MAT 243 NOTES

1. Quantified Statements and Short Proofs

When we write justifications or short proofs for quantified statements, the quantifiers involved will tell us what to do. To prove that a statement is TRUE simply follow the quantifiers involved.

> To prove that $\forall x P(x)$ is TRUE :
>
> first assume that x is an arbitrary element of the universe of discourse and show that x satisfies $P(x)$ or go through all possible x values in the universe of discourse and show x satisfies $P(x)$.
>
> To prove that $\exists x P(x)$ is TRUE
>
> first give an example and then show that the example satisfies $P(x)$.

To prove that a statement is FALSE, first find the negation and then follow those quantifiers involved, and show that the statement is NOT satisfied.

> To prove that $\forall x P(x)$ is FALSE:
>
> $\forall x P(x)$ is false if $\exists x \neg P(x)$ is true, so first give a counter example and then show that it does not satisfy $P(x)$.
>
> To prove that $\exists x P(x)$ is FALSE:
>
> $\exists x P(x)$ is false if $\forall x \neg P(x)$ is true, so first assume that x is an arbitrary element of the universe of discourse and then show that x does not satisfy $P(x)$.

Example 1 : Prove or Disprove the following mathematical statements:

a) For all real number x, $(-x)^2 = x^2$.
Proof: Assume x is an arbitrary real number. Then $(-x)^2 = (-x)(-x) = (-1)(-1)(x)(x) = (x)(x) = x^2$ by the commutative and associative property of real numbers.

b) There exists a real number x, such that $x = x^2$.
Proof: Let $x = 1$. Then $x = 1 = (1)^2 = x^2$

c) For all real number x, $x^2 \geq x$.
Disprove: Let $x = \frac{1}{2}$. Then $x^2 = (\frac{1}{2})^2 = \frac{1}{4} < \frac{1}{2} = x$.

d) There exists an integer x, such that $2x = 1$.
Disprove: Assume x is an arbitrary integer. Then $2x$ is an even number for all x, thus it is not equal to 1 which is not an even number. Thus, $2x \neq 1$ for all integer x.

Example 2 : Prove or Disprove the following mathematical statements. Assume that the universe of discourse is the set of all real numbers.

(1) For all real number x, $6x > x$.
Disprove: Let $x = -1$. Then, $6x = 6(-1) = -6 < -1 = x$

(2) The square of every negative real number is positive.
Proof: Assume x is an arbitrary negative real number. Since the product of any two negative number is positive $x^2 = (x)(x) > 0$.

(3) There exists a non-negative real number whose square is not positive.
Proof: Let $x = 0$. Clearly x is non-negative and $x^2 = 0^2 = 0$ is not positive.

(4) There exist a real number whose square is -1.
Disprove: Assume x is an arbitrary real number. Since the square of any real number is non-negative, $x^2 = (x)(x) \geq 0 \neq -1$. Thus, $x^2 \neq -1$. for all real number x

When we write short proofs for statements with nested quantifiers, statements that are TRUE simply follow the quantifiers involved similarly to the case when only one quantifier was used. When we disprove a statement that is FALSE, first find the negation and then follow those quantifiers involved, and show that the statement is NOT satisfied.

Example 3 : Prove or Disprove the following mathematical statements:

(1) $x + y = 6$ for some real number x and some real number y.
Proof: Let $x = 1$ and $y = 5$. Then $x + y = 1 + 5 = 6$.

(2) Every real number has an additive inverse.
Proof: Assume x is an arbitrary real number and let $y = -x$. Then $x+y = x+(-x) = 0$.

(3) There is a real number such that adding it to any other real number we get the other real number as the sum.
Proof: Let $x = 0$ and assume y is an arbitrary real number. Then $x + y = 0 + y = y$.

(4) For all real numbers x and y, $x + y = y + x$.
Proof: Assume x and y are arbitrary real numbers. By the commutative property of real numbers, $x + y = y + x$.

(5) For every real number x there is an integer n such that $n < x < n + 1$.
Disprove: Let $x = 0$ and assume n is an arbitrary real number. Then for all integers n $0 \leq n$ or $0 \geq n + 1$ which is equivalent to $-1 \geq n$.

(6) There is an integer N that is greater than any real number x.
Disprove: Assume N is an arbitrary integer and let $x = N + 1$. Then $N < N + 1 = x$.

(7) For all integers m and n there is an integer t such that $t = \frac{m+n}{2}$.
Disprove: Let $m = 1$ and $n = 2$ and t be an arbitrary integer. Then $\frac{m+n}{2} = \frac{1+2}{2} = \frac{3}{2}$ which is not equal to any integer, thus $\frac{m+n}{2} \neq t$.