Problem 1. Consider the set $S = \varnothing$. Then what is the power set of the power set of S ?

- $\varnothing$
- $\{\varnothing\}$
- $\{\varnothing, \{\varnothing\}\}$
- $\{\varnothing, \varnothing\}$
- $\{\varnothing, \varnothing, \varnothing\}$
- $\{\varnothing, \{\varnothing, \{\varnothing\}\}\}$
- undefined

Answer(s) submitted:

- a3

(correct)

Correct Answers:

- a3

Problem 2. Suppose $S = \{2,3\}$. Then what is the cartesian product $S^2 = S \times\ S$?

- $\{4,9\}$
- $\{(2,2),(3,3)\}$
- $\{2,2,3,3\}$
- $\{\{2,2\},\{3,3\}\}$
- $\{\{2\},\{3\}\}$
- $\{(2,2),(2,3),(3,2),(3,3)\}$
- $\{\{2,2\},\{2,3\},\{3,2\},\{3,3\}\}$
- None of the above.

Answer(s) submitted:

- a6

(correct)

Correct Answers:

- a6

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}; f(x) = |x|$ be the absolute value function. Determine $f((-2,2))$ and $f^{-1}((0,2))$.

- $f((-2,2)) = [0,2), f^{-1}((0,2)) = (-2,2)$
- $f((-2,2)) = [0,2), f^{-1}((0,2)) = (0,2)$
- $f((-2,2)) = [0,2), f^{-1}((0,2)) = (-2,0) \cup (0,2)$
- $f((-2,2)) = (0,2), f^{-1}((0,2)) = (-2,0) \cup (0,2)$
- $f((-2,2)) = (0,2), f^{-1}((0,2)) = (-2,2)$
- $f((-2,2)) = (0,2), f^{-1}((0,2)) = (0,2)$

Answer(s) submitted:

- a1

(correct)

Correct Answers:

- a1

Problem 4. Let $f : A \to B$ be a function and let $X \subseteq A, Y \subseteq B$. Find the relationship between $f(f^{-1}(Y))$ and Y, and between $f^{-1}(f(X))$ and X.

- $f(f^{-1}(Y)) \subseteq Y, f^{-1}(f(X)) \subseteq X$
- $f(f^{-1}(Y)) \subseteq Y, X \subseteq f^{-1}(f(X))$
- $Y \subseteq f(f^{-1}(Y)), f^{-1}(f(X)) \subseteq X$
- $Y \subseteq f(f^{-1}(Y)), X \subseteq f^{-1}(f(X))$

Answer(s) submitted:

- a2

(correct)

Correct Answers:

- a2

Problem 5. Several students were asked to write proofs from first principle (i.e. based only on the definition) of the theorem that the function $f:[1,3]\to[3,7]; f(x)=2x+1$ is bijective. Only one of them succeeded. Select the name of that student.

Leto:
The function is a straight line with a slope that is not zero. Such functions are always bijective.

Vladimir:
f is injective: suppose a and b are in the domain and a=b. Then f(a) = f(b). Therefore f is injective.
f is surjective: suppose y is in [3,7]. Then f(x) = y for some x in [1,3]. Therefore f is surjective.

Hasimir:
The function's domain contains only two numbers, x=1 and x=3, and the range only contains the two numbers y = 3 and y=7. By definition, f(1) = 3 and f(3)=7. Since every output comes from exactly one input, f is bjective.

Jessica:
f is injective: suppose f(a) = f(b) for some a and b in [3,7]. Since outputs of this function are unique, a = b. Therefore, f is injective.
f is surjective: Let x be in [1,3]. Then 2x is in [2,6] and f(x) = 2x+1 is in [3,7]. That shows that for every x in [1,3] ,f(x) is in [3,7]. Therefore f is surjective.

Paul:
f is injective: suppose f(a) = f(b) for some arbitrary a and b in [3,7]. By applying the inverse function to both sides, we get a=b. Therefore f is injective.
f is surjective: Let y be in [3,7], arbitrary. For that y, pick $x=f^{-1}(y)$. Then by definition of the inverse function, $y=f(x)$. Hence f is surjective.

Chani:
Suppose to get a contradiction that f is not injective. Then we would have f(a) = f(b) for some a, b in [1,3]. By using the definition of f, that means 2a+1 = 2b+1 or a=b. That is a contradiction and proves that f is injective.
f is surjective: Suppose y is in [3,7]. Let x in [1,3] be such that f(x) = y. Since we have found an x whose image is y, we have shown that f is surjective.

Duncan:
Suppose f(a) = f(b) for some a,b in [1,3]. By definition of f, 2a+1 = 2b+1, or a=b. Therefore, f is injective.
f is surjective: Suppose y is in [3,7]. Define $x=(y-1)/2$. Since y-1 is in [2,6], x is in [1,3], so f can be applied and f(x) = y. We have shown that f is surjective.

Feyd:
Suppose f(a) = f(b) for all a,b. By definition of f, that means 2a+1 = 2b+1, which implies or a=b. That proves that f is injective.
f is surjective: Suppose y is in [3,7]. Let $x=(y-1)/2$. Then f(x) = y and we have shown that f is surjective.

Helen:
Since the derivative of f is positive, f is strictly increasing. Hence f is injective.
f is continuous. Since f(1) = 3 and f(3) = 7, every y value between 3 and 7 is the output of some x in (1,3) by the intermediate value theorem of calculus.Hence f is surjective.

- Leto
- Vladimir
- Hasimir
- Jessica
- Paul
- Chani
- Duncan
- Feyd
- Helen

Answer(s) submitted:

- `Chani`

(incorrect)

Correct Answers:

- `Duncan`

Problem 6. Select the smallest k so that $f(x) = x^3 \ln x^4 + x^2$ is big-O of x^k.

- 3
- 4
- 5
- 6
- 7

Answer(s) submitted:

- `4`

(correct)

Correct Answers:

- `4`

Problem 7. Evaluate 7^n mod 11 where $n = 2^{101}$.

- 1
- 2
- 3
- 4
- 5
- 6

Answer(s) submitted:

- `6`

(incorrect)

Correct Answers:

- `5`

Problem 8. How many operations (squaring and mod counts as one) does it take to evaluate 5^n mod 23 where $n = 2^{1000}$ using the fast modular exponentiation algorithm discussed in the lecture?

- 999
- 1000
- 1001
- $2^{1000} - 1$
- 2^{1000}
- $2^{1000} + 1$

Answer(s) submitted:

- `1000`

(correct)

Correct Answers:

- `1000`

Problem 9. Suppose b is an integer that is at least two, and the integer n has the following base-b expansion:

$n = a_k b^k + a_{k-1} b^{k-1} \ldots a_1 b + a_0,$

where k is a non-negative integer.

Then n mod b is equal to:

- $a_k + a_{k-1} + \ldots + a_1 + a_0$
- a_k
- a_0
- a_1
- k
- b

Answer(s) submitted:

- `a5`

(incorrect)

Correct Answers:

- `a3`

Problem 10. Suppose the base-5 expansion of an integer consists of exactly 1000 digits ”4”. Then the number equals

- $5^{999} - 1$
- 5^{999}
- $5^{1000} - 1$
- 5^{1000}
- $5^{1001} - 1$
- 5^{1001}

Answer(s) submitted:

- `5^1000 - 1`

(correct)

Correct Answers:

- `5^1000 - 1`

Problem 11. Several students were asked to write a structural induction proof of the following theorem:
Let S be recursively defined as follows:
1.$(1 \in S)$
2.$\forall x(x \in S \to 2x+1 \in S)$
3.$\forall x(x \in S \to x^2 \in S)$
Then all elements of S are odd. (You may assume without proof that all elements of S are integers.)

Select the name of the one student who succeeded.

Leto:
By applying the generating rules to the initial population, I get 1, 3, 7, 15, 31, etc, and also the squares: 9, 49, etc. Those are all odd numbers.

Vladimir:
The "initial population" consists solely of odd numbers.
Now suppose $x \in S$ arbitrary. Then by definition of odd number, $2x+1$ is odd again, and so is x^2. Therefore, all members of S are odd.

Hasimir: The initial population has only odd numbers.
Now suppose $x \in S$ is an odd number and not in the initial population. Then it must have been generated by rule 2. or 3. If $x = 2y+1$ for some $y \in S$, then y itself must have been odd.
If $x = y^2$ for some $y \in S$, then y must have been odd as well. That completes the structural induction proof that all members of S are odd.

Jessica:
The initial population is all odd.
Now assume $x \in S$ is an arbitrary odd number in S. Rule 2. turns x into $2x+1$. That is is odd simply because x is integer (the fact that x is also odd is not required for that.) Rule 3. turns x, into x^2, which is also odd: we know that x is odd, so $x = 2k+1$ for some integer k. Then $x^2 = (2k+1))^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$. I just verified that x^2 is two times an integer, plus one, so it is odd. That completes the structural induction proof.

Paul:
The initial population is only one number, 1, which is odd.
Now assume $x \in S$ is in the initial population. Therefore, x is odd. By applying rule 2. to x, we get $2x+1$, which is odd for any integer x. By applying rule 3. to x, we get an odd number as well, which we can see as follows: since x is odd, $x = 2k+1$ for some integer k. Thus $x^2 = (2k+1))^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$ which is odd by definition since it is an integer multiple of 2, plus 1. That completes the structural induction proof.

Duncan:
First we verify the statement $P(1)$. The number 1 is odd.
Now let's assume that $P(n)$ has already been proved, so n is

Problem 13. Several students were asked to prove the following theorem by induction.
$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}(1)$ for all positive integers n.
Select the name of the one student who succeeded.

Leto:
Base case: the formula is correct for $n = 1$ because both sides are 1/2 .
Inductive step: $P(n+1) = \sum_{k=1}^{n+1} \frac{1}{k(k+1)} = 1 - \frac{1}{n+2}$.

Vladimir:
Base case $n = 1$ holds because both sides are 1/2 .
Inductive step: $P(n+1) = \sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = 1 - \frac{1}{n+1}$.

Helen:
Base case: $P(1)$ is true because both sides are 1/2.
Inductive step: suppose $P(n)$ holds for some n. By adding $\frac{1}{(n+1)(n+2)}$ to both sides, we get
$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} = 1 - \frac{1}{n+1}\left(1 - \frac{1}{n+2}\right) = 1 - \frac{1}{n+1}\left(\frac{n+2}{n+2} - \frac{1}{n+2}\right) = 1 - \frac{1}{n+1}\frac{n+1}{n+2} = 1 - \frac{1}{n+2}$. This is the statement $P(n+1)$. That completes the proof by induction.

Jessica:
Base case: $P(1)$ is true because both sides are 1/2.
Inductive step: suppose $P(n)$ holds for all n. By adding 1 to both sides, we get
$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} + 1 = 1 - \frac{1}{n+2}$. This is the statement $P(n+1)$. That completes the proof by induction.

Paul:
The partial fraction decomposition of $\frac{1}{k(k+1)}$ is $\frac{1}{k} - \frac{1}{k+1}$. With that, we see that $P(n)$ is a telescoping sum:
$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \ldots + \frac{1}{n} - \frac{1}{n+1}$. All terms cancel except the first and the last. Therefore, $\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} = P(n+1)$.

Duncan:
Base case: $P(1)$ is true because both sides are 1/2.
Inductive step: suppose $P(n)$ holds for all n. By adding $\frac{1}{(n+1)(n+2)}$ to both sides, we get
$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} = 1 - \frac{1}{n+1}\left(1 - \frac{1}{n+2}\right) = 1 - \frac{1}{n+1}\left(\frac{n+2}{n+2} - \frac{1}{n+2}\right) = 1 - \frac{1}{n+1}\frac{n+1}{n+2} = 1 - \frac{1}{n+2}$. We have shown $P(n) \rightarrow P(n+1)$ for all n. That completes the proof by induction.

- Leto
- Vladimir
- Helen
- Jessica
- Paul
- Duncan