LIST OF CONCEPTS – CHAPTER 3

Section 3.1 Vector spaces

Definition:

A vector space is a nonempty set, V, of objects called *vectors*, together with rules for adding any two vectors **x** and **y** in V and for multiplying any vector **x** in V by any scalar α in $\mathbb{R}$.
V must be closed under this vector addition and scalar multiplication so that $\mathbf{x} + \mathbf{y}$ and $\alpha\mathbf{x}$ are both in V.
Moreover, the axioms A1-A8 on page 113 must be satisfied for all vectors **x**, **y**, and **z** in V and all scalars α and β in $\mathbb{R}$. (You <u>do not</u> need to memorize the axioms)

Examples of Vector spaces:

- $\mathbb{R}^n$ (the set of all vectors with *n* components) with the usual operations of addition and scalar multiplication is a vector space for each positive integer *n*.
- $\mathbb{R}^{m\times n}$ (the set of all $m\times n$ matrices) with the usual operations of addition and scalar multiplication is a vector space for all positive integers *m* and *n*.
- We denote by P_n the set of all polynomials of degree strictly less than *n*. P_n is a vector space with the usual addition and scalar multiplication of polynomials.
- We denote by $C([a, b])$ the set of all continuous functions defined on the interval $[a, b]$. $C([a, b])$ is a vector space with the usual addition and scalar multiplication of functions.

Section 3.2 Subspaces

1. A subset S of a vector space V is a ***<u>subspace</u>*** of V if and only if it is nonempty and satisfies the two closure properties (i) and (ii) on page 118. A subspace always contains the zero vector. Examples of subspaces in $\mathbb{R}^3$ are planes through the origin (2-dimensional subspaces) and lines through the origin (1-dimensional subspaces).

2. Let $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_k}$ be vectors in a vector space V. The set of all linear combinations of these vectors is a subspace of V called the ***span*** of the vectors and denoted $\text{span}(\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_k})$. It is the smallest subspace of V containing all the vectors $\mathbf{v}_i$. To determine if a vector $\mathbf{v}$ is in $\text{span}(\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_k})$, one needs to set up the system $\mathbf{v} = x_1\mathbf{v_1} + x_2\mathbf{v_2} + \ldots + x_k\mathbf{v_k}$ and check whether the system is consistent or inconsistent.

3. The **nullspace**, N(**A**), of an $m\times n$ matrix **A** is the subspace of $\mathbb{R}^n$ consisting of all solutions of $\mathbf{Ax} = \mathbf{0}$.
   - To find a basis for N(**A**):
     1. Reduce the matrix **A** to RREF
     2. Assign parameters to the free variables
     3. Write the solution as a linear combination of vectors (with coefficients the parameters from ii.)
     4. The vectors in the linear combination will be a basis for N(**A**).
   - If you are given a vector **v** and you need to determine whether this vector is in N(**A**), all you have to do is evaluate the product **Av**. If this product is the zero vector, then **v** is in N(**A**).

4. A set of vectors is a **spanning set** for the vector space V if their linear combinations fill the space.

Section 3.3: Linear Independence

1. Linear independence is a generalization of the idea that a set of vectors are collinear, coplanar, etc. (see Figs. on pg 130, 131) For example, the vectors $[1, 1]^T$ and $[2, 2]^T$ are multiples of each other and are *linearly dependent* (they are also collinear). The vectors $[1, 0, 0]^T$, $[1, 1, 0]^T$ and $[0, 1, 0]^T$ are also linearly dependent since they all lie in the *xy* plane. It becomes difficult to visualize the notion of collinearity and coplanarity in more than 3 dimensions. Instead, a simpler analytical definition is given:

Let V be a vector space:

Definition: A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ in V is linearly dependent if there exists a dependence relation

(1) $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$ at least one $c_i \neq 0$.

The set is linearly independent if no such dependence relation exists, i.e. equation (1) is satisfied only if $c_1 = c_2 = \ldots = c_n = 0$.

To check if a set of vectors is linearly independent or linearly dependent one can proceed in different ways. First note that equation (1) can be written as $\mathbf{V}\mathbf{c} = \mathbf{0}$ where $\mathbf{V}$ is the matrix whose columns are the vectors $\mathbf{v}_i$ and $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T$

a) One way that works all the time is to set up a system as in Example 1, page 129, Example 3, page 131 or Example 5, page 133, and check whether the zero vector is the only solution or not. If it is, then the vectors are linearly independent.

b) Another way that works only when we have *n* vectors in $\mathbb{R}^n$ is to check whether the determinant of the matrix $\mathbf{V}$ is zero or not. If it is nonzero then the vectors are linearly independent. (See Example 4, page 132).

c) Also remember that any set of more than *n* vectors in $\mathbb{R}^n$ is linearly dependent.

2. A finite list of nonzero vectors in V forms a linearly independent set if and only if no vectors in the list can be expressed as a linear combination of its predecessors.

3. The following statements are equivalent for *n* vectors in $\mathbb{R}^n$
   a) The vectors are linearly independent.
   b) The vectors span $\mathbb{R}^n$
   c) A matrix having the vectors as columns is nonsingular.

4. The definition of dependence and independence can be used to test whether a set of vectors in P_n are independent (see Example 6, page 134). Elements in the vector space of matrices $\mathbb{R}^{m\times n}$ can be also tested for dependence or independence using the definition. Alternatively, we can rewrite the polynomials and the matrices as vectors (as learned in Section 3.4 and 3.5) and then test for dependence or independence of these vectors.

Section 3.4: Basis and Dimension

Let V be a vector space:

1. A set of vectors is a basis for V if it is an independent set of vectors and V is spanned by the set.
2. If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V, each vector in V can be expressed in the form

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$$

   for unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$.
3. Any finite spanning set for V can be reduced (if necessary) to a basis for V by deleting zero vectors, and then casting out those that are linear combinations of the predecessors. See Section 3.6, item 5 below, for a casting out technique in $\mathbb{R}^n$ that can be executed efficiently.
4. All bases of V have the same number of vectors. This number, dim(V), is called the *dimension* of V.
5. Given *n* vectors in $\mathbb{R}^n$ these vectors are a basis for $\mathbb{R}^n$ if and only if the vectors are linearly independent.
6. See the homework and lecture notes for examples of basis of subspaces in $\mathbb{R}^{m\times n}$ and P_n.

Section 3.5: Change of Basis

Let V be a vector space with the ordered basis $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

1. Each vector $\mathbf{x}$ in V has a unique representation as linear combination $\mathbf{x} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$

The vector $[\mathbf{x}]_{\mathcal{V}} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$, for the uniquely determined scalars α_i in the previous equation, is the **coordinate vector** of **x** relative to the basis $\mathcal{V}$. The vector **x** can be written as

$$\mathbf{x} = \mathbf{V}[\mathbf{x}]_{\mathcal{V}}$$

where **V** is the matrix whose columns are the basis vectors. **V** is called the **transition matrix** from the basis $\mathcal{V}$ to the standard basis.

2. Let $\mathcal{V}$ be the basis given in item 1. and $\mathcal{U} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]$ be another ordered basis of $\mathbb{R}^n$. The important equation that relates these basis and the corresponding coordinate vectors is

$$\mathbf{U}[\mathbf{x}]_{\mathcal{U}} = \mathbf{V}\,[\mathbf{x}]_{\mathcal{V}}$$

where **U** is the transition matrix from $\mathcal{U}$ to the standard and basis (the matrix whose columns are the vectors $\mathbf{u}_i$) and **V** is the transition matrix from $\mathcal{V}$ to the standard basis (the matrix whose columns are the vectors $\mathbf{v}_i$).
Solving for $[\mathbf{x}]_{\mathcal{U}}$ gives $[\mathbf{x}]_{\mathcal{U}} = \mathbf{U}^{-1}\,\mathbf{V}\,[\mathbf{x}]_{\mathcal{V}}$ and $\mathbf{S} = \mathbf{U}^{-1}\,\mathbf{V}$ is called the <u>transition matrix from the basis $\mathcal{V}$ to the basis $\mathcal{U}$</u>.
Similarly, solving for $[\mathbf{x}]_{\mathcal{V}}$ gives $[\mathbf{x}]_{\mathcal{V}} = \mathbf{V}^{-1}\,\mathbf{U}\,[\mathbf{x}]_{\mathcal{U}}$ and $\mathbf{T} = \mathbf{V}^{-1}\,\mathbf{U}$ is called the <u>transition matrix from the basis $\mathcal{U}$ to the basis $\mathcal{V}$</u>. (See Examples 4 and 5, page 149-150).

3. See the lecture notes and the homework for examples of coordinate vectors in $\mathbb{R}^{m\times n}$ and in P_n.

Section 3.6: Row space and Column Space

1. Let **A** be an $m\times n$ matrix. The *row space* of **A**, $R(\mathbf{A}^T)$, is the subspace of $\mathbb{R}^n$ spanned by the row vectors of **A**, while the *column space,* $R(\mathbf{A})$, is the subspace of $\mathbb{R}^m$ spanned by the column vectors of A.

2. To determine a basis for the row space of an $m\times n$ matrix **A** proceed as follows:
   a) Reduce **A** to echelon form **U**.
   b) The nonzero rows of **U** constitute a basis for the row space of **A**.

3. To determine a basis for the column space of **A**, R(**A**), proceed as follows:
   a) Reduce **A** to echelon form **U**.
   b) <u>The columns of **A**</u> corresponding to the columns of **U** containing the pivots form a basis for R(**A**).

4. The system **Ax** = **b** has a solution if and only if **b** lies in the column space of **A**. (This follows from the fact that **Ax** = **b** is equivalent to $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = \mathbf{b}$.)

5. Given a set of *n* vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, we can determine a basis for $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n)$. (i.e. we can eliminate nonzero vectors that are linear combinations of the predecessors) by proceeding as follows:
   (i) Form the matrix **A**, with columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
   (ii) Reduce **A** to row echelon form
   (iii) Determine which columns in the RREF contain the pivots. These same columns in **A** will form a basis for the column space of **A** and therefore a basis for $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n)$.

6. The dimension of the row space (which is the same as the dimension of the column space) of an $m\times n$ matrix **A** is called the *rank* of A. The dimension of the nullspace of **A** is called the *nullity* of **A**.

7. Rank-Nullity Theorem: The *nullity* of **A** plus the *rank* of **A** equals the number of columns of **A.**

8. An $n\times n$ matrix **A** is invertible if and only if rank(**A**) = *n*.