PRACTICE PROBLEMS CHAPTER 6

1.

(i) (a) $\det(A-\lambda I)=\lambda^2-1 \Rightarrow \lambda_1=-1,\ \lambda_2=1.$

$A-\lambda_1 I=A+I=\begin{bmatrix}1&1\\1&1\end{bmatrix}\to \text{RREF}\begin{bmatrix}1&1\\0&0\end{bmatrix} \Rightarrow$ e-vector associated to $\lambda_1=-1$ is $\mathbf{x}_1=\begin{bmatrix}1\\-1\end{bmatrix}$,

$A-\lambda_2 I=A-I=\begin{bmatrix}-1&1\\1&-1\end{bmatrix}\to \text{RREF}\begin{bmatrix}1&-1\\0&0\end{bmatrix} \Rightarrow$ e-vector associated to $\lambda_2=1$ is $\mathbf{x}_2=\begin{bmatrix}1\\1\end{bmatrix}$.

(b) $\det(A-\lambda I)=\lambda^2+1 \Rightarrow \lambda_1=i,\ \lambda_2=-i.$

$A-\lambda_1 I=A-iI=\begin{bmatrix}-i&1\\-1&-i\end{bmatrix}\to \text{RREF}\begin{bmatrix}1&i\\0&0\end{bmatrix} \Rightarrow$ e-vector associated to $\lambda_1=i$ is $\mathbf{x}_1=\begin{bmatrix}-i\\1\end{bmatrix}$,

The e-vector associate to $\lambda_2=-i$ is the complex conjugate of $\mathbf{x}_1$: $\mathbf{x}_2=\begin{bmatrix}i\\1\end{bmatrix}$.

(c) $\det(A-\lambda I)=(1-\lambda)^2 \Rightarrow \lambda_1=\lambda_2=1,\quad \lambda_1=1$ (AM = 2)

$A-\lambda_1 I=A-I=\begin{bmatrix}0&0\\0&0\end{bmatrix} \Rightarrow$ two free variables and possible e-vectors are $\mathbf{x}_1=\begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{x}_2=\begin{bmatrix}0\\1\end{bmatrix}$. (GM = 2)

(d) $\det(A-\lambda I)=(1-\lambda)^2 \Rightarrow \lambda_1=\lambda_2=1,\quad \lambda_1=1$ (AM =2)

$A-\lambda_1 I=A-I=\begin{bmatrix}0&1\\0&0\end{bmatrix} \Rightarrow$ e-vector associated to $\lambda_1=1$ is $\mathbf{x}=\begin{bmatrix}1\\0\end{bmatrix}$. (GM =1)

(ii) (a) The matrix is diagonalizable (distinct e-vectors): $X=\begin{bmatrix}1&1\\-1&1\end{bmatrix}$ and $D=\begin{bmatrix}-1&0\\0&1\end{bmatrix}$

(b) The matrix is diagonalizable (distinct e-vectors): $X=\begin{bmatrix}-i&1\\i&1\end{bmatrix}$ and $D=\begin{bmatrix}i&0\\0&-i\end{bmatrix}$

(c) The matrix is diagonalizable (AM = GM =2): $X=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $D=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

(d) Since the matrix has one linearly independent eigenvector (GM =1) associated to the repeated eigenvalue $\lambda=1$ (AM = 2), we have GM < AM and the matrix is defective (non diagonalizable).

2. **(a)** $\det(A-\lambda I)=(1-\lambda)(-\lambda)(3-\lambda) \Rightarrow$ the eigenvalues of A are $\lambda_1=1$, $\lambda_2=0$, $\lambda_3=3$.

The eigenvalues of A^2 are 1, 0, 9 and the eigenvalues of A^n are 1, 0 3^n.

(b) $A-\lambda_1 I=A-I=\begin{bmatrix}0&2&1\\0&-1&6\\0&0&2\end{bmatrix}\to \text{RREF}\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix} \Rightarrow$ e-vector associated to $\lambda_1=1$ is $\mathbf{x}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$,

$A-\lambda_2 I=A-0\text{I}=A=\begin{bmatrix}1&2&1\\0&0&6\\0&0&3\end{bmatrix}\to \text{RREF}\begin{bmatrix}1&2&0\\0&0&1\\0&0&0\end{bmatrix} \Rightarrow$ e-vector associated to $\lambda_2=0$ is $\mathbf{x}_2=\begin{bmatrix}-2\\1\\0\end{bmatrix}$,

$A-\lambda_3 I=A-3\text{I}=\begin{bmatrix}-2&2&1\\0&-3&6\\0&0&0\end{bmatrix}\to \text{RREF}\begin{bmatrix}1&0&-\frac{5}{2}\\0&1&-2\\0&0&0\end{bmatrix} \Rightarrow$ e-vector associated to $\lambda_2=3$ is $\mathbf{x}_3=\begin{bmatrix}5\\4\\2\end{bmatrix}$

The eigenspaces for the matrix A have basis $\mathbf{x}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{x}_2=\begin{bmatrix}-2\\1\\0\end{bmatrix}$, $\mathbf{x}_3=\begin{bmatrix}5\\4\\2\end{bmatrix}$

These are also the basis for the eigenspaces for the matrix A^2 and A^n.

(c) Since $A=XDX^{-1}=\begin{bmatrix}1&-2&5\\0&1&4\\0&0&2\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&0\\0&0&3\end{bmatrix}\begin{bmatrix}1&2&-\frac{13}{2}\\0&1&-2\\0&0&\frac{1}{2}\end{bmatrix}$

we have $A^n = X D^n X^{-1} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & 2 & -\frac{13}{2} \\ 0 & 1 & -2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 2 & -\frac{13}{2}+\frac{5(3^n)}{2} \\ 0 & 0 & 2(3^n) \\ 0 & 0 & 3^n \end{bmatrix}$

(d) Substituting $n = 7$ in the formula from part (c) gives

$$A^7 = \begin{bmatrix} 1 & 2 & -\frac{13}{2}+\frac{5(3^7)}{2} \\ 0 & 0 & 2(3^7) \\ 0 & 0 & 3^7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5461 \\ 0 & 0 & 4374 \\ 0 & 0 & 2187 \end{bmatrix}$$

3.

(a) The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 3$.

(b) The eigenvalues of $A^2+\alpha A+\beta I$ are $2^2+2\alpha+\beta$ and $3^2+3\alpha+\beta$

4. The characteristic equation of the matrix is given by $\lambda^2-2\lambda+9k-35=0$ From the quadratic formula we have that this equation has two distinct solutions if and only if $b^2-4ac=4-4(9k-35)>0$.
Solving the inequality gives $k < 4$.

5. $\mathrm{tr}(A)=\lambda_1+\lambda_2=5, \quad \det(A)=\lambda_1\lambda_2=-14 \quad \Rightarrow \quad \lambda_1=-2, \ \lambda_2=7$

6. (a) Putting the vectors into a matrix: $\left[\begin{array}{cc|c} 1 & 3 & 19 \\ -1 & 2 & 6 \\ -2 & -1 & -13 \end{array}\right]$ RREF $\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$. Thus $\mathbf{v}=4\mathbf{x}_1+5\mathbf{x}_2$

(b) $A\mathbf{v} = A(4\,\mathbf{x}_1 + 5\,\mathbf{x}_2) = 4\,(A\mathbf{x}_1) + 5(A\mathbf{x}_2) = 4\,(\lambda_1\,\mathbf{x}_1) + 5\,(\lambda_2\,\mathbf{x}_2) = (4)(-2)\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + (5)(3)\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 37 \\ 38 \\ 1 \end{bmatrix}$

7. (a) $\det(A-\lambda I)=(3-\lambda)^3$, thus the matrix has a repeated eigenvalue $\lambda = 3$ of Algebraic Multiplicity three

$A-\lambda_1 I=A-3\mathrm{I}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ The matrix is already in RREF and a basis of eigenspace is $\mathbf{x}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,

Since GM = 1 < AM = 3, the matrix is defective and not diagonalizable.

(b) $\det(A-\lambda I)=(4-\lambda)(1-\lambda)^2$. The matrix has the eigenvalue $\lambda_1=1$ with AM =2 and $\lambda_2=4$ with AM = 1.
In order to determine whether the matrix is diagonalizable we need to determine the Geometric Multiplicity of the repeated eigenvalue $\lambda_1=1$.

$A-\lambda_1 I=A-I=\begin{bmatrix} 3 & -3 & 0 \\ 0 & 0 & 0 \\ -4 & 4 & 0 \end{bmatrix} \Rightarrow \mathrm{RREF}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Basis of eigenspace associated to $\lambda_1=1$ is

$\mathbf{x}_1=\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{x}_2=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Thus GM = AM = 2 and the matrix is diagonalizable.

The eigenvalue $\lambda = 4$ has associated eigenvector $[-3, 0, 4]^T$ and a possible diagonalization is:

$$X=\begin{bmatrix} 1 & 0 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 4 \end{bmatrix} \quad \text{and} \quad D=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

8. In Problem 6 part (b) we found that the matrix A is diagonalizable as

$$A = XDX^{-1} = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 4/3 & -4/3 & 1 \\ -1/3 & 1/3 & 0 \end{bmatrix}$$

The matrix B is given by $B = X\sqrt{D}X^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 4/3 & -4/3 & 1 \\ -1/3 & 1/3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ -4/3 & 4/3 & 1 \end{bmatrix}$

9. If $\lambda_1 = 3 + 2i$ is an eigenvalue, then the complex conjugate $\lambda_2 = 3 - 2i$ is also an eigenvalue. If the matrix is singular, then $\lambda_3 = 0$ is an eigenvalue. The sum of the eigenvalues equals the trace of the matrix which is given by $a_{11} + a_{22} + a_{33} + a_{44} = 4$, thus $(3 + 2i) + (3 - 2i) + 0 + \lambda_4 = 4$ which gives $\lambda_4 = -2$.

10.
(a) If $\lambda = 0$ then $\det(A - \lambda I) = \det(A) = 0$ and the matrix is singular which contradicts the assumption, thus $\lambda \neq 0$.
(b) If $A\mathbf{x} = \lambda\mathbf{x}$ then $A^{-1}A\mathbf{x} = \lambda A^{-1}\mathbf{x}$ which gives $\mathbf{x} = \lambda A^{-1}\mathbf{x}$ or $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$ thus $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.with $\mathbf{x}$ the corresponding eigenvector.

11. The matrix has the eigenvalue $\lambda = a$ with algebraic multiplicity 3.The basis of the corresponding eigenspace consists of two linearly independent eigenvectors: $[1, 0, 0]^T$ and $[0, 1, 0]^T$. Since there are only two linearly independent eigenvectors, we have GM = 2 < AM = 3 and the matrix is defective.

12. If A is diagonalizable, then $A = XDX^{-1}$ where D is a diagonal matrix. If B is similar to A, then there exists a nonsingular matrix S such that $B = S^{-1}AS$. It follows that

$$B = S^{-1}(XDX^{-1})S$$
$$= (S^{-1}X)D(S^{-1}X)^{-1}$$

Therefore B is diagonalizable with diagonalizing matrix $S^{-1}X$.

13.
(a) The rank of A is given by the number of nonzero singular values, thus the rank is 3.
(b) An orthonormal basis for $R(A^T)$ is given by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, where $\mathbf{v}_i$ is the ith column of V.
(c) An orthonormal basis is given by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ where $\mathbf{u}_i$ is the ith column of U
(d) The rank-1 matrix B that is the closest matrix of rank-1 to A is given by

$$B = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T = 100\begin{bmatrix} 2/5 \\ 2/5 \\ 2/5 \\ 2/5 \\ 3/5 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 \\ 30 & 30 & 30 & 30 \end{bmatrix}$$

(e) From Theorem 6.5.3 we have that $\|B - A\|_F = \sqrt{\sigma_2^2 + \sigma_3^2} = 10\sqrt{2}$.

(f) $$C = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T = B + 10\begin{bmatrix} -2/5 \\ -2/5 \\ -2/5 \\ 3/5 \\ 2/5 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 18 & 22 & 22 & 18 \\ 18 & 22 & 22 & 18 \\ 18 & 22 & 22 & 18 \\ 23 & 17 & 17 & 23 \\ 32 & 28 & 28 & 32 \end{bmatrix}$$

(g) $\|C - A\|_F = \sigma_3 = 10$.

14. The least squares solution is given by $\hat{\mathbf{x}} = \frac{(\mathbf{u}_1^T\mathbf{b})}{\sigma_1}\mathbf{v}_1 + \frac{(\mathbf{u}_2^T\mathbf{b})}{\sigma_2}\mathbf{v}_2 = -\frac{4.5}{20}\mathbf{v}_1 - \frac{1.5}{15}\mathbf{v}_2 = \begin{bmatrix} 0.12 \\ -.215 \end{bmatrix}$

15. $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad \Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

(This problem is completely worked out on page 5 of the Additional Notes).