Structural Induction

Let S be the set defined recursively as follows:

Basis step: $1 \in S$

Recursive step: If $x \in S$ then $5x \in S$

a. List the elements of S produced by the first 2 applications of the recursive definition.

$$S_0 = \{1\},\ S_1 = \{5\},\ S_2 = \{25\}.$$

The elements of S produced by the first 2 applications of the recursive definition is

$S_0 \cup S_1 \cup S_2 = \{1, 5, 25\}.$

b. **Use structural induction** to prove that $S \subseteq \{5^n | n \in \mathrm{N}_0\}$, where N_0 denotes the set of non-negative integers.

When we use structural induction to show that the elements of a recursively defined set S have a certain property then we need to do the following procedure:

1. **Basis step**: show that all the elements defined in the basis step have the desired property.
2. **Inductive step**: assume that an arbitrary element of the set S has the desired property. This is your inductive hypothesis. Using the inductive hypothesis, prove that, when you create more elements in S by using the recursive definition, these newly created elements of S have the same property.
3. **Conclusion:** state that by structural induction all the elements in S have the same property.

To prove that $S \subseteq \{5^n | n \in \mathrm{N}_0\}$, we have to show that, if $x \in S$ then $x = 5^n$ for some $n \in N_0$.

Basis step: $1 \in S$ by the definition of S and $1 = 5^0$.

Recursive Step: Assume $x \in S$ and $x = 5^n$ for some $n \in N_{0.}$ We need to prove that $5x$

has the same property, that is $5x$ can be expressed as a non-negative integer power of 5.

Proof: Using the inductive hypothesis $5 \cdot x = 5 \cdot 5^n = 5^{n+1}$, where n+1 is a positive integer.

By **structural induction** we have proved that, if $x \in S$ then $x = 5^n$ for some $n \in N_0$,
that is $S \subseteq \{5^n | n \in N_0\}$.