TIME SERIES ANALYSIS

Chapter 5 (Part 2)
Nonzero mean ARMA(p,q) models and invertibility

In the last chapter, we briefly described ways of choosing an appropriate model. However, "model identification" consists merely of selecting the form of the model, but not the numerical values of its parameters. Suppose, for example, we have decided to fit an AR(1) model $X_t = aX_{t-1} + Z_t$. Since the value of the parameter a is not known, it must somehow be estimated from the data. This chapter describes methods of estimating the parameters of ARMA models.

1 AR(P), MA(q) and ARMA(p,q) model with non-zero mean

Suppose that $\{Z_t\} \sim WN(0, \sigma^2)$. Models with zero-mean are

(i) AR(p): $X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t$.

(ii) MA(q): $X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$.

(iii) ARMA(p,q): $X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$.

It is sometimes appropriate to include a constant term δ in a Box-Jenkis model.

Definition 1 : Nonzero mean AR(p): $X_t = \delta + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t$.

Taking expectation on both sides yields

$$EX_t = \delta/(1 - \phi_1 - \cdots - \phi_p).$$

If we define $x_t = X_t - EX_t$, then

$$x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + Z_t.$$

Moreover, we have

$$Cov(x_t, x_s) = Cov(X_t, X_s)$$

Therefore, all the properties of the nonzero-mean AR(p) are the same as those of zero-mean AR(p) in terms of ACF.

Definition 2 : Nonzero mean MA(q): $X_t = \delta + Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$.

Then $EX_t = \delta$. Again if we define $x_t = X_t - EX_t$, then

$$x_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}.$$

We also have $Cov(x_t, x_s) = Cov(X_t, X_s)$.

Definition 3 : Nonzero mean ARMA(p,q):

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = \delta + Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}.$$

Then

$$EX_t = \delta/(1 - \phi_1 - \cdots - \phi_p)$$

If we define $x_t = X_t - EX_t$, then

$$x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}.$$

It follows that

$$Cov(x_t, x_s) = Cov(X_t, X_s).$$

Therefore, all the properties of the nonzero-mean ARMA(p) are the same as those of zero-mean ARMA(p).

Denote the sample mean by $\bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$, which is one possible point estimate of the population mean $\mu = EX_t$. If $\bar{X}$ is statistically (significantly) different from zero, it is reasonable to assume that μ does not equal zero and, therefore, to assume that δ does not equal zero. Let s stand for the standard deviation with $s^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$. One rough rule of thumb is to decide that $\bar{X}$ is statistically different from zero if the absolute value of

$$\frac{\bar{X}}{s/\sqrt{n}},$$

is greater than 1.96.

2 Invertibility

2.1 Motivation

Why do we need the concept of invertibility ? Consider the following example.

Example 1 Evaluate ACF of the following two MA(1) models.

1 . $X_t = Z_t + 1/3Z_{t-1}$.

2 . $X_t = Z_t + 3Z_{t-1}$.

Solution : Consider the first example. Since $EX_t = 0$

$$\gamma_k = Cov(X_t, X_{t+k}) = E[X_t X_{t+k}] = E\Big[(Z_t + 1/3a_{t-1})(Z_{t+k} + 1/3Z_{t+k-1})\Big]$$

$$= E(Z_t Z_{t+k}) + \frac{1}{3}E(Z_t Z_{t+k-1}) + \frac{1}{3}E(Z_{t-1}Z_{t+k}) + \frac{1}{9}E(Z_{t-1}Z_{t+k-1}).$$

This gives $\gamma_1 = 1/3$ and $\gamma_k = 0$ for $k > 1$. $X_t = a_t + 3a_{t-1}$, which further implies $\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{1/3}{1+1/9} = \frac{3}{10}$. Similarly, for the second example, we may obtain $\gamma_1 = 3$ and $\gamma_k = 0$ for $k > 1$ and $\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{3}{10}$. So two different models have the same ACF.

We can not distinguish between these two models by looking at the sample ACFs. Hence we will have to choose only one of them. We now further look at the difference between them.

Example 2 Consider $X_t = Z_t + \theta Z_{t-1}$. It can be written as $(1+\theta B)Z_t = X_t$ so that

$$Z_t = \sum_{j=0}^{\infty}(-\theta)^j B^j X_t = \sum_{j=0}^{\infty}(-\theta)^j X_{t-j}.$$

In other words, we have

$$X_t = Z_t + \theta X_{t-1} - \theta^2 X_{t-2} + \theta^3 X_{t-3} + \cdots - (-\theta)^j X_{t-j} + \cdots .$$

Intuitively speaking, the most recent observations should have higher weight than observations from the more distant past observations on X_t. When $|\theta| < 1$, $|\theta|^j$ becomes smaller as j gets larger. So we should choose model 1 in Example 1 with $\theta = 1/3$.

Definition 4 A time series $\{X_t\}$ is invertible if it can be expressed as an infinite series of past X-observations, i.e.

$$X_t = Z_t + \psi_1 X_{t-1} + \psi_2 X_{t-2} + \cdots ,$$

such that $\sum_{j=1}^{\infty} |\psi_j| < \infty$.

(a) AR(p) model

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t$$

where $\{Z_t\} \sim WN(0, \sigma^2)$, is always invertible.

(b) MA(q) model

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}.$$

The sufficient condition for X_t to be invertible is that

$$\phi(B) = 1 + \theta_1 B + \cdots + \theta_q B^q = 0$$

has all its roots outside the unit circle, i.e. all the roots have modulus (complex norm) greater than 1.

For example, if p = 1, the root of $1 + \theta_1 B = 0$ is

$$B = -1/\theta_1.$$

The condition is then

$$|\theta_1| < 1$$

If p = 2, the condition is

$$-\theta_1 - \theta_2 < 1, \quad -\theta_2 + \theta_1 < 1, \quad |\theta_2| < 1.$$

For $p \geq 3$, there is no clear expression for the conditions.

(c) ARMA(p,q):

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}.$$

The sufficient condition for X_t to be invertible is that

$$\phi(B) = 1 + \theta_1 B + \cdots + \theta_q B^q = 0$$

has all its roots outside the unit circle, i.e. all the roots have modulus (complex norm) greater than 1.

Example 3 Is the model

$$X_t = Z_t + 2Z_{t-1} + Z_{t-2}$$

invertible ? ■