Time-series Analysis Notes

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1 Introduction

1.1 Time-series definitions

Definition 1.1 (Time series). ───────────────────────────
A time-series is a sequence of observations of time-indexed random variables $(X_t)_{t \geq 0}$. Typically, A time-series consists of three components.

- *Trend (T_t).*

- *Cycle (C_t).*

- *Seasonal (S_t).*

Finally, there is a random error (residual) term (R_t). Note that for the sake of this course, we will hardly consider the cycle (C_t) term.

Definition 1.2 (Multiplicative & Additive models). ───────────────────────
*The components of a time-series can be combined either **additively** or multiplicatively:*

- ***Additive model** : $X_t = T_t + C_t + S_t + R_t$.*

- ***Multiplicative model** : $X_t = T_t \times C_t \times S_t \times R_t$.*

1.2 Trend model

Definition 1.3 (Trend model). ───────────────────────────
A trend model is defined as:

$$X_t = T_t + R_t$$

Remark : Consider the linear trend:

$$T_t = \beta_0 + \beta_1 t, \quad \beta_1 \neq 0$$

We can then estimate the time-series $(X_t)_{t \geq 0}$ using the following methods:

- **Method 1** : Least square estimate the trend component T_t by minimizing the sum squared error

$$SSE = \sum_{t \geq 0} (X_t - T_t)^2$$

- **Method 2** : Eliminate the trend component via differencing operator.

Definition 1.4 (Differencing operator). ───────────────────────────
We define the first difference operator ∇ by:

$$\nabla X_t = X_t - X_{t-1} = (I - B)X_t$$

Where I is the identity operator and B is the backward shift operator:

$$BX_t = X_{t-1}, \ B^i X_t = X_{t-i}$$

The ith order differencing operator is:

$$\nabla^i X_t = \nabla(\nabla^{i-1} X_t)$$

Note that $\nabla^0 X_t = I$.

Proposition 1.1: Binomial formula for $\nabla^n X_t$

Given a time-series $(X_t)_{t \geq 0}$, we can write the differenced data to the nth order as followed:

$$\nabla^n X_t = \sum_{k=0}^{n} \binom{n}{k} (-1)^k B^k X_t = \sum_{k=0}^{n} \binom{n}{k} (-1)^k X_{t-k}$$

As a result, we can write:

$$X_t = \nabla^n X_t - \sum_{k=1}^{n} \binom{n}{k} (-1)^k X_{t-k}$$

Proof (Proposition 1.1).
Using the Binomial theorem, we have:

$$\nabla^n X_t = (I - B)^n X_t$$

$$= \sum_{k=0}^{n} \binom{n}{k} (I)^{n-k} (-B)^k X_t \quad (\textit{Binomial theorem})$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k B^k X_t$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k X_{t-k}$$

$$= X_t + \sum_{k=1}^{n} \binom{n}{k} (-1)^k X_{t-k}$$

$$\implies X_t = \nabla^n X_t - \sum_{k=1}^{n} \binom{n}{k} (-1)^k X_{t-k}$$

$\square$.

1.3   Trend + Seasonality model

Definition 1.5 (Trend + Seasonality model).
The model that displays both trend and seasonality is defined as:

$$X_t = T_t + S_t + R_t$$

There are two types of seasonality:

- **Constant seasonality** : *degree of seasonality does not depend on time.*

- **Varying (increasing) seasonality** : *degree of seasonality depends on time.*

Remark : When the time-series displays increasing seasonality, it is a common practice to transform the data so that the series display constant seasonality.

Definition 1.6 (Box-Cox transformation). ―――――――――――――――――
For some $\lambda > 0$, the Box-Cox transformation is defined as:

$$z_t = \frac{x_t^\lambda - 1}{\lambda}$$

Proposition 1.2: Box-Cox as $\lambda \to 0$

As $\lambda \to 0$, $z_t \to \log(x_t)$.

Proof (Proposition 1.2). ――――――――――――――――――――――――
We can write:

$$\lim_{\lambda \to 0} e^{z_t} = \lim_{\lambda \to 0} \lim_{n \to \infty} \left(1 + \frac{z_t}{n} \right)^n$$

$$= \lim_{\lambda \to 0} \left(1 + \lambda z_t \right)^{1/\lambda} \quad \left(Letting\ n = \frac{1}{\lambda} \right)$$

$$= \lim_{\lambda \to 0} \left(1 + (x^\lambda - 1) \right)^{1/\lambda} = x_t$$

$$\implies \lim_{\lambda \to 0} z_t = \log(x_t)$$

$\square$.

1.4   Modelling seasonality using dummy variables

Definition 1.7 (Dummy variables for seasonality). ―――――――――――――――
The seasonal factor (supposing that there are L seasons) expressed using dummy variables can be written as:

$$S_t = \beta_1^{(s)} D_1(t) + \cdots + \beta_{L-1}^{(s)} D_{L-1}(t)$$

Where we have:

$$D_k(t) = \begin{cases} 1, & if\ t\ belongs\ to\ season\ k \\ \\ 0, & otherwise \end{cases}$$

We can write the time-series whose seasonality is modelled using dummy variables as:

$$X_t = T_t + \sum_{k=1}^{L-1} \beta_k^{(s)} D_k(t) + R_t$$

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Remark : Take a time-series with linear trend and seasonality modelled using dummy variables as an example. Suppose that we have 4 seasons, the time-series will take the form:

$$X_t = \underbrace{\beta_0 + \beta_1 t}_{T_t} + \underbrace{\beta_1^{(s)} D_1(t) + \beta_2^{(s)} D_2(t) + \beta_3^{(s)} D_3(t)}_{S_t} + R_t$$

We can estimate the parameters $\beta = \left(\beta_0, \beta_1, \beta_1^{(s)}, \beta_2^{(s)}, \beta_3^{(s)}\right)$ the same way we do for linear regression models:

$$\hat{\beta} = \left(X^T X\right)^{-1} \left(X^T y\right), \quad X \in \mathbb{R}^{T \times 5}$$

Where T is the length of the time-series.

1.5   Autocorrelation & autocorrelation function (ACF)

Definition 1.8 (Autocovariance & Autocorrelation functions). ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯
Given a time-series $(X_t)_{t \geq 0}$, the autocovariance $\gamma(h)$ and autocorrelation $\rho(h)$ are defined as followed:

$$\gamma(h) = Cov\left(X_{t+h}, X_t\right)$$

$$\rho(h) = \frac{\gamma(h)}{Var(X_t)} = \frac{\gamma(h)}{\gamma(0)}$$

When we are given a set of observations $(x_t)_{t=1}^n$, we can estimate γ, ρ by using their sample counterparts:

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (x_{t+h} - \overline{x})(x_t - \overline{x})$$

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} = \frac{\sum_{t=1}^{n-h} (x_{t+h} - \overline{x})(x_t - \overline{x})}{\sum_{t=1}^n (x_t - \overline{x})^2}$$

We have the following useful identities:

- $\gamma_0 = Var(X_t)$.

- $\gamma_k = \gamma_{-k}$ *(symmetric).*

- $\rho_0 = 1$.

- $\rho_k = \rho_{-k}$ *(symmetric).*

Definition 1.9 (Autocorrelation function (ACF) plot). ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯
The autocorrelation function plot, or correlogram is the plot of sample autocorrelation coefficients $\hat{\rho}(h)$ against $h = 0, 1, 2, \ldots$.

2 Stationary time-series

2.1 Strict vs. Weak stationarity

Definition 2.1 (Mean, covariance and variance functions). ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯
Given a time-series $(X_t)_{t \in T}$ (where T is a time domain), we have the following functions:

- *Mean : $\mu(t) = \mathbb{E}[X_t]$, $t \in T$.*

- *Covariance : $\gamma(t, s) = Cov(X_t, X_s)$, $t, s \in T$.*

- *Variance : $\sigma^2(t) = \gamma(t, t) = Var(X_t)$, $t \in T$.*

Definition 2.2 (Strict stationarity). ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯
*A time-series is said to be **strictly stationary** if, $\forall n \in \mathbb{Z}_+, h \in \mathbb{N}$, we have:*

$$(X_1, \ldots, X_n) \stackrel{d}{\simeq} (X_{1+h}, \ldots, X_{n+h})$$

Meaning the two tuples are equal in distribution.

Remark : Note that in practice, strict stationarity is often hard to achieve. Hence, we have some relaxed conditions for stationarity.

Definition 2.3 (Weak stationarity). ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯
*A time-series $(X_t)_{t \geq 0}$ is said to be **weakly stationary** if the following two conditions are met:*

- *$\mu(t)$ does not depend on t.*

- *$\gamma(t, h)$ does not depend on t for all $h \in \mathbb{N}$.*

2.2 Moving Average - MA(q) model

Definition 2.4 (White noise). ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯
*A process $(Z_t)_{t \in T}$ is called **white noise** if it satisfies:*

- *$Cov(Z_t, Z_s)$ for all $s \neq t$.*

- *$Var(Z_t) = \sigma^2$, $\forall t \in T$.*

We denote that $Z_t \sim WN(0, \sigma^2)$.

Definition 2.5 (**MA(q)** model). ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

A List of Definitions

B Important Theorems

C Important Corollaries

D Important Propositions

E References

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