SOLUTION SET 6—DUE 3/12/2008

This solution set is currently only partially complete, to allow a post before the midterm. Please report any errors in this document to Ian Sammis (`isammis@math.berkeley.edu`).

Problem 1 (#5.3.12)**.** *How many bit strings of length 12 contain* **a)** *exactly 3 1s?*
b) *at most three 1s?*
c) *at least thee 1s?*
d) *an equal number of 0s and 1s?*

Solution. In each case, once you know that you want m 1s, there are $\binom{12}{m}$ ways to arrange those 1s (you choose the positions of the 1s). Thus

a) If there are exactly 3 1s, then there are $\binom{12}{3} = 220$ ways to place them into the bit string. Since this is a bit string, once the 1s have been placed the string is fully specified.

b) Now, there are either 0, 1, 2, or 3 ones. Since those cases are disjoint, we can sum the counts, for $\binom{12}{0} + \binom{12}{1} + \binom{12}{2} + \binom{12}{3} = 1 + 12 + 66 + 220 = 299$ such strings.

c) If there are at least three 1s, we'd have to sum over a lot of cases to use the technique of part (b). It'd be much easier to subtract off the cases that we aren't interested in. There are $2^{12} = 4096$ bit strings of length 12, of which we don't care about the ones with 0 ones (1 case), 1 one (12 cases), or 2 ones (66 cases). Thus there are $4096 - 1 - 12 - 66 = 4017$ bit strings of length 12 with at least three 1s.

d) If there are an equal number of 0s and 1s, there are 6 of each. There are $\binom{12}{6} = 924$ such bit strings.

Problem 2 (#5.3.16)**.** *How many subsets with an odd number of elements does a set with 10 elements have?*

Solution. We have to choose 1,3,5,7, or 9 elements of the 10 to be in our subset. There are $\binom{10}{1} + \binom{10}{3} + \binom{10}{5} + \binom{10}{7} + \binom{10}{9}$ ways to do this. We can save ourselves a little bit of work by using the symmetry of the binomial coefficients to write this as $2\binom{10}{1} + 2\binom{10}{3} + \binom{10}{5} = 2(10) + 2(120) + 252 = 512$.

Problem 3 (#5.3.26)**.** *Thirteen people on a softball team show up for a game.*
a) *How many ways are there to choose 10 players to take the field?*
b) *How may ways are there to assign the 10 positions by selecting players from the 13 people who show up?*
c) *Of the 13 people who show up, three are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?*

Solution. If we care about positions, we have a permutation; if we don't we have a combination.

a) Since we're just choosing a 10-element subset here (we don't care about positions), there are $\binom{13}{10} = 286$ ways to choose 10 players.

b) This time we care about positions, so we have $P(13, 10)$ instead of $\binom{13}{10}$. Thus, the positions can be assigned in $13!/3! = 1,037,836,800$ ways.

c) We have either 1 woman and 9 men on the team ($\binom{3}{1}\binom{10}{9} = 30$ ways), or 2 women and 8 men ($\binom{3}{2}\binom{10}{8} = 135$ ways), or 3 women and 7 men ($\binom{3}{3}\binom{10}{7} =$

120 ways), for a total of 285 ways to choose a 10-person team with at least one woman.

Problem 4 (#5.3.32)**.** *How many strings of six lowercase letters from the English alphabet contain*

a) *the letter a?*

b) *the letters a and b?*

c) *the letters a and b is consecutive positions with a preceding b, with all the letters distinct?*

d) *the letters a and b, where a is somewhere to the left of b in the string, with all letters distinct?*

Solution. In the first two problems, it's probably best to count the opposite case. There are 26^6 stings of six lowercase letters. There are 25^6 that don't use a, so there are $26^6 - 25^6$ strings that do (answering a).

For (b), there are 26^6 strings total; 25^6 strings omitting a, 25^6 strings omitting b, and 24^6 strings omitting a and b. Thus, $26^6 - 25^6 - 25^6 + 24^6$ is the number of strings including both a and b. (We have to add the 24^6 back in because we double-counted the strings that include neither a nor b when we subtracted off the words not including a and the words not including b separately.)

For (c), we can treat the ab together as a single letter. We have to choose four more letters in any of the $\binom{24}{4}$ possible ways, then arrange the string in any of the 5! possible ways, for $\binom{24}{4}5!$ possible strings.

For d, choose the positions for the a and the b, in any of the $\binom{6}{2}$ possible ways. Then, choose the other four letters in any of the $\binom{24}{4}$ ways, and finally arrange the other letters in any of the 4! possible ways. Thus, there are $\binom{6}{2}\binom{24}{4}4!$ possible strings in which an a appears somewhere to the left of a b.

Problem 5 (#5.3.42)**.** *How many ways are there for a horse race with 4 horses to finish if ties are possible?*

Solution. If there are no ties, there are $4! = 24$ possible races. If there's a single 2-way tie, there are $\binom{4}{2}$ possibilities for the tied horses, then 3! ways for the three finishers (the two untied horses and the tie) to finish, for 36 possible outcomes. If there's a 3-way tie, there are $\binom{4}{3}$ ways to select the horses in the tie, then 2 ways for the tie and the fourth horse to order themselves, for 8 possible outcomes. There's exactly one 4-way tie.

Finally we need to consider the situation in which two sets of two horses tie. We have to be a bit careful here, lest we overcount. We can choose the leading tied pair in $\binom{4}{2}$ ways. That completely specifies the race, as once we know the leading pair, the other two must be the trailing pair. We needn't pick an order, since the two possible orders of the two tied pairs are both included in that $\binom{4}{2}$ count. Thus, there are 6 ways to have two two-horse ties.

Thus, a four horse race has 24+36+8+1+6=75 possible outcomes.

Problem 6 (#5.4.10)**.** *Give a formula for the coefficient of x^k in the expansion of $(x + 1/x)^{100}$, where k is an integer.*

Solution. By the binomial theorem,

$$\begin{aligned}(x+1/x)^{100} &= \sum_{j=0}^{100} \binom{100}{j} (x)^{100-j} (1/x)^j \\ &= \sum_{j=0}^{100} \binom{100}{j} x^{100-2j}.\end{aligned}$$

Let $k = 100 - 2j$. We have to be careful to only assign coefficients to values of k that match the j in the range $0, 1, \ldots, 100$. If k is an even integer such that $-100 \leq k \leq 100$, then the coefficient of x^k is $\binom{100}{j} = \binom{100}{50-\frac{k}{2}}$. Otherwise the coefficient is zero.

Problem 7 (#5.4.14)**.** *Show that if n is a positive integer, then*

$$1 = \binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil} > \cdots > \binom{n}{n} = 1$$

Proof. First off all, the equalities at the ends may be proven directly: $\binom{n}{0} = \frac{n!}{0!n!} = 1$ and $\binom{n}{n} = \frac{n!}{n!0!} = 1$. For any given $\binom{n}{m}$, we may write

$$\binom{n}{m+1} = \frac{n!}{(m+1)!(n-m-1)!} = \frac{(n-m)}{(m+1)} \frac{n!}{m!(n-m)!} = \frac{n-m}{m+1} \binom{n}{m}$$

Thus, $\binom{n}{m+1} > \binom{n}{m}$ if and only if $\frac{n-m}{m+1} > 1$. This happens when $n - m > m + 1$, or when $m < \frac{n}{2} - \frac{1}{2}$. If n is even, this is equivalent to saying $m < \frac{n}{2}$; for n odd, it means $m < \lfloor \frac{n}{2} \rfloor$. Similarly, $\binom{n}{m+1} < \binom{n}{m}$ for $m > \frac{n}{2} - \frac{1}{2}$; for n even this is for $m \geq \frac{n}{2}$; for n odd it means for $m \geq \lceil \frac{n}{2} \rceil$. Finally, $\binom{n}{m} = \binom{n}{m+1}$ only when $m = \frac{n}{2} - 12$, which can only happen when n is odd; in this case $\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rfloor}$. (When n is even, this is automatically true, as $\lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil$.)

Problem 8 (#5.4.30)**.** *Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$.*

Proof. Consider the problem of selecting an n member committee from amongst n math professors and n CS professors, with a chair who is a math professor. One way to do this is to first choose a math professor to chair the committee (n ways), then choose the remaining $n - 1$ members from amongst the remaining $2n - 1$ professors. This results in $n\binom{2n-1}{n-1}$ ways to seat the committee. Another approach is to first decide the number k of math professors that will sit on the committee (n disjoint cases), then choose the math professors)$\binom{n}{k}$ ways), then the CS professors ($\binom{n}{n-k} = \binom{n}{k}$ ways), then finally choose one of the math professors to chair the committee (k ways). This results in $\sum_{k=1}^{n} k\binom{n}{k}^2$ ways to seat the committee. Since both enumerations are correct, it must be the case that $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$, as desired.

Problem 9 (#5.5.38)**.** *A professors stores 40 journals in 4 boxes, 10 journals per box. How many ways can the journals be divided amongst the boxes if the boxes are (a) distinguishable, and (b) indistinguishable?*

Solution. In case (a), we have to pick 10 journals for the first box ($\binom{40}{10}$ ways), then 10 for the second box ($\binom{30}{10}$ ways), then 10 for the third box ($\binom{20}{10}$ ways). Thus, the number of ways to divide the journals into boxes is

$$\binom{40}{10}\binom{30}{10}\binom{20}{10} = \frac{40!}{30!10!}\frac{30!}{20!10!}\frac{20!}{10!10!} = \frac{40!}{10!10!10!10!}.$$

That last form is sometimes written $\binom{40}{10,10,10,10}$, and called a multinomial coefficient.

In part (b), we notice that for any way of dividing journals into boxes, there are $4! = 24$ divisions that are equivalent but for the specific box numbers. If the boxes are made indistinguishable, those collections of 24 cases are the individual cases, so with indistinguishable boxes we have

$$\frac{\binom{40}{10,10,10,10}}{4!}$$

ways to divide journals into boxes.