Inner Product Spaces

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Inner products in general vector spaces

Definition

An **inner product** on a vector space $(V, +, \cdot)$ is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ (assigns a scalar value to a pair of vectors) with the following properties:

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$
2. $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ for any $\mathbf{v} \neq \mathbf{0}$
3. $\langle \alpha \cdot \mathbf{u} + \beta \cdot \mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{w} \rangle + \beta \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\alpha, \beta \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

A vector space equipped with an inner product is an **inner product space**.

- Inner products are used to define angles between vectors in general vector spaces
- ❶+❸ $\Rightarrow \langle \mathbf{w}, \alpha \cdot \mathbf{u} + \beta \cdot \mathbf{v} \rangle = \alpha \langle \mathbf{w}, \mathbf{u} \rangle + \beta \langle \mathbf{w}, \mathbf{v} \rangle$
- Set $\alpha = \beta = 0$ in ❸ $\Rightarrow \langle \mathbf{0}_V, \mathbf{w} \rangle = 0$ for all $\mathbf{w}$. In particular $\langle \mathbf{0}_V, \mathbf{0}_V \rangle = 0$

Example 1. **Does**

$$(\star) \qquad \langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^T \mathbf{x}$$

define an inner product on $V = \mathbb{R}^n$ **equipped with standard Euclidean operations?**

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 4 \\ -3 \\ 2 \\ -1 \end{bmatrix} \Rightarrow \langle \mathbf{x}, \mathbf{y} \rangle = \begin{bmatrix} 4 & -3 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 4 - 6 + 6 - 4 = 0$$

Check the 3 axioms:

❶ $\langle \mathbf{y}, \mathbf{x} \rangle = \mathbf{x}^T \mathbf{y} = (\mathbf{x}^T \mathbf{y})^T = \mathbf{y}^T \mathbf{x} = \langle \mathbf{x}, \mathbf{y} \rangle \quad \checkmark$

❷ $\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^T \mathbf{x} = \|\mathbf{x}\|_2^2 > 0$ for $\mathbf{x} \neq \mathbf{0} \quad \checkmark$

❸ $\langle \alpha \cdot \mathbf{x} + \beta \cdot \mathbf{y}, \mathbf{z} \rangle = \mathbf{z}^T (\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha\, \mathbf{z}^T \mathbf{x} + \beta\, \mathbf{z}^T \mathbf{y} = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \beta \langle \mathbf{y}, \mathbf{z} \rangle \quad \checkmark$

$\Rightarrow$ $\boxed{\textbf{YES}, (\star) \text{ defines an inner product on } \mathbb{R}^n}$

This inner product is the standard Euclidean inner product on $\mathbb{R}^n$

Example 2. Does

$$(\star) \qquad \langle A, B\rangle = \mathbf{trace}(B^T A)$$

define an inner product on $V = \mathbb{R}^{m\times n}$ equipped with standard operations?

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \Rightarrow \langle A, B\rangle = \text{trace}\left(\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}\right) = -2 + 2 = 0$$

Check the 3 axioms:

❶ $\langle B, A\rangle = \text{trace}(A^T B) = \text{trace}((A^T B)^T) = \text{trace}(B^T A) = \langle A, B\rangle \quad \checkmark$

❷ $\langle A, A\rangle = \text{trace}(A^T A) = \sum_{j=1}^{n}(A^T A)_{jj} = \sum_{j=1}^{n}\sum_{i=1}^{m}(A_{ij})^2 > 0 \text{ for } A \neq 0 \quad \checkmark$

❸ $\langle \alpha A + \beta B, C\rangle = \text{trace}(C^T(\alpha A + \beta B)) = \text{trace}(\alpha C^T A + \beta C^T B)$
$= \alpha\, \text{trace}(C^T A) + \beta\, \text{trace}(C^T B) = \alpha\langle A, C\rangle + \beta\langle B, C\rangle$
using linearity of trace $\quad \checkmark$

$\Rightarrow$ **YES**, $(\star)$ defines an inner product on $\mathbb{R}^{m\times n}$

This inner product is the Frobenius inner product on $\mathbb{R}^{m\times n}$

Example 3. Does

$$(\star) \qquad \langle p,q\rangle = \sum_{i=1}^{n} p(t_i)q(t_i) \quad (t_1<\cdots<t_n)$$

define an inner product on $V=\mathbb{P}_n$ equipped with standard operations?

$$n=2,\quad t_1=-1,\quad t_2=1,\quad p(t)=\tfrac{1}{2}(1+t),\quad q(t)=\tfrac{1}{2}(1-t)$$
$$\Rightarrow\ \langle p,q\rangle = p(-1)q(-1)+p(1)q(1) = 0\cdot 1 + 1\cdot 0 = 0$$

Check the 3 axioms:

❶ $\langle q,p\rangle = \sum_{i=1}^n q(t_i)p(t_i) = \sum_{i=1}^n p(t_i)q(t_i) = \langle p,q\rangle$ ✓

❷ $\langle p,p\rangle = \sum_{i=1}^n p(t_i)^2 \geq 0$ and $\langle p,p\rangle = 0 \Rightarrow p(t_i)=0, i=1,\ldots,n \Rightarrow p=0$
(the only polynomial of degree $< n$ vanishing at n distinct points is 0) ✓

❸ $\langle \alpha p+\beta q, r\rangle = \sum_{i=1}^n (\alpha p+\beta q)(t_i)r(t_i) = \sum_{i=1}^n (\alpha p(t_i)+\beta q(t_i))r(t_i)$
$= \alpha\sum_{i=1}^n p(t_i)r(t_i) + \beta\sum_{i=1}^n q(t_i)r(t_i) = \alpha\langle p,r\rangle + \beta\langle q,r\rangle$ ✓

$\Rightarrow$ **YES**, $(\star)$ defines an inner product on $\mathbb{P}_n$

This inner product is a discrete ℓ^2 inner product on $\mathbb{P}_n$

Example 4. Does

$$(\star) \qquad \langle f, g\rangle = \int_a^b f(t)g(t)\,dt$$

define an inner product on $V = \mathcal{C}([a,b])$ **equipped with standard operations?**

$$[a,b] = [-1,1], f(t) = t, g(t) = e^t \;\Rightarrow\; \langle f, g\rangle = \int_{-1}^{1} te^t\,dt = [(t-1)e^t]_{-1}^{1} = 2e^{-1}$$

Check the 3 axioms:

1. $\langle g, f\rangle = \int_a^b g(t)f(t)\,dt = \int_a^b f(t)g(t)\,dt = \langle f, g\rangle \quad \checkmark$
2. $\langle f, f\rangle = \int_a^b f^2(t)\,dt \geq 0$ and $\langle f, f\rangle = 0$ iff $f(t) = 0$ on $[a,b]$ (f continuous) ✓
3. $\langle \alpha \cdot f + \beta \cdot g, h\rangle = \int_a^b (\alpha f(t) + \beta g(t))h(t)\,dt$
   $= \alpha \int_a^b f(t)h(t)\,dt + \beta \int_a^b g(t)h(t)\,dt = \alpha\langle f, h\rangle + \beta\langle g, h\rangle$ ✓

$\Rightarrow$ **YES**, $(\star)$ defines an inner product on $\mathcal{C}([a,b])$

This inner product is the continuous $L^2([a,b])$ inner product on $\mathcal{C}([a,b])$

Induced norm

Definition

An inner product $\langle \cdot, \cdot \rangle$ on V induces a **norm**

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

on V

- An induced norm is well-defined because $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ by Axiom ❷
- A norm measures vectors in general vector spaces

Examples

- (Euclidean) 2-norm in $\mathbb{R}^n$ is $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$
- Frobenius norm in $\mathbb{R}^{m \times n}$ is $\|A\|_F = \sqrt{\operatorname{trace}(A^T A)} = \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{m} (A_{ij})^2}$
- L_2-norm in $\mathcal{C}([a, b])$ is $\|f\|_2 = \sqrt{\int_a^b [f(t)]^2 \, dt}$

Basic properties of inner product and induced norms

- **Cauchy-Schwarz inequality** $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$
- **Angle** $0 \leq \theta \leq \pi$ **between vectors u** $\neq$ **0 and v** $\neq$ **0** $\cos\theta = \dfrac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|}$
- **Orthogonal vectors** $\mathbf{u} \perp \mathbf{v} \quad \Leftrightarrow \quad \langle \mathbf{u}, \mathbf{v} \rangle = 0$
- **Law of cosines** $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$
- **Triangular inequality** $\|\mathbf{u} - \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|, \quad \|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

- **Pythagorean Law** $\mathbf{u} \perp \mathbf{v} \Leftrightarrow \|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$

Example 5. **Find the angle between $p(t) = \frac{1}{2}(1+t)$ and $q(t) = \frac{1}{2}(1-t)$ in $\mathbb{P}_2$ equipped with standard operations w.r.t. the following inner products:**

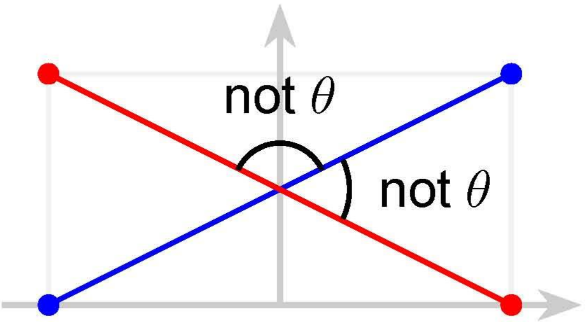
❶ $\langle p, q\rangle = p(-1)q(-1) + p(1)q(1)$

$$\begin{bmatrix} \langle p, q\rangle = 0\cdot 1 + 1\cdot 0 = 0 \\ \|p\| = \sqrt{0^2+1^2} = 1 \\ \|q\| = \sqrt{1^2+0^2} = 1 \end{bmatrix} \Rightarrow \cos\theta = 0 \Rightarrow \boxed{\theta = \frac{\pi}{2}}$$

❷ $\langle p, q\rangle = \int_{-1}^{1} p(t)q(t)\, dt$

$$\begin{bmatrix} \langle p, q\rangle = \int_{-1}^{1} \frac{1-t^2}{4}\, dt = \frac{1}{3} \\ \|p\| = \sqrt{\int_{-1}^{1} \frac{(1+t)^2}{4}\, dt} = \sqrt{\frac{2}{3}} \\ \|q\| = \sqrt{\int_{-1}^{1} \frac{(1-t)^2}{4}\, dt} = \sqrt{\frac{2}{3}} \end{bmatrix} \Rightarrow \cos\theta = \frac{\frac{1}{3}}{\sqrt{\frac{2}{3}}\sqrt{\frac{2}{3}}} = \frac{1}{2} \Rightarrow \boxed{\theta = \frac{\pi}{3}}$$

CAUTION. The “angle” between functions as defined above measures the degree of dependency of the functions. It is **not** related to any angle between the curves, e.g. at intersection points.



Matrix representation of inner products

Let $\mathcal{B}_{\mathbf{u}} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{B}_{\mathbf{v}} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be two bases of V (of dim n) and

$$\mathbf{u} = x_1 \cdot \mathbf{u}_1 + \cdots + x_n \cdot \mathbf{u}_n \quad \longleftrightarrow \quad \mathbf{x} = [\mathbf{u}]_{\mathcal{B}_{\mathbf{u}}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ rep. of } \mathbf{u} \text{ w.r.t. } \mathcal{B}_{\mathbf{u}}$$

$$\mathbf{v} = y_1 \cdot \mathbf{v}_1 + \cdots + y_n \cdot \mathbf{v}_n \quad \longleftrightarrow \quad \mathbf{y} = [\mathbf{v}]_{\mathcal{B}_{\mathbf{v}}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \text{ rep. of } \mathbf{v} \text{ w.r.t. } \mathcal{B}_{\mathbf{v}}$$

Then

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{y}^T C \, \mathbf{x}} \quad \text{with} \quad \boxed{C = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{v}_1 \rangle \\ \vdots & & \vdots \\ \langle \mathbf{u}_1, \mathbf{v}_n \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{v}_n \rangle \end{bmatrix}}$$

C is the **matrix representation of** $\langle \cdot, \cdot \rangle$ **w.r.t.** $\mathcal{B}_{\mathbf{u}}$ **and** $\mathcal{B}_{\mathbf{v}}$

Example 6. **What is the matrix representation of** $\langle p, q\rangle = \int_{-1}^{1} p(t)q(t)\,dt$ **w.r.t. the basis** $\mathcal{B} = \{1, t, t^2\}$ **of** $\mathbb{P}_3$ **equipped with standard operations?**

$$C = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cc} & \begin{array}{ccc} 1 & t & t^2 \end{array} \\ & \begin{bmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{bmatrix} \begin{array}{c} 1 \\ t \\ t^2 \end{array} \end{array}$$

For example

$$C_{13} = C_{31} = \langle 1, t^2\rangle = \int_{-1}^{1} t^2\,dt = \frac{2}{3}$$

Determine the inner product $\langle p, q\rangle$ **with** $p(t) = 1 - t$ **and** $q(t) = 1 + t + t^2$

❶ **directly:**

$$\langle p, q\rangle = \int_{-1}^{1} (1-t)(1+t+t^2)\,dt = \int_{-1}^{1} (1-t^3)\,dt = \boxed{2}$$

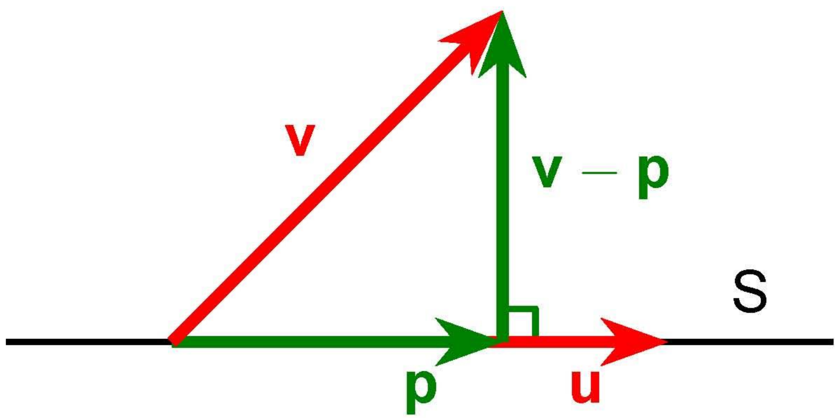
❷ **using representations:**

$$\left.\begin{array}{l} p(t) = 1 - t \leftrightarrow \mathbf{p} = [p]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\begin{array}{c} 1 \\ t \\ t^2 \end{array} \\ q(t) = 1 + t + t^2 \leftrightarrow \mathbf{q} = [q]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{array}{c} 1 \\ t \\ t^2 \end{array} \end{array}\right\} \Rightarrow \langle p, q\rangle = \mathbf{q}^T C \mathbf{p} = \boxed{2}$$

Orthogonal projections onto 1-D subspaces

The **vector orthogonal projection** of a vector $\mathbf{v}$ onto the 1-D subspace $S = \text{span}\{\mathbf{u}\}$ is given by

$$\mathbf{p} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \cdot \mathbf{u} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{u}\|^2} \cdot \mathbf{u}$$



Properties

❶ $\langle \mathbf{v} - \mathbf{p}, \mathbf{u} \rangle = 0$

Proof:

$$\langle \mathbf{v} - \mathbf{p}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle - \langle \mathbf{p}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle - \left\langle \tfrac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \cdot \mathbf{u}, \mathbf{u} \right\rangle = \langle \mathbf{v}, \mathbf{u} \rangle - \tfrac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \langle \mathbf{u}, \mathbf{u} \rangle = 0$$

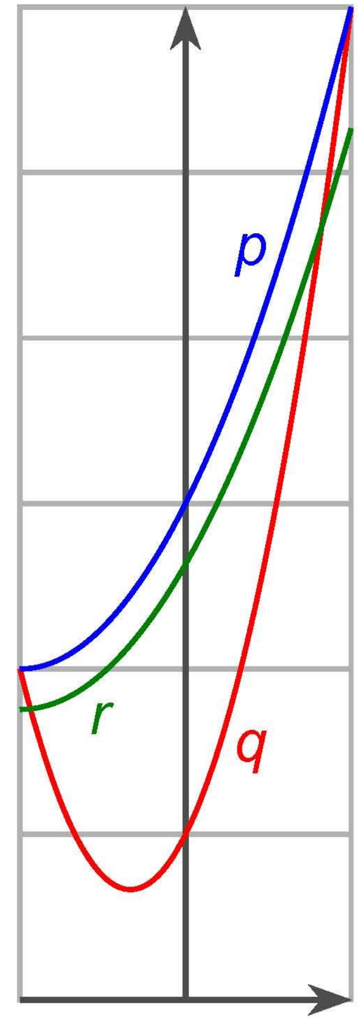
❷ $\mathbf{p} = \mathbf{v}$ iff $\mathbf{v} \in \mathbf{S}$ (i.e. $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$)

❸ $\mathbf{p}$ is the vector of S which minimizes the induced norm $\|\mathbf{v} - \mathbf{q}\|$ over all $\mathbf{q} \in S$.

Example 7. In $\mathbb{P}_3$, with standard operations and inner product

$$\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1),$$

find the orthogonal projection of $q(t) = 3t^2 + 2t + 1$ onto the line L spanned by $p(t) = t^2 + 2t + 3$



$$\begin{array}{lll} p(-1) = 2 & p(0) = 3 & p(1) = 6 \\ q(-1) = 2 & q(0) = 1 & q(1) = 6 \end{array}$$

yield

$$\langle p, q\rangle = 2 \cdot 2 + 3 \cdot 1 + 6 \cdot 6 = 43$$

$$||p||^2 = \langle p, p\rangle = p(-1)^2 + p(0)^2 + p(1)^2$$

$$= 2^2 + 3^2 + 6^2 = 49$$

so that

$$r = \frac{\langle p, q\rangle}{\|p\|^2} \cdot p = \boxed{\frac{43}{49}(t^2 + 2t + 3)}$$

is the orthogonal projection of q onto p.

Check $\langle q - r, p\rangle = (q - r)(-1)p(-1) + (q - r)(0)p(0) + (q - r)(1)p(1)$

$$= \tfrac{12}{49} \cdot 2 + (-\tfrac{80}{49}) \cdot 3 + \tfrac{36}{49} \cdot 6 = 0$$

General norms

Definition

A **norm** on a vector space $(V, +, \cdot)$ is a function $\|\cdot\| : V \to \mathbb{R}$ (assigns a scalar value to a single vector) with the following properties:

❶ $\|\mathbf{v}\| \geq 0$ with equality if and only if $\mathbf{v} = \mathbf{0}$

❷ $\|\alpha \cdot \mathbf{v}\| = |\alpha| \, \|\mathbf{v}\|$ for any $\mathbf{v} \in V$ and $\alpha \in \mathbb{R}$

❸ $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ for all $\mathbf{u}, \mathbf{v} \in V$ (**triangular inequality**)

A vector space V equipped with a norm is a **normed space**.

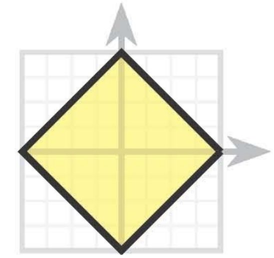
- Recall that norms are used to measure the size of vectors in general vector spaces, whereas inner products are used to measure angles between vectors.
- A norm induced by an inner product automatically satisfies ❶, ❷, and ❸. In particular an inner product space is also a normed space.
- Not all norms are induced by an inner product, i.e., a normed space is not necessarily an inner product space.

The p-norm in $\mathbb{R}^n$

If $p \geq 1$ then $\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$ is the **p-norm** of $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

- $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$ is the 1-norm of $\mathbf{x}$

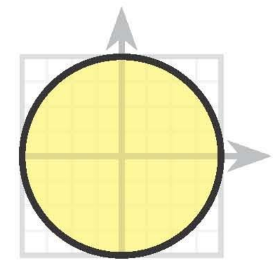
  $\left\| \begin{bmatrix} 4 \\ -3 \end{bmatrix} \right\|_1 = |4| + |-3| = 7$



$\|\mathbf{x}\|_1 = |x_1| + |x_2| \leq 1$

- $\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} = \sqrt{\mathbf{x}^T\mathbf{x}}$ is the 2-norm of $\mathbf{x}$

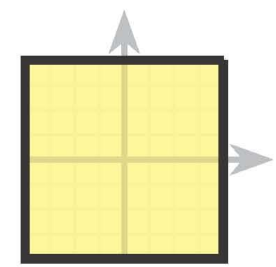
  $\left\| \begin{bmatrix} 4 \\ -3 \end{bmatrix} \right\|_2 = \sqrt{|4|^2 + |-3|^2} = 5$



$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2} \leq 1$

- $\|\mathbf{x}\|_\infty = \lim_{p \to \infty} \|\mathbf{x}\|_p = \max_{1 \leq i \leq n} |x_i|$ is the “infinite” norm of $\mathbf{x}$

  $\left\| \begin{bmatrix} 4 \\ -3 \end{bmatrix} \right\|_\infty = \max(|4|, |-3|) = 4$



$\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|\} \leq 1$

The 2-norm is the only p-norm induced by an inner product.