

# High Dimensional Probability Notes

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# 1 Random variables

## 1.1 Basic inequalities

First, we revisit the definition of a random variable as well as some basic inequalities that we learned in introductory statistics.

### Definition 1.1.

*Random variable Let  $(\Omega, \Sigma, \mathbb{P})$  be a probability space. A random variable  $X$  is defined as a mapping from the sample space  $\Omega$  to  $\mathbb{R}$ :*

$$X : \Omega \rightarrow \mathbb{R} \quad (1)$$

$\Sigma$  is the  $\sigma$ -algebra containing the possible events (collection of subsets of  $\Omega$ ) and  $\mathbb{P}$  is a probability measure that assigns events with probabilities:

$$\mathbb{P} : \Sigma \rightarrow [0, 1] \quad (2)$$

For a given probability space  $(\Omega, \Sigma, \mathbb{P})$  and a random variable  $X : \Omega \rightarrow \mathbb{R}$ , we will use the following basic notations throughout this note:

- $\|X\|_{L^p}$  - The  $p^{th}$  root of the  $p^{th}$  moment of the random variable  $X$ .

$$\|X\|_{L^p} = (\mathbb{E}|X|^p)^{1/p}, \quad p \in (0, \infty) \quad (3)$$

$$\|X\|_{L^\infty} = \text{ess sup } |X| \quad (4)$$

- $L^p(\Omega, \Sigma, \mathbb{P})$  - The space of random variables  $X$  satisfying:

$$L^p(\Omega, \Sigma, \mathbb{P}) = \left\{ X : \Omega \rightarrow \mathbb{R} \mid \|X\|_{L^p} < \infty \right\} \quad (5)$$

Some basic inequalities and identities:

- **1. Jensen's Inequality** - For a random variable  $X$  and a convex function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ , we have:

$$\varphi(\mathbb{E}X) \leq \mathbb{E}\varphi(X) \quad (6)$$

- **2. Monotonicity of  $L^p$  norm** - For a random variable  $X$ :

$$\|X\|_{L^p} \leq \|X\|_{L^q}, \quad 0 \leq p \leq q \leq \infty \quad (7)$$

- **3. Minkowski's Inequality** - For  $1 \leq p \leq \infty$  and two random variables  $X, Y$  in  $L^p(\Omega, \Sigma, \mathbb{P})$  space:

$$\|X + Y\|_{L^p} \leq \|X\|_{L^p} + \|Y\|_{L^p} \quad (8)$$

- **4. Holder's Inequality** - For  $p, q \in [1, \infty]$  such that  $1/p + 1/q = 1$ . Then, for random variables  $X \in L^p(\Omega, \Sigma, \mathbb{P})$  and  $Y \in L^q(\Omega, \Sigma, \mathbb{P})$ , we have:

$$|\mathbb{E}XY| \leq \|X\|_{L^p} \cdot \|Y\|_{L^q} \quad (9)$$

- **5. Markov's Inequality** - For a non-negative random variable  $X$  and  $t > 0$ , we have:

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}X}{t} \quad (10)$$

- **6. Chebyshev's Inequality** - For a random variable  $X$  with mean  $\mu$  and variance  $\sigma^2$ . Then, for any  $t > 0$ , we have:

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2} \quad (11)$$

- **7. Integral Identity** - Let  $X$  be a non-negative random variable, we have:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt \quad (12)$$

## Exercises

### Exercise 1.1.1: Generalized Integral Identity

Let  $X$  be a random variable (not necessarily non-negative). Prove the following identity:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt - \int_{-\infty}^0 \mathbb{P}(X < t)dt \quad (13)$$

**Solution** (Exercise 1.1.1). \_\_\_\_\_  
□.

### Exercise 1.1.2: $p^{th}$ -moments via tails

Let  $X$  be a random variable and  $p \in (0, \infty)$ . Show that:

$$\mathbb{E}|X|^p = \int_0^\infty pt^{p-1}\mathbb{P}(|X| > t)dt \quad (14)$$

**Solution** (Exercise 1.1.2). \_\_\_\_\_  
□.

## 1.2 Limit Theorems

## B List of Definitions

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## B Important Theorems

## C Important Corollaries

## D Important Propositions

## E References