High Dimensional Probability Notes

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Contents

1 Random variables

1.1 Basic Inequalities

First, we revisit the definition of a random variable as well as some basic inequalities that we learned in introductory statistics.

Definition 1.1 (Random variable).
Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. A random variable X is defined as a mapping from the sample space Ω to $\mathbb{R}$:

$$X : \Omega \to \mathbb{R} \tag{1}$$

Σ is the σ-algebra containing the possible events (collection of subsets of Ω) and $\mathbb{P}$ is a probability measure that assigns events with probabilities:

$$\mathbb{P} : \Sigma \to [0, 1] \tag{2}$$

For a given probability space $(\Omega, \Sigma, \mathbb{P})$ and a random variable $X : \Omega \to \mathbb{R}$, we will use the following basic notations throughout this note:

- $\|X\|_{L^p}$ - The pth root of the pth moment of the random variable X.

$$\|X\|_{L^p} = (\mathbb{E}|X|^p)^{1/p}, \ p \in (0, \infty) \tag{3}$$
$$\|X\|_{L^\infty} = \operatorname{ess\,sup} |X| \tag{4}$$

- $L^p(\Omega, \Sigma, \mathbb{P})$ - The space of random variables X satisfying:

$$L^p(\Omega, \Sigma, \mathbb{P}) = \left\{ X : \Omega \to \mathbb{R} \,\middle|\, \|X\|_{L^p} < \infty \right\} \tag{5}$$

Some basic inequalities and identities:

- **1. Jensen's Inequality** - For a random variable X and a convex function $\varphi : \mathbb{R} \to \mathbb{R}$, we have:

$$\varphi(\mathbb{E}X) \leqslant \mathbb{E}\varphi(X) \tag{6}$$

- **2. Monotonicity of L^p norm** - For a random variable X:

$$\|X\|_{L^p} \leqslant \|X\|_{L^q}, \ 0 \leqslant p \leqslant q \leqslant \infty. \tag{7}$$

- **3. Minkowski's Inequality** - For $1 \leqslant p \leqslant \infty$ and two random variables X, Y in $L^p(\Omega, \Sigma, \mathbb{P})$ space:

$$\|X + Y\|_{L^p} \leqslant \|X\|_{L^p} + \|Y\|_{L^p}. \tag{8}$$

- **4. Holder's Inequality** - For $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$. Then, for random variables $X \in L^p(\Omega, \Sigma, \mathbb{P})$ and $Y \in L^q(\Omega, \Sigma, \mathbb{P})$, we have:

$$|\mathbb{E}XY| \leqslant \|X\|_{L^p} \cdot \|Y\|_{L^q}. \tag{9}$$

- **5. Markov's Inequality** - For a non-negative random variable X and $t > 0$, we have:

$$\mathbb{P}(X \geqslant t) \leqslant \frac{\mathbb{E}X}{t}. \tag{10}$$

We can also generalize Markov's Inequality for p^{th} moment:

$$\mathbb{P}(|X| \geqslant t) \leqslant \frac{\mathbb{E}[|X|^p]}{t^p}, \forall t > 0, k \in [2, \infty). \tag{11}$$

- **6. Chebyshev's Inequality** - For a random variable X with mean μ and variance σ^2. Then, for any $t > 0$, we have:

$$\mathbb{P}(|X - \mu| \geqslant t) \leqslant \frac{\sigma^2}{t^2}. \tag{12}$$

- **7. Integral Identity** - Let X be a non-negative random variable, we have:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt. \tag{13}$$

Exercises

Exercise 1.1.1: Generalized Integral Identity

Let X be a random variable (not necessarily non-negative). Prove the following identity:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt - \int_{-\infty}^0 \mathbb{P}(X < t)dt. \tag{14}$$

Solution (Exercise 1.1.1). ─────────────────────────────────────
For $x \in \mathbb{R}$, using the basic integral indentity, we have:

$$|x| = \int_0^\infty \mathbf{1}\{t < |x|\}dt$$

We consider the following cases:

- *When $x < 0 \implies x = -|x|$:*

$$x = -\int_0^\infty \mathbf{1}\{t < |x|\}dt = -\int_0^\infty \mathbf{1}\{t < -x\}dt = -\int_0^\infty \mathbf{1}\{-t > x\}dt = -\int_{-\infty}^0 \mathbf{1}\{t > x\}dt.$$

- *When $x \geqslant 0 \implies x = |x|$:*

$$x = \int_0^\infty \mathbf{1}\{t < |x|\}dt = \int_0^\infty \mathbf{1}\{t < x\}dt.$$

Therefore, for $x \in \mathbb{R}$, we can write:

$$x = \int_0^\infty \mathbf{1}\{t < x\}dt - \int_{-\infty}^0 \mathbf{1}\{t > x\}dt.$$

Therefore, for a random variable X not necessarily non-negative, we have:

$$\mathbb{E}X = \mathbb{E}\left[\int_0^\infty \mathbf{1}\{t < X\}dt - \int_{-\infty}^0 \mathbf{1}\{t > X\}dt\right]$$

$$= \mathbb{E}\int_0^\infty \mathbf{1}\{t < X\}dt - \mathbb{E}\int_{-\infty}^0 \mathbf{1}\{t > X\}dt$$

$$= \int_0^\infty \mathbb{E}\mathbf{1}\{t < X\}dt - \int_{-\infty}^0 \mathbb{E}\mathbf{1}\{t > X\}dt$$

$$= \int_0^\infty \mathbb{P}(t < X)dt - \int_{-\infty}^0 \mathbb{P}(t > X)dt.$$

$\square.$

3

Exercise 1.1.2: p^{th}-moments via tails

Let X be a random variable and $p \in (0, \infty)$. Show that:

$$\mathbb{E}|X|^p = \int_0^\infty pt^{p-1}\mathbb{P}(|X| > t)dt. \tag{15}$$

Solution (Exercise 1.1.2). ─────────────────────────────

Let X be a random variable that is not necessarily non-negative. Using the integral identity, we have:

$$\mathbb{E}|X|^p = \int_0^\infty \mathbb{P}(u < |X|^p)du.$$

Let $t^p = u \implies pt^{p-1}dt = du$. Since we integrate u from $0 \to \infty$, we also integrate t from $0 \to \infty$ when changing the variables. Hence, we have:

$$\mathbb{E}|X|^p = \int_0^\infty \mathbb{P}(t^p < |X|^p)pt^{p-1}dt = \int_0^\infty \mathbb{P}(t < |X|)pt^{p-1}dt.$$

Hence, we obtained the desired identity. □.

1.2 Limit Theorems

1.2.1 Weak Law of Large Numbers

Theorem 1.1: Weak Law of Large Numbers (WLLN)

Let $X_1, \ldots, X_N$ be *i.i.d* random variables with mean μ. Consider the sum:

$$S_N = X_1 + \cdots + X_N$$

Then, the sample mean **converges to μ in probability** $(S_N/N \xrightarrow{p} \mu)$:

$$\lim_{N \to \infty} \mathbb{P}\left(|S_N/N - \mu| > \epsilon\right) = 0, \ \forall \epsilon > 0 \tag{16}$$

Proof (Weak Law of Large Numbers (**WLLN**)). ─────────────────────

We split the proof into two sections corresponding to the assumptions of finite variance and non-finite variance.

1. **Finite variance case:** *Suppose that* $\mathrm{Var}X_i = \sigma^2 < \infty$ *for all* $1 \leqslant i \leqslant N$. *Let* $\bar{X} = S_N/N$. *Then, $\bar{X}$ is a random variable with the following mean and variance:*

$$\mathbb{E}\bar{X} = \mu \quad and \quad \mathrm{Var}\bar{X} = \frac{\sigma^2}{N}.$$

Hence, by the Chebyshev's inequality, we have:

$$\mathbb{P}\left(|S_N/N - \mu| > \epsilon\right) = \mathbb{P}\left(|\bar{X} - \mu| > \epsilon\right) \leqslant \frac{\sigma^2}{N\epsilon^2}.$$

Therefore, we have:

$$\lim_{N \to \infty} \mathbb{P}\left(|S_N/N - \mu| > \epsilon\right) \leqslant \lim_{N \to \infty} \frac{\sigma^2}{N\epsilon^2} = 0.$$

Hence, we have $\lim_{N \to \infty} \mathbb{P}\left(|S_N/N - \mu| > \epsilon\right) = 0$ *and we obtained* (**WLLN**).

2. **Non-finite variance case**: *In this case, we rely on the Levy Continuity Theorem* (**LCT**), *which relies on the convergence of the characteristic function. For $n \geqslant 1$, define the sequence of random variable $Y_n = S_n/n$. Hence, we have:*

$$\begin{aligned}
\varphi_{Y_n}(t) &= \varphi_{S_n/n}(t) \\
&= \varphi_{S_n}(t/n) \\
&= \prod_{i=1}^{n} \varphi_{X_i}(t/n) = \left[\varphi_X(t/n)\right]^n,
\end{aligned}$$

Where $X = X_1 = \cdots = X_n$. By Taylor's expansion, we have:

$$\varphi_X(t/n) = 1 + \frac{it\mathbb{E}[X]}{n} + \mathcal{O}(1/n^2) = 1 + \frac{it\mu}{n} + \mathcal{O}(1/n^2).$$

Hence, we have:

$$\lim_{n\to\infty} \varphi_{Y_n}(t) = \lim_{n\to\infty} \left(1 + \frac{it\mu}{n} + \mathcal{O}(1/n^2)\right)^n = e^{it\mu}.$$

Therefore, by (**LCT**), *we have $Y_n \xrightarrow{p} \mu$.*

Remark 1.1 (Taylor expansion of Moment Generating and Characteristic Functions). _____
Given a random variable X. For reference, the following are the Taylor expansions of the Moment Generating Function $M_X(t)$ and the Characteristic Function $\varphi_X(t)$:

$$\begin{aligned}
M_X(t) &= \mathbb{E}[e^{tX}] = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!}\mathbb{E}[X^n], \\
\varphi_X(t) &= \mathbb{E}[e^{itX}] = 1 + \sum_{n=1}^{\infty} \frac{(it)^n}{n!}\mathbb{E}[X^n].
\end{aligned} \tag{17}$$

For the sake of my laziness, here are the Taylor expansion for the first three terms of both the MGF and the CF:

$$\begin{aligned}
M_X(t) &= 1 + t\mathbb{E}[X] + \frac{t^2}{2}\mathbb{E}[X^2] + \mathcal{O}(t^3), \\
\varphi_X(t) &= 1 + it\mathbb{E}[X] - \frac{t^2}{2}\mathbb{E}[X^2] + \mathcal{O}(t^3).
\end{aligned} \tag{18}$$

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Theorem 1.2: Levy Continuity Theorem (**LCT**)

Let $X_1, X_2, \ldots$ be *i.i.d* random variables. Then:

$$\forall t \in \mathbb{R} : \lim_{n\to\infty} \varphi_{X_n}(t) = \varphi_X(t) \iff X_n \xrightarrow{d} X, \tag{19}$$

for some random variable X. In a special case where $X = c$ for some $c \in \mathbb{R}$, we have:

$$\forall t \in \mathbb{R} : \lim_{n\to\infty} \varphi_{X_n}(t) = e^{itc} \iff X_n \xrightarrow{p} c. \tag{20}$$

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Proof (Levy Continuity Theorem (**LCT**)). _____
The proof for (**LCT**) *can be found in Gut 2004, Section 9.1, Theorem 9.1 and Collorary 9.1* □.

1.2.2 Strong Law of Large Numbers

> **Theorem 1.3: Strong Law of Large Numbers** (SLLN)
>
> Let $X_1, \ldots, X_N$ be *i.i.d* random variables with mean $\mu < \infty$. Consider the sum:
>
> $$S_N = X_1 + \cdots + X_N$$
>
> Then, the sample mean **converges to μ almost surely** ($S_N/N \xrightarrow{a.s} \mu$):
>
> $$\mathbb{P}\left(\limsup_{N \to \infty} |S_N/N - \mu| > \epsilon \right) = 0, \ \forall \epsilon > 0 \tag{21}$$

Proof (Strong Law of Large Numbers (**SLLN**)). ──────────────

*For the sake of simplicity, we will present the proof for (**SLLN**) with an additional assumption that $\mathbb{E}[|X_n|^4] < \infty, \forall n \geqslant 1$. The proof for the general case of (**SLLN**) (also called the Kolmogorov Strong Law) can be found in Gut 2004, Section 6, Theorem 6.1. For convenience, we assume the following:*

1. $\mathbb{E}[|X_n|^4] = K < \infty$.

2. $\mathbb{E}[X_n] = 0$. *For non-zero mean case, we can set $Y_n = X_n - \mu$ and repeat the same arguments made below.*

We aim to prove that $\mathbb{P}\left(\limsup_{N \to \infty} |S_N/N| > \epsilon \right) = 0$ for any $\epsilon > 0$. Firstly, use the Multinomial formula to expand $\mathbb{E}[S_n]$. The expansion will contain the terms in the following forms:

$$X_i^2, X_i^3 X_j, X_i^2 X_j^2, X_i^2 X_j X_k, X_i X_j X_k X_\ell,$$

where i, j, k, ℓ are distinct indices. By independence, we have:

$$\mathbb{E}[X_i^3 X_j] = \mathbb{E}[X_i^2 X_j X_k] = \mathbb{E}[X_i X_l X_k X_\ell] = 0.$$

As a result, we have the following remaining terms by the Multinomial formula:

$$\begin{aligned}
\mathbb{E}[S_n^4] &= \sum_{i=1}^{n} \mathbb{E}[X_i^4] + \binom{4}{2} \sum_{1 \leqslant i < j \leqslant n} \mathbb{E}[X_i^2 X_j^2] \\
&= \sum_{i=1}^{n} \mathbb{E}[X_i^4] + 6 \underbrace{\sum_{1 \leqslant i < j \leqslant n} \mathbb{E}[X_i^2 X_j^2]}_{n(n-1)/2 \ terms} \\
&= nK + 3n(n-1)\mathbb{E}[X_i^2 X_j^2].
\end{aligned}$$

By independence, we have $\mathbb{E}[X_i^2 X_j^2] = \mathbb{E}[X_i^2]\mathbb{E}[X_j^2]$ and for any $1 \leqslant i \leqslant n$. Furthermore, we have $\mathbb{E}[X_i^2] = \mathrm{Var}(X_i) + \mu^2 = \sigma^2 + \mu^2$. Therefore:

$$\mathbb{E}[S_n^4] = nK + 3n(n-1)(\sigma^2 + \mu^2) < nK + 3n^2(\sigma^2 + \mu^2).$$

Applying Markov's Inequality with the fourth moment, we have:

$$\begin{aligned}
\mathbb{P}(|S_n/n| \geqslant \epsilon) = \mathbb{P}(|S_n| \geqslant n\epsilon) \\
\leqslant \frac{\mathbb{E}[S_n^4]}{n^4 \epsilon^4} \\
< \frac{K}{n^3 \epsilon^4} + \frac{3(\sigma^2 + \mu^2)}{n^2}.
\end{aligned}$$

6

Therefore, we have:

$$\sum_{n=1}^{\infty} \mathbb{P}(|S_n/n| \geq \epsilon) < \frac{K}{\epsilon^4} \sum_{n=1}^{\infty} n^{-3} + 3(\sigma^2 + \mu^2) \sum_{n=1}^{\infty} n^{-2} < \infty \qquad (22)$$

Finally, by the Borel-Cantelli Lemma (**BCL**)*, we have:*

$$\mathbb{P}\left(\limsup_{n \to \infty} |S_n/n| \geq \epsilon \right) = 0, \quad \forall \epsilon > 0.$$

□.

Theorem 1.4: Borel-Cantelli Lemma (BCL)

1. First Borel-Cantelli Lemma: Given a probability space $(X, \mathcal{S}, \mathbb{P})$ and a sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{S}$. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, we have:

$$\mathbb{P}\left(\limsup_{n \to \infty} A_n \right) = 0. \qquad (23)$$

2. Second Borel-Cantelli Lemma: On the other hand, if $\sum_{n=1}^{\infty} \mathbb{P}(E_n) = \infty$, we have:

$$\mathbb{P}\left(\limsup_{n \to \infty} A_n \right) = 1. \qquad (24)$$

Proof (Borel-Cantelli Lemma (**BCL**)). ————————————————————
We focus on proving the first Borel-Cantelli lemma. We define another sequence of $\mathcal{S}$-measurable sets $\{B_n\}_{n=1}^{\infty}$ such that:

$$B_n = \bigcup_{k=n}^{\infty} A_n.$$

Hence, we have $B_{\ell+1} \subset B_\ell$ for every $\ell \geq 1$. In other words, B_n is a decreasing sequence of $\mathcal{S}$-measurable sets. By continuity of measure, we have:

$$\mathbb{P}\left(\lim_{n \to \infty} B_n \right) = \lim_{n \to \infty} \mathbb{P}(B_n)$$

$$= \lim_{n \to \infty} \sum_{k=n}^{\infty} \mathbb{P}(A_n) \quad \textit{(By additivity)}$$

$$= \sum_{i=1}^{\infty} \mathbb{P}(A_i) - \lim_{n \to \infty} \sum_{k=1}^{n} \mathbb{P}(A_n)$$

$$= 0.$$

Furthermore, we have:

$$\mathbb{P}\left(\lim_{n \to \infty} B_n \right) = \mathbb{P}\left(\lim_{n \to \infty} \bigcup_{k=n}^{\infty} \right) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_n \right) = \mathbb{P}\left(\limsup_{n \to \infty} A_n \right).$$

Hence proved the first Borel-Cantelli Lemma. To prove the second Borel-Cantelli Lemma, we prove the following:

$$1 - \mathbb{P}\left(\limsup_{n \to \infty} A_n \right) = \mathbb{P}\left(\left\{ \limsup_{n \to \infty} A_n \right\}^c \right)$$

$$= \mathbb{P}\left(\liminf_{n \to \infty} A_n^c \right) = 0.$$

□.

1.2.3 Uniform Law of Large Numbers

The Uniform Law of Large Numbers (**ULLN**) provides a convergence result for collection of estimators where the convergence is uniform in the parameters space.

Theorem 1.5: Uniform Law of Large Numbers (ULLN)

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random sample from a distribution $p_{\boldsymbol{\theta}}$ over $\mathcal{X}$ that depends on the true parameters $\boldsymbol{\theta}$. Let $\Theta \subset \mathbb{R}^m$ be the parameters space and $f_{\theta} : \mathcal{X} \to \mathbb{R}$ be a function indexed by $\theta \in \Theta$ that satisfies the following conditions:

1. $\boldsymbol{\theta} \in \Theta$ and Θ is compact.

2. $\mathbb{E}_{\boldsymbol{\theta}}[\sup_{\theta \in \Theta} |f_{\theta}(X)|] < \infty^{a}$.

3. $f_{\theta}(x)$ is continuous in θ for all $x \in \mathcal{X}$.

Then, we have:

$$\sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^{n} f_{\theta}(X_i) - \mathbb{E}_{\boldsymbol{\theta}}[f_{\theta}(X)] \right| \xrightarrow{p} 0. \tag{25}$$

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[a]$\mathbb{E}_{\boldsymbol{\theta}}$ denotes the expectation taken over the distribution described by $p_{\boldsymbol{\theta}}$.

Proof (Theorem 1.5). ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

A nice proof is provided in Ferguson 1996, Theorem 16, Page 111. However, we will conduct our own version of the proof relying on results in learning theory. For $\theta \in \Theta$, define the following function:

$$\phi_{\theta}(\mathbf{X}) = \left| \frac{1}{n} \sum_{i=1}^{n} \left(f_{\theta}(X_i) - \mathbb{E}_{\boldsymbol{\theta}}[f_{\theta}(X)] \right) \right| \tag{26}$$

We have to prove that $\mathbb{P}(\sup_{\theta \in \Theta} |\phi_{\theta}(\mathbf{X})| \geq \epsilon) \to 0$ for all $\epsilon > 0$. For $\epsilon > 0$, we have:

$$\mathbb{P}\left(\sup_{\theta \in \Theta} |\phi_{\theta}(\mathbf{X})| \geq \epsilon \right) = \mathbb{P}\left(\left\{ \sup_{\theta \in \Theta} \phi_{\theta}(\mathbf{X}) \geq \epsilon \right\} \cup \left\{ \sup_{\theta \in \Theta} (-\phi_{\theta}(\mathbf{X})) \geq \epsilon \right\} \right)$$

$$\leq \mathbb{P}\left(\sup_{\theta \in \Theta} \phi_{\theta}(\mathbf{X}) \geq \epsilon \right) + \mathbb{P}\left(\sup_{\theta \in \Theta} (-\phi_{\theta}(\mathbf{X})) \geq \epsilon \right)$$

$$\leq \frac{1}{\epsilon} \left(\mathbb{E}\left[\sup_{\theta \in \Theta} \phi_{\theta}(\mathbf{X}) \right] + \mathbb{E}\left[\sup_{\theta \in \Theta} (-\phi_{\theta}(\mathbf{X})) \right] \right) \quad \textit{(Markov's Inequality)}$$

Now, we need to bound the expectations on the right-hand-side. To do so, we use the symmetrization trick. Specifically, given a sequence of i.i.d random variables $S = \{Z_1, \ldots, Z_n\} \sim \rho^n$ sampled from a distribution ρ and a class of functions $\mathcal{F}$. Let $S' = \{Z'_1, \ldots, Z'_n\} \sim \rho^n$ be a another sample

from the same distribution ρ (which we called the phantom sample), we have:

$$\mathbb{E}_S\left[\sup_{f\in\mathcal{F}}\frac{1}{n}\sum_{i=1}^n(f(Z_i)-\mathbb{E}_S[f(Z_i)])\right]$$

$$=\mathbb{E}_S\left[\sup_{f\in\mathcal{F}}\frac{1}{n}\sum_{i=1}^n(f(Z_i)-\mathbb{E}_{S'}[f(Z_i')])\right]$$

$$=\mathbb{E}_S\left[\sup_{f\in\mathcal{F}}\frac{1}{n}\sum_{i=1}^nf(Z_i)-\frac{1}{n}\sum_{i=1}^n\mathbb{E}_{S'}[f(Z_i')]\right]$$

$$=\mathbb{E}_S\left[\sup_{f\in\mathcal{F}}\mathbb{E}_{S'}\left[\frac{1}{n}\sum_{i=1}^n(f(Z_i)-f(Z_i'))\right]\right]$$

$$\leqslant\mathbb{E}_{S,S'}\left[\sup_{f\in\mathcal{F}}\frac{1}{n}\sum_{i=1}^n(f(Z_i)-f(Z_i'))\right]\quad(\textit{Jensen's Inequality})$$

$$=\mathbb{E}_{S,S',\sigma}\left[\sup_{f\in\mathcal{F}}\frac{1}{n}\sum_{i=1}^n\sigma_i(f(Z_i)-f(Z_i'))\right]\quad(\sigma\sim\mathrm{Rad}^n,f(Z_i)-f(Z_i')\textit{ is symmetric})$$

$$\leqslant\mathbb{E}_{S,\sigma}\left[\sup_{f\in\mathcal{F}}\frac{1}{n}\sum_{i=1}^n\sigma_if(Z_i)\right]+\mathbb{E}_{S',\sigma}\left[\sup_{f\in\mathcal{F}}\frac{1}{n}\sum_{i=1}^n(-\sigma_i)f(Z_i')\right]$$

$$=\mathbb{E}_{S,\sigma}\left[\sup_{f\in\mathcal{F}}\frac{1}{n}\sum_{i=1}^n\sigma_if(Z_i)\right]+\mathbb{E}_{S',\sigma}\left[\sup_{f\in\mathcal{F}}\frac{1}{n}\sum_{i=1}^n\sigma_if(Z_i')\right]\quad(\textit{Rademacher variables are symmetric})$$

$$=2\mathbb{E}_{S,\sigma}\left[\sup_{f\in\mathcal{F}}\frac{1}{n}\sum_{i=1}^n\sigma_if(Z_i)\right]\quad(S\textit{ and }S'\textit{ are identically distributed})$$

$$=2\mathfrak{R}_n(\mathcal{F}).$$

$$(27)$$

Using the same argument, we can also have:

$$\mathbb{E}_S\left[\sup_{f\in\mathcal{F}}\frac{1}{n}\sum_{i=1}^n(\mathbb{E}_S[f(Z_i)]-f(Z_i))\right]\leqslant2\mathfrak{R}_n(\mathcal{F}).\tag{28}$$

Using equations 27 and 28 to bound $\mathbb{P}(\sup_{\theta\in\Theta}|\phi_\theta(S)|\geqslant\epsilon)$, we have:

$$\mathbb{P}\left(\sup_{\theta\in\Theta}|\phi_\theta(S)|\geqslant\epsilon\right)\leqslant\frac{4\mathfrak{R}_n(\mathcal{F}_\Theta)}{\epsilon},\ \ where\ \mathcal{F}_\Theta=\Big\{f_\theta:\theta\in\Theta\Big\}.$$

To complete the proof, we have to show that the Rademacher complexity $\mathfrak{R}_n(\mathcal{F}_\Theta)\to 0$ as $n\to\infty$. By the definition of Rademacher Complexity, we have:

$$\mathfrak{R}_n(\mathcal{F}_\Theta)=\mathbb{E}_{\mathbf{X}\sim p_\theta^n}\mathbb{E}_\sigma\Big[\hat{\mathfrak{R}}_{\mathbf{X}}(\mathcal{F}_\Theta)\Big],$$

It is sufficient to prove that the Empirical Rademacher Complexity $\hat{\mathfrak{R}}_{\mathbf{X}}(\mathcal{F}_\Theta)\to 0$ as $n\to\infty$. Using Dudley's Entropy Integral Bartlett et al. 2017, Lemma A.5, we have:

$$\hat{\mathfrak{R}}_{\mathbf{X}}(\mathcal{F}_\Theta)\leqslant\frac{12}{\sqrt{n}}\int_\alpha^M\sqrt{\log\mathcal{N}\Big(\mathcal{F}_\Theta,\epsilon,L_2(\mathbf{X})\Big)}d\epsilon,\quad\alpha>0,\tag{29}$$

where $M<\infty$ is a constant such that $|f_\theta(x)|\leqslant M$ p_θ-(almost everywhere) on $\mathcal{X}$ for all $\theta\in\Theta$[1]. Now, we need to construct a cover in L_2 norm for the class $\mathcal{F}_\Theta$. Since $f_\theta(x)$ is continuous in θ for all $x\in\mathcal{X}$, for all $\epsilon>0$, there exists $\delta_x>0$ such that $\|\theta-\bar{\theta}\|_2<\delta_x\implies|f_\theta(x)-f_{\bar{\theta}}(x)|<\epsilon$. Define $\delta>0$ as follows:

$$\delta=\min_{1\leqslant i\leqslant n}\delta_{X_i},\quad X_i\in\mathbf{X}.\tag{30}$$

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[1]M exists because we have $\mathbb{E}_{\boldsymbol{\theta}}[\sup_{\theta\in\Theta}|f_\theta(X)|]<\infty$.

Due to continuity, constructing an ϵ-cover in $\mathcal{F}_\Theta$ is equivalent to constructing a δ-cover for Θ with respect to the Euclidean norm. Since Θ is compact, there exists $N \in \mathbb{N}$ and a sequence of open balls $\{\mathcal{B}(y_j, r_j)\}_{j=1}^N$ where $y_j \in \Theta$ for all $1 \leqslant j \leqslant N$ such that:

$$\Theta \subseteq \bigcup_{j=1}^N \mathcal{B}(y_j, r_j). \tag{31}$$

By Long et al. 2020, Lemma A.8, we have:

$$\log \mathcal{N}\left(\mathcal{B}(y_j, r_j), \delta, \|.\|_2\right) \leqslant m \log\left(\frac{3r_j}{\delta}\right). \tag{32}$$

Therefore, we have:

$$\log \mathcal{N}\left(\Theta, \delta, \|.\|_2\right) \leqslant \sum_{j=1}^N \log \mathcal{N}\left(\mathcal{B}(y_j, r_j), \delta, \|.\|_2\right) \leqslant mN \log\left(\frac{\prod_{j=1}^N r_j}{\delta^N}\right). \tag{33}$$

From the above covering number bound, we can see that $\log \mathcal{N}\left(\Theta, \delta, \|.\|_2\right)$ does not grow with n and therefore, $\log \mathcal{N}\left(\mathcal{F}_\Theta, \epsilon, L_2(\mathbf{X})\right)$ does not grow with n. Therefore, we have $\hat{\mathfrak{R}}_{\mathbf{X}}(\mathcal{F}_\Theta) \in \mathcal{O}(1/\sqrt{n})$ and $\hat{\mathfrak{R}}_{\mathbf{X}}(\mathcal{F}_\Theta) \to 0$ as $n \to \infty$. □

1.2.4 Central Limit Theorem

> **Theorem 1.6: Central Limit Theorem** (CLT)
>
> Let $X_1, \ldots X_n$ be a sequence of $i.i.d$ random variables with expected value μ and finite variance σ^2. Then, we have:
>
> $$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1) \quad as \quad n \to \infty, \tag{34}$$
>
> where $\bar{X}_n = S_n/n$ and $\mathcal{N}(0, 1)$ is the standard normal distribution.

Proof (Central Limit Theorem (CLT)). ────────────────────
We prove this via the Characteristic Function. Let $\bar{Z}_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$, notice that:

$$\bar{Z}_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{X_i - \mu}{\sigma},$$

Let $Z_i = X_i - \mu$ for $1 \leqslant i \leqslant n$ and suppose $Z = Z_1 = \cdots = Z_n$, we have:

$$\begin{aligned} \varphi_{\bar{Z}_n}(t) = \varphi_{\sum_{i=1}^n Z_i}\left(\frac{t}{\sqrt{n}}\right) &= \left[\varphi_Z\left(\frac{t}{\sqrt{n}}\right)\right]^n \\ &= \left[1 + \frac{it\mathbb{E}[Z]}{\sqrt{n}} - \frac{t^2}{2n}\mathbb{E}[Z^2] + \mathcal{O}(1/n)\right]^n \quad (\textit{Taylor's Expansion}) \\ &= \left[1 - \frac{t^2}{2n} + \mathcal{O}(1/n)\right]^n. \end{aligned}$$

The final equality comes from the fact that $\mathbb{E}[Z] = 0$ and $\mathbb{E}[Z^2] = \mathbb{E}[Z]^2 + \mathrm{Var}(Z) = 1$. Finally, we have:

$$\lim_{n \to \infty} \varphi_{\bar{Z}_n}(t) = \lim_{n \to \infty} \left[1 - \frac{t^2}{2n} + \mathcal{O}(1/n)\right]^n = e^{-t^2/2}.$$

Since $e^{-t^2/2}$ is the Characteristic Function of the standard normal distribution, by (**LCT**), we have $\bar{Z}_n \xrightarrow{d} \mathcal{N}(0,1)$. □.

1.3   Convergence of Random Variables

In this section, we revise the modes of convergence in random variables.

1.3.1   Convergence in Distribution

Definition 1.2 (Convergence in Distribution). ─────────────────────────────
Given a sequence of real-valued random variables $X_1, X_2, \ldots$ with CDFs $F_1, F_2, \ldots$. We say that the sequence converges in distribution to a random variable X with CDF F, denoted $X_n \xrightarrow{d} X$ if:

$$\lim_{n \to \infty} F_n(x) = F(x), \tag{35}$$

for all $x \in \mathbb{R}$ at which F is continuous. Convergence in distribution can also be referred to as <u>*weak convergence*</u> *in measure theory.*

Theorem 1.7: Slutsky's Theorem (SLUTSKY)

Let X_n and Y_n be two sequences of random variables such that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} c$[a] where $c < \infty$ is a constant. Then, we have:

1. $X_n + Y_n \xrightarrow{d} X + c$.

2. $X_n Y_n \xrightarrow{d} cX$.

3. $X_n/Y_n \xrightarrow{d} X/c$ if $c \neq 0$.

───────────────────────────────────────

[a]In the next section, we will see that $Y_n \xrightarrow{d} c$ implies $Y_n \xrightarrow{p} c$ for a constant c.

Proof (Theorem 1.7). ──────────────────────────────────

1. $X_n + Y_n \xrightarrow{d} X + c$. *Let $\epsilon > 0$ be any positive constant, we have:*

$$\begin{aligned}
F_{X_n+Y_n}(t) &= \mathbb{P}(X_n + Y_n \leqslant t) \\
&= \mathbb{P}(X_n + Y_n \leqslant t, |Y_n - c| \leqslant \epsilon) + \mathbb{P}(X_n + Y_n \leqslant t, |Y_n - c| > \epsilon) \\
&\leqslant \mathbb{P}(X_n + Y_n \leqslant t, |Y_n - c| \leqslant \epsilon) + \underbrace{\mathbb{P}(|Y_n - c| > \epsilon)}_{approaches\ 0\ as\ n \to \infty}.
\end{aligned}$$

In the event that $|Y_n - c| \leqslant \epsilon$, we have $Y_n \geqslant c - |Y_n - c| \geqslant c - \epsilon$. Therefore, we have:

$$F_{X_n+Y_n}(t) \leqslant \mathbb{P}(X_n \leqslant t - c + \epsilon) + \mathbb{P}(|Y_n - c| > \epsilon).$$

Similarly, we have:

$$F_{X_n+Y_n}(t) \geqslant \mathbb{P}(X_n \leqslant t - c - \epsilon) - \mathbb{P}(|Y_n - c| > \epsilon).$$

Taking limits of both inequalities, we have:

$$F_X(t - c - \epsilon) \leqslant \lim_{n \to \infty} F_{X_n+Y_n}(t) \leqslant F_X(t - c + \epsilon).$$

Let $\epsilon \to 0$, we have $\lim_{n \to \infty} F_{X_n+Y_n}(t) \to F_X(t - c) = F_{X+c}(t)$ as $n \to \infty$.

11

2. $X_n Y_n \xrightarrow{d} cX$. (Apply the same proof method as (1)).

3. $X_n / Y_n \xrightarrow{d} X/c$ if $c \neq 0$. (Apply the same proof method as (1)).

<div align="right">□.</div>

1.3.2 Convergence in Probability

Definition 1.3 (Convergence in Probability). ─────────────────────
Given a sequence of real-valued random variables $X_1, X_2, \ldots$. We say that the sequence converges in probability to a random variable X, denoted $X_n \xrightarrow{p} X$ if:

$$\lim_{n \to \infty} \mathbb{P}\left(|X_n - X| \geqslant \epsilon\right) = 0, \quad \forall \epsilon > 0. \tag{36}$$

We also refer to convergence in probability as convergence in measure in measure theory.

Proposition 1.1: $X_n \xrightarrow{p} X \implies X_n \xrightarrow{d} X$

Let X and the sequence $X_1, X_2, \ldots$ be real-valued random variables. If $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{d} X$.

Proof (Proposition 1.1). ─────────────────────
We first prove the following claim: Let X, Y be random variables, $a \in \mathbb{R}$ and $\epsilon > 0$, the inequality $\mathbb{P}(Y \leqslant a) \leqslant \mathbb{P}(X \leqslant a + \epsilon) + \mathbb{P}(|Y - X| \geqslant \epsilon)$ holds. We have:

$$
\begin{aligned}
\mathbb{P}(Y \leqslant a) &= \mathbb{P}(Y \leqslant a, X \leqslant a + \epsilon) + \mathbb{P}(Y \leqslant a, X \geqslant a + \epsilon) \\
&\leqslant \mathbb{P}(X \leqslant a + \epsilon) + \mathbb{P}(Y - X \leqslant a - X, a - X \leqslant -\epsilon) \\
&\leqslant \mathbb{P}(X \leqslant a + \epsilon) + \mathbb{P}(Y - X \leqslant -\epsilon) \\
&\leqslant \mathbb{P}(X \leqslant a + \epsilon) + \mathbb{P}(Y - X \leqslant -\epsilon) + \mathbb{P}(Y - X \geqslant \epsilon) \\
&= \mathbb{P}(X \leqslant a + \epsilon) + \mathbb{P}(|Y - X| \geqslant \epsilon).
\end{aligned}
$$

Using the above inequality, we have:

$$\mathbb{P}(X \leqslant a - \epsilon) - \mathbb{P}(|X_n - X| \geqslant \epsilon) \leqslant \mathbb{P}(X_n \leqslant a) \leqslant \mathbb{P}(X \leqslant a + \epsilon) + \mathbb{P}(|X_n - X| \geqslant \epsilon).$$

Taking limits as $n \to \infty$ from both sides, we have:

$$F_X(a - \epsilon) \leqslant \lim_{n \to \infty} F_{X_n}(a) \leqslant F_X(a + \epsilon).$$

Taking $\epsilon \to 0^+$, we have $\lim_{n \to \infty} F_{X_n}(a) = F_X(a)$. □.

Proposition 1.2: $X_n \xrightarrow{d} c \iff X_n \xrightarrow{p} c$

Let $c \in \mathbb{R}$ be a constant and $X_1, X_2, \ldots$ be a sequence of real-valued random variables. Then, $X_n \xrightarrow{d} c \iff X_n \xrightarrow{p} c$.

Proof (Proposition 1.2, Pishro-Nik 2014). ─────────────────────
Since $X_n \xrightarrow{d} c$, we immediately have the following:

$$\lim_{n \to \infty} F_{X_n}(c - \epsilon) = 0,$$

$$\lim_{n \to \infty} F_{X_n}(c + \epsilon/2) = 1.$$

<div align="center">12</div>

Then, for any $\epsilon > 0$, we have:

$$\begin{aligned}
\lim_{n\to\infty}(|X_n - c| \geqslant \epsilon) &= \lim_{n\to\infty}\mathbb{P}\Big[\mathbb{P}(X_n \leqslant c - \epsilon) + \mathbb{P}(X_n \geqslant c + \epsilon)\Big] \\
&= \underbrace{\lim_{n\to\infty}F_{X_n}(c - \epsilon)}_{=0} + \lim_{n\to\infty}\mathbb{P}(X_n \geqslant c + \epsilon) \\
&\leqslant \lim_{n\to\infty}\mathbb{P}(X_n \geqslant c + \epsilon/2) \\
&= 1 - \underbrace{\lim_{n\to\infty}F_{X_n}(c + \epsilon/2)}_{=1} \\
&= 0.
\end{aligned}$$

From the above, we have $\lim_{n\to\infty}\mathbb{P}(|X_n - c| \geqslant \epsilon) = 0$ and $X_n \xrightarrow{p} c$. $\qquad\square$.

1.3.3 Convergence in L^p norm

Definition 1.4 (Convergence in L^p norm). ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯
Given a sequence of random variables $X_1, X_2, \ldots$ and a real number $p \in [1, \infty)$. We say that the sequence converges in L^p norm to a random variable X, denoted as $X_n \xrightarrow{L^p} X$ if:

$$\lim_{n\to\infty}\mathbb{E}|X_n - X|^p = 0. \tag{37}$$

Proposition 1.3: $X_n \xrightarrow{L^p} X \implies X_n \xrightarrow{p} X$

Let $p \geqslant 1$ and $X_1, X_2, \ldots$ be a sequence of real-valued random variables. Let X be a random variable, then, $X_n \xrightarrow{L^p} X \implies X_n \xrightarrow{p} X$.

Proof (Proposition 1.3). ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯
Let $\epsilon > 0$, we have:

$$\begin{aligned}
\mathbb{P}(|X_n - X| \geqslant \epsilon) &= \mathbb{P}(|X_n - X|^p \geqslant e^p) \quad (p \geqslant 1) \\
&\leqslant \frac{\mathbb{E}|X_n - X|^p}{\epsilon^p}. \quad (\textit{Markov's Inequality})
\end{aligned}$$

Taking the limits from both sides, we have $\lim_{n\to\infty}\mathbb{P}(|X_n - X| \geqslant \epsilon) = 0$ and $X_n \xrightarrow{p} X$. $\qquad\square$.

1.3.4 Almost-sure Convergence

Definition 1.5 (Convergence almost-surely). ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯
Let $X_1, X_2, \ldots$ be a sequence of real-valued random variables that map from a sample space Ω. Let X also be a real-valued random variable. We say that X_n converges almost surely to X, denoted as $X_n \xrightarrow{a.s} X$, if:

$$\mathbb{P}\Big(\limsup_{n\to\infty}E_n\Big) = 0 \quad \textit{where} \quad E_n = \Big\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geqslant \epsilon\Big\}.$$

Remark 1.2 (Consequence of (**BCL**)). ────────────────────────────────

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty \implies X_n \xrightarrow{a.s} X. \tag{38}$$

Proposition 1.4: $X_n \xrightarrow{a.s} X \implies X_n \xrightarrow{p} X$

Let $X_1, X_2, \ldots$ be a sequence of real-valued random variables and also let X be a real valued random variables. If $X_n \xrightarrow{a.s} X$ then $X_n \xrightarrow{p} X$.

Proof (Proposition 1.4). ────────────────────────────────
Let $f_n : \Omega \to \mathbb{R}_+$ be a sequence of nonnegative Borel-measurable functions such that $f_n(\omega) = |X_n(\omega) - X(\omega)|$. By Fatou's Lemma (reverse), we have:

$$\underbrace{\mathbb{P}\left(\limsup_{n\to\infty} \{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geqslant \epsilon\} \right)}_{=0} = \int f_n d\mathbb{P}$$

$$\geqslant \limsup_{n\to\infty} \mathbb{P}(|X_n - X| \geqslant \epsilon)$$

$$\geqslant \lim_{n\to\infty} \mathbb{P}(|X_n - X| \geqslant \epsilon).$$

Hence, we have $\lim_{n\to\infty} \mathbb{P}(|X_n - X| \geqslant \epsilon) = 0$ and $X_n \xrightarrow{p} X$. ▫.

Theorem 1.8: Continuous Mapping Theorem (CMT)

Let $f : \mathbb{R} \to \mathbb{R}$ be a <u>continuous</u> function and $X_1, X_2, \ldots$ be a sequence of real-valued random variables. Then, the following statements hold true:

1. $X_n \xrightarrow{d} X \implies f(X_n) \xrightarrow{d} f(X)$.

2. $X_n \xrightarrow{p} X \implies f(X_n) \xrightarrow{p} f(X)$.

3. $X_n \xrightarrow{a.s} X \implies f(X_n) \xrightarrow{a.s} f(X)$.

Proof (Continuous Mapping Theorem (**CMT**)). ────────────────────
Since almost-sure convergence implies the other two modes of convergence, we only have to handle the almost-sure convergence case. Since h is continuous, for any $\omega \in \Omega$ such that $X_n(\omega) \to X(\omega)$, we have $f(X_n(\omega)) \to f(X(\omega))$. Therefore, we have:

$$\left\{ \omega \in \Omega : X_n(\omega) \to X(\omega) \right\} \subseteq \left\{ \omega \in \Omega : f(X_n(\omega)) \to f(X(\omega)) \right\}.$$

Therefore, we have:

$$\mathbb{P}\left(\limsup_{n\to\infty} \left\{ \omega \in \Omega : |f(X_n(\omega)) - f(X(\omega))| \leqslant \epsilon \right\} \right)$$

$$\geqslant \mathbb{P}\left(\limsup_{n\to\infty} \left\{ \omega \in \Omega : |X_n(\omega) - X(\omega)| \leqslant \epsilon \right\} \right) = 1,$$

for all $\epsilon > 0$. Therefore, we have $f(X_n) \xrightarrow{a.s} f(X)$. ▫.

14

2 Statistical Inference

2.1 Sufficiency & Likelihood Principles

2.1.1 Sufficiency

Definition 2.1 (Sufficient Statistics). ────────────────────────────
Let $\mathbf{X} = (X_1, \ldots, X_n) \sim p(.; \boldsymbol{\theta})$ be a random sample drawn i.i.d from a distribution with parameters $\boldsymbol{\theta}$. Let $\mathbf{U} = T(\mathbf{X})$ be a statistic, then it is called a <u>sufficient statistic</u> if the conditional distribution $p_{\mathbf{X}|\mathbf{U}}$ does not depend on $\boldsymbol{\theta}$.

Example 2.1 (Bernoulli random variables). ────────────────────────────
Let $\mathbf{X} = (X_1, \ldots, X_n) \sim \text{Bernoulli}(\theta)$ be a random sample from the Bernoulli distribution. Let $\mathbf{U} = \frac{1}{n} \sum_{i=1}^{n} X_i$, then $\mathbf{U}$ is a sufficient statistic of θ. To illustrate this, suppose that $\mathbf{x} = (x_1, \ldots, x_n)$ is an observation of the random sample $\mathbf{X}$ and $\mathbf{u} = \frac{1}{n} \sum_{i=1}^{n} x_i$. We have:

$$
\begin{aligned}
\mathbb{P}(\mathbf{X} = \mathbf{x}|\mathbf{U} = \mathbf{u}) &= \frac{\mathbb{P}(\mathbf{X} = \mathbf{x}, \mathbf{U} = \mathbf{u})}{\mathbb{P}(\mathbf{U} = \mathbf{u})} \\
&= \frac{\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n, \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} x_i)}{\mathbb{P}(\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} x_i)} \\
&= \frac{\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n)}{\mathbb{P}(\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} x_i)} \\
&= \frac{\theta^{\sum_{i=1}^{n} x_i} (1 - \theta)^{n - \sum_{i=1}^{n} x_i}}{\mathbb{P}(\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} x_i)}.
\end{aligned}
$$

Now, setting $k = \sum_{i=1}^{n} x_i$, The denominator is basically the probability that the Bernoulli variables sums up to k. Hence, we can calculate the denominator as follows:

$$
\mathbb{P}\left(\sum_{i=1}^{n} X_i = k \right) = \binom{n}{k} \theta^k (1 - \theta)^{n-k}.
$$

Therefore, we have:

$$
\mathbb{P}(\mathbf{X} = \mathbf{x}|\mathbf{U} = \mathbf{u}) = \frac{\theta^k (1 - \theta)^{n-k}}{\binom{n}{k} \theta^k (1 - \theta)^{n-k}} = \frac{1}{\binom{n}{k}}.
$$

Therefore, the conditional distribution does not depend on θ and $\mathbf{U}$ is a sufficient statistic.

Definition 2.2 (Sufficiency Principle). ────────────────────────────
If $\mathbf{U} = T(\mathbf{X})$ is a sufficient statistic for $\boldsymbol{\theta}$, then any inference about $\boldsymbol{\theta}$ should only depend on the sample $\mathbf{X}$ through $\mathbf{U}$. *In other words, if we estimate $\boldsymbol{\theta}$ using an estimator $\hat{\boldsymbol{\theta}}$, only $\mathbf{U}$ shows up in the formula of $\hat{\boldsymbol{\theta}}$, not the sample $\mathbf{X}$ itself. We will see why this is the case in the Factorisation Theorem (**FacT**), which states that we can factorise the density function into a function of $\mathbf{U}, \boldsymbol{\theta}$ and a function of the observations $\mathbf{x}$ and thus, the inference about $\boldsymbol{\theta}$ is independent of the observations $\mathbf{x}$.*

> **Theorem 2.1: Factorisation Theorem (FacT)**
>
> Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random sample with joint density function $p(\mathbf{x}; \boldsymbol{\theta})$ over $\mathcal{X}^n$. The statistic $\mathbf{U} = T(\mathbf{X})$ is sufficient for the parameters $\boldsymbol{\theta}$ if and only if we can find functions h, g such that:
>
> $$p(\mathbf{x}; \boldsymbol{\theta}) = g(T(\mathbf{x}), \boldsymbol{\theta}) h(\mathbf{x}),$$
>
> for all $\mathbf{x} \in \mathbb{R}^n$ and $\boldsymbol{\theta} \in \Theta$.

Proof (Factorisation Theorem (**FacT**)). ─────────────────────────────
We have to conduct the proof in both directions.

- *$T(\mathbf{X})$ is sufficient $\implies$ Factorisation exists: Let $\mathbf{U} = T(\mathbf{X})$ be a sufficient statistics and $\mathbf{u} = T(\mathbf{x})$ be the statistics evaluated on the observations $\mathbf{x}$. Then, we have:*

$$p(\mathbf{x}; \boldsymbol{\theta}) = \mathbb{P}(\mathbf{X} = \mathbf{x}; \boldsymbol{\theta})$$
$$= \mathbb{P}(\mathbf{X} = \mathbf{x} | \mathbf{U} = \mathbf{u}; \boldsymbol{\theta}) \mathbb{P}(\mathbf{U} = \mathbf{u}; \boldsymbol{\theta}).$$

  Since $\mathbf{U} = T(\mathbf{X})$ is a sufficient statistics, $\mathbb{P}(= \mathbf{x} | \mathbf{U} = \mathbf{u}; \boldsymbol{\theta})$ does not depend on $\boldsymbol{\theta}$. Hence, we denote $h(\mathbf{x}) = \mathbb{P}(\mathbf{X} = \mathbf{x} | \mathbf{U} = \mathbf{u}; \boldsymbol{\theta})$. Furthermore, $\mathbb{P}(\mathbf{U} = \mathbf{u}; \boldsymbol{\theta})$ is a function of $\mathbf{u}$ and $\boldsymbol{\theta}$. We denote this function as $g(\mathbf{u}, \boldsymbol{\theta})$ and conclude that the factorisation $p(\mathbf{x}; \boldsymbol{\theta}) = h(\mathbf{x}) g(T(\mathbf{x}), \boldsymbol{\theta})$ indeed exists.

- *Factorisation exists $\implies$ $T(\mathbf{X})$ is sufficient: Suppose that there exists g, h such that $p(\mathbf{x}; \boldsymbol{\theta}) = g(T(\mathbf{x}), \boldsymbol{\theta}) h(\mathbf{x})$. We then have:*

$$\mathbb{P}(\mathbf{X} = \mathbf{x} | \mathbf{U} = \mathbf{u}; \boldsymbol{\theta}) = \frac{p(\mathbf{x}; \boldsymbol{\theta})}{\mathbb{P}(\mathbf{U} = \mathbf{u}; \boldsymbol{\theta})} = \frac{g(\mathbf{u}, \boldsymbol{\theta}) h(\mathbf{x})}{\mathbb{P}(\mathbf{U} = \mathbf{u}; \boldsymbol{\theta})}.$$

  We denote $A_{\mathbf{u}} = \left\{ \tilde{\mathbf{x}} \in \mathcal{X}^n : T(\tilde{\mathbf{x}}) = \mathbf{u} \right\}$. We have:

$$\mathbb{P}(\mathbf{U} = \mathbf{u}; \boldsymbol{\theta}) = \sum_{\tilde{\mathbf{x}} \in A_{\mathbf{u}}} \mathbb{P}(\mathbf{X} = \tilde{\mathbf{x}})$$
$$= \sum_{\tilde{\mathbf{x}} \in A_{\mathbf{u}}} p(\tilde{\mathbf{x}}; \boldsymbol{\theta}) = \sum_{\tilde{\mathbf{x}} \in A_{\mathbf{u}}} g(T(\tilde{\mathbf{x}}), \boldsymbol{\theta}) h(\tilde{\mathbf{x}})$$
$$= g(\mathbf{u}, \boldsymbol{\theta}) \sum_{\tilde{\mathbf{x}} \in A_{\mathbf{u}}} h(\tilde{\mathbf{x}}).$$

  From the above, we have:

$$\mathbb{P}(\mathbf{X} = \mathbf{x} | \mathbf{U} = \mathbf{u}; \boldsymbol{\theta}) = \frac{h(\mathbf{x})}{\sum_{\tilde{\mathbf{x}} \in A_{\mathbf{u}}} h(\tilde{\mathbf{x}})},$$

  and the above expression does not depend on $\boldsymbol{\theta}$. Hence, $T(\mathbf{X})$ is a sufficient statistics.

2.1.2   Likelihood

Definition 2.3 (Likelihood Function). ─────────────────────────────
Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random sample from a distribution $p(.|\theta)$ that depends on parameters $\theta \in \Theta$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be an observation of the random sample $\mathbf{X}$. Then, the likelihood function $L(\theta; \mathbf{x})$ is defined as follows:

$$L(\theta; \mathbf{x}) = \prod_{i=1}^{n} p(x_i; \theta), \quad \theta \in \Theta. \tag{39}$$

In some cases, we also use the log-likelihood function:

$$\ell(\theta; \mathbf{x}) = \log L(\theta; \mathbf{x}) = \sum_{i=1}^{n} \log p(x_i; \theta), \quad \theta \in \Theta. \tag{40}$$

Essentially, $L(\theta; \mathbf{x})$ quantifies the likelihood that θ generates the observations $\mathbf{x}$. In a way, it is the inverse of probability density (mass) functions, we can see the contrast as follows:

- **Probability Density Function**: The parameters are fixed but the observations are random.

- **Likelihood Function**: The observations are fixed but the parameters are variable.

Definition 2.4 (Maximum Likelihood Estimator). ──────────────────────
Given $\mathbf{X} = (X_1, \ldots, X_n)$ be a random sample from a distribution $p(.; \theta)$ that depends on $\theta \in \Theta$ and $\mathbf{x} = (x_1, \ldots, x_n)$ be an observation of $\mathbf{X}$. The _Maximum Likelihood Estimator_ $\theta_{MLE} \in \Theta$ is the parameter that maximizes the likelihood function:

$$\theta_{MLE} = \arg\max_{\theta \in \Theta} L(\theta; \mathbf{x}). \tag{41}$$

In the subsequent propositions, we will discuss some of the key properties of MLE.

Proposition 2.1: Consistency of MLE

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random sample from a distribution $p(.; \boldsymbol{\theta})$ over $\mathcal{X}$ dependent on a true set of parameters $\boldsymbol{\theta}$. Let Θ be the parameters space. Then, the Maximum Likelihood Estimator $\theta_{MLE} = \arg\max_{\theta \in \Theta} L(\theta; \mathbf{X})$, which is a random variable, is consistent, meaning $\theta_{MLE} \xrightarrow{p} \boldsymbol{\theta}$, provided that the following conditions are met:

1. $\boldsymbol{\theta} \in \Theta$ and Θ is a compact space.

2. If $\theta_1 \neq \theta_2$ then $p(x; \theta_1) \neq p(x; \theta_2), \quad \forall x \in \mathcal{X}$.

3. $\log p(x; \theta)$ is continuous in θ for almost all $x \in \mathcal{X}$.

4. $\mathbb{E}_{\boldsymbol{\theta}}[\sup_{\theta \in \Theta} |\log(X; \theta)|] < \infty$.

Furthermore, we can also show that θ_{MLE} is asymptotically unbiased. In other words, $\lim_{n \to \infty} \mathbb{E}[\theta_{MLE}] = \boldsymbol{\theta}$.

Proof (Proposition 2.1). ──────────────────────
A proof for consistency of MLE can be found in _"Chapter 36 Large sample estimation and hypothesis testing" 1994_, Theorem 2.5. In this section, we briefly go through the proof again.   □.

Proposition 2.2: Asymptotic Normality of MLE

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random sample from a distribution $p(.; \boldsymbol{\theta})$ dependent on a true set of parameters $\boldsymbol{\theta}$. Let Θ be the parameters space. Then, the Maximum Likelihood Estimator $\theta_{MLE} = \arg\max_{\theta \in \Theta} L(\theta; \mathbf{X})$ is asymptotically normal:

$$\frac{\theta_{MLE} - \boldsymbol{\theta}}{\sqrt{\mathrm{Var}(\theta_{MLE})/n}} \xrightarrow{d} \mathcal{N}(0, 1). \tag{42}$$

Proof (Proposition 2.2). _____

$\square.$

2.2   Point Estimation

2.2.1   Bias, Variance, Consistency and MSE

2.2.2   Sufficient Statistics & Rao-Blackwell Theorem

> **Theorem 2.2: Rao-Blackwell Theorem (RB)**

2.2.3   Estimator Variance & Cramer-Rao Lower Bound

Definition 2.5 (Fisher Information). _____
Let $\mathbf{X} = (X_1, \ldots, X_n) \sim p(.; \boldsymbol{\theta})$ be a random sample from a distribution parameterized by $\boldsymbol{\theta}$. The
(total) Fisher Information about θ in the random sample $\mathbf{X}$ is defined as follows:

$$\mathcal{I}_{\mathbf{X}}(\boldsymbol{\theta}) = \mathbb{E}_{\mathbf{X}}\left[\left(\frac{\partial}{\partial \theta} \log L(\theta; \mathbf{X}) \right)^2 \ \middle| \ \boldsymbol{\theta} \right]. \tag{43}$$

The Fisher Informaton is the total information about $\boldsymbol{\theta}$ contained in the sample $\mathbf{X}$.

> **Theorem 2.3: Cramer-Rao Lower Bound (CRLB)**

2.2.4   Maximum Likelihood Estimation (MLE)

A List of Definitions

B Important Theorems

C Important Propositions

↰ *Return to Contents*

D   References

References

Bartlett, Peter L., Dylan J. Foster, and Matus Telgarsky (2017). "Spectrally-normalized margin bounds for neural networks". In: *Conference on Neural Information Processing Systems*.

"Chapter 36 Large sample estimation and hypothesis testing" (1994). In: vol. 4. Handbook of Econometrics. Elsevier, pp. 2111–2245. DOI: https://doi.org/10.1016/S1573-4412(05)80005-4. URL: https://www.sciencedirect.com/science/article/pii/S1573441205800054.

Durrett, Rick (2010). *Probability: Theory and Examples*. 4th. USA: Cambridge University Press. ISBN: 0521765390.

Ferguson, Thomas S. (1996). *A Course in Large Sample Theory*. Chapman & Hall.

Gut, Allan (2004). *A Graduate Course in Probability*. Graduate Text in Mathematics.

Long, Philip M. and Hanie Sedghi (2020). "Generalization Bounds for Deep Convolutional Neural Networks". In: *International Conference on Learning Representation*.

Pishro-Nik, Hossein (2014). *Introduction to Probability, Statistics and Random Processes*. Kappa Research, LLC.

undefinedinlar, Erhan (2011). *Probability and Stochastics*. Springer New York. ISBN: 9780387878591. URL: http://dx.doi.org/10.1007/978-0-387-87859-1.

Wikipedia (2023). *Vitali set — Wikipedia, The Free Encyclopedia*. http://en.wikipedia.org/w/index.php?title=Vitali%20set&oldid=1187241923. [Online; accessed 24-December-2023].