High Dimensional Probability Notes

Nong Minh Hieu[1]

[1] School of Physical and Mathematical Sciences, Nanyang Technological University (NTU - Singapore)

Contents

1 Random variables

1.1 Basic inequalities

First, we revisit the definition of a random variable as well as some basic inequalities that we learned in introductory statistics.

Definition 1.1 (Random variable).

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. A random variable X is defined as a mapping from the sample space Ω to $\mathbb{R}$:

$$X : \Omega \to \mathbb{R} \tag{1}$$

Σ is the σ-algebra containing the possible events (collection of subsets of Ω) and $\mathbb{P}$ is a probability measure that assigns events with probabilities:

$$\mathbb{P} : \Sigma \to [0, 1] \tag{2}$$

For a given probability space $(\Omega, \Sigma, \mathbb{P})$ and a random variable $X : \Omega \to \mathbb{R}$, we will use the following basic notations throughout this note:

- $\|X\|_{L^p}$ - The p^{th} root of the p^{th} moment of the random variable X.

$$\|X\|_{L^p} = (\mathbb{E}|X|^p)^{1/p},\ p \in (0, \infty) \tag{3}$$
$$\|X\|_{L^\infty} = \operatorname{ess\,sup} |X| \tag{4}$$

- $L^p(\Omega, \Sigma, \mathbb{P})$ - The space of random variables X satisfying:

$$L^p(\Omega, \Sigma, \mathbb{P}) = \left\{ X : \Omega \to \mathbb{R} \middle| \|X\|_{L^p} < \infty \right\} \tag{5}$$

Some basic inequalities and identities:

- **1. Jensen's Inequality** - For a random variable X and a convex function $\varphi : \mathbb{R} \to \mathbb{R}$, we have:

$$\varphi(\mathbb{E}X) \leqslant \mathbb{E}\varphi(X) \tag{6}$$

- **2. Monotonicity of L^p norm** - For a random variable X:

$$\|X\|_{L^p} \leqslant \|X\|_{L^q},\ 0 \leqslant p \leqslant q \leqslant \infty. \tag{7}$$

- **3. Minkowski's Inequality** - For $1 \leqslant p \leqslant \infty$ and two random variables X, Y in $L^p(\Omega, \Sigma, \mathbb{P})$ space:

$$\|X + Y\|_{L^p} \leqslant \|X\|_{L^p} + \|Y\|_{L^p}. \tag{8}$$

- **4. Holder's Inequality** - For $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$. Then, for random variables $X \in L^p(\Omega, \Sigma, \mathbb{P})$ and $Y \in L^q(\Omega, \Sigma, \mathbb{P})$, we have:

$$|\mathbb{E}XY| \leqslant \|X\|_{L^p} \cdot \|Y\|_{L^q}. \tag{9}$$

- **5. Markov's Inequality** - For a non-negative random variable X and $t > 0$, we have:

$$\mathbb{P}(X \geqslant t) \leqslant \frac{\mathbb{E}X}{t}. \tag{10}$$

  We can also generalize Markov's Inequality for p^{th} moment:

$$\mathbb{P}(|X| \geqslant t) \leqslant \frac{\mathbb{E}[|X|^p]}{t^p}, \forall t > 0, k \in [2, \infty). \tag{11}$$

- **6. Chebyshev's Inequality** - For a random variable X with mean μ and variance σ^2. Then, for any $t > 0$, we have:

$$\mathbb{P}(|X - \mu| \geqslant t) \leqslant \frac{\sigma^2}{t^2}. \tag{12}$$

- **7. Integral Identity** - Let X be a non-negative random variable, we have:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt. \tag{13}$$

Exercises

Exercise 1.1.1: Generalized Integral Identity

Let X be a random variable (not necessarily non-negative). Prove the following identity:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt - \int_{-\infty}^0 \mathbb{P}(X < t)dt. \tag{14}$$

Solution (Exercise 1.1.1).

For $x \in \mathbb{R}$, using the basic integral indentity, we have:

$$|x| = \int_0^\infty \mathbf{1}\{t < |x|\}dt$$

We consider the following cases:

- *When $x < 0 \implies x = -|x|$:*

$$x = -\int_0^\infty \mathbf{1}\{t < |x|\}dt = -\int_0^\infty \mathbf{1}\{t < -x\}dt = -\int_0^\infty \mathbf{1}\{-t > x\}dt = -\int_{-\infty}^0 \mathbf{1}\{t > x\}dt.$$

- *When $x \geqslant 0 \implies x = |x|$:*

$$x = \int_0^\infty \mathbf{1}\{t < |x|\}dt = \int_0^\infty \mathbf{1}\{t < x\}dt.$$

Therefore, for $x \in \mathbb{R}$, we can write:

$$x = \int_0^\infty \mathbf{1}\{t < x\}dt - \int_{-\infty}^0 \mathbf{1}\{t > x\}dt.$$

Therefore, for a random variable X not necessarily non-negative, we have:

$$\begin{aligned}
\mathbb{E}X &= \mathbb{E}\left[\int_0^\infty \mathbf{1}\{t < X\}dt - \int_{-\infty}^0 \mathbf{1}\{t > X\}dt\right] \\
&= \mathbb{E}\int_0^\infty \mathbf{1}\{t < X\}dt - \mathbb{E}\int_{-\infty}^0 \mathbf{1}\{t > X\}dt \\
&= \int_0^\infty \mathbb{E}\mathbf{1}\{t < X\}dt - \int_{-\infty}^0 \mathbb{E}\mathbf{1}\{t > X\}dt \\
&= \int_0^\infty \mathbb{P}(t < X)dt - \int_{-\infty}^0 \mathbb{P}(t > X)dt.
\end{aligned}$$

$\square$.

Exercise 1.1.2: p^{th}-moments via tails

Let X be a random variable and $p \in (0, \infty)$. Show that:

$$\mathbb{E}|X|^p = \int_0^\infty pt^{p-1}\mathbb{P}(|X| > t)dt. \tag{15}$$

Solution (Exercise 1.1.2).

Let X be a random variable that is not necessarily non-negative. Using the integral identity, we have:

$$\mathbb{E}|X|^p = \int_0^\infty \mathbb{P}(u < |X|^p)du.$$

Let $t^p = u \implies pt^{p-1}dt = du$. Since we integrate u from $0 \to \infty$, we also integrate t from $0 \to \infty$ when changing the variables. Hence, we have:

$$\mathbb{E}|X|^p = \int_0^\infty \mathbb{P}(t^p < |X|^p)pt^{p-1}dt = \int_0^\infty \mathbb{P}(t < |X|)pt^{p-1}dt.$$

Hence, we obtained the desired identity. □.

1.2 Limit Theorems

1.2.1 Weak Law of Large Numbers

Theorem 1.1: Weak Law of Large Numbers (WLLN)

Let $X_1, \dots, X_N$ be *i.i.d* random variables with mean μ. Consider the sum:

$$S_N = X_1 + \cdots + X_N$$

Then, the sample mean **converges to μ in probability** ($S_N/N \xrightarrow{p} \mu$):

$$\lim_{N\to\infty} \mathbb{P}\Big(|S_N/N - \mu| > \epsilon\Big) = 0, \ \forall \epsilon > 0 \tag{16}$$

Proof (Weak Law of Large Numbers (**WLLN**)).

We split the proof into two sections corresponding to the assumptions of finite variance and non-finite variance.

1. ***Finite variance case**: Suppose that* $\mathrm{Var}X_i = \sigma^2 < \infty$ *for all* $1 \leqslant i \leqslant N$*. Let* $\bar{X} = S_N/N$*. Then,* $\bar{X}$ *is a random variable with the following mean and variance:*

$$\mathbb{E}\bar{X} = \mu \quad \text{and} \quad \mathrm{Var}\bar{X} = \frac{\sigma^2}{N}.$$

Hence, by the Chebyshev's inequality, we have:

$$\mathbb{P}\Big(|S_N/N - \mu| > \epsilon\Big) = \mathbb{P}\Big(|\bar{X} - \mu| > \epsilon\Big) \leqslant \frac{\sigma^2}{N\epsilon^2}.$$

Therefore, we have:

$$\lim_{N\to\infty} \mathbb{P}\Big(|S_N/N - \mu| > \epsilon\Big) \leqslant \lim_{N\to\infty} \frac{\sigma^2}{N\epsilon^2} = 0.$$

Hence, we have $\lim_{N\to\infty} \mathbb{P}\Big(|S_N/N - \mu| > \epsilon\Big) = 0$ *and we obtained* (**WLLN**).

2. ***Non-finite variance case***: *In this case, we rely on the Levy Continuity Theorem* (**LCT**)*, which relies on the convergence of the characteristic function. For* $n \geqslant 1$*, define the sequence of random variable* $Y_n = S_n/n$*. Hence, we have:*

$$\begin{aligned} \varphi_{Y_n}(t) &= \varphi_{S_n/n}(t) \\ &= \varphi_{S_n}(t/n) \\ &= \prod_{i=1}^{n} \varphi_{X_i}(t/n) = \Big[\varphi_X(t/n)\Big]^n, \end{aligned}$$

Where $X = X_1 = \cdots = X_n$*. By Taylor's expansion, we have:*

$$\varphi_X(t/n) = 1 + \frac{it\mathbb{E}[X]}{n} + \mathcal{O}(1/n^2) = 1 + \frac{it\mu}{n} + \mathcal{O}(1/n^2).$$

Hence, we have:

$$\lim_{n\to\infty} \varphi_{Y_n}(t) = \lim_{n\to\infty} \left(1 + \frac{it\mu}{n} + \mathcal{O}(1/n^2)\right)^n = e^{it\mu}.$$

Therefore, by (**LCT**)*, we have* $Y_n \xrightarrow{p} \mu$*.*

Remark 1.1 (Taylor expansion of Moment Generating and Characteristic Functions)**.**
Given a random variable X*. For reference, the following are the Taylor expansions of the Moment Generating Function* $M_X(t)$ *and the Characteristic Function* $\varphi_X(t)$*:*

$$\begin{aligned} M_X(t) &= \mathbb{E}[e^{tX}] = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!}\mathbb{E}[X^n], \\ \varphi_X(t) &= \mathbb{E}[e^{itX}] = 1 + \sum_{n=1}^{\infty} \frac{(it)^n}{n!}\mathbb{E}[X^n]. \end{aligned} \tag{17}$$

□.

Theorem 1.2: Levy Continuity Theorem (LCT)

Let $X_1, X_2, \ldots$ be $i.i.d$ random variables. Then:

$$\forall t \in \mathbb{R} : \lim_{n\to\infty} \varphi_{X_n}(t) = \varphi_X(t) \iff X_n \xrightarrow{d} X, \tag{18}$$

for some random variable X. In a special case where $X = c$ for some $c \in \mathbb{R}$, we have:

$$\forall t \in \mathbb{R} : \lim_{n\to\infty} \varphi_{X_n}(t) = e^{itc} \iff X_n \xrightarrow{p} c. \tag{19}$$

Proof (Levy Continuity Theorem (**LCT**))**.**
The proof for (**LCT**) *can be found in Gut 2004, Section 9.1, Theorem 9.1 and Collorary 9.1* □.

1.2.2 Strong Law of Large Numbers

Theorem 1.3: Strong Law of Large Numbers (SLLN)

Let $X_1, \dots, X_N$ be $i.i.d$ random variables with mean μ. Consider the sum:

$$S_N = X_1 + \cdots + X_N$$

Then, the sample mean **converges to μ almost surely** ($S_N/N \xrightarrow{a.s} \mu$):

$$\mathbb{P}\left(\limsup_{N\to\infty} |S_N/N - \mu| > \epsilon\right) = 0,\ \forall \epsilon > 0 \tag{20}$$

Proof (Strong Law of Large Numbers (**SLLN**)).
For the sake of simplicity, we will present the proof for (**SLLN**) *with an additional assumption that* $\mathbb{E}[|X_n|^4] < \infty, \forall n \geqslant 1$. *The proof for the general case of* (**SLLN**) *(also called the Kolmogorov Strong Law) can be found in Gut 2004, Section 6, Theorem 6.1. For convenience, we assume the following:*

1. $\mathbb{E}[|X_n|^4] = K < \infty$.
2. $\mathbb{E}[X_n] = 0$. *For non-zero mean case, we can set* $Y_n = X_n - \mu$ *and repeat the same arguments made below.*

Then, use the Binomial formula to expand $\mathbb{E}[S_n]$. *The expansion will contain the terms in the following forms:*

$$X_i^2, X_i^3 X_j, X_i^2 X_j^2, X_i^2 X_j X_k, X_i X_j X_k X_\ell,$$

where i, j, k, ℓ *are distinct indices. By independence, we have:*

$$\mathbb{E}[X_i^3 X_j] = \mathbb{E}[X_i^2 X_j] = \mathbb{E}[X_i X_l X_k X_\ell] = 0.$$

As a result, we have the following remaining terms by the Binomial formula:

$$\mathbb{E}[S_n^4]$$

□.

1.2.3 Central Limit Theorem

B List of Definitions

B Important Theorems

C Important Corollaries

D Important Propositions

E References

References

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