Name: _____

Math 243, Spring 2006, Professor Callahan
Test #1, Thu–Fri, Feb. 23–24.

Note 1: This test is closed book. You may use one $8\frac{1}{2} \times 11$ sheet of notes (both sides).

Note 2: Show your work. Clarity counts. If I can't follow your reasoning I can't give credit.

Problem 1: Given that $F(x, y) =$"x is a friend of y" and $L(x, y) =$"x loves y," express the following using quantifiers:

(a): There is someone who is a friend of everyone.

Answer: $\exists x \forall y F(x, y)$.

(b): Everyone loves someone who is not their friend.

Answer: $\forall x \exists y (L(x, y) \wedge \neg F(y, x))$.

(c): There are two people, each of whom loves all of the other's friends.

Answer: $\exists x \exists y (x \neq y) \wedge (\forall z (F(z, x) \rightarrow L(y, z))) \wedge (\forall z (F(z, y) \rightarrow L(x, z)))$.

Problem 2: Mary only waits for the bus if her car is broken. If she waits for the bus and it's raining then she gets wet, and vice versa. If she doesn't get wet then she doesn't get a cold. Mary gets a cold. Prove that her car is broken.

Answer: Let C = "Mary's **c**ar is broken," B = "Mary waits for the **b**us," R = "It's **r**aining," W = "Mary gets **w**et," and D = "Mary gets a col**d**." Then we have

$$B \to C, \qquad (B \wedge R) \leftrightarrow W, \qquad \neg W \to \neg D, \qquad D.$$

We know that D is true. Because $\neg W \to \neg D$, we know that W is true. Because $(B \wedge R) \leftrightarrow W$, we know that B and R are true. Because $B \to C$, we know that C is true.

Problem 3: Prove that if x^3 is irrational then x is irrational.

Answer: We prove this indirectly. Suppose that x is rational, so that $x = a/b$ with a and b integers and $b \neq 0$. Then $x^3 = a^3/b^3$, where a^3 and b^3 are integers and $b \neq 0$. Thus x^3 is rational. This completes the proof.

Problem 4: Find a *simple* function $g(x)$ such that $f(x)$ is $O(g(x))$, where

(a): $f(x) = n\log(n^2+1) + n^2\log n$

Answer: f is $O(n^2 \log n)$.

(b): $f(x) = (n\log n + n)^2 + (\log n + 1)(n^3+1)$

Answer: f is $O(n^3 \log n)$.