TIME SERIES ANALYSIS

Chapter 2: Stationary time series

A key role in time series analysis is played by processes whose properties, or some of them, do not vary time. Such a property is illustrated in the following important concept, stationarity. We then introduce the most commonly used stationary linear time series models–the autoregressive moving average (ARMA) models. These models have assumed great importance in modeling real-world processes.

1 Strong and weak stationary

Loosely speaking, a time series $\{X_t, t = 0, \pm 1, \cdots\}$ is said to be stationary if it has statistical properties similar to those of the "time shifted" series $\{X_{t+h}, t = 0, \pm 1, \cdots\}$ for each integer h. We can make this idea precise with the following definitions.

DEFINITION 1 The expectation function of X is defined as

$$\mu(t) = E[X_t], \;\; t \in T.$$

And the covariance function of X is given by

$$\begin{aligned} \gamma(t,s) &= \operatorname{cov}(X_t, X_s) \\ &= E[(X_t - \mu_X(t))(X_s - \mu_X(s))] \end{aligned}$$

for all $t, s \in T$.

The variance function is defined by

$$\sigma_X^2(t) = \gamma(t,t) = \operatorname{var}(X_t).$$

Thus $\mu(t)$, $\operatorname{var}(X_t)$ and $\gamma(t,s)$ are just real functions of t and (t,s) respectively.

EXAMPLE 1 . Consider the Gaussian process $(X_t, t \in [0,1])$ of i.i.d. $N(0,1)$ random variables X_t.

Its expectation and covariance functions are given by

$$\begin{aligned}
\mu_X(t) &= 0 \text{ and} \\
\gamma(t,s) &= 1 \text{ if } t = s;\ 0 \text{ otherwise.}
\end{aligned}$$

DEFINITION 2 A time series is said to be strictly stationary if, for any $n \in \mathbb{Z}^+$ and all integers h, $(X_1, ..., X_n)$ and $(X_{1+h}, \cdots, X_{n+h}), s \in \mathbb{Z}\}$ have the same distributions.

DEFINITION 3 Denote

$$\mu_t = EX_t \quad \text{and} \quad \gamma(t,k) = \text{cov}(X_t, X_{t+k}), \quad t, k \in \mathbb{Z}.$$

A time series is said to be weakly stationary if (a) $\mu_t = \mu$ is independent of t; and (b) $\gamma(t,k)$ is independent of t for each k.

Weak stationarity is also called stationarity directly since we are mainly interested in this type of stationarity. The relationship between strict stationarity and weak stationarity is as follows. Strict stationarity and finiteness of second moment ensure weak stationarity. However, generally speaking, weak stationarity does not imply strict stationarity. An exceptional case is that the strict stationarity is the same as the weak stationarity when X_t is Gaussian process. Note that some distributions may have infinite second moment such as Cauchy distribution.

REMARK 1 . Usually, we first have a look at whether either

$$\mu(t) = c_0 \quad \text{or} \quad \text{var}(X_t) = c_1. \tag{1.1}$$

If so, we need to look at whether the following holds

$$\gamma(t,s) = h(t-s). \tag{1.2}$$

If (1.1) doesn't hold, then we will conclude that X_t is not weakly stationary. The verification of (1.2) is then not required. ■

EXAMPLE 2 Consider

$$X_t = \beta_0 + \beta_1 t + e_t,$$

where e_t's are i.i.d. with mean zero and variance one, and $\beta_1 \neq 0$. Is y_t stationary ?

Solution: It is not stationary because

$$EX_t = \beta_0 + \beta_1 t,$$

dependent on time t. ■

Example 3 Consider a random walk of the form

$$X_t = X_{t-1} + e_t, \ t = 1, 2, \ldots,$$

where e_t is a sequence of i.i.d. random variables with $E[e_t] = 0$ and $E[e_t^2] = 1$. Let $X_0 = 0$.

▷ Justify whether X_t is stationary.

▷ Is $Z_t = X_t - X_{t-1}$ stationary ?

2 Autocovariance and autocorrelation functions

In view of condition (b) in Definition 3, whenever we use the term covariance function with reference to a stationary process $\{X_t\}$, $\gamma(t, k)$ is a function of one variable, time lag k and hence denote it by γ_k.

Definition 4 Let $X(t)$ be a stationary time series. The autocovariance function at lag k is

$$\gamma_k = \text{cov}(X_t, X_{t+k}) = E(X_t - \mu)(X_{t+k} - \mu), \quad k \in \mathbb{Z};$$

and its autocorrelation function (ACF) is

$$\rho_k = \frac{\text{cov}(X_t, X_{t+k})}{\sqrt{\text{var}(X_t)\text{var}(X_{t+k})}} = \gamma_k/\gamma_0 \quad k \in \mathbb{Z}.$$

The two functions have the following properties,

▷ $\gamma_0 = \text{var}(X_t)$; $\rho_0 = 1$.

▷ $\gamma_k = \gamma_{-k}$; $\rho_k = \rho_{-k}$.
Therefore, the ACFs are often plotted only for the nonnegative lags.

Example 4 If Z_t is i.i.d with variance $Var(Z_t) = \sigma^2$ then for any s, t with $s \neq t$,

$$Cov(Z_t, Z_s) = 0$$

and $\{Z_t\}$ is stationary. ■

Definition 5 The process Z_t is said to be white noise if it is a stationary with $Cov(Z_t, Z_s) = 0$ for $s \neq t$, and $Var(Z_t) = \sigma^2$. Denote it by $z_t \sim WN(0, \sigma_z^2)$.

Example 5 We can build time series with white noise sequence. Suppose $Z_t \sim WN(0, \sigma^2)$. Let

$$X_t = Z_t + \theta Z_{t-1} \qquad (MA(1) model)$$

Then

$$E(X_t) = E(Z_t + \theta Z_{t-1}) = EZ_t + \theta EZ_{t-1} = 0.$$

and

$$\begin{aligned} Var(X_t) &= Var(Z_t + \theta Z_{t-1}) \\ &= Var(Z_t) + \theta^2 Var(Z_{t-1}) \\ &= \sigma^2 + \theta^2\sigma^2 = (1+\theta^2)\sigma^2 \end{aligned}$$

What about $Cov(X_t, X_{t-1})$ and $Cov(X_t + h, X_t), h > 1$? Is it stationary ? ■

3 Sample Autocovariance and autocorrelation functions

Although we have just seen how to compute the autocorrelation function for a few simple time series model, in reality, we do not start with a model but with observations $x_1, x_2, \cdots, x_n$. Suppose that $\{x_t\}$ is stationary, then we have to esitamte μ_x, $\gamma_x(h)$ and $\rho_x(h)$, with $h = 0, 1, 2, \cdots$. This estimate may suggest which of the many possible stationary models is a suitable candidate for representing the dependence in the data. For example, a sample ACF that is close to zero for all nonzero lags suggests that an appropriate model for the data might be i.i.d noise.

Definition 6 The sample mean is

$$\hat{\mu}_x = \bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t.$$

The sample autocovariance function (SACVF) is

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(x_{t+h} - \bar{x})(x_t - \bar{x}).$$

The sample autocorrelation function (SACF) at lag h is

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} = \frac{\sum_{t=1}^{n-h}(x_{t+h} - \bar{x})(x_t - \bar{x})}{\sum_{t=1}^{n}(x_t - \bar{x})^2}.$$

4 The Moving average

Definition 7 : The process $\{X_t\}$ defined by

$$X_t = Z_t + \sum_{j=1}^{q} \theta_j Z_{t-j} = \sum_{j=0}^{q} \theta_j Z_{t-j}, \ \theta_0 = 1,$$

where $Z_t \sim WN(0, \sigma^2)$, is called a *moving average process of order q* and is denoted by MA(q). We always assume $\theta_i = 0$ for $i \leq -1$.

Properties of MA(q) are as follows.

(a) $E[X_t] = 0$ since $E[Z_t] = 0$.

(b) Its variance is

$$\begin{aligned} \text{var}(X_t) &= \text{var}(Z_t + \sum_{j=1}^{q} \theta_j Z_{t-j}) \\ &= \sigma^2 + \sum_{j=1}^{q} \theta_j^2 \sigma^2 \\ &= \sigma^2 \left(1 + \sum_{j=1}^{q} \theta_j^2\right), \end{aligned}$$

where $\sigma^2 = \text{var}(Z_t)$.

(c) Its covariance functions are

$$\begin{aligned} &\text{cov}(X_{t-k}, X_t) = \text{cov}(\sum_{j=0}^{q} \theta_j Z_{t-k-j}, \sum_{l=0}^{q} \theta_l Z_{t-l}) \\ &= \left(\theta_k + \theta_1 \theta_{k+1} + \cdots + \theta_{q-k} \theta_q\right) \sigma^2 \quad \text{for } k \leq q. \end{aligned}$$

When $k > q$, $\text{cov}(X_{t-k}, X_t) = 0$.

(d) Its correlation functions are

$$\rho(k) = \begin{cases} 1, & k = 0 \\ \frac{\theta_k + \theta_1 \theta_{k+1} + \cdots + \theta_{q-k} \theta_q}{1 + \sum_{j=1}^{q} \theta_j^2}, & 1 \leq k \leq q \\ 0, & k > q \end{cases} \tag{4.1}$$

(e) **Generally, the ACF for a MA(q) time series <u>cut-off</u> after lag q**

(f) Since the mean and covariance function are both independent of time t we conclude that MA(q) is stationary.

EXAMPLE 6 (MA(2) process): If $Z_t \sim WN(0, \sigma^2)$, then $\{X_t\}$ defined by

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$$

is an MA(2) process. Find its ACFs,

SOLUTION : It follows that the ACFs are given by

$$\rho(1) = \frac{\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2},$$

$$\rho(2) = \frac{\theta_2}{1 + \theta_1^2 + \theta_2^2},$$

$$\rho(k) = 0 \ \forall \ k \geq 3.$$

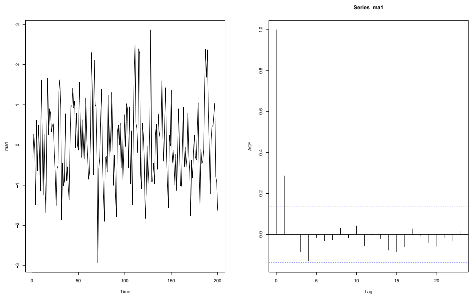


Figure 1: Plot of $MA(1)$ with $\theta_1 = 0.2$ and its acf

Below are R codes for simulating MA(1) and plotting ACF.

```
ma1=arima.sim(list(order=c(0,0,1),ma=0.2), n = 200)
ts.plot(ma1)
acf(ma1)
```

EXAMPLE 7 Let $Z_t \sim WN(0, \sigma^2)$. Consider $\nabla^k Z_t$, which appeared when applying the differencing operator to produce stationary process. Prove that it is stationary.

SOLUTION It is easily seen that $\nabla Z_t = Z_t - Z_{t-1}$ is a MA(1) and $\nabla^2 Z_t = Z_t - 2Z_{t-1} + Z_{t-2}$ is a MA(2). Similarly one may see that $\nabla^k Z_t$ is a MA(k) model. Therefore it is stationary.

DEFINITION 8 Let

$$X_t = \sum_{j=1}^{\infty} \psi_j Z_{t-j}$$

where $\psi_0 = 1$ and $Z_t \sim WN(0, \sigma^2)$ and

$$\sum_{j=0}^{\infty} |\psi_j| < \infty.$$

$\{X_t\}$ is called a **general linear process** or a **MA**(∞) process

5 Autoregressive models

DEFINITION 9 : Assume that $\{Z_t\}$ is white noise. Then the process $\{X_t\}$ defined by

$$X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + Z_t$$

is called an autoregressive process of order p, denoted by $AR(p)$.

Recall the following backshift operator B,

$$BX_t = X_{t-1} \quad \text{and} \quad B^i X_t = X_{t-i}$$

for $i = 1, 2, \ldots$.

Define the p^{th} degree polynomial,

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p.$$

We now can rewrite the model of X_t by

$$\phi(B) X_t = Z_t,$$

where $\phi(B) = 1 - \sum_{j=1}^{p} \phi_j B^j$. It is called the AR *characteristic polynomial.*

THEOREM 1 $AR(p)$ is stationary if all the roots of the polynomial $\phi(z)$ are outside the unit circle.

Properties of stationary AR(p) are listed below.

(a) $EX_t = 0$

(b) $E(Z_t X_{t-k}) = 0$ for any integer $k > 0$

(c) for $k \geq 1$, $\gamma(k) = \phi_1\gamma(k-1) + \phi_2\gamma(k-2) + \cdots + \phi_p\gamma(k-p)$.

> THEOREM 2 The Yule-Walker equations are
>
> $$\left\{ \rho(k) = \phi_1\rho(k-1) + \phi_2\rho(k-2) + \cdots + \phi_p\rho(k-p). \right.$$

PROOF Fact (c) may be obtained by first multiplying X_{t-k} on both sides of $AR(p)$ model, then taking expectation on both sides and eventually using (b) and the fact that $\gamma(k) = Cov(X_t, X_{t-k}) = E(X_t X_{t-k})$ because $EX_t = 0$. ■

EXAMPLE 8 (AR(1) process continued): Consider an AR(1) model of the form

$$X_t = \phi_1 X_{t-1} + Z_t, \quad Z_t \sim WN(0, \sigma^2).$$

- ▷ (i) Show that X_t is stationary if $|\phi_1| < 1$.
- ▷ (ii) Find the ACFs.

SOLUTION : (i)

Part a. If $|\phi_1| < 1$, then the root to $1 - \phi_1 z = 0$ is $z = 1/\phi_1$ which lies outside the unit circle. Hence $\{X_t\}$ is stationary.

(ii)

By the Yule-Walker equations

$$\rho_k = \phi_1\rho_{k-1}.$$

This yields

$$\rho_k = \phi_1^k\rho_0 = \phi_1^k.$$

Hence

$$|\rho(k)| = |\phi_1|^k = e^{-k\log(|\phi_1|^{-1})} \longrightarrow 0$$

exponentially. This fact is very useful in identifying an AR(1) model. Note that the exponential behaviour of the ACF of AR(1).

6 Stationary ARMA models

Definition 10 : (The ARMA(p,q) Process). The process $\{X_t\}$ is said to be an auto-regressive moving average or ARMA(p,q) process if $\{X_t\}$ is stationary and for every t,

$$\begin{aligned} & X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} \\ & = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}, \end{aligned} \tag{6.1}$$

where $Z_t \sim WN(0, \sigma^2)$.

We say that $\{X_t\}$ is an ARMA(p,q) process with mean μ if $\{X_t - \mu\}$ is an ARMA(p,q) process.

Model (6.1) can be written symbolically in the more compact form

$$\phi(B)X_t = \theta(B)Z_t, \ t = 0, \pm 1, \pm 2, \ldots,$$

where ϕ and θ are the p^{th} and q^{th} degree polynomials

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$$

and

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$$

and B is the backward shift operator defined by

$$B^j X_t = X_{t-j}, \ j = 0, \pm 1, \pm 2, \ldots.$$

Example 9 (The AR(p) Process). If $\theta(z) \equiv 1$ then

$$\phi(B)X_t = Z_t$$

and the process is said to be an autoregressive process of order p (or AR(p)). This model can also be written as

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t.$$

Example 10 (The MA(q) Process). If $\phi(z) \equiv 1$ then

$$X_t = \theta(B)Z_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

and the process is said to be a moving-averaging process of order q (or MA(q)).■

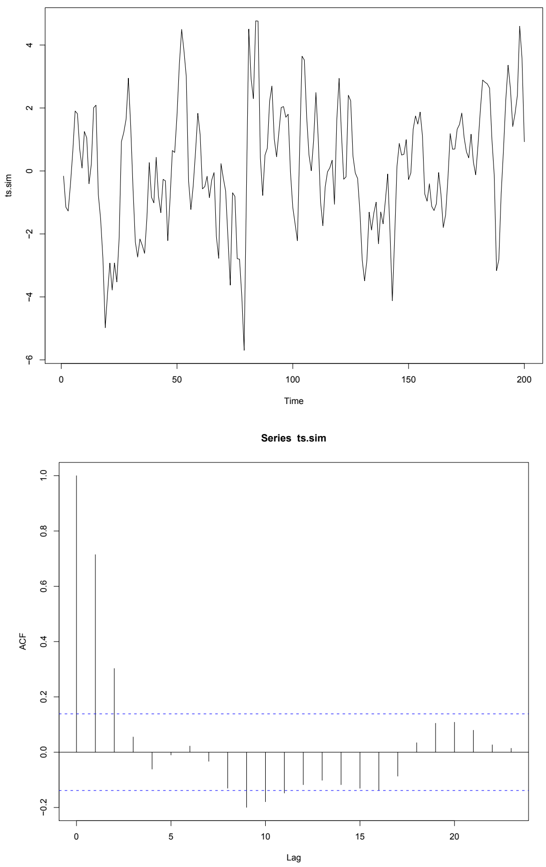


Figure 2: Plot of $ARMA(1, 1)$ with $\phi_1 = 0.5$, $\theta_1 = 0.8$ and its acf

Theorem 3 The ARMA process is stationary if all roots of the polynomial of $\phi(z)$ are outside the unit root circle.

Example 11 Consider the model

$$X_t = X_{t-1} + Z_t.$$

Note that $\phi(z) = 1 - z$. Its root is one which does not lie outside the unit circle. It follows that it is not stationary. ■

7 The Partial autocorrelation coefficient function (PACF)

The partial autocorrelation coefficient function, like the autocorrelation function, conveys vital information regarding the dependence structure of a stationary process.

7.1 Motivation of PACF

To motivate the idea, consider an $AR(1)$ model,

$$X_t = \phi X_{t-1} + Z_t.$$

Then

$$\begin{aligned}\gamma(2) &= cov(X_t, X_{t-2}) = cov(\phi X_{t-1} + Z_t, X_{t-2}) \\ &= cov(\phi^2 X_{t-2} + \phi Z_{t-1} + Z_t, X_{t-2}) = \phi^2 \gamma(0),\end{aligned}$$

because Z_t and Z_{t-1} are both uncorrelated with X_{t-2} (so called causal property). The correlation between X_t and X_{t-2} is not equal to zero because X_t is dependent on X_{t-2} through X_{t-1}. Suppose we break this chain of dependence by removing X_{t-1}. That is, consider the correlation between $X_t - \phi X_{t-1}$ and $X_{t-2} - \phi X_{t-1}$ because the correlation between X_t and X_{t-2} with the linear dependence of each on X_{t-1} removed. In this way, we have broken the dependence chain between X_t and X_{t-2}. Indeed,

$$\begin{aligned}&cov(X_t - \phi X_{t-1}, X_{t-2} - \phi X_{t-1}) \\ &= cov(Z_t, X_{t-2} - \phi X_{t-1}) = 0.\end{aligned}$$

7.2 Definition of PACF

The aim of introducing PACF is to detect the relation between x_t and x_{t-h} with the effect of $x_{t-1}, \cdots, x_{t-h+1}$ removed.

$$x_t, \underbrace{x_{t-1}, \cdots, x_{t-h+1}}, x_{t-h}.$$

DEFINITION 11 For a stationary process $\{X_t\}$, the partial autocorrelation function (PACF) is defined as

$$\phi_{11} = corr(X_1, X_0) \quad = \quad \rho(1)$$
$$\phi_{kk} = corr(X_k - f_{k-1}, X_0 - f_{k-1}), \;\; k \geq 2$$

where $f_{k-1} = f(X_{k-1}, \cdots, X_1)$ minimizes the mean square linear prediction error

$$E(X_k - f_{k-1})^2.$$

Usually, $f(X_{k-1}, \cdots, X_1) = \beta_1 X_1 + \cdots + \beta_{k-1} X_{k-1}$.

For a stationary process $\{X_t\}$, the partial autocorrelation function (PACF) at lag k can be interpreted as the correlation function between X_0 and X_k after removing their linear dependency on $X_1, ..., X_{k-1}$.

EXAMPLE 12 If $x_t \sim AR(p)$

$$x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + a_t,$$

then its PACFs of are

$$\begin{aligned} \phi_{1,1} &= \rho(1), \\ \phi_{p,p} &= \phi_p, \\ \phi_{k,k} &= 0 \text{ for } \;\; k > p. \end{aligned}$$

When $k > p$, the linear combination minimizing the mean square linear prediction error is

$$f_p = \phi_1 x_{k-1} + \cdots + \phi_p x_{k-p}$$

(obtaining such an expression is complicated and beyond the scope of the course). Then PACF at lag $k > p$ is

$$\phi_{k,k} = corr(X_k - f_p, X_0 - f_p) = corr(Z_k, X_0 - f_p) = 0,$$

because Z_k is uncorrelated with the past values of X_{k-j} with $j > 0$. The calculation of $\phi_{p,p} = \phi_p$ is also beyond the scope of the course. ■

THEOREM 4 Generally, the PACF for an AR(p) time series cut-off after lag p.

The general formula of ϕ_{kk} of an ARMA process is the last component of

$$\phi_h = \Gamma_h^{-1} g_h$$

where the matrix $\Gamma_h = [\gamma(i-j)]_{i,j=1}^{h}$ and $g_h = [\gamma(1), \gamma(2), \cdots, \gamma(h)]^T$ (this formula is for your reference only).

Summary

Identification of ARMA model

model	ACF	PACF
AR(p)	dies down	cut-off after lag p
MA(q)	cut-off after lag q	dies down
ARMA(p,q)	dies down	dies down

8 ARIMA models for nonstationary time series

We have already discussed the class of ARMA models for representing stationary series. A generalization of this class, which incorporates a wide range of non-stationary series, is provided by the ARIMA processes, which, after differencing finitely many times, reduces to ARMA models.

DEFINITION 12 (Auto–regressive Integrated Moving Average (ARIMA) Process): A process $\{X_t\}$ is ARIMA(p,d,q) iff $\{\bigtriangledown^d X_t\}$ is a stationary ARMA(p,q) process.

ARIMA processes are **not stationary** when $d > 0$.

EXAMPLE 13 Suppose that $\{X_t\}$ is a stationary process. Let $Y_t = X_t + c_1 + c_2 t + c_3 t^2$. Is $\{Y_t\}$ stationary? $(1-B)Y_t$? $(1-B)^2 Y_t$?
Answer: $(1-B)^2 Y_t$ is stationary. ■

EXAMPLE 14 **ARIMA(0,1,0):** Consider

$$(1-B)X_t = Z_t, \text{ or } X_t = X_0 + \sum_{i=1}^{t} Z_i,$$

where Z_t is white noise. The above special case is also called random walk. It can be classified as ARIMA(0,1,0). ■

Example 15 (ARIMA(0, 1, 1)): Consider a time series model of the form

$$X_t = X_{t-1} + Z_t + \theta Z_{t-1}. \tag{8.1}$$

where $\{Z_t\}$ is a white noise.

Model (8.1) is an ARIMA(0, 1, 1) since

$$\triangledown X_t = Z_t + \theta Z_{t-1}$$

defines $\{\triangledown X_t\}$ an ARMA(0, 1)=MA(1). ■

Aside: Notation: If p, d or q are zero, names can be abbreviated in the obvious way. e.g., ARMA(1,0) ≡ AR(1); ARIMA(0,0,3) ≡ MA(3).

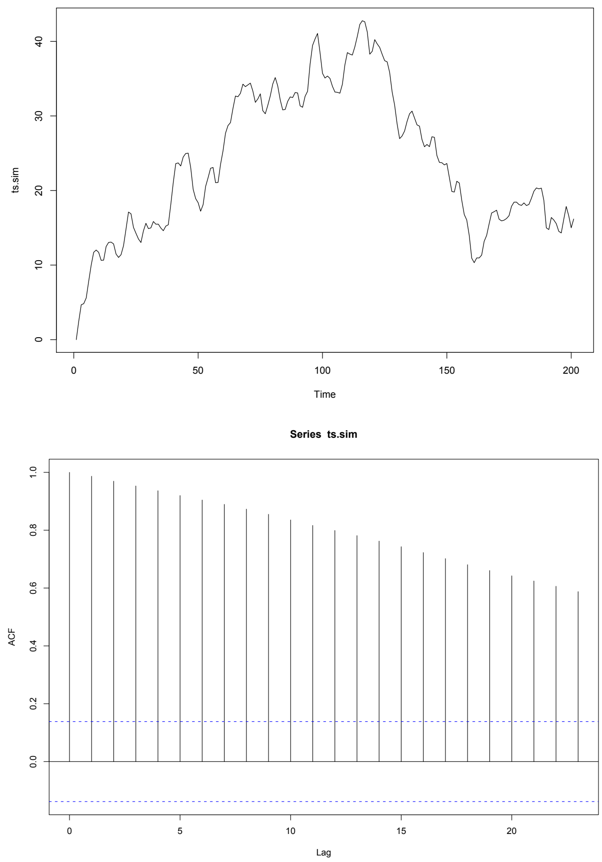


Figure 3: Plot of $ARIMA(0, 1, 1)$ with $\theta_1 = 0.8$ and its acf

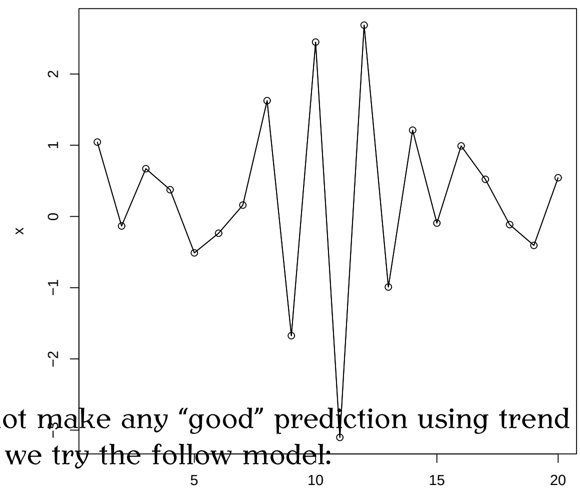
The R codes for generating the above the plots are as follows.

```
ts.sim<- arima.sim(list(order = c(0,1,1), ma = 0.7), n = 200)
ts.plot(ts.sim)
acf(ts.sim)
```

9 Why do we investigate AR, MA model?

Example 16 Consider the following data set of x_t :

1.0445, -0.1338, 0.6706, 0.3755, -0.5110, -0.2352, 0.1595, 1.6258, -1.6739, 2.4478, -3.1019, 2.6860, -0.9905, 1.2113, -0.0929, 0.9905, 0.5213, -0.1139, -0.4062, 0.5438



We cannot make any "good" prediction using trend and seasonality model.

Suppose we try the follow model:

$$x_t = 0.40 - 0.82x_{t-1}$$

This is a AR(1) model (we will explain how we get 0.4 and -0.82 in the subsequent courses).

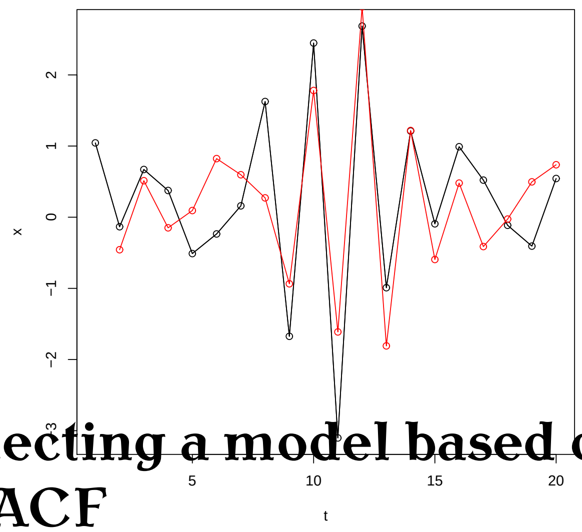
The fitted values are (time: prediction) 2: -0.4569, 3: 0.5130, 4: -0.1491, 5: 0.0938, 6: 0.8235, 7: 0.5965, 8: 0.2716, 9: -0.9354, 10: 1.7809, 11: -1.6121, 12: 2.9564, 13: -1.8082, 14: 1.2183, 15: -0.5942, 16: 0.47939, 17: -0.4124, 18: -0.0262, 19: 0.4966, 20: 0.73730

which can make reasonable prediction. ■

AR model is USEFUL in prediction

For the reason above (and more), we need to investigate models: AR(p), MA(q) .... More generally, stationary time series. To this end, we need answer the following questions.

- ▹ How to check stationarity
- ▹ ACF; PACF?
- ▹ Properties of AR, MA, ARMA models.
- ▹ How to calculate ACF and PACF for a given ARMA model.



10 Selecting a model based on SACF and SPACF

What will the SACF look like if the time series is not stationary ? Here is an example.

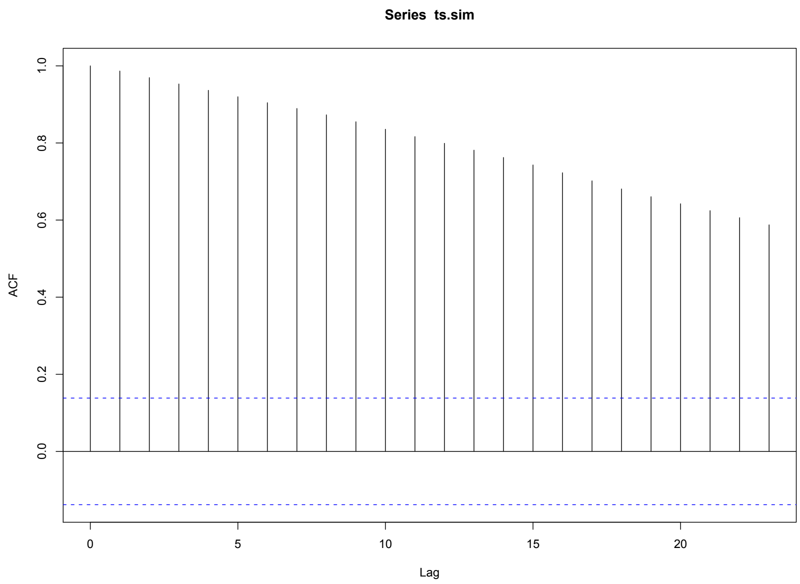


Figure 4: Plot of $ARIMA(0,1,1)$ with $\theta_1 = 0.8$ and its acf

For stationary TS (time series), the SAC will die down very quickly or even cuts off. Otherwise, the time series is not stationary.

Given observations of a time series, one approach to the fitting of a model to the data is to match the sample ACF and sample PACF of the data with the ACF and PACF of the model, respectively. Suppose that $z_1, \cdots, z_n$ is a realization (sample, observations) from a stationary time series. The sample ACF (SACF) at lag h is

$$r_h = \frac{\sum_{t=1}^{n-h}(z_t - \bar{z})(z_{t+h} - \bar{z})}{\sum_{t=1}^{n}(z_t - \bar{z})^2}$$

and the sample PACF (SPACF) is

$$\begin{aligned} r_{1,1} &= r_1 \\ r_{k,k} &= \frac{r_k - \sum_{j=1}^{k-1} r_{k-1,j} r_{k-j}}{1 - \sum_{j=1}^{k-1} r_{k-1,j} r_j}, \quad \text{for} \quad k = 2, 3, \cdots \\ r_{k,j} &= r_{k-1,j} - r_{k,k} r_{k-1,k-j}, \\ & \qquad\qquad\qquad\qquad \text{for} \quad j = 1, 2, \cdots, k-1 \end{aligned}$$

where $\bar{z} = \sum_{i=1}^{n} z_t / n$.

The $\underline{\textit{standard error of } r_h}$ is

$$s_{r_h} = \left(\frac{1 + 2\sum_{j=1}^{h} r_j^2}{n} \right)^{1/2}.$$

and The $\underline{\textit{standard error of } r_{hh}}$ is

$$s_{r_{hh}} = \left(\frac{1}{n} \right)^{1/2}.$$

Recall that

model	ACF	PACF
AR(p)	dies down quickly	cut-off after lag p
MA(q)	cut-off after lag q	dies down quickly
ARMA(p,q)	dies down quickly	dies down quickly
Nonstationary	dies down slowly	—

THEOREM 5 If the SACF r_h is significantly different from zero for $0 \leq h \leq q$ and negligible for $h > q$, then an $MA(q)$ model might be suitable for the data.

How do we classify ACF as negligible ? If (with probability approximately 0.95)

$$|r_h| \leq 1.96 s_{r_h},$$

then SACFs are considered as negligible. In practice, we frequently use the more stringent values $1.96/\sqrt{n}$ as the bound.

THEOREM 6 If the SPACF r_{hh} is significantly different from zero for $0 \leq h \leq p$ and negligible for $h > p$, then an $AR(p)$ model might be suitable for the data.

How do we classify PACF as negligible ? If (with probability approximately 0.95)

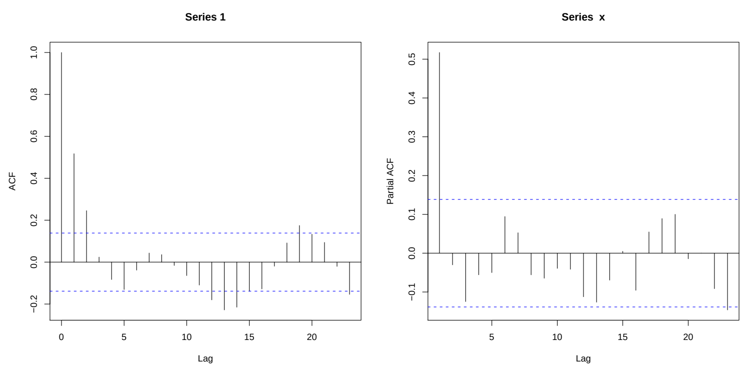
$$|r_{hh}| \leq \frac{1.96}{\sqrt{n}},$$

then SPACFs are considered as negligible.

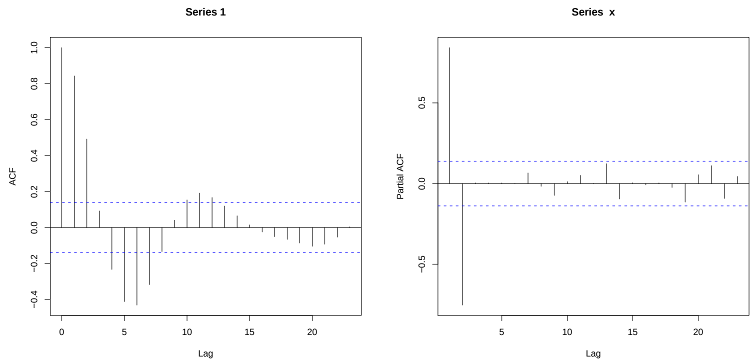
There is no theoretical model for which the ACF has nonzero values at lags $1, \cdots, q$ and cuts off after lag q, and the PACF has nonzero values at lags $1, \cdots, p$ and cuts off at all lags after lag p. However, in practice, sometimes for the time series values SACF has spikes at lags $1, \cdots, q$ and cuts off after lag q, and SPACF has spikes at lags $1, \cdots, p$ and cuts off after lag p. If this occurs, experience indicates that we should attempt to determine which of the SACF or SPACF is cutting off more abruptly. If the SACF cuts off more abruptly, then an MA(q) might be appropriate.

Here are examples. Can you identify appropriate models for them ?

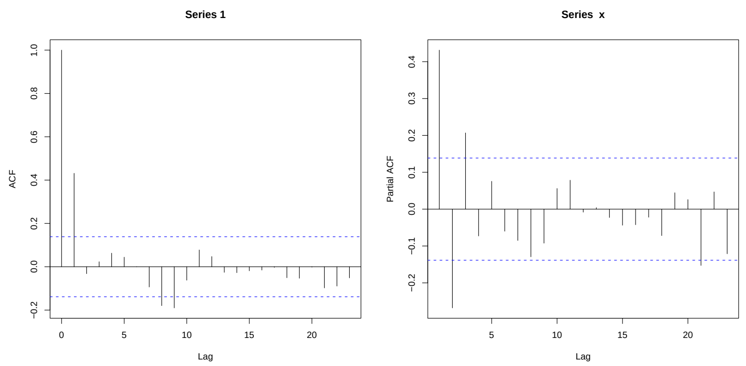
Time series 1



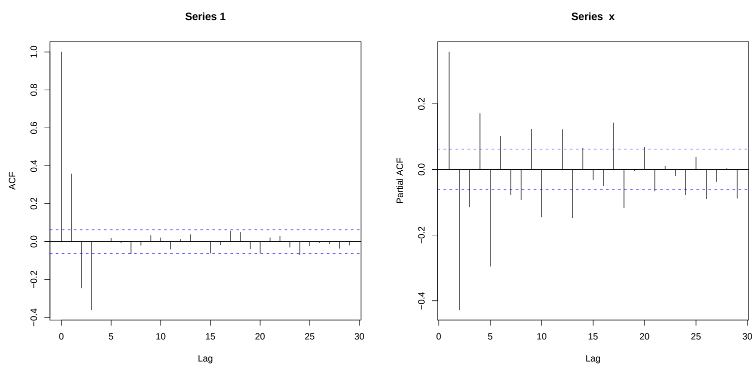
Time series 2



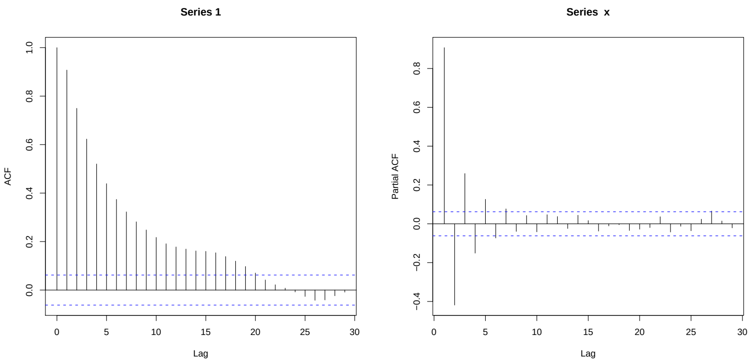
Time series 3



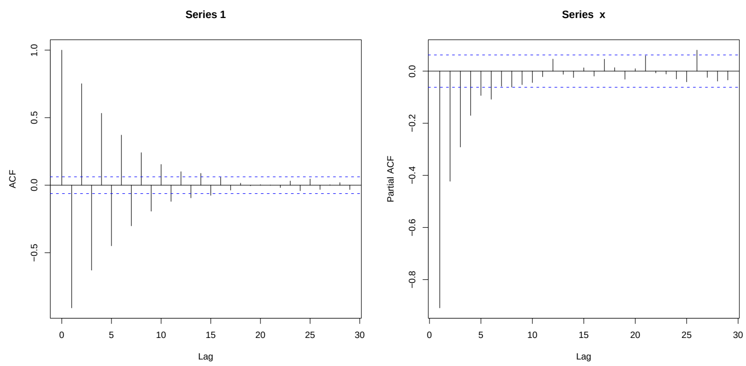
Time series 4



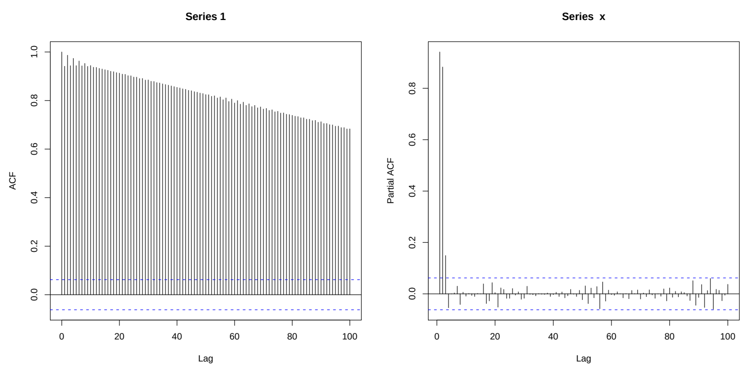
Time series 5



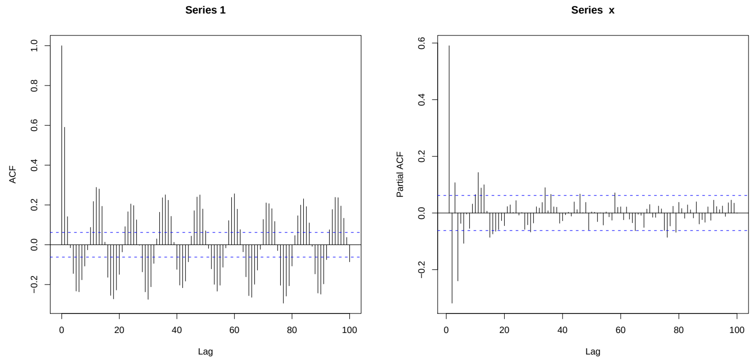
Time series 6



Time series 7



Time series 8



Time series 9