Mathematical Induction

Use induction to prove that

$$\sum_{i=1}^{n}(2i-1)^2 = 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

for all positive integers n.

Let P(n) denote the proposition

$\sum_{i=1}^{n}(2i-1)^2 = 1^2+3^2+5^2+\cdots+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$, where n is a positive integer.

BASIS STEP: P(1) is true since $\sum_{i=1}^{1}(2i-1)^2 = 1^2 = 1$ and $1 = \frac{1(2-1)(2+1)}{3}$

INDUCTIVE STEP: Let us assume P(n), that is

$$\sum_{i=1}^{n}(2i-1)^2 = 1^2+3^2+5^2+\cdots+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

is true for an arbitrary positive integer n . This is our inductive hypothesis.

We have to show the statement P(n+1),

$$\sum_{i=1}^{n+1}(2i-1)^2 = 1^2+3^2+5^2+\cdots+(2n-1)^2+(2(n+1)-1)^2$$

$$= \frac{(n+1)(2(n+1)-1)(2(n+1)+1)}{3} = \frac{(n+1)(2n+1)(2n+3)}{3}$$

is true assuming the inductive hypothesis P(n). Note that $(2(n+1)-1)^2 = (2n+1)^2$

Proof:

$$\sum_{i=1}^{n+1}(2i-1)^2 = \sum_{i=1}^{n}(2i-1)^2 + (2n+1)^2 = 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 + (2n+1)^2 =$$

$\frac{n(2n-1)(2n+1)}{3} + (2n+1)^2$ using the inductive hypothesis.

Now we have to show that

$$\frac{n(2n-1)(2n+1)}{3} + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

$$\frac{n(2n-1)(2n+1)}{3} + (2n+1)^2 = \frac{n(2n-1)(2n+1)+3(2n+1)^2}{3} =$$

$$\frac{(2n+1)(n(2n-1)+3(2n+1))}{3} = \frac{(2n+1)(2n^2+5n+3)}{3}$$

$$= \frac{(2n+1)(n+1)(2n+3)}{3} = \frac{(n+1)(2n+1)(2n+3)}{3}$$

By the Principle of Mathematical Induction (Basis Step and Inductive Step together)

$\sum_{i=1}^{n}(2i-1)^2 = 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ for all positive integers n.