Mathematical Induction

Use induction to prove that

$\sum_{i=1}^{n} \frac{1}{i^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ **for all positive integers $n \geq 2$.**

Let $P(n)$ denote the proposition that

$\sum_{i=1}^{n} \frac{1}{i^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$, where n is a positive integer $n \geq 2$.

BASIS STEP: $P(2)$ is true since $1 + \frac{1}{4} < 2 - \frac{1}{2}$ and $1.25 < 1.5$.

INDUCTIVE STEP: Let us assume $P(n)$, that is

$\sum_{i=1}^{n} \frac{1}{i^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ is true for an arbitrary positive integer $n \geq 2$. This is our inductive hypothesis.

We have to show that $P(n+1)$, $\sum_{i=1}^{n+1} \frac{1}{i^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} < 2 - \frac{1}{n+1}$ is true assuming the inductive hypothesis $P(n)$.

Proof:

$$\sum_{i=1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} < 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$$

using the inductive hypothesis.

Now we have to show that $2 - \frac{1}{n} + \frac{1}{(n+1)^2} < 2 - \frac{1}{n+1}$ when $n \geq 2$.

Equivalently, $-\frac{1}{n} + \frac{1}{(n+1)^2} < -\frac{1}{n+1}$.

Equivalently, $\frac{1}{(n+1)^2} < \frac{1}{n} - \frac{1}{n+1}$.

Equivalently, $\frac{1}{(n+1)^2} < \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} = \frac{1}{n(n+1)}$.

Equivalently, $\frac{1}{n+1} < \frac{1}{n}$.

Equivalently, $n < n + 1$ true for all positive integers n**.**

By the **Principle of Mathematical Induction** (Basis Step and Inductive Step together) $\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \frac{\mathbf{1}}{\boldsymbol{i}^{\mathbf{2}}} = 1 + \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{9}} + \cdots + \frac{\mathbf{1}}{\boldsymbol{n}^{\mathbf{2}}} < 2 - \frac{\mathbf{1}}{\boldsymbol{n}}$ for all positive integers $n \geq 2$.