PRACTICE PROBLEMS CHAPTER 6

1. (i) Find eigenvalue and bases for the associated eigenspace for the following matrices:

(a) $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ (b) $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ (c) $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ (d) $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$

(ii) For each of the matrix in (a)-(d) determine whether or not the matrix is diagonalizable. If it is, give an invertible matrix X and a diagonal matrix D such that $D = X^{-1} A X$.

2. Consider the matrix $A=\begin{bmatrix}1 & 2 & 1\\ 0 & 0 & 6\\ 0 & 0 & 3\end{bmatrix}$.

(a) Find the eigenvalues of the matrices A, A^2 and A^n.
(b) Find the associated eigenspaces (for the matrices A, A^2, and A^n).
(c) Find formulas for the entries of A^n.
(d) Find A^7

3. Let $A=\begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}$

(a) Find the eigenvalues of A.
(b) Find the eigenvalues of 5A.
(c) Find the eigenvalues of $A^2 + \alpha A + \beta I$ where I is the 2×2 identity matrix and α and β are scalars.

4. Determine the values of k for which the matrix $\mathbf{A}=\begin{bmatrix}7 & k\\ -9 & -5\end{bmatrix}$ has two distinct eigenvalues.

5. Let A be a 2×2 matrix with $\mathrm{tr}(A) = 5$ and $\det(A) = -14$. Find the eigenvalues of A.

6. Let $\mathbf{x}_1=\begin{bmatrix}1\\ -1\\ -2\end{bmatrix}$, and $\mathbf{x}_2=\begin{bmatrix}3\\ 2\\ -1\end{bmatrix}$ be eigenvectors of a matrix A with corresponding eigenvalues $\lambda_1 = -2$, and $\lambda_2 = 3$, respectively.

(a) Let $\mathbf{v}=\begin{bmatrix}19\\ 6\\ -13\end{bmatrix}$. Write $\mathbf{v}$ as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$.

(b) Find $A\mathbf{v}$.

7. In (a)-(b) determine whether the given matrix is diagonalizable. If it is, find a diagonalizing matrix X and a diagonal matrix D such that $D = X^{-1}$ A X. If not, explain why.

(a) $A=\begin{bmatrix}3 & 1 & 0\\ 0 & 3 & 1\\ 0 & 0 & 3\end{bmatrix}$ (b) $A=\begin{bmatrix}4 & -3 & 0\\ 0 & 1 & 0\\ -4 & 4 & 1\end{bmatrix}$

8. Given $A=\begin{bmatrix}4 & -3 & 0\\ 0 & 1 & 0\\ -4 & 4 & 1\end{bmatrix}$, find a matrix B such that $B^2 = A$.

9. Let A be a 4×4 matrix with real entries that has all 1's on the main diagonal (i.e. $a_{11} = a_{22} = a_{33} = a_{44} = 1$). If A is singular and $\lambda_1 = 3 + 2i$ is an eigenvalue of A, then what, if anything, is it possible to conclude about the values of the remaining eigenvalues λ_2, λ_3, and λ_4 ?

10. Let A be a nonsingular $n \times n$ matrix and let λ be an eigenvalue of A.
(a) Show that $\lambda \neq 0$.
(b) Show that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

11. Show that if A is a matrix of the form $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$ then A must be defective.

12. Let A be a diagonalizable $n \times n$ matrix. Prove that if B is any matrix that is similar to A, then B is diagonalizable.

13. Let A be a matrix whose singular value decomposition is given by

$$\begin{bmatrix} \frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & \frac{3}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{2}{5} & \frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} \\ \frac{3}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Make use of the singular value decomposition to do each of the following:
(a) Determine the rank of A.
(b) Find an orthonormal basis for $R(A^T)$.
(c) Find an orthonormal basis for $R(A)$.
(d) Find the rank 1 matrix B that is the closest matrix of rank 1 to A (distance between matrices is measured using the Frobenius norm).
(e) Let B be the matrix asked for in part (d). Use the singular values of A to determine the distance between A and B (i.e. use the singular values of A to determine the value of $\|B - A\|_F$).
(f) Find the rank 2 matrix C that is the closest matrix of rank 2 to A.
(g) Use the singular values of A to determine the distance between A and C (i.e. use the singular values of A to determine the value of $\|B - C\|_F$).

14. A singular value decomposition of a matrix A is as follows:

$$\begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 20 & 0 \\ 0 & 15 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$$

Use the SVD to find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} -1 \\ -1 \\ 5 \\ 2 \end{bmatrix}$.

15. Find the SVD decomposition of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$