Structural Induction

Let S be the set of ordered pairs of integers defined recursively as follows:

Basis step: $(0,0) \in S$

Recursive step: If $(a,b) \in S$ then $(a+2, b+1) \in S$, $(a+1, b+2) \in S$, $(a+3, b) \in S$ and $(a, b+3) \in S$.

a. List the elements of S produced by the first 2 applications of the recursive definition.

$$S_0 = \{(0,0)\}, S_1 = \{(2,1),(1,2),(3,0),(0,3)\}, S_2 = \{(4,2),(3,3),(5,1),(2,4),(1,5),(6,0),(0,6)\}.$$

The elements of S produced by the first 2 applications of the recursive definition are

$$S_0 \cup S_1 \cup S_2 = \{(0,0),(2,1),(1,2),(3,0),(0,3),(4,2),(3,3),(5,1),(2,4),(1,5),(6,0),(0,6)\}.$$

b. Use structural induction to prove that if $(\boldsymbol{a},\boldsymbol{b}) \in \boldsymbol{S}$ then $\boldsymbol{a}+\boldsymbol{b}$ is divisible by 3.

When we use structural induction to show that the elements of a recursively defined set S have a certain property, then we need to do the following procedure:

1. **Basis step**: show all the elements defined in the basis step have the desired property.
2. **Inductive step**: assume that an arbitrary element of the set S has the desired property . This is your inductive hypothesis. Using the inductive hypothesis, prove that, when you create more elements in S by using the recursive definition, these newly created elements of S have the same property .
3. **Conclusion:** state that by the principle of structural induction all the elements in S have the same property.

Basis step: $(0, 0) \in S$ and $0 + 0 = 0$ is divisible by 3 the definition of divisibility. $(3 \cdot 0 = 0)$.

Recursive Step: Assume $(a, b) \in S$ with the property that $a + b$ is divisible by $3,$ that is $a + b = 3k$ for some integer k. We need to prove that the following elements of S, created by using the recursive definition,

$(a + 2, b + 1)$, $(a + 1, b + 2)$, $(a + 3, b)$ and $(a, b + 3)$ have the same property. That is, the sum of the first and the second coordinate is divisible by 3.

Proof: Using the inductive hypothesis,

Case 1: $(a + 2) + b + 1 = (a + b) + 3 = 3 \cdot k + 3 = 3 \cdot (k + 1)$, where $k + 1$ is an integer.

Case 2: $(a + 1) + (b + 2) = (a + b) + 3 = 3 \cdot k + 3 = 3 \cdot (k + 1)$, where $k + 1$ is an integer.

Case 3: $(a + 3) + b = (a + b) + 3 = 3 \cdot k + 3 = 3 \cdot (k + 1)$, where $k + 1$ is an integer.

Case 4: $a + (b + 3) = (a + b) + 3 = 3 \cdot k + 3 = 3 \cdot (k + 1)$, where $k + 1$ is an integer.

Thus, in all four cases the sum of the first and the second coordinate is divisible by 3 by definition of divisibility.

By **structural induction** we have proved that if $(a, b) \in S$ then $a + b$ is divisible by 3.