

High Dimensional Probability Notes

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1 Random variables

1.1 Basic inequalities

First, we revisit the definition of a random variable as well as some basic inequalities that we learned in introductory statistics.

Definition 1.1 (Random variable).

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. A random variable X is defined as a mapping from the sample space Ω to \mathbb{R} :

$$X : \Omega \rightarrow \mathbb{R} \quad (1)$$

Σ is the σ -algebra containing the possible events (collection of subsets of Ω) and \mathbb{P} is a probability measure that assigns events with probabilities:

$$\mathbb{P} : \Sigma \rightarrow [0, 1] \quad (2)$$

For a given probability space $(\Omega, \Sigma, \mathbb{P})$ and a random variable $X : \Omega \rightarrow \mathbb{R}$, we will use the following basic notations throughout this note:

- $\|X\|_{L^p}$ - The p^{th} root of the p^{th} moment of the random variable X .

$$\|X\|_{L^p} = (\mathbb{E}|X|^p)^{1/p}, \quad p \in (0, \infty) \quad (3)$$

$$\|X\|_{L^\infty} = \text{ess sup } |X| \quad (4)$$

- $L^p(\Omega, \Sigma, \mathbb{P})$ - The space of random variables X satisfying:

$$L^p(\Omega, \Sigma, \mathbb{P}) = \left\{ X : \Omega \rightarrow \mathbb{R} \mid \|X\|_{L^p} < \infty \right\} \quad (5)$$

Some basic inequalities and identities:

- **1. Jensen's Inequality** - For a random variable X and a convex function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, we have:

$$\varphi(\mathbb{E}X) \leq \mathbb{E}\varphi(X) \quad (6)$$

- **2. Monotonicity of L^p norm** - For a random variable X :

$$\|X\|_{L^p} \leq \|X\|_{L^q}, \quad 0 \leq p \leq q \leq \infty \quad (7)$$

- **3. Minkowski's Inequality** - For $1 \leq p \leq \infty$ and two random variables $X, Y \in L^p(\Omega, \Sigma, \mathbb{P})$:

$$\|X + Y\|_{L^p} \leq \|X\|_{L^p} + \|Y\|_{L^p} \quad (8)$$

- **4. Holder's Inequality** - For $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$. Then, for $X \in L^p(\Omega, \Sigma, \mathbb{P})$ and $Y \in L^q(\Omega, \Sigma, \mathbb{P})$, we have:

$$|\mathbb{E}XY| \leq \|X\|_{L^p} \cdot \|Y\|_{L^q} \quad (9)$$

- **5. Integral identity** - Let X be a non-negative random variable, we have:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t) dt \quad (10)$$

A List of Definitions

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B Important Theorems

C Important Corollaries

D Important Propositions

E References

References

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