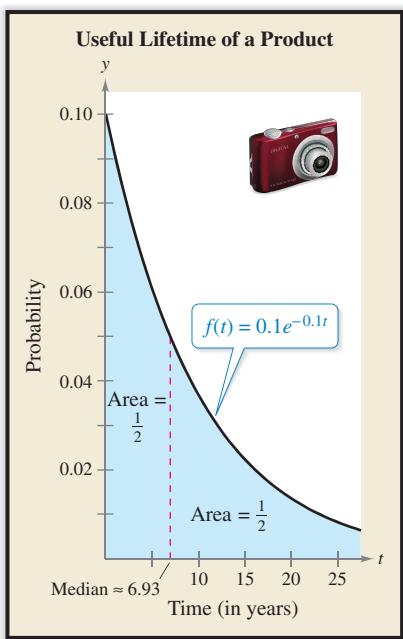


16 Probability



Example 4 on page 1175 shows how calculus can be used to find the mean and median useful lifetimes of a product.

- 16.1** Counting Principles
- 16.2** Probability
- 16.3** Discrete and Continuous Random Variables
- 16.4** Expected Value and Variance
- 16.5** Mathematical Induction
- 16.6** The Binomial Theorem

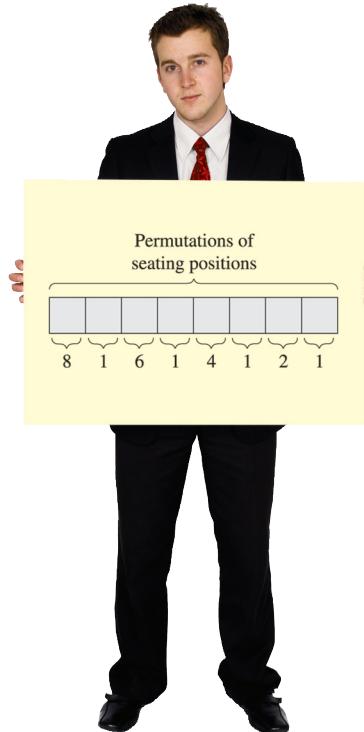


takayuki/www.shutterstock.com
Kurhan/www.shutterstock.com



16.1 Counting Principles

- Solve simple counting problems.
- Use the Fundamental Counting Principle to solve counting problems.
- Use permutations to solve counting problems.
- Use combinations to solve counting problems.



In Exercise 25 on page 1144, you will use counting principles to find the number of ways four couples can sit at a concert.

Simple Counting Problems

This section and Section 16.2 offer a brief introduction to some of the basic counting principles and their application to probability.



Example 1 Selecting Pairs of Numbers at Random

Eight pieces of paper are numbered from 1 to 8 and placed in a box. One piece of paper is drawn from the box, its number is written down, and the piece of paper is *replaced in the box*. Then, a second piece of paper is drawn from the box, and its number is written down. Finally, the two numbers are added together. In how many different ways can a sum of 12 be obtained?

SOLUTION To solve this problem, count the different ways that a sum of 12 can be obtained using two numbers from 1 to 8.

First number	4	5	6	7	8
Second number	8	7	6	5	4

From this list, you can see that a sum of 12 can occur in five different ways.

✓ Checkpoint 1

In Example 1, in how many different ways can a sum of 14 be obtained? □



Example 2 Selecting Pairs of Numbers at Random

Eight pieces of paper are numbered from 1 to 8 and placed in a box. One piece of paper is drawn from the box and, *without replacing the paper*, a second piece of paper is drawn. The numbers on the pieces of paper are written down and totaled. In how many different ways can a sum of 12 be obtained?

SOLUTION To solve this problem, count the different ways that a sum of 12 can be obtained using two *different* numbers from 1 to 8.

First number	4	5	7	8
Second number	8	7	5	4

So, a sum of 12 can be obtained in four different ways.

✓ Checkpoint 2

In Example 2, in how many different ways can a sum of 14 be obtained? □

The counting problems in Examples 1 and 2 can be distinguished by saying that the random selection in Example 1 occurs **with replacement**, whereas the random selection in Example 2 occurs **without replacement**, which eliminates the possibility of choosing two 6's.

Counting Principles

Examples 1 and 2 describe simple counting problems in which you can *list* each possible way that an event can occur. When it is possible, this is the best way to solve a counting problem. Some events, however, can occur in so many different ways that it is not feasible to write out the entire list. In such cases, you must rely on formulas and counting principles. The most important of these is the **Fundamental Counting Principle**.

Fundamental Counting Principle

Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \cdot m_2$.

The Fundamental Counting Principle can be extended to three or more events. For instance, the number of ways that three events E_1 , E_2 , and E_3 can occur is $m_1 \cdot m_2 \cdot m_3$.



Example 3 Using the Fundamental Counting Principle

How many different pairs of letters from the English alphabet are possible?

SOLUTION This experiment has two events. The first event is the choice of the first letter, and the second event is the choice of the second letter. Because the English alphabet contains 26 letters, it follows that the number of pairs of letters is $26 \cdot 26 = 676$.

✓ Checkpoint 3

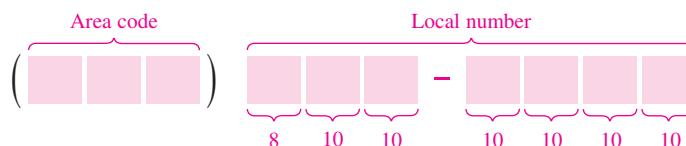
The extension for a file name on a computer typically consists of three letters from the English alphabet. How many different extensions are possible? ■



Example 4 Using the Fundamental Counting Principle

Telephone numbers in the United States currently have 10 digits. The first three are the *area code* and the next seven are the *local telephone number*. How many different local telephone numbers are possible within each area code? (Note that at this time, a local telephone number cannot begin with 0 or 1.)

SOLUTION Because the first digit cannot be 0 or 1, there are only eight choices for the first digit. For each of the other six digits, there are 10 choices.



So, the number of local telephone numbers that are possible within each area code is $8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8,000,000$.

✓ Checkpoint 4

The catalog number for a particular product consists of one letter from the English alphabet followed by a five-digit number. How many different catalog numbers are possible? ■

Permutations

One important application of the Fundamental Counting Principle is in determining the number of ways that n elements can be arranged (in order). An ordering of n elements is called a **permutation** of the elements.

Definition of Permutation

A **permutation** of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

Example 5 Finding the Number of Permutations of n Elements

How many permutations of the letters

A, B, C, D, E, and F

are possible?

SOLUTION Consider the following reasoning.

First position: Any of the six letters.

Second position: Any of the remaining five letters.

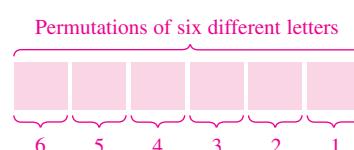
Third position: Any of the remaining four letters.

Fourth position: Any of the remaining three letters.

Fifth position: Either of the remaining two letters.

Sixth position: The one remaining letter.

So, the numbers of choices for the six positions are shown below.



The total number of permutations of the six letters is

$$\begin{aligned} 6! &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 720. \end{aligned}$$

Checkpoint 5

How many permutations of the letters

W, X, Y, and Z

are possible? □

Number of Permutations of n Elements

The number of permutations of n distinct elements is given by

$$n \cdot (n - 1) \cdot \dots \cdot 4 \cdot 3 \cdot 2 \cdot 1 = n!.$$

In other words, there are $n!$ different ways that n distinct elements can be ordered.

It is useful, on occasion, to order a *subset* of a collection of elements rather than the entire collection. For example, you might want to choose (and order) r elements out of a collection of n elements. Such an ordering is called a **permutation of n elements taken r at a time**.



Example 6 Counting Kidney Donors

A patient with end-stage kidney (renal) disease has eight family members who are potential kidney donors. How many *possible* orders are there for a best match, a second-best match, and a third-best match?

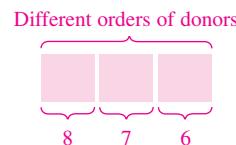
SOLUTION Here are the different possibilities.

Best match (first position): *Eight* choices

Second-best match (second position): *Seven* choices

Third-best match (third position): *Six* choices

The numbers of choices for the three positions are shown below.



So, using the Fundamental Counting Principle, you can determine that the number of different orders is $8 \cdot 7 \cdot 6 = 336$.

Checkpoint 6

In how many different ways can the top three potential donors in Example 6 be ordered when six family members are potential donors? ■

Permutations of n Elements Taken r at a Time

The number of permutations of n distinct elements taken r at a time is

$$\begin{aligned} {}_n P_r &= \frac{n!}{(n - r)!} \\ &= n(n - 1)(n - 2) \cdots (n - r + 1). \end{aligned}$$

Using this formula, you can rework Example 6 to find the number of permutations of the eight potential donors taken three at a time.

$$\begin{aligned} {}_8 P_3 &= \frac{8!}{(8 - 3)!} \\ &= \frac{8!}{5!} \\ &= \frac{(8 \cdot 7 \cdot 6) \cdot 5!}{5!} \\ &= 336 \end{aligned}$$

The answer is 336 different orders, the same answer obtained in Example 6.

Remember that for permutations, order is important. So, when finding the possible permutations of the letters A, B, C, and D taken three at a time, the permutations

ABD and BAD

would be different because the *order* of the elements is different.

Consider, however, the possible permutations of the letters A, A, B, and C. The total number of permutations of these four letters would be

$${}_4P_4 = 4!$$

Not all of these arrangements, however, would be *distinguishable* because there are two A's in the list. To find the number of distinguishable permutations, you can use the following formula.

Distinguishable Permutations

Suppose a set of n objects has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on, with

$$n = n_1 + n_2 + n_3 + \dots + n_k.$$

Then the number of **distinguishable permutations** of the n objects is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \dots \cdot n_k!}.$$

Example 7 Distinguishable Permutations

In how many distinguishable ways can the letters in BANANA be written?

SOLUTION This word has six letters, so $n = 6$, but you have to take into account that there are three A's, two N's, and one B. So, the number of distinguishable ways the letters can be written is

$$\begin{aligned}\frac{n!}{n_1! \cdot n_2! \cdot n_3!} &= \frac{6!}{3! \cdot 2! \cdot 1!} \\ &= \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 2!} \\ &= 60.\end{aligned}$$

The 60 different distinguishable permutations are as follows.

AAABNN	AAANBN	AAANNB	AABANN	AABNAN	AABNNA
AANABN	AANANB	AANBAN	AANBNA	AANNAB	AANNBA
ABAANN	ABANAN	ABANNA	ABNAAN	ABNANA	ABNNAA
ANAABN	ANAANB	ANABAN	ANABNA	ANANAB	ANANBA
ANBAAN	ANBANA	ANBNAA	ANNAAB	ANNABA	ANNBAA
BAAANN	BAANAN	BAANNA	BANAAN	BANANA	BANNAA
BNAAAN	BNAANA	BNANAA	BNNAAA	NAAABN	NAAANB
NAABAN	NAABNA	NAANAB	NAANBA	NABAAN	NABANA
NABNAA	NANAAB	NANABA	NANBAA	NBAAAN	NBAANA
NBANAA	NBNAAA	NNAAAB	NNAABA	NNABAA	NNBAAA

Checkpoint 7

In how many distinguishable ways can the letters in MITOSIS be written? ■

Combinations

When you count the number of possible permutations of a set of elements, *order* is important. As a final topic in this section, you will look at a method of selecting subsets of a larger set in which order is *not important*. Such subsets are called **combinations of n elements taken r at a time**. For instance, the combinations

$$\{A, B, C\} \text{ and } \{B, A, C\}$$

are equivalent because both sets contain the same three elements, and the order in which the elements are listed is *not important*. So, you would count only one of the two sets. A common example of how a combination occurs is a card game in which the player is free to reorder the cards after they have been dealt.

Example 8 Combinations of n Elements Taken r at a Time

In how many different ways can three letters be chosen from the letters A, B, C, D, and E? (The order of the three letters is not important.)

SOLUTION The following subsets represent the different combinations of three letters that can be chosen from the five letters.

$\{A, B, C\}$	$\{A, B, D\}$
$\{A, B, E\}$	$\{A, C, D\}$
$\{A, C, E\}$	$\{A, D, E\}$
$\{B, C, D\}$	$\{B, C, E\}$
$\{B, D, E\}$	$\{C, D, E\}$

From this list, you can conclude that there are 10 different ways that three letters can be chosen from the five letters.

✓ Checkpoint 8

In how many different ways can two letters be chosen from the letters J, K, L, M, and N? (The order of the two letters is not important.)

Combinations of n Elements Taken r at a Time

The number of combinations of n elements taken r at a time is

$${}_nC_r = \frac{n!}{(n - r)!r!}.$$

Note that the formula for ${}_nC_r$ is the same one given for binomial coefficients. To see how this formula is used, solve the counting problem in Example 8. In that problem, you are asked to find the number of combinations of five elements taken three at a time. So, $n = 5$, $r = 3$, and the number of combinations is

$$\begin{aligned} {}_5C_3 &= \frac{5!}{2!3!} \\ &= \frac{5 \cdot 4 \cdot 3 \cdot 2!}{2! \cdot 3 \cdot 2 \cdot 1} \\ &= 10 \end{aligned}$$

which is the same answer obtained in Example 8.

TECH TUTOR



Most graphing utilities have keys that will evaluate the formulas for the number of permutations or combinations of n elements taken r at a time, as shown in the screen below. Consult the user's guide for your graphing utility for specific instructions on how to evaluate permutations and combinations.

8 nPr 3	336
5 nCr 3	10


Example 9 Counting Card Hands

A standard poker hand consists of five cards dealt from a deck of 52. How many different poker hands are possible? (After the cards are dealt the player may reorder them, and therefore order is not important.)

SOLUTION Using the formula for the number of combinations of 52 elements taken five at a time, the number of different poker hands is

$${}_{52}C_5 = \frac{52!}{47!5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960 \text{ different hands.}$$

✓ **Checkpoint 9**

In three-card poker, each player is dealt three cards from a deck of 52. How many different three-card poker hands are possible? (Order is not important.) □


Example 10 Choosing Between Permutations and Combinations

Decide whether each scenario should be counted using permutations or combinations.

- a. Number of different arrangements of three types of flowers from an array of 20 types
- b. Number of different three-digit pin numbers for a debit card

SOLUTION To determine which counting principle is necessary to solve each problem, ask yourself two questions: (1) Is the order of the elements important? If yes, then you should solve the problem using permutations. (2) Are the chosen elements a subset of a larger set in which order is not important? If yes, then you should solve the problem using combinations.

- a. The order of the flowers is not important. So, the number of possible flower arrangements should be counted using combinations.
- b. The order of the digits in the pin number matters. So, the number of possible pin numbers should be counted using permutations.

✓ **Checkpoint 10**

Should the number of different nine-digit social security numbers be counted using permutations or combinations? □


SUMMARIZE (Section 16.1)

1. State the Fundamental Counting Principle (*page 1137*). For examples of using the Fundamental Counting Principle, see Examples 3 and 4.
2. State the definition of permutation and explain how to find the number of permutations of n elements (*page 1138*), the number of permutations of n elements taken r at a time (*page 1139*), and the number of distinguishable permutations of n elements (*page 1140*). For examples of finding numbers of permutations, see Examples 5, 6, and 7.
3. Explain how to find the number of combinations of n elements taken r at a time (*page 1141*). For examples of finding numbers of combinations, see Examples 8 and 9.

SKILLS WARM UP 16.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2 and 15.1.

In Exercises 1–4, evaluate the expression.

1. $13 \cdot 8^2 \cdot 2^3$

2. $10^2 \cdot 9^3 \cdot 4$

3. $\frac{12!}{2!(7!)(3!)}$

4. $\frac{25!}{22!}$

In Exercises 5–8, simplify the expression.

5. $\frac{n!}{(n-4)!}$

7. $\frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n)}{2^n}$

6. $\frac{(2n)!}{4(2n-3)!}$

8. $\frac{3 \cdot 6 \cdot 9 \cdot 12 \cdots (3n)}{3^n}$

Exercises 16.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Random Selection In Exercises 1–10, determine the number of ways a computer can randomly generate the specified integer(s) from 1 through 15. See Examples 1 and 2.

1. An odd integer
2. An even integer
3. A prime integer
4. An integer that is greater than 9
5. An integer that is divisible by 4
6. An integer that is divisible by 5
7. Two integers whose sum is 20
8. Two integers whose sum is 16
9. Two *distinct* integers whose sum is 20
10. Two *distinct* integers whose sum is 16

11. **Job Applicants** A small college needs two additional faculty members: a chemist and a statistician. There are five applicants for the chemistry position and six applicants for the statistics position. In how many ways can these positions be filled?
12. **Dissections** In a physiology class, a student must dissect three different organisms. The student can select one of nine earthworms, one of four frogs, and one of seven fetal pigs. In how many ways can the student select the specimens?
13. **Toboggan Ride** Six people are lining up for a ride on a toboggan, but only two of the six are willing to take the first position. With that constraint, in how many ways can the six people be seated on the toboggan? Draw a diagram to illustrate the number of ways that the people can sit.

14. **Course Schedule** A college student needs to schedule five prescribed courses for next semester. Only three of the five courses are able to be scheduled as the first class of the day. With that constraint, in how many ways can she select her schedule? Draw a diagram to illustrate the number of ways she can select her schedule.
15. **License Plate Numbers** In a certain state, each automobile license plate number consists of two letters followed by a four-digit number. How many distinct license plate numbers can be formed?
16. **License Plate Numbers** In a certain state, each automobile license plate number consists of two letters followed by a four-digit number. To avoid confusion between “O” and “zero” and “T” and “one,” the letters “O” and “I” are not used. How many distinct license plate numbers can be formed?
17. **True-False Exam** In how many ways can a 10-question true-false exam be answered? (Assume that no questions are omitted.)
18. **True-False Exam** In how many ways can a 15-question true-false exam be answered? (Assume that no questions are omitted.)
19. **Three-Digit Numbers** How many three-digit numbers can be formed under each condition?
 - (a) The leading digit cannot be 0.
 - (b) The leading digit cannot be 0 and no repetition of digits is allowed.
 - (c) The leading digit cannot be 0 and the number must be a multiple of 5.

- 20. Four-Digit Numbers** How many four-digit numbers can be formed under each condition?
- The leading digit cannot be 0.
 - The leading digit cannot be 0 and no repetition of digits is allowed.
 - The leading digit cannot be 0 and the number must be a multiple of 5.
- 21. Combination Lock** A combination lock will open when the right choice of three numbers (from 1 to 40, inclusive) is selected. How many different lock combinations are possible?
- 22. Combination Lock** A combination lock will open when the right choice of three numbers (from 1 to 50, inclusive) is selected. How many different lock combinations are possible?
- 23. Chemistry Lab** In a chemistry lab, five oxidation-reduction reactions can be performed in any order. How many different orders are possible for the five reactions?
- 24. Biology Lab** In a plant biology lab, seven different experiments can be performed in any order. How many different orders are possible for the seven experiments?
- 25. Concert Seats** Four couples have reserved seats in a given row for a concert. In how many different ways can they be seated, given the following conditions?
- There are no restrictions.
 - The two members of each couple wish to sit together.
- 26. Single File** In how many orders can five girls and four boys walk through a doorway in single file, given the following conditions?
- There are no restrictions.
 - The boys go before the girls.
 - The girls go before the boys.

Finding a Permutation In Exercises 27–36, evaluate nP_r using the formula on page 1139.

27. ${}_4P_3$ 28. ${}_5P_4$
 29. ${}_4P_4$ 30. ${}_7P_7$
 31. ${}_6P_2$ 32. ${}_{10}P_5$
 33. ${}_8P_0$ 34. ${}_{11}P_0$
 35. ${}_{12}P_4$ 36. ${}_{48}P_3$

Finding a Permutation Using a Graphing Utility In Exercises 37–40, evaluate nP_r using a graphing utility.

37. ${}_{20}P_6$ 38. ${}_{10}P_8$
 39. ${}_{120}P_4$ 40. ${}_{100}P_3$

Finding Distinguishable Permutations In Exercises 41–48, find the number of distinguishable permutations of the group of letters. See Example 7.

41. B, R, O, O, M 42. V, I, V, I, D
 43. A, S, S, E, T, S 44. A, C, C, R, U, A, L
 45. S, P, E, C, I, E, S 46. M, A, M, M, A, L
 47. A, L, G, E, B, R, A 48. M, I, S, S, I, S, S, I, P, P, I

Finding a Combination In Exercises 49–58, evaluate $_nC_r$ using the formula on page 1141. See Examples 8 and 9.

49. ${}_4C_3$ 50. ${}_5C_4$
 51. ${}_4C_4$ 52. ${}_7C_7$
 53. ${}_6C_2$ 54. ${}_{10}C_5$
 55. ${}_8C_0$ 56. ${}_{11}C_0$
 57. ${}_{12}C_4$ 58. ${}_{48}C_3$

 **Finding a Combination Using a Graphing Utility** In Exercises 59–62, evaluate $_nC_r$ using a graphing utility.

59. ${}_{20}C_5$ 60. ${}_{10}C_7$
 61. ${}_{42}C_5$ 62. ${}_{50}C_6$

Simplifying Expressions In Exercises 63–66, simplify the expression.

63. ${}_nP_3$ 64. ${}_nC_3$
 65. ${}_nP_{n-1}$ 66. ${}_nC_{n-1}$

Choosing Between Permutations and Combinations

In Exercises 67–70, decide whether the scenario should be counted using permutations or combinations. Explain your reasoning. See Example 10.

67. Number of ways 10 people can line up in a row for concert tickets
 68. Number of ways three different roles can be filled by 10 people auditioning for a play
 69. Number of different three-topping pizzas that can be made from an assortment of 15 different toppings
 70. Number of ways a jury of 12 people can be selected from a group of 50 people

71. Choosing Officers From a pool of 15 candidates, the offices of president, vice-president, secretary, and treasurer will be filled. In how many different ways can the offices be filled if each of the 15 candidates can hold any office?

72. Bike Race There are 10 bicyclists entered in a race. In how many different ways can the top three places be decided?

73. Forming an Experimental Group In order to conduct an experiment, five students are randomly selected from a class of 30. How many different groups of five students are possible?

74. Lab Practical A student may answer any 15 questions from a total of 20 questions on a biology lab practical. In how many ways can the student select the questions?



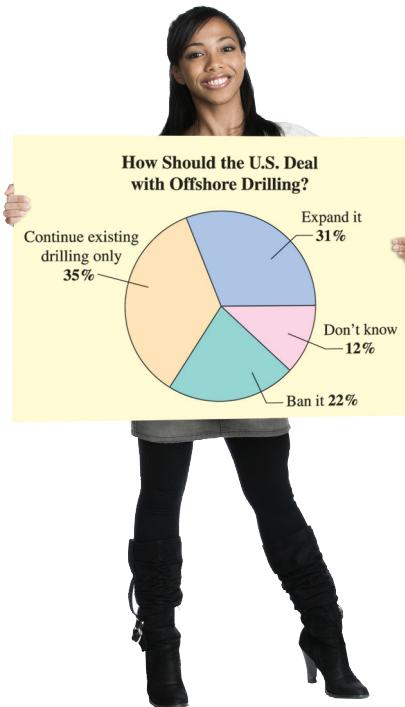
86.

HOW DO YOU SEE IT?

HOW DO YOU SEE IT? Without calculating, determine whether the value of ${}_nP_r$ is greater than the value of ${}_nC_r$ for the values of n and r given in the table. Complete the table using yes (Y) or no (N). Is the value of ${}_nP_r$ always greater than the value of ${}_nC_r$? Explain.

16.2 Probability

- Determine the sample space of an experiment and find the probability of an event.
- Find the probability of mutually exclusive events.
- Find the probability of independent events.



In Exercise 84 on page 1156, you will find probabilities based on the attitudes of Americans about offshore drilling for gas and oil.

The Probability of an Event

In measuring the uncertainties of everyday life, ambiguous terminology, such as “fairly certain” and “highly unlikely,” is often used. Mathematics attempts to assign a number to the likelihood of the occurrence of an event. This measurement is called the **probability** that the event will occur. For instance, when a fair coin is tossed, the probability that it will land heads up is $\frac{1}{2}$.

Any happening whose result is uncertain is called an **experiment**. The possible results of an experiment are **outcomes**, the set of all possible outcomes of an experiment is the **sample space**, and any subcollection of a sample space is an **event**.



Example 1 Finding the Sample Space

Find the sample space for each experiment.

- One coin is tossed.
- Two coins are tossed.

SOLUTION

- Because the coin will land either heads up (denoted by H) or tails up (denoted by T), the sample space is $S = \{H, T\}$.
- Because either coin can land heads up or tails up, the possible outcomes are

HH = heads up on both coins

HT = heads up on first coin and tails up on second coin

TH = tails up on first coin and heads up on second coin

TT = tails up on both coins.

The sample space is $S = \{HH, HT, TH, TT\}$. This list distinguishes between the two cases HT and TH , even though these two outcomes appear to be similar.

Another way to list the possible outcomes is by using a **tree diagram**, as shown in Figure 16.1.

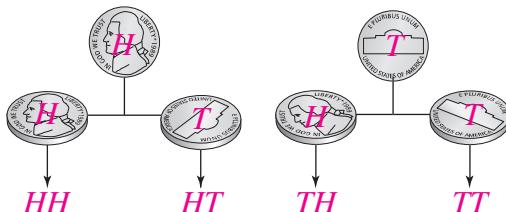


FIGURE 16.1

✓ Checkpoint 1

Find the sample space when a six-sided die is tossed.

When an event E has equally likely outcomes and its sample space S has equally likely outcomes, the probability of event E is

$$P(E) = \frac{\text{Number of outcomes in event } E}{\text{Number of outcomes in sample space } S}.$$

More formally, the number of outcomes in event E is denoted by $n(E)$, and the number of outcomes in the sample space S is denoted by $n(S)$.

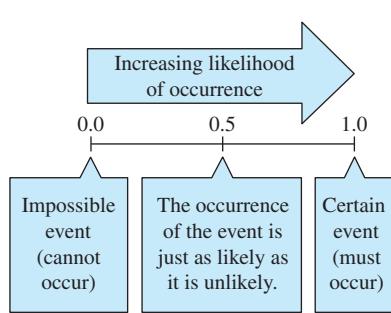


FIGURE 16.2

The Probability of an Event

If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, then the **probability** of event E is

$$P(E) = \frac{n(E)}{n(S)}.$$

Because the number of outcomes in an event must be less than or equal to the number of outcomes in the sample space, the probability of an event must be a number between 0 and 1, as shown in Figure 16.2. So, for any event E , $0 \leq P(E) \leq 1$.

Properties of the Probability of an Event

Let E be an event that is a subset of a finite sample space S .

1. $0 \leq P(E) \leq 1$
2. If $P(E) = 0$, then E cannot occur and is called an **impossible event**.
3. If $P(E) = 1$, then E must occur and is called a **certain event**.



Example 2 Finding the Probability of an Event

- a. Two coins are tossed. What is the probability that both land heads up?
- b. A card is drawn at random from a standard deck of playing cards (see Figure 16.3). What is the probability that it is an ace?

SOLUTION

- a. Following the procedure in Example 1(b), let $E = \{HH\}$ and $S = \{HH, HT, TH, TT\}$. The probability of getting two heads is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}.$$

- b. Because there are 52 cards in a standard deck of playing cards and there are four aces (one in each suit), the probability of drawing an ace is

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{52} = \frac{1}{13}.$$

Checkpoint 2

A card is drawn at random from a standard deck of playing cards. What is the probability that the card is a diamond? ■

A ♥	A ♦	A ♣	A ♠
2 ♥	2 ♦	2 ♣	2 ♠
3 ♥	3 ♦	3 ♣	3 ♠
4 ♥	4 ♦	4 ♣	4 ♠
5 ♥	5 ♦	5 ♣	5 ♠
6 ♥	6 ♦	6 ♣	6 ♠
7 ♥	7 ♦	7 ♣	7 ♠
8 ♥	8 ♦	8 ♣	8 ♠
9 ♥	9 ♦	9 ♣	9 ♠
10 ♥	10 ♦	10 ♣	10 ♠
J ♥	J ♦	J ♣	J ♠
Q ♥	Q ♦	Q ♣	Q ♠
K ♥	K ♦	K ♣	K ♠

Standard Deck of Playing Cards

FIGURE 16.3

A probability can be written as a fraction, a decimal, or a percent. For instance, in Example 2(a), the probability of getting two heads can be written as $\frac{1}{4}$, 0.25, or 25%.

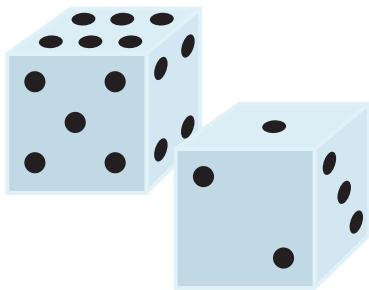


FIGURE 16.4

**Example 3** Finding the Probability of an Event

Two six-sided dice are tossed. What is the probability that the total of the two dice is 7? (See Figure 16.4.)

SOLUTION Because there are six possible outcomes on each die, you can use the Fundamental Counting Principle to conclude that there are $6 \cdot 6$ or 36 different outcomes when two dice are tossed. To find the probability of rolling a total of 7, you must first count the number of ways this can occur.

First Die	1	2	3	4	5	6
Second Die	6	5	4	3	2	1

TECH TUTOR

You can use the *random integer* feature of a graphing utility to simulate the tossing of a die or set of dice. For instance, the result of rolling a pair of dice five times and summing the outcomes is shown below.

```
randInt(1,6,5)+randInt(1,6,5)
{11 7 4 10 5}
```

So, a total of 7 can be rolled in six ways, which means that the probability of rolling a total of 7 is

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$

✓ Checkpoint 3

Two six-sided dice are tossed. What is the probability that the total of the two dice is 4?

**Example 4** Finding the Probability of an Event

Twelve-sided dice can be constructed (in the shape of regular dodecahedrons) such that each of the numbers from 1 to 6 appears twice on each die, as shown in Figure 16.5. Can these dice be used in any game requiring ordinary six-sided dice without changing the probabilities of different outcomes?

SOLUTION For an ordinary six-sided die, each of the numbers

1, 2, 3, 4, 5, and 6

occurs only once, so the probability of any particular number coming up is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}.$$

For a twelve-sided die, each number occurs twice, so the probability of any particular number coming up is

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{12} = \frac{1}{6}.$$

So, the twelve-sided dice can be used in any game requiring ordinary six-sided dice without changing the probabilities of different outcomes.

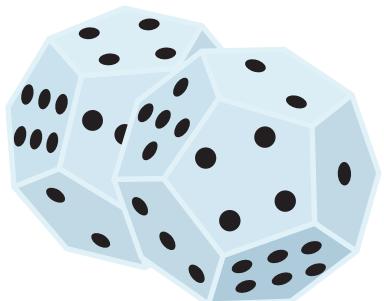


FIGURE 16.5

✓ Checkpoint 4

Two twelve-sided dice (as described in Example 4) are tossed. What is the probability that the total of the two dice is less than 5?

In Examples 2, 3, and 4, you simply counted the outcomes in the desired events. For larger sample spaces, however, you should use the counting principles discussed in Section 16.1.


Example 5 The Probability of Winning a Lottery

In Louisiana's Lotto game, a player chooses six different two-digit numbers from 01 to 40. To win the jackpot, the six numbers must match the six numbers drawn (in any order) by the lottery commission. What is the probability of winning the jackpot?

SOLUTION Because the order of the six numbers does not matter, use the formula for the number of combinations of 40 elements taken six at a time.

$$n(S) = {}_{40}C_6 = \frac{40 \cdot 39 \cdot 38 \cdot 37 \cdot 36 \cdot 35}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 3,838,380$$

For a person buying only one ticket, the probability of winning the jackpot is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{3,838,380}.$$

✓ **Checkpoint 5**

In Pennsylvania's Cash 5 game, a player chooses five different numbers from 1 to 43. To win the jackpot, the five numbers must match the five numbers drawn (in any order) by the lottery commission. What is the probability of winning the jackpot? ■


Example 6 Random Selection

The numbers of colleges and universities in various regions of the United States in 2010 are shown in Figure 16.6. One institution is selected at random. What is the probability that the institution is in one of the three southern regions? (*Source: U.S. National Center for Education Statistics*)

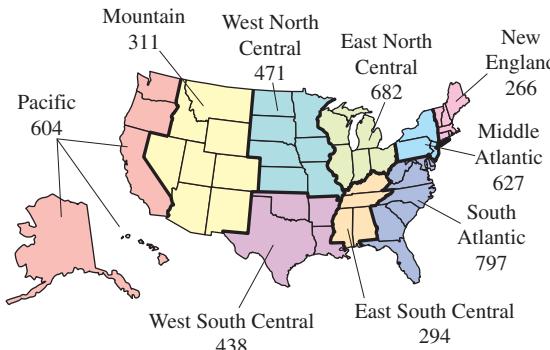


FIGURE 16.6

SOLUTION From the figure, the total number of colleges and universities is 4490. Because there are

$$797 + 294 + 438 = 1529$$

colleges and universities in the three southern regions, the probability that the institution is in one of these regions is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1529}{4490} \approx 0.341.$$

✓ **Checkpoint 6**

In Example 6, what is the probability that the randomly selected institution is in the Pacific region? ■



Mutually Exclusive Events

Two events A and B (from the same sample space) are **mutually exclusive** when A and B have no outcomes in common. In the terminology of sets, the **intersection of A and B** is the empty set and $P(A \cap B) = 0$. For instance, when two dice are tossed, the event A of rolling a total of 6 and the event B of rolling a total of 9 are mutually exclusive. To find the probability that one or the other of two mutually exclusive events will occur, you can *add* their individual probabilities.

Probability of the Union of Two Events

If A and B are events in the same sample space, then the probability of A or B occurring is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$



Example 7 The Probability of a Union

One card is selected at random from a standard deck of 52 playing cards. What is the probability that the card is either a heart or a face card?

SOLUTION Because the deck has 13 hearts, the probability of selecting a heart (event A) is

$$P(A) = \frac{13}{52}.$$

Similarly, because the deck has 12 face cards, the probability of selecting a face card (event B) is

$$P(B) = \frac{12}{52}.$$

Because three of the cards are hearts and face cards (see Figure 16.7), it follows that the events A and B are *not* mutually exclusive, and that

$$P(A \cap B) = \frac{3}{52}.$$

Finally, applying the formula for the probability of the union of two events, you can conclude that the probability of selecting a heart or a face card is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} \\ &\approx 0.423. \end{aligned}$$

Checkpoint 7

One card is selected at random from a standard deck of 52 playing cards. What is the probability that the card is either a spade or an eight? ■


Example 8 Probability of Mutually Exclusive Events

The human resources department of a company has compiled data showing the number of years of service for each employee. The results are shown in the table.

Years of service	Number of employees
0–4	157
5–9	89
10–14	74
15–19	63
20–24	42
25–29	38
30–34	37
35–39	21
40–44	8

An employee is chosen at random. What is the probability that the employee has had 9 or fewer years of service?

SOLUTION To begin, add the number of employees and find that the total is 529. Next, let event A represent choosing an employee with 0 to 4 years of service. Because there are 157 employees with 0 to 4 years of service, the probability of choosing an employee who has 4 or fewer years of service is

$$P(A) = \frac{157}{529}.$$

Let event B represent choosing an employee with 5 to 9 years of service. Because there are 89 employees with 5 to 9 years of service, the probability of choosing one of these employees is

$$P(B) = \frac{89}{529}.$$

Because A and B have no outcomes in common, you can conclude that these two events are mutually exclusive and that

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{157}{529} + \frac{89}{529} \\ &= \frac{246}{529} \\ &\approx 0.465. \end{aligned}$$

So, the probability of choosing an employee who has had 9 or fewer years of service is about 0.465.

✓ Checkpoint 8

In Example 8, what is the probability that the employee chosen at random has had 30 or more years of service? □

Independent Events

Two events are **independent** when the occurrence of one has no effect on the occurrence of the other. For instance, rolling a total of 12 with two six-sided dice has no effect on the outcome of future rolls of the dice. To find the probability that two independent events will occur, *multiply* the probabilities of the events.

Probability of Independent Events

If A and B are independent events, then the probability that both A and B will occur is

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

This definition can be extended to any number of independent events.



Example 9 Probability of Independent Events

- The probability that a particular surgery is successful is 0.90. What is the probability that two surgeries are successful? (Assume the surgeries are on different patients.)
- Two six-sided dice are tossed three times. What is the probability that the total for each toss is 7?

SOLUTION

- The probability that two surgeries are successful is

$$P(A) \cdot P(A) = (0.90)(0.90) = 0.81.$$

- From Example 3, you know that the probability of tossing two dice once and getting a total of 7 is $P(A) = \frac{1}{6}$. So, when two dice are tossed three times, the probability that the total is 7 every time is

$$P(A) \cdot P(A) \cdot P(A) = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{216} \approx 0.005.$$

✓ Checkpoint 9

The probability that a particular surgery is successful is 0.65. What is the probability that two surgeries are successful? (Assume the surgeries are on different patients.) □

SUMMARIZE (Section 16.2)

- State the definition of sample space (page 1146). For an example of finding the sample space for an experiment, see Example 1.
- State the definition of the probability of an event (page 1147). For examples of finding the probability of an event, see Examples 2, 3, 4, 5, and 6.
- State the definitions of mutually exclusive events and the probability of the union of two events (page 1150). For examples of finding the probability of the union of two events, see Examples 7 and 8.
- State the definitions of independent events and the probability of independent events (page 1152). For an example of finding the probability of independent events, see Example 9.



SKILLS WARM UP 16.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 0.3, 15.1, and 16.1.

In Exercises 1–8, evaluate the expression.

1. $\frac{1}{4} + \frac{5}{8} - \frac{5}{16}$

2. $\frac{4}{15} + \frac{3}{5} - \frac{1}{3}$

3. $\frac{5 \cdot 4}{5!}$

4. $\frac{5!22!}{27!}$

5. $\frac{4!}{8!12!}$

6. $\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{9!}$

7. $\frac{{}^5C_3}{{}^{10}C_3}$

8. $\frac{{}^{10}C_2 \cdot {}^{10}C_2}{{}^{20}C_4}$

In Exercises 9 and 10, evaluate the expression. (Round your answer to three decimal places.)

9. $\left(\frac{99}{100}\right)^{100}$

10. $1 - \left(\frac{89}{100}\right)^{50}$

Exercises 16.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Finding the Sample Space In Exercises 1–6, find the sample space for the experiment. See *Example 1*.

1. A coin and a six-sided die are tossed.
2. A six-sided die is tossed twice and the sum of the points is recorded.
3. A taste tester ranks three varieties of yogurt (A, B, and C) according to preference.
4. Two marbles are selected from a sack containing two red marbles, two blue marbles, and one black marble. The color of each marble is recorded.
5. Two county supervisors are selected from five supervisors (A, B, C, D and E) to study a recycling plan.
6. A sales representative makes presentations of a product in three homes. In each home, there may be a sale (denote by *S*) or there may be no sale (denote by *F*).

Using a Tree Diagram In Exercises 7–10, make a tree diagram that shows the possible outcomes that make up the sample space for the experiment.

7. Three coins are tossed.
8. A committee rates three programs A, B, and C in order of importance.
9. Two of four vendors A, B, C, and D are given contracts for the upcoming year.
10. Team A plays team B in one semifinal game and team C plays team D in the other semifinal game to determine the two teams in the championship game.

Finding the Probability of an Event In Exercises 11–14, a computer randomly generates an integer from 1 through 50. Find the probability of the event. See *Examples 2, 3, and 4*.

11. An odd number is generated.

12. A number less than 25 is generated.
13. A prime number is generated.
14. A multiple of 5 is generated.

Finding the Probability of an Event In Exercises 15–18, a coin is tossed three times. Find the probability of the event. See *Examples 2, 3, and 4*.

15. Getting exactly one head
16. Getting a tail on the second toss
17. Getting at least one tail
18. Getting at least two heads

Finding the Probability of an Event In Exercises 19–24, two six-sided dice are tossed. Find the probability of the event. See *Examples 2, 3, and 4*.

19. The sum is 5.
20. The sum is less than 10.
21. The sum is at least 7.
22. The sum is 2, 3, or 8.
23. The sum is odd and no more than 7.
24. The sum is odd or a prime number.

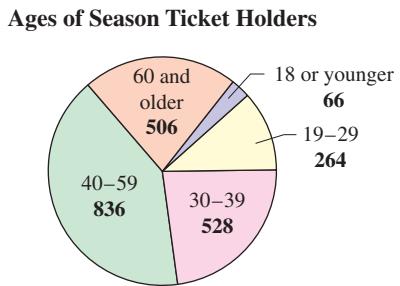
Finding the Probability of an Event In Exercises 25–30, a card is selected at random from a standard deck of 52 playing cards. Find the probability of the event. See *Examples 2, 3, and 4*.

25. Getting a red card
26. Getting a face card
27. Not getting a face card
28. Getting a red card that is not a face card
29. Getting a card that is a 6 or less (aces are low)
30. Getting a card that is a 9 or higher (aces are low)

Using Combinations In Exercises 31–36, two marbles are drawn randomly (without replacement) from a bag containing two green, three yellow, and four red marbles. Find the probability of the event.

31. Drawing exactly one red marble
32. Drawing exactly one yellow marble
33. Drawing two green marbles
34. Drawing two red marbles
35. Drawing none of the yellow marbles
36. Drawing marbles of different colors

Random Selection In Exercises 37–40, one of a team's 2200 season ticket holders is selected at random to win a prize. The circle graph shows the ages of the season ticket holders. Find the probability of the event. See Example 6.



37. The winner is younger than 19 years old.
38. The winner is older than 39 years old.
39. The winner is 19 to 39 years old.
40. The winner is younger than 19 years old or older than 59 years old.

The Probability of a Union In Exercises 41–44, one card is selected at random from a standard deck of 52 playing cards. Use a formula to find the probability of the union of the two events. See Example 7.

41. The card is a club or a king.
42. The card is a face card or a black card.
43. The card is a face card or a 2.
44. The card is a heart or a spade.

Probability of Mutually Exclusive Events In Exercises 45–48, use the table, which shows the age groups of students in a college sociology class. See Example 8.

Age	Number of students
18–19	11
20–21	18
22–30	2
31–40	1

A student from the class is chosen randomly for a project. Find the probability that the student is the given age.

45. 20 or 21 years old
46. 18 to 21 years old
47. Older than 21 years old
48. Younger than 31 years old

Probability of Independent Events In Exercises 49–52, a random number generator selects three numbers from 1 through 10. Find the probability of the event. See Example 9.

49. All three numbers are even.
50. All three numbers are less than or equal to 4.
51. Two numbers are less than 5 and the other number is 10.
52. One number is 2, 4, or 6, and the other two numbers are odd.

Finding the Probability of a Complement The complement of an event A is the collection of all outcomes in the sample space that are *not* in A . If the probability of A is $P(A)$, then the probability of the complement A' is given by $P(A') = 1 - P(A)$. In Exercises 53–56, you are given the probability that an event *will* happen. Find the probability that the event *will not* happen.

53. $P(E) = 0.75$
54. $P(E) = 0.32$
55. $P(E) = \frac{2}{3}$
56. $P(E) = \frac{7}{8}$

Using the Probability of a Complement In Exercises 57–60, you are given the probability that an event *will not* happen. Find the probability that the event *will* happen.

57. $P(E') = 0.12$
58. $P(E') = 0.84$
59. $P(E') = \frac{13}{20}$
60. $P(E') = \frac{61}{100}$

61. **Winning an Election** Three people have been nominated for president of a college class. From a small poll, it is estimated that the probability of Jane winning the election is 0.46, and the probability of Larry winning the election is 0.32. What is the probability of the third candidate winning the election?

62. **Winning an Election** Taylor, Moore, and Jenkins are candidates for a public office. It is estimated that Moore and Jenkins have about the same probability of winning, and Taylor is believed to be twice as likely to win as either of the others. Find each candidate's probability of winning the election.

63. **College Bound** In a high school graduating class of 198 students, 43 are on the honor roll. Of these, 37 are going on to college, and of the other 155 students, 102 are going on to college. A student is selected at random from the class. What is the probability that the person chosen is (a) going to college, (b) not going to college, and (c) on the honor roll, but not going to college?

- 64. Alumni Association** The alumni office of a college sends a survey to selected members of the class of 2010. The numbers of graduates, and the numbers of graduates who attended graduate school, are shown in the table. An alumnus is selected at random. What is the probability that the person is (a) female, (b) male, and (c) female and did not attend graduate school?

	Number of graduates	Number of graduates who attended graduate school
Women	672	124
Men	582	198

- 65. Random Number Generator** Two integers (from 1 to 30, inclusive) are chosen by a random number generator on a computer. What is the probability that (a) both numbers are even, (b) one number is even and one is odd, (c) both numbers are less than 10, and (d) the same number is chosen twice?
- 66. Preparing for a Test** A biology instructor gives her class a list of eight study problems, from which she will select five to be answered on an exam. A student knows how to solve six of the problems. Find the probability that the student will be able to answer all five questions on the exam.
- 67. Drawing Cards from a Deck** Two cards are selected at random from a standard deck of 52 playing cards. Find the probability that two hearts are selected under each condition.
- The cards are drawn in sequence, with the first card being replaced and the deck reshuffled prior to the second drawing.
 - The two cards are drawn consecutively, without replacement.
- 68. Game Show** On a game show, you are given five different digits to arrange in the proper order to represent the price of a car. If you are correct, then you win the car. Find the probability of winning under each condition.
- You must guess the position of each digit.
 - You know the first digit but must guess the remaining four.
 - You know the first and last digits but must guess the remaining three.
- 69. Game Show** On a game show, you are given six different digits to arrange in the proper order to represent the price of a house. If you are correct, then you win the house. Find the probability of winning under each condition.
- You must guess the position of each digit.
 - You know the first digit but must guess the remaining five.
 - You know the first and last digits but must guess the remaining four.

- 70. Letter Mix-Up** Five letters and envelopes are addressed to five different people. The letters are inserted randomly into the envelopes. What is the probability that (a) exactly one is inserted in the correct envelope and (b) at least one is inserted in the correct envelope?
- 71. Payroll Mix-Up** Three paychecks and envelopes are addressed to three different people. The paychecks get mixed up and are inserted randomly into the envelopes.
- What is the probability that exactly one is inserted in the correct envelope?
 - What is the probability that at least one is inserted in the correct envelope?
- 72. Poker Hand** Five cards are drawn randomly from a standard deck of 52 playing cards. What is the probability of getting a full house? (A full house consists of three of one kind and two of another. For example, A-A-A-5-5 and K-K-K-10-10 are full houses.)
- 73. Poker Hand** Five cards are drawn randomly from a standard deck of 52 playing cards. What is the probability of getting a straight flush? (A straight flush consists of five cards that are in order and of the same suit. For example, A♥, 2♥, 3♥, 4♥, 5♥ and 10♠, J♠, Q♠, K♠, A♠ are straight flushes.)
- 74. Defective Units** A shipment of 1000 compact disc players contains four defective units. A retail outlet has ordered 20 units.
- What is the probability that all 20 units are good?
 - What is the probability that at least one unit is defective?
- 75. Defective Units** A shipment of 12 stereos contains three defective units. Four of the units are shipped to a retail store. What is the probability that (a) all four units are good, (b) exactly two units are good, and (c) at least two units are good?
- 76. Houses for Sale by Region** In 2009, about 8.29% of all houses for sale in the United States were in the northeast, about 14.44% were in the midwest, about 54.01% were in the south, and about 23.26% were in the west. A house is selected at random from the houses for sale in the United States. What is the probability that the house is in the midwest or the south? (*Source: U.S. Census Bureau*)
- 77. Making a Sale** A sales representative makes sales on approximately one-third of all calls. On a given day, the representative calls on four offices. What is the probability that sales are made at (a) all four offices, (b) none of the offices, and (c) at least one office?
- 78. A Boy or a Girl?** Assume that the probability of the birth of a child of a particular gender is 50%. In a family with four children, what is the probability that (a) all four children are boys, (b) all four children are of the same gender, and (c) there is at least one boy?

Probability In Exercises 79–81, consider n independent trials of an experiment in which each trial has two possible outcomes, called success and failure. The probability of a success on each trial is p , and the probability of a failure is $q = 1 - p$. In this context, the term ${}_nC_k p^k q^{n-k}$ in the expansion of $(p + q)^n$ gives the probability of k successes in the n trials of the experiment.

79. A fair coin is tossed eight times. To find the probability of obtaining five heads, evaluate the term

$${}_8C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^3$$

in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^8$.

80. The probability of a baseball player getting a hit during any given at bat is $\frac{1}{5}$. To find the probability that the player will get four hits during the next 10 at bats, evaluate the term

$${}_{10}C_4 \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^6$$

in the expansion of $\left(\frac{1}{5} + \frac{4}{5}\right)^{10}$.

81. The probability of a sales representative making a sale with any one customer is $\frac{1}{4}$. The sales representative makes 10 contacts a day. To find the probability of making four sales, evaluate the term

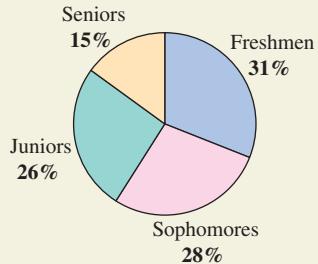
$${}_{10}C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^6$$

in the expansion of $\left(\frac{1}{4} + \frac{3}{4}\right)^{10}$.

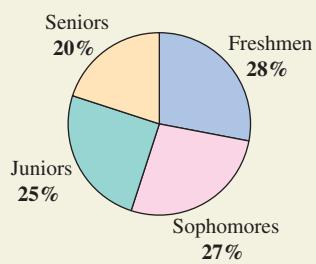


HOW DO YOU SEE IT? The circle graphs show the percents of undergraduate students by class level at two colleges. A student is chosen at random from the combined undergraduate population of the two colleges. The probability that the student is a freshman, sophomore, or junior is 81%. Which college has a greater number of undergraduate students? Explain your reasoning.

College A



College B

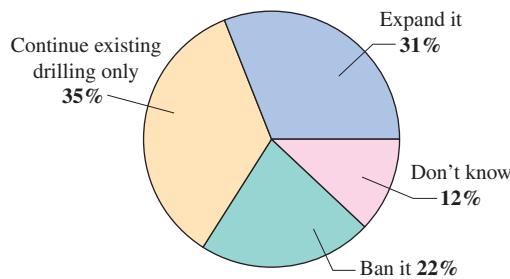


83. **Writing** Write a paragraph describing in your own words the difference between mutually exclusive events and independent events.



84. **Offshore Drilling** In a survey, Americans were asked how they felt about offshore drilling for oil and gas. The results are shown in the circle graph. Two people from the survey were chosen at random. (*Source: Pew Research Center*)

How Should the U.S. Deal with Offshore Drilling?

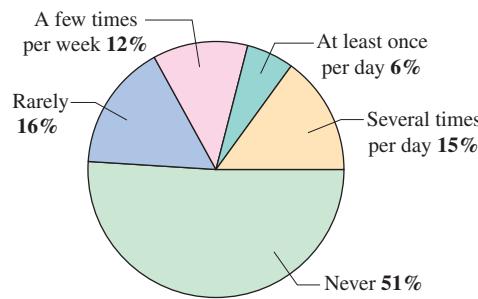


- (a) What is the probability that neither person felt the U.S. should expand offshore drilling?
 (b) What is the probability that both people felt the U.S. should participate in offshore drilling?



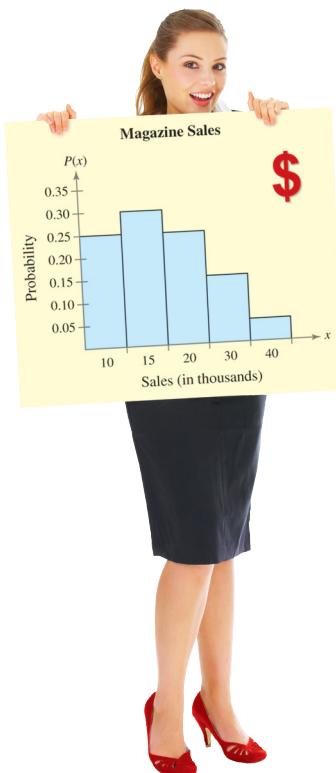
85. **Texting** In a survey, adults with cell phones were asked how often they sent or received text messages related to work. The results are shown in the circle graph. Two people from the survey were chosen at random. (*Source: Pew Research Center's Internet and American Life Project*)

Frequency of Work-Related Texts



- (a) What is the probability that both people sent or received work-related texts *at least* a few times per week?
 (b) What is the probability that neither person sent or received work-related texts?

16.3 Discrete and Continuous Random Variables



In Exercise 25 on page 1168, you will find the expected value and standard deviation of the sales of a weekly magazine.

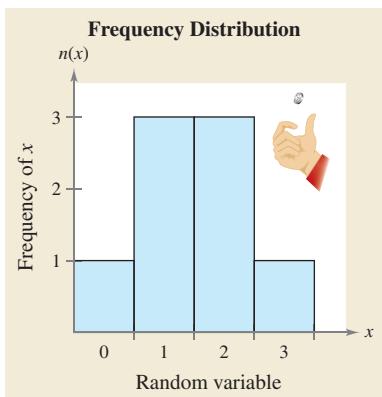


FIGURE 16.8

- Assign values to, and form frequency distributions for, discrete random variables.
- Find the probability distributions for discrete random variables.
- Find the expected values or means of discrete random variables.
- Find the variances and standard deviations of discrete random variables.
- Verify continuous probability density functions and use continuous probability density functions to find probabilities.

Discrete Random Variables

A function that assigns a numerical value to each of the outcomes in a sample space is called a **random variable**. For instance, when two coins are tossed, the sample space is $S = \{HH, HT, TH, TT\}$. These possible outcomes can be assigned the numbers 2, 1, and 0, depending on the number of heads in the outcome.

Definition of Discrete Random Variable

Let S be a sample space. A **random variable** is a function x that assigns a numerical value to each outcome in S . When the set of values taken on by the random variable is finite, the random variable is **discrete**. The number of times a specific value of x occurs is the **frequency** of x and is denoted by $n(x)$.

Example 1 Finding Frequencies

Three coins are tossed. A random variable assigns the number 0, 1, 2, or 3 to each possible outcome, depending on the number of heads in the outcome.

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓
 3 2 2 1 2 1 1 0

Find the frequencies of 0, 1, 2, and 3. Then use a bar graph to represent the result.

SOLUTION To find the frequencies, simply count the number of occurrences of each value of the random variable, as shown in the table.

Random variable, x	0	1	2	3
Frequency of x , $n(x)$	1	3	3	1

This table is called a **frequency distribution** of the random variable. The result is shown graphically by the bar graph in Figure 16.8.

Checkpoint 1

Two coins are tossed. A random variable assigns the number 0, 1, or 2 to each possible outcome, depending on the number of heads in the outcome.

$$S = \{HH, HT, TH, TT\}$$

↓ ↓ ↓ ↓
 2 1 1 0

Find the frequencies of 0, 1, and 2. Then use a bar graph to represent the result.

Discrete Probability

The probability of a random variable x is

$$P(x) = \frac{\text{Frequency of } x}{\text{Number of outcomes in } S} = \frac{n(x)}{n(S)}$$

where $n(S)$ is the number of equally likely outcomes in the sample space. By this definition, it follows that the probability of an event must be a number between 0 and 1. That is, $0 \leq P(x) \leq 1$.

The collection of probabilities corresponding to the values of the random variable is called the **probability distribution** of the random variable. If the range of a discrete random variable consists of m different values

$$\{x_1, x_2, x_3, \dots, x_m\}$$

then the sum of the probabilities of x_i is 1. This can be written as

$$P(x_1) + P(x_2) + P(x_3) + \dots + P(x_m) = 1.$$



Example 2 Finding a Probability Distribution

Five coins are tossed. Graph the probability distribution, where the random variable represents the number of heads in each possible outcome.

SOLUTION Let x be the random variable that represents the number of heads in each possible outcome. The possible outcomes are shown below.

x	Outcome	$n(x)$
0	TTTTT	1
1	HTTTT, THTTT, TTHTT, TTTHT, TTTTH	5
2	HHTTT, HTHTT, HTTHT, HTTTT, THHTT THTHT, THTTH, TTHHT, TTHTH, TTTHH	10
3	HHHTT, HHTHT, HHTTH, HTHHT, HTHTH HTTHH, THHHT, THHTH, THTHH, TTHHH	10
4	HHHHT, HHHTH, HHTHH, HTTHH, THHHH	5
5	HHHHH	1

The number of outcomes in the sample space is $n(S) = 32$. The probability of each value of the random variable is shown in the table.

Random variable, x	0	1	2	3	4	5
Probability, $P(x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

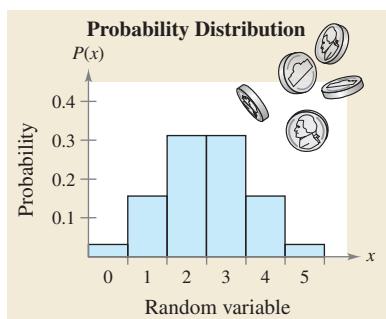


FIGURE 16.9

A graph of this probability distribution is shown in Figure 16.9. Note that values of the random variable are represented by intervals on the x -axis. Observe that the sum of the probabilities is 1.

Checkpoint 2

Two six-sided dice are tossed. Graph the probability distribution, where the random variable represents the sum of the points in each possible outcome.

Expected Value

Suppose you repeated the coin-tossing experiment in Example 2 several times. On the average, how many heads would you expect to turn up? From Figure 16.9, it seems reasonable that the average number of heads would be $2\frac{1}{2}$. This “average” is the **expected value** of the random variable.

Definition of Expected Value

If the range of a discrete random variable consists of m different values $\{x_1, x_2, x_3, \dots, x_m\}$, then the **expected value** of the random variable is

$$E(x) = x_1P(x_1) + x_2P(x_2) + x_3P(x_3) + \dots + x_mP(x_m).$$

The expected value is also called the **mean** of the random variable and is usually denoted by μ (the lowercase Greek letter mu). Because the mean often occurs near the center of the values in the range of the random variable, it is called a **measure of central tendency**.



Example 3 Finding an Expected Value

Five coins are tossed. Find the expected value of the number of heads that will turn up.

SOLUTION Using the results of Example 2, you obtain the expected value as shown.

$\underbrace{0 \text{ Heads}}_{E(x) = (0)\left(\frac{1}{32}\right)} \quad \underbrace{1 \text{ Head}}_{(1)\left(\frac{5}{32}\right)} \quad \underbrace{2 \text{ Heads}}_{(2)\left(\frac{10}{32}\right)} \quad \underbrace{3 \text{ Heads}}_{(3)\left(\frac{10}{32}\right)} \quad \underbrace{4 \text{ Heads}}_{(4)\left(\frac{5}{32}\right)} \quad \underbrace{5 \text{ Heads}}_{(5)\left(\frac{1}{32}\right)}$

$$E(x) = (0)\left(\frac{1}{32}\right) + (1)\left(\frac{5}{32}\right) + (2)\left(\frac{10}{32}\right) + (3)\left(\frac{10}{32}\right) + (4)\left(\frac{5}{32}\right) + (5)\left(\frac{1}{32}\right) = \frac{80}{32} = 2.5$$

Checkpoint 3

Two six-sided dice are tossed. Find the expected value of the sum of the points. ■



Example 4 Finding an Expected Value

Over a period of 1 year (225 selling days), a sales representative sold from zero to six units per day, as shown in Figure 16.10. From these data, what is the average number of units per day the sales representative should expect to sell?

SOLUTION One way to answer this question is to calculate the expected value of the number of units.

$$\begin{aligned} E(x) &= (0)\left(\frac{34}{225}\right) + (1)\left(\frac{46}{225}\right) + (2)\left(\frac{53}{225}\right) + (3)\left(\frac{47}{225}\right) + (4)\left(\frac{25}{225}\right) + (5)\left(\frac{12}{225}\right) + (6)\left(\frac{8}{225}\right) \\ &\approx \frac{501}{225} \\ &\approx 2.23 \text{ units per day} \end{aligned}$$

Checkpoint 4

Over a period of 1 year, a salesperson worked 6 days a week (312 selling days) and sold from zero to six units per day. Using the data in the table shown below, what is the average number of units per day the sales representative should expect to sell? ■

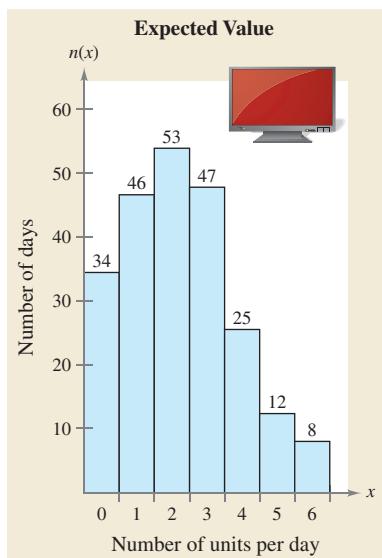


FIGURE 16.10

Number of units per day	0	1	2	3	4	5	6
Number of days	39	60	75	62	48	18	10

Variance and Standard Deviation

The expected value or mean gives a measure of the average value assigned by a random variable. But the mean does not tell the whole story. For instance, all three of the distributions shown below have a mean of 2.

Distribution 1

Random variable, x	0	1	2	3	4
Frequency of x , $n(x)$	2	2	2	2	2

Distribution 2

Random variable, x	0	1	2	3	4
Frequency of x , $n(x)$	0	3	4	3	0

Distribution 3

Random variable, x	0	1	2	3	4
Frequency of x , $n(x)$	5	0	0	0	5

Even though each distribution has the same mean, the patterns of the distributions are quite different. In the first distribution, each value has the same frequency [see Figure 16.11(a)]. In the second, the values are clustered about the mean [see Figure 16.11(b)]. In the third distribution, the values are far from the mean [see Figure 16.11(c)].

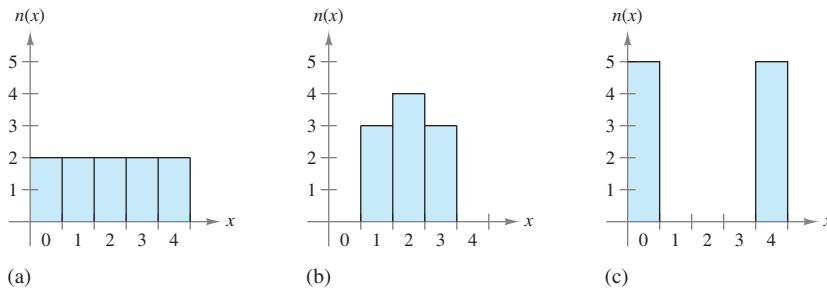


FIGURE 16.11

Definitions of Variance and Standard Deviation

Consider a random variable whose range is $\{x_1, x_2, x_3, \dots, x_m\}$ with a mean of μ . The **variance** of the random variable is

$$V(x) = (x_1 - \mu)^2 P(x_1) + (x_2 - \mu)^2 P(x_2) + \dots + (x_m - \mu)^2 P(x_m).$$

The **standard deviation** of the random variable is

$$\sigma = \sqrt{V(x)}$$

(σ is the lowercase Greek letter sigma).

When the standard deviation is small, most of the values of the random variable are clustered near the mean. As the standard deviation becomes larger, the distribution becomes more and more spread out. For instance, in the three distributions shown in Figure 16.11, you would expect the second to have the smallest standard deviation and the third to have the largest. This is confirmed in Example 5.

Example 5 Finding Variance and Standard Deviation

Find the variance and standard deviation of each of the three distributions shown on the preceding page.

SOLUTION

- a. For Distribution 1, the mean is $\mu = 2$, the variance is

$$\begin{aligned} V(x) &= (0 - 2)^2\left(\frac{2}{10}\right) + (1 - 2)^2\left(\frac{2}{10}\right) + (2 - 2)^2\left(\frac{2}{10}\right) + \\ &\quad (3 - 2)^2\left(\frac{2}{10}\right) + (4 - 2)^2\left(\frac{2}{10}\right) \\ &= 2 \quad \text{Variance} \end{aligned}$$

and the standard deviation is $\sigma = \sqrt{2} \approx 1.41$.

- b. For Distribution 2, the mean is $\mu = 2$, the variance is

$$\begin{aligned} V(x) &= (0 - 2)^2\left(\frac{0}{10}\right) + (1 - 2)^2\left(\frac{3}{10}\right) + (2 - 2)^2\left(\frac{4}{10}\right) + \\ &\quad (3 - 2)^2\left(\frac{3}{10}\right) + (4 - 2)^2\left(\frac{0}{10}\right) \\ &= 0.6 \quad \text{Variance} \end{aligned}$$

and the standard deviation is $\sigma = \sqrt{0.6} \approx 0.77$.

- c. For Distribution 3, the mean is $\mu = 2$, the variance is

$$\begin{aligned} V(x) &= (0 - 2)^2\left(\frac{5}{10}\right) + (1 - 2)^2\left(\frac{0}{10}\right) + (2 - 2)^2\left(\frac{0}{10}\right) + \\ &\quad (3 - 2)^2\left(\frac{0}{10}\right) + (4 - 2)^2\left(\frac{5}{10}\right) \\ &= 4 \quad \text{Variance} \end{aligned}$$

and the standard deviation is $\sigma = \sqrt{4} = 2$.

As you can see in Figure 16.12, the second distribution has the smallest standard deviation and the third distribution has the largest.

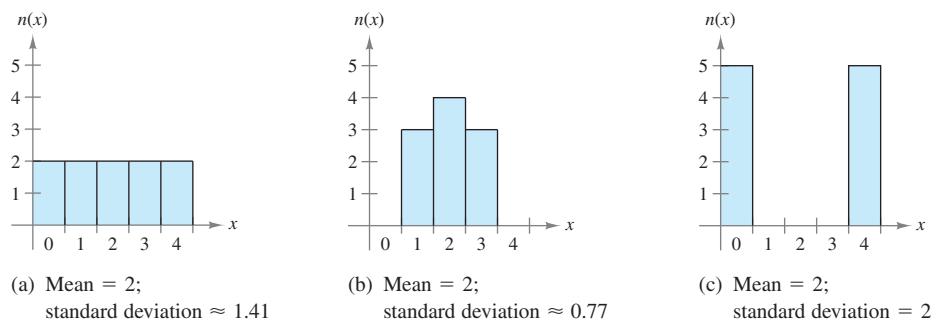


FIGURE 16.12

Checkpoint 5

Find the variance and standard deviation of the distribution shown in the table. Then graph the distribution.

Random variable, x	0	1	2	3	4
Frequency of x , $n(x)$	1	2	4	2	1

Continuous Random Variables

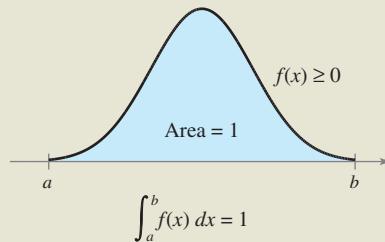
In many applications of probability, it is useful to consider a random variable whose range is an interval on the real number line. Such a random variable is **continuous**. For instance, the random variable that measures the height of a person in a population is continuous.

To define the probability of an event involving a continuous random variable, you cannot simply count the number of ways the event can occur (as you can with a discrete random variable). Rather, you need to define a function f called a **probability density function**.

Definition of Probability Density Function

Consider a function f of a continuous random variable x whose range is the interval $[a, b]$. The function is a **probability density function** when it is nonnegative and continuous on the interval $[a, b]$ and when

$$\int_a^b f(x) dx = 1. \quad \text{See Figure 16.13.}$$



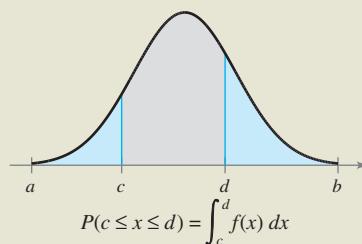
Probability Density Function

FIGURE 16.13

The probability that x lies in the interval $[c, d]$ is

$$P(c \leq x \leq d) = \int_c^d f(x) dx. \quad \text{See Figure 16.14.}$$

When the range of the continuous random variable is an infinite interval, the integrals are improper integrals (see Example 7).



Probability that x lies in the interval $[c, d]$

FIGURE 16.14

Example 6 Verifying a Probability Density Function

Show that

$$f(x) = 12x(1 - x)^2$$

is a probability density function over the interval $[0, 1]$.

SOLUTION Begin by observing that f is continuous and nonnegative on the interval $[0, 1]$.

$$f(x) = 12x(1 - x)^2 \geq 0, \quad 0 \leq x \leq 1$$

$f(x)$ is nonnegative on $[0, 1]$.

Next, evaluate the integral below.

$$\begin{aligned} \int_0^1 12x(1 - x)^2 dx &= 12 \int_0^1 (x^3 - 2x^2 + x) dx && \text{Expand polynomial.} \\ &= 12 \left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} \right]_0^1 && \text{Integrate.} \\ &= 12 \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right) && \text{Apply Fundamental} \\ &= 1 && \text{Theorem of Calculus.} \\ &&& \text{Simplify.} \end{aligned}$$

Because this value is 1, you can conclude that f is a probability density function over the interval $[0, 1]$. The graph of f is shown in Figure 16.15.

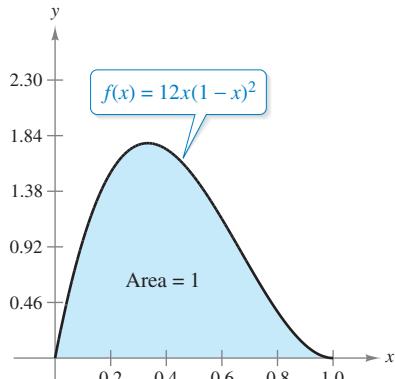


FIGURE 16.15

✓ Checkpoint 6

Show that $f(x) = \frac{1}{2}x$ is a probability density function over the interval $[0, 2]$.

The next example deals with an infinite interval and its corresponding improper integral.

Example 7 Verifying a Probability Density Function

Show that

$$f(t) = 0.1e^{-0.1t}$$

is a probability density function over the infinite interval $[0, \infty)$.

SOLUTION Begin by observing that f is continuous and nonnegative on the interval $[0, \infty)$.

$$f(t) = 0.1e^{-0.1t} \geq 0, \quad t \geq 0$$

$f(t)$ is nonnegative on $[0, \infty)$.

Next, evaluate the integral below.

$$\begin{aligned} \int_0^\infty 0.1e^{-0.1t} dt &= \lim_{b \rightarrow \infty} \left[-e^{-0.1t} \right]_0^b && \text{Improper integral} \\ &= \lim_{b \rightarrow \infty} (-e^{-0.1b} + 1) && \text{Evaluate limit.} \\ &= 1 \end{aligned}$$

Because this value is 1, you can conclude that f is a probability density function over the interval $[0, \infty)$. The graph of f is shown in Figure 16.16.

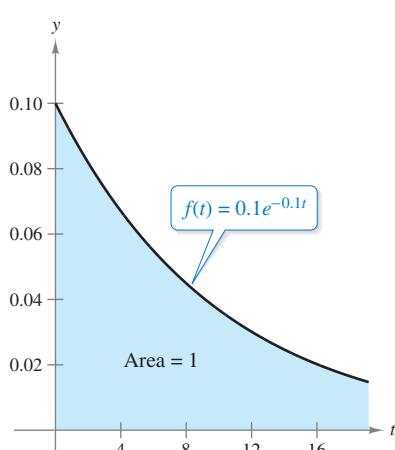


FIGURE 16.16

✓ Checkpoint 7

Show that $f(x) = 2e^{-2x}$ is a probability density function over the interval $[0, \infty)$.

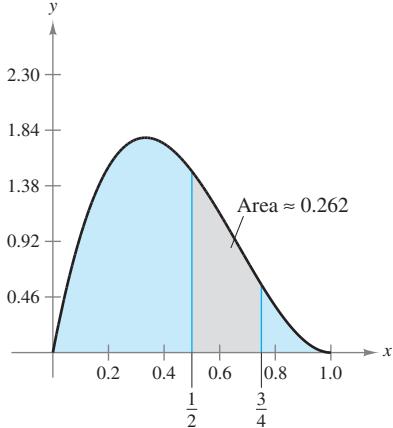
Example 8 Finding a Probability

For the probability density function in Example 7

$$f(x) = 12x(1 - x)^2$$

find the probability that x lies in the interval $\frac{1}{2} \leq x \leq \frac{3}{4}$.

SOLUTION



$$\begin{aligned} P\left(\frac{1}{2} \leq x \leq \frac{3}{4}\right) &= 12 \int_{1/2}^{3/4} x(1 - x)^2 dx && \text{Integrate } f(x) \text{ over } \left[\frac{1}{2}, \frac{3}{4}\right]. \\ &= 12 \int_{1/2}^{3/4} (x^3 - 2x^2 + x) dx && \text{Expand polynomial.} \\ &= 12 \left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} \right]_{1/2}^{3/4} && \text{Integrate.} \\ &= 12 \left[\frac{\left(\frac{3}{4}\right)^4}{4} - \frac{2\left(\frac{3}{4}\right)^3}{3} + \frac{\left(\frac{3}{4}\right)^2}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{2\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{2}\right)^2}{2} \right] \\ &\approx 0.262 && \text{Approximate.} \end{aligned}$$

So, the probability that x lies in the interval $\left[\frac{1}{2}, \frac{3}{4}\right]$ is approximately 0.262 or 26.2%, as indicated in Figure 16.17.

FIGURE 16.17

Checkpoint 8

Find the probability that x lies in the interval $\frac{1}{2} \leq x \leq 1$ for the probability density function in Checkpoint 6.

In Example 8, the probability that x lies in any of the intervals $\frac{1}{2} < x < \frac{3}{4}$, $\frac{1}{2} \leq x < \frac{3}{4}$, or $\frac{1}{2} < x \leq \frac{3}{4}$ is the same. In other words, the inclusion of either endpoint adds nothing to the probability. This demonstrates an important difference between discrete and continuous random variables. For a continuous random variable, the probability that x will be precisely one value (such as 0.5) is considered to be zero, because

$$P(0.5 \leq x \leq 0.5) = \int_{0.5}^{0.5} f(x) dx = 0.$$

You should not interpret this result to mean that it is impossible for the continuous random variable x to have the value 0.5. It simply means that the probability that x will have this *exact* value is insignificant.

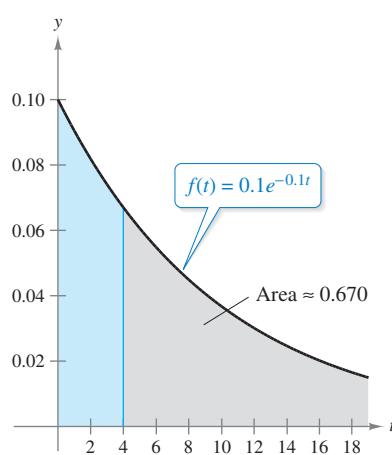
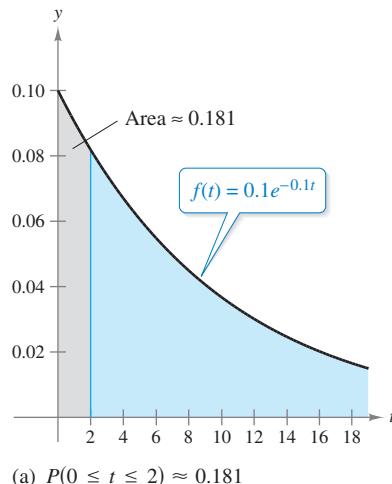
Example 9 Finding a Probability

Consider a probability density function defined over the interval $[0, 5]$. The probability that x lies in the interval $[0, 2]$ is 0.7. What is the probability that x lies in the interval $[2, 5]$?

SOLUTION Because the probability that x lies in the interval $[0, 5]$ is 1, you can conclude that the probability that x lies in the interval $[2, 5]$ is $1 - 0.7 = 0.3$.

Checkpoint 9

A probability density function is defined over the interval $[0, 4]$. The probability that x lies in $[0, 1]$ is 0.6. What is the probability that x lies in $[1, 4]$?

**FIGURE 16.18****Application****Example 10****Modeling the Lifetime of a Product**

The useful lifetime (in years) of a product is modeled by the probability density function $f(t) = 0.1e^{-0.1t}$ for $0 \leq t < \infty$. Find the probability that a randomly selected unit will have a lifetime falling in each interval.

- a. No more than 2 years b. More than 4 years

SOLUTION

- a. The probability that the unit will last no more than 2 years is

$$\begin{aligned} P(0 \leq t \leq 2) &= 0.1 \int_0^2 e^{-0.1t} dt && \text{Integrate } f(t) \text{ over } [0, 2]. \\ &= \left[-e^{-0.1t} \right]_0^2 && \text{Find antiderivative.} \\ &= -e^{-0.2} + 1 && \text{Apply Fundamental Theorem of Calculus.} \\ &\approx 0.181. && \text{Approximate.} \end{aligned}$$

- b. The probability that the unit will last more than 4 years is

$$\begin{aligned} P(4 < t < \infty) &= 0.1 \int_4^\infty e^{-0.1t} dt && \text{Integrate } f(t) \text{ over } [4, \infty). \\ &= \lim_{b \rightarrow \infty} \left[-e^{-0.1t} \right]_4^b && \text{Improper integral} \\ &= \lim_{b \rightarrow \infty} (-e^{-0.1b} + e^{-0.4}) && \text{Evaluate limit.} \\ &= e^{-0.4} && \\ &\approx 0.670. && \text{Approximate.} \end{aligned}$$

These two probabilities are illustrated graphically in Figure 16.18.

✓ Checkpoint 10

For the function in Example 10, find the probability that a randomly selected unit will have a lifetime of more than 2 years, but no more than 4 years.


SUMMARIZE (Section 16.3)

- State the definition of a discrete random variable (page 1157). For an example of finding the frequencies of a discrete random variable, see Example 1.
- State the definition of a probability distribution (page 1158). For an example of finding a probability distribution, see Example 2.
- State the definition of expected value (page 1159). For examples of finding an expected value, see Examples 3 and 4.
- State the definitions of variance and standard deviation (page 1160). For an example of finding variance and standard deviation, see Example 5.
- State the definition of a probability density function (page 1162). For examples of finding a probability, see Examples 6, 7, 8, and 9.

SKILLS WARM UP 16.3

The following warm-up exercises involve skills that were covered in a previous course. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 7.2, 11.4, and 12.5.

In Exercises 1 and 2, solve for x .

1. $\frac{1}{9} + \frac{2}{3} + \frac{2}{9} = x$

2. $\frac{1}{3} + \frac{5}{12} + \frac{1}{8} + \frac{1}{12} + \frac{x}{24} = 1$

In Exercises 3 and 4, evaluate the expression.

3. $0\left(\frac{1}{16}\right) + 1\left(\frac{3}{16}\right) + 2\left(\frac{8}{16}\right) + 3\left(\frac{3}{16}\right) + 4\left(\frac{1}{16}\right)$

4. $(0 - 1)^2\left(\frac{1}{4}\right) + (1 - 1)^2\left(\frac{1}{2}\right) + (2 - 1)^2\left(\frac{1}{4}\right)$

In Exercises 5–8, write the fraction as a percent. Round your answers to 2 decimal places, if necessary.

5. $\frac{3}{8}$

6. $\frac{9}{11}$

7. $\frac{13}{24}$

8. $\frac{112}{256}$

In Exercises 9–12, determine whether f is continuous and nonnegative on the given interval.

9. $f(x) = \frac{1}{x}, [1, 4]$

10. $f(x) = x^2 - 1, [0, 1]$

11. $f(x) = 3 - x, [1, 5]$

12. $f(x) = e^{-x}, [0, 1]$

In Exercises 13–15, evaluate the definite integral.

13. $\int_0^4 \frac{1}{4} dx$

14. $\int_1^2 \frac{2-x}{2} dx$

15. $\int_0^\infty 3e^{-3t} dt$

Exercises 16.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Finding Frequency and Probability Distributions In Exercises 1 and 2, (a) find the frequency distribution for the random variable and (b) find the probability distribution for the random variable. See Examples 1 and 2.

- Coin Toss** Four coins are tossed. A random variable assigns the number 0, 1, 2, 3, or 4 to each possible outcome, depending on the number of heads in the outcome.
- Exam** Three students answer a true-false question on an examination. A random variable assigns the number 0, 1, 2, or 3 to each possible outcome, depending on the number of answers of *true* among the three students.
- Random Selection** In a class of 72 students, 44 are girls and, of these, 12 are going to college. Of the 28 boys in the class, 9 are going to college. A student is selected at random from the class. What is the probability that the person chosen is
 - going to college?
 - not going to college?
 - a girl who is not going to college?

- Random Selection** A card is chosen at random from a standard 52-card deck of playing cards. What is the probability that the card is
 - red?
 - a 5?
 - black and a face card?

Determining a Missing Probability In Exercises 5 and 6, find the missing value of the probability distribution.

5.

x	0	1	2	3	4
$P(x)$	0.20	0.35	0.15	?	0.05

6.

x	0	1	2	3	4	5
$P(x)$	0.05	?	0.25	0.30	0.15	0.10

Identifying Probability Distributions In Exercises 7–10, determine whether the table represents a probability distribution. If it is a probability distribution, sketch its graph. If it is not a probability distribution, state any properties that are not satisfied.

7.

x	0	1	2	3
$P(x)$	0.10	0.45	0.30	0.15

8.

x	0	1	2	3	4	5
$P(x)$	0.05	0.30	0.10	0.40	0.15	0.20

9.

x	0	1	2	3	4
$P(x)$	$\frac{12}{50}$	$\frac{20}{50}$	$\frac{8}{50}$	$\frac{10}{50}$	$-\frac{5}{50}$

10.

x	0	1	2	3	4	5
$P(x)$	$\frac{8}{30}$	$\frac{2}{30}$	$\frac{6}{30}$	$\frac{3}{30}$	$\frac{4}{30}$	$\frac{7}{30}$

Using Probability Distributions In Exercises 11–14, sketch a graph of the probability distribution and find the required probabilities.

11.

x	0	1	2	3	4
$P(x)$	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{6}{20}$	$\frac{6}{20}$	$\frac{4}{20}$

- (a) $P(1 \leq x \leq 3)$
- (b) $P(x \geq 2)$

12.

x	0	1	2	3	4
$P(x)$	$\frac{8}{20}$	$\frac{6}{20}$	$\frac{3}{20}$	$\frac{2}{20}$	$\frac{1}{20}$

- (a) $P(x \leq 2)$
- (b) $P(x > 2)$

13.

x	0	1	2	3	4	5
$P(x)$	0.041	0.189	0.247	0.326	0.159	0.038

- (a) $P(x \leq 3)$
- (b) $P(x > 3)$

14.

x	0	1	2	3
$P(x)$	0.027	0.189	0.441	0.343

- (a) $P(1 \leq x \leq 2)$
- (b) $P(x < 2)$

Rob Byron/www.shutterstock.com



15. **Health** The table shows the probability distribution of the numbers of AIDS cases diagnosed in the United States in 2009 by age group. (*Source: Centers for Disease Control and Prevention*)

Age, a	14 and under	15–24	25–34	35–44
$P(a)$	0.004	0.198	0.262	0.255

Age, a	45–54	55–64	65 and over
$P(a)$	0.195	0.069	0.017

- (a) Sketch the probability distribution.
- (b) Find the probability that an individual diagnosed with AIDS was from 15 to 44 years of age.
- (c) Find the probability that an individual diagnosed with AIDS was at least 35 years of age.
- (d) Find the probability that an individual diagnosed with AIDS was at most 24 years of age.



16. **Children** The table shows the probability distribution of the numbers of children per family in the United States in 2009. (*Source: U.S. Census Bureau*)

Children, x	0	1	2	3 or more
$P(x)$	0.548	0.193	0.167	0.092

- (a) Sketch the probability distribution.
- (b) Find the probability that a family has at least 2 children.
- (c) Find the probability that a family has at most 2 children.
- (d) Find the probability that a family has at least 1 child.

- 17. Biology** Consider a couple who have four children. Assume that it is equally likely that each child is a girl or a boy.

- (a) Complete the set to form the sample space consisting of 16 elements.

$$S = \{gggg, gggb, ggbg, \dots\}$$

- (b) Complete the table, in which the random variable x is the number of girls in the family.

x	0	1	2	3	4
$P(x)$					

- (c) Use the table in part (b) to sketch the graph of the probability distribution.
 (d) Use the table in part (b) to find the probability that at least one of the children is a boy.

- 18. Die Toss** Consider the experiment of tossing a 6-sided die twice.

- (a) Complete the set to form the sample space of 36 elements. Note that each element is an ordered pair in which the entries are the numbers of points on the first and second tosses, respectively.

$$S = \{(1, 1), (1, 2), \dots, (2, 1), (2, 2), \dots\}$$

- (b) Complete the table, in which the random variable x is the sum of the two rolls.

x	2	3	4	5	6	7	8	9	10	11	12
$P(x)$											

- (c) Use the table in part (b) to sketch the graph of the probability distribution.
 (d) Use the table in part (b) to find $P(10 \leq x \leq 12)$.

Finding Expected Value, Variance, and Standard Deviation In Exercises 19–22, find the expected value, variance, and standard deviation for the given probability distribution. See Examples 3, 4, and 5.

x	1	2	3	4	5
$P(x)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{8}{16}$	$\frac{3}{16}$	$\frac{1}{16}$

x	1	2	3	4	5
$P(x)$	$\frac{4}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

x	-3	-1	0	3	5
$P(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

x	-5000	-2500	300
$P(x)$	0.008	0.052	0.940

Finding Mean, Variance, and Standard Deviation In Exercises 23 and 24, find the mean, variance, and standard deviation of the discrete random variable x . See Examples 3, 4, and 5.

- 23. Die Toss** x is (a) the number of points when a four-sided die is tossed once and (b) the sum of the points when the four-sided die is tossed twice.

- 24. Coin Toss** x is the number of heads when a coin is tossed four times.

- 25. Revenue** A publishing company introduces a new weekly magazine that sells for \$4.95 on the newsstand. The marketing group of the company estimates that sales x (in thousands) will be approximated by the following probability function.

x	10	15	20	30	40
$P(x)$	0.25	0.30	0.25	0.15	0.05

- (a) Find $E(x)$ and σ .

- (b) Find the expected revenue.

- 26. Personal Income** The probability distribution of the random variable x , the annual income of a family (in thousands of dollars) in a certain section of a large city, is shown in the table. Find $E(x)$ and σ .

x	30	40	50	60	80
$P(x)$	0.10	0.20	0.50	0.15	0.05

- 27. Insurance** An insurance company needs to determine the annual premium required to break even on fire protection policies with a face value of \$90,000. The random variable x is the claim size on these policies, and the analysis is restricted to the losses \$30,000, \$60,000, and \$90,000. The probability distribution of x is as shown in the table. What premium should customers be charged for the company to break even?

x	0	30,000	60,000	90,000
$P(x)$	0.995	0.0036	0.0011	0.0003

- 28. Insurance** An insurance company needs to determine the annual premium required to break even for collision protection for cars with a value of \$10,000. The random variable x is the claim size on these policies, and the analysis is restricted to the losses \$1000, \$5000, and \$10,000. The probability distribution of x is as shown in the table. What premium should customers be charged for the company to break even?

x	0	1000	5000	10,000
$P(x)$	0.936	0.040	0.020	0.004

- 29. Baseball** A baseball fan examined the record of a favorite baseball player's performance during his last season. The numbers of games in which the player had zero, one, two, three, and four hits are recorded in the table shown below.

Number of hits	0	1	2	3	4
Frequency	40	61	40	17	2

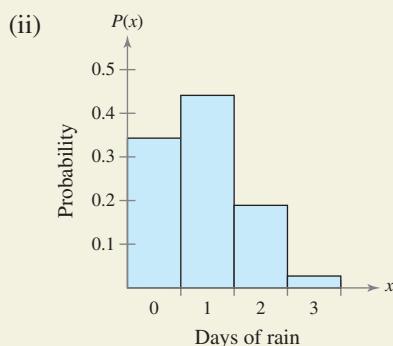
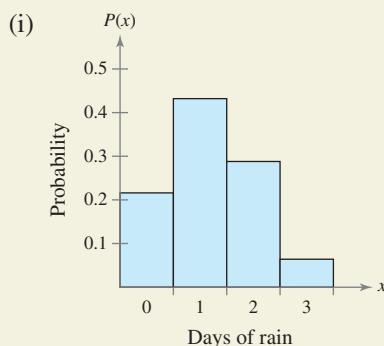
- (a) Complete the table below, where x is the number of hits.

x	0	1	2	3	4
$P(x)$					

- (b) Use the table in part (a) to sketch the graph of the probability distribution.
(c) Use the table in part (a) to find $P(1 \leq x \leq 3)$.
(d) Determine the mean, variance, and standard deviation. Explain your results.



- 30. HOW DO YOU SEE IT?** News Station A says that there is a 40% chance of rain for the next three days. News Station B says that there is a 30% chance of rain for the next three days. Determine which graph represents the probability distribution for each news station. Explain your reasoning.



Games of Chance If x is a player's net gain in a game of chance, then $E(x)$ is usually negative. This value gives the average amount per game the player can expect to lose over the long run. In Exercises 31 and 32, find the player's expected net gain for one play of the specified game.

- 31. Roulette** In roulette, the wheel has the 38 numbers 00, 0, 1, 2, . . . , 34, 35, and 36, marked on equally spaced slots. If a player bets \$1 on a number and wins, then the player keeps the dollar and receives an additional \$35. Otherwise, the dollar is lost.

- 32. Raffle** A service organization is selling \$2 raffle tickets as part of a fundraising program. The first prize is a boat valued at \$2950, and the second prize is a camping tent valued at \$400. In addition to the first and second prizes, there are 25 \$20 gift certificates to be awarded. The number of tickets sold is 3000.

- 33. Market Analysis** A sporting goods company has decided on two possible cities in which to open a new store. Management estimates that city 1 will yield \$20 million in revenues if successful and will lose \$4 million if not, whereas city 2 will yield \$50 million in revenues if successful and will lose \$9 million if not. City 1 has a 0.3 probability of being successful and city 2 has a 0.2 probability of being successful. In which city should the sporting goods company open the new store with respect to the expected return from each store?

- 34. Market Analysis** Repeat Exercise 33 for the case in which the probabilities of city 1 and city 2 being successful are 0.4 and 0.25, respectively.



Identifying Probability Density Functions In Exercises 35–44, use a graphing utility to graph the function. Then determine whether the function f represents a probability density function over the given interval. If f is not a probability density function, identify the condition(s) that is (are) not satisfied. See Examples 6 and 7.

35. $f(x) = \frac{1}{8}$, $[0, 8]$

36. $f(x) = \frac{1}{5}$, $[0, 4]$

37. $f(x) = \frac{4-x}{8}$, $[0, 4]$

38. $f(x) = \frac{x}{18}$, $[0, 6]$

39. $f(x) = \frac{1}{7}e^{-x/7}$, $[0, 3]$

40. $f(x) = \frac{1}{6}e^{-x/6}$, $[0, \infty)$

41. $f(x) = 2\sqrt{4-x}$, $[0, 2]$

42. $f(x) = 12x^2(1-x)$, $[0, 2]$

43. $f(x) = \frac{2}{9}x(3-x)$, $[0, 3]$

44. $f(x) = 0.4e^{-0.4x}$, $[0, \infty)$

Making a Probability Density Function In Exercises 45–50, find the constant k such that the function f is a probability density function over the given interval.

45. $f(x) = kx$, $[1, 4]$ 46. $f(x) = kx^3$, $[0, 4]$

47. $f(x) = k(4 - x^2)$, $[-2, 2]$

48. $f(x) = k\sqrt{x}(1 - x)$, $[0, 1]$

49. $f(x) = ke^{-x/2}$, $[0, \infty)$

50. $f(x) = \frac{k}{b - a}$, $[a, b]$

Finding a Probability In Exercises 51–58, sketch the graph of the probability density function over the indicated interval and find the indicated probabilities. See Examples 8 and 9.

51. $f(x) = \frac{1}{3}$, $[0, 3]$

- (a) $P(0 < x < 2)$ (b) $P(1 < x < 2)$
 (c) $P(1 < x < 3)$ (d) $P(x = 2)$

52. $f(x) = \frac{1}{10}$, $[0, 10]$

- (a) $P(0 < x < 6)$ (b) $P(4 < x < 6)$
 (c) $P(8 < x < 10)$ (d) $P(x \geq 2)$

53. $f(x) = \frac{x}{50}$, $[0, 10]$

- (a) $P(0 < x < 6)$ (b) $P(4 < x < 6)$
 (c) $P(8 < x < 10)$ (d) $P(x \geq 2)$

54. $f(x) = \frac{2x}{25}$, $[0, 5]$

- (a) $P(0 < x < 3)$ (b) $P(1 < x < 3)$
 (c) $P(3 < x < 5)$ (d) $P(x \geq 1)$

55. $f(x) = \frac{3}{16}\sqrt{x}$, $[0, 4]$

- (a) $P(0 < x < 2)$ (b) $P(2 < x < 4)$
 (c) $P(1 < x < 3)$ (d) $P(x \leq 3)$

56. $f(x) = \frac{5}{4(x + 1)^2}$, $[0, 4]$

- (a) $P(0 < x < 2)$ (b) $P(2 < x < 4)$
 (c) $P(1 < x < 3)$ (d) $P(x \leq 3)$

57. $f(t) = \frac{1}{3}e^{-t/3}$, $[0, \infty)$

- (a) $P(t < 2)$ (b) $P(t \geq 2)$
 (c) $P(1 < t < 4)$ (d) $P(t = 3)$

58. $f(t) = \frac{3}{256}(16 - t^2)$, $[-4, 4]$

- (a) $P(t < -2)$ (b) $P(t > 2)$
 (c) $P(-1 < t < 1)$ (d) $P(t > -2)$

Waiting Time Buses arrive and depart from a college every 30 minutes. The probability density function for the waiting time t (in minutes) for a person arriving at the bus stop is

$$f(t) = \frac{1}{30}, \quad [0, 30].$$

Find the probabilities that the person will wait (a) no more than 5 minutes and (b) at least 18 minutes.

60. Demand The daily demand for gasoline x (in millions of gallons) in a city is described by the probability density function

$$f(x) = 0.41 - 0.08x, \quad [0, 4].$$

Find the probabilities that the daily demand for gasoline will be (a) no more than 3 million gallons and (b) at least 2 million gallons.

61. Learning Theory The time t (in hours) required for a new employee to learn to successfully operate a machine in a manufacturing process is described by the probability density function

$$f(t) = \frac{5}{324}t\sqrt{9 - t}, \quad [0, 9].$$

Find the probabilities that a new employee will learn to operate the machine (a) in less than 3 hours and (b) in more than 4 hours but less than 8 hours.

62. Meteorology A meteorologist predicts that the amount of rainfall x (in inches) expected for a certain coastal community during a hurricane has the probability density function

$$f(x) = \frac{\pi}{30} \sin \frac{\pi x}{15}, \quad 0 \leq x \leq 15.$$

Find and interpret the probabilities.

- (a) $P(0 \leq x \leq 10)$ (b) $P(10 \leq x \leq 15)$
 (c) $P(0 \leq x < 5)$ (d) $P(12 \leq x \leq 15)$

Using the Exponential Density Function In Exercises 63–66, find the required probabilities using the *exponential density function*

$$f(t) = \frac{1}{\lambda}e^{-t/\lambda}, \quad [0, \infty).$$

63. Waiting Time The waiting time t (in minutes) for service at the checkout at a grocery store is exponentially distributed with $\lambda = 3$. Find the probabilities of waiting (a) less than 2 minutes, (b) more than 2 minutes but less than 4 minutes, and (c) at least 2 minutes.

64. Waiting Time The length of time t (in hours) required to unload trucks at a depot is exponentially distributed with $\lambda = \frac{3}{4}$. Find the probabilities that the trucks can be unloaded (a) in less than 1 hour, (b) in more than 1 hour but less than 2 hours, and (c) in at most 3 hours.

65. Useful Life The lifetime t (in years) of a battery is exponentially distributed with $\lambda = 5$. Find the probabilities that the lifetime of a given battery will be (a) less than 6 years, (b) more than 2 years but less than 6 years, and (c) more than 8 years.

66. Useful Life The time t (in years) until failure of a component in a machine is exponentially distributed with $\lambda = 3.5$. Find the probabilities that the lifetime of a given component will be (a) less than 1 year, (b) more than 2 years but less than 4 years, and (c) at least 5 years.

QUIZ YOURSELFSee www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

- How many different five-digit passwords can be formed if the leading digit cannot be 0 and the number must be odd?
- An ecology quiz consists of 12 multiple choice questions with four possible answers for each question. In how many ways can the quiz be completed? (Assume that no question is omitted.)

In Exercises 3 and 4, evaluate each expression.

- | | | | |
|------------------|------------------|---------------------|------------------|
| 3. (a) ${}_9P_2$ | (b) ${}_{70}P_3$ | 4. (a) ${}_{11}C_4$ | (b) ${}_{66}C_5$ |
|------------------|------------------|---------------------|------------------|
- A shipment of 2500 microwave ovens contains 20 defective units. You order three units from the shipment. What is the probability that all three are good?
 - The weather report calls for 55% chance of rain. According to this report, what is the probability that it will not rain?
 - One card is selected at random from a standard deck of 52 playing cards. What is the probability that the card is either an ace or a diamond?
 - Seventy-five percent of the freshmen at a college go on to graduate. What is the probability that three freshmen chosen at random from the college will all graduate?

In Exercises 9 and 10, sketch a graph of the probability distribution and find the indicated probabilities.

x	1	2	3	4
$P(x)$	$\frac{3}{16}$	$\frac{7}{16}$	$\frac{1}{16}$	$\frac{5}{16}$

x	7	8	9	10	11
$P(x)$	0.21	0.13	0.19	0.42	0.05

- (a) $P(x < 3)$ (b) $P(x \geq 3)$ (a) $P(7 \leq x \leq 10)$ (b) $P(x > 8)$

In Exercises 11 and 12, find the expected value, variance, and standard deviation for the given probability distribution.

11.

x	0	1	2	3
$P(x)$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{4}{10}$	$\frac{3}{10}$

12.

x	-2	-1	0	1	2
$P(x)$	0.141	0.305	0.257	0.063	0.234



In Exercises 13 and 14, use a graphing utility to graph the function. Then determine whether the function f represents a probability density function over the given interval. If f is not a probability density function, identify the condition(s) that is (are) not satisfied.

13. $f(x) = \frac{\pi}{2} \sin \pi x, [0, 1]$ 14. $f(x) = \frac{3-x}{6}, [-1, 1]$

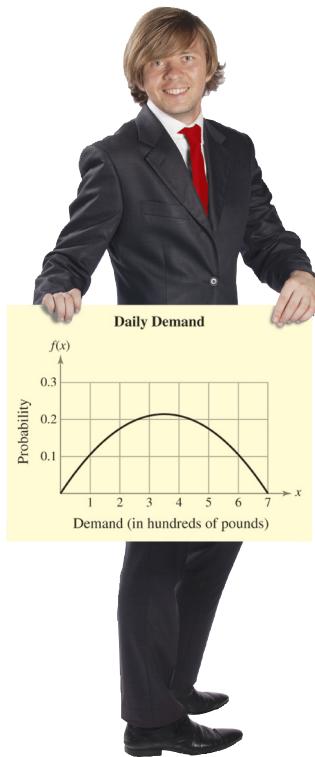
In Exercises 15–17, find the indicated probabilities for the probability density function.

15. $f(x) = \frac{x}{32}, [0, 8]$ (a) $P(1 \leq x \leq 4)$ (b) $P(3 \leq x \leq 6)$

16. $f(x) = 4(x - x^3), [0, 1]$ (a) $P(0 < x < 0.5)$ (b) $P(0.25 \leq x < 1)$

17. $f(x) = 2xe^{-x^2}, [0, \infty)$ (a) $P(x < 1)$ (b) $P(x \geq 1)$

16.4 Expected Value and Variance



In Exercise 39 on page 1180, you will use integration to find the mean, standard deviation, and median of the daily demand for a product.

- Find the expected values or means of continuous probability density functions.
- Find the variances and standard deviations of continuous probability density functions.
- Find the medians of continuous probability density functions.
- Use special probability density functions to answer questions about real-life situations.

Expected Value

In Section 16.3, you studied the concepts of expected value (or mean), variance, and standard deviation of *discrete* random variables. In this section, you will extend these concepts to *continuous* random variables.

Definition of Expected Value

If f is a probability density function of a continuous random variable x over the interval $[a, b]$, then the **expected value** or **mean** of x is

$$\mu = E(x) = \int_a^b xf(x) dx.$$



Example 1 Finding Average Weekly Demand

The weekly demand for a product is modeled by the probability density function

$$f(x) = \frac{1}{36}(-x^2 + 6x), \quad 0 \leq x \leq 6$$

where x is the number of units sold (in thousands). Find the expected weekly demand for this product.

SOLUTION

$$\begin{aligned} \mu &= E(x) \\ &= \frac{1}{36} \int_0^6 x(-x^2 + 6x) dx \\ &= \frac{1}{36} \int_0^6 (-x^3 + 6x^2) dx \\ &= \frac{1}{36} \left[\frac{-x^4}{4} + 2x^3 \right]_0^6 \\ &= 3 \end{aligned}$$

In Figure 16.19, you can see that an expected value of 3 seems reasonable because the region is symmetric about the line $x = 3$.

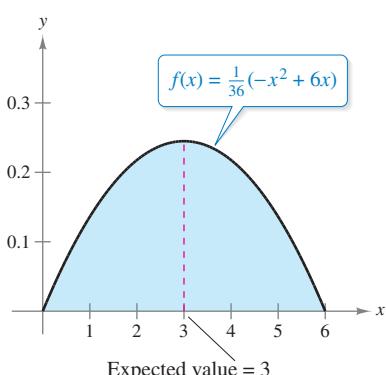


FIGURE 16.19

✓ Checkpoint 1

Find the expected value of the probability density function

$$f(x) = \frac{1}{32}(3x)(4 - x)$$

on the interval $[0, 4]$.

Variance and Standard Deviation

Definitions of Variance and Standard Deviation

If f is a probability density function of a continuous random variable x over the interval $[a, b]$, then the **variance** of x is

$$V(x) = \int_a^b (x - \mu)^2 f(x) dx$$

where μ is the mean of x . The **standard deviation** of x is

$$\sigma = \sqrt{V(x)}.$$

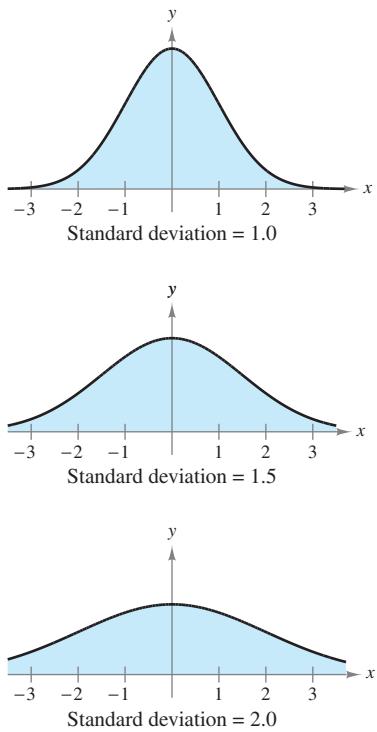


FIGURE 16.20

Recall from Section 16.3 that distributions that are clustered about the mean tend to have smaller standard deviations than distributions that are more dispersed. For instance, the probability density distributions shown in Figure 16.20 have a mean of $\mu = 0$, but they have different standard deviations. Because the first distribution is clustered more toward the mean, its standard deviation is the smallest of the three.

Example 2 Finding Variance and Standard Deviation

Find the variance and standard deviation of the probability density function

$$f(x) = 2 - 2x, \quad 0 \leq x \leq 1.$$

SOLUTION Begin by finding the mean.

$$\begin{aligned} \mu &= \int_0^1 x(2 - 2x) dx \\ &= \int_0^1 (2x - 2x^2) dx \\ &= \left[x^2 - \frac{2x^3}{3} \right]_0^1 \\ &= \frac{1}{3} \end{aligned} \quad \text{Mean}$$

Next, apply the formula for variance.

$$\begin{aligned} V(x) &= \int_0^1 \left(x - \frac{1}{3} \right)^2 (2 - 2x) dx \\ &= \int_0^1 \left(-2x^3 + \frac{10x^2}{3} - \frac{14x}{9} + \frac{2}{9} \right) dx \\ &= \left[-\frac{x^4}{2} + \frac{10x^3}{9} - \frac{7x^2}{9} + \frac{2x}{9} \right]_0^1 \\ &= \frac{1}{18} \end{aligned} \quad \text{Variance}$$

Finally, you can conclude that the standard deviation is

$$\sigma = \sqrt{\frac{1}{18}} \approx 0.236. \quad \text{Standard deviation}$$

Checkpoint 2

Find the variance and standard deviation of the probability density function in Checkpoint 1.

The integral for variance can be difficult to evaluate. The following alternative formula is sometimes simpler.



Alternative Formula for Variance

If f is a probability density function of a continuous random variable x over the interval $[a, b]$, then the **variance** of x is

$$V(x) = \int_a^b x^2 f(x) dx - \mu^2$$

where μ is the mean of x .

Example 3 Using the Alternative Formula

Find the standard deviation of the probability density function

$$f(x) = \frac{2}{\pi(x^2 - 2x + 2)}, \quad 0 \leq x \leq 2.$$

What percent of the distribution lies within one standard deviation of the mean?

SOLUTION Begin by using a symbolic integration utility to find the mean.

$$\begin{aligned} \mu &= \int_0^2 \left[\frac{2}{\pi(x^2 - 2x + 2)} \right] (x) dx \\ &= 1 \end{aligned} \quad \text{Mean}$$

Next, use a symbolic integration utility to find the variance.

$$\begin{aligned} V(x) &= \int_0^2 \left[\frac{2}{\pi(x^2 - 2x + 2)} \right] (x^2) dx - 1^2 \\ &\approx 0.273 \end{aligned} \quad \text{Variance}$$

This implies that the standard deviation is

$$\begin{aligned} \sigma &\approx \sqrt{0.273} \\ &\approx 0.522. \end{aligned} \quad \text{Standard deviation}$$

To find the percent of the distribution that lies within one standard deviation of the mean, integrate the probability density function between

$$\mu - \sigma = 0.478$$

and

$$\mu + \sigma = 1.522.$$

Using a symbolic integration utility, you obtain

$$\int_{0.478}^{1.522} \frac{2}{\pi(x^2 - 2x + 2)} dx \approx 0.613.$$

So, about 61.3% of the distribution lies within one standard deviation of the mean. This result is illustrated in Figure 16.21.

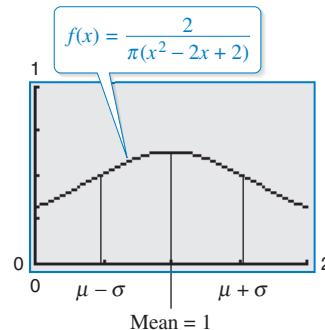


FIGURE 16.21

Checkpoint 3

Use a symbolic integration utility to find the percent of the distribution in Example 3 that lies within 1.5 standard deviations of the mean.

Median

The mean of a probability density function is an example of a *measure of central tendency*. Another useful measure of central tendency is the **median**.

Definition of Median

If f is a probability density function of a continuous random variable x over the interval $[a, b]$, then the **median** of x is the number m such that

$$\int_a^m f(x) dx = 0.5.$$



Example 4 Comparing Mean and Median

In Example 10 in Section 16.3, the probability density function

$$f(t) = 0.1e^{-0.1t}, \quad 0 \leq t < \infty$$

was used to model the useful lifetime of a product. Find the mean and median useful lifetimes.

SOLUTION Using integration by parts or a symbolic integration utility, you can find the mean to be

$$\begin{aligned} \mu &= \int_0^\infty 0.1te^{-0.1t} dt \\ &= 10 \text{ years.} \end{aligned}$$

Mean

The median is given by

$$\begin{aligned} \int_0^m 0.1e^{-0.1t} dt &= 0.5 \\ \left[-e^{-0.1t} \right]_0^m &= 0.5 \\ -e^{-0.1m} + 1 &= 0.5 \\ -e^{-0.1m} &= -0.5 \\ e^{-0.1m} &= 0.5 \\ -0.1m &= \ln 0.5 \\ m &= -10 \ln 0.5 \\ m &\approx 6.93 \text{ years.} \end{aligned}$$

Median

From this, you can see that the mean and median of a probability distribution can be quite different. Using the mean, the “average” lifetime of a product is 10 years, but using the median, the “average” lifetime is approximately 6.93 years. In Figure 16.22, note that half of the products have useful lifetimes of approximately 6.93 years or less. You can confirm this by evaluating the integral

$$\int_0^{-10 \ln 0.5} 0.1e^{-0.1t} dt$$

using a symbolic integration utility.

Checkpoint 4

Find the mean and median of the probability density function

$$f(x) = 2e^{-2x}, \quad 0 \leq x < \infty.$$

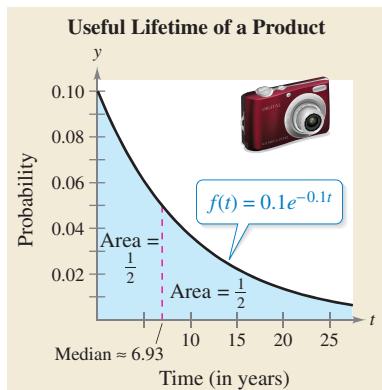


FIGURE 16.22

Special Probability Density Functions

The remainder of this section describes three common types of probability density functions: uniform, exponential, and normal. The **uniform probability density function** is defined as

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b. \quad \text{Uniform probability density function}$$

This probability density function represents a continuous random variable for which each outcome is equally likely.

Example 5 Analyzing a Probability Density Function

Find the expected value and standard deviation of the uniform probability density function

$$f(x) = \frac{1}{8}, \quad 0 \leq x \leq 8.$$

SOLUTION The expected value (or mean) is

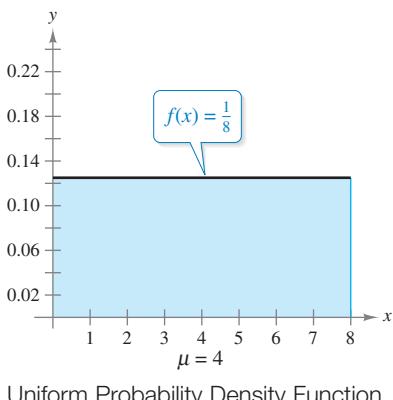
$$\begin{aligned} \mu &= \int_0^8 \frac{1}{8} x \, dx \\ &= \left[\frac{x^2}{16} \right]_0^8 \\ &= 4. \end{aligned} \quad \text{Expected value}$$

The variance is

$$\begin{aligned} V(x) &= \int_0^8 \frac{1}{8} x^2 \, dx - 4^2 \\ &= \left[\frac{x^3}{24} \right]_0^8 - 16 \\ &\approx 5.333. \end{aligned} \quad \text{Variance}$$

The standard deviation is

$$\begin{aligned} \sigma &\approx \sqrt{5.333} \\ &\approx 2.309. \end{aligned} \quad \text{Standard deviation}$$



Uniform Probability Density Function

FIGURE 16.23

The graph of f is shown in Figure 16.23.

Checkpoint 5

Find the expected value and standard deviation of the uniform probability density function

$$f(x) = \frac{1}{2}, \quad 0 \leq x \leq 2.$$

It can be shown that the uniform probability density function has an expected value of

$$\mu = \frac{1}{2}(a + b) \quad \text{Expected value}$$

and a variance of

$$V(x) = \frac{1}{12}(b - a)^2. \quad \text{Variance}$$

Try using these formulas to confirm the results of Example 5.

The second special type of probability density function is the **exponential probability density function**, which has the form

$$f(x) = ae^{-ax}, \quad 0 \leq x < \infty.$$

Exponential probability density function, $a > 0$

This function has an expected value of

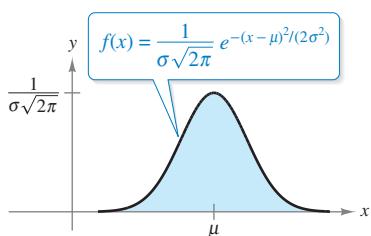
$$\mu = \frac{1}{a}$$

Expected value

and a variance of

$$V(x) = \frac{1}{a^2}.$$

Variance



Normal Probability Density Function

FIGURE 16.24

The function in Example 4 is an exponential probability density function. Use the formula $\mu = 1/a$ to confirm the mean found in Example 4.

The third special type of probability density function (and the most widely used) is the **normal probability density function** given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}, \quad -\infty < x < \infty.$$

Normal probability density function

The expected value of this function is μ , and the standard deviation is σ . Figure 16.24 shows the graph of a typical normal probability density function.

A normal probability density function for which

$$\mu = 0 \quad \text{and} \quad \sigma = 1$$

is called a **standard normal probability density function**.



Example 6 Finding a Probability

In 2010, the scores on the Graduate Management Admission Test (GMAT) could be modeled by a normal probability density function with a mean of $\mu = 540.4$ and a standard deviation of $\sigma = 120.72$. A person who took the GMAT is selected at random. What is the probability that the person scored between 600 and 700? What is the probability that the person scored between 700 and 800? (*Source: Graduate Management Admission Council*)

SOLUTION Using a symbolic integration utility, the first probability is

$$P(600 \leq x \leq 700) = \int_{600}^{700} \frac{1}{120.72\sqrt{2\pi}} e^{-(x-540.4)^2/(2 \cdot 120.72^2)} dx \\ \approx 0.218.$$

So, the probability of choosing a person who scored between 600 and 700 is about 21.8%. In a similar way, you can find the probability of choosing a person who scored between 700 and 800. By evaluating the integral

$$P(700 \leq x \leq 800) = \int_{700}^{800} \frac{1}{120.72\sqrt{2\pi}} e^{-(x-540.4)^2/(2 \cdot 120.72^2)} dx \\ \approx 0.077.$$

you can see that the probability is about 7.7%.

✓ Checkpoint 6

Find the probability that a person selected at random scored between 400 and 600 on the test described in Example 6.

TECH TUTOR



Finding the antiderivative of the normal probability density function analytically is beyond the scope of this text. You can use a graphing utility, a symbolic integration utility, or a spreadsheet to find this antiderivative. Use one of these tools to evaluate the definite integrals in Example 6.

STUDY TIP

In Example 6, note that probabilities connected with normal distributions can be evaluated with a table of values, with a symbolic integration utility, or with a spreadsheet.



In addition to being popular zoo animals, rhesus monkeys are commonly used in medical and behavioral research. Research on rhesus monkeys led to the discovery of the Rh factor in human red blood cells.



Example 7 Modeling Body Weight

Assume the weights of adult male rhesus monkeys are normally distributed with a mean of 17 pounds and a standard deviation of 4 pounds. In a typical population of adult male rhesus monkeys, what percent of the monkeys would have weights within one standard deviation of the mean?

SOLUTION For this population, $\mu = 17$ and $\sigma = 4$. So, the normal probability density function is

$$f(x) = \frac{1}{4\sqrt{2\pi}} e^{-(x-17)^2/32}.$$

The probability that a randomly chosen adult male rhesus monkey will weigh between 13 and 21 pounds (that is, within 4 pounds of 17 pounds) is

$$\begin{aligned} P(13 \leq x \leq 21) &= \int_{13}^{21} \frac{1}{4\sqrt{2\pi}} e^{-(x-17)^2/32} dx \\ &\approx 0.683. \end{aligned}$$

So, about 68% of the adult male rhesus monkeys have weights that lie within one standard deviation of the mean, as shown in Figure 16.25.

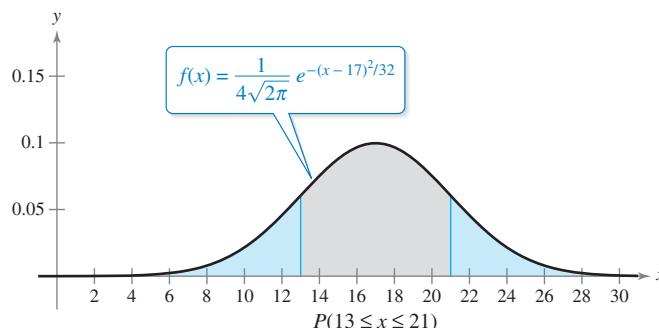


FIGURE 16.25

✓ Checkpoint 7

Use the results of Example 7 to find the probability that an adult male rhesus monkey chosen at random will weigh more than 21 pounds. □

The result described in Example 7 can be generalized to all normal distributions. That is, in any normal distribution, the probability that x lies within one standard deviation of the mean is about 68%. For normal distributions, 95.4% of the x -values lie within two standard deviations of the mean, and almost all (99.7%) of the x -values lie within three standard deviations of the mean.



SUMMARIZE (Section 16.4)

- State the definition of expected value for a continuous random variable (page 1172). For an example of finding an expected value, see Example 1.
- State the definitions of variance and standard deviation for a continuous random variable (page 1173). For an example of finding variance and standard deviation, see Example 2.
- List the three special probability density functions discussed in this section (pages 1176 and 1177). For examples of these functions, see Examples 4, 5, 6, and 7.

SKILLS WARM UP 16.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 11.4 and 16.3.

In Exercises 1 and 2, solve for m .

1. $\int_0^m \frac{1}{16} dx = 0.5$

2. $\int_0^m \frac{1}{3} e^{-t/3} dt = 0.5$

In Exercises 3 and 4, evaluate the definite integral.

3. $\int_0^2 \frac{x^2}{2} dx$

4. $\int_1^2 x(4 - 2x) dx$

In Exercises 5 and 6, find the indicated probability using the given probability density function.

5. $f(x) = \frac{1}{8}$, $[0, 8]$

(a) $P(x \leq 2)$

(b) $P(3 < x < 7)$

6. $f(x) = 6x - 6x^2$, $[0, 1]$

(a) $P\left(x \leq \frac{1}{2}\right)$

(b) $P\left(\frac{1}{4} \leq x \leq \frac{3}{4}\right)$

Exercises 16.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Finding Mean, Variance, and Standard Deviation In Exercises 1–12, use the given probability density function over the indicated interval to find the (a) mean, (b) variance, and (c) standard deviation of the random variable. (d) Then sketch the graph of the density function and locate the mean on the graph. See Examples 1, 2, and 3.

1. $f(x) = \frac{1}{3}$, $[0, 3]$

2. $f(x) = \frac{1}{4}$, $[0, 4]$

3. $f(t) = \frac{t}{18}$, $[0, 6]$

4. $f(x) = \frac{x}{8}$, $[0, 4]$

5. $f(x) = \frac{4}{3x^2}$, $[1, 4]$

6. $f(x) = \frac{5}{2}x^{3/2}$, $[0, 1]$

7. $f(x) = 1 - \frac{1}{2}x$, $[0, 2]$

8. $f(x) = \frac{1}{2} - \frac{1}{8}x$, $[0, 4]$

9. $f(x) = 6x(1 - x)$, $[0, 1]$

10. $f(x) = \frac{3}{32}x(4 - x)$, $[0, 4]$

11. $f(x) = \frac{3}{16}\sqrt{4 - x}$, $[0, 4]$

12. $f(x) = \frac{1}{18}\sqrt{9 - x}$, $[0, 9]$

Finding the Mean and Median In Exercises 13–18, find the mean and median of the probability density function. See Example 4.

13. $f(x) = \frac{1}{11}$, $[0, 11]$

14. $f(x) = 0.05$, $[0, 20]$

15. $f(x) = 4(1 - 2x)$, $[0, \frac{1}{2}]$

16. $f(x) = \frac{4}{3} - \frac{2}{3}x$, $[0, 1]$

17. $f(t) = \frac{1}{9}e^{-t/9}$, $[0, \infty)$

18. $f(t) = \frac{2}{5}e^{-2t/5}$, $[0, \infty)$

Special Probability Density Functions In Exercises 19–24, identify the probability density function. Then find the mean, variance, and standard deviation without integrating. See Example 5.

19. $f(x) = \frac{1}{10}$, $[0, 10]$

20. $f(x) = \frac{1}{6}$, $[0, 6]$

21. $f(x) = \frac{1}{8}e^{-x/8}$, $[0, \infty)$

22. $f(x) = \frac{5}{3}e^{-5x/3}$, $[0, \infty)$

23. $f(x) = \frac{1}{11\sqrt{2\pi}}e^{-(x-100)^2/242}$, $(-\infty, \infty)$

24. $f(x) = \frac{1}{6\sqrt{2\pi}}e^{-(x-30)^2/72}$, $(-\infty, \infty)$



Finding Probabilities Using Technology In Exercises 25–30, use a symbolic integration utility to find the mean, standard deviation, and given probability.

*Function**Probability*

25. $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$P(0 \leq x \leq 0.85)$

26. $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$P(-1.21 \leq x \leq 1.21)$

27. $f(x) = \frac{1}{6} e^{-x/6}$

$P(x \geq 2.23)$

28. $f(x) = \frac{3}{4} e^{-3x/4}$

$P(x \geq 0.27)$

29. $f(x) = \frac{1}{2\sqrt{2\pi}} e^{-(x-8)^2/8}$

$P(3 \leq x \leq 13)$

30. $f(x) = \frac{1}{1.5\sqrt{2\pi}} e^{-(x-2)^2/4.5}$

$P(-2.5 \leq x \leq 2.5)$



Finding Probabilities In Exercises 31 and 32, let x be a random variable that is normally distributed with the given mean and standard deviation. Find the indicated probabilities using a symbolic integration utility. See Example 6.

31. $\mu = 50, \sigma = 10$

- (a) $P(x > 55)$
- (b) $P(x > 60)$
- (c) $P(x < 60)$
- (d) $P(30 < x < 55)$

32. $\mu = 70, \sigma = 14$

- (a) $P(x > 65)$
- (b) $P(x < 98)$
- (c) $P(x < 49)$
- (d) $P(56 < x < 75)$

33. **Transportation** The arrival time t (in minutes) of a bus at a bus stop is uniformly distributed between 10:00 A.M. and 10:10 A.M.

- (a) Find the probability density function for the random variable t .
- (b) Find the mean and standard deviation of the arrival times.
- (c) What is the probability that you will miss the bus if you arrive at the bus stop at 10:03 A.M.?

34. **Transportation** Repeat Exercise 33 for a bus that arrives between 10:00 A.M. and 10:05 A.M.

35. **Useful Life** The time t (in years) until failure of an appliance is exponentially distributed with a mean of 2 years.

- (a) Find the probability density function for the random variable t .
- (b) Find the probability that the appliance will fail in less than 1 year.

36. **Useful Life** The time t (in years) until failure of a printer is exponentially distributed with a mean of 4 years.

- (a) Find the probability density function for the random variable t .
- (b) Find the probability that the printer will fail in more than 1 year but less than 3 years.

37. **Waiting Time** The waiting time t (in minutes) for service in a store is exponentially distributed with a mean of 5 minutes.

- (a) Find the probability density function for the random variable t .
- (b) Find the probability that the waiting time is within one standard deviation of the mean.

38. **License Renewal** The time t (in minutes) spent at a driver's license renewal center is exponentially distributed with a mean of 15 minutes.

- (a) Find the probability density function for the random variable t .
- (b) Find the probability that the time spent is within one standard deviation of the mean.

39. **Demand** The daily demand x for a certain product (in hundreds of pounds) is a random variable with the probability density function

$$f(x) = \frac{6}{343}x(7-x), \quad [0, 7].$$

- (a) Find the mean and standard deviation of the demand.
- (b) Find the median of the demand.
- (c) Find the probability that the demand is within one standard deviation of the mean.

40. **Demand** Repeat Exercise 39 for the probability density function

$$f(x) = \frac{3}{256}(x-2)(10-x), \quad [2, 10].$$

41. **Consumer Trends** The number of coupons x used by a customer in a grocery store is a random variable with the probability density function

$$f(x) = \frac{2x+1}{12}, \quad [0, 3].$$

Find the expected number of coupons a customer will use.

42. **Cost** The daily cost x (in dollars) of electricity in a city is a random variable with the probability density function

$$f(x) = 0.28e^{-0.28x}, \quad [0, \infty).$$

Find the median daily cost of electricity.

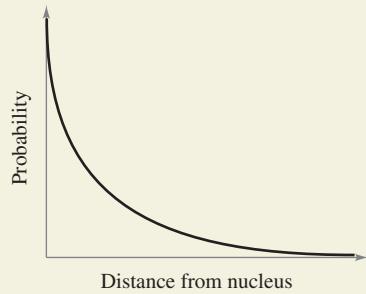
- 43. Demand** The daily demand x for water (in millions of gallons) in a town is a random variable with the probability density function

$$f(x) = \frac{1}{9}xe^{-x/3}, \quad [0, \infty).$$

- (a) Find the mean and standard deviation of the demand.
 (b) Find the probability that the demand is greater than 4 million gallons on a given day.

44.

HOW DO YOU SEE IT? In chemistry, the probability of finding an electron at a particular position is greatest close to the nucleus and drops off rapidly as the distance from the nucleus increases. The graph displays the probability of finding the electron at points along a line drawn from the nucleus outward in any direction for the hydrogen $1s$ orbital. Make a sketch of this graph, and add to your sketch an indication of where you think the median might be. (*Source: Adapted from Zumdahl, Chemistry, Seventh Edition*)



- 45. Useful Life** The lifetime of a certain type of battery is normally distributed with a mean of 400 hours and a standard deviation of 24 hours. You purchased one of the batteries, and its useful life was 340 hours.



- (a) How far, in standard deviations, did the useful life of your battery fall short of the expected life?
 (b) Use a symbolic integration utility to approximate the percent of all other batteries of this type with useful lives that exceed that of your battery.

- 46. Education** The scores on a national exam are normally distributed with a mean of 150 and a standard deviation of 16. You scored 174 on the exam.



- (a) By how much, in standard deviations, did your score exceed the national mean?
 (b) Use a symbolic integration utility to approximate the percent of those who took the exam who had scores lower than yours.

- 47. Useful Life** A storage battery has an expected lifetime of 4.5 years with a standard deviation of 0.5 year. Assume that the useful lives of these batteries are normally distributed.

- (a) Use a computer or graphing utility and Simpson's Rule (with $n = 12$) to approximate the probability that a given battery will last for 4 to 5 years.
 (b) Will 10% of the batteries last less than 3 years?

- 48. Wages** The employees of a large corporation are paid an average wage of \$14.50 per hour with a standard deviation of \$1.50. Assume that these wages are normally distributed.

- (a) Use a computer or graphing utility and Simpson's Rule (with $n = 10$) to approximate the percent of employees who earn hourly wages of \$11 to \$14.
 (b) Are 20% of the employees paid more than \$16 per hour?

- 49. Medical Science** A medical research team has determined that for a group of 500 females, the length of pregnancy from conception to birth varies according to an approximately normal distribution with a mean of 266 days and a standard deviation of 16 days.

- (a) Use a graphing utility to graph the distribution.
 (b) Use a symbolic integration utility to approximate the probability that a pregnancy will last from 240 days to 280 days.
 (c) Use a symbolic integration utility to approximate the probability that a pregnancy will last more than 280 days.

- 50. Education** For high school graduates from 2008 through 2010, the scores on the ACT Test can be modeled by a normal probability density function with a mean of 21.1 and a standard deviation of 5.1. (*Source: ACT, Inc.*)

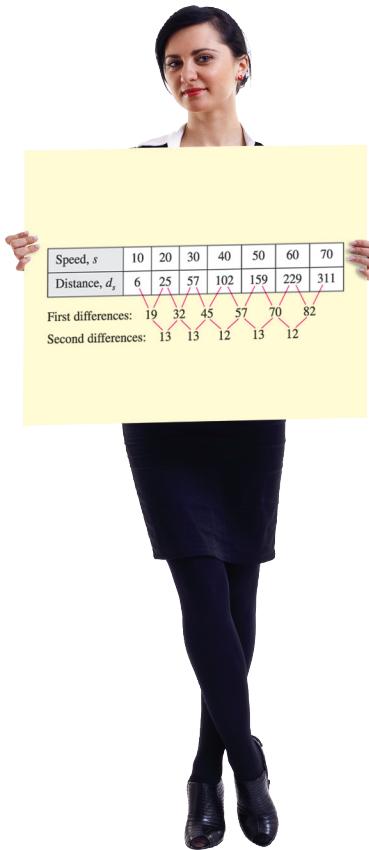
- (a) Use a graphing utility to graph the distribution.
 (b) Use a symbolic integration utility to approximate the probability that a person who took the ACT scored between 24 and 36.
 (c) Use a symbolic integration utility to approximate the probability that a person who took the ACT scored more than 26.

- 51. Fuel Mileage** Assume the fuel mileages of all 2011 model vehicles are normally distributed with a mean of 21.0 miles per gallon and a standard deviation of 5.6 miles per gallon. (*Source: U.S. Environmental Protection Agency*)

- (a) Use a graphing utility to graph the distribution.
 (b) Use a symbolic integration utility to approximate the probability that a vehicle's fuel mileage is between 25 and 30 miles per gallon.
 (c) Use a symbolic integration utility to approximate the probability that a vehicle's fuel mileage is less than 18 miles per gallon.

16.5 Mathematical Induction

- Use mathematical induction to prove a formula.
- Find a sum of powers of integers.
- Find a formula for a finite sum.
- Use finite differences to find a linear or quadratic model.



In Exercise 76 on page 1192, you will use finite differences to determine whether the distances required to stop a truck at various speeds are better represented by a linear or a quadratic model.

Introduction

In this section, you will study a form of mathematical proof called **mathematical induction**. It is important that you clearly see the logical need for it, so take a look at the problem below.

$$\begin{aligned} S_1 &= 1 = 1^2 \\ S_2 &= 1 + 3 = 2^2 \\ S_3 &= 1 + 3 + 5 = 3^2 \\ S_4 &= 1 + 3 + 5 + 7 = 4^2 \\ S_5 &= 1 + 3 + 5 + 7 + 9 = 5^2 \end{aligned}$$

Judging from the pattern formed by these first five sums, it appears that the sum of the first n odd integers is

$$S_n = 1 + 3 + 5 + 7 + 9 + \dots + (2n - 1) = n^2.$$

Although this particular formula *is* valid, it is important for you to see that recognizing a pattern and then simply *jumping to the conclusion* that the pattern must be true for all values of n is *not* a logically valid method of proof. There are many examples in which a pattern appears to be developing for small values of n and then at some point the pattern fails. One of the most famous cases of this was the conjecture by the French mathematician Pierre de Fermat (1601–1665), who speculated that all numbers of the form

$$F_n = 2^{2^n} + 1, \quad n = 0, 1, 2, \dots$$

are prime. For $n = 0, 1, 2, 3$, and 4, the conjecture is true.

$$F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65,537$$

The size of the next Fermat number ($F_5 = 4,294,967,297$) is so great that it was difficult for Fermat to determine whether it was prime or not. However, another well-known mathematician, Leonhard Euler (1707–1783), later found the factorization

$$\begin{aligned} F_5 &= 4,294,967,297 \\ &= 641(6,700,417) \end{aligned}$$

which proved that F_5 is not prime and therefore Fermat's conjecture was false.

Just because a rule, pattern, or formula seems to work for several values of n , you cannot simply decide that it is valid for all values of n without going through a *legitimate proof*. Mathematical induction is one method of proof.

The Principle of Mathematical Induction

Let P_n be a statement involving the positive integer n . If

1. P_1 is true, and
 2. the truth of P_k implies the truth of P_{k+1} for every positive integer k ,
- then P_n must be true for all positive integers n .

It is important to recognize that both parts of the Principle of Mathematical Induction are necessary. To apply the Principle of Mathematical Induction, you need to be able to determine the statement P_{k+1} for a given statement P_k .

Example 1 Using P_k to Find P_{k+1}

Find P_{k+1} for each statement.

a. $P_k: S_k = \frac{k^2(k+1)^2}{4}$

b. $P_k: S_k = 1 + 5 + 9 + \dots + [4(k-1) - 3] + (4k - 3)$

c. $P_k: 3^k \geq 2k + 1$

SOLUTION

a. $P_{k+1}: S_{k+1} = \frac{(k+1)^2(k+1+1)^2}{4}$ Replace k by $k+1$.
 $= \frac{(k+1)^2(k+2)^2}{4}$ Simplify.

b. $P_{k+1}: S_{k+1} = 1 + 5 + 9 + \dots + \{4[(k+1)-1] - 3\} + [4(k+1) - 3]$
 $= 1 + 5 + 9 + \dots + (4k-3) + (4k+1)$

c. $P_{k+1}: 3^{k+1} \geq 2(k+1) + 1$
 $3^{k+1} \geq 2k + 3$

✓ Checkpoint 1

Find P_{k+1} for $P_k: S_k = k(2k+1)$.



FIGURE 16.26

A well-known illustration used to explain why the Principle of Mathematical Induction works is the unending line of dominoes shown in Figure 16.26. If the line actually contains infinitely many dominoes, then it is clear that you could not knock the entire line down by knocking down only *one domino* at a time. However, suppose it were true that each domino would knock down the next one as it fell. Then you could knock them all down simply by pushing the first one and starting a chain reaction. Mathematical induction works in the same way. If the truth of P_k implies the truth of P_{k+1} and if P_1 is true, then the chain reaction proceeds as follows: P_1 implies P_2 , P_2 implies P_3 , P_3 implies P_4 , and so on.

Using Mathematical Induction

Example 2 Using Mathematical Induction

Use mathematical induction to prove the formula

$$S_n = 1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2$$

for all integers $n \geq 1$.

SOLUTION Mathematical induction consists of two distinct parts. First, you must show that the formula is true when $n = 1$.

- When $n = 1$, the formula is valid, because

$$S_1 = 1 = 1^2.$$

The second part of mathematical induction has two steps. The first step is to assume that the formula is valid for some integer k . The second step is to use this assumption to prove that the formula is valid for the next integer, $k + 1$.

- Assuming that the formula

$$S_k = 1 + 3 + 5 + 7 + \dots + (2k - 1) = k^2$$

is true, you must show that the formula $S_{k+1} = (k + 1)^2$ is true.

$$\begin{aligned} S_{k+1} &= 1 + 3 + 5 + 7 + \dots + (2k - 1) + [2(k + 1) - 1] \\ &= [1 + 3 + 5 + 7 + \dots + (2k - 1)] + (2k + 2 - 1) \\ &= S_k + (2k + 1) \quad \text{Group terms to form } S_k. \\ &= k^2 + 2k + 1 \quad \text{Replace } S_k \text{ by } k^2. \\ &= (k + 1)^2 \quad \text{Factor.} \end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for all integers $n \geq 1$.

STUDY TIP

When using mathematical induction to prove a *summation* formula (such as the one in Example 2), it is helpful to think of S_{k+1} as $S_{k+1} = S_k + a_{k+1}$, where a_{k+1} is the $(k + 1)$ th term of the original sum.

Checkpoint 2

Use mathematical induction to prove the formula

$$S_n = 4 + 5 + 6 + 7 + \dots + (n + 3) = \frac{n(n + 7)}{2}$$

for all integers $n \geq 1$.

It occasionally happens that a statement involving natural numbers is not true for the first $k - 1$ positive integers but is true for all values of $n \geq k$. In these instances, you use a slight variation of the Principle of Mathematical Induction in which you verify P_k rather than P_1 . This variation is called the **Extended Principle of Mathematical Induction**. To see the validity of this variation, note from Figure 16.26 that all but the first $k - 1$ dominoes can be knocked down by knocking over the k th domino. This suggests that you can prove a statement P_n to be true for $n \geq k$ by showing that P_k is true and that P_k implies P_{k+1} . In Exercises 35–38 of this section, you are asked to apply this extension of mathematical induction.

Example 3 Using Mathematical Induction

Use mathematical induction to prove the formula

$$\begin{aligned} S_n &= 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 \\ &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

for all integers $n \geq 1$.

SOLUTION

- 1.** When $n = 1$, the formula is valid, because

$$S_1 = 1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{1(2)(3)}{6}.$$

- 2.** Assuming that

$$S_k = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

you must show that

$$S_{k+1} = \frac{(k+1)(k+1+1)[2(k+1)+1]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

To do this, write the following.

$$\begin{aligned} S_{k+1} &= S_k + a_{k+1} \\ &= (1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2) + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{By assumption} \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} && \text{Add expressions.} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} && \text{Factor out } (k+1). \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} && \text{Simplify.} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} && \text{Completely factor.} \end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for *all* integers $n \geq 1$.

✓ Checkpoint 3

Use mathematical induction to prove the formula

$$S_n = 3 + 6 + 9 + 12 + \cdots + 3n = \frac{3n(n+1)}{2}$$

for all integers $n \neq 1$.

When proving a formula by mathematical induction, the only statement that you *need* to verify is P_1 . As a check, however, it is a good idea to try verifying some of the other statements. For instance, in Example 3, try verifying S_2 and S_3 .

Sums of Powers of Integers

The formula in Example 3 is one of a collection of useful summation formulas. This and other formulas dealing with the sums of various powers of the first n positive integers are shown.

Sums of Powers of Integers

1. $1 + 2 + 3 + 4 + \dots + n = \frac{n(n + 1)}{2}$
2. $1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$
3. $1^3 + 2^3 + 3^3 + 4^3 + \dots + n^3 = \frac{n^2(n + 1)^2}{4}$
4. $1^4 + 2^4 + 3^4 + 4^4 + \dots + n^4 = \frac{n(n + 1)(2n + 1)(3n^2 + 3n - 1)}{30}$
5. $1^5 + 2^5 + 3^5 + 4^5 + \dots + n^5 = \frac{n^2(n + 1)^2(2n^2 + 2n - 1)}{12}$

Each of these formulas for sums can be proven by mathematical induction.

Example 4 Finding a Sum of Powers of Integers

Find $\sum_{n=1}^7 n^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$.

SOLUTION Using the formula for the sum of the cubes of the first n positive integers, you obtain the following.

$$\begin{aligned}\sum_{n=1}^7 n^3 &= 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 \\ &= \frac{7^2(7 + 1)^2}{4} \\ &= \frac{49(64)}{4} \\ &= 784\end{aligned}$$

Check this sum by adding the numbers, as shown below.

$$1 + 8 + 27 + 64 + 125 + 216 + 343 = 784$$

✓ Checkpoint 4

Find $\sum_{n=1}^{10} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2$.



Example 5 Proving an Inequality by Mathematical Induction

Prove that $n < 2^n$ for all positive integers n .

SOLUTION

- For $n = 1$ and $n = 2$, the formula is true, because

$$1 < 2^1 \text{ and } 2 < 2^2.$$

- Assuming that

$$k < 2^k$$

you need to show that $k + 1 < 2^{k+1}$. For $n = k$, you have

$$2^{k+1} = 2(2^k) > 2(k) = 2k. \quad \text{By assumption}$$

Because $2k = k + k > k + 1$ for all $k > 1$, it follows that

$$2^{k+1} > 2k > k + 1$$

or

$$k + 1 < 2^{k+1}.$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that $n < 2^n$ for all positive integers n .

✓ Checkpoint 5

Prove that $n! > 2n$ for all integers $n > 3$.

Pattern Recognition

Although choosing a formula on the basis of a few observations does *not* guarantee the validity of the formula, pattern recognition *is* important. Once you have a pattern that you think works, you can try using mathematical induction to prove your formula.

Finding a Formula for the n th Term of a Sequence

To find a formula for the n th term of a sequence, consider these guidelines.

- Calculate the first several terms of the sequence. It is often a good idea to write the terms in both simplified and factored forms.
- Try to find a recognizable pattern for the terms and write a formula for the n th term of the sequence. This is your *hypothesis* or *conjecture*. You might try computing one or two more terms in the sequence to test your hypothesis.
- Use mathematical induction to prove your hypothesis.

Example 6 Finding a Formula for a Finite Sum

Find a formula for the finite sum.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{n(n+1)}$$

SOLUTION Begin by writing out the first few sums.

$$S_1 = \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1+1}$$

$$S_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{4}{6} = \frac{2}{3} = \frac{2}{2+1}$$

$$S_3 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{9}{12} = \frac{3}{4} = \frac{3}{3+1}$$

$$S_4 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} = \frac{48}{60} = \frac{4}{5} = \frac{4}{4+1}$$

From this sequence, it appears that the formula for the k th sum is

$$\begin{aligned} S_k &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{k(k+1)} \\ &= \frac{k}{k+1}. \end{aligned}$$

To prove the validity of this hypothesis, use mathematical induction, as shown below. Note that you have already verified the formula for $n = 1$, so you can begin by assuming that the formula is valid for $n = k$ and trying to show that it is valid for $n = k + 1$.

$$\begin{aligned} S_{k+1} &= \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{k(k+1)} \right] + \frac{1}{(k+1)(k+2)} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} && \text{By assumption} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} && \text{Add fractions.} \\ &= \frac{k^2+2k+1}{(k+1)(k+2)} && \text{Distributive Property} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} && \text{Factor numerator.} \\ &= \frac{k+1}{k+2} && \text{Simplify.} \end{aligned}$$

So, the hypothesis is valid.

Checkpoint 6

Find a formula for the finite sum.

$$3 + 5 + 7 + 9 + \cdots + (2n+1)$$



Finite Differences

The **first differences** of a sequence are found by subtracting consecutive terms. The **second differences** are found by subtracting consecutive first differences. The first and second differences of the sequence 3, 5, 8, 12, 17, 23, . . . are as shown.

$n:$	1	2	3	4	5	6
$a_n:$	3	5	8	12	17	23
First differences:	2	3	4	5	6	
Second differences:	1	1	1	1	1	

For this sequence, the second differences are all the same nonzero number. When this happens, the sequence has a perfect *quadratic* model. When the first differences are all the same, the sequence has a *linear* model. That is, it is arithmetic.

Example 7 Finding an Appropriate Model

Decide whether the sequence 6, 12, 26, 48, 78, 116, . . . can be represented perfectly by a linear or a quadratic model. If so, find the model.

SOLUTION Find the first and second differences of the sequence.

$n:$	1	2	3	4	5	6
$a_n:$	6	12	26	48	78	116
First differences:	6	14	22	30	38	
Second differences:	8	8	8	8	8	

Because the second differences are all the same, the sequence can be represented perfectly by a quadratic model of the form

$$a_n = an^2 + bn + c.$$

By substituting 1, 2, and 3 for n , you can obtain a system of three linear equations in three variables.

$$a_1 = a(1)^2 + b(1) + c = 6 \quad \text{Substitute 1 for } n.$$

$$a_2 = a(2)^2 + b(2) + c = 12 \quad \text{Substitute 2 for } n.$$

$$a_3 = a(3)^2 + b(3) + c = 26 \quad \text{Substitute 3 for } n.$$

You now have a system of three equations in a , b , and c .

$$\begin{cases} a + b + c = 6 & \text{Equation 1} \\ 4a + 2b + c = 12 & \text{Equation 2} \\ 9a + 3b + c = 26 & \text{Equation 3} \end{cases}$$

Using the techniques discussed in Chapter 5, you can find that the solution of this system is

$$a = 4, b = -6, \text{ and } c = 8.$$

So, the quadratic model is

$$a_n = 4n^2 - 6n + 8.$$

✓ Checkpoint 7

Decide whether the sequence

$$3, 8, 17, 30, 47, 68, \dots$$

can be represented perfectly by a linear or a quadratic model. If so, find the model. ■

Example 8 Finding an Appropriate Model

Decide whether the sequence 2, 9, 16, 23, 30, 37, . . . can be represented perfectly by a linear or a quadratic model. If so, find the model.

SOLUTION Find the first differences of the sequence.

$n:$	1	2	3	4	5	6
$a_n:$	2	9	16	23	30	37
First differences:	7	7	7	7	7	7

Because, the first differences are all the same, the sequence can be represented perfectly by a linear model of the form

$$a_n = an + b.$$

By substituting 1 and 2 for n , you can obtain a system of two linear equations in two variables.

$$\begin{aligned} a_1 &= a(1) + b = 2 && \text{Substitute 1 for } n. \\ a_2 &= a(2) + b = 9 && \text{Substitute 2 for } n. \end{aligned}$$

You now have a system of two equations in a and b .

$$\begin{cases} a + b = 2 \\ 2a + b = 9 \end{cases} \quad \begin{array}{l} \text{Equation 1} \\ \text{Equation 2} \end{array}$$

Using the techniques discussed in Chapter 5, you can find that the solution of this system is $a = 7$ and $b = -5$. So, the linear model is

$$a_n = 7n - 5.$$

✓ Checkpoint 8

Decide whether the sequence 6, 10, 14, 18, 22, 26, . . . can be represented perfectly by a linear or a quadratic model. If so, find the model.



SUMMARIZE (Section 16.5)

1. State the principle of mathematical induction (*page 1183*). For examples of using the principle of mathematical induction to prove a formula, see Examples 2 and 3.
2. Make a list of the formulas for the sums of powers of integers (*page 1186*). For an example that uses a formula to find a sum of powers of integers, see Example 4.
3. Describe the guidelines for finding a formula for the n th term of a sequence (*page 1187*). For an example of finding a formula for the n th term of a sequence, see Example 6.
4. Describe how to use the finite differences of a sequence to determine whether the sequence can be represented perfectly by a linear or a quadratic model (*page 1189*). For examples of using finite differences to find models, see Examples 7 and 8.

SKILLS WARM UP 16.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3, 0.4, 0.7, and 15.1.

In Exercises 1–4, find the sum.

1. $\sum_{k=3}^6 (2k - 3)$

2. $\sum_{j=1}^5 (j^2 - j)$

3. $\sum_{k=2}^5 \frac{1}{k}$

4. $\sum_{i=1}^2 \left(1 + \frac{1}{i}\right)$

In Exercises 5–10, simplify the expression.

5. $\frac{2(k+1) + 3}{5}$

6. $\frac{3(k+1) - 2}{6}$

7. $2 \cdot 2^{2(k+1)}$

8. $\frac{3^{2k}}{3^{2(k+1)}}$

9. $\frac{k+1}{k^2+k}$

10. $\frac{\sqrt{32}}{\sqrt{50}}$

Exercises 16.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Using P_k to find P_{k+1} In Exercises 1–4, find P_{k+1} for the given P_k . See Example 1.

1. $P_k = \frac{5}{k(k+1)}$

2. $P_k = \frac{3}{(k+2)(k+3)}$

3. $P_k = \frac{k^2(k-1)^2}{4}$

4. $P_k = \frac{k^2}{2(k+1)^2}$

Using Mathematical Induction In Exercises 5–18, use mathematical induction to prove the formula for every positive integer n . See Examples 2 and 3.

5. $2 + 4 + 6 + 8 + \dots + 2n = n(n+1)$

6. $3 + 7 + 11 + 15 + \dots + (4n-1) = n(2n+1)$

7. $2 + 7 + 12 + 17 + \dots + (5n-3) = \frac{n}{2}(5n-1)$

8. $1 + 4 + 7 + 10 + \dots + (3n-2) = \frac{n}{2}(3n-1)$

9. $1 + 2 + 2^2 + 2^3 + \dots + 2^{n-1} = 2^n - 1$

10. $2(1 + 3 + 3^2 + 3^3 + \dots + 3^{n-1}) = 3^n - 1$

11. $1 + 2 + 3 + 4 + \dots + n = \frac{n(n+1)}{2}$

12. $1^3 + 2^3 + 3^3 + 4^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$

13. $\frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

14. $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \dots \left(1 + \frac{1}{n}\right) = n + 1$

15. $\sum_{i=1}^n i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

16. $\sum_{i=1}^n i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

17. $\sum_{i=1}^n i(i+1) = \frac{n(n+1)(n+2)}{3}$

18. $\sum_{i=1}^n \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$

Finding a Sum of Powers of Integers In Exercises 19–28, find the sum using the formulas for the sums of powers of integers. See Example 4.

19. $\sum_{n=1}^{20} n$

20. $\sum_{n=1}^{50} n$

21. $\sum_{n=1}^6 n^2$

22. $\sum_{n=1}^{10} n^3$

23. $\sum_{n=1}^5 n^4$

24. $\sum_{n=1}^8 n^5$

25. $\sum_{n=1}^6 (n^2 - n)$

26. $\sum_{n=1}^{10} (n^3 - n^2)$

27. $\sum_{i=1}^6 (6i - 8i^3)$

28. $\sum_{j=1}^5 (2j^3 - j^2)$

Checking Solutions In Exercises 29–34, determine whether the inequality is true for $n = 1, 2, 3$, and 4.

29. $n^2 < n^3$

30. $4n^4 \leq n^6$

31. $(n+1)^2 \leq n^3$

32. $(n-2)^3 < n^2$

33. $n-1 \geq \frac{n+1}{n^2}$

34. $\frac{(n-1)^2}{n} < n! - n$

Proving an Inequality by Mathematical Induction In Exercises 35–38, prove the inequality for the indicated integer values of n . See Example 5.

35. $n! > 2^n, n \geq 4$

36. $\left(\frac{4}{3}\right)^n > n, n \geq 7$

37. $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n}, n \geq 2$

38. $\left(\frac{x}{y}\right)^{n+1} < \left(\frac{x}{y}\right)^n, n \geq 1 \text{ and } 0 < x < y$

Finding a Formula for a Finite Sum In Exercises 39–44, find a formula for the sum of the first n terms of the sequence. See Example 6.

39. $1, 5, 9, 13, \dots$

40. $25, 22, 19, 16, \dots$

41. $1, \frac{9}{10}, \frac{81}{100}, \frac{729}{1000}, \dots$

42. $3, -\frac{9}{2}, \frac{27}{4}, -\frac{81}{8}, \dots$

43. $\frac{1}{4}, \frac{1}{12}, \frac{1}{24}, \frac{1}{40}, \dots, \frac{1}{2n(n+1)}, \dots$

44. $\frac{1}{2 \cdot 3}, \frac{1}{3 \cdot 4}, \frac{1}{4 \cdot 5}, \frac{1}{5 \cdot 6}, \dots, \frac{1}{(n+1)(n+2)}, \dots$

Proving Properties In Exercises 45–52, use mathematical induction to prove the given property for all positive integers n .

45. $(ab)^n = a^n b^n$

46. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

47. If $x_1 \neq 0, x_2 \neq 0, \dots, x_n \neq 0$, then $(x_1 x_2 x_3 \cdots x_n)^{-1} = x_1^{-1} x_2^{-1} x_3^{-1} \cdots x_n^{-1}$.

48. If $x_1 > 0, x_2 > 0, \dots, x_n > 0$, then $\ln(x_1 x_2 x_3 \cdots x_n) = \ln x_1 + \ln x_2 + \ln x_3 + \cdots + \ln x_n$.

49. Generalized Distributive Law:

$x(y_1 + y_2 + \cdots + y_n) = xy_1 + xy_2 + \cdots + xy_n$

50. $(a + bi)^n$ and $(a - bi)^n$ are complex conjugates for all $n \geq 1$.

51. A factor of $(n^3 + 3n^2 + 2n)$ is 3.

52. A factor of $(2^{2n-1} + 3^{2n-1})$ is 5.

53. Writing In your own words, explain what is meant by a proof by mathematical induction.

54. Think About It What conclusion can be drawn from the given information about the sequence of statements P_n ?

(a) P_3 is true and P_k implies P_{k+1} .(b) $P_1, P_2, P_3, \dots, P_{50}$ are all true.(c) P_1, P_2 , and P_3 are all true, but the truth of P_k does not imply the truth of P_{k+1} .(d) P_2 is true and P_{2k} implies P_{2k+2} .

Finding an Appropriate Model In Exercises 55–60, decide whether the sequence can be represented perfectly by a linear or a quadratic model. If so, find the model. See Examples 7 and 8.

55. $5, 13, 21, 29, 37, 45, \dots$

56. $2, 9, 16, 23, 30, 37, \dots$

57. $6, 15, 30, 51, 78, 111, \dots$

58. $0, 6, 16, 30, 48, 70, \dots$

59. $-2, 1, 6, 13, 22, 33, \dots$

60. $-1, 8, 23, 44, 71, 104, \dots$

Using Finite Differences In Exercises 61–70, write the first five terms of the sequence, where $a_1 = f(1)$. Then calculate the first and second differences of the sequence. Does the sequence have either a linear model or a quadratic model? If so, find the model.

61. $f(1) = 0$

$a_n = a_{n-1} + 3$

62. $f(1) = 2$

$a_n = n - a_{n-1}$

63. $f(1) = 3$

$a_n = a_{n-1} - n$

64. $f(1) = -3$

$a_n = -2a_{n-1}$

65. $a_0 = 0$

$a_n = a_{n-1} + n$

66. $a_0 = 2$

$a_n = (a_{n-1})^2$

67. $f(1) = 2$

$a_n = a_{n-1} + 2$

68. $f(1) = 0$

$a_n = a_{n-1} + 2n$

69. $a_0 = 1$

$a_n = a_{n-1} + n^2$

70. $a_0 = 0$

$a_n = a_{n-1} - 1$

Finding a Quadratic Model In Exercises 71–74, find a quadratic model for the sequence with the indicated terms.

71. $a_0 = 3, a_1 = 3, a_4 = 15$

72. $a_0 = 7, a_1 = 6, a_3 = 10$

73. $a_1 = 0, a_2 = 8, a_4 = 30$

74. $a_0 = -3, a_2 = -5, a_6 = -57$

75. Think About It When the second differences of a sequence are all zero, can the sequence be represented perfectly by a linear model? Explain.



76.

HOW DO YOU SEE IT?

The table shows the distance d_s (in feet) required to stop a truck from the moment the brakes are applied at a speed of s miles-per-hour on dry, level pavement. The first and second differences of the sequence of terms d_s are shown below the table. (Source: *Virginia Transportation Research Council*)

Speed, s	10	20	30	40	50	60	70
Distance, d_s	6	25	57	102	159	229	311

First differences: 19 32 45 57 70 82

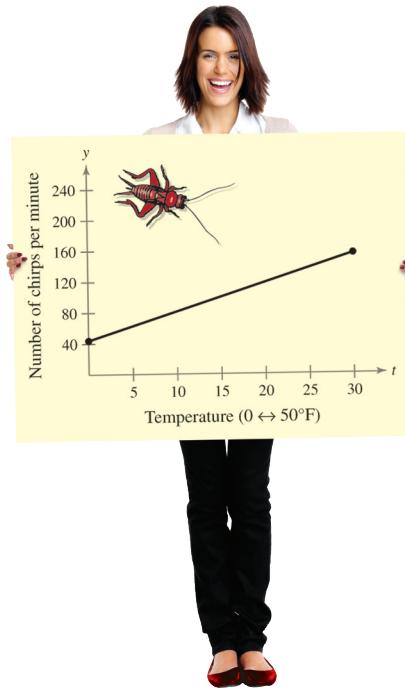
Second differences: 13 13 12 13 12

(a) Will a linear or a quadratic model fit the data best? Explain.

(b) When you drive a truck, do you assume an equal risk for each 10 mile-per-hour increase in speed? Explain your reasoning.

16.6 The Binomial Theorem

- Find a binomial coefficient.
- Use Pascal's Triangle to determine a binomial coefficient.
- Expand a binomial using binomial coefficients and find a specific term in a binomial expansion.



In Exercise 72 on page 1199, you will use the Binomial Theorem to rewrite a function that approximates the numbers of chirps a cricket makes each minute at various temperatures.

Binomial Coefficients

Recall that a *binomial* is a polynomial that has two terms. In this section, you will study a formula that gives a quick method of raising a binomial to a power. To begin, look at the expansion of

$$(x + y)^n$$

for several values of n .

$$(x + y)^0 = 1$$

$$(x + y)^1 = x + y$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

There are several observations you can make about these expansions.

1. In each expansion, there are $n + 1$ terms.
2. In each expansion, x and y have symmetrical roles. The powers of x decrease by 1 in successive terms, whereas the powers of y increase by 1.
3. The degree of each term is n . For instance, in the expansion of

$$(x + y)^5$$

the degree of each term is 5.

$$\underbrace{4+1=5}_{(x+y)^5} \quad \underbrace{3+2=5}_{(x+y)^5}$$

$$(x + y)^5 = x^5 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

4. The coefficients increase and then decrease in a symmetric pattern.

The coefficients of a binomial expansion are called **binomial coefficients**. To find them, you can use the **Binomial Theorem**.

The Binomial Theorem

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$, the $(r + 1)$ th term, is given by

$${}_nC_r = \frac{n!}{(n - r)!r!}.$$

The notation $\binom{n}{r}$ is often used in place of ${}_nC_r$ to denote a binomial coefficient.

Example 1 Finding Binomial Coefficients

a. ${}_8C_2 = \frac{8!}{6! \cdot 2!} = \frac{(8 \cdot 7) \cdot 6!}{6! \cdot 2!} = \frac{8 \cdot 7}{2 \cdot 1} = 28$

Coefficient of third term
in expansion of $(x + y)^8$

b. ${}_{10}C_3 = \frac{10!}{7! \cdot 3!} = \frac{(10 \cdot 9 \cdot 8) \cdot 7!}{7! \cdot 3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$

Coefficient of fourth term
in expansion of $(x + y)^{10}$

c. ${}_7C_0 = \frac{7!}{7! \cdot 0!} = 1$

Coefficient of first term
in expansion of $(x + y)^7$

d. ${}_8C_8 = \frac{8!}{0! \cdot 8!} = 1$

Coefficient of ninth term
in expansion of $(x + y)^8$

✓ Checkpoint 1

Find each binomial coefficient.

a. ${}_5C_2$ b. ${}_6C_6$

When $r \neq 0$ and $r \neq n$, as in parts (a) and (b) of Example 1, there is a simple pattern for evaluating binomial coefficients.

$${}_8C_2 = \frac{\overbrace{8 \cdot 7}^{2 \text{ factors}}}{\underbrace{2 \cdot 1}_{2 \text{ factorial}}} \quad \text{and} \quad {}_{10}C_3 = \frac{\overbrace{10 \cdot 9 \cdot 8}^{3 \text{ factors}}}{\underbrace{3 \cdot 2 \cdot 1}_{3 \text{ factorial}}}$$

Example 2 Finding Binomial Coefficients

Find each binomial coefficient using the pattern shown above.

a. ${}_7C_3$ b. $\binom{7}{4}$ c. ${}_{12}C_1$ d. $\binom{12}{11}$

SOLUTION

a. ${}_7C_3 = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$

Coefficient of fourth term
in expansion of $(x + y)^7$

b. $\binom{7}{4} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} = 35$

Coefficient of fifth term
in expansion of $(x + y)^7$

c. ${}_{12}C_1 = \frac{12}{1} = 12$

Coefficient of second term
in expansion of $(x + y)^{12}$

d. $\binom{12}{11} = \frac{12!}{1! \cdot 11!} = \frac{(12) \cdot 11!}{1! \cdot 11!} = \frac{12}{1} = 12$

Coefficient of twelfth term
in expansion of $(x + y)^{12}$

✓ Checkpoint 2

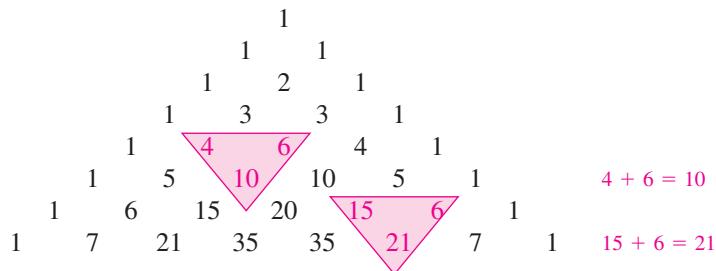
Find each binomial coefficient using the pattern shown before Example 2.

a. ${}_9C_2$ b. $\binom{6}{4}$

It is not a coincidence that the results in parts (a) and (b) of Example 2 are the same and that the results in parts (c) and (d) are the same. In general, ${}_nC_r = {}_nC_{n-r}$.

Pascal's Triangle

There is a convenient way to remember the pattern for binomial coefficients. By arranging the coefficients in a triangular pattern, you obtain the following array, which is called **Pascal's Triangle**. This triangle is named after the famous French mathematician Blaise Pascal (1623–1662).



The first and last numbers in each row of Pascal's Triangle are 1. Every other number in each row is formed by adding the two numbers immediately above the number. Pascal noticed that the numbers in this triangle are precisely the same numbers that are the coefficients of binomial expansions, as follows.

$$\begin{aligned}
 (x+y)^0 &= 1 && \text{0th row} \\
 (x+y)^1 &= 1x + 1y && \text{1st row} \\
 (x+y)^2 &= 1x^2 + 2xy + 1y^2 && \text{2nd row} \\
 (x+y)^3 &= 1x^3 + 3x^2y + 3xy^2 + 1y^3 && \text{3rd row} \\
 (x+y)^4 &= 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 && \vdots \\
 (x+y)^5 &= 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5 \\
 (x+y)^6 &= 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6 \\
 (x+y)^7 &= 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7
 \end{aligned}$$

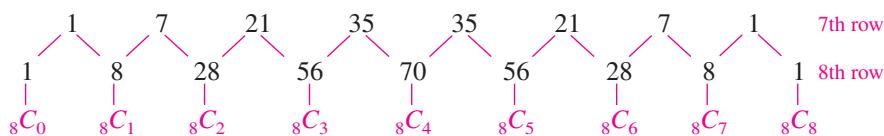
The top row in Pascal's Triangle is called the *zero row* because it corresponds to the binomial expansion $(x+y)^0 = 1$. Similarly, the next row is called the *first row* because it corresponds to the binomial expansion $(x+y)^1 = 1x + 1y$. In general, the *nth row* in Pascal's Triangle gives the coefficients of $(x+y)^n$.

Example 3 Using Pascal's Triangle

Use the seventh row of Pascal's Triangle to find the binomial coefficients.

$${}_8C_0 \quad {}_8C_1 \quad {}_8C_2 \quad {}_8C_3 \quad {}_8C_4 \quad {}_8C_5 \quad {}_8C_6 \quad {}_8C_7 \quad {}_8C_8$$

SOLUTION



Checkpoint 3

Use the eighth row of Pascal's Triangle to find the binomial coefficients.

$${}_9C_3 \quad {}_9C_5 \quad {}_9C_7$$

Binomial Expansions

As mentioned at the beginning of this section, when you write out the coefficients for a binomial that is raised to a power, you are **expanding a binomial**. Pascal's Triangle and the formulas for binomial coefficients give you a way to expand binomials, as demonstrated in the next two examples.

Example 4 Expanding a Binomial

Write the expansion of the expression

$$(x + 1)^3.$$

SOLUTION The binomial coefficients from the third row of Pascal's Triangle are

$$1, 3, 3, 1.$$

So, the expansion is

$$\begin{aligned}(x + 1)^3 &= (1)x^3 + (3)x^2(1) + (3)x(1^2) + (1)(1^3) \\ &= x^3 + 3x^2 + 3x + 1.\end{aligned}$$

✓ Checkpoint 4

Write the expansion of the expression

$$(x + 2)^5.$$

To expand binomials representing *differences*, rather than sums, you alternate signs. Here are two examples.

$$\begin{aligned}(x - 1)^3 &= [x + (-1)]^3 \\ &= x^3 - 3x^2 + 3x - 1 \\ (x - 1)^4 &= [x + (-1)]^4 \\ &= x^4 - 4x^3 + 6x^2 - 4x + 1\end{aligned}$$

Example 5 Expanding a Binomial

Write the expansion of each expression.

a. $(2x - 3)^4$ b. $(x - 2y)^4$

SOLUTION The binomial coefficients from the fourth row of Pascal's Triangle are

$$1, 4, 6, 4, 1.$$

The expansions are shown below.

$$\begin{aligned}\text{a. } (2x - 3)^4 &= (1)(2x)^4 - (4)(2x)^3(3) + (6)(2x)^2(3^2) - (4)(2x)(3^3) + (1)(3^4) \\ &= 16x^4 - 96x^3 + 216x^2 - 216x + 81 \\ \text{b. } (x - 2y)^4 &= (1)x^4 - (4)x^3(2y) + (6)x^2(2y)^2 - (4)x(2y)^3 + (1)(2y)^4 \\ &= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4\end{aligned}$$

✓ Checkpoint 5

Write the expansion of the expression

$$(3x + 1)^5.$$

Sometimes you will need to find a specific term in a binomial expansion. Instead of writing out the entire expansion, you can use the fact that, from the Binomial Theorem, the $(r + 1)$ th term is

$${}_nC_r x^{n-r} y^r.$$

Note that this formula is for the $(r + 1)$ th term. So, r is one less than the number of the term you need. For instance, to find the third term of the expression in Example 5(a), you could use this formula with $r = 2$ and $n = 4$ to obtain

$${}_4C_2(2x)^{4-2}(-3)^2 = 6(4x^2)(9) = 216x^2.$$

Example 6 Finding a Term or Coefficient in a Binomial Expansion

- Find the sixth term of $(a + 2b)^8$.
- Find the coefficient of the term a^6b^5 in the expansion of $(3a - 2b)^{11}$.

SOLUTION

- Because the formula ${}_nC_r x^{n-r} y^r$ is for the $(r + 1)$ th term, r is one less than the number of the term you need. So, to find the sixth term in this binomial expansion, use $r = 5$, $n = 8$, $x = a$, and $y = 2b$.

$$\begin{aligned} {}_nC_r x^{n-r} y^r &= {}_8C_5 a^{8-5}(2b)^5 \\ &= 56 \cdot a^3 \cdot (32b^5) \\ &= 1792a^3b^5 \end{aligned}$$

- Note that $(3a - 2b)^{11} = [3a + (-2b)]^{11}$. So, $n = 11$, $r = 5$, $x = 3a$, and $y = -2b$. Substitute these values to obtain

$$\begin{aligned} {}_nC_r x^{n-r} y^r &= {}_{11}C_5 (3a)^6(-2b)^5 \\ &= (462)(729a^6)(-32b^5) \\ &= -10,777,536a^6b^5. \end{aligned}$$

So, the coefficient is $-10,777,536$.

✓ Checkpoint 6

Find the specified term in the expansion of each binomial.

- $(2a + b)^{10}$, sixth term
- $(3a + 5b)^6$, fourth term

SUMMARIZE (Section 16.6)

- State the Binomial Theorem (page 1193). For examples of using the Binomial Theorem to find binomial coefficients, see Examples 1 and 2.
- Explain how Pascal's Triangle can be used to find binomial coefficients (page 1195). For an example of using Pascal's Triangle, see Example 3.
- Explain what is meant by expanding a binomial (page 1196). For examples of expanding a binomial, see Examples 4 and 5.
- Explain how to find a specific term in a binomial expansion (page 1197). For an example of finding a specific term in a binomial expansion, see Example 6.



SKILLS WARM UP 16.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3, 0.5, and 15.1.

In Exercises 1–6, perform the indicated operations and simplify.

1. $5x^2(x^3 + 3)$

2. $(x + 5)(x^2 - 3)$

3. $(x + 4)^2$

4. $(2x - 3)^2$

5. $x^2y(3xy^{-2})$

6. $(-2z)^5$

In Exercises 7–10, evaluate the expression.

7. $5!$

8. $\frac{8!}{5!}$

9. $\frac{10!}{7!}$

10. $\frac{6!}{3!3!}$

Exercises 16.6

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Finding Binomial Coefficients In Exercises 1–12, find the binomial coefficient. See Examples 1 and 2.

1. ${}_6C_2$

2. ${}_9C_4$

3. ${}_{10}C_4$

4. ${}_{10}C_6$

5. ${}_{12}C_0$

6. ${}_{12}C_{12}$

7. ${}_{20}C_{15}$

8. ${}_{20}C_5$

9. $\binom{7}{7}$

10. $\binom{20}{0}$

11. $\binom{100}{98}$

12. $\binom{100}{2}$

Using Pascal's Triangle In Exercises 13–20, evaluate using Pascal's Triangle. See Example 3.

13. ${}_4C_2$

14. ${}_4C_3$

15. ${}_7C_4$

16. ${}_6C_3$

17. $\binom{8}{5}$

18. $\binom{9}{8}$

19. $\binom{10}{8}$

20. $\binom{7}{5}$

Expanding a Binomial In Exercises 21–40, use the Binomial Theorem to expand the expression. Simplify your answer. See Examples 4 and 5.

21. $(x + 1)^4$

22. $(x + 1)^6$

23. $(x + 6)^4$

24. $(x + 5)^5$

25. $(x - 2)^4$

26. $(x - 2)^5$

27. $(x + y)^5$

28. $(x + y)^6$

29. $(3s + 2)^4$

30. $(2s + 3)^5$

31. $(3 - 2s)^6$

32. $(2 - 3s)^4$

33. $(x - y)^5$

34. $(2x - y)^5$

35. $(1 - 2x)^3$

36. $(1 - 3x)^4$

37. $(x^2 - y^2)^5$

38. $(x^2 + y^2)^6$

39. $\left(\frac{1}{x} + y\right)^5$

40. $\left(x - \frac{3}{y}\right)^3$

Expanding Complex Numbers In Exercises 41–46, use the Binomial Theorem to expand the complex number. Simplify your answer by using the fact that $i^2 = -1$.

41. $(1 + i)^4$

42. $(2 - i)^5$

43. $(3 - \sqrt{-4})^6$

44. $(5 + \sqrt{-16})^3$

45. $\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)^3$

46. $\left(\frac{3}{4} - \frac{\sqrt{2}}{4}i\right)^4$

Using Pascal's Triangle In Exercises 47–50, expand the expression by using Pascal's Triangle to determine the coefficients.

47. $(2t - 1)^5$

48. $(x + 4)^5$

49. $(3 + 2z)^4$

50. $(3y + 2)^6$

Finding a Term in a Binomial Expansion In Exercises 51–58, find the specified n th term in the expansion of the binomial. See Example 6.

Binomial	Term
51. $(x + y)^{10}$	4th term
52. $(x - y)^6$	7th term
53. $(x - 6y)^5$	3rd term
54. $(x - 10z)^7$	4th term
55. $(4x + 3y)^9$	8th term
56. $(5a + 6b)^5$	5th term
57. $(10x - 3y)^{12}$	10th term
58. $(7x + 2y)^{15}$	7th term

Finding a Coefficient in a Binomial Expansion In Exercises 59–66, find the coefficient a of the term in the expansion of the binomial. See Example 6.

Binomial	Term
59. $(x + 2)^{10}$	ax^5
60. $(x + 3)^{12}$	ax^8
61. $(x - 2y)^{10}$	ax^8y^2

Binomial	Term
62. $(4x - y)^{10}$	ax^3y^7
63. $(3x - 2y)^9$	ax^5y^4
64. $(2x - 3y)^{11}$	ax^3y^8
65. $(x^2 - 1)^8$	ax^6
66. $(x^2 + 2)^{12}$	ax^{10}

Expanding a Binomial In Exercises 67–70, use the Binomial Theorem to expand the expression.

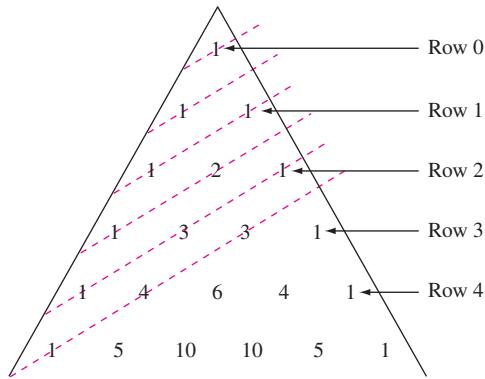
67. $\left(\frac{1}{2} + \frac{1}{2}\right)^7$

68. $\left(\frac{1}{4} + \frac{3}{4}\right)^{10}$

69. $(0.6 + 0.4)^5$

70. $(0.35 + 0.65)^6$

71. Finding a Pattern Describe the pattern formed by the sums of the numbers along the diagonal lines shown in Pascal's Triangle (see figure).

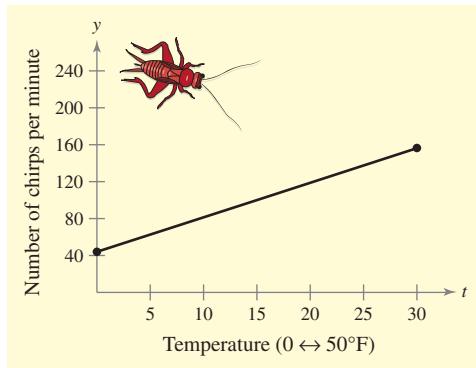


- 72. Crickets** You conduct an experiment and determine that the numbers of chirps $f(t)$ that a cricket makes each minute at various temperatures can be approximated by

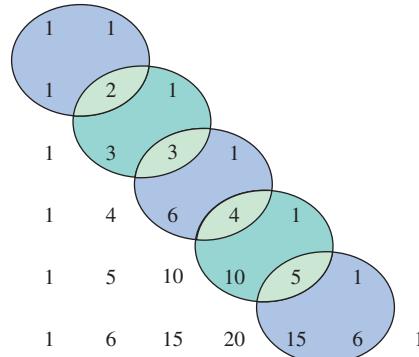
$$f(t) = 0.004t^2 + 3.73t + 44, \quad 0 \leq t \leq 30$$

where $t = 0$ represents 50°F (see figure).

- (a) You want to adjust this model so that $t = 0$ corresponds to 60°F rather than 50°F . To do this, you shift the graph of f 10 units to the left and obtain $g(t) = f(t + 10)$. Write $g(t)$ in standard form.
- (b) Use a graphing utility to graph f and g in the same viewing window.



- 73. Finding a Pattern** Use each of the encircled groups of numbers in the figure to form a 2×2 matrix. Find the determinant of each matrix. Describe the pattern.



- 74. HOW DO YOU SEE IT?** The expansions of $(x + y)^4$, $(x + y)^5$, and $(x + y)^6$ are shown below.

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

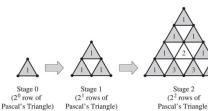
$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

- (a) Explain how the exponent of a binomial is related to the number of terms in its expansion.
 (b) How many terms are in the expansion of $(x + y)^n$?

- 75. Project: Fractals** To work an extended application involving fractals, visit this text's website at www.cengagebrain.com.

Section 16.6 Project: Fractals

Project: Fractals Consider Stages 0, 1, and 2 shown below of a pattern created using dots and the rows of Pascal's Triangle. The number of rows of Pascal's Triangle in each stage of the pattern is given by $r_n = 2^n$, where n is the stage number. The dots in each stage are connected to form equilateral triangles, and the triangles containing odd numbers are shaded.



- (a) How many rows of Pascal's Triangle are in Stage 3 of the pattern?
 (b) Create Stage 3 of the pattern.
 (c) The area of each equilateral triangle formed by connecting the dots is 1 square unit. Write a formula for s_n , the total area of the shaded triangles in Stage n .
 (d) Write a formula for t_n , the total area of the pattern in Stage n .
 (e) Write a formula for a_n , the ratio of the total area s_n of the shaded triangles in Stage n to the total area t_n of the pattern in Stage n .
 (f) What kind of sequence is a_n ? Does the sum of the sequence exist? Explain what this means in the context of the problem.
 (g) The pattern found above is similar to the Sierpinski Triangle, which is a fractal created using equilateral triangles. Use a library or the Internet to research the Sierpinski Triangle, and explain how it differs from the work above.

[Download Data](#)

ALGEBRA TUTOR**Using Counting Principles**

In discrete probability, one of the basic skills is being able to count the number of ways an event can happen. To do this, the strategies below can be helpful.

- 1. The Fundamental Counting Principle:** The number of ways that two or more events can occur is the product of the numbers of ways each event can occur by itself. These ways can be listed graphically using a tree diagram.
- 2. Permutations:** The number of permutations of n elements is $n!$.
- 3. Combinations:** The number of combinations of n elements taken r at a time is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

In the strategies above, note that $n!$ (read as “ n factorial”) is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$$

where n is a positive integer. As a special case, zero factorial is defined as

$$0! = 1.$$

TECH TUTOR

Most graphing utilities have a *factorial* key. Consult your user’s manual for specific keystrokes for your graphing utility.

Example 1 Counting the Ways an Event Can Happen

- a. How many ways can you form a five-letter password when no letter is used more than once?
- b. Your class is divided into five work groups containing three, four, four, three, and five people. How many ways can you poll one person from each group?
- c. In how many orders can seven runners finish a race when there are no ties?
- d. You have 12 phone calls to return. In how many orders can you return them?

SOLUTION

- a. For the first letter of the password, you have 26 choices. For the second letter, you have 25 choices. For the third letter, you have 24 choices, and so on.

$$\begin{aligned}\text{Number of ways} &= 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 && \text{Counting Principle} \\ &= 7,893,600 && \text{Multiply.}\end{aligned}$$

- b. When choosing one person from the first group, you have 3 choices. From the second group, you have 4 choices, and so on.

$$\begin{aligned}\text{Number of ways} &= 3 \cdot 4 \cdot 4 \cdot 3 \cdot 5 && \text{Counting Principle} \\ &= 720 && \text{Multiply.}\end{aligned}$$

- c. The solution is given by the number of permutations of the seven runners.

$$\begin{aligned}\text{Number of ways} &= 7! && \text{Permutation} \\ &= 5040 && \text{Use a calculator.}\end{aligned}$$

- d. The solution is given by the number of permutations of the 12 phone calls.

$$\begin{aligned}\text{Number of ways} &= 12! && \text{Permutation} \\ &= 479,001,600 && \text{Use a calculator.}\end{aligned}$$

Example 2 Counting the Ways an Event Can Happen

How many different ways can you choose a three-person group

- from a class of 20 people?
- from a class of 40 people?

SOLUTION

- The number of ways to choose a three-person group from a class of 20 is given by the number of combinations of 20 elements taken three at a time.

$$\begin{aligned}
 \text{Number of ways} &= {}_{20}C_3 && \text{Combination} \\
 &= \frac{20!}{17! \cdot 3!} && \text{Formula for combination} \\
 &= \frac{20 \cdot 19 \cdot 18 \cdot 17!}{17! \cdot 3!} \\
 &= \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} && \text{Divide out common factors.} \\
 &= 20 \cdot 19 \cdot 3 && \text{Divide out common factors.} \\
 &= 1140 && \text{Multiply.}
 \end{aligned}$$

- The number of ways to choose a three-person group from a class of 40 is given by ${}_{40}C_3$, which is 9880. You can verify this answer using a graphing utility, as shown in the figure.

TECH TUTOR

Most graphing utilities have a *combination* key. Consult your user's manual for specific keystrokes for your graphing utility.



40 nCr 3 9880

Example 3 Counting the Ways an Event Can Happen

To test for defective units, you are choosing a sample of 10 from a manufacturing production of 2000 units. How many different samples of 10 are possible?

SOLUTION The solution is given by the number of combinations of 2000 elements taken 10 at a time.

$$\begin{aligned}
 \text{Number of ways} &= {}_{2000}C_{10} && \text{Combination} \\
 &= \frac{2000!}{1990! \cdot 10!} && \text{Formula for combination} \\
 &\approx 2.76 \times 10^{26} && \text{Use a calculator.} \\
 &= 276,000,000,000,000,000,000,000,000,000
 \end{aligned}$$

From these examples, you can see that combinations and permutations can be very large numbers.

SUMMARY AND STUDY STRATEGIES

After studying this chapter, you should have acquired the following skills.

The exercise numbers are keyed to the Review Exercises that begin on page 1204.

Answers to odd-numbered Review Exercises are given in the back of the text.*

Section 16.1

Review Exercises

- Use the Fundamental Counting Principle to solve counting problems.

1–4

Fundamental Counting Principle: Event E_1 can occur in m_1 ways.

After event E_1 occurs, event E_2 can occur in m_2 ways. The number of ways that both events can occur is $m_1 \cdot m_2$.

- Use permutations to solve counting problems.

5–12

There are $n!$ different ways that n distinct elements can be ordered.

The number of permutations of n distinct elements taken r at a time is

$${}_n P_r = \frac{n!}{(n - r)!}$$

- Use combinations to solve counting problems.

13, 14

The number of combinations of n elements taken r at a time is ${}_n C_r = \frac{n!}{(n - r)!r!}$.

Section 16.2

- Determine the sample space of an experiment and find the probability of an event.

15–18, 29–34

$$P(E) = \frac{n(E)}{n(S)}$$

- Find the probability of mutually exclusive events.

19–26, 32–36

A and B mutually exclusive: $P(A \cup B) = P(A) + P(B)$

A and B not mutually exclusive: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

- Find the probability of independent events.

27, 28, 35, 36

A and B independent: $P(A \text{ and } B) = P(A) \cdot P(B)$

Section 16.3

- Form frequency distributions for discrete random variables.
- Find probability distributions for discrete random variables.
- Determine whether tables represent probability distributions.
- Use probability distributions to find probabilities.
- Find the expected values (or means), variances, and standard deviations of discrete random variables.

37, 38

39, 40

41, 42

43, 44

45–52

$$\mu = E(x) = x_1 P(x_1) + x_2 P(x_2) + x_3 P(x_3) + \dots + x_m P(x_m)$$

$$V(x) = (x_1 - \mu)^2 P(x_1) + \dots + (x_m - \mu)^2 P(x_m), \quad \sigma = \sqrt{V(x)}$$

- Verify probability density functions.

53–60

- Use probability density functions to find probabilities.

61–66

$$P(c \leq x \leq d) = \int_c^d f(x) dx$$

* A wide range of valuable study aids are available to help you master the material in this chapter.

The *Student Solutions Manual* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at www.cengagebrain.com offers algebra help and a *Graphing Technology Guide*, which contains step-by-step commands and instructions for a wide variety of graphing calculators.

Section 16.4**Review Exercises**

- Find the means, variances, and standard deviations of continuous probability density functions. 69–74

$$\mu = E(x) = \int_a^b xf(x) dx, \quad V(x) = \int_a^b (x - \mu)^2 f(x) dx, \quad \sigma = \sqrt{V(x)}$$

- Find the medians of probability density functions. 75–78

$$\int_a^m f(x) dx = 0.5$$

- Find the means, variances, and standard deviations of special probability density functions. 79–84
- Use continuous probability density functions to answer questions about real-life situations. 85–90

Section 16.5

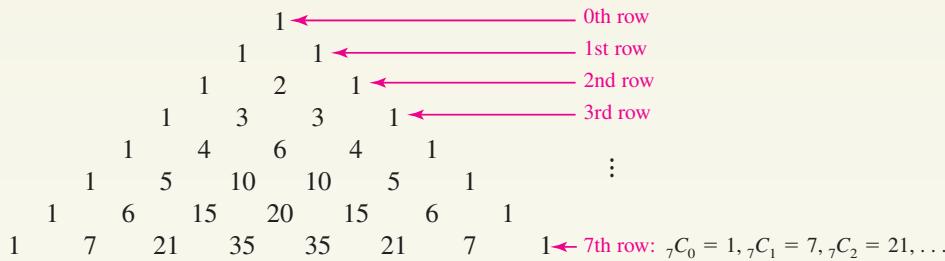
- Use mathematical induction to prove a formula. 91–94
- Find a sum of powers of integers. 95–98
- Find a formula for a finite sum. 99, 100
- Use finite differences to find a linear or quadratic model. 101–104

Section 16.6

- Find a binomial coefficient. 105–108

Binomial Theorem: In the expansion of $(x + y)^n$, the coefficient of $x^{n-r}y^r$, the $(r + 1)$ th term, is given by ${}_nC_r = \frac{n!}{(n - r)!r!}$.

- Use Pascal's Triangle to determine a binomial coefficient. 109, 110



- Expand a binomial using binomial coefficients and find a specific term in a binomial expansion. 111–120

Study Strategies

- **Studying Relationships Between Concepts** To reinforce your understanding of the concepts presented in this chapter, review how they are related. For instance, look back at the similarities and distinctions between concepts such as combinations and binomial coefficients, combinations and permutations, counting methods and probability, and so on.
- **Using Mathematical Induction** Proofs can be complicated. When using mathematical induction, write out a plan to identify how you will meet the two conditions necessary for the proof. If you get lost in the proof, review the plan to get yourself back on track. Revise the plan if you need to.
- **Using Technology** Integrals that arise with continuous probability density functions tend to be difficult to evaluate by hand. When evaluating such integrals, we suggest that you use a symbolic integration utility or that you use a numerical integration technique such as Simpson's Rule with a programmable calculator.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

- Computer Systems** A customer in a computer store can choose one of six monitors, one of five keyboards, and one of seven computers. All of the choices are compatible. How many different systems can be chosen?
- Chemistry Experiment** To set up an experiment, a chemist can use one of two beakers, one of two graduated cylinders, one of six hot plates, and one of three stirring rods. How many different equipment setups are possible for the experiment?
- Four-Digit Numbers** How many four-digit numbers can be formed under each condition?
 - The leading digit cannot be 0, 1, or 2.
 - The leading digit cannot be 0, 1, or 2, and no repetition of digits is allowed.
 - The leading digit cannot be 0 and the number must be divisible by 2.
- Five-Digit Numbers** How many five-digit numbers can be formed under each condition?
 - The leading digit cannot be 0.
 - The leading digit cannot be 0 and no repetition of digits is allowed.
 - The leading digit cannot be 0 and the number must be odd.
- Genetics** A geneticist is using gel electrophoresis to analyze DNA. The DNA is divided into five samples. Each sample is treated with a different restriction enzyme and then injected into one of five wells that have been formed in a bed of gel. In how many orders can the five samples be injected into the wells?
- Aircraft Boarding** Ten people are boarding an aircraft. Two have tickets for first class and board before the other eight people. In how many orders can the ten people board the aircraft?

Finding a Permutation In Exercises 7–10, evaluate ${}_nP_r$ using the formula on page 1139.

7. ${}_9P_3$ 8. ${}_{30}P_1$ 9. ${}_8P_8$ 10. ${}_{15}P_2$

Finding Distinguishable Permutations In Exercises 11 and 12, find the number of distinguishable permutations of the group of letters.

11. E, N, T, R, E, P, R, E, N, E, U, R
12. P, H, O, T, O, S, Y, N, T, H, E, S, I, S

13. **Zebrafish** A researcher has 24 zebrafish embryos and must choose a sample of them for study. How many different samples of each size are possible?
(a) 5 embryos (b) 10 embryos (c) 18 embryos

- Job Applicants** An employer interviews 10 people for five identical job openings in the marketing department of a company. All 10 applicants are qualified and seven of them are women. For each situation below, how many different groups of five people are possible?
 - The people are chosen at random.
 - Exactly three of those hired are women.
 - All five of those hired are women.

Finding the Sample Space In Exercises 15 and 16, find the sample space for the experiment.

15. Guessing the last digit in a telephone number
16. Tossing four coins

Finding the Probability of an Event In Exercises 17 and 18, a card is selected at random from a standard deck of 52 playing cards. Find the probability of the event.

17. Getting a 10, jack, or queen
18. Getting a black card that is not a face card

Probability of Mutually Exclusive Events In Exercises 19–22, use the table, which shows the weights of the players on a college football team.

Weight (in pounds)	Number of players
160–199	12
200–239	31
240–279	34
280–319	8

A player from the team is chosen randomly to compete in a tug of war for charity. Find the probability that the player's weight is in the given range.

19. 280 to 319 pounds
20. 240 to 319 pounds
21. Greater than 199 pounds
22. Less than 280 pounds

The Probability of a Union In Exercises 23 and 24, one card is selected at random from a deck of 52 playing cards. Use a formula to find the probability of the union of the two events.

23. The card is red or an ace.
24. The card is a face card or a heart.

Finding the Probability of a Complement In Exercises 25 and 26, you are given the probability p that an event *will* happen. Find the probability that the event *will not* happen.

25. $p = 0.57$

26. $p = 0.23$

Probability of Independent Events In Exercises 27 and 28, a random number generator selects three numbers from 1 through 20. Find the probability of the event.

27. All three numbers are odd.

28. Two numbers are even and the third number is 11.

29. **Education** In a high school graduating class of 72 students, 28 are on the honor roll. Of these, 18 are going on to college, and of the other 44 students, 12 are going on to college. A student is selected at random from the class. What is the probability that the person chosen is (a) going to college, (b) not going to college, and (c) on the honor roll, but not going to college?

30. **Game Show** On a game show, you are given four different digits to arrange in the proper order to represent the price of a trip. If you are correct, then you win the trip. Find the probability of winning under each condition.

(a) You must guess the position of each digit.

(b) You know the first digit, but must guess the remaining three.

31. **Poker Hand** Five cards are drawn randomly from a standard deck of 52 playing cards. What is the probability of getting two pair? (Two pair consists of two of one kind, two of another kind, and one of a third kind. For example, K-K-9-9-5 is two pair.)

32. **Preparing for a Test** An instructor gives a class a list of 12 study problems, from which eight will be selected to construct an exam. A student knows how to solve 10 of the problems. Find the probability that the student will be able to answer (a) all eight questions on the exam, (b) exactly seven questions on the exam, and (c) at least seven questions on the exam.

33. **Payroll Mix-Up** Four paychecks and envelopes are addressed to four different people. The paychecks get mixed up and are inserted randomly into the envelopes.

(a) What is the probability that exactly one will be inserted in the correct envelope?

(b) What is the probability that at least one will be inserted in the correct envelope?

34. **Defective Units** A shipment of 1000 radar detectors contains five defective units. A retail outlet has ordered 50 units.

(a) What is the probability that all 50 units are good?

(b) What is the probability that at least one unit is defective?

35. **A Boy or a Girl?** Assume that the probability of the birth of a child of a particular gender is 50%. In a family with five children, what is the probability that (a) all the children are boys, (b) all the children are of the same gender, and (c) there is at least one boy?

36. **A Boy or a Girl?** Assume that the probability of the birth of a child of a particular gender is 50%. In a family with seven children, what is the probability that (a) all the children are girls, (b) all the children are of the same gender, and (c) there is at least one girl?

Frequency Distributions In Exercises 37 and 38, complete the table to form the frequency distribution of the random variable x . Then construct a bar graph to represent the result.

37. **Bar Code** A computer randomly selects a three-digit bar code. Each digit can be 0 or 1, and x is the number of 1's in the bar code.

x	0	1	2	3
$n(x)$				

38. **Kittens** A cat has a litter of four kittens. Let x represent the number of male kittens.

x	0	1	2	3	4
$n(x)$					

39. **Bar Code** Use the frequency distribution in Exercise 37 to find the probability distribution for the random variable.

40. **Kittens** Use the frequency distribution in Exercise 38 to find the probability distribution for the random variable.

Identifying Probability Distributions In Exercises 41 and 42, determine whether the table represents a probability distribution. If it is a probability distribution, sketch its graph. If it is not a probability distribution, state any properties that are not satisfied.

x	0	1	2	3
$P(x)$	0.25	0.45	0.20	0.15

x	0	1	2	3	4	5
$P(x)$	$\frac{1}{25}$	$\frac{3}{25}$	$\frac{5}{25}$	$\frac{7}{25}$	$\frac{8}{25}$	$\frac{1}{25}$

Using Probability Distributions In Exercises 43 and 44, sketch a graph of the probability distribution and find the required probabilities.

43.

x	1	2	3	4	5
$P(x)$	$\frac{1}{18}$	$\frac{7}{18}$	$\frac{5}{18}$	$\frac{3}{18}$	$\frac{2}{18}$

(a) $P(2 \leq x \leq 4)$

(b) $P(x \geq 3)$

44.

x	-2	-1	1	3	5
$P(x)$	$\frac{1}{11}$	$\frac{2}{11}$	$\frac{4}{11}$	$\frac{3}{11}$	$\frac{1}{11}$

(a) $P(x < 0)$

(b) $P(x > 1)$

Finding Expected Value, Variance, and Standard Deviation In Exercises 45–48, find the expected value, variance, and standard deviation for the given probability distribution.

45.

x	0	1	2	3	4
$P(x)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{10}$

46.

x	1	2	3	4	5
$P(x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

47.

x	0	1	2	3
$P(x)$	0.006	0.240	0.614	0.140

48.

x	0	1	2
$P(x)$	0.310	0.685	0.005

49. **Revenue** A publishing company introduces a new weekly magazine that sells for \$3.95. The marketing group of the company estimates that sales x (in thousands) will be approximated by the following probability function.

x	10	15	20	30	40
$P(x)$	0.10	0.20	0.50	0.15	0.05

(a) Find $E(x)$ and σ .

(b) Find the expected revenue.

50. **Raffle** A service organization is selling \$5 raffle tickets as part of a fundraising program. The first and second prizes are \$3000 and \$1000, respectively. In addition to the first and second prizes, there are 50 \$20 gift certificates to be awarded. The number of tickets sold is 2000. Find the player's expected net gain when one ticket is purchased.

51. **Consumer Trends** The probability distribution of the random variable x , the number of cars per household in a city, is shown in the table. Find the expected value, variance, and standard deviation of x .

x	0	1	2	3	4	5
$P(x)$	0.10	0.28	0.39	0.17	0.04	0.02

52. **Vital Statistics** The probability distribution of the random variable x , the number of children per family in a city, is shown in the table. Find the expected value, variance, and standard deviation of x .

x	0	1	2	3	4
$P(x)$	0.12	0.31	0.43	0.12	0.02



Identifying Probability Density Functions In Exercises 53–60, use a graphing utility to graph the function. Then determine whether the function f represents a probability density function over the given interval. If f is not a probability density function, identify the condition(s) that is (are) not satisfied.

53. $f(x) = \frac{1}{12}, [0, 12]$ 54. $f(x) = \frac{1}{8}, [1, 8]$

55. $f(x) = \frac{1}{4}(3 - x), [0, 4]$

56. $f(x) = \frac{3}{4}x^2(2 - x), [0, 2]$

57. $f(x) = \frac{1}{4\sqrt{x}}, [1, 9]$

58. $f(x) = 8.75x^{3/2}(1 - x), [0, 2]$

59. $f(x) = \frac{1}{8}e^{-x/8}, [0, 8]$

60. $f(x) = \frac{1}{5}e^{-x/5}, [0, \infty)$

Finding a Probability In Exercises 61–66, sketch the graph of the probability density function over the indicated interval and find the indicated probabilities.

61. $f(x) = \frac{1}{16}, [0, 16]$

(a) $P(0 < x < 5)$

(b) $P(12 < x < 13)$

(c) $P(x \geq 5)$

(d) $P(8 < x < 12)$

62. $f(x) = \frac{x}{32}, [0, 8]$

(a) $P(0 < x < 3)$

(b) $P(7 < x < 8)$

(c) $P(4 < x < 6)$

(d) $P(x \geq 6)$

63. $f(x) = \frac{1}{50}(10 - x)$, $[0, 10]$

- (a) $P(0 < x < 2)$
- (b) $P(x \geq 7)$
- (c) $P(x \leq 5)$
- (d) $P(8 < x < 9)$

64. $f(x) = \frac{1}{36}(9 - x^2)$, $[-3, 3]$

- (a) $P(x < 2)$
- (b) $P(x > -2)$
- (c) $P(-1 < x < 2)$
- (d) $P(x > 2)$

65. $f(x) = \frac{2}{(x + 1)^2}$, $[0, 1]$

- (a) $P\left(0 < x < \frac{1}{2}\right)$
- (b) $P\left(\frac{1}{4} < x < \frac{3}{4}\right)$
- (c) $P\left(x \geq \frac{1}{2}\right)$
- (d) $P\left(\frac{1}{10} < x < \frac{3}{10}\right)$

66. $f(x) = \frac{3}{128}\sqrt{x}$, $[0, 16]$

- (a) $P(4 < x < 9)$
- (b) $P(4 < x < 16)$
- (c) $P(x < 9)$
- (d) $P(0 < x < 12)$

67. Waiting Time The waiting time t (in minutes) for patients arriving at a health clinic is described by the probability density function

$$f(t) = \frac{1}{12}e^{-t/12}, \quad [0, \infty).$$

Find the probabilities that a patient will wait

- (a) no more than 5 minutes.
- (b) between 9 and 12 minutes.

68. Medicine The time t (in days) until recovery after a certain medical procedure is described by the probability density function

$$f(t) = \frac{1}{4\sqrt{t-4}}, \quad [5, 13].$$

Find the probability that a patient will take (a) no more than 6 days to recover and (b) at least 8 days to recover.

Finding Mean, Variance, and Standard Deviation In Exercises 69–74, use the given probability density function over the indicated interval to find the (a) mean, (b) variance, and (c) standard deviation of the random variable. (d) Then sketch the graph of the density function and locate the mean on the graph.

69. $f(x) = \frac{1}{7}$, $[0, 7]$

70. $f(x) = \frac{x}{72}$, $[0, 12]$

71. $f(x) = \frac{3}{2x^2}$, $[1, 3]$

72. $f(x) = \frac{8-x}{32}$, $[0, 8]$

73. $f(x) = \frac{2}{9}x(3-x)$, $[0, 3]$

74. $f(x) = \frac{3}{16}\sqrt{x}$, $[0, 4]$

Finding the Median In Exercises 75–78, find the median of the probability density function.

75. $f(x) = \frac{1}{14}$, $[0, 14]$

76. $f(x) = \frac{3}{2} - x$, $[0, 1]$

77. $f(x) = 0.25e^{-x/4}$, $[0, \infty)$

78. $f(x) = \frac{5}{6}e^{-5x/6}$, $[0, \infty)$

Special Probability Density Functions In Exercises 79–84, identify the probability density function. Then find the mean, variance, and standard deviation without integrating.

79. $f(x) = \frac{1}{2}$, $[0, 2]$

80. $f(x) = \frac{1}{9}$, $[0, 9]$

81. $f(x) = \frac{1}{6}e^{-x/6}$, $[0, \infty)$

82. $f(x) = \frac{4}{5}e^{-4x/5}$, $[0, \infty)$

83. $f(x) = \frac{1}{3\sqrt{2\pi}}e^{-(x-16)^2/18}$, $(-\infty, \infty)$

84. $f(x) = \frac{1}{5\sqrt{2\pi}}e^{-(x-40)^2/50}$, $(-\infty, \infty)$

85. Waiting Time The waiting time t (in minutes) for service at the checkout at a grocery store is exponentially distributed with a mean of 15 minutes.

- (a) Find the probability density function for the random variable t .

- (b) Find the probability that the waiting time is less than 10 minutes.

86. Useful Life The time t (in hours) until failure of a mechanical unit is exponentially distributed with a mean of 350 hours.

- (a) Find the probability density function for the random variable t .

- (b) Find the probability that the mechanical unit will fail in more than 400 hours but less than 500 hours.

87. Transportation The arrival time t (in minutes) of a train at a train station is uniformly distributed between 7:00 A.M. and 7:20 A.M.

- (a) Find the probability density function for the random variable t .

- (b) Find the mean and standard deviation of the arrival time.

- (c) What is the probability that you will miss the train if you arrive at the train station at 7:04 A.M.?

- 88. Transportation** Repeat Exercise 87 for a train that arrives between 7:00 A.M. and 7:15 A.M.
- 89. Demand** The daily demand x for a certain product (in hundreds of pounds) is a random variable with the probability density function

$$f(x) = \frac{3}{256}x(8-x), [0, 8].$$

- (a) Find the mean and standard deviation of the demand.
 (b) Find the median of the demand.
 (c) Find the probability that the demand is within one standard deviation of the mean.

- 90. Heart Transplants** Assume the waiting times for heart transplants are normally distributed with a mean of 168 days and a standard deviation of 25 days.
(Source: Organ Procurement and Transplant Network)



- (a) Use a graphing utility to graph the distribution.
 (b) Use a symbolic integration utility to approximate the probability that a waiting time is between 70 and 105 days.
 (c) Use a symbolic integration utility to approximate the probability that a waiting time is more than 120 days.

Using Mathematical Induction In Exercises 91–94, use mathematical induction to prove the formula for every positive integer n .

91. $1 + 4 + \dots + (3n - 2) = \frac{n}{2}(3n - 1)$

92. $1 + \frac{3}{2} + 2 + \frac{5}{2} + \dots + \frac{1}{2}(n+1) = \frac{n}{4}(n+3)$

93. $\sum_{i=0}^{n-1} ar^i = \frac{a(1-r^n)}{1-r}$

94. $\sum_{k=0}^{n-1} (a + kd) = \frac{n}{2}[2a + (n-1)d]$

Finding a Sum of Powers of Integers In Exercises 95–98, find the sum using the formulas for the sums of powers of integers.

95. $\sum_{n=1}^{30} n$

96. $\sum_{n=1}^5 n^2$

97. $\sum_{n=1}^8 n^3$

98. $\sum_{n=1}^6 (n^2 + n)$

Finding a Formula for a Finite Sum In Exercises 99 and 100, find a formula for the sum of the first n terms of the sequence.

99. $1, \frac{3}{5}, \frac{9}{25}, \frac{27}{125}, \dots$

100. $5, -\frac{5}{2}, \frac{5}{4}, -\frac{5}{8}, \dots$

Finding an Appropriate Model In Exercises 101 and 102, decide whether the sequence can be represented perfectly by a linear or a quadratic model. If so, find the model.

101. $5, 10, 15, 25, 30, 35, \dots$

102. $1, 6, 13, 22, 33, 46, \dots$

Using Finite Differences In Exercises 103 and 104, write the first five terms of the sequence, where $a_1 = f(1)$. Then calculate the first and second differences of the sequence. Does the sequence have either a linear model or a quadratic model? If so, find the model.

103. $a_0 = 2; a_n = a_{n-1} + n$

104. $f(1) = 1; a_n = a_{n-1} + 2n$

Finding Binomial Coefficients In Exercises 105–108, find the binomial coefficient.

105. ${}_7C_4$

106. ${}_{12}C_8$

107. $\binom{20}{20}$

108. $\binom{20}{10}$

Using Pascal's Triangle In Exercises 109 and 110, evaluate using Pascal's Triangle.

109. ${}_{12}C_5$

110. $\binom{7}{5}$

Expanding a Binomial In Exercises 111–116, use the Binomial Theorem to expand the expression. Simplify your answer.

111. $(x+1)^6$

112. $(x-1)^7$

113. $(3x+y)^5$

114. $(2x-y)^6$

115. $(x-2y)^8$

116. $(x^2+4)^5$

Finding a Term in a Binomial Expansion In Exercises 117 and 118, find the specified n th term in the expansion of the binomial.

117. $(a-3b)^5, n=3$

118. $(2x+y)^{10}, n=8$

Finding a Coefficient in a Binomial Expansion In Exercises 119 and 120, find the coefficient a of the term in the expansion of the binomial.

Binomial *Term*

119. $(x+2)^{12}$

ax^7

120. $(2x-5y)^9$

ax^4y^5

TEST YOURSELFSee www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

1. A customer in an electronics store can choose one of eight digital cameras, one of four memory cards, one of two compact tripods, and one of ten cases. How many different camera packages are possible?
2. In how many ways can three toppings for a pizza be selected from a list of 12 toppings?
3. Evaluate (a) ${}_8C_6$ and (b) ${}_{10}P_{10}$.
4. On a game show, the digits 7, 8, and 9 must be arranged in the proper order to form the price of an appliance. If the digits are arranged correctly, the contestant wins the appliance. What is the probability of winning if the contestant knows that the price is at least \$800?
5. You roll three six-sided dice. What is the probability that you roll a five on each die?
6. A coin is tossed four times.
 - (a) Write the sample space and frequency distribution for the possible outcomes.
 - (b) What is the probability that at least two heads occur?

In Exercises 7 and 8, find $E(x)$, and σ for the given probability distribution. Then find the indicated probability.

7.

x	1	2	3	4
$P(x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

$$P(2 \leq x \leq 4)$$

8.

x	0	1	2	3	4
$P(x)$	0.2	0.2	0.2	0.2	0.2

$$P(x > 0)$$

In Exercises 9–11, find the mean, variance, and standard deviation of the probability density function. Then find the indicated probability.

9. $f(x) = \frac{1}{14}$, $[0, 14]$; $P(x \geq 7)$
10. $f(x) = 3x - \frac{3}{2}x^2$, $[0, 1]$; $P\left(\frac{1}{2} < x \leq 1\right)$
11. $f(x) = e^{-x}$, $[0, \infty)$; $P(x \leq 1)$



12. An *intelligence quotient* or IQ is a number that is meant to measure intelligence. The IQs of students in a school are normally distributed with a mean of 110 and a standard deviation of 10. Use a symbolic integration utility to find the probability that a student selected at random will have an IQ within one standard deviation of the mean.

13. Use mathematical induction to prove the following formula.

$$5 + 10 + 15 + \dots + 5n = \frac{5n(n + 1)}{2}$$

14. Find $\sum_{n=1}^{10} (n^3 + n)$ using the formulas for the sums of powers of integers.
15. Use the Binomial Theorem to expand $(2x - 3)^5$.
16. Determine the coefficient of the x^7 term in the expansion of $(x - 3)^{12}$.