Mat 243 – Ionascu – Test II – Review — Solutions —

1. Fill in the blank in the statements below:

   (a) A function f : A → B is one-to-one if and only if $\forall x, y \in dom(f)\ (f(x) = f(y) \Rightarrow x = y)$
   (or, equiv'ly: $\forall x, y \in dom(f)\ (x \neq y \rightarrow f(x) \neq f(y))$)

   (b) A function f : A → B is onto if and only if $\forall y \in codom(f)\ \exists x \in dom(f): y = f(x)$
   (equiv'ly: range(f) = codomain(f))

   (c) A function f : A → B is bijective if and only if is one-to-one and onto (i.e., injective and surjective)

   (c) A sequence is a function w/ domain $\{0, 1, 2, \ldots\}$ (or $\{1, 2, \ldots\}$)

   (d) For a function f : A → B and Y a subset of B, the preimage of Y through f is
   $f^{-1}(Y) = \{x \in A \mid f(x) \in Y\}$

   (e) A function f(x) is big-O of g(x) if and only if $\exists M$ and k so that
   $|f(x)| \leq M|g(x)|, \forall x > k$

   (e) A function f(x) is of the order of function g(x) if and only if
   $f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$

   (f) An integer a divides integer b if and only if $\exists k \in \mathbb{Z}: b = a \cdot k$

   (g) Integers a and b are congruent mod m if and only if $m \mid a - b$
   (or, equiv'ly, a and b have the same remainder when div. by m)

   (h) If gcd(a, b) = 1 then we say that a and b are relatively prime.

2. Prove that f : Z → Z; f(n) = 2n + 3 is one-to-one but not onto.

   a) $f(n) = f(m) \Longrightarrow 2n+3 = 2m+3 \Longrightarrow 2n = 2m \Longrightarrow n = m$ ✓ ($\Rightarrow f = 1\text{-}1$)

   b) no even number k is an $f(n)$; since $f(n) = 2n+3 = 2n+2+1$ ✓
   $= 2(n+1)+1 = \text{ODD}$
   ($\Rightarrow f$ not onto)

3. Prove that f : Z → Z; g(n) = $\left\lceil \frac{n-1}{2} \right\rceil$ is not one-to-one but it is onto.

   a) $g(1) = \left\lceil \frac{1-1}{2} \right\rceil = \lceil 0 \rceil = 0, \quad g(0) = \left\lceil \frac{0-1}{2} \right\rceil = \lceil -.5 \rceil = 0$
   so $g(1) = g(0)$, but $1 \neq 0 \Longrightarrow f = \text{not } 1\text{-}1$

   b) for any $m \in \mathbb{Z}$, let $n = 2m+1 \in \mathbb{Z}$; then
   $f(n) = \left\lceil \frac{(2m+\cancel{1})\cancel{-1}}{2} \right\rceil = \left\lceil \frac{2m}{2} \right\rceil = \lceil m \rceil = m$ (since $m \in \mathbb{Z}$) $\Rightarrow f = \text{onto}$

4. Let $S = \mathbb{R} - \{0\}$. Prove that the function $f: S \to S$; $f(x) = \frac{1}{x}$ is one-to-one and onto.

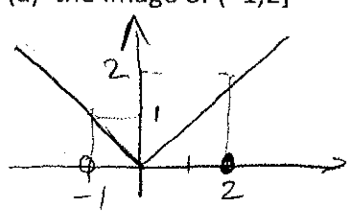
a) $f(x) = f(y) \implies \frac{1}{x} = \frac{1}{y} \implies x = y \implies f = 1\text{-}1$.

b) for $y \in S$ (ie; $y \neq 0$) $\Rightarrow$ let $x = \frac{1}{y} \in S \implies$

$$f(x) = f\left(\tfrac{1}{y}\right) = \frac{1}{\left(\frac{1}{y}\right)} = y \implies f = \text{onto}$$

5. If $f(x) = |x|$ is the absolute value function defined on the real numbers, find

(a) the image of $(-1,2]$

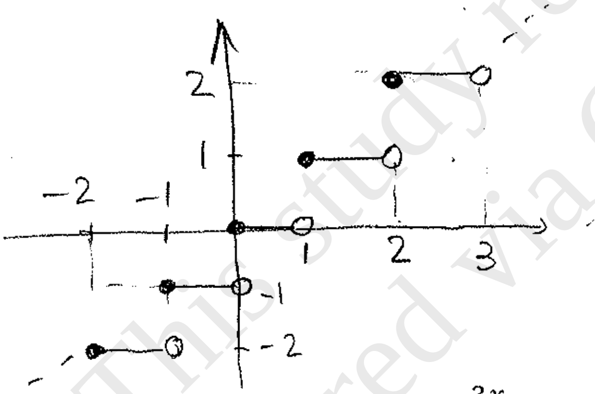


$$f((-1,2]) = [0,2]$$

(b) the preimage of $(0,2]$.

$$f^{-1}((0,2]) = [-2,0) \cup (0,2]$$

6. Repeat problem **5** for $f(x)$ = the floor function defined on $\mathbb{R}$. $f(x) = \lfloor x \rfloor$



a) $f((-1,2]) = \{-1, 0, 1, 2\}$

b) $f^{-1}((0,2]) = [1, 3)$

only integers here are 1 and 2

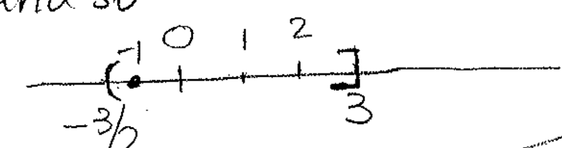
7. Repeat problem **5** for $g(x) = \lceil \frac{3x}{2} \rceil$.

a) $g((-1,2]) = \left\{ \left\lceil \frac{3x}{2} \right\rceil \;\middle|\; -1 < x \leq 2 \right\} = \{-2, -1, 0, 1, 2, 3\}$

if $-1 < x \leq 2$, then

$-3 < 3x \leq 6$, and so

$-\frac{3}{2} < \frac{3x}{2} \leq 3$



8. Evaluate the sum and simplify as much as possible. Show all your work. Calculator answers will not be accepted.

$$\sum_{n=100}^{200} \frac{5^{2n+3}}{3^{2n+1}} = \sum_{n=100}^{200} \frac{5^3 \cdot 5^{2n}}{3 \cdot 3^{2n}} = \sum_{n=100}^{200} \frac{125}{3}\left(\frac{5}{3}\right)^{2n} = \sum_{n=100}^{200} \frac{125}{3}\left(\frac{25}{9}\right)^{n}$$

= sum of consecutive terms of a geometric sequence:

1st term $= \frac{125}{3}\left(\frac{25}{9}\right)^{100}$; ratio $= \frac{25}{9}$, # of terms $= (200-100)+1 = 101$

9. Prove that for any positive integer n, $\sum_{i=1}^{n}(6k-1) = 3n^2 + 2n$.

$$\sum_{K=1}^{n}(6K-1) = \sum_{K=1}^{n} 6K - \sum_{K=1}^{n} 1 = 6\sum_{K=1}^{n} K - \sum_{K=1}^{n} 1 =$$

$$6 \cdot \frac{n(n+1)}{2} - 1 \cdot n = 3n(n+1) - n = 3n^2 + 3n - n = 3n^2 + 2n$$ ✓

done!

10. Let $a_n = 2^n + 5 \cdot 3^n$ for $\underbrace{\text{nonnegative integers } n}_{n \geq 0}$. Find a_0, a_1, a_2 and a_3.

$a_0 = 2^0 + 5 \cdot 3^0 = 1 + 5 \cdot 1 = 6$

$a_1 = 2 + 5 \cdot 3 = 2 + 15 = 17$

$a_2 = 2^2 + 5 \cdot 3^2 = 4 + 45 = 49$

$a_3 = 2^3 + 5 \cdot 3^3 = 8 + 5 \cdot 27 = 8 + 135 = 143$

11. Consider the sequence $\{a_n\}_n$ defined recursively by

$a_1 = 3$, and $a_{n+1} = 3a_n - 5$ for $n \geq 1$.

Find the values of a_2 a_3 and a_4.

$a_2 = 3a_1 - 5 = 3 \cdot 3 - 5 = 9 - 5 = 4$

$a_3 = 3a_2 - 5 = 3 \cdot 4 - 5 = 12 - 5 = 7$

$a_4 = 3 \cdot a_3 - 5 = 3 \cdot 7 - 5 = 21 - 5 = 16$

Cont'd #8

$$\text{Sum} = \boxed{\frac{125}{3}\left(\frac{25}{9}\right)^{100} \cdot \frac{\left(\frac{25}{9}\right)^{101} - 1}{\frac{25}{9} - 1}}$$

you can leave it in this form (no meaningful simplification exists!)

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12. For each of the given the functions find the best big-O estimate. Explain all your steps by referring to the theorems you are using.

a) $f(x) = x^5 + x\log(x^7) + \frac{x^5+4x}{x^3+1000}$ $= \underbrace{x^5}_{O(x^5)} + \underbrace{7x\log(x)}_{\in O(x\log(x))} + \underbrace{\frac{x^5+4x}{x^3+1000}}_{\in O(x^2)}$ $\sim \frac{x^5}{x^3} = x^2$

$\boxed{O(x^5)}$

$\in O(x^5)$ since

$x\log(x) \in O(x^5)$

$x^2 \in O(x^5)$

b) $f(x) = \underbrace{(x^3 + \log(x))}_{\in O(x^3)}\underbrace{(\log(x) + 17)}_{\in O(\log(x))}$

$\in \boxed{O(x^3\log(x))}$

c) $g(n) = \underbrace{(2^n + \log(n))}_{\in O(2^n)}(\underbrace{\log(n!)}_{\in \log(n!)} + n^2)$

$\in \boxed{O(2^n \cdot \log(n!))}$

13. Write the following functions in order such that each of them is big-O of the next one; justify your answer:

$\underset{= 7n\log(n)}{n\log(n^7)}$, $\underset{\sim n^2}{\frac{n^5+4n}{n^3+1000}}$, $\sqrt{n}\log(n)$, 2^n, $\underset{= n\log(5) \leftarrow \text{contant}}{\log(5^n)}$, $n^3\log(n)$, $\frac{n^3}{1000000}$, $7000\sqrt{n}$

$7000\sqrt{n}$, $\sqrt{n}\log(n)$, $\log(5^n)$, $n\log(n^7)$; $\frac{n^3}{1000000}$, $n^3\log(n)$, 2^n

14. Prove or disprove:

a) If a divides b then a divides b^2.

Proof

Assume $a|b$. Then, by definition, there is a $k \in \mathbb{Z}$ so that $b = ak$. Then $b^2 = a^2k^2 = a\cdot(ak^2)$ and $ak^2 \in \mathbb{Z}$, Hence $a|b^2$. ∎

b) If a divides b^2 then a divides b.

FALSE.

contraexample: $9 | \underset{= 6^2}{36}$, but $9 \nmid 6$

15. a) Convert $(1011000010001)_2$ from binary to decimal.

$2^{12} + 2^{10} + 2^9 + 2^4 + 1 = 4096 + 1024 + 512 + 16 + 1$

$= \boxed{5649}$

b) Convert 3124 from decimal to binary.

$3124 = 2048 + \underbrace{1024 + 32 + 16}_{1076} + 4 =$ 52

$= 2^{11} + 2^{10} + 2^5 + 2^4 + 2^2 = \longrightarrow \boxed{(110000110100)_2}$

base 8 base 16

c) Convert 1000111111 from binary to octal and hexadecimal.

$(1077)_8 = (22F)_{16}$

d) Calculate 100101 · 1101 where both numbers are in binary representation form.

$$\begin{array}{r} 100101 \\ 1101 \\ \hline 100101 \\ 100101 \\ + \quad 100101 \\ \hline 111100001 \end{array}$$

$\rightarrow (11110000)_2$

e) Evaluate $310563_7 + 23104_7$ without changing to base 10 first.

$$\begin{array}{r} 310563 \\ + \quad 23104 \\ \hline 334000 \end{array}$$

$\rightarrow (334000)_7$

16. a) Using the Euclidean algorithm, find gcd(1386, 490). $= \boxed{14}$

$1386 = 490 \cdot 2 + 406$
$490 = 406 \cdot 1 + 84$
$406 = 84 \cdot 4 + 70$
$84 = 70 \cdot 1 + 14$
$70 = 14 \cdot 5 + 0$

b) Find lcm(1386, 490).

$$\frac{1386 \cdot 490}{14} = \boxed{48510}$$

17. a) Prove that for any positive integers *a* and *b*, gcd(*a*,*b*) · lcm(*a*,*b*) = *a*·*b*.

$a = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ $\quad k_i, l_i \geq 0$, p_i = primes
$b = p_1^{l_1} p_2^{l_2} \cdots p_m^{l_m}$
$gcd(a,b) = p_1^{\min(k_1,l_1)} p_2^{\min(k_2,l_2)} \cdots p_m^{\min(k_m,l_m)}$
$lcm(a,b) = p_1^{\max(k_1,l_1)} p_2^{\max(k_2,l_2)} \cdots p_m^{\max(k_m,l_m)}$

$\rightarrow$ multiply and use that $\max(s,t) + \min(s,t) = s + t$

b) If the product of two integers is $2^7 \cdot 3^8 \cdot 5^2 \cdot 7^2$ and their greatest common divisor is $2^3 \cdot 3^4 \cdot 5$ what is their least common multiple?

$$\rightarrow \frac{2^7 \cdot 3^8 \cdot 5^2 \cdot 7^2}{2^5 \cdot 3^4 \cdot 5} = \boxed{2^2 \cdot 3^4 \cdot 5 \cdot 7^2} \text{ or } = 79380$$

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18. Evaluate the following quantities.

a) - 45 **mod** 8 $= \boxed{3}$

$-45 = 8\cdot(-6) + 3$

b) 33 **mod** 7 $= \boxed{5}$

$33 = 7\cdot 4 + 5$

c) 45 **div** 7 $= \boxed{6}$

$45 = 7\cdot 6 + 3$

d) $((\underbrace{18 \textbf{ mod } 14}_{4}) + (\underbrace{-35 \textbf{ mod } 7}_{0})) \textbf{ mod } 8 = \boxed{4}$

e) 6^{23456} **mod** 5 $= \boxed{1}$

$6^{23456} = 6^{2\cdot 11728} = (6^2)^{11728} = 36^{11728} \equiv 1^{11728} \pmod 5$

$1^{11728} = 1$

f) 4^{12345} **mod** 5 $= \boxed{4}$

$4^{12345} = 4^{12344+1} = 4^{12344}\cdot 4 = 4^{2\cdot 6172}\cdot 4 =$

$= (4^2)^{6172}\cdot 4 = 16^{6172}\cdot 4 \equiv 1\cdot 4 \pmod 5$

g) Find two *(integer)* values for *c* such that $\underbrace{11 \equiv c \mod 5}$.

$c = 11,\ c = 16$ (others: $c = -6,\ c = -21$ etc)

19. Compute 4^{1033} **mod** 9 using fast modular exponentiation. Show and explain all your steps.

$4^{1033} \mod 9 = \boxed{4}$

$1033 = 1024 + 8 + 1 = 2^{10} + 2^3 + 1$

$4^{1033} = 4^{2^{10}}\, 4^{2^3}\, 4 \equiv 4\cdot 7\cdot 4 = 112$

$\equiv 4 \pmod 9$

$4^{2^{10}} \equiv 4, \quad 4^{2^3} \equiv 7$

Aside

$4^2 = 16 \equiv 7 \pmod 9$

$4^{2^2} = (4^2)^2 \equiv 49 \equiv 4 \pmod 9$

$4^{2^3} = (4^{2^2})^2 \equiv 16 \equiv 7 \pmod 9$

$4^{2^4} \equiv 49 \equiv 4 \pmod 9$

$4^{2^5} \equiv 16 \equiv 7 \pmod 9$

$\vdots$

$4^{2^{10}} \equiv 4 \pmod 9$