Chapter 6: Information channel

Presented notions

- Physical medium to send information from transmitter to receiver
- Channel has additive noise which changes the propagated signal
- Channel has different types:
  - Discrete channel
    - Memoryless channel
    - Memory channel
    - Binary channel
  - Continuous channel
- Input source, noise source, output are considered random variables
  - Output = Input + Noise

6.1. Channel model

- Discrete channel: discrete input signal and discrete output signal
  - Channel model:
    - Random variable represents discrete source
      - X = {x1, x2,…xn}
      - P(X) = {P(x1), P(x2),…,P(xn)}
    - Alphabet of output
      - Y = {y1,y2,…,yn}
        - Yi = xj (alphabet of output is identical to alphabet of input)
      - Generally, number of inputs (r) is different with number of outputs (s).
      - This course only focuses: r = s = n

6.1. Channel Model (Cont.)

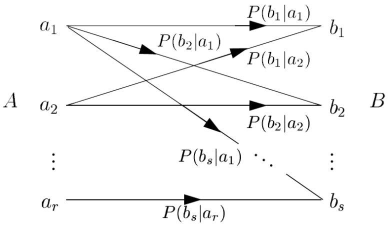
- Discrete channel:
    - This course only focus on memoryless channel
    - Matrix channel for memoryless channel: consists of all transfer probabilities
        - P(Y|X) = {P(yj|xi)}

$$\begin{bmatrix} P(y_1|x_1) & P(y_2|x_1) & \dots & P(y_n|x_1) \\ P(y_1|x_2) & P(y_2|x_2) & \dots & P(y_n|x_2) \\ \dots & \dots & \dots & \dots \\ P(y_1|x_n) & P(y_2|x_n) & \dots & P(y_n|x_n) \end{bmatrix}$$

        - Elements in main diagonal is correct probability
        - Elements outside off main diagonal is wrong probability
        - If all correct probabilities are equal, all wrong probabilities are equal: uniform channel
        - Sum of all elements in one row = 1
        - Identity matrix (unit matrix): transmission in channel not wrong
        - If all elements are equal, the channel wrong: output independents with input
        - Elements of the matrix are symmetric across the main diagonal (symmetric matrix): symmetric channel

6.1. Channel Model (Cont.)

- Transition diagram of the discrete channel:
  - r input points. Each point represents a input signal
  - s output points. Each point represents a output signal
  - Connection line between a input point and a output point represent transfer probability



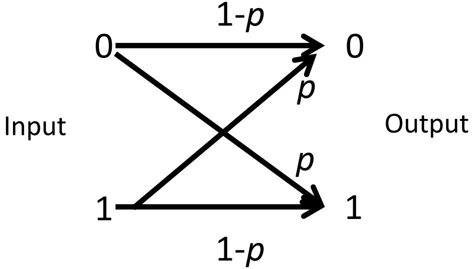
6.1. Channel Model (Cont.)

- Discrete Binary Symmetric Channel (BSC)
  - Binary channel: input signal and output signal are binary
    - For example: 0 and 1
  - Symmetric channel: The channel matrix is symmetric across the main diagonal
    - Channel matrix: contain transmission probability *P(y|x)*
      - *P(y|x)*: conditional probability of receiving output signal *y* when input signal *x* is sent
- Binary symmetric channel = binary channel + symmetric channel
  - Example: channel matrix of one BSC

$$P(Y|X) = \begin{array}{cc} \left|\begin{array}{cc} 3/4 & 1/4 \\ 1/4 & 3/4 \end{array}\right| & \begin{array}{c} x1 \\ x2 \end{array} \\ \begin{array}{cc} y1 & y2 \end{array} & \end{array}$$

6.1. Channel Model (Cont.)

Binary symmetric channel (BSC) model



$$P[Y = 0|X = 1] = P[Y = 1|X = 0] = p$$
$$P[Y = 1|X = 1] = P[Y = 0|X = 0] = 1 - p$$

6.1. Channel Model (Cont.)

- Continuous channel: continuous input signal and continuous output signal
  - Channel model:
    - Random variable represents continuous source
      - X = {x}   xmin ≤ x ≤ xmax
      - P_X(x)   probability density function of source X
    - Output:
      - Y = {y}   ymin ≤ y ≤ ymax
      - P(y|x)   conditional probability density function of output Y when input X has determined values

6.2. Mutual information of discrete channel (Cont.)

- According chapter 2: mutual information between two random variables X, Y:

$$I(X;Y) = \sum\sum P(x,y) \log\frac{P(x,y)}{P(x)P(y)}$$

$$= H(X) + H(Y) - H(X,Y)$$

- Because: $\frac{P(x,y)}{P(y)} = P(x|y)$

  So : $I(X;Y) = \sum\sum P(x,y) \log\frac{P(x|y)}{P(x)}$

  $= H(X) - H(X|Y)$

- Corollary:
  - Mutual information is a average information (entropy) that a symbol may transmit through channel
  - Mutual information equals average input information (H(X)) minus average loss information (H(X|Y)).
  - Average loss information is generated by noise: H(X|Y) $= H(N)_X$ with noise source N and source X are inputs

6.2. Mutual information of discrete channel (Cont.)

- Because: $\frac{P(x,y)}{P(x)}$ = *P(y|x)*

  So : $I(X;Y) = \sum\sum P(x,y) \log\frac{P(y|x)}{P(y)}$

  $= H(Y) – H(Y|X)$

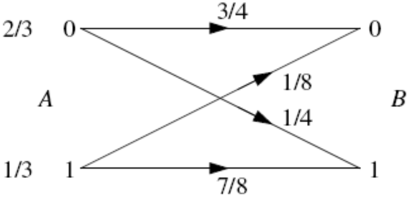
- Corollary:
  - Mutual information is a average information (entropy) that a symbol may transmit through channel
  - Mutual information equals average input information (H(Y)) minus average loss information (H(Y|X)).
  - Average loss information is generated by noise: H(Y|X) =$H(N)_Y$ with noise source N and source Y are inputs

6.2. Mutual information of discrete channel (Cont.)

- Information transmitted through channel always equals input information minus loss information
- Loss information: noise information generated by noise
- Information transmitted though channel in both ways are equals
- $0 \le I(X;Y) \le H(X)$ when no noise, $X = Y$

6.2. Mutual information of discrete channel (Cont.)

- H(A), H(A|B)?
- Mutual information?



6.3. Mutual information of continuous channel

- As presented in chapter 2, entropy of continuous source are calculated using integral and probability density function
  - So, mutual information is also calculated using integral

$$I(X;Y) = \iint_{x,y} P_{x,y}(\mathrm{x,y}) \log\frac{P_{x,y}(x,y)}{P_x(x)P_y(y)}\, d_x\, d_y$$

$$= H(X) + H(Y) - H(X,Y)$$

$$= \iint_{x,y} P_{x,y}(\mathrm{x,y}) \log\frac{P_{x|y}(x|y)}{P_x(x)}\, d_x\, d_y$$

$$= H(X) - H(X|Y)$$

$$= \iint_{x,y} P_{x,y}(\mathrm{x,y}) \log\frac{P_{y|x}(y|x)}{P_y(y)}\, d_x\, d_y$$

$$= H(Y) - H(Y|X)$$

- Properties of mutual information in continuous channel are similar to the properties of mutual information in discrete channel

6.4. Channel capacity

- Maximum average amount of information that channel may transmit in a unit of time without error
- Denoted by C
- Calculated by number of transmitted symbols n_o multiply maximum average amount of information that a symbol can transmit through channel (mutual information)
  - C = $n_o \times$ I(X;Y)max
- Remark: in information theory, n_o is physical parameter so that n_o is considered as an unit

  → C = I(X;Y)max

6.4. Channel capacity (Cont.)

- Discrete channel:
  - Number of transmitted symbols (n_o) is bandwidth of channel (Δf)
    - So, C = Δf × I(X;Y)max
  - C = Δf $\times (H(X) - H(X|Y))$max = Δf $\times (H(X) - H(N)_X)max$
  - Without noise: $H(N)_X$ = 0
    - So: C = Δf $\times H(X)$ max = Δf $\times \log|X| =$ Δf $\times \log L$ (L is number of different symbols of source X)

6.4. Channel capacity (Cont.)

- Continuous channel:
  - n_o is calculated as the number of samples of the discrete source equivalent to the continuous source
  - According to sampling theorem:
    - n_o = 2 Fmax (Fmax is maximum frequency of source connected to channel)
    - Fmax = Δf (bandwidth of channel)
  - C = 2 Δf × I(X;Y)max

    = 2 Δf × (H(Y)- H(Y|X))max

6.4. Channel capacity (Cont.)

- Continuous channel:
  - Normally, source has Gaussian distribution and noise has also Gaussian distribution:
    - H(Y|X) = $H(N)_Y$ = $\log\sqrt{2\Pi e P_N}$
      - P_N : average power (variance) of the noise source
    - H(Y) = $\log\sqrt{2\Pi e P_Y}$
      - P_Y : average power (variance) of the output
    - $P_Y = P_X + P_N$
      - P_x: average power (variance) of input source
      - The variance of the sum of the Gaussian random variables is equal to the sum of the variances of each variable
    - C = 2 Δf ×($\log\sqrt{2\Pi e P_Y}$ - $\log\sqrt{2\Pi e P_N}$)

$$= 2\Delta f \times log\sqrt{\frac{P_Y}{P_N}}$$

$$= \Delta f \times log\frac{P_Y}{P_N} = \Delta f \times \log\frac{P_X + P_N}{P_N} = \Delta f \times \log(1 + \frac{P_X}{P_N})$$

→C = $\Delta f \times \log(1 + \frac{S}{N})$

$\frac{S}{N}$: signal/noise rate: calculated by the ratio of the average power of the input signal with the average of the noise measured at the output of the channel