Mathematical Induction

Use induction to prove that

$$\sum_{i=1}^{n} f_i^2 = f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$$

for all positive integers n, where f_n denotes the nth Fibonacci number.

Let P(n) denote the proposition $\sum_{i=1}^{n} f_i^2 = f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$, where n is a positive integer.

Recall: $f_{n+1} = f_n + f_{n-1},$ where $f_0 = 0$ and $f_1 = 1$.

BASIS STEP: P(1) is true since $\sum_{i=1}^{1} f_i^2 = f_1^2 = 1$ and $f_1 f_2$=1

INDUCTIVE STEP:

Let us assume P(n), that is $\sum_{i=1}^{n} f_i^2 = f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$, is true for an arbitrary positive integer n. This is our inductive hypothesis.

We have to show the statement P(n+1),
$\sum_{i=1}^{n+1} f_i^2 = f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2 = f_{n+1} f_{n+2}$ is true assuming the inductive hypothesis P(n).

Proof:

$\sum_{i=1}^{n+1} f_i^2 = \sum_{i=1}^{n} f_i^2 + f_{n+1}^2 = f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2 = f_n f_{n+1} + f_{n+1}^2$ using the inductive hypothesis.

$f_n f_{n+1} + f_{n+1}^2 = \mathbf{f}_{n+1}(\mathbf{f}_n + \mathbf{f}_{n+1}) = \mathbf{f}_{n+1}\mathbf{f}_{n+2}$, since $\mathbf{f}_{n+2} = \mathbf{f}_n + \mathbf{f}_{n+1}$ by the definition of the Fibonacci numbers.

By the Principle of Mathematical Induction (Basis Step and Inductive Step together) $\sum_{i=1}^{n} f_i^2 = f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ for all positive integers n.