Examples of Theorems with Proofs

Theorem 1: Every even number can be written as the sum of two odd numbers whose difference is at most two.

Theorem 2: For any natural number n, there is an odd integer k such that $n^2 < k < (n+1)^2$.

Theorem 3: For every real number x, there exist an integer n and a real number $\mathrm{r} \in [0,1)$ such that

$$x = n + r.$$

Theorem 4: If $m + n$ and $n + p$ are even integers, where m, n and p are integers, then $m + p$ is even.

The proofs are on the next page. You won't get anything out of this if you just read the proofs. You must at least seriously attempt to prove the theorems yourself before you read the supplied proofs.

Proof of theorem 1: Suppose n is an arbitrary even number. By definition of even number, that means that there exists an integer k such that $n = 2k$.

Case 1: k is odd. Then $n = k + k$ is the representation of n as a sum of two odd integers with difference less than or equal two that we are seeking.

Case 2: k is even. Then $k + 1$ and $k - 1$ are odd. Therefore $n = (k + 1) + (k - 1)$ is the representation of n as a sum of two odd integers with difference less than or equal two that we are seeking.

We have shown that either way, we can write n as a sum of two odd integers whose difference is less than or equal to two. Since n was arbitrary, we have proved the theorem.

(Some comments which are not part of the proof:

The most critical thing here is that you get the fundamental logical structure of a "for all, there exists.." type of proof correct.

The goal is to show that for all even numbers, these two other numbers exist. Therefore, the proof has to start by assuming that we have an arbitrary even number n. Nothing more must be assumed.

The existence of the two odd numbers that add up to n and are no more than two away from each other must be <u>*shown.*</u> *Saying "let p and q be odd such that $n = p + q$ and $|p - q| \leq 2$" or similar does not show it, it simply assumes it. You cannot proof existence by postulating it. To prove that a unicorn exists, you can't say "let Barney be a unicorn..". You have to produce one.*

That's what's happening in this proof after the n is introduced. It demonstrate how, depending on n, you can find these two odd numbers.)

Proofs of theorem 2:

Proof 1:

Let n be an arbitrary natural number.

Case 1: n is even. Then $k = n^2 + 1$ is odd, and $n^2 < n^2 + 1 < n^2 + 2n + 1$. The last inequality is true because $n \geq 1$ by definition of natural number. Therefore, $n^2 < k < (n + 1)^2$.

Case 2: n is odd. Then $k = n^2 + 2$ is odd, and $n^2 < n^2 + 2 \leq n^2 + 2n$ because $n \geq 1$ by definition of natural number. Therefore, $n^2 < n^2 + 2 < n^2 + 2n + 1$ or $n^2 < k < (n + 1)^2$.

We have shown that in either case, an odd number k exists with $n^2 < k < (n+1)^2$. That proves the theorem.

Proof 2:

Let n be an arbitrary natural number. Then $(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$. By definition of natural number, $n \geq 1$, hence,

$$(n+1)^2 - n^2 \geq 3$$

It follows that there are at least two consecutive integers strictly between n^2 and $(n+1)^2$: $n^2 + 1$ and $n^2 + 2$. Out of two consecutive integers, one must be odd. Let k be that odd integer. Then $n^2 < k < (n+1)^2$. We have proved that for any natural number n, an odd number k exists such that $n^2 < k < (n+1)^2$.

Proof 3:

Let n be an arbitrary natural number. Define $k = n^2 + n + 1 = n(n+1) + 1$. Since a product of two consecutive natural numbers is even, $n(n+1)$ is even and k is odd. Furthermore, $n^2 < k = n^2 + n + 1 < n^2 + 2n + 1$. The last inequality follows from the fact that $n \geq 1$. Hence, for any natural number n, we have found an odd number k between n^2 and $n^2 + 2n + 1 = (n+1)^2$.

Proof of theorem 3:

Assume $x \in \mathbb{R}$, arbitrary. Define $n = \lfloor x \rfloor$. By properties of the floor function, $n \in \mathbb{Z}$, and $x - 1 < n \leq x$. By subtracting x from the inequality, we get $-1 < n - x \leq 0$. Multiplying this by (-1), we get $0 \leq x - n < 1$. Thus, if we define $r = x - n$, then $r \in [0,1)$, and $x = n + r$. We have shown that for each $x \in \mathbb{R}$, there is an integer n and $r \in [0,1)$ such that $x = n + r$.

Proof of theorem 4:

Suppose that $m + n$ and $n + p$ are even. Then $m + n = 2k$ for some integer k and $n + p = 2q$ for some integer q. By adding these equations, we obtain $(m + n) + (n + p) = 2k + 2q$. Simplifying, we get $m + p + 2n = 2(k + q)$, or equivalently, $m + p = 2(k + q - n)$. By definition of an even integer, that means that $m + p$ is even.