$, we have $\bigcup_{n=1}^{N} F_n$ is Lebesgue measurable because $\mathcal{M}$ is an algebra. Hence, we have:

$$\begin{aligned}\mu^*(A) &= \mu^*\left(A \cap \left[\bigcup_{n=1}^{N} F_n\right]\right) + \mu^*\left(A \cap \left[\bigcup_{n=1}^{N} F_n\right]^c\right)\\ &\geq \mu^*\left(A \cap \left[\bigcup_{n=1}^{N} F_n\right]\right) + \mu^*\left(A \cap \left[\bigcup_{n=1}^{\infty} F_n\right]^c\right) = \mu^*\left(A \cap \left[\bigcup_{n=1}^{N} F_n\right]\right) + \mu^*(A \cap E^c)\end{aligned}$$

By proposition 1.3, we have:

$$\mu^*(A) \geq \sum_{n=1}^{N} \mu^*(A \cap F_n) + \mu^*(A \cap E^c)$$

Taking $N \to \infty$, we have:

$$\begin{aligned}\mu^*(A) &\geq \sum_{n=1}^{\infty} \mu^*(A \cap F_n) + \mu^*(A \cap E^c)\\ &= \mu^*\left(A \cap \left[\bigcup_{n=1}^{\infty} F_n\right]\right) + \mu^*(A \cap E^c) = \mu^*(A \cap E) + \mu^*(A \cap E^c)\end{aligned}$$

Hence, $\mathcal{M}$ is closed under countable union and is a σ-algebra.

$\square$.

Proposition 1.5: Translation invariance of Lebesgue measure

The Lebesgue (outer) measure is translation invariant.

Proof (Proposition 1.5).
Let $\mu^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ be the outer measure. We have to prove that for every $E \in \mathcal{M}$, we have $\mu * (E) = \mu^*(E + x)$.

Claim 1 : $\mu^*(E) \geq \mu^*(E + x)$
Let $\{I_n\}_{n=1}^{\infty}$ be the collection of open intervals that covers E. Then, $\{I_n + x\}_{n=1}^{\infty}$ covers $E + x$. Hence, we have:

$$\begin{aligned}\mu^*(E + x) &\leq \mu^*\left(\bigcup_{n=1}^{\infty} (I_n + x)\right)\\ &\leq \sum_{n=1}^{\infty} \mu^*(I_n + x)\\ &= \sum_{n=1}^{\infty} \mu^*(I_n) = \mu^*(E)\end{aligned}$$

Claim 2 : $\mu^*(E) \leq \mu^*(E+x)$
Let $\{I_n\}_{n=1}^{\infty}$ be the collection of open intervals that covers $E+x$. $\{I_n - x\}_{n=1}^{\infty}$ covers E. Hence, we have:

$$\begin{aligned}
\mu^*(E) &\leq \mu^*\left(\bigcup_{n=1}^{\infty}(I_n - x)\right) \\
&\leq \sum_{n=1}^{\infty}\mu^*(I_n - x) \\
&= \sum_{n=1}^{\infty}\mu^*(I_n) = \mu^*(E+x)
\end{aligned}$$

*From **Claim 1** and **Claim 2**, we have $\mu^*(E) = \mu^*(E+x) \quad \forall E \in \mathcal{M}, x \in \mathbb{R}$. Hence, the Lebesgue (outer) measure is translation invariant.* $\square$.

1.1.5 Borel Measure

Definition 1.10 (Borel measure)**.**
*Let E be a topological space and $\mathcal{B}(E)$ be the Borel-σ-algebra generated from the open sets of E. Then, any measure defined on $(E, \mathcal{B}(E))$ is called a **Borel measure**.*

Proposition 1.6: Non-Borel Lebesgue-measurable sets

We know that all open intervals are Lebesgue measurable. Hence, $\mathcal{B} \subset \mathcal{M}$, this implies the existence of **Non-Borel Lebesgue measurable sets**.

Proof (Proposition 1.6)**.**
*Define $\mathcal{C}$ as the **Cantor set** and $c : [0,1] \to [0,1]$ be the **Cantor function**. We define the following function $f : [0,1] \to [0,2]$ as:*

$$f(x) = c(x) + x$$

The function f is strictly increasing defined on the unit interval. Hence, it maps Borel sets to Borel sets ([2], exercises 45-47, chapter 2).

Note that $f(\mathcal{C})$ has positive measure. Therefore, we can always choose non-measurable subsets from $f(\mathcal{C})$.

Define a non-Borel-measurable subset $N \subset [0,2]$ such that $f^{-1}(N) \subset \mathcal{C}$. Since the Cantor set has zero measure, $f^{-1}(N)$ has zero measure and is Lebesgue measurable.

*However, $f^{-1}(N)$ is not Borel measurable because then $f(f^{-1}(N)) = N$ has to be Borel measurable, which is not true. Therefore $f^{-1}(N)$ **is Lebesgue measurable but not Borel measurable**.* $\square$.

In the following section about the **Caratheodory Extension Theorem**, we will use it to prove the following results about Borel measures (Corollary 1.1):

- There exists a unique Borel measure μ on $\mathbb{R}$ such that $\mu([a,b]) = b - a$.
- That unique Borel measure is the Lebesgue measure.

1.2 Dynkin's π-λ Theorem

Before diving into the theorem, we should familiarise ourselves with the relevant definitions. Specifically, what is a π-system and what is a λ-system.

1.2.1 π-system and λ-system

Definition 1.11 (π-system).
Given a set X. A collection $\mathcal{P}$ of subsets of X is called a π-system if it is **closed under intersection.**

The simplest example of a π-system is the set of any single elements of X or the set that contains only the empty set. However, we are more interested in some of the more non-trivial examples of π-system:

- The set of half-open intervals (from the left) : $\{(-\infty, a] : a \in \mathbb{R}\}$.
- The set of half-open intervals (from the right) : $\{[a, \infty) : a \in \mathbb{R}\}$.
- The set of closed intervals are also a π-system if the empty set is included : $\{[a, b] : a, b \in \mathbb{R}; a \leq b\} \cup \{\emptyset\}$.
- If $\mathcal{P}_1, \mathcal{P}_1$ are π-systems over X_1, X_2 then the Cartesian products $\mathcal{P}_1 \times \mathcal{P}_2 = \{A_1 \times A_2 : A_1 \in \mathcal{P}_1, A_2 \in \mathcal{P}_2\}$ is also a π-system over $X_1 \times X_2$.
- Any σ-algebra is a π-system.

Definition 1.12 (λ-system).
Given a set X. A collection of $\mathcal{D}$ of subsets of X is called a λ-system if it satisfies the following conditions:

- $X \in \mathcal{D}$
- ***Closure under relative complement*** *: If $A, B \in \mathcal{D}$ and $A \subseteq B \implies B \setminus A \in \mathcal{D}$.*
- ***Closure under countable disjoint union*** *: If there exists a countable collection of disjoint sets $\{A_n\}_{n=1}^{\infty} \subset \mathcal{D}$. Then, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$.*

Now that we see that λ-system is actually a slightly more complicated algebraic structure than a π-system. However, one thing we can notice is that any σ-algebra is also a λ-system. More generally, we have the following proposition.

Proposition 1.7: σ-algebra $=$ π-system $+$ λ-system

Every σ-algebra is both a π-system and a λ-system.

Proof (Proposition 1.7).
Given a set X and let $\mathcal{A}$ be a σ-algebra on X.

Claim 1 : $\mathcal{A}$ is a π-system over X
We have to prove that $\mathcal{A}$ is closed under (finite) intersection. We know that $\mathcal{A}$ is closed under countable intersection. Hence, for all countable collection of sets $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{A}$, we have:

$$A = \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$$

For all $N \geq 1$, we have:

$$\begin{aligned}\bigcap_{n=1}^{N} A_n &= A \setminus \bigcap_{m=N+1}^{\infty} A_m \\ &= A \cap \left(\bigcup_{m=N+1}^{\infty} A_m^c \right) = \bigcup_{m=N+1}^{\infty} (A \cap A_m^c)\end{aligned}$$

For all $m \geq N+1$, $A \cap A_m^c$ is a countable intersection of sets in $\mathcal{A}$. Hence, $\bigcup_{m=N+1}^{\infty}(A \cap A_m^c)$ is a countable union of sets in $\mathcal{A}$. Hence, $\bigcap_{n=1}^{N} A_n \in \mathcal{A}$.
Therefore, $\mathcal{A}$ is closed under finite intersection and is a π-system.

Claim 2 : $\mathcal{A}$ is a λ-system over X
We have to prove that:

- *$X \in \mathcal{A}$: Trivial.*
- *$\mathcal{A}$ is closed under relative complement : For $A, B \in \mathcal{A}$ and $A \subseteq B$, we have $B \setminus A = B \cap A^c \in \mathcal{A}$ (By closure under intersection).*
- *$\mathcal{A}$ is closed under countable disjoint union : Trivial since $\mathcal{A}$ is already closed under countable union.*

Hence, $\mathcal{A}$ is a λ-system over X.

$\square$.

1.2.2 Theorem and proof

Theorem 1.1: Dynkin's π-λ Theorem

If $\mathcal{D}$ is a λ-system containing the π-system $\mathcal{P}$. Then, it also contains the σ-algebra generated by $\mathcal{P}$.

$$\mathcal{P} \subseteq \mathcal{D} \implies \sigma(\mathcal{P}) \subseteq \mathcal{D}$$

In other words, if a λ-system contains a π-system, it also contains the smallest σ-algebra containing the π-system.

Proof (Theorem 1.1).
We will prove the theorem by using the smallest λ-system generated from $\mathcal{P}$. We will prove that:

- *(i) $\lambda(\mathcal{P})$ is a σ-algebra.*
- *(ii) $\sigma(\mathcal{P}) \subseteq \lambda(\mathcal{P})$.*
- *(iii) Since $\lambda(\mathcal{P}) \subseteq \mathcal{D} \implies \sigma(\mathcal{P}) \subseteq \lambda(\mathcal{P}) \subseteq \mathcal{D}$.*

Obviously, we know that $\lambda(\mathcal{P})$ is a λ-system, we have to prove that it is also a π-system. Meaning, $\forall A, B \in \lambda(\mathcal{P}) \implies A \cap B \in \lambda(\mathcal{P})$. Let $A \in \lambda(\mathcal{P})$ be an arbitrary set and define:

$$\mathcal{L}_A = \left\{ E : A \cap E \in \lambda(\mathcal{P}) \right\}$$

Claim 1 : $\mathcal{L}_A$ is a λ-system
We have:

- *$X \in \mathcal{L}_A$ because $A \cap X = A \in \lambda(\mathcal{P})$.*
- *$\forall P, Q \in \mathcal{L}_A, P \subseteq Q \implies Q - P \in \mathcal{L}_A$ because we have:*
  - *$A \cap (Q - P) = (A \cap Q) - (A \cap P)$.*

- $(A\cap Q),(A\cap P)\in\lambda(\mathcal{P})$ *and we have* $A\cap P\subseteq A\cap Q$. *Hence,* $(A\cap Q)-(A\cap P)\in\lambda(\mathcal{P})$.

- $\forall\{E_n\}_{n=1}^{\infty}\subseteq\mathcal{L}_A$ *be a disjoint collection, we have* $\bigcup_{n=1}^{\infty}E_n\in\mathcal{L}_A$ *because:*
  - $A\cap\bigcup_{n=1}^{\infty}E_n=\bigcup_{n=1}^{\infty}\{A\cap E_n\}$.
  - *For all* E_n, $A\cap E_n\in\lambda(\mathcal{P})$ *and disjoint, hence* $\bigcup_{n=1}^{\infty}\{A\cap E_n\}\in\mathcal{L}_A$.

Claim 2 : $A\in\mathcal{P}\implies\lambda(\mathcal{P})\subseteq\mathcal{L}_A$
We have:

- $\forall C\in\mathcal{P}:A\cap C\in\mathcal{P}$ *because both* $A,C\in\mathcal{P}$.

  $\implies A\cap C\in\lambda(\mathcal{P})$.

  $\implies\forall C\in\mathcal{P}:C\in\mathcal{L}_A$.

  $\implies\mathcal{P}\subseteq\mathcal{L}_A$ *(Meaning* $\mathcal{L}_A$ *is a* λ*-system generated by* $\mathcal{P}$*).*

- *But, we already stated that* $\lambda(\mathcal{P})$ *is the smallest* λ*-system generated by* $\mathcal{P}$. *Hence, we have* $\lambda(\mathcal{P})\subseteq\mathcal{L}_A$.

Claim 3 : $B\in\lambda(\mathcal{P})\implies\lambda(\mathcal{P})\subseteq\mathcal{L}_B$

- *We already proved that* $\forall A\in\mathcal{P}:\lambda(\mathcal{P})\subseteq\mathcal{L}_A$.
- *Hence, for an arbitrary* $B\in\lambda(\mathcal{P})\implies B\in\mathcal{L}_A$.
- *In other words :* $\forall A\in\mathcal{P}:A\cap B\in\lambda(\mathcal{P})\implies\forall A\in\mathcal{P}:A\in\mathcal{L}_B$.

  $\implies\mathcal{P}\subseteq\mathcal{L}_B$.

  $\implies\lambda(\mathcal{P})\subseteq\mathcal{L}_B$ *(Again,* $\lambda(\mathcal{P})$ *is the smallest* λ*-system containing* $\mathcal{P}$*).*

From ***Claim 3****, we can conclude that for two arbitrary sets* $A,B\in\lambda(\mathcal{P})\implies A\in\mathcal{L}_B$ *(and vice versa). Therefore,* $A\cap B\in\lambda(\mathcal{P})$ *and* $\lambda(\mathcal{P})$ *is also a* π*-system. Finally, we conclude that* $\lambda(\mathcal{P})$ *is a* σ*-algebra* $\square$.

(ii) We have proven that $\lambda(\mathcal{P})$ *is a* σ*-algebra. We also have that* $\sigma(\mathcal{P})$ *is the smallest* σ*-algebra generated by* $\mathcal{P}$. *Hence,* $\sigma(\mathcal{P})\subseteq\lambda(\mathcal{P})$ $\square$.

(iii) Since $\lambda(\mathcal{P})$ *is the smallest* λ*-system containing* $\mathcal{P}$. *We have* $\lambda(\mathcal{P})\subseteq\mathcal{D}$. *Finally, we have* $\sigma(\mathcal{P})\subseteq\lambda(\mathcal{P})\subseteq D$. $\square$.

1.3 Caratheodory Extension Theorem

1.3.1 Semi-ring of sets

Definition 1.13 (Semi-ring of sets)**.**
Given a set X, a collection of subsets of X - $\mathcal{A}$ is called a semi-ring of subsets in X if it satisfies the following conditions:

- $\emptyset \in \mathcal{A}$.
- ***Closure under intersection*** *:* $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$.
- *If $A, B \in \mathcal{A}$, there exists a **finite disjoint** collection of subsets $\{I_n\}_{n=1}^{N} \subset \mathcal{A}$ such that:*

$$A \setminus B = \bigcup_{n=1}^{N} I_n$$

Remark : Notice that any semi-ring of sets over a set X is also a π-system over X.

1.3.2 σ-finiteness of measure

Definition 1.14 (σ-finite measure)**.**
*Let $(X, \mathcal{A})$ be a measurable space and $\mu : \mathcal{A} \to [0, \infty]$ be a measure (or pre-measure) defined on it. Then μ is called σ-finite if X can be **covered by countably many measurable sets with finite measure**. In other words, There exists $\{S_n\}_{n=1}^{\infty} \subset \mathcal{A}$, $\mu(S_n) < \infty$ such that $X = \bigcup_{n=1}^{\infty} S_n$.*

1.3.3 Theorem and proof

The Caratheodory Extension Theorem involves the uniqueness and existence of extension of pre-measures. Before diving into the theorem, we look at the following lemma **??**, which will help us prove the uniqueness.

Proposition 1.8: Uniqueness of measure

Suppose that μ_1, μ_2 are measures on $(X, \mathcal{E})$ such that $\mu_1(X) = \mu_2(X) < \infty$. Let $\mathcal{A} \subset \mathcal{E}$ be a π-system over X such that $\mathcal{E} = \sigma(\mathcal{A})$. Then,

$$\mu_1\Big|_{\mathcal{A}} := \mu_2\Big|_{\mathcal{A}} \implies \mu_1 := \mu_2$$

In other words, it two measures agree on a π-system, they also agree on the σ-algebra generated by that π-system.

Proof (Proposition 1.8)**.**
Let $\mathcal{D}$ be the set where μ_1, μ_2 agrees:

$$\mathcal{D} = \Big\{ A \in \mathcal{E} : \mu_1(A) = \mu_2(A) \Big\}$$

Hence, we have $\mathcal{A} \subseteq \mathcal{D}$.

Claim : $\mathcal{D}$ is a λ-system

- *$E \in \mathcal{D}$ by assumption.*
- *If $A, B \in \mathcal{D}$ and $A \subseteq B$. Since $B = (B \setminus A) \cup A$, we have:*

$$\mu_1(B) = \mu_1(A) + \mu_1(B \setminus A)$$
$$\mu_2(B) = \mu_2(A) + \mu_2(B \setminus A)$$

  But we have $\mu_1(A) = \mu_2(A)$ and $\mu_1(B) = \mu_2(B)$. Hence, $\mu_1(B \setminus A) = \mu_2(B \setminus A)$ and $B \setminus A \in \mathcal{D}$.
- *Let $\{A_n\}_{n=1}^{\infty}$ be a countable disjoint sets in $\mathcal{D}$. Since $\mu_1(A_n) = \mu_2(A_n)$ for all $n \geq 1$. Hence:*

$$\sum_{n=1}^{\infty} \mu_1(A_n) = \sum_{n=1}^{\infty} \mu_2(A_n) \implies \mu_1\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu_2\left(\bigcup_{n=1}^{\infty} A_n\right)$$

  Therefore, we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$.

Now that we have proved that $\mathcal{D}$ is a λ-system that contains a π-system, by Dynkin's π-λ theorem 1.1, we have $\mathcal{A} \subseteq \sigma(\mathcal{A}) \subseteq \mathcal{D}$. Therefore:

$$\mu_1\Big|_{\sigma(\mathcal{A})} := \mu_2\Big|_{\sigma(\mathcal{A})} \text{ or } \mu_1\Big|_{\mathcal{E}} := \mu_2\Big|_{\mathcal{E}}$$

$\square$.

Theorem 1.2: Caratheodory Extension Theorem

Let X be a set and $\mathcal{A}$ be a semi-ring of sets over X. Let $\mu : \mathcal{A} \to [0, \infty]$ be a pre-measure defined on the semi-ring of sets. Then,

- There exists an extension of μ, $\tilde{\mu} : \sigma(\mathcal{A}) \to [0, \infty]$, which is a measure on the σ-algebra generated by the semi-ring.
- If μ is σ-finite, then $\tilde{\mu}$ is unique.

Proof (Theorem 1.2).
We have to prove both the existence and uniqueness of $\tilde{\mu}$.

(*i*) ***Existence of $\tilde{\mu}$***
We start by defining the outer measure $\mu^* : \mathcal{P}(X) \to [0, \infty]$ of the pre-measure as followed:

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(E_n) : E_n \in \mathcal{A}, E \subseteq \bigcup_{n=1}^{\infty} E_n\right\}$$

Now we restrict μ^* to the set of Caratheodory-measurable subsets only:

$$\mathcal{M} = \left\{E \subseteq X : \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c),\ \forall A \subseteq X\right\}$$

The strategy is to prove the following:

- *$\mathcal{M}$ is a σ-algebra $\implies$ μ^* is a measure on $\mathcal{M}$.*
- *$\mathcal{A} \subset \mathcal{M} \implies \sigma(\mathcal{A}) \subset \mathcal{M}$.*
- *Finally, conclude that $\mu^* : \sigma(\mathcal{A}) \to [0, \infty]$ is an extension of μ to $\sigma(\mathcal{A})$.*

Claim 1 :** $\mathcal{M}$ **is a** σ**-algebra (In other words,** μ^* **is indeed a measure on** $\mathcal{M}$**)
It is trivial to prove closure under complement because of the symmetry in Caratheodory criterion. Hence, we will focus on proving closure under countable union.

Let $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$*, we have to prove that* $E = \bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$*. To do this, we use the same technique that we used to prove proposition 1.4 for Lebesgue measurable subsets. We make use of the following lemmas for the proof:*

- ***Proposition 1.1** : In an algebra* $\mathcal{M}$*, for any countable collection* $\{E_n\}_{n=1}^{\infty}$*, there exists a countable disjoint collection* $\{F_n\}_{n=1}^{\infty}$ *such that* $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$*.*
- ***Proposition 1.3** : For any finite disjoint collection of measurable sets* $\{E_n\}_{n=1}^{N}$*:*

$$\mu^*\left(A \cap \bigcup_{n=1}^{N} E_n\right) = \sum_{n=1}^{N} \mu^*(A \cap E_n)$$

***Claim 2 :** $\mathcal{A} \subset \mathcal{M}$*
For any $E \in \mathcal{A}$*, we have to show that for all* $A \subseteq X$*, we have:*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

By the definition of the outer measure μ^**, for all* $\epsilon > 0$*, we can always find a countable collection* $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ *such that:*

$$A \subset \bigcup_{n=1}^{\infty} A_n \text{ and } \mu^*(A) + \epsilon \geq \sum_{n=1}^{\infty} \mu^*(A_n)$$

Then, we also have:

$$A \cap E \subseteq \bigcup_{n=1}^{\infty} (A_n \cap E)$$
$$A \cap E^c \subseteq \bigcup_{n=1}^{\infty} (A_n \cap E^c)$$

Hence, we have:

$$\begin{aligned}\mu^*(A \cap E) + \mu^*(A \cap E^c) &\leq \mu^*\left(\bigcup_{n=1}^{\infty} (A_n \cap E)\right) + \mu^*\left(\bigcup_{n=1}^{\infty} (A_n \cap E^c)\right) \\ &= \mu^*\left(\bigcup_{n=1}^{\infty} (A_n \cap E)\right) + \mu^*\left(\bigcup_{n=1}^{\infty} (A_n \setminus E)\right)\end{aligned}$$

Since we have $A_n \cap E \in \mathcal{A}$ *for all* $n \geq 1$*, we have:*

$$\mu^*\left(\bigcup_{n=1}^{\infty} (A_n \cap E)\right) = \sum_{n=1}^{\infty} \mu^*(A_n \cap E)$$

Furthermore, we can always write $A_n \setminus E$ *as a finite disjoint union of elements in* $\mathcal{A}$*, we also have:*

$$\mu^*\left(\bigcup_{n=1}^{\infty} (A_n \setminus E)\right) = \sum_{n=1}^{\infty} \mu^*(A_n \setminus E)$$

Therefore, we can rewrite the above inequality as:

$$\begin{aligned}
\mu^*(A \cap E) + \mu^*(A \cap E^c) &\leq \sum_{n=1}^{\infty} \mu^*(A_n \cap E) + \sum_{n=1}^{\infty} \mu^*(A_n \setminus E) \\
&= \sum_{n=1}^{\infty} \Big(\mu^*(A_n \cap E) + \mu^*(A_n \cap E^c) \Big) \\
&= \sum_{n=1}^{\infty} \mu^*(A_n) \\
&\leq \mu^*(A) + \epsilon
\end{aligned}$$

Taking $\epsilon \to 0$, we have $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$. Hence, E is measurable and we conclude that $\mathcal{A} \subset \mathcal{M}$.

Claim 3 : $\sigma(\mathcal{A}) \subset \mathcal{M}$

*This is a direct consequence of **Claim 2** due to the Dynkin's π-λ theorem 1.1.*

*With **Claim 1**, **Claim 2** and **Claim 3**, we define $\tilde{\mu}$ as a measure such that $\tilde{\mu}\Big|_{\sigma(\mathcal{A})} := \mu^*\Big|_{\sigma(\mathcal{A})}$ and conclude the proof for existence of an extension measure.*

(ii) Uniqueness of $\tilde{\mu}$

Suppose that μ_1 and μ_2 are two measures defined on $(X, \sigma(\mathcal{A})$ such that $\mu_1\Big|_{\mathcal{A}} := \mu_2\Big|_{\mathcal{A}}$. By proposition 1.8, since μ_1, μ_2 agrees on a π-system and $\mu_1(X) = \mu_2(X) < \infty$, μ_1 and μ_2 are identical and thus conclude the proof for uniqueness.

$\square$.

1.3.4 Important corollaries

Corollary 1.1: Unique Borel measure on $\mathbb{R}$

Let $\mathcal{B}$ be the Borel-σ-algebra generated from open intervals in $\mathbb{R}$. Then:

- There exists a unique Borel measure μ on $(\mathbb{R}, \mathcal{B})$ such that $\mu([a, b)) = b - a$ for all half-open intervals $[a, b)$.
- That Borel measure is the Lebesgue measure.

Proof (Corollary 1.1).

Define the collection of half-open intervals as:

$$\mathcal{I} = \Big\{ [a, b) : a, b \in \mathbb{R}; a \leq b \Big\}$$

It is clear that $\mathcal{I}$ is a semi-ring of sets. Define the following pre-measure:

$$l([a, b)) = b - a$$

Which defined the length of all half-open intervals. It is easy to show that μ is σ-finite.

Claim 1 : $\sigma(\mathcal{I}) = \mathcal{B}$

- ***Prove that*** $\mathcal{B} \subseteq \sigma(\mathcal{I})$ *: For all open interval* $(a,b), a \leq b$*, we can write that open interval as a countable union of half-open intervals:*

$$(a,b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b\right)$$

  Hence, $\sigma(\mathcal{I})$ *contains all open intervals. But again, we have that* $\mathcal{B}$ *is the smallest* σ*-algebra generated by open intervals. Therefore,* $\mathcal{B} \subseteq \sigma(\mathcal{I})$*.*

- ***Prove that*** $\sigma(\mathcal{I}) \subseteq \mathcal{B}$ *: Since all open intervals can be written as countable unions of half-open intervals, the Borel-*σ*-algebra also contain all half-open intervals. By Dynkin's* π*-*λ *theorem 1.1, we have:*

$$\mathcal{I} \subseteq \mathcal{B} \implies \sigma(\mathcal{I}) \subseteq \mathcal{B}$$

Claim 2 : The extension of $l : \mathcal{I} \to [0, \infty]$ ***on*** $\sigma(\mathcal{I})$ ***is the Lebesgue measure.***
By Caratheodory Extension Theorem 1.2, since l *is* σ*-finite, there exists a unique extension on* $\sigma(\mathcal{I})$*. Let* $\tilde{\mu} : \sigma(\mathcal{I}) \to [0, \infty]$ *be that unique extension and* $\mu : \mathcal{M} \to [0, \infty]$ *be the Lebgesgue measure. We have:*

$$\tilde{\mu}\Big|_{\mathcal{I}} := \mu\Big|_{\mathcal{I}}$$

Since the Lebesgue measure extends interval's length. By proposition 1.8, since μ *and* $\tilde{\mu}$ *agrees on a* π*-system, they also agree on the* σ*-algebra generated by that* π*-system. Therefore,*

$$\tilde{\mu}\Big|_{\sigma(\mathcal{I})} := \mu\Big|_{\sigma(\mathcal{I})}$$

Hence, the extension $\tilde{\mu}$ *is the Lebesgue measure.*

$\square$.

2 Chapter 2 - Measurable functions & Integration

2.1 Measurable functions

Definition 2.1 (Function (Mapping)).
*A **mapping** or a **function** $f : E \to F$ from E to F is a rule that assigns every element $f(x) \in F$ to an element $x \in E$. Given a subset $B \subset F$, the pre-image of B under a function $f : E \to F$ is given by:*

$$f^{-1}(B) = \Big\{x \in E : f(x) \in B\Big\}$$

Proposition 2.1: Properties of functions

Let $f : E \to F$ be a function, the following properties hold:

- $f^{-1}(\emptyset) = \emptyset$.
- $f^{-1}(F) = E$.
- $f^{-1}(B \setminus C) = f^{-1}(B) \setminus f^{-1}(C)$.
- $f^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(B_n)$.
- $f^{-1}\left(\bigcap_{n=1}^{\infty} B_n\right) = \bigcap_{n=1}^{\infty} f^{-1}(B_n)$.

Definition 2.2 (Measurable function).
*Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces, $f : E \to F$ be a function. Then, f is called **measurable relative to $\mathcal{E}$ and $\mathcal{F}$** (or $(\mathcal{E}, \mathcal{F})$-measurable) if:*

$$\forall B \in \mathcal{F} : f^{-1}(B) \in \mathcal{E}$$

Proposition 2.2: σ-algebra of pre-images

Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be two measurable spaces and $f : E \to F$ be a function (**not necessarily $(\mathcal{E}, \mathcal{F})$-measurable**). Then, the following collection:

$$\mathcal{G} = \Big\{A \subseteq F : f^{-1}(A) \in \mathcal{E}\Big\}$$

is a σ-algebra.

Proof (Proposition 2.2).
We have to prove that $\mathcal{G}$ is closed under complement and countable union:

- ***Closure under complement*** *: Let $A \in \mathcal{G} \implies f^{-1}(A) \in \mathcal{E}$, we have:*

$$\begin{aligned} f^{-1}(A^c) &= f^{-1}(F \setminus A) = f^{-1}(F) \setminus f^{-1}(A) \\ &= E \setminus f^{-1}(A) = \Big[f^{-1}(A)\Big]^c \in \mathcal{E} \\ \implies A^c \in \mathcal{G} \end{aligned}$$

- ***Closure under countable union*** *: Let $\{A_n\}_{n=1}^{\infty}$ be a countable collection in $\mathcal{G}$. Hence, $f^{-1}(A_n) \in \mathcal{E},\ \forall n \geq 1$. We have:*

$$f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n-1}^{\infty} f^{-1}(A_n) \in \mathcal{E}$$
$$\implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{G}$$

From the above, we conclude that $\mathcal{G}$ is a σ-algebra. $\square$.

Proposition 2.3: Measurability criterion of functions

Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be two measurable spaces and $\mathcal{G} = \sigma(\tilde{\mathcal{G}})$ for some collection $\tilde{\mathcal{G}}$ of subsets of G. We have:

$$\forall A \in \tilde{\mathcal{G}} : f^{-1}(A) \in \mathcal{E} \implies \forall A \in \sigma(\tilde{\mathcal{G}}) : f^{-1}(A) \in \mathcal{E}$$

In other words, to show that f is measurable, we only have to show that $f^{-1}(A) \in \mathcal{E}$ for A belonging to a small collection that generates $\mathcal{G}$.

Proof (Proposition 2.3).

Define the following collection :

$$\mathcal{G}^* = \left\{A \subseteq G : f^{-1}(A) \in \mathcal{E}\right\}$$

We know that $\tilde{\mathcal{G}} \subseteq \mathcal{G}^*$. By proposition 2.2, we know that $\mathcal{G}^*$ is a σ-algebra. Therefore, we have $\sigma(\tilde{\mathcal{G}}) \subseteq \mathcal{G}^*$ because $\sigma(\tilde{\mathcal{G}})$ is the smallest σ-algebra generated by $\tilde{\mathcal{G}}$.

The result follows and we have:

$$\forall A \in \sigma(\tilde{\mathcal{G}}) : f^{-1}(A) \in \mathcal{E}$$

Therefore, f is $(\mathcal{E}, \mathcal{F})$-measurable. $\square$.

A List of Definitions

B Important Theorems

C Important Corollaries

D Important Propositions

E References

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