High Dimensional Probability Notes

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Contents

1 Random variables

1.1 Basic inequalities

First, we revisit the definition of a random variable as well as some basic inequalities that we learned in introductory statistics.

Definition 1.1 (Random variable).

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. A random variable X is defined as a mapping from the sample space Ω to $\mathbb{R}$:

$$X : \Omega \to \mathbb{R} \tag{1}$$

Σ is the σ-algebra containing the possible events (collection of subsets of Ω) and $\mathbb{P}$ is a probability measure that assigns events with probabilities:

$$\mathbb{P} : \Sigma \to [0, 1] \tag{2}$$

For a given probability space $(\Omega, \Sigma, \mathbb{P})$ and a random variable $X : \Omega \to \mathbb{R}$, we will use the following basic notations throughout this note:

- $\|X\|_{L^p}$ - The p^{th} root of the p^{th} moment of the random variable X.

$$\|X\|_{L^p} = (\mathbb{E}|X|^p)^{1/p}, \ p \in (0, \infty) \tag{3}$$

$$\|X\|_{L^\infty} = \operatorname{ess\,sup} |X| \tag{4}$$

- $L^p(\Omega, \Sigma, \mathbb{P})$ - The space of random variables X satisfying:

$$L^p(\Omega, \Sigma, \mathbb{P}) = \left\{ X : \Omega \to \mathbb{R} \middle| \|X\|_{L^p} < \infty \right\} \tag{5}$$

Some basic inequalities and identities:

- **1. Jensen's Inequality** - For a random variable X and a convex function $\varphi : \mathbb{R} \to \mathbb{R}$, we have:

$$\varphi(\mathbb{E}X) \leqslant \mathbb{E}\varphi(X) \tag{6}$$

- **2. Monotonicity of L^p norm** - For a random variable X:

$$\|X\|_{L^p} \leqslant \|X\|_{L^q}, \ 0 \leqslant p \leqslant q \leqslant \infty \tag{7}$$

- **3. Minkowski's Inequality** - For $1 \leqslant p \leqslant \infty$ and two random variables X, Y in $L^p(\Omega, \Sigma, \mathbb{P})$ space:

$$\|X + Y\|_{L^p} \leqslant \|X\|_{L^p} + \|Y\|_{L^p} \tag{8}$$

- **4. Holder's Inequality** - For $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$. Then, for random variables $X \in L^p(\Omega, \Sigma, \mathbb{P})$ and $Y \in L^q(\Omega, \Sigma, \mathbb{P})$, we have:

$$|\mathbb{E}XY| \leqslant \|X\|_{L^p} \cdot \|Y\|_{L^q} \tag{9}$$

- **5. Markov's Inequality** - For a non-negative random variable X and $t > 0$, we have:

$$\mathbb{P}(X \geqslant t) \leqslant \frac{\mathbb{E}X}{t} \tag{10}$$

- **6. Chebyshev's Inequality** - For a random variable X with mean μ and variance σ^2. Then, for any $t > 0$, we have:

$$\mathbb{P}(|X - \mu| \geqslant t) \leqslant \frac{\sigma^2}{t^2} \tag{11}$$

- **7. Integral Identity** - Let X be a non-negative random variable, we have:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt \tag{12}$$

Exercises

Exercise 1.1.1: Generalized Integral Identity

Let X be a random variable (not necessarily non-negative). Prove the following identity:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt - \int_{-\infty}^0 \mathbb{P}(X < t)dt \tag{13}$$

Solution (Exercise 1.1.1).

For $x \in \mathbb{R}$, using the basic integral indentity, we have:

$$|x| = \int_0^\infty \mathbf{1}\{t < |x|\}dt$$

We consider the following cases:

- *When $x < 0 \implies x = -|x|$:*

$$x = -\int_0^\infty \mathbf{1}\{t < |x|\}dt = -\int_0^\infty \mathbf{1}\{t < -x\}dt = -\int_0^\infty \mathbf{1}\{-t > x\}dt = -\int_{-\infty}^0 \mathbf{1}\{t > x\}dt$$

- *When $x \geqslant 0 \implies x = |x|$:*

$$x = \int_0^\infty \mathbf{1}\{t < |x|\}dt = \int_0^\infty \mathbf{1}\{t < x\}dt$$

Therefore, for $x \in \mathbb{R}$, we can write:

$$x = \int_0^\infty \mathbf{1}\{t < x\}dt - \int_{-\infty}^0 \mathbf{1}\{t > x\}dt$$

Therefore, for a random variable X not necessarily non-negative, we have:

$$\begin{aligned}
\mathbb{E}X &= \mathbb{E}\left[\int_0^\infty \mathbf{1}\{t < X\}dt - \int_{-\infty}^0 \mathbf{1}\{t > X\}dt\right] \\
&= \mathbb{E}\int_0^\infty \mathbf{1}\{t < X\}dt - \mathbb{E}\int_{-\infty}^0 \mathbf{1}\{t > X\}dt \\
&= \int_0^\infty \mathbb{E}\mathbf{1}\{t < X\}dt - \int_{-\infty}^0 \mathbb{E}\mathbf{1}\{t > X\}dt \\
&= \int_0^\infty \mathbb{P}(t < X)dt - \int_{-\infty}^0 \mathbb{P}(t > X)dt
\end{aligned}$$

$\square$.

Exercise 1.1.2: p^{th}-moments via tails

Let X be a random variable and $p \in (0, \infty)$. Show that:

$$\mathbb{E}|X|^p = \int_0^\infty pt^{p-1}\mathbb{P}(|X| > t)dt \tag{14}$$

Solution (Exercise 1.1.2).

Let X be a random variable that is not necessarily non-negative. Using the integral identity, we have:

$$\mathbb{E}|X|^p = \int_0^\infty \mathbb{P}(u < |X|^p)du$$

Let $t^p = u \implies pt^{p-1}dt = du$. Since we integrate u from $0 \to \infty$, we also integrate t from $0 \to \infty$ when changing the variables. Hence, we have:

$$\mathbb{E}|X|^p = \int_0^\infty \mathbb{P}(t^p < |X|^p)pt^{p-1}dt = \int_0^\infty \mathbb{P}(t < |X|)pt^{p-1}dt$$

Hence, we obtained the desired identity. $\square$.

1.2 Limit Theorems

Theorem 1.1: Weak Law of Large Numbers (WLLN)

Let $X_1, \ldots, X_N$ be $i.i.d$ random variables with mean μ. Consider the sum:

$$S_N = X_1 + \cdots + X_N$$

Then, the sample mean **converges to μ in probability** ($S_N/N \xrightarrow{p} \mu$):

$$\lim_{N\to\infty} \mathbb{P}\Big(|S_N/N - \mu| > \epsilon\Big) = 0, \ \forall\epsilon > 0 \tag{15}$$

Proof (Weak Law of Large Numbers (**WLLN**)).

Suppose that $\text{Var}X_i = \sigma^2$ *for all* $1 \leqslant i \leqslant N$. *Let* $\bar{X} = S_N/N$. *Then,* $\bar{X}$ *is a random variable with the following mean and variance:*

$$\mathbb{E}\bar{X} = \mu \quad \textit{and} \quad \text{Var}\bar{X} = \frac{\sigma^2}{N}.$$

Hence, by the Chebyshev's inequality, we have:

$$\mathbb{P}\Big(|S_N/N - \mu| > \epsilon\Big) = \mathbb{P}\Big(|\bar{X} - \mu| > \epsilon\Big) \leqslant \frac{\sigma^2}{N\epsilon^2}.$$

Therefore, we have:

$$\lim_{N\to\infty} \mathbb{P}\Big(|S_N/N - \mu| > \epsilon\Big) \leqslant \lim_{N\to\infty} \frac{\sigma^2}{N\epsilon^2} = 0.$$

Hence, we have $\lim_{N\to\infty} \mathbb{P}\Big(|S_N/N - \mu| > \epsilon\Big) = 0$ *and we obtained* (**WLLN**). $\square$.

Theorem 1.2: Strong Law of Large Numbers (SLLN)

Let $X_1, \dots, X_N$ be *i.i.d* random variables with mean μ. Consider the sum:

$$S_N = X_1 + \cdots + X_N$$

Then, the sample mean **converges to μ almost surely** ($S_N/N \xrightarrow{a.s} \mu$):

$$\mathbb{P}\left(\limsup_{N\to\infty} |S_N/N - \mu| > \epsilon\right) = 0, \ \forall \epsilon > 0 \tag{16}$$

Proof (Strong Law of Large Numbers (**SLLN**)).

We revisit the ***Borel-Cantelli*** *lemma: Given a sequence of events* $\{E_n\}_{n=1}^{\infty}$ *such that its sum of probabilities* $\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty$*. Then, we have:*

$$P\left(\limsup_{n\to\infty} E_n\right) = 0 \tag{17}$$

Let $\epsilon > 0$ *be given. Denote the sequence of events* $\{E_n\}_{n=1}^{\infty}$ *as follows:*

$$E_n = \left\{|S_n/n - \mu| > \epsilon\right\}$$

One of our goal in this proof is to prove that $\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty$ *to apply the Borel-Cantelli lemma.*

1. ***Attempt 1****: Suppose that* X_i *have finite variance. Denote* $\bar{S}_n = S_n/n$*. Then,* $\bar{S}_n$ *is a random variable with* $\mathbb{E}\bar{S}_n = \mu$ *and* $Var(\bar{S}_n) = \sigma^2/n$*. For* $n \geqslant 1$*, we have:*

$$\mathbb{P}(E_n) = \mathbb{P}\left(|\bar{S}_n - \mu| > \epsilon\right) \leqslant \frac{\sigma^2}{n\epsilon^2} \quad \textit{(Chebyshev's Inequality)}$$

However, we cannot use the above inequality because we can only conclude:

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) \leqslant \frac{\sigma^2}{\epsilon^2} \sum_{n=1}^{\infty} n^{-1} \quad \textit{(Divergent sum)}$$

Hence, we move on to our next attemp.

2. ***Attempt 2****: Now, we change the strategy to bound* $\mathbb{P}(E_n)$*. Specifically, for any* $s > 0$*, using the Chernoff bound, we have:*

$$\begin{aligned} P\left(|\bar{S}_n - \mu| > \epsilon\right) &\leqslant M_{|\bar{S}_n - \mu|}(s) e^{-s\epsilon} \\ &= \mathbb{E}\left[e^{s|\bar{S}_n - \mu|}\right] e^{-s\epsilon} \\ &\leqslant e^{s\mathbb{E}|\bar{S}_n - \mu|} e^{-s\epsilon} \end{aligned}$$

Now, we have to bound $\mathbb{E}|\bar{S}_n - \mu|$*. Using the integral identity, we have:*

$$\begin{aligned} \mathbb{E}|\bar{S}_n - \mu| &= \int_0^{\infty} \mathbb{P}(|\bar{S}_n - \mu| > t) dt \\ &\leqslant \frac{\sigma^2}{n} \int_0^{\infty} t^{-2} dt \quad \textit{(Chebyshev's Inequality)} \end{aligned}$$

This does not work either because the above integral is divergent. □.

B List of Definitions

B Important Theorems

C Important Corollaries

D Important Propositions

E References