# High Dimensional Probability Notes

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# 1 Random variables

## 1.1 Basic inequalities

First, we revisit the definition of a random variable as well as some basic inequalities that we learned in introductory statistics.

**Definition 1.1.**
*Random variable Let $(\Omega, \Sigma.\mathbb{P})$ be a probability space. A random variable $X$ is defined as a mapping from the sample space $\Omega$ to $\mathbb{R}$:*

$$X : \Omega \to \mathbb{R} \tag{1}$$

*$\Sigma$ is the $\sigma$-algebra containing the possible events (collection of subsets of $\Omega$) and $\mathbb{P}$ is a probability measure that assigns events with probabilities:*

$$\mathbb{P} : \Sigma \to [0, 1] \tag{2}$$

For a given probability space $(\Omega, \Sigma, \mathbb{P})$ and a random variable $X : \Omega \to \mathbb{R}$, we will use the following basic notations throughout this note:

- $\|X\|_{L^p}$ - The $p^{th}$ root of the $p^{th}$ moment of the random variable $X$.

$$\|X\|_{L^p} = (\mathbb{E}|X|^p)^{1/p},\ p \in (0, \infty) \tag{3}$$
$$\|X\|_{L^\infty} = \operatorname{ess\,sup}|X| \tag{4}$$

- $L^p(\Omega, \Sigma, \mathbb{P})$ - The space of random variables $X$ satisfying:

$$L^p(\Omega, \Sigma, \mathbb{P}) = \left\{ X : \Omega \to \mathbb{R} \middle| \|X\|_{L^p} < \infty \right\} \tag{5}$$

Some basic inequalities and identities:

- **1. Jensen's Inequality** - For a random variable $X$ and a convex function $\varphi : \mathbb{R} \to \mathbb{R}$, we have:

$$\varphi(\mathbb{E}X) \leqslant \mathbb{E}\varphi(X) \tag{6}$$

- **2. Monotonicity of $L^p$ norm** - For a random variable $X$:

$$\|X\|_{L^p} \leqslant \|X\|_{L^q},\ 0 \leqslant p \leqslant q \leqslant \infty \tag{7}$$

- **3. Minkowski's Inequality** - For $1 \leqslant p \leqslant \infty$ and two random variables $X, Y$ in $L^p(\Omega, \Sigma, \mathbb{P})$ space:

$$\|X + Y\|_{L^p} \leqslant \|X\|_{L^p} + \|Y\|_{L^p} \tag{8}$$

- **4. Holder's Inequality** - For $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$. Then, for random variables $X \in L^p(\Omega, \Sigma, \mathbb{P})$ and $Y \in L^q(\Omega, \Sigma, \mathbb{P})$, we have:

$$|\mathbb{E}XY| \leqslant \|X\|_{L^p} \cdot \|Y\|_{L^q} \tag{9}$$

- **5. Markov's Inequality** - For a non-negative random variable $X$ and $t > 0$, we have:

$$\mathbb{P}(X \geqslant t) \leqslant \frac{\mathbb{E}X}{t} \tag{10}$$

- **6. Chebyshev's Inequality** - For a random variable $X$ with mean $\mu$ and variance $\sigma^2$. Then, for any $t > 0$, we have:

$$\mathbb{P}(|X - \mu| \geqslant t) \leqslant \frac{\sigma^2}{t^2} \tag{11}$$

- **7. Integral Identity** - Let $X$ be a non-negative random variable, we have:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt \tag{12}$$

## Exercises

**Exercise 1.1.1: Generalized Integral Identity**

Let $X$ be a random variable (not necessarily non-negative). Prove the following identity:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt - \int_{-\infty}^0 \mathbb{P}(X < t)dt \tag{13}$$

**Solution** (Exercise 1.1.1).

□.

# B List of Definitions



# B Important Theorems

# C Important Corollaries

# D Important Propositions

## E References