Mathematical Induction

Use induction to prove that

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n.

Let P(n) denote the proposition $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$, where n is a positive integer.

BASIS STEP: P(1) is true since $\sum_{i=1}^{1} i^2 = 1^2 = 1$ and $\frac{1(1+1)(2+1)}{6} = 1$

INDUCTIVE STEP:

Let us assume P(n), that is $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ is true for an arbitrary positive integer n . This is our inductive hypothesis.

We have to show the statement P(n+1),
$\sum_{i=1}^{n+1} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$ is true assuming the inductive hypothesis P(n).

Proof:

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

using the inductive hypothesis.

Now we have to show that

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)((n+2)(2n+3)}{6}$$

Proof:

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} = \frac{n(n+1)(2n+1)+6(n+1)^2}{6} = \frac{(n+1)\big(n(2n+1)+6(n+1)\big)}{6} = \frac{(n+1)\big(2n^2+7n+6\big)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

By the Principle of Mathematical Induction (Basis Step and Inductive Step together)

$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all positive integers n.