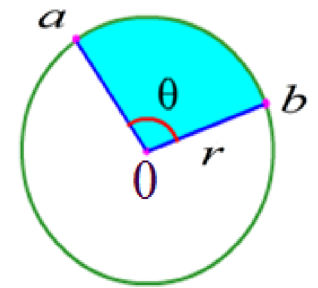
Areas and Lengths in Polar Coordinates

Areas and lengths in polar coordinates - Area

Consider two points *a* and *b* on a circle of radius *r* with center at O:



The area of the sector O*ab* is $\frac{1}{2}\theta r^2$ where is θ the central angle.

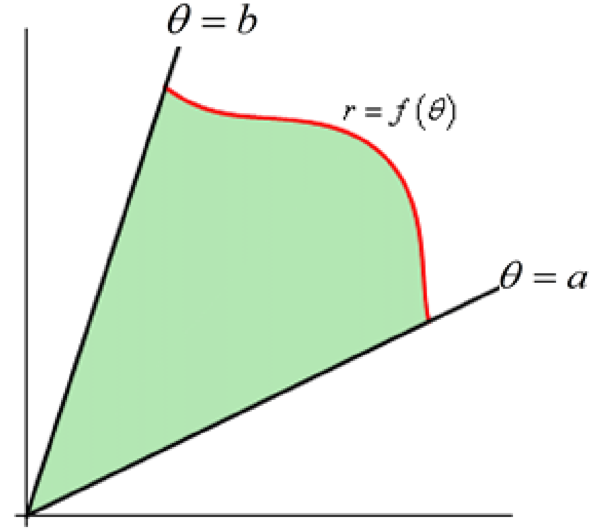
Now suppose $r = f(\theta)$ is a positive continuous function which is defined for $a \le \theta \le b$ with $0 \le b - a \le 2\pi$.

Goal: Determine the **<u>area</u>** bounded by the graphs of

$$r = f(\theta)$$

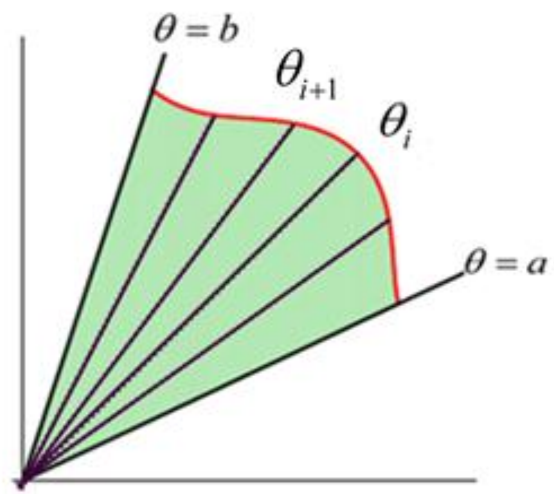
$$\theta = a$$

$$\theta = b$$



Areas and lengths in polar coordinates - Area

Partition the interval [*a*, *b*] into *n* subintervals of equal width $\Delta\theta$ with endpoints $\theta_0, \theta_1, \cdots, \theta_n$
The rays $\theta = \theta_i$ divide the region into *n* smaller regions with central angle $\Delta\theta$



In each subinterval $[\theta_i, \theta_{i+1}]$ pick a point θ_i^* and draw sectors of circles with center at O, radius $f\left(\theta_i^*\right)$ and central angle $\Delta\theta$

The area of each of these sectors is $\frac{1}{2} f\left(\theta_i^*\right)^2 \Delta\theta$

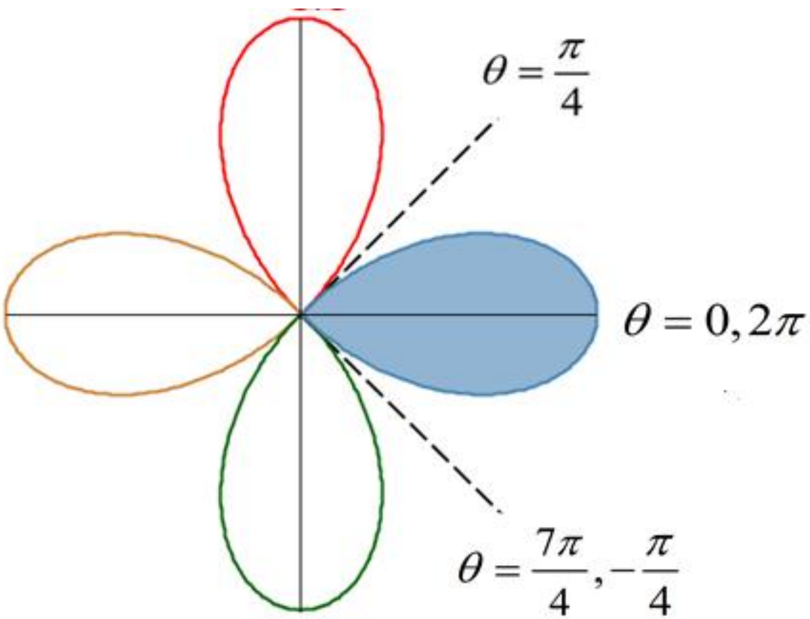
The area of the region: $$A = \lim_{n\to\infty} \frac{1}{2}\sum_{i=1}^{n} f\left(\theta_i^*\right)^2 \Delta\theta = \frac{1}{2}\int_a^b f(\theta)^2\, d\theta$$

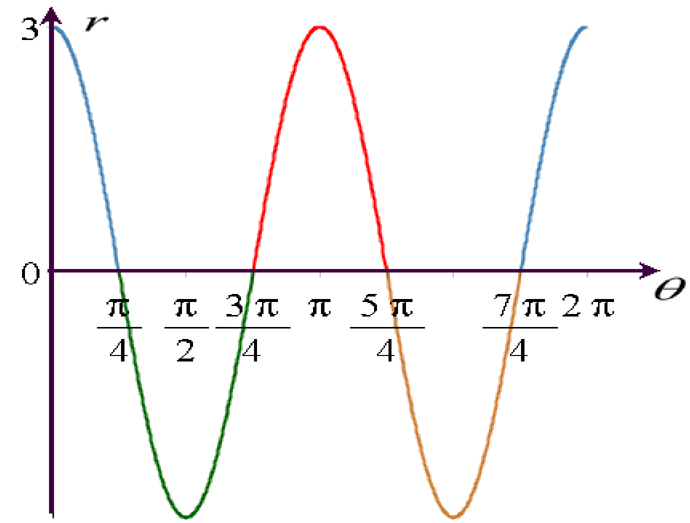
Area bounded by the graphs of $r = f(\theta)$, $\theta = a$ and $\theta = b$ with $a \le \theta \le b$:

$$\boxed{A = \frac{1}{2}\int_a^b r^2\, d\theta}$$

Areas and Lengths in Polar coordinates – Areas Example 1

Find the area enclosed by one loop of the four-leaved rose: $r = 3\cos(2\theta)$





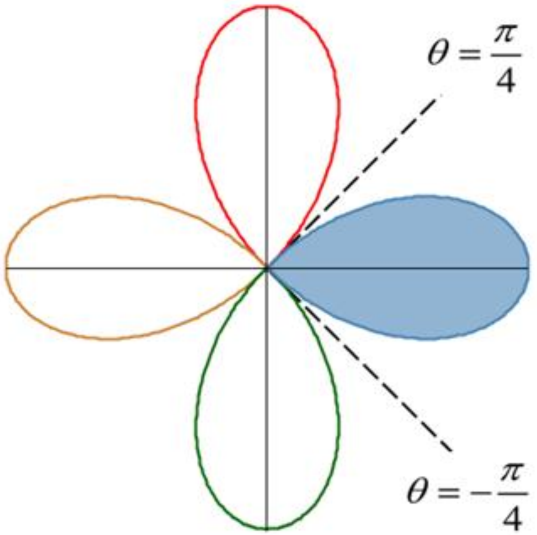
Limits of Integration

$$\cos(2\theta) = 0 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

The blue leaf is swept out as θ rotates from 0 to $\frac{\pi}{4}$ and from $\frac{7\pi}{4}$ to 2π or, equivalently, from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$.

In the formula for the area, the lower limit of integration MUST be smaller than the upper limit. Choosing the limits as $\frac{7\pi}{4}$ and $\frac{\pi}{4}$ is WRONG!

Areas and Lengths in polar coordinates – Areas Example 1 cont.



$$A = \frac{1}{2}\int_a^b r^2\,d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4}\left(3\cos(2\theta)\right)^2 d\theta$$

$$= \frac{9}{2}\int_{-\pi/4}^{\pi/4}\frac{1}{2}\left(1+\cos(4\theta)\right)d\theta$$

$$= \frac{9}{4}\left(\theta + \frac{1}{4}\sin(4\theta)\right)\Bigg|_{-\pi/4}^{\pi/4}$$

$$= \frac{9\pi}{8}$$

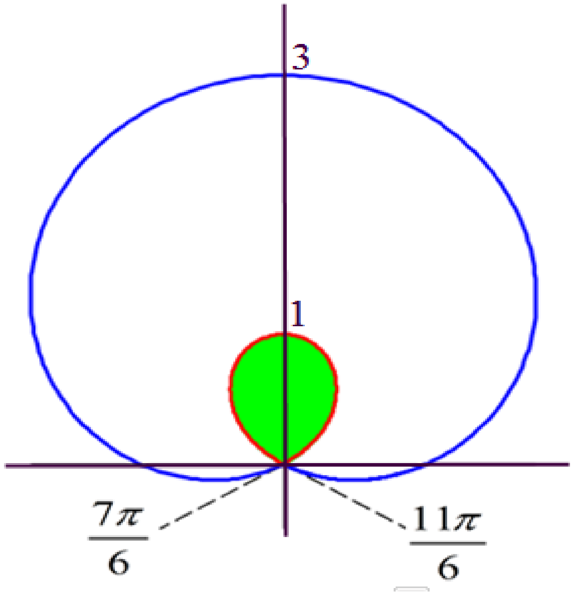
NOTES:

- Always verify that the computed area is positive.
- Using symmetry with respect to the *x* axis:

$$A = 2\cdot\frac{1}{2}\int_0^{\pi/4}\left(3\cos(2\theta)\right)^2 d\theta$$

Areas and Lengths in Polar coordinates - Areas Example 2

Find the area of the inner loop of $r = 1 + 2\sin(\theta)$



Limits of Integration:

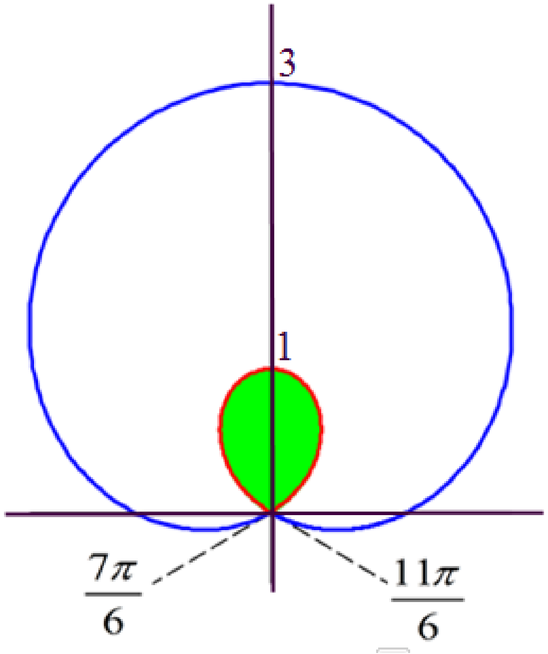
Set r=0:

$$1 + 2\sin(\theta) = 0$$

$$\sin(\theta) = -\frac{1}{2}$$

$$\theta = \frac{7\pi}{6}, \ \theta = \frac{11\pi}{6}$$

Areas and Lengths in Polar coordinates – Area Example 2 cont.



$$A = \frac{1}{2}\int_a^b r^2\,d\theta = \frac{1}{2}\int_{7\pi/6}^{11\pi/6}\left(1+2\sin(\theta)\right)^2 d\theta$$

$$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6}\left(1+4\sin(\theta)+4\sin^2(\theta)\right)d\theta$$

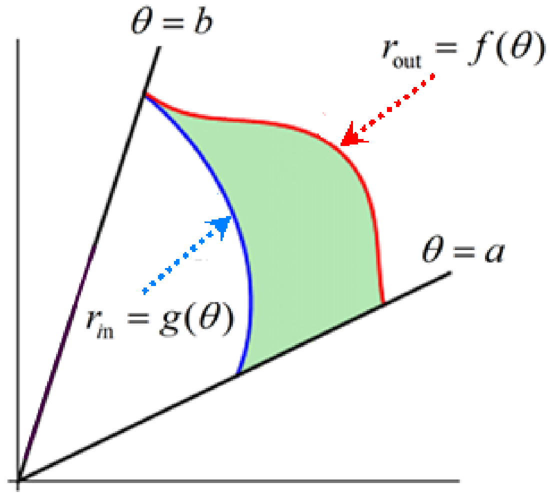
$$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6}\left(1+4\sin(\theta)+2-2\cos(2\theta)\right)d\theta$$

$$= \frac{1}{2}\left(3\theta-4\cos(\theta)-\sin(2\theta)\right)\Big|_{7\pi/6}^{11\pi/6}$$

$$= \pi - \frac{3\sqrt{3}}{2} \approx 0.54$$

Areas and Lengths in polar coordinates

Consider the region bounded by the polar curves $r = f(\theta),\ r = g(\theta),\ \theta = a,$ and $\theta = b.$



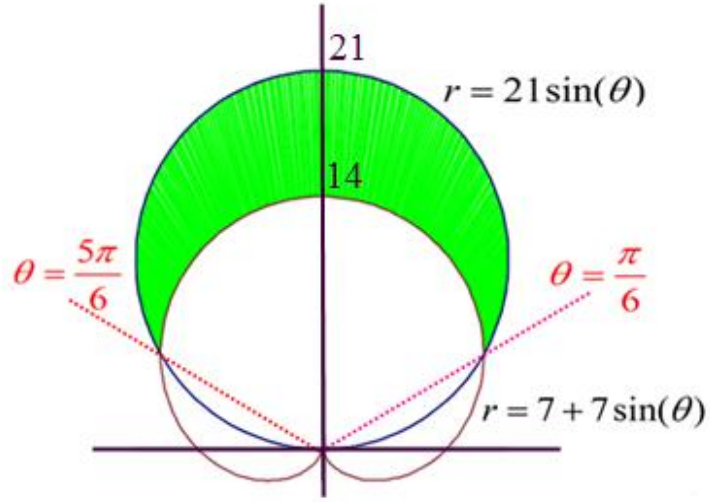
Assume $f(\theta) \geq g(\theta) \geq 0$ and $0 \leq b - a \leq 2\pi$

The area of the region is the area: $A = \dfrac{1}{2}\int_a^b f(\theta)^2\, d\theta - \dfrac{1}{2}\int_a^b g(\theta)^2\, d\theta$

$$\boxed{A = \frac{1}{2}\int_a^b \left(f(\theta)^2 - g(\theta)^2 \right) d\theta = \frac{1}{2}\int_a^b \left(r_{\text{out}}{}^2 - r_{\text{in}}{}^2 \right) d\theta}$$

Areas and Lengths in Polar Coordinates Example 2

Find the area of the region outside $r = 7 + 7\sin(\theta)$ and inside $r = 21\sin(\theta)$



Limits of Integration

$$7 + 7\sin(\theta) = 21\sin(\theta)$$

$$\sin(\theta) = \frac{1}{2} \qquad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$A = \frac{1}{2}\int_a^b \left(r_{\text{out}}{}^2 - r_{\text{in}}{}^2\right) d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(\left(21\sin(\theta)\right)^2 - \left(7 + 7\sin(\theta)\right)^2\right) d\theta$$

$$= 49\pi$$

Areas and Lengths in Polar coordinates - Length

Consider the curve with polar equation $r = f(\theta)$ for $a \le \theta \le b$

Recall the arc length formula for parametric curves: $$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta}$$

Write the curve in parametric form (θ is the parameter)

$$x = r\cos(\theta) = f(\theta)\cos(\theta), \quad y = r\sin(\theta) = f(\theta)\sin(\theta)$$

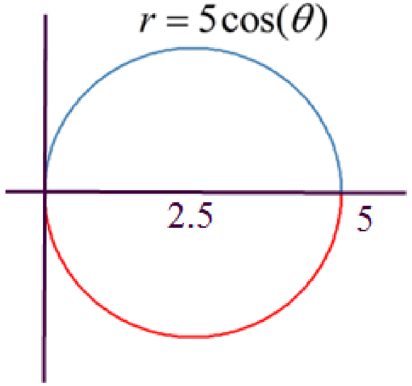
$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos(\theta) - r\sin(\theta), \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)$$

Arc Length of the curve with polar equation $r = f(\theta)$ for $a \le \theta \le b$

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta}$$

Areas and Lengths in Polar Coordinates – Lengths Example 3

Find the length of the polar curve $r = 5\cos(\theta)$ for $0 \le \theta \le \pi$



$$L = \int_0^{\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{25\sin^2(\theta) + 25\cos^2(\theta)}\, d\theta$$

$$= 5\int_0^{\pi} \sqrt{\sin^2(\theta) + \cos^2(\theta)}\, d\theta$$

$$= 5\int_0^{\pi} d\theta$$

$$= 5\pi$$

NOTES:

- The curve is a circle of radius 5/2, thus the answer is the circumference of the circle: $2\pi R = 2\pi \frac{5}{2} = 5\pi$
- If θ goes from 0 to 2π then the circle is described twice and the corresponding length would be twice the circumference.