Propositional Equivalence

Tautologies and contradictions

A compound proposition that is always true, regardless of the truth values of the individual propositions involved, is called a **tautology**.

Example: $p \vee \neg p$ is a tautology.

A compound proposition that is always false, regardless of the truth values of the individual propositions involved, is called a **contradiction**.

Example: $p \wedge \neg p$ is a contradiction.

A compound statement that is neither a tautology nor a contradiction is called a **contingency**.

Logical equivalence

We call two algebraic expressions equal if they have the same value for each possible value of the input variables. For example, we say

$$x^2 - 1 = (x + 1)(x - 1)$$

because for all real numbers x, the left side and the right side have the same value.

Correspondingly, we should call two compound statements p and q "equal" if they always share the same truth value. However, the convention is to call them **logically equivalent** instead, and to use the symbol $\equiv$ to represent logical equivalence.

Using the biconditional and the concept of a **tautology** that we just introduced, we can formally define logical equivalence as follows:

$p \equiv q$ means that $p \leftrightarrow q$ is a tautology.

An alternative definition is that $p \equiv q$ means that p and q share the same truth table.

An important logical equivalence

$$p \to q \equiv \neg p \vee q$$

When we introduced the conditional, we discussed the idea that the conditional should not only be true when the conclusion is true, but also when the premise is false. This explains the equivalence above. A more formal justification can be given using a truth table:

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The Laws of De Morgan

The two laws of De Morgan explain how negation interacts with conjunction and disjunction. We will illustrate the principle with an example first:

It is not true that Joe is rich and famous.

This does not mean that Joe is neither rich nor famous. An *and* statement is true if both individual statements are true; so if the *and* statement is false, it means that at least one of the two statements is false. Therefore, the statement above is logically equivalent to

Joe is not rich or Joe is not famous (perhaps neither).

We conclude that the **negation of a conjunction must be the disjunction of the negations**. There is also a symmetric law of De Morgan which is obtained by switching the roles of conjunction and disjunction. Both laws can be formally proved by truth table.

$$\neg(p \land q) \equiv \neg p \lor \neg q$$
$$\neg(p \lor q) \equiv \neg p \land \neg q$$

De Morgan and Double Inequalities

The laws of de Morgan explain how to properly negate a double inequality. In the previous presentation, we stated that the correct negation of $1 \leq x \leq 2$ is

$$x < 1 \vee x > 2$$

To understand this, consider that the original double inequality is a conjunction of two inequalities:

$$1 \leq x \leq 2 \equiv 1 \leq x \wedge x \leq 2.$$

Thereby, by De Morgan, the negation is the disjunction of the individual inequalities, which leads to $x < 1 \vee x > 2$.

Other Logical Equivalences involving conjunction, disjunction and negation

Equivalences	Name of the Equivalences
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$p \vee (q \vee r) \equiv (p \vee q) \vee r$ $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	Associate laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive Laws
$p \vee F \equiv p$ $p \wedge T \equiv p$	Identity Laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination Laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent Laws
$\neg(\neg p) \equiv p$	Double Negation Law
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption Laws

Proving Logical Equivalences

All the logical equivalences on the previous slide can be formally justified by truth table. This is a tedious exercise and provides little true insight. We shall give an example of a perhaps more meaningful explanation for one of the absorption laws:

$$p \wedge (p \vee q) \equiv p$$

The $\vee\ q$ inside the parentheses has no effect if p is true already. It can only upgrade a false p to true, but then the conjunction with p reverts that back to false. Therefore, the left side always has the same truth value as p.

Simplifying compound statements using logical equivalences

Just like we can use the algebraic laws of numbers to simplify algebraic expressions, we can use the logical equivalences just discussed to rewrite compound statements as equivalent (and hopefully, simpler) compound statements. This helps us understand compound statements, and to identify tautologies and contradictions. The next several slides present examples of such simplifications.

Example 1

Show $(p \to q) \land (p \to r) \equiv p \to q \land r$

We will rewrite the left side using logical equivalences until we have transformed it into the right side.

$$(p \to q) \land (p \to r)$$

$$\equiv (\neg p \lor q) \land (\neg p \lor r) \quad \text{(conditional as disjunction)}$$

$$\equiv \neg p \lor (q \land r) \quad \text{(by the distributive law)}$$

$$\equiv p \to (q \land r) \quad \text{(conditional as disjunction)}$$

Example 2

Show $(p \to q) \to p \equiv p$

Again we rewrite the left side using logical equivalences until we get the right side.

$$
\begin{aligned}
&(p \to q) \to p \\
&\equiv \neg(\neg p \lor q) \lor p && \text{(conditional as disjunction)} \\
&\equiv (\mathrm{p} \land \neg q) \lor p && \text{(De Morgan, double negation)} \\
&\equiv p && \text{(by the absorption law)}
\end{aligned}
$$

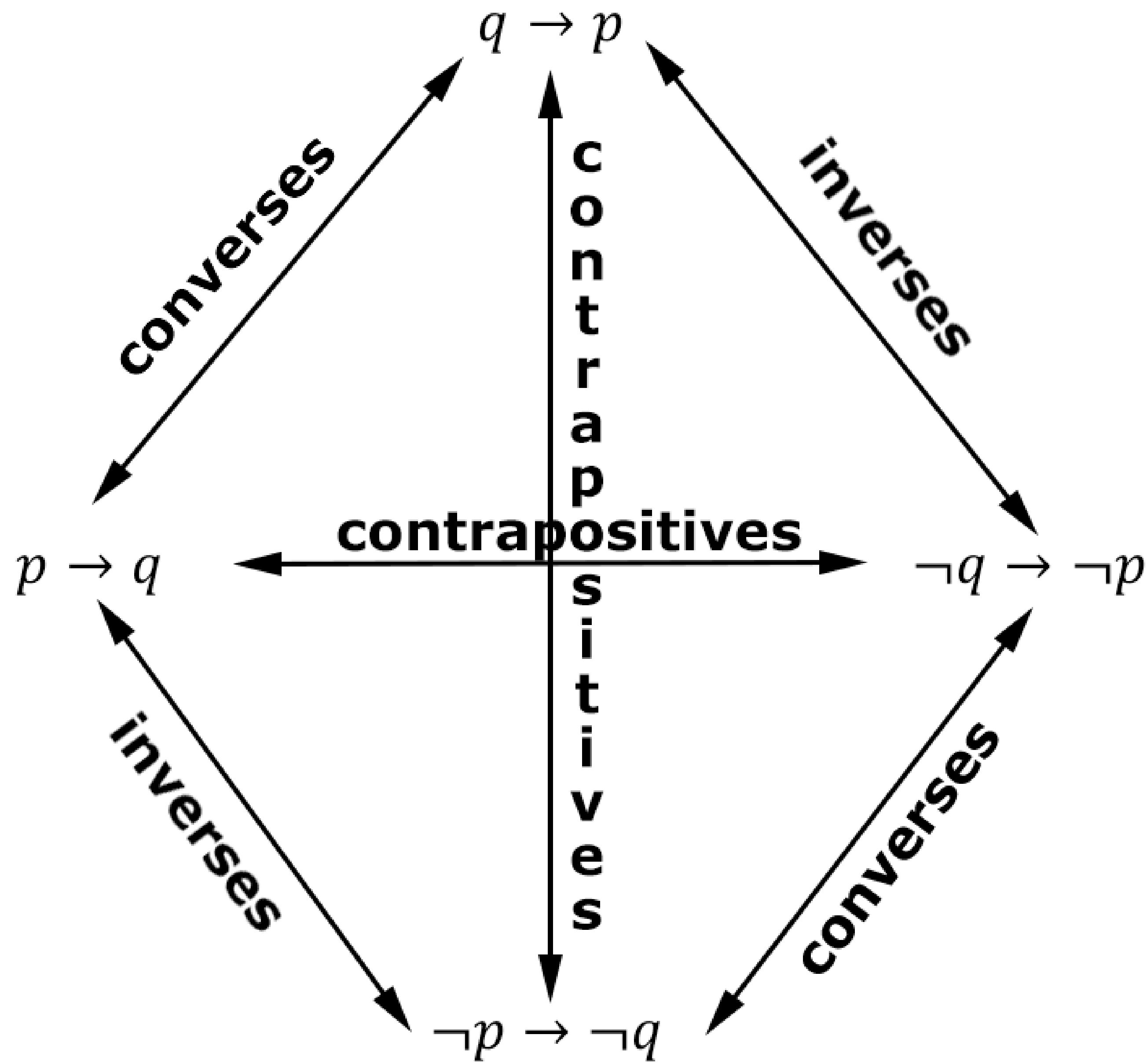
Logical Equivalence of Conditionals

It is an important fact that a conditional is logically equivalent to its contrapositive, but **not** to its inverse or converse. We can prove this by truth table or by using the logical equivalences we just studied.

$$p \to q \equiv \neg q \to \neg p$$

Since the inverse is the contrapositive of the converse, inverse and converse are logically equivalent to *each other*:

$$q \to p \equiv \neg p \to \neg q$$



$q \to p$
$p \to q$
$\neg q \to \neg p$
$\neg p \to \neg q$
converses
inverses
contrapositives
contrapositives
inverses
converses

Equivalences Involving The Biconditional

We defined the biconditional $\mathrm{p} \leftrightarrow q$ as

$$(p \rightarrow q) \wedge (q \rightarrow p)$$

We shall explore a consequence of that definition. Rewriting the conditionals as disjunctions and then using the distributive law, we get

$$(p \rightarrow q) \wedge (q \rightarrow p) \equiv (\neg p \vee q) \wedge (\neg q \vee p)$$
$$\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge p) \vee (q \wedge \neg q) \vee (q \wedge p)$$
$$\equiv (\neg p \wedge \neg q) \vee (q \wedge p)$$

This confirms what we already learned from the truth table: that $\mathrm{p} \leftrightarrow q$ is true if either both p and q are true or both p and q are false.