Structural Induction

Let S be the set of order pairs of integers defined recursively as follows:

Basis step: $(0, 3) \in S$

Recursive step: If $(x, y) \in S$ then $(x + 2, y - 1) \in S$ $(x - 3, y) \in S$.

a. List the elements of S produced by the first 2 applications of the recursive definition.

$$S_0 = \{(0,3)\},\ S_1 = \{(2,2),(-3,3)\},\ S_2 = \{(4,1),(-1,2),(-6,3)\}.$$

The elements of S produced by the first 2 applications of the recursive definition is

$$S_0 \cup S_1 \cup S_2 = \{(0,3),(2,2),(-3,3),(4,1),(-1,2),(-6,3)\}.$$

b. Use structural induction to prove that if $(\boldsymbol{x},\boldsymbol{y}) \in \boldsymbol{S}$ then $\boldsymbol{x} \equiv \boldsymbol{y}$ mod 3.

When we use structural induction to show that the elements of a recursively defined set S have a certain property, then we need to do the following procedure:

1. **Basis step**: show all the elements defined in the basis step have the desired property.
2. **Inductive step**: assume that an arbitrary element of the set S has the desired property . This is your inductive hypothesis. Using the inductive hypothesis, prove that, when you create more elements in S by using the recursive definition, these newly created elements of S have the same property.
3. **Conclusion:** state that by the principle of structural induction all the elements in S have the same property.

Recall: $\boldsymbol{a}$ **is divisible by** $\boldsymbol{m}$ **if there exists and integer** $\boldsymbol{k}$ **such** that $\boldsymbol{a = k \cdot m}$

Recall: $\boldsymbol{x \equiv y}$ **mod** $\boldsymbol{m}$ **if and only if** $\boldsymbol{x - y}$ **is divisible by** $\boldsymbol{m}$.

Basis step: $(0{,}3) \in S$ and $0 \equiv 3$ mod 3 since $0 - 3 = -3$ is divisible by 3, since $-3 = 3 \cdot (-1)$.

Recursive Step: Assume $(x, y) \in S$ with the property that $x \equiv y$ mod 3, that is $x - y = 3k$ for some integer k. We need to prove that the following elements of S, created by using the recursive definition, $(x + 2, y - 1)$ and $(x - 3, y)$ have the same property.

That is, $x + 2 \equiv y - 1 \text{ mod } 3$ and $x - 3 \equiv y \text{ mod } 3$.

Proof: Using the inductive hypothesis,

Case 1: $(x + 2) - (y - 1) = (x - y) + 3 = 3{\cdot}k + 3 = 3{\cdot}(k + 1)$, where $k + 1$ is an integer.

Thus, $x + 2 \equiv y - 1 \text{ mod } 3$.

Case 2: $(x - 3) - y = (x - y) - 3 = 3{\cdot}k - 3 = 3{\cdot}(k - 1)$, where $k - 1$ is an integer.

Thus, $x - 3 \equiv y \text{ mod } 3$

By **structural induction** we have proved that if $(x, y) \in S$ then $x \equiv y$ mod 3.