List of Concepts for Chapter 4

Section 4.1:

- A transformation $L: \mathrm{V} \to \mathrm{W}$, is linear if
  (i) $L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})$ for every **x** and **y** in V
  (ii) $L(\alpha\mathbf{x}) = \alpha L(\mathbf{x})$ for every scalar α and every **x** in V
  Make sure you know how to determine whether a given transformation is linear or not.

- A linear transformation $L: \mathrm{V} \to \mathrm{W}$ maps the zero vector in V into the zero vector in W.

- Given a linear transformation $L: \mathrm{V} \to \mathrm{W}$,
  - The Kernel of *L* , Ker(*L*) is the set of vectors $\mathbf{v} \in \mathrm{V}$ such that *L*(**v**) =0.
  - The Image in *L* of the subspace S is the set of vectors $\mathbf{w} \in \mathrm{W}$ such that **w** = *L*(**v**) for some $\mathbf{v} \in \mathrm{S}$. In particular, the Range of *L* is the image of the whole space V.

Section 4.2:

PART I: **Matrix representations w.r.t. standard bases**

- **Theorem 4.2.1:** If L is a linear transformation $L: \mathbb{R}^n \to \mathbb{R}^m$, then there is an $m \times n$ matrix **A** such that *L*(**x**) = **Ax** for each $\mathbf{x} \in \mathbb{R}^n$. The *j*-th column vector of **A** is given by $\mathbf{a}_j = L(\mathbf{e}_j)$, $j = 1, 2 .. n$ where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are the standard basis vectors in $\mathbb{R}^n$.
  The matrix **A** is called the standard matrix representation of *L*.

- If *L*(**x**) = **Ax**, then Ker(*L*) = N(**A**) and Range of *L* = R(**A**) (column space of the matrix **A**).

- Make sure you know how to find the matrix representations of basic transformations such as rotations, reflections and projections. In particular, remember that the matrix that rotates a vector in $\mathbb{R}^2$ by an angle θ in the counterclockwise direction is given by $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.
  The rotation matrix that rotates by an angle θ in the clockwise direction is given by
  $R^{-1} = R^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

PART II: Matrix representations w.r.t. general bases

- **Theorem 4.2.2**: If $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $\mathcal{W} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ are ordered bases for vector spaces *V* and *W*, respectively, then there is an $m \times n$ matrix **B** such that
  $$[L(\mathbf{v})]_{\mathcal{W}} = \mathbf{B}[\mathbf{v}]_{\mathcal{V}} \text{ for each } \mathbf{v} \in V$$
  The *j*-th column of **B** is given by $\mathbf{b}_j = [L(\mathbf{v}_j)]_{\mathcal{W}}$, i.e., it is the vector of coefficients of $L(\mathbf{v}_j)$ as a linear combination of the vectors in $\mathcal{W}$.

- **Property**: Let $L: \mathbb{R}^n \to \mathbb{R}^m$, and let **A** be the matrix representation of *L* w.r.t the standard bases i.e. *L*(**x**) = **Ax**. Let $\mathcal{V}$ and $\mathcal{W}$ be bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, that are not standard, and let **B** be the matrix representation of *L* w.r.t. these bases, i.e. $[L(\mathbf{x})]_{\mathcal{W}} = \mathbf{B}[\mathbf{x}]_{\mathcal{V}}$. Then
  $$\mathbf{A} = \mathbf{W}\,\mathbf{B}\,\mathbf{V}^{-1}$$
  where **V** and **W** are the transition matrices from the bases $\mathcal{V}$ and $\mathcal{W}$ to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.