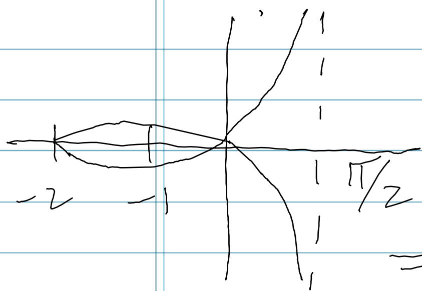
Example: The curve $x = 2 - 4\cos^2 t$ & $y = \tan t(1 - 2\cos^2 t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$, crosses at some point. Find the equations of the tangent lines at that point.

Solution:



Assume the curve crosses at $t = t_1$ and $t = t_2$ $(t_1 \neq t_2)$

$$2 - 4\cos^2 t_1 = 2 - 4\cos^2 t_2$$

$$\Rightarrow 1 - 2\cos^2 t_1 = 1 - 2\cos^2 t_2$$

$$\tan t_1 (1 - 2\cos^2 t_1) = \tan t_2 (1 - 2\cos^2 t_2)$$

$$= \tan t_2 (1 - 2\cos^2 t_1)$$

$$(\tan t_1 - \tan t_2)(1 - 2\cos^2 t_1) = 0$$

$-\frac{\pi}{2} < t < \frac{\pi}{2}$, $\tan t_1 \neq \tan t_2$.

$$1 - 2\cos^2 t_1 = 0$$

$$\Rightarrow \cos^2 t_1 = \frac{1}{2} \text{ on } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$t_1 = -\frac{\pi}{4} \text{ and } t_2 = \frac{\pi}{4}$$

$$\frac{dy}{dx} = \frac{\sec^2 t(1 - 2\cos^2 t) + 4\tan t \cos t \sin t}{8\cos t \sin t}$$

$t = \pi/4$

$$\frac{dy}{dx} = \frac{\sec^2\left(\frac{\pi}{4}\right)\left(1 - 2\cos^2\left(\frac{\pi}{4}\right)\right) + 4\tan\frac{\pi}{4}\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)}{8\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)}$$

$$= \frac{1}{2}$$

at $t = -\frac{\pi}{4}$,

$$\frac{dy}{dx} = -\frac{1}{2}$$

Also observe at $t = \frac{\pi}{4}$

$$(x, y) = (0, 0)$$

$$\left.\begin{array}{l} y = \frac{1}{2}x \\ \text{and } y = -\frac{1}{2}x \end{array}\right\} \text{Equations of tangents.}$$