List of Concepts for Chapter 6

Section 6.1

Let A be an $n\times n$ matrix.

- A scalar λ is said to be an **eigenvalue** of A if there exists a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is said to be an **eigenvector** belonging to λ.
- λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$. If we expand $\det(A - \lambda \mathbf{I})$ we obtain a polynomial $p(\lambda)$ called the characteristic polynomial of A. The equation $\det(A - \lambda \mathbf{I}) = 0$ is called the characteristic equation of A, thus to find the eigenvalues of A we need to solve the characteristic equation and to find the eigenvectors we need to find a basis for the nullspace of $A - \lambda \mathbf{I}$.
- Complex e-values: If $\lambda = a + ib$ is an eigenvalue of A, then $\bar{\lambda}=a-ib$ is also an eigenvalue. Also the eigenvectors occur in conjugate pairs.
- The product of the eigenvalues equals the determinant of A: $\lambda_1\cdot\lambda_2\cdots\lambda_n=\det(A)$
- The sum of the eigenvalues is equal to the sum of the diagonal elements of A: $\sum_{i=1}^{n}\lambda_i=\sum_{i=1}^{n}a_{ii}=\operatorname{tr}(A)$
- Theorem: Similar matrices: B is similar to A if there exists a non singular matrix S such that $B = S^{-1} A S$. Two similar matrices have the same eigenvalues.

Section 6.3:

- Theorem: If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigenvalues of an $n\times n$ matrix, then the corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots \mathbf{x}_k$ are linearly independent.
- An $n\times n$ matrix A is **diagonalizable** if there exists a non singular matrix X and a diagonal matrix D such that $A = X D X^{-1}$. We say that X diagonalizes A.
- Theorem: A is *diagonalizable* if and only if it has n linearly independent eigenvectors. The columns of the matrix X are given by the n linearly independent eigenvectors and the entries of the matrix D are the corresponding eigenvalues (note that some of the eigenvalues may be repeated). Note that X is not unique: we can reorder its columns (the entries of D must then be reordered accordingly) or multiply them by a scalar.
- If A has n distinct eigenvalues, then A is diagonalizable. If the eigenvalues are not distinct, it may or may not be diagonalizable depending on whether A has n linearly independent eigenvectors. If A has fewer than n linearly independent eigenvectors, we say that A is **defective**. A defective matrix is not diagonalizable.
- If A is diagonalizable, it is easy to evaluate powers of A: $A^k = X D^k X^{-1}$.

Section 6.5:

- An $n\times n$ matrix Q is called an **orthogonal** matrix if the the columns of Q are orthonormal. Note that this implies $Q^TQ = I$ and since Q has an inverse it follows that $Q^{-1} = Q^T$.
- Assume A is $m\times n$, then A can be factored as $A=U\Sigma V^T$, where U is an $m\times m$ orthogonal matrix, V is an $n\times n$ orthogonal matrix and Σ is an $m\times n$ matrix with off diagonal entries all 0's and diagonal elements $\sigma_1\geq\sigma_2\geq\ldots\geq\sigma_n\geq 0$. The σi's are called **singular values** of A and the factorization $A=U\Sigma V^T$ is called the **Singular Value Decomposition** of A, or SVD.

- Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ be the columns of U and $\mathbf{v}_1, \mathbf{v}_2, .., \mathbf{v}_n$ be the columns of V. We then have:

$$A = U\Sigma V^T = \sigma_1 \mathbf{u_1 v_1^T} + \sigma_2 \mathbf{u_2 v_2^T} + \cdots + \sigma_n \mathbf{u_n v_n^T}$$

- The singular values are given by $\sigma_i = \sqrt{\lambda_i}$ where λ_i are the eigenvalues of $A^T A$. The columns of V are the corresponding orthonormal eigenvectors. If $\sigma_1, \sigma_2, ..\ \sigma_r \neq 0$, then the columns of U are $\mathbf{u}_i = \frac{1}{\sigma_i} A\, \mathbf{v}_i$ for i = 1, 2, ... , r. The remaining columns, $\mathbf{u}_{r+1}, \mathbf{u}_{r+2}, ..\ \mathbf{u}_m$, are an orthonormal basis of $N(A^T)$.

- Since $A = U\Sigma V^T = \sigma_1 \mathbf{u_1 v_1^T} + \sigma_2 \mathbf{u_2 v_2^T} + \cdots \sigma_n \mathbf{u_n v_n^T}$ and each of the matrices $\mathbf{u}_i \mathbf{v}_i^T$ have rank 1, we have that the <u>rank</u> <u>of A is given by the number of non zero singular values of A</u>.

- If A has rank r then $\mathbf{u}_1, \mathbf{u}_2, ..\ \mathbf{u}_r$ are a basis for the column space of A, and $\mathbf{v}_1, \mathbf{v}_2, ..\ ,\mathbf{v}_r$ are a basis for the row space of A.

- If A has rank n, then $A' = \sigma_1 \mathbf{u_1 v_i^T} + \sigma_2 \mathbf{u_2 v_2^T} + \cdots + \sigma_{n-1} \mathbf{u_{n-1} v_{n-1}^T}$ is the <u>matrix of rank $n-1$ that is closest to A</u> with respect to the Frobenius norm, i.e., $\| A - A' \|_F$ = minimum among all matrices A' of rank n-1.

- If **A** is nonsingular $n \times n$, then A' is singular and $\|A - A'\|_F = \sigma_n$. Thus σ_n may be taken as a measure of how close a square matrix is to be singular.

  In general, $B = \sigma_1 \mathbf{u_1 v_i^T} + \sigma_2 \mathbf{u_2 v_2^T} + \cdots + \sigma_k \mathbf{u_k v_k^T}$ is the matrix of rank k that is closest to A with respect to the Frobenius norm. and $\|A - B\|_F = \sqrt{\sigma_{k+1}^2 + \sigma_{k+2}^2 + \ldots \sigma_n^2}$

- If $A = U\Sigma V^T$, the <u>least squares solution</u> of $A\mathbf{x} = \mathbf{b}$ is given by $\hat{\mathbf{x}} = \frac{(\mathbf{u_1^T b})}{\sigma_1}\mathbf{v_1} + \frac{(\mathbf{u_2^T b})}{\sigma_2}\mathbf{v_2} + \ldots + \frac{(\mathbf{u_n^T b})}{\sigma_n}\mathbf{v_n}$