Solutions to Assignment 1

Math 217, Fall 2002

1.1.25 Find an equation involving g, h, and k that makes this augmented matrix correspond to a consistent system:

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}.$$

To see if this matrix is consistent, we put it in row reduced form. First label the rows R_1, R_2, and R_3. Then

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$$

$$\updownarrow$$

$$R_3 \to R_3 + 2R_1 \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & k+2g \end{bmatrix}$$

$$\updownarrow$$

$$R_3 \to R_3 + R_2 \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & k+2g+h \end{bmatrix}.$$

We see that this matrix will be consistent if and only if $k + 2g + h = 0$.

1.1.27 Suppose that the system below is consistent for all possible values of f and g. What can you say about the coefficients c and d? Justify your answer.

$$\begin{aligned} x_1 + 3x_2 &= f \\ cx_1 + dx_2 &= g. \end{aligned}$$

This system corresponds to the augmented matrix:

$$\begin{bmatrix} 1 & 3 & f \\ c & d & g \end{bmatrix}$$

which has row echelon form:

$$\begin{bmatrix} 1 & 3 & f \\ 0 & d-3c & g-cf \end{bmatrix}.$$

If $d = 3c$, then this matrix is inconsistent whenever $g - cf \neq 0$ (take $g = 1$, $f = 0$, for instance). Because this matrix is supposed to be consistent for all f and g, we can conclude that $d \neq 3c$.

1.2.20 Choose h and k such that the system below has (a) no solution, (b) a unique solution, and (c) many solutions.

$$\begin{aligned} x_1 + 3x_2 &= 2 \\ 3x_1 + hx_2 &= k. \end{aligned}$$

This system corresponds to the augmented matrix:

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & h & k \end{bmatrix}.$$

which has row echelon form:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{bmatrix}.$$

(a) If $h = 9$ and $k = 7$, then A becomes

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

which is inconsistent. Thus for $h = 9$ and $k = 7$, the system above has no solution. More generally, this system has a no solution whenever $h = 9$ and $k \neq 6$.

(b) If $k = 10$, then A becomes

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & k-6 \end{bmatrix},$$

and because every column contains a pivot row, this gives a unique solution for all choices of k (for instance, $k = 0$ works just fine). Formally, the system given above has a unique solution for $h = 10$, $k = 0$. More generally, this system has a unique solution for all k and h such that $h \neq 9$.

(c) Finally, if $h = 9$ and $k = 6$, then A becomes

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

which is consistent and contains a free column. Thus the system given above has a free variable, and hence, infinitely many solutions.

1.2.28 What would you have to know about the pivot columns in an augmented matrix in order to know that the linear system is consistent and has a unique solution?

To know that a system has a unique solution, you need to know that it is consistent (so no pivot in the last column of the augmented matrix), and that there are no free variables (so a pivot position in each column of the coefficient matrix). Note that this also means there can not be less rows then columns (see theorem 8, pg. 69).

1.3.17 Let $\mathbf{a_1} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $\mathbf{a_2} = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$, and $\mathbf{a_3} = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$. For what values(s) of h is $\mathbf{b}$ in the plane spanned by $\mathbf{a_1}$ and $\mathbf{a_2}$?

The vector $\mathbf{b}$ is in the span of $\mathbf{a_1}$ and $\mathbf{a_2}$ if there is a solution to the linear system of equations

$$\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} x_1 + \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} x_2 = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}.$$

This system corresponds to the augmented matrix:

$$\begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix},$$

which has row echelon form

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 0 & h+17 \end{bmatrix}.$$

This matrix is consistent if and only if $h = -17$. Note that the back of the book contains a typo for this problem (it says $h = 19$).

1.3.24 True/False

(a) Any list of five real numbers is a vector in $\mathbb{R}^5$. **True**. You'll find this statement on pg. 28 of the text. However, I think this is a poorly written problem, so good explanations will be given full credit.

(b) The vector $\mathbf{u}$ results when a vector $\mathbf{u} - \mathbf{v}$ is added to the vector $\mathbf{v}$. **True**. On the table on pg 32 there is a list of certain algebraic properties which hold for vectors. By rule (ii) $(\mathbf{u} - \mathbf{v}) + \mathbf{v} = \mathbf{u} + (-\mathbf{v} + \mathbf{v})$. By rule (iv), $\mathbf{u} + (-\mathbf{v} + \mathbf{v}) = \mathbf{u} + \mathbf{0}$. Finally, by rule (iii), $\mathbf{u} + \mathbf{0} = \mathbf{u}$ as required.

(c) The weights $c_1, \ldots, c_p$ in a linear combination $c_1 v_1 + \cdots + c_p v_p$ cannot all be zero. **False**. Take a look at page 32.

(d) When $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, Span$\{\mathbf{u}, \mathbf{v}\}$ contains the line through $\mathbf{u}$ and the origin. **True**. Take a look at page 35.

(e) Asking whether the linear system corresponding to an augmented matrix $[\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{b}]$ has a solution amounts to asking whether $\mathbf{b}$ is in Span$\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$. **True**. It says this on page 35, just below the definition of Span.

1.4.16 Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Show that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for all possible $\mathbf{b}$, and describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ does have a solution.

We put the augmented matrix

$$\begin{bmatrix} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{bmatrix}$$

in row reduced echelon form.

$$\begin{bmatrix} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{bmatrix}$$

$$\updownarrow$$

$$R_2 \to R_2 + 3R_1, R_3 \to R_3 - 5R_1 \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 14 & 12 & b_3 - 5b_1 \end{bmatrix}$$

$$\updownarrow$$

$$R_3 \to R_3 + 2R_2 \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 0 & 0 & b_3 + b_1 + 2b_2 \end{bmatrix}$$

$$\updownarrow$$

$$R_2 \to R_2/(-7) \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & 1 & 6/7 & b_2/7 + 3b_1/7 \\ 0 & 0 & 0 & b_3 + b_1 + 2b_2 \end{bmatrix}$$

$$\updownarrow$$

$$R_1 \to R_1 + 3R_2 \begin{bmatrix} 1 & 0 & -10/7 & b_1 + 3b_2/7 + 9b_1/7 \\ 0 & 1 & 6/7 & b_2/7 + 3b_1/7 \\ 0 & 0 & 0 & b_3 + b_1 + 2b_2 \end{bmatrix}.$$

We note that if $b_1 + 2b_2 + b_3 \neq 0$ (say, if $b_1 = 1$, and $b_2 = b_3 = 0$), then this matrix is inconsistent, so the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for all $\mathbf{b}$. If, however, $b_1 + 2b_2 + b_3 = 0$ then this matrix is consistent. So we have to describe all vectors $\mathbf{b}$ such that $b_1 + 2b_2 + b_3 = 0$. First solve for b_1 in terms of b_2 and b_3. This is straightforward enough, $b_1 = -2b_2 - b_3$. So if $\mathbf{b} = \begin{bmatrix} -2b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix}$, then $A\mathbf{x} = \mathbf{b}$ has a solution.

1.4.26 Let $\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$. It can be shown that $3\mathbf{u} - 5\mathbf{v} - \mathbf{w} = \mathbf{0}$. Use this fact (and no row operations) to find x_1 and x_2 that satisfy the equation

$$\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}.$$

Rewriting the linear system above in terms of vectors we get $[\mathbf{u}, \mathbf{v}] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{w}$, or, multiplying the left side out, $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{w}$. But we already know that $3\mathbf{u} - 5\mathbf{v} - \mathbf{w} = \mathbf{0}$, or rearranging things, $3\mathbf{u} - 5\mathbf{v} = \mathbf{w}$. This means, of course, that $x_1 = 3$ and $x_2 = -5$ satisfy the linear system above.

1.4.34 Suppose that A is a 3×3 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^3$ with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Explain why the columns of A must span $\mathbb{R}^3$.

By theorem 4, to show that the columns of A span $\mathbb{R}^3$ it is enough to show that A has a pivot in every column. So put the augmented matrix $[A\ \mathbf{b}]$ in row reduced echelon form. Because there is exactly one solution, there must be no free variables. This implies that there are no free columns, so there must be a pivot in every column of the coefficient matrix (which is A). That does it.

1.5.26 Suppose that $A\mathbf{x} = \mathbf{b}$ has a solution. Explain why the solution is unique precisely when $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

First show that if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution then $A\mathbf{x} = \mathbf{b}$ has a unique solution. Suppose there are vectors $\mathbf{u}$ and $\mathbf{v}$ such that $A\mathbf{u} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{b}$. The $A\mathbf{u} - A\mathbf{v} = \mathbf{b} - \mathbf{b} = \mathbf{0}$, so $A(\mathbf{u} - \mathbf{v}) = \mathbf{0}$. Because $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, this means that $\mathbf{u} - \mathbf{v} = \mathbf{0}$, or $\mathbf{u} = \mathbf{v}$. Thus, $A\mathbf{x} = \mathbf{b}$ has a unique solution.

On the other hand, suppose that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Let $\mathbf{u}$ be the unique vector such that $A\mathbf{u} = \mathbf{b}$ and let $\mathbf{v}$ be such that $A\mathbf{v} = \mathbf{0}$. Now $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}$, so $\mathbf{u} + \mathbf{v}$ is a solution to $A\mathbf{x} = \mathbf{b}$. We know that $\mathbf{u}$ is the unique solution such that $A\mathbf{x} = \mathbf{b}$, so we conclude that $\mathbf{u} + \mathbf{v} = \mathbf{u}$, or $\mathbf{v} = 0$. Thus all solutions to $A\mathbf{x} = \mathbf{0}$ turn out to be the trivial solution.

1.5.24 True/False

(a) If $\mathbf{x}$ is a nontrivial solution of $A\mathbf{x} = \mathbf{0}$, then every entry in $\mathbf{x}$ is nonzero.
False. Here is a counter-example. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and let $\mathbf{x} = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Then $\mathbf{x} \neq \mathbf{0}$, $A\mathbf{x} = \mathbf{0}$, and not all the entries of $\mathbf{x}$ are non-zero.

(b) The equation $\mathbf{x} = \mathbf{x}_2\mathbf{u} + x_3\mathbf{v}$ with x_2 and x_3 free (and neither $\mathbf{u}$ nor $\mathbf{v}$ a multiple of the other), describes a plane through the origin **True**. Take a look at pg 51.

(c) The equation $A\mathbf{x} = \mathbf{b}$ is homogeneous if the zero vector is a solution. **True**. If $A\mathbf{0} = \mathbf{b}$ is true, then $\mathbf{b} = \mathbf{0}$.

(d) The effect of adding $\mathbf{p}$ to a vector is to move the vector in a direction parallel to $\mathbf{p}$. **True**. Take a look at page 53.

(e) The solution set of $A\mathbf{x} = \mathbf{b}$ is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$. **False**. This one is a little tricky, because it would be true if we knew that $A\mathbf{x} = \mathbf{b}$ was consistent. If it is inconsistent, however, theorem 6 does not hold. So in the general case we can't say that the solutions must be a translation.

1.5.28 Suppose that A is a 3×3 matrix and $\mathbf{y}$ is a vector in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{y}$ does not have a solution. Does there exist a vector $\mathbf{z}$ in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{z}$ has a unique solution?

The answer is no. If $A\mathbf{x} = \mathbf{y}$ does not have a solution, we know that the row reduced form of A has a zero row (because the row reduced augmented matrix must have a row of the form $[0, \ldots, 0, 1]$; see theorem 2 pg 24). Since A is 3×3, this means that one of the columns of A does not have a pivot (because one of the rows doesn't). Now if $A\mathbf{x} = \mathbf{z}$ has a unique solution that means that the row reduced form of A does not have any free columns, because otherwise the system $[A\ \mathbf{z}]$ would have free variables. This means that all the columns of A must be pivot columns. This is a contradiction, because we already decided that A must have a free column. We conclude that $A\mathbf{x} = \mathbf{z}$ must not have a unique solution.

1.6.3 Consider an economy with three sectors, Chemicals & Metals, Fuels & Power, and Machinery. Chemicals sells 30% of its output to Fuels and 50% to Machinery and retains the rest. Fuels sells 80% of its output to Chemicals and 10% to Machinery and retains the rest. Machinery sells 40% to Chemicals and 40% to Fuels and retains the rest. a) Construct the exchange table for this economy. b) Develop a system of equations that leads to prices at which each sector's income matches its expenses. Then write the augmented matrix that can be row reduced to find these prices. c) Find a set of equilibrium prices when the price for the Machinery output is 100 units.

(a) The exchange table for this economy is:

Distribution of Output from:			
Chemicals	Fuels	Machinery	Purchased by:
.2	.8	.4	Chemicals
.3	.1	.4	Fuels
.5	.1	.2	Machinery

(b) Let x_1 be the total price of all of the Chemical sector's output, x_2 be the total price of all of the Fuel sector's output, and x_3 be the total price of all of the Machinery sector's output. Then the system of linear equations that leads to prices at which each sector's income matches its expenses is:

$$\begin{aligned} .2x_1 + .8x_2 + .4x_3 &= x_1 \\ .3x_1 + .1x_2 + .4x_3 &= x_2 \\ .5x_1 + .1x_2 + .2x_3 &= x_3. \end{aligned}$$

We can turn this into a homogeneous system by moving all the variables to the left side. The result is:

$$\begin{aligned} -.8x_1 + .8x_2 + .4x_3 &= 0 \\ .3x_1 - .9x_2 + .4x_3 &= 0 \\ .5x_1 + .1x_2 - .8x_3 &= 0. \end{aligned}$$

The augmented matrix which can be row reduced to find these prices is: $\begin{bmatrix} -.8 & .8 & .4 & 0 \\ .3 & -.9 & .4 & 0 \\ .5 & .1 & -.8 & 0 \end{bmatrix}$.

(c) The row reduced echelon form of the matrix given in part (b) is, according to my trusty TI-83, $\begin{bmatrix} 1 & 0 & -1.417 & 0 \\ 0 & 1 & -.917 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which means that a solution is $\begin{bmatrix} 1.417x_3 \\ .917x_2 \\ x_3 \end{bmatrix}$. So if the price for the Machinery output is 100 units, then the price for the Fuel output will be $1.417(100) = 141.7$ units and the price for the Chemical output will be $.917(100) = 91.7$ units.