MAT 275 Laboratory 4
MATLAB solvers for First-Order IVP

In this laboratory session we will learn how to

1. Use MATLAB solvers for solving scalar IVP
2. Use MATLAB solvers for solving higher order ODEs and systems of ODES.

First-Order Scalar IVP

Consider the IVP

$$\begin{cases} y' = t - y, \\ y(0) = 1. \end{cases} \tag{L4.1}$$

The exact solution is $y(t) = t - 1 + 2e^{-t}$. A numerical solution can be obtained using various MATLAB solvers. The standard MATLAB ODE solver is `ode45`. The function `ode45` implements 4/5th order Runge-Kutta method. Type `help ode45` to learn more about it.

Basic `ode45` Usage

The basic usage of ode45 requires a function (the right-hand side of the ODE), a time interval on which to solve the IVP, and an initial condition. For scalar first-order ODEs the function may often be specified using the `inline` MATLAB command. A complete MATLAB solution would read:

```
1 f = inline('t-y','t','y');
2 [t,y] = ode45(f,[0,3],1);
3 plot(t,y)
```

(line numbers are not part of the commands!)

- Line 1 defines the function `f` as a function of t and y, i.e., $f(t, y) = t - y$. This is the right-hand side of the ODE (L4.1).
- Line 2 solves the IVP numerically using the `ode45` solver. The first argument is the function `f`, the second one determines the time interval on which to solve the IVP in the form [initial time, final time], and the last one specifies the initial value of y. The output of `ode45` consists of two arrays: an array `t` of discrete times at which the solution has been approximated, and an array `y` with the corresponding values of y. These values can be listed in the Command Window as

```
    [t,y]
ans =
         0    1.0000
    0.0502    0.9522
    0.1005    0.9093
    0.1507    0.8709
    0.2010    0.8369
    ........
    2.9010    2.0109
    2.9257    2.0330
    2.9505    2.0551
    2.9752    2.0773
    3.0000    2.0996
```

Since the output is quite long we printed only some selected values.
For example the approximate solution at $t \simeq 2.9257$ is $y \simeq 2.0330$. Unless specific values of y are needed it is better in practice to simply plot the solution to get a sense of the behavior of the solution.

- Line 3 thus plots y as a function of t in a figure window. The plot is shown in Figure L4a.

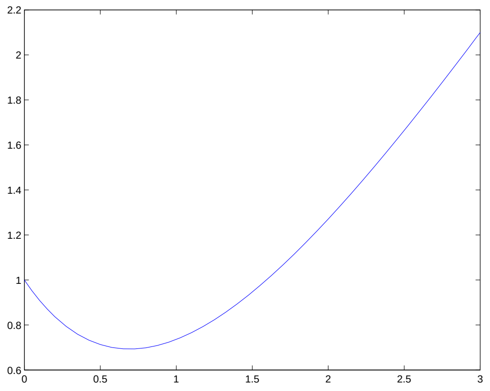


Figure L4a: Solution of (L4.1).

Error Plot, Improving the Accuracy

Error plots are commonly used to show the accuracy in the numerical solution. Here the error is the difference between the exact solution $y(t) = t-1+2e^{-t}$ and the numerical approximation obtained from `ode45`. Since this approximation is only given at specified time values (contained in the array `t`) we only evaluate this error at these values of t:

```
err = t-1+2*exp(-t)-y
err =
  1.0e-005 *
         0
    0.0278
    0.0407
    0.0162
   -0.0042
    ......
   -0.0329
   -0.0321
   -0.0313
   -0.0305
   -0.0298
```

(in practice the exact solution is unknown and this error is estimated, for example by comparing the solutions obtained by different methods). Again, since the error vector is quite long we printed only a few selected values. Note the `1.0e-005` at the top of the error column. This means that each component of the vector `err` is less than 10^{-5} in absolute value.
A plot of `err` versus `t` is more revealing. To do this note that errors are usually small so it is best to use a logarithmic scale in the direction corresponding to `err` in the plot. To avoid problems with negative numbers we plot the absolute value of the error (values equal to 0, e.g. at the initial time, are not plotted):

```
semilogy(t,abs(err)); grid on;
```

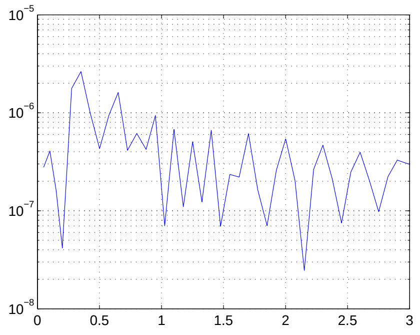


Figure L4b: Error in the solution of (L4.1) computed by `ode45`.

See Figure L4b. Note that the error level is about 10^{-6}. It is sometimes important to reset the default accuracy `ode45` uses to determine the approximation. To do this use the MATLAB `odeset` command prior to calling `ode45`, and include the result in the list of arguments of `ode45`:

```
f = inline('t-y','t','y');
options = odeset('RelTol',1e-10,'AbsTol',1e-10);
[t,y] = ode45(f,[0,3],1,options);
err = t-1+2*exp(-t)-y;
semilogy(t,abs(err))
```

See Figure L4c.

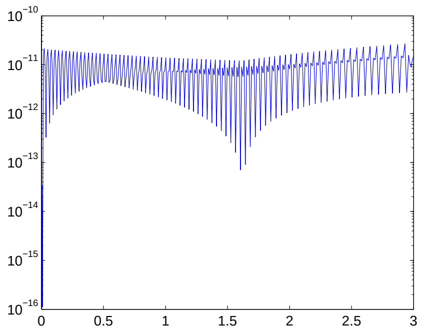


Figure L4c: Error in the solution of (L4.1) computed by `ode45` with a better accuracy.

Parameter-Dependent ODE

When the function defining the ODE is complicated, rather than entering it as `inline`, it is more convenient to enter it as a separate `function` file, included in the same file as the calling sequence of ode45 (as below) or saved in a separate `m`-file.
In this Section we will look at an example where the ODE depends on a parameter.
Consider the IVP

$$\begin{cases} y' = -a\,(y - e^{-t}) - e^{-t}, \\ y(0) = 1. \end{cases} \tag{L4.2}$$

with exact solution $y(t) = e^{-t}$ (independent of the parameter a!). An implementation of the MATLAB solution in the interval $[0, 3]$ follows.

```
1 function ex_with_param
2 t0 = 0; tf = 3; y0 = 1;
3 a = 1;
4 [t,y] = ode45(@f,[t0,tf],y0,[],a);
5 disp(['y(' num2str(t(end)) ') = ' num2str(y(end))])
6 disp(['length of y = ' num2str(length(y))])
7 %-------------------------------------------------
8 function dydt = f(t,y,a)
9 dydt = -a*(y-exp(-t))-exp(-t);
```

- Line 1 must start with `function`, since the file contains at least two functions (a driver + a function).
- Line 2 sets the initial data and the final time.
- Line 3 sets a particular value for the parameter a.
- In line 4 the parameter is passed to `ode45` as the 5th argument (the 4th argument is reserved for setting options such as the accuracy using `odeset`, see page 3, and the placeholder `[]` must be used if default options are used).
  Correspondingly the function f defined in lines 8-9 must include a 3rd argument corresponding to the value of the parameter. Note the `@f` in the argument of `ode45`. This is necessary when the function is not defined as `inline`. See the help on `ode45` for more information.
- On line 5 the value of $y(3)$ computed by `ode45` is then displayed in a somewhat fancier form than the one obtained by simply entering `y(end)`. The command `num2string` converts a number to a string so that it can be displayed by the `disp` command.
  The `m`-file `ex_with_param.m` is executed by entering `ex_with_param` at the MATLAB prompt. The output is

  ```
  >> ex_with_param
  y(3) = 0.049787
  length of y = 45
  ```

- The additional line 6 in the file lists the length of the array `y` computed by `ode45`. It is interesting to check the size of `y` obtained for larger values of a. For example for $a = 1000$ we obtain

  ```
  >> ex_with_param
  y(3) = 0.049792
  length of y = 3621
  ```

  This means that `ode45` needed to take smaller step sizes to cover the same time interval compared to the case $a = 1$, even though the exact solution is the same!

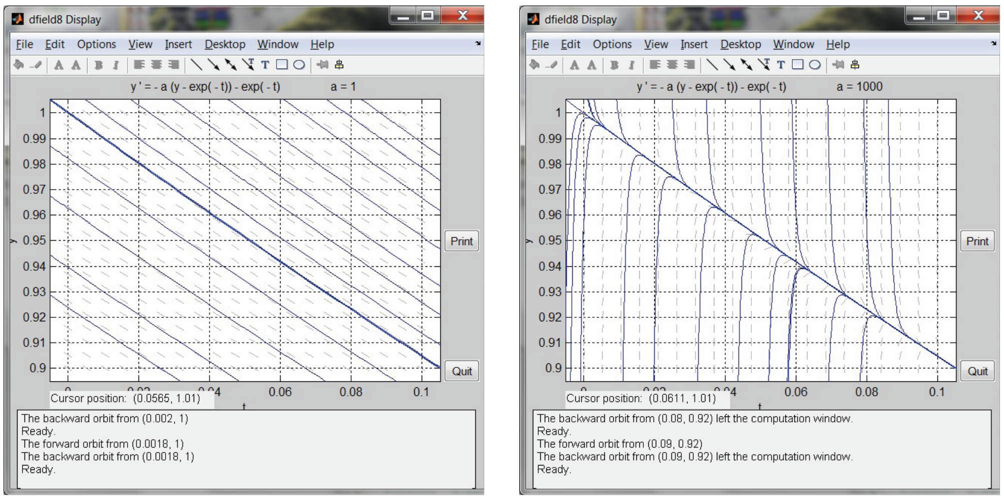
  Not all problems with a common solution are the same! Some are easier to solve than others.

  When a is large the ODE in (L4.2) is said to be *stiff*. Stiffness has to do with how fast nearby solutions approach the solution of (L4.2), see Figure L4d.

★ Other MATLAB ODE solvers are designed to better handle stiff problems. For example replace `ode45` with `ode15s` in `ex_with_param.m` (without changing anything else) and set $a = 1000$:

```
4 [t,y] = ode15s(@f,[t0,tf],y0,[],a);
```

```
>> ex_with_param
y(3) = 0.049787
length of y = 18
```


dfield8 Display
File Edit Options View Insert Desktop Window Help
y ' = - a (y - exp(- t)) - exp(- t)   a = 1
Print
Quit
Cursor position: (0.0565, 1.01)
dfield8 Display
y ' = - a (y - exp(- t)) - exp(- t)   a = 1000
Cursor position: (0.0611, 1.01)

```
The backward orbit from (0.002, 1)
Ready.
The forward orbit from (0.0018, 1)
The backward orbit from (0.0018, 1)
Ready.
```

```
The backward orbit from (0.08, 0.92) left the computation window.
Ready.
The forward orbit from (0.09, 0.92)
The backward orbit from (0.09, 0.92) left the computation window.
Ready.
```

Figure L4d: Direction field and sample solutions in the t-y window $[0, 0.1] \times [0.9, 1]$ as obtained using DFIELD8: $a = 1$ (left) and $a = 1000$ (right).

A solution exhibiting blow up in finite time

Consider the Differential Equation

$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0$$

The exact solution of this IVP is $y = \tan t$ and the domain of validity is $[0, \frac{\pi}{2})$. Let's see what happens when we try to implement this IVP using `ode45` in the interval $[0, 3]$.

```
>> f=inline('1+y^2','t','y');
>> [t,y]=ode45(f,[0,3],0);
Warning: Failure at t=1.570781e+000.  Unable to meet integration
tolerances without reducing the step size below the smallest value
allowed (3.552714e-015) at time t.
> In ode45 at 371
```

The MATLAB ode solver gives a warning message when the value of $t = 1.570781$ is reached. This is extremely close to the value of $\pi/2$ where the vertical asymptote is located.
If we enter `plot(t,y)` we obtain Figure L4e on the left (note the scale on the y-axis), however, if we use `xlim([0,1.5])`, we can recognize the graph of $y = \tan t$.

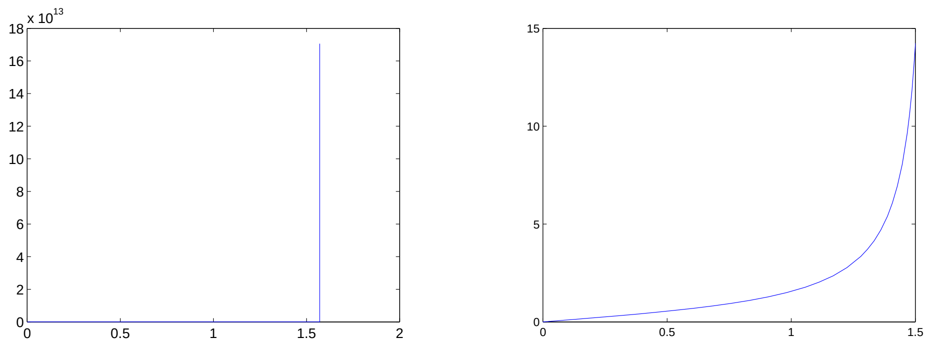


Figure L4e: Solution of $y' = 1 + y^2$, $y(0) = 0$ without restrictions on the axis, and with `xlim([0,1.5])`

Higher-Order and Systems of IVPs

We show here how to extend the use of `ode45` to systems of first-order ODEs (the same holds for other solvers such as `ode15s`). Higher-order ODEs can first be transformed into a system of first-order ODEs to fit into this framework. We will see later how to do this.

As an example consider the predator-prey system (Lotka-Volterra) representing the evolution of two populations. $u_1 = u_1(t)$ and $u_2 = u_2(t)$:

$$\begin{cases} \dfrac{du_1}{dt} = au_1 - bu_1u_2, \\ \dfrac{du_2}{dt} = -cu_2 + du_1u_2 \end{cases} \tag{L4.3}$$

with initial populations $u_1(0) = 10$ and $u_2(0) = 60$. The parameters a, b, c, and d are set to $a = 0.8$, $b = 0.01$, $c = 0.6$, and $d = 0.1$. The time unit depends on the type of populations considered.

Although the ODE problem is now defined with two equations, the MATLAB implementation is very similar to the case of a single ODE, except that vectors must now be used to describe the unknown functions.

```
 1 function ex_with_2eqs
 2 t0 = 0; tf = 20; y0 = [10;60];
 3 a = .8; b = .01; c = .6; d = .1;
 4 [t,y] = ode45(@f,[t0,tf],y0,[],a,b,c,d);
 5 u1 = y(:,1); u2 = y(:,2);    % y in output has 2 columns corresponding to u1 and u2
 6 figure(1);
 7 subplot(2,1,1); plot(t,u1,'b-+'); ylabel('u1');
 8 subplot(2,1,2); plot(t,u2,'ro-');  ylabel('u2');
 9 figure(2)
10 plot(u1,u2); axis square; xlabel('u_1'); ylabel('u_2');  % plot the phase plot
11 %-----------------------------------------------------------------
12 function dydt = f(t,y,a,b,c,d)
13 u1 = y(1); u2 = y(2);
14 dydt = [ a*u1-b*u1*u2 ; -c*u2+d*u1*u2 ];
```

- In line 2 the 2×1 vector `y0` defines the initial condition for both u_1 and u_2.
- In line 4 the parameters are passed to the ODE solver `ode45` as extra arguments (starting from the 5th), as many as there are parameters in the problem (4 here).
  The output array `y` of `ode45` now has 2 columns, corresponding to approximations for u_1 and u_2, respectively, instead of a single one.
- In line 5 these quantities are therefore retrieved and stored in arrays `u1` and `u2`, which are descriptive names.
- The part of the program defining the ODE system includes lines 11-14. Note that all the parameters appearing as arguments of `ode45` must appear as arguments of the function `f`. For a specific value of `t` the input `y` to `f` is a 2×1 vector, whose coefficients are the values of u_1 and u_2 at time t. Rather than referring to `y(1)` and `y(2)` in the definition of the equations on line 14, it is best again to use variable names which are easier to identify, e.g., `u1` and `u2`.
- Line 14 defines the right-hand sides of the ODE system as a 2×1 vector: the first coefficient is the first right-hand side ($\frac{du_1}{dt}$) and the second coefficient the second right-hand side ($\frac{du_2}{dt}$).
- Lines 6-10 correspond to the visualization of the results. To plot the time series of $u1$ and $u2$, we create a 2×1 array of subplots. Because the scales of $u1$ and $u2$ are different, it is best using two different graphs for $u1$ and $u2$ here. Type `help subplot` to learn more about it. On a different figure, we then plot the *phase plot* representing the evolution of u_2 in terms of u_1. Note that u_1 and u_2 vary cyclically. The periodic evolution of the two populations becomes clear from the closed curve u_2 vs. u_1 in the phase plot.

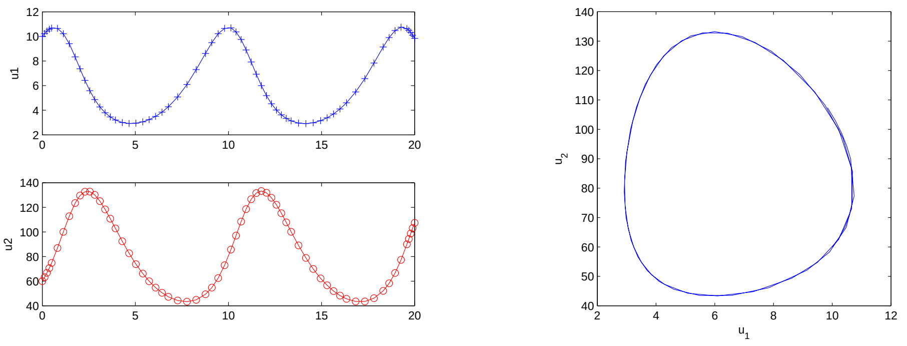


Figure L4f: Lotka-Volterra example.

Reducing a Higher-Order ODE

Numerical solution to IVPs involving higher order ODEs – homogeneous or not, linear or not, can be obtained using the same MATLAB commands as in the first-order by rewriting the ODE in the form of a system of first order ODEs.

Let's start with an example. Consider the IVP

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = \cos t, \quad \text{with} \quad y(0) = -1, \frac{dy}{dt}(0) = 0. \tag{L4.4}$$

To reduce the order of the ODE we introduce the intermediate unknown function $v = \frac{dy}{dt}$. As a result $\frac{dv}{dt} = \frac{d^2y}{dt^2}$ so that the ODE can be written $\frac{dv}{dt} + 4v + 3y = \cos t$. This equation only involves first-order derivatives, but we now have two unknown functions $y = y(t)$ and $v = v(t)$ with two ODEs. For MATLAB implementations it is necessary to write these ODEs in the form $\frac{d*}{dt} = \ldots$. Thus

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = \cos t \quad \Leftrightarrow \quad \begin{cases} \frac{dy}{dt} = v, \\ \frac{dv}{dt} = \cos t - 4v - 3y. \end{cases} \tag{L4.5}$$

Initial conditions from (L4.4) must also be transformed into initial conditions for y and v. Simply,

$$y(0) = -1, \frac{dy}{dt}(0) = 0 \quad \Leftrightarrow \quad \begin{cases} y(0) = -1, \\ v(0) = 0. \end{cases} \tag{L4.6}$$

EXERCISES

Instructions: For your lab write-up follow the instructions of LAB 1.

1. (a) Modify the function `ex_with_2eqs` to solve the IVP (L4.4) for $0 \le t \le 40$ using the MATLAB routine `ode45`. Call the new function `LAB04ex1`.
Let `[t,Y]` (note the upper case `Y`) be the output of `ode45` and `y` and `v` the unknown functions. Use the following commands to define the ODE:

```
function dYdt= f(t,Y)
y=Y(1); v=Y(2);
dYdt = [v; cos(t)-4*v-3*y];
```

Plot $y(t)$ and $v(t)$ in the same window (do not use `subplot`), and the phase plot showing v vs y in a separate window.
Add a legend to the first plot. (Note: to display $v(t) = y'(t)$, use `'v(t)=y''(t)'`).
Add a grid. Use the command `ylim([-1.5,1.5])` to adjust the y-limits for both plots. Adjust the x-limits in the phase plot so as to reproduce the pictures in Figure L4g.
Include the M-file in your report.

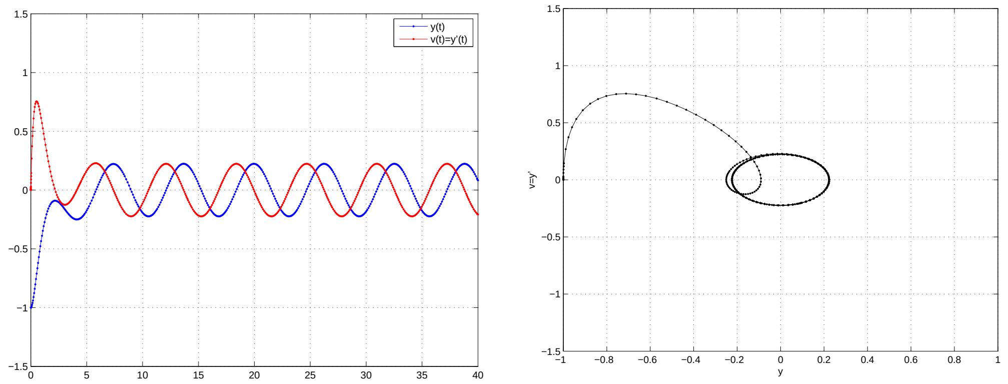


Figure L4g: Time series $y = y(t)$ and $v = v(t) = y'(t)$ (left), and phase plot $v = y'$ vs. y for (L4.4).

(b) For what (approximate) value(s) of t does y reach a local maximum in the window $0 \le t \le 40$? Check by reading the matrix `Y` and the vector `t`. Note that, because the M-file `LAB04ex1.m` is a function file, all the variables are local and thus not available on the Command Window. To read the matrix `Y` and the vector `t`, you need to modify the M-file by adding the line
`[t Y].`
Do not include the whole output in your lab write-up. Include only the values necessary to answer the question.

(c) What seems to be the long term behavior of y?

(d) Modify the initial conditions to $y(0) = 1.5$, $v(0) = 5$ and run the file `LAB04ex1.m` with the modified initial conditions. Based on the new graphs, determine whether the long term behavior of the solution changes. Explain. Include the pictures with the modified initial conditions to support your answer.

Nonlinear Problems

Nonlinear problems do not present any additional difficulty from an implementation point of view (they may present new numerical challenges for integration routines like `ode45`).

EXERCISES

2. (a) Consider the modified problem

$$\frac{d^2y}{dt^2} + 4y^2\frac{dy}{dt} + 3y = \cos t, \quad \text{with} \quad y(0) = -1, \frac{dy}{dt}(0) = 0. \qquad \text{(L4.7)}$$

The ODE (L4.7) is very similar to (L4.4) except for the y^2 term in the left-hand side. Because of the factor y^2 the ODE (L4.7) is nonlinear, while (L4.4) is linear. There is however very little to change in the implementation of (L4.4) to solve (L4.7). In fact, the only thing that needs to be modified is the ODE definition.
Modify the function defining the ODE in `LAB04ex1.m`. Call the revised file `LAB04ex2.m`. The new function M-file should reproduce the pictures in Fig L4h.
Include in your report the changes you made to `LAB04ex1.m` to obtain `LAB04ex2.m`.

(b) Compare the output of Figs L4g and L4h. Describe the changes in the behavior of the solution in the short term.

(c) Compare the long term behavior of both problems (L4.4) and (L4.7), in particular the amplitude of oscillations.

(d) Modify `LAB04ex2.m` so that it solves (L4.7) using Euler's method with $N = 400$ in the interval $0 \le t \le 40$ (use the file `euler.m` from LAB 3 to implement Euler's method; do

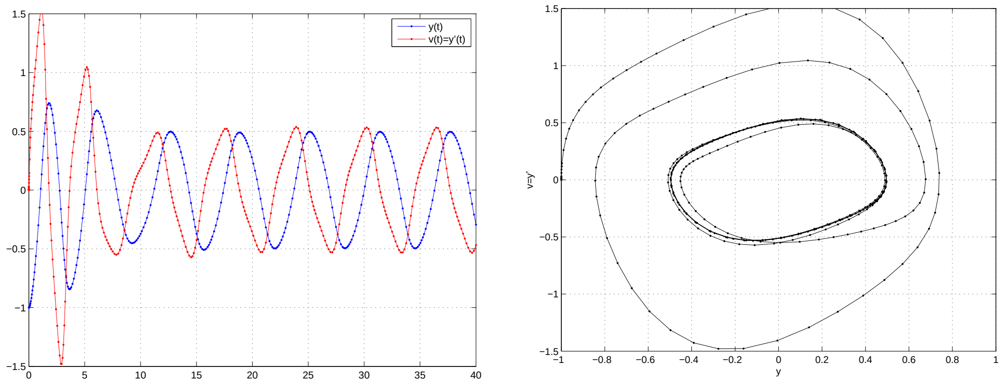


Figure L4h: Time series $y = y(t)$ and $v = v(t) = y'(t)$ (left), and phase plot $v = y'$ vs. y for (L4.7).

not delete the lines that implement `ode45`). Let `[te,Ye]` be the output of `euler`, and note that `Ye` is a matrix with two columns from which the Euler's approximation to $y(t)$ must be extracted. Plot the approximation to the solution $y(t)$ computed by `ode45` (in black) and the approximation computed by `euler` (in red) in the same window (you do not need to plot $v(t)$ nor the phase plot). Are the solutions identical? Comment.
Include the modified M-file in your report.

3. Solve numerically the IVP

$$\frac{d^2y}{dt^2} + 4y\frac{dy}{dt} + 3y = \cos t, \quad \text{with} \quad y(0) = -1, \frac{dy}{dt}(0) = 0$$

in the interval $0 \le t \le 40$. Include the M-file in your report.
Is the behavior of the solution significantly different from that of the solution of (L4.7)?
Is MATLAB giving any warning message? Comment.

A Third-Order Problem

Consider the third-order IVP

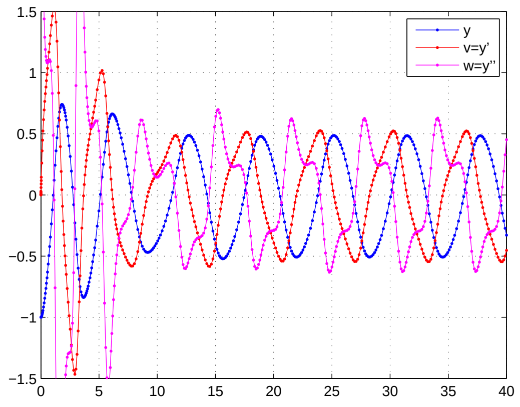
$$\frac{d^3y}{dt^3} + 4y^2\frac{d^2y}{dt^2} + 8y\left(\frac{dy}{dt}\right)^2 + 3\frac{dy}{dt} = -\sin t, \quad \text{with} \quad y(0) = -1, \frac{dy}{dt}(0) = 0, \frac{d^2y}{dt^2}(0) = 4. \tag{L4.8}$$

Introducing $v = \frac{dy}{dt}$ and $w = \frac{dy^2}{dt^2}$ we obtain $\frac{dv}{dt} = w$ and $\frac{dw}{dt} = \frac{d^3y}{dt^3} = -\sin t - 4y^2w - 8yv^2 - 3v$. Moreover, $v(0) = \frac{dy}{dt}(0) = 0$ and $w(0) = \frac{d^2y}{dt^2}(0) = 4$. Thus (L4.8) is equivalent to

$$\left\{\begin{array}{l} \frac{dy}{dt} = v, \\ \frac{dv}{dt} = w, \\ \frac{dw}{dt} = -\sin t - 4y^2w - 8yv^2 - 3v \end{array}\right. \quad \text{with} \quad \left\{\begin{array}{l} y(0) = -1, \\ v(0) = 0, \\ w(0) = 4. \end{array}\right. \tag{L4.9}$$

4. (a) Write a function M-file that implements (L4.8) in the interval $0 \le t \le 40$. Note that the initial condition must now be in the form `[y0,v0,w0]` and the matrix `Y`, output of `ode45`, has now three columns (from which y, v and w must be extracted). On the same figure, plot the three time series and, on a separate window, plot the phase plot using

```
figure(2); plot3(y,v,w,'k.-');
grid on;  view([-40,60])
xlabel('y'); ylabel('v=y'''); zlabel('w=y''''');
```



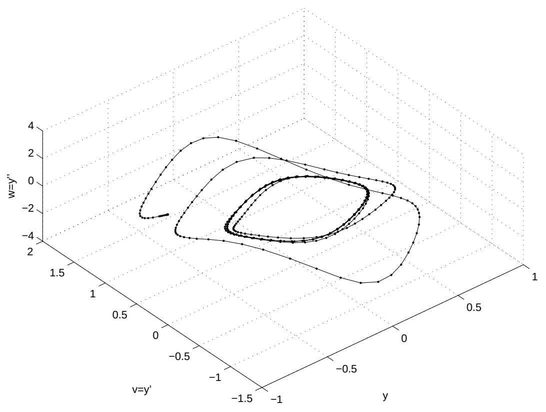


Figure L4i: Time series $y = y(t)$, $v = v(t) = y'(t)$, and $w = w(t) = y'(t)$ (left), and 3D phase plot $v = y'$ vs. y vs $w = y''$ for (L4.8) (rotated with `view = [-40,60]`).

Do not forget to modify the function defining the ODE.
The output is shown in Fig. L4i. The limits in the vertical axis of the plot on the left were deliberately set to the same ones as in Fig. L4h for comparison purposes using the MATLAB command `ylim([-1.5,1.5])`.
Include the M-file in you report.

(b) Compare the output of Figs L4h and L4i. What do you notice?

(c) Differentiate the ODE in (L4.7) and compare to the ODE in (L4.8).

(d) Explain why the solution of (L4.7) also satisfies the initial conditions in (L4.8). *Hint:* Substitute $t = 0$ into (L4.7) and use the initial conditions for y and v.