Error Propagation: Volumes

Introduction

Physics is an empirical science. Every theoretical concept has to be supported with the experiment to be fully accepted. Experience has shown that no measurement is exact. All measurements have some degree of uncertainty due to the limits of instruments and the people using them. In science it is critically important to specify that uncertainty and include its reliable estimate in the final statement of the result of the measurement. The concept of the error propagation becomes a vital part of each physics experiment.

There are two concepts associated with measurements: accuracy and precision. The accuracy of the measurement refers to how close the measured value is to the true or accepted value. For example, if you used a balance to find the mass of a known standard 100.00 g mass, and you got a reading of 78.55 g, your measurement would not be very accurate because the percent discrepancy between the measured and "true value" is 21%. Precision refers to how close together a group of measurements actually are to each other. In the example above, if the three measurements of mass that obtained are 78.55g, 78.60 g and 78.50 g the measurements can be considered precise. Precision has nothing to do with the true or accepted value of a measurement, so it is quite possible to be very precise and totally inaccurate as in the example with an object's mass.

In many cases, when precision is high and accuracy is low, the fault can lie with the instrument. If a balance or a thermometer is not working correctly, they might consistently give inaccurate answers, resulting in high precision and low accuracy. One important distinction between accuracy and precision is that accuracy can be determined by only one measurement, while precision can only be determined with multiple measurements. A measurement can be accurate but not precise, precise but not accurate, neither, or both. For example, if an experiment contains a systematic error*, then increasing the sample size generally increases precision but does not improve accuracy. Eliminating the systematic error improves accuracy but does not change precision.

Commonly the best estimate of the true value of the measured quantity x can be obtained by calculating the mean (average) value ($\bar{x}$) of some significant number of measurements (*N*) of that quantity repeated using the same equipment and procedures:

$$\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N} \tag{1}$$

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* Systematic errors: Commonly caused by a flaw in the experimental apparatus. For example bad calibration in the instrumentation will give systematic error.

In such case the measure of the uncertainty (random error*) in the mean value will be the standard deviation of the mean:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}} \qquad (2)$$

where $\sigma_{\bar{x}}$ is the standard deviation of the mean, σ – is the standard deviation and N – is the number of measurements. The standard form for reporting a measurement of a physical quantity x is:

$$x = \bar{x} \pm \sigma_{\bar{x}} \qquad (3)$$

This statement expresses our confidence that the correct value of x probably lies in (or close to) the range from $x = \bar{x} - \sigma_{\bar{x}}$ to $x = \bar{x} + \sigma_{\bar{x}}$

There are two rules that will be followed in this course for stating the final result: a) uncertainty should be rounded to one sig.fig. unless that sig. fig. equals one; if the very first significant figure is one than the following digit should be reported; b) The last sig.fig. in the mean value should be in the same decimal place as the first significant figure in uncertainty. For example: $l = (24.\underline{245} \pm .\underline{006})$ mm

To estimate the accuracy of the final result the percent discrepancy or percent difference is calculated.

$$\%discr = \frac{\left| x_{theor} - \bar{x}_{\exp er} \right|}{x_{theor}} * 100\% \qquad (4)$$

$$\%differ = \frac{\left| \bar{x}_{\exp er1} - \bar{x}_{\exp er2} \right|}{x_{aver}} * 100\% \qquad (5)$$

A measure of the accuracy can only be determined if some prior knowledge of the true value is available.

The precision of the measurements shows the quality of the measurements and can be expressed as the relative uncertainty (the standard deviation of the mean divided by the mean value):

$$\text{relative uncertainty} = \frac{\sigma_{\bar{x}}}{\bar{x}} * 100\% \qquad (6)$$

* Random or statistical error: Class of errors produced by unpredictable or unknown variations in the measuring process. The effect of the random error may be reduced by repetition of the experiment.

The mean value of the physical quantity, as well as the standard deviation of the mean, can be evaluated after a relatively large number of independent similar measurements have been carried out. These measured quantities in many cases serve as a basis for the calculation of other physical quantities of interest. In such situations the uncertainties associated with the directly measured quantities affect the overall uncertainty in the final quantity of interest. To estimate the end uncertainty we use the rules of error propagation. For example, the volume of a box is $V = LWH$. If we measure the width W, height H, and length L many times to establish errors for each quantity: ΔW,ΔH,ΔL, what will the error in the volume ΔV be when we multiply the average values of the three quantities together? Or, in the other words, how the error of volume propagates.

In this lab error analysis will be practiced by measuring volumes of a few regular shapes objects. The partial derivatives approach will be used through the course to propagate the error:

$$\Delta f^2 = \left(\frac{\partial f}{\partial x}\Delta x\right)^2 + \left(\frac{\partial f}{\partial y}\Delta y\right)^2 \qquad (8)$$

where x and y are the measured quantities with the standard deviation of the mean (also called standard errors)Δx and Δy.

Objectives:

- to practice error propagation concept and reporting experimental result in the correct format.

Equipment: hollow cylinder, vernier caliper.

Procedure:

A student was asked to use Vernier caliper to measure the necessary dimensions of hollow cylinder, bullet-shapes object, 3D rectangular wood object to calculate the volume of each object and its error. The Vernier caliper is precise to 0.05 mm, so many of the measurements gave the same results. Each dimension was measured 6 times. The collected data is provided in table below.

Table Hollow cylinder

Height, mm	Thickness, mm	Inner diameter, mm
127.90	5.10	75.75
128.00	5.05	76.20
127.95	4.70	76.00
127.90	5.00	76.10
128.05	4.90	76.15
128.00	5.10	76.10

Based on the acquired data, use Logger Pro to find the mean value of each dimension and the standard deviation. Using the above measurements to calculate the standard error (standard deviation of the mean) for each dimension[1]; the mean value of volume for hollow cylinder and its standard error. The final results should be reported in centimeters, using the correct format.

[1]**How to find the mean value of each dimension and the standard deviation of the mean, using the Logger Pro:**
Create columns in Logger Pro and label them appropriately. Copy the acquired data from the table and paste it to Logger Pro.
Hint: The default file of Logger Pro consists of a two-column table and a graph area. Double click on the column name to change its name. Enter the corresponding name and units for the measured quantity in the appropriate box. Create in this way columns for each dimension of the object. Name data set by the name of the object.
Use the statistics feature of Logger Pro to find the mean values for each dimension as well as the standard deviation.
Hint: To get the statistical values (mean, standard deviation etc.) for a set of data you need to put the same data on both axes of the graph and next press STAT button  on the tool bar. Unless noted otherwise when you present any data in a graph form avoid lines connecting the data points → in Logger Pro you change graph features by double-click on the graph window. To add a new graph: INSERT→GRAPH.
To paste the data for the other object add manual columns; paste data and show its statistics.
Calculate standard deviation of the mean for each of the measured dimensions using equation (2). Use those values to propagate the error in volume with the partial derivative approach.

Complete the provided Template "Error Propagation exercise" to submit your work for grading.