Structural Induction

Let S be the set defined recursively as follows:

Basis step: $\frac{2}{7} \in S$

Recursive step: If $x \in S$ then $x + 2 \in S$ and $5x \in S$

a. **List the elements of S produced by the first 2 applications of the recursive definition.**

$$S_0 = \left\{\frac{2}{7}\right\},\ S_1 = \left\{\frac{16}{7}, \frac{10}{7}\right\},\ S_2 = \left\{\frac{30}{7}, \frac{24}{7}, \frac{80}{7}, \frac{50}{7}\right\}$$

The elements of S produced by the first 2 applications of the recursive definition are

$$S_0 \cup S_1 \cup S_2 = \left\{\frac{2}{7}, \frac{16}{7}, \frac{10}{7}, \frac{30}{7}, \frac{24}{7}, \frac{80}{7}, \frac{50}{7}\right\}.$$

b. **Use structural induction to prove that** if $x \in S$ then x is a fraction whose numerator is even and the denominator is 7.

When we use structural induction to show that the elements of a recursively defined set S have a certain property, then we need to do the following procedure:

1. **Basis step**: show all the elements defined in the basis step have the desired property.
2. **Inductive step**: assume that an arbitrary element of the set S has the desired property. This is your inductive hypothesis. Using the inductive hypothesis, prove that, when you create more elements in S by using the recursive definition, these newly created elements of S have the same property.
3. **Conclusion:** state that by the principle of structural induction all the elements in S have the same property.

Basis step: $\frac{2}{7} \in S$ by the definition of S and 2 is even by the definition of even numbers $(2 = 2 \cdot 1)$ and the denominator 7.

Recursive Step: Assume $x \in S$ and x has the given property. Thus, $x = \frac{2k}{7}$ for some integer k. We need to prove that $x + 2$ and $5x$ have the same property, that is, in both cases the numerators are even and the denominators are 7 .

Proof: Using the inductive hypothesis,

$Case1$: $x + 2 = \frac{2k}{7} + 2 = \frac{2(k+7)}{7}$, where $k + 7$ is an integer.

$Case2$: $5 \cdot x = 5 \cdot \frac{2k}{7} = \frac{2(5k)}{7}$, where $5k$ is an integer.

In both cases, the numerator is even and the denominator is 7.

By **structural induction** we have proved that, if $x \in S$ then x is a fraction whose numerator is even and the denominator is 7.