Math 243, Spring 2006, Professor Callahan

Test #2, Thu–Fri, Apr. 13–14.

Note 1: This test is closed book. You may use one $8\frac{1}{2} \times 11$ sheet of notes (both sides).

Note 2: Show your work. Clarity counts. If I can't follow your reasoning I can't give credit.

Problem 1: Evaluate 7^{1172} mod 31.

Answer: We know that 31 is prime, so by Fermat's Little Theorem $7^{30} \equiv 1 \pmod{31}$. Knowing that $1172 = 30 \cdot 39 + 2$, we get

$$7^{1172} = 7^{30 \cdot 39 + 2} = 7^{30 \cdot 39} \cdot 7^2 = \left(7^{30}\right)^{39} \cdot 7^2 \equiv 1^{39} \cdot 7^2 \equiv 49 \equiv 18 \pmod{31}.$$

Problem 2: Find the value of x between 0 and 280 that satisfies

$$\begin{aligned} x &\equiv 2 \pmod{5} \\ x &\equiv 2 \pmod{7} \\ x &\equiv 3 \pmod{8}. \end{aligned}$$

Answer: We use the Chinese Remainder Theorem. With $m_1 = 5$, $m_2 = 7$ and $m_3 = 8$ we have

$$M_1 = m_2 m_3 = 56, \qquad M_2 = m_1 m_3 = 40, \qquad M_3 = m_1 m_2 = 35.$$

Then

$$M_1 \bmod m_1 = 56 \bmod 5 = 1,$$

so the inverse is $y_1 = 1$.

$$M_2 \bmod m_2 = 40 \bmod 7 = 5.$$

We note that $3 \cdot 5 = 15 \equiv 1 \pmod{7}$, so the inverse is $y_2 = 3$.

$$M_3 \bmod m_3 = 35 \bmod 8 = 3.$$

We note that $3 \cdot 3 = 9 \equiv 1 \pmod{8}$, so the inverse is $y_3 = 3$. Then we take

$$a_1 M_1 y_1 + a_2 M_2 y_2 + a_3 M_3 y_3 = 2 \cdot 56 \cdot 1 + 2 \cdot 40 \cdot 3 + 3 \cdot 35 \cdot 3 = 667 \equiv 107 \pmod{280}.$$

Thus $x = 107$.

Problem 3: Use mathematical induction to show that 5 divides $n^5 - n$ whenever n is a nonnegative integer.

Answer: First we check the smallest case, with $n = 0$. Then $n^5 - n = 0$, which is divisible by 5.

Now we assume that $n^5 - n$ is divisible by 5. Then

$$\begin{aligned}(n+1)^5 - (n+1) &= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - n - 1 \\ &= (n^5 - n) + 5n^4 + 10n^3 + 10n^2 + 5n \\ &= (n^5 - n) + 5(n^4 + 2n^3 + 2n^2 + n).\end{aligned}$$

The first part is a multiple of 5 by our inductive hypothesis, and the second part is obviously a multiple of 5, so the whole thing is a multiple of 5.

Problem 4: Use the geometric sum formula to evaluate

$$2 + \frac{4}{3} + \frac{8}{9} + \cdots + 2\left(\frac{2}{3}\right)^8.$$

Answer: This is

$$2 + 2 \cdot \left(\frac{2}{3}\right) + 2 \cdot \left(\frac{2}{3}\right)^2 + \cdots + 2\left(\frac{2}{3}\right)^8 = \frac{2(2/3)^9 - 2}{(2/3) - 1} = \frac{38342}{6561} \approx 5.84393.$$