TIME SERIES ANALYSIS

Chapter 4: Forecasting and SARIMA models

Forecasting the future values of an observed time series is an important problem in many areas including economics, production planning and sales forecasting and stock control.

1 Simple exponential smoothing

Exponential smoothing is the name given to the a general class of forecasting procedures that rely on simple updating equations to calculate forecasts. The most basic form, introduced here, is called **simple exponential smoothing**, but this should only be used for non-seasonal time series.

Given a non-seasonal time series, say $x_1, x_2, \cdots, x_N$, with no systematic trend, it is natural to forecast x_{N+1} by means of a weighted sum of the past observations

$$\hat{x}_N(1) = c_0 x_N + c_1 x_{N-1} + c_2 x_{N-2} + \cdots \tag{1.1}$$

where $\{c_i\}$ are weights. It seems sensible to give more weight to recent observations and less weight to observations further in the past. An intuitively appealing set of weights are geometric weights, which decrease by a constant ratio for every unit increase in the lag. Therefore in order that the weights sum to one we take

$$c_i = \alpha(1-\alpha)^i, \quad i = 0, 1, \cdots,$$

where α is a constant such that $0 < \alpha < 1$. Then equation (1.1) becomes

$$\begin{aligned}
\hat{x}_N(1) &= \alpha x_N + \alpha(1-\alpha)x_{N-1} + \alpha(1-\alpha)^2 x_{N-2} + \cdots \\
&= \alpha x_N + (1-\alpha)\Big[\alpha x_{N-1} + \alpha(1-\alpha)x_{N-2} + \cdots\Big] \\
&= \alpha x_N + (1-\alpha)\hat{x}_{N-1}(1).
\end{aligned}$$

If we set $\hat{x}_1(1) = x_1$, then the above equation can be used recursively to compute forecasts. Weights are exponentially decaying (hence the name). Choose α by minimizing squared one-step prediction error.

2 Invertibility

2.1 Motivation

Why do we need the concept of invertibility ? Consider the following example.

EXAMPLE 1 Evaluate ACF of the following two MA(1) models.

1 . $X_t = Z_t + 1/3Z_{t-1}$.

2 . $X_t = Z_t + 3Z_{t-1}$.

SOLUTION : Consider the first example. Since $EX_t = 0$

$$\begin{aligned}\gamma_k &= Cov(X_t, X_{t+k}) = E[X_t X_{t+k}] = E\Big[(Z_t + 1/3a_{t-1})(Z_{t+k} + 1/3Z_{t+k-1})\Big] \\ &= E(Z_t Z_{t+k}) + \frac{1}{3}E(Z_t Z_{t+k-1}) + \frac{1}{3}E(Z_{t-1}Z_{t+k}) + \frac{1}{9}E(Z_{t-1}Z_{t+k-1}).\end{aligned}$$

This gives $\gamma_1 = 1/3$ and $\gamma_k = 0$ for $k > 1$. $X_t = a_t + 3a_{t-1}$, which further implies $\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{1/3}{1+1/9} = \frac{3}{10}$. Similarly, for the second example, we may obtain $\gamma_1 = 3$ and $\gamma_k = 0$ for $k > 1$ and $\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{3}{10}$. So two different models have the same ACF.

We can not distinguish between these two models by looking at the sample ACFs. Hence we will have to choose only one of them. We now further look at the difference between them.

EXAMPLE 2 Consider $X_t = Z_t + \theta Z_{t-1}$. It can be written as $(1+\theta B)Z_t = X_t$ so that

$$Z_t = \sum_{j=0}^{\infty}(-\theta)^j B^j X_t = \sum_{j=0}^{\infty}(-\theta)^j X_{t-j}.$$

In other words, we have

$$X_t = Z_t + \theta X_{t-1} - \theta^2 X_{t-2} + \theta^3 X_{t-3} + \cdots - (-\theta)^j X_{t-j} + \cdots .$$

Intuitively speaking, the most recent observations should have higher weight than observations from the more distant past observations on X_t. When $|\theta| < 1$, $|\theta|^j$ becomes smaller as j gets larger. So we should choose model 1 in Example 1 with $\theta = 1/3$.

DEFINITION 1 A time series $\{X_t\}$ is invertible if it can be expressed as an infinite series of past X-observations, i.e.

$$X_t = Z_t + \psi_1 X_{t-1} + \psi_2 X_{t-2} + \cdots ,$$

such that $\sum_{j=1}^{\infty} |\psi_j| < \infty$.

(a) AR(p) model

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t$$

where $\{Z_t\} \sim WN(0, \sigma^2)$, is always invertible.

(b) MA(q) model

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}.$$

The sufficient condition for X_t to be invertible is that

$$\phi(B) = 1 + \theta_1 B + \cdots + \theta_q B^q = 0$$

has all its roots outside the unit circle, i.e. all the roots have modulus (complex norm) greater than 1.

For example, if p = 1, the root of $1 + \theta_1 B = 0$ is

$$B = -1/\theta_1.$$

The condition is then

$$|\theta_1| < 1$$

If p = 2, the condition is

$$-\theta_1 - \theta_2 < 1, \quad -\theta_2 + \theta_1 < 1, \quad |\theta_2| < 1.$$

For $p \geq 3$, there is no clear expression for the conditions.

(c) ARMA(p,q):

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}.$$

The sufficient condition for X_t to be invertible is that

$$\phi(B) = 1 + \theta_1 B + \cdots + \theta_q B^q = 0$$

has all its roots outside the unit circle, i.e. all the roots have modulus (complex norm) greater than 1.

Example 3 Is the model

$$X_t = Z_t + 2Z_{t-1} + Z_{t-2}$$

invertible ? ∎

3 Seasonal ARIMA models

In practice, it may not be reasonable to assume that the seasonality component repeats itself precisely in the same way cycle after cycle. Seasonal ARIMA models to be introduced allow for randomness in the seasonal pattern from one cycle to the next.

Suppose that we have r years of monthly data which we tabulate as follows:

	Month			
Year	1	2	$\cdots$	12
1	X_1	X_2	$\cdots$	X_{12}
2	X_{13}	X_{14}	$\cdots$	X_{24}
3	X_{25}	X_{26}	$\cdots$	X_{36}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r	$X_{1+12(r-1)}$	$X_{2+12(r-1)}$	$\cdots$	$X_{12+12(r-1)}$

Each column may be viewed as a time series. Suppose that each one of these time series is generated by the same ARMA(P,Q) model. Then at the j–th month and year t, we have

$$\begin{aligned} X_{j+12t} &= \Phi_1 X_{j+12(t-1)} + \cdots + \Phi_P X_{j+12(t-P)} + U_{j+12t} \\ &+ \Theta_1 U_{j+12(t-1)} + \cdots + \Theta_Q U_{j+12(t-Q)}, \end{aligned} \tag{3.1}$$

for $j = 1, 2, \cdots, 12$ and $t = 0, 1, \cdots, r-1$, where $\{U_s\} \sim WN(0, \sigma_U^2)$.

Since the same ARMA(P,Q) model is assumed to apply to each month, model (3.1) can be written for all t as

$$\begin{aligned} X_t &= \Phi_1 X_{t-12} + \cdots + \Phi_P X_{t-12P} + U_t \\ &+ \Theta_1 U_{t-12} + \cdots + \Theta_Q U_{t-12Q}, \end{aligned} \tag{3.2}$$

which is equivalent to

$$\Phi(B^{12})X_t = \Theta(B^{12})U_t. \tag{3.3}$$

Since it is likely that the 12 series corresponding to the different months are correlated. To incorporate dependence between these series we assume that $\{U_t\}$ follows an ARMA(p,q) model of the form

$$\phi(B)U_t = \theta(B)Z_t, \tag{3.4}$$

where $Z_t \sim WN(0, \sigma^2)$.

Combining models (3.3) and (3.4) implies

$$\phi(B)\Phi(B^{12})X_t = \theta(B)\Theta(B^{12})Z_t, \tag{3.5}$$

which motives us to introduce a definition for general seasonal ARIMA models.

Definition 2 : Consider a general seasonal ARIMA (SARIMA) model of the form

$$\phi_p(B)\Phi_P(B^s)Y_t = \theta_q(B)\Theta_Q(B^s)Z_t, \tag{3.6}$$

where B denotes the backward shift operator, ϕ_p, Φ_P, θ_q and Θ_Q are polynomials of order p, P, q and Q, respectively, $\{Z_t\} \sim WN(0, \sigma^2)$, and

$$Y_t = \bigtriangledown^d \bigtriangledown_s^D X_t = (1-B)^d(1-B^s)^D X_t$$

denotes the differenced series. This model is called a SARIMA model of order $(p, d, q) \times (P, D, Q)_s$ for $\{X_t\}$.

Remark 1 : Note that $\bigtriangledown^d$ is used to remove the trend while $\bigtriangledown_s^D$ is to remove seasonality.

Note also that

$$\{Y_t\} \sim \text{ARMA}(p + sP, q + sQ)$$

and

$$W_t = \bigtriangledown_s^D X_t \sim \text{ARIMA}(p, d, q).$$

When there is no seasonality ($s = D = 0$), $X_t \sim \text{ARIMA}(p, d, q)$.

Example 4 : SARIMA models look rather complicated at first sight, so let us consider some simple cases.

Note that when $d = D = 1$ and $s = 12$, $\{Y_t\}$ becomes

$$\begin{aligned} Y_t &= \bigtriangledown \bigtriangledown_{12} X_t = \bigtriangledown_{12} X_t - \bigtriangledown_{12} X_{t-1} \\ &= (X_t - X_{t-12}) - (X_{t-1} - X_{t-13}). \end{aligned}$$

Also consider a SARIMA model of order $(1, 0, 0) \times (0, 1, 1)_{12}$. This means that we have one non–seasonal AR term, one seasonal MA term and one seasonal difference. We then may simplify (3.6) as

$$(1 - \alpha B)Y_t = (1 + \Theta B^{12})Z_t, \tag{3.7}$$

where $Y_t = \bigtriangledown_{12} X_t$. This implies that model (3.7) is equivalent to

$$X_t - X_{t-12} - \alpha(X_{t-1} - X_{t-13}) = Z_t + \Theta Z_{t-12}. \tag{3.8}$$

Implementation:

General guidelines for identifying model SARIMA(p,d,q,P,D,Q,s) are as follows.

(a) Identify d and D so as to make the differenced observations

$$Y_t = \nabla^d \nabla_s^D X_t = (1-B)^d(1-B^s)^D X_t$$

stationary in appearance.

(b) Examine the SACF and SPACF of Y_t at lags which are multiples of s in order to identify the orders of P and Q in the model. For example, P and Q should be chosen so that $r(ks), k = 1, 2, \cdots$ is compatible with the ACF of ARMA(P,Q).

(c) Examine the SACF and SPACF of Y_t at lags $1, \cdots, s-1$. The p and q should be chosen so that they are compatible with ARMA(p,q).

(d) Ultimately, the AIC and diagnostic checking are used to identify the best SARIMA model among competing alternatives.

Typical R codes are the following.

```
> fit = arima(y, order = c(p,d,q), seasonal = list(order=c(P,D,Q), period=s))
```

e.g.

```
> tsdiag(fit)
```

e.g.

```
> predict(fit, n.ahead=10)
```

Example 5 Quarterly Propane gas bills for the Farmer's Bureau Co-op X_t: 344.39, 246.63, 131.53, 288.87, 313.45, 189.76, 179.1, 221.1, 246.84, 209, 51.21, 133.89, 277.01, 197.98, 50.68, 218.08, 365.1, 207.51, 54.63, 214.09, 267, 230.28, 230.32, 426.41, 467.06, 306.03, 253.23, 279.46, 336.56, 196.67, 152.15, 319.67, 440, 315.04, 216.42, 339.78, 434.66, 399.66, 330.8, 539.78. ■

In this data, the seasonal period is s = 4.

Figure 1, the acf of X_t dies down very slowly. We need to take differences.

$$u_t = X_t - X_{t-1}$$

and

$$v_t = X_t - X_{t-4}.$$

The acf of u_t dies down slowly and thus it is not stationary.

The acf of v_t dies down quickly. Based on figure 2, we can try models

$$\text{model A:} \quad v_t = (1+\theta_1 B)(1+\theta_4 B^4)Z_t$$

or

$$\text{model B::} \quad (1+\phi_1 B)v_t = (1+\theta_4 B^4)Z_t.$$

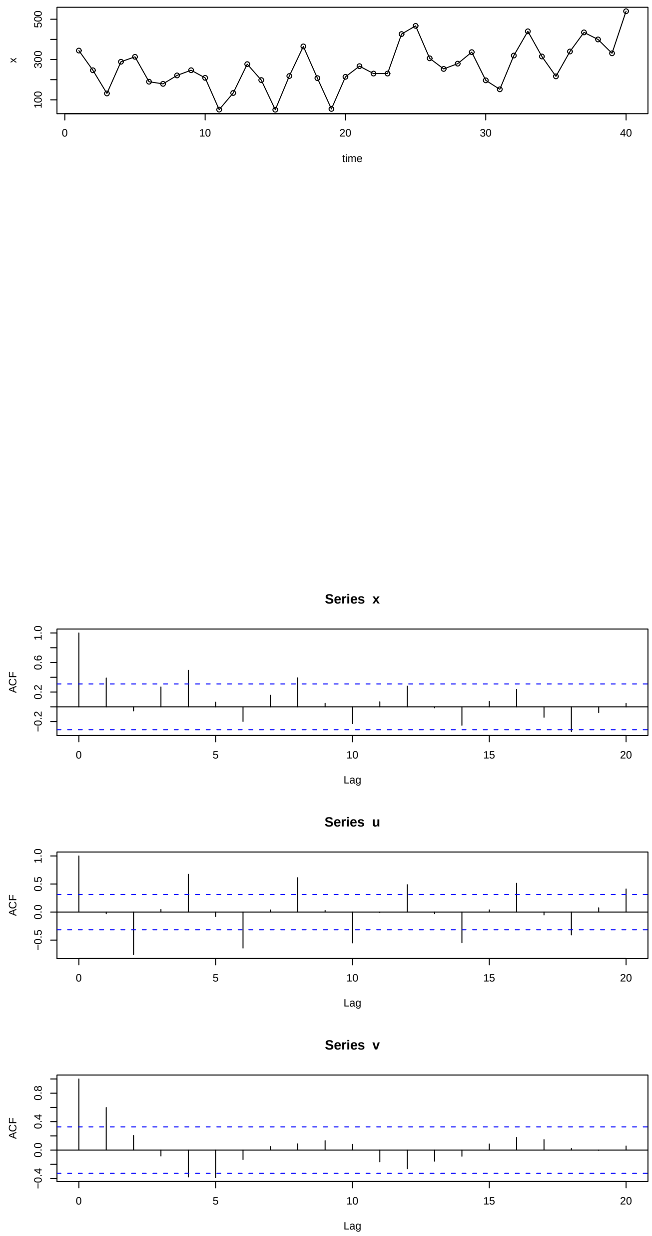


Figure 1:

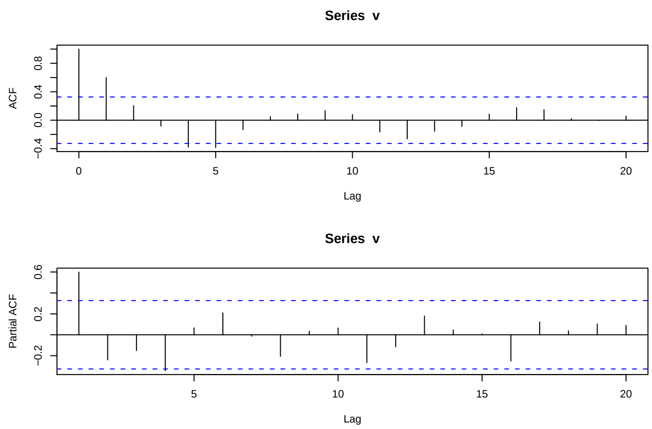


Figure 2:

i.e.

$$\text{model A: SARIMA}(0{,}0{,}1,\ 0{,}1{,}1,\ 4):$$
$$(1 - B^4)X_t = (1 + \theta_1 B)(1 + \theta_4 B^4)Z_t$$

or

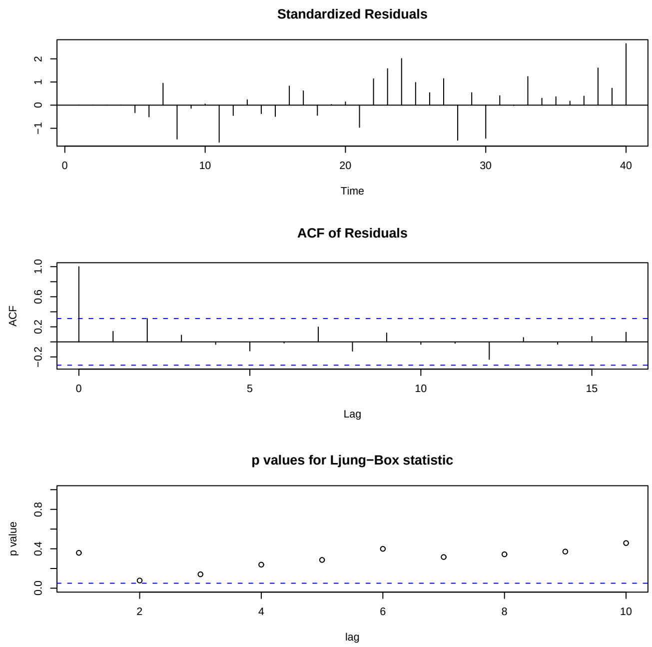
$$\text{model B: SARIMA}(1{,}0{,}0,\ 0{,}1{,}1,\ 4):$$
$$(1 + \phi_1 B)(1 - B^4)X_t = (1 + \theta_4 B^4)Z_t.$$

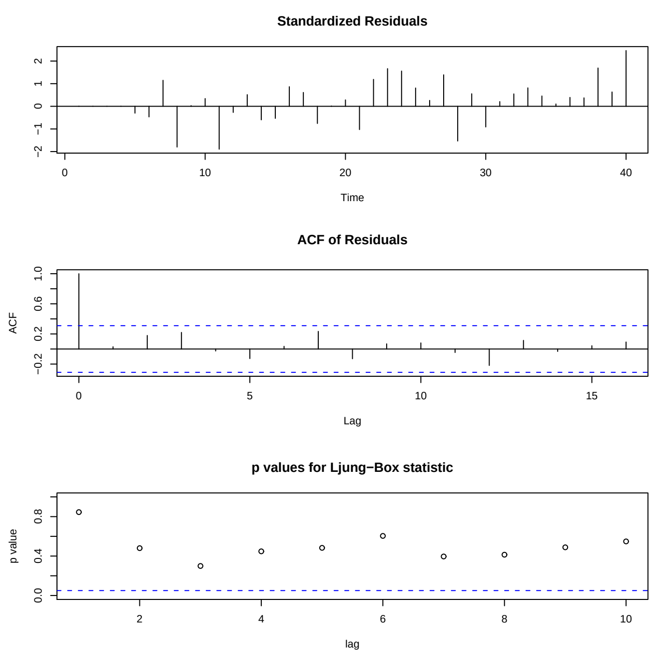
Model A:
Try model SARIMA(0,0,1, 0,1,1, 4).

```
> fita1 = arima(x, order=c(0,0,1), seasonal=list(order=c(0,1,1),period=4))
> tsdiag(fita1)
```

Try model SARIMA(0,0,2, 0,1,1, 4).

```
> fita2 = arima(x, order=c(0,0,2), seasonal=list(order=c(0,1,1),period=4))
> tsdiag(fita2)
```


Standardized Residuals
Time
ACF of Residuals
ACF
Lag
p values for Ljung−Box statistic
p value
lag


Standardized Residuals
Time
ACF of Residuals
ACF
Lag
p values for Ljung−Box statistic
p value
lag

Write down the fitted model.

> fita2

Call: arima(x = x, order = c(0, 0, 2), seasonal = list(order = c(0, 1, 1), period = 4))

Coefficients:

	ma1	ma2	sma1
	0.8810	0.2739	-0.6132
s.e.	0.1756	0.1662	0.1874

σ^2 estimated as 4130: log likelihood = -202.24, aic = 412.47

The fitted model is

$$(1 - B^4)X_t = (1 + 0.881B + 0.2739B^2)(1 - 0.6132B^4)Z_t.$$