Structural Induction

Let S be the set defined recursively as follows:

Basis step: $1 \in S$ and $2 \in S$

Recursive step: If $x \in S$ then $x + 3 \in S$

a. **List the elements of S produced by the first 3 applications of the recursive definition.**

$$S_0 = \{1, 2\},\ S_1 = \{4, 5\},\ S_2 = \{7, 8\},\ S_3 = \{10, 11\}$$

The elements of S produced by the first 3 applications of the recursive definition is

$S_0 \cup S_1 \cup S_2 \cup S_3 = \{1, 2, 4, 5, 7, 8, 10, 11\}.$

b. **Use structural induction to prove that** $S \subseteq \{3n + 1, 3m + 2 | n \in N_0 \text{ and } m \in N_0\}$, where N_0 denotes the set of non-negative integers.

Note that $\{3n + 1, 3m + 2 | n \in \mathrm{N}_0$ and $m \in \mathrm{N}_0 \}$ is the set of positive integers that are not divisible by 3.

When we use structural induction to show that the elements of a recursively defined set S have a certain property, then we need to do the following procedure:

1. **Basis step**: show all the elements defined in the basis step have the desired property.
2. **Inductive step**: assume that an arbitrary element of the set S has the desired property. This is your inductive hypothesis. Using the inductive hypothesis, prove that, when you create more elements in S by using the recursive definition, these newly created elements of S have the same property.
3. **Conclusion:** state that by structural induction all the elements in S have the same property .

We have to show that, if $x \in S$ then $x = 3n + 1$ for some $n \in \mathrm{N}_0$ or $x = 3m + 2$ for some m $\in \mathrm{N}_0$. We use the recursive definition of S and structural induction to prove this statement.

Basis step: $1 \in S$ by the definition of S and $1 = 3{\cdot}0 + 1$.

$2 \in S$ by the definition of S and $2 = 3{\cdot}0 + 2$.

Recursive Step: Assume $x \in S$ and $x = 3n + 1$ for some $n \in \mathrm{N}_0$ or $x = 3m + 2$ for some m $\in \mathrm{N}_0$. We need to prove that $x + 3$ has the same property, that is $x + 3$ can be expressed in the above described form.

Proof: Using the inductive hypothesis,
$Case1$: $x + 3 = 3n + 1 + 3 = 3(n + 1) + 1$, where n+1 is a positive integer.
Case2: $x + 3 = 3m + 2 + 3 = 3(m + 1) + 2$, where m+1 is a positive integer.

By **structural induction** we have proved that, if $x \in S$ then $x = 3n + 1$ for some $n \in N_0$ or $x = 3m + 2$ for some m $\in N_0$, that is $S \subseteq \{3n + 1, 3m + 2 | n \in \mathrm{N}_0$ and $m \in \mathrm{N}_0\}$.