MAT 243 ADDITIONAL PRACTICE UNIT 5

Exercise #1. Prove or disprove, assuming that x, y and z are real numbers.

(1) $\exists x$ such that $\forall y$ and $\forall z$, $x + y = z$.
(2) $\exists x$ such that $\forall y, \exists z$ $(x + y = z)$.
(3) $\forall y$ and $\forall z$, $\exists x$ such that $x + y = z$.
(4) $\forall x$ and $\forall y$, $\exists z$ such that $xz = y$.
(5) $\exists x$ such that $\forall y$ and $\forall z$, $(z > y \to z > x + y)$.
(6) $\forall x, \exists y$ such that $\forall z$, $(z > y \to z > x + y)$.

Exercise #2. Prove that n is even if and only if $7n + 4$ is even, where the domain is all integers.

Exercise #3. For each of the conditional statements below, state what needs to be assumed and what needs to be shown to prove the statment (i) using direct proof, (ii) using proof by contraposition, and (iii) using proof by contradiction.

- If integers x or y is even, then xy is even.
- If x and y are odd, xy is odd.

Exercise #1 solutions. Prove or disprove, assuming that x, y and z are real numbers.

(1) $\exists x$ such that $\forall y, \exists z$ such that $x + y = z$.
Proof: Let $x = 0$. Assume y is arbitrary. Let $z = y$. Then $x + y = 0 + y = y = z$. $\square$

Note: $x = 0$ and $z = y$ are not the only choices for a proof here. $x = 1$ and $z = y + 1$ could be another good choice. There are many more different choices.

(2) $\exists x$ such that $\forall y$ and $\forall z$, $x + y = z$.
Disproof: We show that $\forall x$, $\exists y$ and $\exists z$ such that $x + y \neq z$. Assume x be arbitrary. Let $y = -x$ and $z = 1$. Then $x + y = 0 \neq 1 = z$. $\square$

(3) $\forall y$ and $\forall z$, $\exists x$ such that $x + y = z$.
Proof: Assume y and z are arbitrary. Let $x = z - y$. Then $x + y = (z - y) + y = z$. $\square$

(4) $\forall x$ and $\forall y$, $\exists z$ such that $xz = y$.
Disproof: We show that $\exists x$ and $\exists y$, $\forall z$ such that $xz \neq y$. Let $x = 0$ and $y = 1$. Assume z is arbitrary. Then $xz = 0 \neq 1 = y$. $\square$

(5) $\exists x$ such that $\forall y$ and $\forall z$, $z > y$ implies that $z > x + y$.
Proof: Pick $x = -1$ and assume y and z are arbitrary. Then $z > y$ implies that $z > y > y - 1 = -1 + y = x + y$. $\square$

(6) $\forall x, \exists y$ such that $\forall z$, $(z > y \to z > x + y)$.
Disproof: We show that $\exists x, \forall y$ such that $\exists z$, $(z > y \wedge z \leq x + y)$. Let $x = 2$. Assume y is arbitrary and let $z = y+1$. Then $z > y \wedge z = y+1 \leq y+2 = y+x = x+y$. $\square$

Exercise #2 solutions. Prove that n is even if and only if $7n+4$ is even, where the domain is all positive integers.

- proof : $\to$
  Assume n is even. Then there exists an integer k such that $n = 2k$. Then
  7n+4=7(2k)+4=14k+4=2(7k+2). Since k is an integer, $m = 7k + 2$ is also an integer. Thus $7n + 4 = 2m$ for some integer m. Thus, by definition of even, we conclude that $7n + 4$ is even.
- proof : $\leftarrow$
  We will use proof by contraposition so we assume that n is odd. Hence, there exists some integer k such that $n = 2k + 1$. Then $7n + 4 = 7(2k + 1) + 4 = 14k + 7 + 4 = 14k + 10 + 1 = 2(7k + 5) + 1$. Since k is an integer, $m = 7k + 5$ is an integer. Thus,

$7n + 4 = 2m + 1$ Therefore, by definition of odd, we conclude that $7n + 4$ is odd. Thus we proved that if n is odd then $7n + 4$ is odd, which is the contrapositive of if $7n + 4$ is even then n is even. □

Exercise #3 solutions.

(1) Direct proof: Assume that x or y is even. Need to show that xy is even.
Indirect proof: Assume that xy is an odd integer. Need to show that x and y are odd integer.
Contradiction: Assume that x or y is even integers and that xy is odd. Find a contradiction.

(2) Direct proof: Assume that x and y are odd integers. Need to show that xy is odd.
Indirect proof: Assume that xy is an even integer. Need to show that x or y is an even integer.
Contradiction: Assume that x and y are odd integers and xy is an even integer. Find a contradiction.