

# High Dimensional Probability Notes

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# 1 Random variables

## 1.1 Basic inequalities

First, we revisit the definition of a random variable as well as some basic inequalities that we learned in introductory statistics.

**Definition 1.1** (Random variable).

*Let  $(\Omega, \Sigma, \mathbb{P})$  be a probability space. A random variable  $X$  is defined as a mapping from the sample space  $\Omega$  to  $\mathbb{R}$ :*

$$X : \Omega \rightarrow \mathbb{R} \quad (1)$$

$\Sigma$  is the  $\sigma$ -algebra containing the possible events (collection of subsets of  $\Omega$ ) and  $\mathbb{P}$  is a probability measure that assigns events with probabilities:

$$\mathbb{P} : \Sigma \rightarrow [0, 1] \quad (2)$$

For a given probability space  $(\Omega, \Sigma, \mathbb{P})$  and a random variable  $X : \Omega \rightarrow \mathbb{R}$ , we will use the following basic notations throughout this note:

- $\|X\|_{L^p}$  - The  $p^{th}$  root of the  $p^{th}$  moment of the random variable  $X$ .

$$\|X\|_{L^p} = (\mathbb{E}|X|^p)^{1/p}, \quad p \in (0, \infty) \quad (3)$$

$$\|X\|_{L^\infty} = \text{ess sup } |X| \quad (4)$$

- $L^p(\Omega, \Sigma, \mathbb{P})$  - The space of random variables  $X$  satisfying:

$$L^p(\Omega, \Sigma, \mathbb{P}) = \left\{ X : \Omega \rightarrow \mathbb{R} \mid \|X\|_{L^p} < \infty \right\} \quad (5)$$

Some basic inequalities and identities:

- **1. Jensen's Inequality** - For a random variable  $X$  and a convex function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ , we have:

$$\varphi(\mathbb{E}X) \leq \mathbb{E}\varphi(X) \quad (6)$$

- **2. Monotonicity of  $L^p$  norm** - For a random variable  $X$ :

$$\|X\|_{L^p} \leq \|X\|_{L^q}, \quad 0 \leq p \leq q \leq \infty \quad (7)$$

- **3. Minkowski's Inequality** - For  $1 \leq p \leq \infty$  and two random variables  $X, Y$  in  $L^p(\Omega, \Sigma, \mathbb{P})$  space:

$$\|X + Y\|_{L^p} \leq \|X\|_{L^p} + \|Y\|_{L^p} \quad (8)$$

- **4. Holder's Inequality** - For  $p, q \in [1, \infty]$  such that  $1/p + 1/q = 1$ . Then, for random variables  $X \in L^p(\Omega, \Sigma, \mathbb{P})$  and  $Y \in L^q(\Omega, \Sigma, \mathbb{P})$ , we have:

$$|\mathbb{E}XY| \leq \|X\|_{L^p} \cdot \|Y\|_{L^q} \quad (9)$$

- **5. Markov's Inequality** - For a non-negative random variable  $X$  and  $t > 0$ , we have:

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}X}{t} \quad (10)$$

- **6. Chebyshev's Inequality** - For a random variable  $X$  with mean  $\mu$  and variance  $\sigma^2$ . Then, for any  $t > 0$ , we have:

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2} \quad (11)$$

- **7. Integral Identity** - Let  $X$  be a non-negative random variable, we have:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t) dt \quad (12)$$

## Exercises

### Exercise 1.1.1: Generalized Integral Identity

Let  $X$  be a random variable (not necessarily non-negative). Prove the following identity:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t) dt - \int_{-\infty}^0 \mathbb{P}(X < t) dt \quad (13)$$

**Solution** (Exercise 1.1.1).

For  $x \in \mathbb{R}$ , using the basic integral identity, we have:

$$|x| = \int_0^\infty \mathbf{1}\{t < |x|\} dt$$

We consider the following cases:

- When  $x < 0 \implies x = -|x|$ :

$$x = - \int_0^\infty \mathbf{1}\{t < |x|\} dt = - \int_0^\infty \mathbf{1}\{t < -x\} dt = - \int_0^\infty \mathbf{1}\{-t > x\} dt = - \int_{-\infty}^0 \mathbf{1}\{t > x\} dt$$

- When  $x \geq 0 \implies x = |x|$ :

$$x = \int_0^\infty \mathbf{1}\{t < |x|\} dt = \int_0^\infty \mathbf{1}\{t < x\} dt$$

Therefore, for  $x \in \mathbb{R}$ , we can write:

$$x = \int_0^\infty \mathbf{1}\{t < x\} dt - \int_{-\infty}^0 \mathbf{1}\{t > x\} dt$$

Therefore, for a random variable  $X$  not necessarily non-negative, we have:

$$\begin{aligned} \mathbb{E}X &= \mathbb{E} \left[ \int_0^\infty \mathbf{1}\{t < X\} dt - \int_{-\infty}^0 \mathbf{1}\{t > X\} dt \right] \\ &= \mathbb{E} \int_0^\infty \mathbf{1}\{t < X\} dt - \mathbb{E} \int_{-\infty}^0 \mathbf{1}\{t > X\} dt \\ &= \int_0^\infty \mathbb{E} \mathbf{1}\{t < X\} dt - \int_{-\infty}^0 \mathbb{E} \mathbf{1}\{t > X\} dt \\ &= \int_0^\infty \mathbb{P}(t < X) dt - \int_{-\infty}^0 \mathbb{P}(t > X) dt \end{aligned}$$

□.

**Exercise 1.1.2:  $p^{th}$ -moments via tails**

Let  $X$  be a random variable and  $p \in (0, \infty)$ . Show that:

$$\mathbb{E}|X|^p = \int_0^\infty pt^{p-1}\mathbb{P}(|X| > t)dt \quad (14)$$

**Solution** (Exercise 1.1.2). \_\_\_\_\_

Let  $X$  be a random variable that is not necessarily non-negative. Using the integral identity, we have:

$$\mathbb{E}|X|^p = \int_0^\infty \mathbb{P}(u < |X|^p)du$$

Let  $t^p = u \implies pt^{p-1}dt = du$ . Since we integrate  $u$  from  $0 \rightarrow \infty$ , we also integrate  $t$  from  $0 \rightarrow \infty$  when changing the variables. Hence, we have:

$$\mathbb{E}|X|^p = \int_0^\infty \mathbb{P}(t^p < |X|^p)pt^{p-1}dt = \int_0^\infty \mathbb{P}(t < |X|)pt^{p-1}dt$$

Hence, we obtained the desired identity.  $\square$ .

**1.2 Limit Theorems****1.2.1 Weak Law of Large Numbers****Theorem 1.1: Weak Law of Large Numbers (WLLN)**

Let  $X_1, \dots, X_N$  be *i.i.d* random variables with mean  $\mu$ . Consider the sum:

$$S_N = X_1 + \dots + X_N$$

Then, the sample mean **converges to  $\mu$  in probability** ( $S_N/N \xrightarrow{p} \mu$ ):

$$\lim_{N \rightarrow \infty} \mathbb{P}\left(|S_N/N - \mu| > \epsilon\right) = 0, \quad \forall \epsilon > 0 \quad (15)$$

**Proof** (Weak Law of Large Numbers (WLLN)). \_\_\_\_\_

We split the proof into two sections corresponding to the assumptions of finite variance and non-finite variance.

1. **Finite variance case:** Suppose that  $\text{Var}X_i = \sigma^2 < \infty$  for all  $1 \leq i \leq N$ . Let  $\bar{X} = S_N/N$ . Then,  $\bar{X}$  is a random variable with the following mean and variance:

$$\mathbb{E}\bar{X} = \mu \quad \text{and} \quad \text{Var}\bar{X} = \frac{\sigma^2}{N}.$$

Hence, by the Chebyshev's inequality, we have:

$$\mathbb{P}\left(|S_N/N - \mu| > \epsilon\right) = \mathbb{P}\left(|\bar{X} - \mu| > \epsilon\right) \leq \frac{\sigma^2}{N\epsilon^2}.$$

Therefore, we have:

$$\lim_{N \rightarrow \infty} \mathbb{P}\left(|S_N/N - \mu| > \epsilon\right) \leq \lim_{N \rightarrow \infty} \frac{\sigma^2}{N\epsilon^2} = 0.$$

Hence, we have  $\lim_{N \rightarrow \infty} \mathbb{P}\left(|S_N/N - \mu| > \epsilon\right) = 0$  and we obtained **(WLLN)**.

2. **Non-finite variance case:** In this case, we rely on the Levy Continuity Theorem (**LCT**), which relies on the convergence of the characteristic function. For  $n \geq 1$ , define the sequence of random variable  $Y_n = S_n/n$ . Hence, we have:

$$\varphi_{Y_n}(t) =$$

□.

#### Theorem 1.2: Levy Continuity Theorem (**LCT**)

Let  $X_1, X_2, \dots$  be *i.i.d* random variables. Then:

$$\forall t \in \mathbb{R} : \lim_{n \rightarrow \infty} \varphi_{X_n}(t) = \varphi_X(t) \iff X_n \xrightarrow{d} X, \quad (16)$$

for some random variable  $X$ . In a special case where  $X = c$  for some  $c \in \mathbb{R}$ , we have:

$$\forall t \in \mathbb{R} : \lim_{n \rightarrow \infty} \varphi_{X_n}(t) = e^{itc} \iff X_n \xrightarrow{p} c. \quad (17)$$

**Proof** (Levy Continuity Theorem (**LCT**)).

The proof for (**LCT**) can be found in Gut 2004, Section 9.1, Theorem 9.1 and Corollary 9.1 □.

### 1.2.2 Strong Law of Large Numbers

#### Theorem 1.3: Strong Law of Large Numbers (**SLLN**)

Let  $X_1, \dots, X_N$  be *i.i.d* random variables with mean  $\mu$ . Consider the sum:

$$S_N = X_1 + \dots + X_N$$

Then, the sample mean **converges to  $\mu$  almost surely** ( $S_N/N \xrightarrow{a.s} \mu$ ):

$$\mathbb{P} \left( \limsup_{N \rightarrow \infty} |S_N/N - \mu| > \epsilon \right) = 0, \quad \forall \epsilon > 0 \quad (18)$$

**Proof** (Strong Law of Large Numbers (**SLLN**)).

□.

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## B Important Theorems

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1.3 Strong Law of Large Numbers ( <b>SLLN</b> ) . . . . .	5

## C Important Corollaries

## D Important Propositions

## E References

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