Example: Find a power series representation for $f(x) = \frac{x^3}{(1-2x)^2}$ and determine the radius of convergence.

Solution:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n, \quad |2x| < 1$$

$$= \sum_{n=0}^{\infty} 2^n x^n, \quad |x| < \frac{1}{2}$$

$$(R = \tfrac{1}{2})$$

$$\frac{2}{(1-2x)^2} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} 2^n x^n\right)$$

$$= \sum_{n=0}^{\infty} 2^n \cdot \frac{d}{dx}(x^n)$$

$$= \sum_{n=1}^{\infty} 2^n \cdot n \cdot x^{n-1} \quad (R = \tfrac{1}{2})$$

$$\frac{1}{(1-2x)^2} = \frac{1}{2}\sum_{n=1}^{\infty} 2^n \cdot n \cdot x^{n-1}$$

$$= \sum_{n=1}^{\infty} 2^{n-1} \cdot n \cdot x^{n-1}$$

$$f(x) = \frac{x^3}{(1-2x)^2} = x^3\left(\sum_{n=1}^{\infty} 2^{n-1} \cdot n \cdot x^{n-1}\right)$$

$$= \sum_{n=1}^{\infty} 2^{n-1} \cdot n \cdot x^{n-1} \cdot x^3$$

$$= \sum_{n=1}^{\infty} 2^{n-1} \cdot n \cdot x^{n+2} \quad (R = \tfrac{1}{2}).$$