

High Dimensional Probability Notes

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1 Random variables

1.1 Basic Inequalities

First, we revisit the definition of a random variable as well as some basic inequalities that we learned in introductory statistics.

Definition 1.1 (Random variable).

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. A random variable X is defined as a mapping from the sample space Ω to \mathbb{R} :

$$X : \Omega \rightarrow \mathbb{R} \quad (1)$$

Σ is the σ -algebra containing the possible events (collection of subsets of Ω) and \mathbb{P} is a probability measure that assigns events with probabilities:

$$\mathbb{P} : \Sigma \rightarrow [0, 1] \quad (2)$$

For a given probability space $(\Omega, \Sigma, \mathbb{P})$ and a random variable $X : \Omega \rightarrow \mathbb{R}$, we will use the following basic notations throughout this note:

- $\|X\|_{L^p}$ - The p^{th} root of the p^{th} moment of the random variable X .

$$\|X\|_{L^p} = (\mathbb{E}|X|^p)^{1/p}, \quad p \in (0, \infty) \quad (3)$$

$$\|X\|_{L^\infty} = \text{ess sup } |X| \quad (4)$$

- $L^p(\Omega, \Sigma, \mathbb{P})$ - The space of random variables X satisfying:

$$L^p(\Omega, \Sigma, \mathbb{P}) = \left\{ X : \Omega \rightarrow \mathbb{R} \mid \|X\|_{L^p} < \infty \right\} \quad (5)$$

Some basic inequalities and identities:

- **1. Jensen's Inequality** - For a random variable X and a convex function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, we have:

$$\varphi(\mathbb{E}X) \leq \mathbb{E}\varphi(X) \quad (6)$$

- **2. Monotonicity of L^p norm** - For a random variable X :

$$\|X\|_{L^p} \leq \|X\|_{L^q}, \quad 0 \leq p \leq q \leq \infty. \quad (7)$$

- **3. Minkowski's Inequality** - For $1 \leq p \leq \infty$ and two random variables X, Y in $L^p(\Omega, \Sigma, \mathbb{P})$ space:

$$\|X + Y\|_{L^p} \leq \|X\|_{L^p} + \|Y\|_{L^p}. \quad (8)$$

- **4. Holder's Inequality** - For $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$. Then, for random variables $X \in L^p(\Omega, \Sigma, \mathbb{P})$ and $Y \in L^q(\Omega, \Sigma, \mathbb{P})$, we have:

$$|\mathbb{E}XY| \leq \|X\|_{L^p} \cdot \|Y\|_{L^q}. \quad (9)$$

- **5. Markov's Inequality** - For a non-negative random variable X and $t > 0$, we have:

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}X}{t}. \quad (10)$$

We can also generalize Markov's Inequality for p^{th} moment:

$$\mathbb{P}(|X| \geq t) \leq \frac{\mathbb{E}[|X|^p]}{t^p}, \quad \forall t > 0, p \in [2, \infty). \quad (11)$$

- **6. Chebyshev's Inequality** - For a random variable X with mean μ and variance σ^2 . Then, for any $t > 0$, we have:

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}. \quad (12)$$

- **7. Integral Identity** - Let X be a non-negative random variable, we have:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t) dt. \quad (13)$$

Exercises

Exercise 1.1.1: Generalized Integral Identity

Let X be a random variable (not necessarily non-negative). Prove the following identity:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t) dt - \int_{-\infty}^0 \mathbb{P}(X < t) dt. \quad (14)$$

Solution (Exercise 1.1.1).

For $x \in \mathbb{R}$, using the basic integral identity, we have:

$$|x| = \int_0^\infty \mathbf{1}\{t < |x|\} dt$$

We consider the following cases:

- When $x < 0 \implies x = -|x|$:

$$x = - \int_0^\infty \mathbf{1}\{t < |x|\} dt = - \int_0^\infty \mathbf{1}\{t < -x\} dt = - \int_0^\infty \mathbf{1}\{-t > x\} dt = - \int_{-\infty}^0 \mathbf{1}\{t > x\} dt.$$

- When $x \geq 0 \implies x = |x|$:

$$x = \int_0^\infty \mathbf{1}\{t < |x|\} dt = \int_0^\infty \mathbf{1}\{t < x\} dt.$$

Therefore, for $x \in \mathbb{R}$, we can write:

$$x = \int_0^\infty \mathbf{1}\{t < x\} dt - \int_{-\infty}^0 \mathbf{1}\{t > x\} dt.$$

Therefore, for a random variable X not necessarily non-negative, we have:

$$\begin{aligned} \mathbb{E}X &= \mathbb{E} \left[\int_0^\infty \mathbf{1}\{t < X\} dt - \int_{-\infty}^0 \mathbf{1}\{t > X\} dt \right] \\ &= \mathbb{E} \int_0^\infty \mathbf{1}\{t < X\} dt - \mathbb{E} \int_{-\infty}^0 \mathbf{1}\{t > X\} dt \\ &= \int_0^\infty \mathbb{E} \mathbf{1}\{t < X\} dt - \int_{-\infty}^0 \mathbb{E} \mathbf{1}\{t > X\} dt \\ &= \int_0^\infty \mathbb{P}(t < X) dt - \int_{-\infty}^0 \mathbb{P}(t > X) dt. \end{aligned}$$

□.

Exercise 1.1.2: p^{th} -moments via tails

Let X be a random variable and $p \in (0, \infty)$. Show that:

$$\mathbb{E}|X|^p = \int_0^\infty pt^{p-1}\mathbb{P}(|X| > t)dt. \quad (15)$$

Solution (Exercise 1.1.2). _____

Let X be a random variable that is not necessarily non-negative. Using the integral identity, we have:

$$\mathbb{E}|X|^p = \int_0^\infty \mathbb{P}(u < |X|^p)du.$$

Let $t^p = u \implies pt^{p-1}dt = du$. Since we integrate u from $0 \rightarrow \infty$, we also integrate t from $0 \rightarrow \infty$ when changing the variables. Hence, we have:

$$\mathbb{E}|X|^p = \int_0^\infty \mathbb{P}(t^p < |X|^p)pt^{p-1}dt = \int_0^\infty \mathbb{P}(t < |X|)pt^{p-1}dt.$$

Hence, we obtained the desired identity. \square .

1.2 Limit Theorems**1.2.1 Weak Law of Large Numbers****Theorem 1.1: Weak Law of Large Numbers (WLLN)**

Let X_1, \dots, X_N be *i.i.d* random variables with mean μ . Consider the sum:

$$S_N = X_1 + \dots + X_N$$

Then, the sample mean **converges to μ in probability** ($S_N/N \xrightarrow{p} \mu$):

$$\lim_{N \rightarrow \infty} \mathbb{P}\left(|S_N/N - \mu| > \epsilon\right) = 0, \quad \forall \epsilon > 0 \quad (16)$$

Proof (Weak Law of Large Numbers (WLLN)). _____

We split the proof into two sections corresponding to the assumptions of finite variance and non-finite variance.

1. **Finite variance case:** Suppose that $\text{Var}X_i = \sigma^2 < \infty$ for all $1 \leq i \leq N$. Let $\bar{X} = S_N/N$. Then, \bar{X} is a random variable with the following mean and variance:

$$\mathbb{E}\bar{X} = \mu \quad \text{and} \quad \text{Var}\bar{X} = \frac{\sigma^2}{N}.$$

Hence, by the Chebyshev's inequality, we have:

$$\mathbb{P}\left(|S_N/N - \mu| > \epsilon\right) = \mathbb{P}\left(|\bar{X} - \mu| > \epsilon\right) \leq \frac{\sigma^2}{N\epsilon^2}.$$

Therefore, we have:

$$\lim_{N \rightarrow \infty} \mathbb{P}\left(|S_N/N - \mu| > \epsilon\right) \leq \lim_{N \rightarrow \infty} \frac{\sigma^2}{N\epsilon^2} = 0.$$

Hence, we have $\lim_{N \rightarrow \infty} \mathbb{P}\left(|S_N/N - \mu| > \epsilon\right) = 0$ and we obtained **(WLLN)**.

2. **Non-finite variance case:** In this case, we rely on the Levy Continuity Theorem (**LCT**), which relies on the convergence of the characteristic function. For $n \geq 1$, define the sequence of random variable $Y_n = S_n/n$. Hence, we have:

$$\begin{aligned}\varphi_{Y_n}(t) &= \varphi_{S_n/n}(t) \\ &= \varphi_{S_n}(t/n) \\ &= \prod_{i=1}^n \varphi_{X_i}(t/n) = \left[\varphi_X(t/n) \right]^n,\end{aligned}$$

Where $X = X_1 = \dots = X_n$. By Taylor's expansion, we have:

$$\varphi_X(t/n) = 1 + \frac{it\mathbb{E}[X]}{n} + \mathcal{O}(1/n^2) = 1 + \frac{it\mu}{n} + \mathcal{O}(1/n^2).$$

Hence, we have:

$$\lim_{n \rightarrow \infty} \varphi_{Y_n}(t) = \lim_{n \rightarrow \infty} \left(1 + \frac{it\mu}{n} + \mathcal{O}(1/n^2) \right)^n = e^{it\mu}.$$

Therefore, by (**LCT**), we have $Y_n \xrightarrow{p} \mu$.

Remark 1.1 (Taylor expansion of Moment Generating and Characteristic Functions). —————
Given a random variable X . For reference, the following are the Taylor expansions of the Moment Generating Function $M_X(t)$ and the Characteristic Function $\varphi_X(t)$:

$$\begin{aligned}M_X(t) &= \mathbb{E}[e^{tX}] = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \mathbb{E}[X^n], \\ \varphi_X(t) &= \mathbb{E}[e^{itX}] = 1 + \sum_{n=1}^{\infty} \frac{(it)^n}{n!} \mathbb{E}[X^n].\end{aligned}\tag{17}$$

For the sake of my laziness, here are the Taylor expansion for the first three terms of both the MGF and the CF:

$$\begin{aligned}M_X(t) &= 1 + t\mathbb{E}[X] + \frac{t^2}{2}\mathbb{E}[X^2] + \mathcal{O}(t^3), \\ \varphi_X(t) &= 1 + it\mathbb{E}[X] - \frac{t^2}{2}\mathbb{E}[X^2] + \mathcal{O}(t^3).\end{aligned}\tag{18}$$

□.

Theorem 1.2: Levy Continuity Theorem (**LCT**)

Let X_1, X_2, \dots be *i.i.d* random variables. Then:

$$\forall t \in \mathbb{R} : \lim_{n \rightarrow \infty} \varphi_{X_n}(t) = \varphi_X(t) \iff X_n \xrightarrow{d} X,\tag{19}$$

for some random variable X . In a special case where $X = c$ for some $c \in \mathbb{R}$, we have:

$$\forall t \in \mathbb{R} : \lim_{n \rightarrow \infty} \varphi_{X_n}(t) = e^{itc} \iff X_n \xrightarrow{p} c.\tag{20}$$

Proof (Levy Continuity Theorem (**LCT**)). —————
The proof for (**LCT**) can be found in Gut 2004, Section 9.1, Theorem 9.1 and Corollary 9.1 □.

1.2.2 Strong Law of Large Numbers

Theorem 1.3: Strong Law of Large Numbers (SLLN)

Let X_1, \dots, X_N be *i.i.d* random variables with mean μ . Consider the sum:

$$S_N = X_1 + \dots + X_N$$

Then, the sample mean **converges to μ almost surely** ($S_N/N \xrightarrow{a.s} \mu$):

$$\mathbb{P}\left(\limsup_{N \rightarrow \infty} |S_N/N - \mu| > \epsilon\right) = 0, \quad \forall \epsilon > 0 \quad (21)$$

Proof (Strong Law of Large Numbers (SLLN)).

For the sake of simplicity, we will present the proof for (SLLN) with an additional assumption that $\mathbb{E}[|X_n|^4] < \infty, \forall n \geq 1$. The proof for the general case of (SLLN) (also called the Kolmogorov Strong Law) can be found in Gut 2004, Section 6, Theorem 6.1. For convenience, we assume the following:

1. $\mathbb{E}[|X_n|^4] = K < \infty$.
2. $\mathbb{E}[X_n] = 0$. For non-zero mean case, we can set $Y_n = X_n - \mu$ and repeat the same arguments made below.

We aim to prove that $\mathbb{P}\left(\limsup_{N \rightarrow \infty} |S_N/N| > \epsilon\right) = 0$ for any $\epsilon > 0$. Firstly, use the Multinomial formula to expand $\mathbb{E}[S_n]$. The expansion will contain the terms in the following forms:

$$X_i^2, X_i^3 X_j, X_i^2 X_j^2, X_i^2 X_j X_k, X_i X_j X_k X_\ell,$$

where i, j, k, ℓ are distinct indices. By independence, we have:

$$\mathbb{E}[X_i^3 X_j] = \mathbb{E}[X_i^2 X_j X_k] = \mathbb{E}[X_i X_j X_k X_\ell] = 0.$$

As a result, we have the following remaining terms by the Multinomial formula:

$$\begin{aligned} \mathbb{E}[S_n^4] &= \sum_{i=1}^n \mathbb{E}[X_i^4] + \binom{4}{2} \sum_{1 \leq i < j \leq n} \mathbb{E}[X_i^2 X_j^2] \\ &= \sum_{i=1}^n \mathbb{E}[X_i^4] + 6 \underbrace{\sum_{1 \leq i < j \leq n} \mathbb{E}[X_i^2 X_j^2]}_{n(n-1)/2 \text{ terms}} \\ &= nK + 3n(n-1)\mathbb{E}[X_i^2 X_j^2]. \end{aligned}$$

By independence, we have $\mathbb{E}[X_i^2 X_j^2] = \mathbb{E}[X_i^2] \mathbb{E}[X_j^2]$ and for any $1 \leq i \leq n$. Furthermore, we have $\mathbb{E}[X_i^2] = \text{Var}(X_i) + \mu^2 = \sigma^2 + \mu^2$. Therefore:

$$\mathbb{E}[S_n^4] = nK + 3n(n-1)(\sigma^2 + \mu^2) < nK + 3n^2(\sigma^2 + \mu^2).$$

Applying Markov's Inequality with the fourth moment, we have:

$$\begin{aligned} \mathbb{P}(|S_n/n| \geq \epsilon) &= \mathbb{P}(|S_n| \geq n\epsilon) \\ &\leq \frac{\mathbb{E}[S_n^4]}{n^4 \epsilon^4} \\ &< \frac{K}{n^3 \epsilon^4} + \frac{3(\sigma^2 + \mu^2)}{n^2}. \end{aligned}$$

Therefore, we have:

$$\sum_{n=1}^{\infty} \mathbb{P}(|S_n/n| \geq \epsilon) < \frac{K}{\epsilon^4} \sum_{n=1}^{\infty} n^{-3} + 3(\sigma^2 + \mu^2) \sum_{n=1}^{\infty} n^{-2} < \infty \quad (22)$$

Finally, by the Borel-Cantelli Lemma (**BCL**), we have:

$$\mathbb{P}\left(\limsup_{n \rightarrow \infty} |S_n/n| \geq \epsilon\right) = 0, \quad \forall \epsilon > 0.$$

□.

Theorem 1.4: Borel-Cantelli Lemma (**BCL**)

1. First Borel-Cantelli Lemma: Given a probability space $(X, \mathcal{S}, \mathbb{P})$ and a sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{S}$. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, we have:

$$\mathbb{P}\left(\limsup_{n \rightarrow \infty} A_n\right) = 0. \quad (23)$$

2. Second Borel-Cantelli Lemma: On the other hand, if $\sum_{n=1}^{\infty} \mathbb{P}(E_n) = \infty$, we have:

$$\mathbb{P}\left(\limsup_{n \rightarrow \infty} A_n\right) = 1. \quad (24)$$

Proof (Borel-Cantelli Lemma (**BCL**)).

We focus on proving the first Borel-Cantelli lemma. We define another sequence of \mathcal{S} -measurable sets $\{B_n\}_{n=1}^{\infty}$ such that:

$$B_n = \bigcup_{k=n}^{\infty} A_k.$$

Hence, we have $B_{\ell+1} \subset B_{\ell}$ for every $\ell \geq 1$. In other words, B_n is a decreasing sequence of \mathcal{S} -measurable sets. By continuity of measure, we have:

$$\begin{aligned} \mathbb{P}\left(\lim_{n \rightarrow \infty} B_n\right) &= \lim_{n \rightarrow \infty} \mathbb{P}(B_n) \\ &= \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) \quad (\text{By additivity}) \\ &= \sum_{i=1}^{\infty} \mathbb{P}(A_i) - \lim_{n \rightarrow \infty} \sum_{k=1}^n \mathbb{P}(A_k) \\ &= 0. \end{aligned}$$

Furthermore, we have:

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} B_n\right) = \mathbb{P}\left(\lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k\right) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = \mathbb{P}\left(\limsup_{n \rightarrow \infty} A_n\right).$$

Hence proved the first Borel-Cantelli Lemma. To prove the second Borel-Cantelli Lemma, we prove the following:

$$\begin{aligned} 1 - \mathbb{P}\left(\limsup_{n \rightarrow \infty} A_n\right) &= \mathbb{P}\left(\left\{\limsup_{n \rightarrow \infty} A_n\right\}^c\right) \\ &= \mathbb{P}\left(\liminf_{n \rightarrow \infty} A_n^c\right) = 0. \end{aligned}$$

□.

1.2.3 Central Limit Theorem

Theorem 1.5: Central Limit Theorem (CLT)

Let X_1, \dots, X_n be a sequence of *i.i.d* random variables with expected value μ and finite variance σ^2 . Then, we have:

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1) \quad \text{as } n \rightarrow \infty, \quad (25)$$

where $\bar{X}_n = S_n/n$ and $\mathcal{N}(0, 1)$ is the standard normal distribution.

Proof (Central Limit Theorem (CLT)).

We prove this via the Characteristic Function. Let $\bar{Z}_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$, notice that:

$$\bar{Z}_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{X_i - \mu}{\sigma},$$

Let $Z_i = X_i - \mu$ for $1 \leq i \leq n$ and suppose $Z = Z_1 = \dots = Z_n$, we have:

$$\begin{aligned} \varphi_{\bar{Z}_n}(t) &= \varphi_{\sum_{i=1}^n Z_i} \left(\frac{t}{\sqrt{n}} \right) = \left[\varphi_Z \left(\frac{t}{\sqrt{n}} \right) \right]^n \\ &= \left[1 + \frac{it\mathbb{E}[Z]}{\sqrt{n}} - \frac{t^2}{2n} \mathbb{E}[Z^2] + \mathcal{O}(1/n) \right]^n \quad (\text{Taylor's Expansion}) \\ &= \left[1 - \frac{t^2}{2n} + \mathcal{O}(1/n) \right]^n. \end{aligned}$$

The final equality comes from the fact that $\mathbb{E}[Z] = 0$ and $\mathbb{E}[Z^2] = \mathbb{E}[Z]^2 + \text{Var}(Z) = 1$. Finally, we have:

$$\lim_{n \rightarrow \infty} \varphi_{\bar{Z}_n}(t) = \lim_{n \rightarrow \infty} \left[1 - \frac{t^2}{2n} + \mathcal{O}(1/n) \right]^n = e^{-t^2/2}.$$

Since $e^{-t^2/2}$ is the Characteristic Function of the standard normal distribution, by (LCT), we have $\bar{Z}_n \xrightarrow{d} \mathcal{N}(0, 1)$. \square .

1.3 Convergence of Random Variables

In this section, we revise the modes of convergence in random variables.

1.3.1 Convergence in Distribution

Definition 1.2 (Convergence in Distribution).

Given a sequence of real-valued random variables X_1, X_2, \dots with CDFs F_1, F_2, \dots . We say that the sequence converges in distribution to a random variable X with CDF F , denoted $X_n \xrightarrow{d} X$ if:

$$\lim_{n \rightarrow \infty} F_n(x) = F(x), \quad (26)$$

for all $x \in \mathbb{R}$ at which F is continuous. Convergence in distribution can also be referred to as weak convergence in measure theory.

1.3.2 Convergence in Probability

Definition 1.3 (Convergence in Probability). _____

Given a sequence of real-valued random variables X_1, X_2, \dots . We say that the sequence converges in probability to a random variable X , denoted $X_n \xrightarrow{p} X$ if:

$$\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| \geq \epsilon) = 0, \quad \forall \epsilon > 0. \quad (27)$$

We also refer to convergence in probability as convergence in measure in measure theory.

Proposition 1.1: $X_n \xrightarrow{p} X \implies X_n \xrightarrow{d} X$

Let X and the sequence X_1, X_2, \dots be real-valued random variables. If $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{d} X$.

Proof (Proposition 1.1). _____

We first prove the following claim: Let X, Y be random variables, $a \in \mathbb{R}$ and $\epsilon > 0$, the inequality $\mathbb{P}(Y \leq a) \leq \mathbb{P}(X \leq a + \epsilon) + \mathbb{P}(|Y - X| \geq \epsilon)$ holds. We have:

$$\begin{aligned} \mathbb{P}(Y \leq a) &= \mathbb{P}(Y \leq a, X \leq a + \epsilon) + \mathbb{P}(Y \leq a, X \geq a + \epsilon) \\ &\leq \mathbb{P}(X \leq a + \epsilon) + \mathbb{P}(Y - X \leq a - X, a - X \leq -\epsilon) \\ &\leq \mathbb{P}(X \leq a + \epsilon) + \mathbb{P}(Y - X \leq -\epsilon) \\ &\leq \mathbb{P}(X \leq a + \epsilon) + \mathbb{P}(Y - X \leq -\epsilon) + \mathbb{P}(Y - X \geq \epsilon) \\ &= \mathbb{P}(X \leq a + \epsilon) + \mathbb{P}(|Y - X| \geq \epsilon). \end{aligned}$$

Using the above inequality, we have:

$$\mathbb{P}(X \leq a - \epsilon) - \mathbb{P}(|X_n - X| \geq \epsilon) \leq \mathbb{P}(X_n \leq a) \leq \mathbb{P}(X \leq a + \epsilon) + \mathbb{P}(|X_n - X| \geq \epsilon).$$

Taking limits as $n \rightarrow \infty$ from both sides, we have:

$$F_X(a - \epsilon) \leq \lim_{n \rightarrow \infty} F_{X_n}(a) \leq F_X(a + \epsilon).$$

Taking $\epsilon \rightarrow 0^+$, we have $\lim_{n \rightarrow \infty} F_{X_n}(a) = F_X(a)$. □.

Proposition 1.2: $X_n \xrightarrow{d} c \iff X_n \xrightarrow{p} c$

Let $c \in \mathbb{R}$ be a constant and X_1, X_2, \dots be a sequence of real-valued random variables. Then, $X_n \xrightarrow{d} c \iff X_n \xrightarrow{p} c$.

Proof (Proposition 1.2, Pishro-Nik 2014). _____

Since $X_n \xrightarrow{d} c$, we immediately have the following:

$$\begin{aligned} \lim_{n \rightarrow \infty} F_{X_n}(c - \epsilon) &= 0, \\ \lim_{n \rightarrow \infty} F_{X_n}(c + \epsilon/2) &= 1. \end{aligned}$$

Then, for any $\epsilon > 0$, we have:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - c| \geq \epsilon) &= \lim_{n \rightarrow \infty} \mathbb{P}[\mathbb{P}(X_n \leq c - \epsilon) + \mathbb{P}(X_n \geq c + \epsilon)] \\
&= \underbrace{\lim_{n \rightarrow \infty} F_{X_n}(c - \epsilon)}_{=0} + \lim_{n \rightarrow \infty} \mathbb{P}(X_n \geq c + \epsilon) \\
&\leq \lim_{n \rightarrow \infty} \mathbb{P}(X_n \geq c + \epsilon/2) \\
&= 1 - \underbrace{\lim_{n \rightarrow \infty} F_{X_n}(c + \epsilon/2)}_{=1} \\
&= 0.
\end{aligned}$$

From the above, we have $\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - c| \geq \epsilon) = 0$ and $X_n \xrightarrow{P} c$. \square .

1.3.3 Convergence in L^p norm

Definition 1.4 (Convergence in L^p norm).

Given a sequence of random variables X_1, X_2, \dots and a real number $p \in [1, \infty)$. We say that the sequence converges in L^p norm to a random variable X , denoted as $X_n \xrightarrow{L^p} X$ if:

$$\lim_{n \rightarrow \infty} \mathbb{E}|X_n - X|^p = 0. \quad (28)$$

Proposition 1.3: $X_n \xrightarrow{L^p} X \implies X_n \xrightarrow{P} X$

Let $p \geq 1$ and X_1, X_2, \dots be a sequence of real-valued random variables. Let X be a random variable, then, $X_n \xrightarrow{L^p} X \implies X_n \xrightarrow{P} X$.

Proof (Proposition 1.3).

Let $\epsilon > 0$, we have:

$$\begin{aligned}
\mathbb{P}(|X_n - X| \geq \epsilon) &= \mathbb{P}(|X_n - X|^p \geq \epsilon^p) \quad (p \geq 1) \\
&\leq \frac{\mathbb{E}|X_n - X|^p}{\epsilon^p}. \quad (\text{Markov's Inequality})
\end{aligned}$$

Taking the limits from both sides, we have $\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| \geq \epsilon) = 0$ and $X_n \xrightarrow{P} X$. \square .

1.3.4 Almost-sure Convergence

Definition 1.5 (Convergence almost-surely).

Let X_1, X_2, \dots be a sequence of real-valued random variables that map from a sample space Ω . Let X also be a real-valued random variable. We say that X_n converges almost surely to X , denoted as $X_n \xrightarrow{a.s} X$, if:

$$\mathbb{P}\left(\limsup_{n \rightarrow \infty} E_n\right) = 0 \quad \text{where} \quad E_n = \left\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \epsilon\right\}.$$

Remark 1.2 (Consequence of **(BCL)**).

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty \implies X_n \xrightarrow{a.s.} X. \quad (29)$$

Proposition 1.4: $X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{p} X$

Let X_1, X_2, \dots be a sequence of real-valued random variables and also let X be a real valued random variables. If $X_n \xrightarrow{a.s.} X$ then $X_n \xrightarrow{p} X$.

Proof (Proposition 1.4).

Let $f_n : \Omega \rightarrow \mathbb{R}_+$ be a sequence of nonnegative Borel-measurable functions such that $f_n(\omega) = |X_n(\omega) - X(\omega)|$. By Fatou's Lemma (reverse), we have:

$$\begin{aligned} \underbrace{\mathbb{P}\left(\limsup_{n \rightarrow \infty} \{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \epsilon\}\right)}_{=0} &= \int f_n d\mathbb{P} \\ &\geq \limsup_{n \rightarrow \infty} \mathbb{P}(|X_n - X| \geq \epsilon) \\ &\geq \lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| \geq \epsilon). \end{aligned}$$

Hence, we have $\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| \geq \epsilon) = 0$ and $X_n \xrightarrow{p} X$. \square .

Theorem 1.6: Continuous Mapping Theorem (CMT)

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and X_1, X_2, \dots be a sequence of real-valued random variables. Then, the following statements hold true:

1. $X_n \xrightarrow{d} X \implies f(X_n) \xrightarrow{d} f(X)$.
2. $X_n \xrightarrow{p} X \implies f(X_n) \xrightarrow{p} f(X)$.
3. $X_n \xrightarrow{a.s.} X \implies f(X_n) \xrightarrow{a.s.} f(X)$.

Proof (Continuous Mapping Theorem **(CMT)**).

Since almost-sure convergence implies the other two modes of convergence, we only have to handle the almost-sure convergence case. Since f is continuous, for any $\omega \in \Omega$ such that $X_n(\omega) \rightarrow X(\omega)$, we have $f(X_n(\omega)) \rightarrow f(X(\omega))$. Therefore, we have:

$$\{\omega \in \Omega : X_n(\omega) \rightarrow X(\omega)\} \subseteq \{\omega \in \Omega : f(X_n(\omega)) \rightarrow f(X(\omega))\}.$$

Therefore, we have:

$$\begin{aligned} &\mathbb{P}\left(\limsup_{n \rightarrow \infty} \left\{\omega \in \Omega : |f(X_n(\omega)) - f(X(\omega))| \leq \epsilon\right\}\right) \\ &\geq \mathbb{P}\left(\limsup_{n \rightarrow \infty} \left\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \leq \epsilon\right\}\right) = 1, \end{aligned}$$

for all $\epsilon > 0$. Therefore, we have $f(X_n) \xrightarrow{a.s.} f(X)$. \square .

2 Statistical Inference

2.1 Sufficiency & Likelihood

2.1.1 Sufficiency

Definition 2.1 (Sufficient Statistics).

Let $\mathbf{X} = (X_1, \dots, X_n) \sim p(\cdot; \theta)$ be a random sample drawn i.i.d from a distribution with parameters θ . Let $\mathbf{U} = T(\mathbf{X})$ be a statistic, then it is called a sufficient statistic if the conditional distribution $p_{\mathbf{X}|\mathbf{U}}$ does not depend on θ .

Example 2.1 (Bernoulli random variables).

Let $\mathbf{X} = (X_1, \dots, X_n) \sim \text{Bernoulli}(\theta)$ be a random sample from the Bernoulli distribution. Let $\mathbf{U} = \frac{1}{n} \sum_{i=1}^n X_i$, then \mathbf{U} is a sufficient statistic of θ . To illustrate this, suppose that $\mathbf{x} = (x_1, \dots, x_n)$ is an observation of the random sample \mathbf{X} and $\mathbf{u} = \frac{1}{n} \sum_{i=1}^n x_i$. We have:

$$\begin{aligned} \mathbb{P}(\mathbf{X} = \mathbf{x} | \mathbf{U} = \mathbf{u}) &= \frac{\mathbb{P}(\mathbf{X} = \mathbf{x}, \mathbf{U} = \mathbf{u})}{\mathbb{P}(\mathbf{U} = \mathbf{u})} \\ &= \frac{\mathbb{P}(X_1 = x_1, \dots, X_n = x_n, \sum_{i=1}^n X_i = \sum_{i=1}^n x_i)}{\mathbb{P}(\sum_{i=1}^n X_i = \sum_{i=1}^n x_i)} \\ &= \frac{\mathbb{P}(X_1 = x_1, \dots, X_n = x_n)}{\mathbb{P}(\sum_{i=1}^n X_i = \sum_{i=1}^n x_i)} \\ &= \frac{\theta^{\sum_{i=1}^n x_i} (1 - \theta)^{n - \sum_{i=1}^n x_i}}{\mathbb{P}(\sum_{i=1}^n X_i = \sum_{i=1}^n x_i)}. \end{aligned}$$

Now, setting $k = \sum_{i=1}^n x_i$, The denominator is basically the probability that the Bernoulli variables sums up to k . Hence, we can calculate the denominator as follows:

$$\mathbb{P}\left(\sum_{i=1}^n X_i = k\right) = \binom{n}{k} \theta^k (1 - \theta)^{n-k}.$$

Therefore, we have:

$$\mathbb{P}(\mathbf{X} = \mathbf{x} | \mathbf{U} = \mathbf{u}) = \frac{\theta^k (1 - \theta)^{n-k}}{\binom{n}{k} \theta^k (1 - \theta)^{n-k}} = \frac{1}{\binom{n}{k}}.$$

Therefore, the conditional distribution does not depend on θ and \mathbf{U} is a sufficient statistic.

Definition 2.2 (Sufficiency Principle).

If $\mathbf{U} = T(\mathbf{X})$ is a sufficient statistic for θ , then any inference about θ should only depend on the sample \mathbf{X} through \mathbf{U} . In other words, if we estimate θ using an estimator $\hat{\theta}$, only \mathbf{U} shows up in the formula of $\hat{\theta}$, not the sample \mathbf{X} itself. We will see why this is the case in the Factorisation Theorem (**FacT**), which states that we can factorise the density function into a function of \mathbf{U}, θ and a function of the observations \mathbf{x} and thus, the inference about θ is independent of the observations \mathbf{x} .

Theorem 2.1: Factorisation Theorem (FactT)

Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random sample with joint density function $p(\mathbf{x}; \boldsymbol{\theta})$ over \mathcal{X}^n . The statistic $\mathbf{U} = T(\mathbf{X})$ is sufficient for the parameters $\boldsymbol{\theta}$ if and only if we can find functions h, g such that:

$$p(\mathbf{x}; \boldsymbol{\theta}) = g(T(\mathbf{x}), \boldsymbol{\theta})h(\mathbf{x}),$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $\boldsymbol{\theta} \in \Theta$.

Proof (Factorisation Theorem (FactT)).

We have to conduct the proof in both directions.

- $T(\mathbf{X})$ is sufficient \implies Factorisation exists: Let $\mathbf{U} = T(\mathbf{X})$ be a sufficient statistics and $\mathbf{u} = T(\mathbf{x})$ be the statistics evaluated on the observations \mathbf{x} . Then, we have:

$$\begin{aligned} p(\mathbf{x}; \boldsymbol{\theta}) &= \mathbb{P}(\mathbf{X} = \mathbf{x}; \boldsymbol{\theta}) \\ &= \mathbb{P}(\mathbf{X} = \mathbf{x} | \mathbf{U} = \mathbf{u}; \boldsymbol{\theta}) \mathbb{P}(\mathbf{U} = \mathbf{u}; \boldsymbol{\theta}). \end{aligned}$$

Since $\mathbf{U} = T(\mathbf{X})$ is a sufficient statistics, $\mathbb{P}(\mathbf{X} = \mathbf{x} | \mathbf{U} = \mathbf{u}; \boldsymbol{\theta})$ does not depend on $\boldsymbol{\theta}$. Hence, we denote $h(\mathbf{x}) = \mathbb{P}(\mathbf{X} = \mathbf{x} | \mathbf{U} = \mathbf{u}; \boldsymbol{\theta})$. Furthermore, $\mathbb{P}(\mathbf{U} = \mathbf{u}; \boldsymbol{\theta})$ is a function of \mathbf{u} and $\boldsymbol{\theta}$. We denote this function as $g(\mathbf{u}, \boldsymbol{\theta})$ and conclude that the factorisation $p(\mathbf{x}; \boldsymbol{\theta}) = h(\mathbf{x})g(T(\mathbf{x}), \boldsymbol{\theta})$ indeed exists.

- Factorisation exists $\implies T(\mathbf{X})$ is sufficient: Suppose that there exists g, h such that $p(\mathbf{x}; \boldsymbol{\theta}) = g(T(\mathbf{x}), \boldsymbol{\theta})h(\mathbf{x})$. We then have:

$$\mathbb{P}(\mathbf{X} = \mathbf{x} | \mathbf{U} = \mathbf{u}; \boldsymbol{\theta}) = \frac{p(\mathbf{x}; \boldsymbol{\theta})}{\mathbb{P}(\mathbf{U} = \mathbf{u}; \boldsymbol{\theta})} = \frac{g(\mathbf{u}, \boldsymbol{\theta})h(\mathbf{x})}{\mathbb{P}(\mathbf{U} = \mathbf{u}; \boldsymbol{\theta})}.$$

We denote $A_{\mathbf{u}} = \{\tilde{\mathbf{x}} \in \mathcal{X}^n : T(\tilde{\mathbf{x}}) = \mathbf{u}\}$. We have:

$$\begin{aligned} \mathbb{P}(\mathbf{U} = \mathbf{u}; \boldsymbol{\theta}) &= \sum_{\tilde{\mathbf{x}} \in A_{\mathbf{u}}} \mathbb{P}(\mathbf{X} = \tilde{\mathbf{x}}) \\ &= \sum_{\tilde{\mathbf{x}} \in A_{\mathbf{u}}} p(\tilde{\mathbf{x}}; \boldsymbol{\theta}) = \sum_{\tilde{\mathbf{x}} \in A_{\mathbf{u}}} g(T(\tilde{\mathbf{x}}), \boldsymbol{\theta})h(\tilde{\mathbf{x}}) \\ &= g(\mathbf{u}, \boldsymbol{\theta}) \sum_{\tilde{\mathbf{x}} \in A_{\mathbf{u}}} h(\tilde{\mathbf{x}}). \end{aligned}$$

From the above, we have:

$$\mathbb{P}(\mathbf{X} = \mathbf{x} | \mathbf{U} = \mathbf{u}; \boldsymbol{\theta}) = \frac{h(\mathbf{x})}{\sum_{\tilde{\mathbf{x}} \in A_{\mathbf{u}}} h(\tilde{\mathbf{x}})},$$

and the above expression does not depend on $\boldsymbol{\theta}$. Hence, $T(\mathbf{X})$ is a sufficient statistics.

2.1.2 Likelihood

Definition 2.3 (Likelihood Function).

Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random sample from a distribution $p(\cdot | \theta)$ that depends on parameters $\theta \in \Theta$. Let $\mathbf{x} = (x_1, \dots, x_n)$ be an observation of the random sample \mathbf{X} . Then, the likelihood function $L(\theta; \mathbf{x})$ is defined as follows:

$$L(\theta; \mathbf{x}) = \prod_{i=1}^n p(x_i; \theta), \quad \theta \in \Theta. \quad (30)$$

In some cases, we also use the log-likelihood function:

$$\ell(\theta; \mathbf{x}) = \log L(\theta; \mathbf{x}) = \sum_{i=1}^n \log p(x_i; \theta), \quad \theta \in \Theta. \quad (31)$$

Essentially, $L(\theta; \mathbf{x})$ quantifies the likelihood that θ generates the observations \mathbf{x} . In a way, it is the inverse of probability density (mass) functions, we can see the contrast as follows:

- **Probability Density Function:** The parameters are fixed but the observations are random.
- **Likelihood Function:** The observations are fixed but the parameters are variable.

Definition 2.4.

Given $\mathbf{X} = (X_1, \dots, X_n)$ be a random sample from a distribution $p(\cdot; \theta)$ that depends on $\theta \in \Theta$ and $\mathbf{x} = (x_1, \dots, x_n)$ be an observation of \mathbf{X} . The Maximum Likelihood Estimator $\theta_{MLE} \in \Theta$ is the parameter that maximizes the likelihood function:

$$\theta_{MLE} = \arg \max_{\theta \in \Theta} L(\theta; \mathbf{x}). \quad (32)$$

Proposition 2.1: Properties of Maximum Likelihood Estimator

Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random sample from a distribution dependent on a true set of parameters $\boldsymbol{\theta}$. Then, the estimator

$$\theta_{MLE} = \arg \max_{\theta \in \Theta} L(\theta; \mathbf{X}), \quad (33)$$

which is a random variable, has the following properties:

1. θ_{MLE} is asymptotically consistent, meaning:

$$\lim_{n \rightarrow \infty} \mathbb{P}(|\theta_{MLE} - \boldsymbol{\theta}| \geq \epsilon) = 0, \quad \forall \epsilon > 0. \quad (34)$$

2. θ_{MLE} is asymptotically unbiased:

$$\lim_{n \rightarrow \infty} \mathbb{E}[\theta_{MLE}] = \boldsymbol{\theta}. \quad (35)$$

3. θ_{MLE} is asymptotically normal:

$$\frac{\theta_{MLE} - \boldsymbol{\theta}}{\sqrt{\text{Var}(\theta_{MLE})}} \xrightarrow{d} \mathcal{N}(0, 1). \quad (36)$$

Proof (Proposition 2.1).

For properties (1) and (2), we can just prove the consistency of MLE because consistency implies asymptotic unbiasedness (We will provide a proof for this claim as a lemma below).

1. **Consistency:** By **(SLLN)**, for any $\theta \in \Theta$, we have the following

$$n^{-1} \log L(\theta; \mathbf{X}) = \frac{1}{n} \sum_{i=1}^n \log p(X_i; \theta) \xrightarrow{a.s.} \mathbb{E}_{\boldsymbol{\theta}}[\log p(X; \theta)], \quad (37)$$

where $\mathbb{E}_{\boldsymbol{\theta}}$ denotes the expectation taken over the distribution corresponding to the true parameters $\boldsymbol{\theta}$. We denote $q(\theta_2|\theta_1)$ as $\mathbb{E}_{\theta_1}[\log p(X; \theta_2)]$. Then, we want to prove that

$\mathbb{E}_{\boldsymbol{\theta}}[\log p(X; \boldsymbol{\theta})] \geq \mathbb{E}_{\boldsymbol{\theta}}[\log p(X; \theta)]$ for all $\theta \in \Theta$ and $X \sim p(.|\boldsymbol{\theta})$. To do so, we prove the following inequality:

$$\mathbb{E}_{\boldsymbol{\theta}} \left[\frac{\log p(X; \theta)}{\log p(X; \boldsymbol{\theta})} \right] \leq 0, \quad \forall \theta \in \Theta. \quad (38)$$

□.

2.2 Point Estimation

2.2.1 Bias, Variance, Consistency and MSE

2.2.2 Sufficient Statistics & Rao-Blackwell Theorem

Theorem 2.2: Rao-Blackwell Theorem (RB)

2.2.3 Estimator Variance & Cramer-Rao Lower Bound

Definition 2.5 (Fisher Information).

Let $\mathbf{X} = (X_1, \dots, X_n) \sim p(.; \boldsymbol{\theta})$ be a random sample from a distribution parameterized by $\boldsymbol{\theta}$. The (total) Fisher Information about θ in the random sample \mathbf{X} is defined as follows:

$$\mathcal{I}_{\mathbf{X}}(\boldsymbol{\theta}) = \mathbb{E}_{\mathbf{X}} \left[\left(\frac{\partial}{\partial \theta} \log L(\theta; \mathbf{X}) \right)^2 \mid \boldsymbol{\theta} \right]. \quad (39)$$

The Fisher Informaton is the total information about $\boldsymbol{\theta}$ contained in the sample \mathbf{X} .

Theorem 2.3: Cramer-Rao Lower Bound (CRLB)

2.2.4 Maximum Likelihood Estimation (MLE)

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