Mathematical Induction

Use induction to prove that

$\sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \frac{n}{2} + 1$ **for all positive integers** n.

Let $P(n)$ denote the proposition $\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \frac{\mathbf{1}}{\boldsymbol{i}} = 1 + \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} + \cdots + \frac{\mathbf{1}}{\boldsymbol{n}} < \frac{\boldsymbol{n}}{\mathbf{2}} + \mathbf{1}$, where n is a positive integer.

BASIS STEP: $P(1)$ is true since 1$< \frac{\mathbf{1}}{\mathbf{2}}$+1 and $1 < 1.5$.

INDUCTIVE STEP: Let us assume $P(n)$, that is $\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \frac{\mathbf{1}}{\boldsymbol{i}}$=$1 + \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} + \cdots + \frac{\mathbf{1}}{\boldsymbol{n}} < \frac{\boldsymbol{n}}{\mathbf{2}} + 1$ is true for an arbitrary positive integer n. This is our inductive hypothesis.

We have to show that $P(n+1)$, $\sum_{i=1}^{n+1} \frac{1}{i}$=$1 + \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} + \cdots + \frac{\mathbf{1}}{\boldsymbol{n}} + \frac{\mathbf{1}}{\boldsymbol{n+1}} < \frac{\boldsymbol{n+1}}{\mathbf{2}} + 1$ is true assuming the inductive hypothesis $P(n)$.

Proof:

$\sum_{i=1}^{n+1} \frac{1}{i} = \sum_{i=1}^{n} \frac{1}{i} + \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} < \frac{n}{2} + 1 + \frac{1}{n+1}$ using the inductive hypothesis.

Now we have to show that $\frac{n}{2} + 1 + \frac{1}{n+1} \leq \frac{n+1}{2} + 1.$

Equivalently, $\frac{n}{2} + \frac{1}{n+1} \leq \frac{n+1}{2}.$

Equivalently, $\frac{1}{n+1} \leq \frac{n+1}{2} - \frac{n}{2}.$

Equivalently, $\frac{1}{n+1} \leq \frac{1}{2}.$

Equivalently, $2 \leq n + 1$ is true for all positive integers n.

By the **Principle of Mathematical Induction** (Basis Step and Inductive Step together) $\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \frac{\mathbf{1}}{\boldsymbol{i}}$ $=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}<\frac{n}{2}+1$ for all positive integers n.