MAT 343 **Applied Linear Algebra** **/200**

August 8, 2008 **Final Examination** **name**____________

Show your work - explain what you are doing. Scattered formulas without clear logical order or without explanations will be ignored (ZERO CREDIT!) It is YOUR responsibility to demonstrate that you have mastered the material of this class.

1	2	3	4	5	6
/40	/40	/30	/10	/30	/50

1. For the matrix A given on the right,

a. calculate the characteristic polynomial and all eigenvalues,

b. find, if possible, an invertible matrix V and a diagonal matrix D such that $A = VDV^{-1}$.

c. Explain why it is clear *"at first sight"* that the matrix V cannot be orthogonal.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

2. a. Find an orthonormal basis for the column space $C(A)$ of A.

b. Use your answer from **a.** to find the vector p in the column space $C(A)$ that is closest to b.

Bonus. Use a projection onto the left null-space of A to calculate p.

3. a. For the matrix A given below find a basis for each of its four fundamental subspaces.

b. For each of the vectors $v_0, v_1, \ldots v_4$ given below decide in which of the four subspaces it lies. (*Each vector may lie in none, one, or more than one of the spaces – check all possible pairings.*)

c. Whenever one of the vectors lies in one of the subspaces write it as a linear combination of your basis vectors from **a.**

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 1 & 3 & 1 & 7 \\ 2 & 6 & 0 & 4 \end{pmatrix}, \quad v_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, v_1 = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 302 \\ 317 \\ 604 \end{pmatrix}. v_3 = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 15 \end{pmatrix}, v_4 = \begin{pmatrix} -3002 \\ 1000 \\ -5 \\ 1 \end{pmatrix}.$$

4. Prove one of the following:

a. Eigenvectors v, w of a symmetric (real) matrix A corresponding to different eigenvalues $\lambda \neq \mu$ are orthogonal.

b. Eigenvectors v, w of a matrix A corresponding to different eigenvalues $\lambda \neq \mu$ are linearly independent.

5. Suppose A is an $m \times n$ matrix and $b \in \mathrm{R}^m$.

a. Using language of chapters II and III, state two conditions that guarantee that $Ax = b$ has at least one solution.

b. Assuming that $m = n$ state ten, as diverse as possible, conditions that guarantee existence of a unique solution.

6. For each of the following decide whether true or false. Enter **T** or **F** in the box on this page.
For at least three true statements give a brief reason why it is true.
For at least three false statements give an explicit counterexample.

☐ **a.** If A is not square and A^TA is invertible, then AA^T cannot be invertible.

☐ **b.** If $A^TAx = 0$ then $Ax = 0$. (Hint: In which of the fundamental subspaces of A does Ax lie?)

☐ **c.** If all diagonal entries of a matrix A are zero, then A is not invertible.

☐ **d.** The transpose A^T of any square matrix has the same eigenvalues and the same eigenvectors as A.

☐ **e.** If $A = LU$ is the LU-decomposition of A, then $\det A$ is the product of the diagonal entries of U.

☐ **f.** If $A = LU$ is the LU-decomposition of A, then the eigenvalues of A are on the diagonal of U.

☐ **g.** If Q is symmetric and orthogonal, then $Q^2 = I$ and hence Q is diagonal with $\pm$ on the diagonal.

☐ **h.** If u is a unit (column) vector then $H = I - 2uu^T$ is orthogonal.

☐ **i.** If u, v are column vectors then it does not matter in which order one multiplies $(uu^T)v$ or $u(u^Tv)$.

☐ **j.** If it takes 30 seconds to solve a $5,000 \times 5,000$ system, then it will take about an hour to solve a $50,000 \times 50,000$ system (assuming no problems with limited memory).

☐ **Bonus.** For every square matrix A, the matrix e^A is invertible. (*Hint: Find a candidate for the inverse!*)