High Dimensional Probability Notes

Nong Minh Hieu[1]

[1] School of Physical and Mathematical Sciences, Nanyang Technological University (NTU - Singapore)

Contents

1 Random variables

1.1 Basic inequalities

First, we revisit the definition of a random variable as well as some basic inequalities that we learned in introductory statistics.

Definition 1.1.
Random variable Let $(\Omega, \Sigma . \mathbb{P})$ *be a probability space. A random variable* X *is defined as a mapping from the sample space* Ω *to* $\mathbb{R}$*:*

$$X : \Omega \to \mathbb{R} \tag{1}$$

Σ *is the* σ*-algebra containing the possible events (collection of subsets of* Ω*) and* $\mathbb{P}$ *is a probability measure that assigns events with probabilities:*

$$\mathbb{P} : \Sigma \to [0, 1] \tag{2}$$

For a given probability space $(\Omega, \Sigma, \mathbb{P})$ and a random variable $X : \Omega \to \mathbb{R}$, we will use the following basic notations throughout this note:

- $\|X\|_{L^p}$ - The p^{th} root of the p^{th} moment of the random variable X.

$$\|X\|_{L^p} = (\mathbb{E}|X|^p)^{1/p},\ p \in (0, \infty) \tag{3}$$

$$\|X\|_{L^\infty} = \operatorname{ess\,sup} |X| \tag{4}$$

- $L^p(\Omega, \Sigma, \mathbb{P})$ - The space of random variables X satisfying:

$$L^p(\Omega, \Sigma, \mathbb{P}) = \left\{ X : \Omega \to \mathbb{R} \middle| \|X\|_{L^p} < \infty \right\} \tag{5}$$

Some basic inequalities and identities:

- **1. Jensen's Inequality** - For a random variable X and a convex function $\varphi : \mathbb{R} \to \mathbb{R}$, we have:

$$\varphi(\mathbb{E}X) \leq \mathbb{E}\varphi(X) \tag{6}$$

- **2. Monotonicity of L^p norm** - For a random variable X:

$$\|X\|_{L^p} \leq \|X\|_{L^q},\ 0 \leq p \leq q \leq \infty \tag{7}$$

- **3. Minkowski's Inequality** - For $1 \leq p \leq \infty$ and two random variables X, Y in $L^p(\Omega, \Sigma, \mathbb{P})$ space:

$$\|X + Y\|_{L^p} \leq \|X\|_{L^p} + \|Y\|_{L^p} \tag{8}$$

- **4. Holder's Inequality** - For $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$. Then, for random variables $X \in L^p(\Omega, \Sigma, \mathbb{P})$ and $Y \in L^q(\Omega, \Sigma, \mathbb{P})$, we have:

$$|\mathbb{E}XY| \leq \|X\|_{L^p} \cdot \|Y\|_{L^q} \tag{9}$$

- **5. Integral Identity** - Let X be a non-negative random variable, we have:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt \tag{10}$$

A List of Definitions

B Important Theorems

C Important Corollaries

D Important Propositions

E References

References

[1] Rick Durrett. *Probability: Theory and Examples*. 4th. USA: Cambridge University Press, 2010. ISBN: 0521765390.

[2] Erhan undefinedinlar. *Probability and Stochastics*. Springer New York, 2011. ISBN: 9780387878591. DOI: 10.1007/978-0-387-87859-1. URL: http://dx.doi.org/10.1007/978-0-387-87859-1.

[3] Wikipedia. *Vitali set — Wikipedia, The Free Encyclopedia*. http://en.wikipedia.org/w/index.php?title=Vitali%20set&oldid=1187241923. [Online; accessed 24-December-2023]. 2023.