Mathematical Induction

Use induction to prove that

$$\sum_{i=1}^{n} f_{2i-1} = f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$$

for all positive integers n, where f_n denotes the nth Fibonacci number.

Let P(n) denote the proposition $\sum_{i=1}^{n} \boldsymbol{f}_{2i-1} = \boldsymbol{f}_1 + \boldsymbol{f}_3 + \cdots + \boldsymbol{f}_{2n-1} = \boldsymbol{f}_{2n}$, where n is a positive integer.

Recall: $f_{n+1} = f_n + f_{n-1},$ where $f_0 = 0$ and $f_1 = 1$.

BASIS STEP: P(1) is true since $\sum_{i=1}^{1} \boldsymbol{f}_{2i-1} = f_1 = 1$ and $f_2 = 1$

INDUCTIVE STEP: Let us assume P(n), that is

$$\sum_{i=1}^{n} \boldsymbol{f}_{2i-1} = \boldsymbol{f}_1 + \boldsymbol{f}_3 + \cdots + \boldsymbol{f}_{2n-1} = \boldsymbol{f}_{2n}$$

is true for an arbitrary positive integer n . This is our inductive hypothesis.

We have to show the statement P(n+1),

$$\sum_{i=1}^{n+1} \boldsymbol{f}_{2i-1} = \boldsymbol{f}_1 + \boldsymbol{f}_3 + \cdots + \boldsymbol{f}_{2n-1} + \boldsymbol{f}_{2(n+1)-1} =$$

$$\boldsymbol{f}_1 + \boldsymbol{f}_3 + \cdots + \boldsymbol{f}_{2n-1} + \boldsymbol{f}_{2n+1} = \boldsymbol{f}_{2(n+1)} = \boldsymbol{f}_{2n+2}$$

is true assuming the inductive hypothesis P(n).

Proof:

$$\sum_{i=1}^{n+1} \boldsymbol{f}_{2i-1} = \sum_{i=1}^{n} \boldsymbol{f}_{2i-1} + \boldsymbol{f}_{2n+1} = \boldsymbol{f}_1 + \boldsymbol{f}_3 + \cdots + \boldsymbol{f}_{2n-1} + \boldsymbol{f}_{2n+1} =$$

$\boldsymbol{f}_{2n} + \boldsymbol{f}_{2n+1} = \boldsymbol{f}_{2n+2}$ using the inductive hypothesis and the definition of the Fibonacci numbers.

By the Principle of Mathematical Induction (Basis Step and Inductive Step together) $\sum_{i=1}^{n} \boldsymbol{f}_{2i-1} = f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ for all positive integers n.