LU Factorization Example

EXAMPLE. Find an LU factorization of $A = \begin{bmatrix} -1 & 1 & 2 & -2 \\ 3 & -3 & -6 & 12 \\ -2 & 4 & 3 & 2 \\ -5 & 1 & 17 & -24 \end{bmatrix}$.

SOLUTION. We proceed to factor A as

$$\begin{bmatrix} 1&0&0&0 \\ 3&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 & -2 \\ 3 & -3 & -6 & 12 \\ -2 & 4 & 3 & 2 \\ -5 & 1 & 17 & -24 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 & -2 \\ 0 & 0 & 0 & 6 \\ -2 & 4 & 3 & 2 \\ -5 & 1 & 17 & -24 \end{bmatrix}$$

$$\begin{bmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix} \begin{bmatrix} 1&0&0&0 \\ 3&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 & -2 \\ 3 & -3 & -6 & 12 \\ -2 & 4 & 3 & 2 \\ -5 & 1 & 17 & -24 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 & -2 \\ -5 & 1 & 17 & -24 \\ -2 & 4 & 3 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

In full this is

$$\begin{bmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix} \begin{bmatrix} 1&0&0&0 \\ 3&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 & -2 \\ 3 & -3 & -6 & 12 \\ -2 & 4 & 3 & 2 \\ -5 & 1 & 17 & -24 \end{bmatrix} = \begin{bmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 & -2 \\ 0 & 0 & 0 & 6 \\ -2 & 4 & 3 & 2 \\ -5 & 1 & 17 & -24 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 & -2 \\ -5 & 1 & 17 & -24 \\ -2 & 4 & 3 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Then (with no intermediate steps) we get

$$\begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&1/2&1&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ -2&0&1&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0&0 \\ -5&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix} \begin{bmatrix} 1&0&0&0 \\ 3&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 & -2 \\ 3 & -3 & -6 & 12 \\ -2 & 4 & 3 & 2 \\ -5 & 1 & 17 & -24 \end{bmatrix}.$$

$$= \begin{bmatrix} -1 & 1 & 2 & -2 \\ 0 & -4 & 7 & -14 \\ 0 & 0 & 5/2 & -1 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Solving Matrix Equations with LU Factorizations For Computers

LU factorizations are used to compute solutions of matrix equations of the form $A\mathbf{x} = \mathbf{y}$ where y is known and $\mathbf{x}$ is unknown. Here A needs to have an inverse; otherwise, there may not be a unique solution.

Suppose that we have an LU factorization like the one we just computed. Here there are five matrices $E_1, \dots, E_5$ used in the factorization:

$$E_5E_4E_3E_2E_1A = U.$$

This is the last line on the preceding page with

$$E_1 = \begin{bmatrix} 1&0&0&0 \\ 3&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}, E_2 = \begin{bmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 1&0&0&0 \end{bmatrix}, \dots, E_5 = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&1/2&1&0 \\ 0&0&0&1 \end{bmatrix}$$

etc. Since all the elmentary matrices $E_1, \dots, E_5$ are invertible, the matrix U must be invertible (since U is the product of invertible matrices and thus invertible). So we have that

$$E_5E_4E_3E_2E_1\mathbf{y} = E_5E_4E_3E_2E_1A\mathbf{x} = U\mathbf{x}.$$

Here $A\mathbf{x} = \mathbf{y}$ is known. For the sake of illustration, suppose we want to solve

$$\begin{bmatrix} -1 & 1 & 2 & -2 \\ 3 & -3 & -6 & 12 \\ -2 & 4 & 3 & 2 \\ -5 & 1 & 17 & -24 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix}.$$

Here $A\mathbf{x} = (1, -1, 3, 2)^T$ and the column vector $\mathbf{x}$ is unknown.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} = Ux.$$

We then run throuugh the calculations as transformations on the vector

$$\begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} \xrightarrow[\text{to R2}]{\text{add } 3\times\text{R1}} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \xrightarrow[\text{\& R2}]{\text{inter R4}} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \xrightarrow[+\text{R2}]{-5\times\text{R1}} \begin{bmatrix} 1 \\ -3 \\ 3 \\ 2 \end{bmatrix} \xrightarrow[+\text{R3}]{-2\times\text{R1}} \begin{bmatrix} 1 \\ -3 \\ 1 \\ 2 \end{bmatrix} \xrightarrow[\text{to R3}]{\text{add } 1/2 \text{ R2}} \begin{bmatrix} 1 \\ -3 \\ -1/2 \\ 2 \end{bmatrix}..$$

So now we must solve

$$\begin{bmatrix} 1 \\ -3 \\ -1/2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 & -2 \\ 0 & -4 & 7 & -14 \\ 0 & 0 & 5/2 & -1 \\ 0 & 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

This system can be solved by the usual backward substitution but we can factor U into $\eta-$ matrices (eta–matrices).

Eta Matrices

We can perform an additional factorization on the upper triangular matrix U instead of using backward substitution. Each of the factors in U are eta matrices whose inverse is easy to compute.

DEFINITION. An $n \times n$ matrix is said to be an <u>eta matrix</u> if all its columns are identical with the corresponding columns of the identity matrix except for perhaps one column.

The matrix

$$E_2(v) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an eta matrix. Here the subscript 2 on E indicates the column that is not the identity and the column 2 is the vector

$$v = \begin{bmatrix} 0 \\ 1 \\ 1/2 \\ 0 \end{bmatrix}.$$

In addition to being easy to invert, an eta matrix is easy to store in computer memory. For example, the matrix $E_2(v)$ is stored as (2,0,1,1/2,0) which consists of 5 terms instead of the 16 terms ordinarily needed by a 4×4 matrix.

THEOREM. An eta matrix $E_k(v)$ is invertible if and only if v_k (the kth component of v) is nonzero and the inverse is

$$E_k(v)^{-1} = E_k(v')$$

where

$$v' = (-\,v_1/v_k,\ \dots\ ,\ -\,v_{k-1}/v_k,\ 1/v_k,\ -\,v_{k+1}/v_k,\ \dots\ ,\ -\,v_n/v_k)^T.$$

PROOF. Direct computation. Q.E.D.

EXAMPLE. For the previous eta matrix, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/1 & 0 & 0 \\ 0 & (1/2)/(-1) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now we find a factorization of the U in the product of eta matrices. This is easy to do since it can be done by inspection as a sort of separation of the columns, viz.,

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 & -2 \\ 0 & -4 & 7 & -14 \\ 0 & 0 & 5/2 & -1 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

So now we must solve the equation

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = Ux = \begin{bmatrix} 1 \\ -3 \\ -1/2 \\ 2 \end{bmatrix}.$$

We do this by running through the eta list as follows:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

so that

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -1/2 \\ 2 \end{bmatrix}$$

which gives by backward substitution

$$\begin{aligned} &z_4 = 1/3, && \\ &z_3 - z_4 = (-1/2) && \Rightarrow z_3 = (-1/6), \\ &z_2 - 14z_4 = -3 && \Rightarrow z_2 = -3 + 14z_4 = 5/3, \\ &z_1 - 2z_4 = 1 && \Rightarrow z_1 = 5/3. \end{aligned}$$

Repeating this, we get

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

with

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 5/3 \\ -1/6 \\ 1/3 \end{bmatrix}$$

so that

$$\begin{aligned} w_4 &= 1/3, \\ w_3 = (2/5)(-1/6) &= -1/15, \\ w_2 = (5/3) + (7/15) &= 32/15, \\ w_1 = (5/3) - 2w_3 = (5/3) + (2/15) &= 27/15 = 9/5. \end{aligned}$$

Now, we get

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 9/5 \\ 32/15 \\ -1/15 \\ 1/3 \end{bmatrix}.$$

The first two t's we can write down immediately as

$$\begin{aligned} t_4 &= 1/3 \\ t_3 &= -1/15, \end{aligned}$$

and the other t's are easy to obtain as

$$\begin{aligned} t_2 = (-1/4)(32/15) &= -8/15, \\ t_1 = (9/5) - (-8/15) &= 35/15 = 7/3. \end{aligned}$$

Finally, we get that

$$x_4 = 1/3,\ x_3 = -1/15,\ x_2 = -8/15,\ x_1 = -7/3$$

since we do not have to write down the last backward substitution. This is the solution of the problem.

.

Application of LU Factorization

We have already obtained an LU factorization of

$$\begin{bmatrix} -1 & 1 & 2 & -2 \\ 3 & -3 & -6 & 12 \\ -2 & 4 & 3 & 2 \\ -5 & 1 & 17 & -24 \end{bmatrix}.$$

Suppose that we wish to factor the matrix

$$B = \begin{bmatrix} -1 & 2 & -2 & 1 \\ 3 & -6 & 12 & -1 \\ -2 & 3 & 2 & 3 \\ -5 & 17 & -24 & 2 \end{bmatrix}$$

Here the second column of the original matrix has been deleted and the third and fourth columns have been moved forward and a new column placed in the last column. To find an LU factorization of this matrix we just need to update the previous factorization of the original matrix found on page 1 Here we use the following lemma.

LEMMA. Let A and B be matrices. Let the columns of B be $\mathbf{b_1}, \ldots, \mathbf{b_p}$ respectively. Then the columns of the product AB are $A\mathbf{b_1}, \ldots, A\mathbf{b_p}$ respectively.

Using this, we have that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & -2 & 1 \\ 3 & -6 & 12 & -1 \\ -2 & 3 & 2 & 3 \\ -5 & 17 & -24 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 & 1 \\ 0 & 7 & -14 & -3 \\ 0 & 5/2 & -1 & -1/2 \\ 0 & 0 & 6 & 2 \end{bmatrix} = V.$$

In terms of the Lemma, the A is the matrix product $E_5E_4 \ldots E_1$ and the B is the matrix given on page 6 and the matrix AB is V. In terms of the Lemma, the columns 1, 2, and 3 of V are exactly the columns 1, 3, and 4 of U respectively. The last column of V is exactly

$$E_5E_4 \ldots E_1(1, -1, 3, 2)^T.$$

We chose the last column of B in the way we did because we already calculated $E_5E_4 \ldots E_1(1, -1, 3, 2)^T$ on page 2.

So now the update consists of bringing the right hand side above to upper triangular form by the elementary operations (which must be lower triangular matrices or permutations) as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5/14 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & -2 & 1 \\ 0 & 7 & -14 & -3 \\ 0 & 5/2 & -1 & -1/2 \\ 0 & 0 & 6 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 & 1 \\ 0 & 7 & -14 & -3 \\ 0 & 0 & 4 & 4/7 \\ 0 & 0 & 6 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3/2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & -2 & 1 \\ 0 & 7 & -14 & -3 \\ 0 & 0 & 4 & 4/7 \\ 0 & 0 & 6 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 & 1 \\ 0 & 7 & -14 & -3 \\ 0 & 0 & 4 & 4/7 \\ 0 & 0 & 0 & 8/7 \end{bmatrix}$$

and the complete factorization is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5/14 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & -2 & 1 \\ 3 & -6 & 12 & -1 \\ -2 & 3 & 2 & 3 \\ -5 & 17 & -24 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 & 1 \\ 0 & 7 & -14 & -3 \\ 0 & 0 & 4 & 4/7 \\ 0 & 0 & 0 & 8/7 \end{bmatrix}$$

We see the new factorization just strings the two extra factors in front of the factorization that we have already obtained and gets a new upper triangular matrix out of the matrix V that is slightly different form an upper triangular matrix. The fact that V is nearly upper triangular makes the new factors easy to obtain.

Certain problems require computing **x** in A**x** = **y** for given **y** over and over as the matrix A changes by the addition of one new column and the deletion of an existing column. Factorization makes the information that we have previously obtained available for the next factorization. However, it has the disadvantage of stringing more factors than might be necessary. So when the factor file becomes too large, the matrix in question is often refactored.

Homework Exercise

1. Find **x** in A**x** = **y**, where **y** is the the column $(1, 2, 3, 1)^T$ and A is the matrix on page 1. Use the method on page 2 ff. and the factorization on page 1.
2. Find **x** in B**x** = **y**, where **y** is the the column $(0, 2, -2, 0)^T$ and B is the matrix on page 1. Use the method on page 2 ff. and the factorization of B on page 6-7.
3. If an upper triangular matrix U with the final row equal to (0, 0, … ,0) is obtained from the LU factorization of a matrix C (i.e., $E_n E_{n-1} \dots E_1 C = U$), show that C has no inverse.