# §6.5
# Singular Value Decomposition

## Preliminaries

1. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$, then

$$A = \mathbf{x}\mathbf{y}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \ldots & y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & \ldots & x_1y_n \\ x_2y_1 & x_2y_2 & & x_2y_n \\ \vdots & \vdots & & \vdots \\ x_ny_1 & x_ny_2 & & x_ny_n \end{bmatrix} =$$

$$= \begin{bmatrix} y_1\mathbf{x} & y_2\mathbf{x} & \ldots & y_n\mathbf{x} \end{bmatrix} = \begin{bmatrix} x_1\mathbf{y}^T \\ x_2\mathbf{y}^T \\ \vdots \\ x_n\mathbf{y}^T \end{bmatrix}$$

$\mathbf{x}\mathbf{y}^T$ is called the <u>OUTER PRODUCT EXPANSION</u> of $\mathbf{x}$ and $\mathbf{y}$ and it is a matrix of **rank 1** since $R(A) = \text{span}(\mathbf{x})$ (and $R(A^T) = \text{span}(\mathbf{y})$).
We can say more: *every rank one matrix* has the special form $A = \mathbf{x}\mathbf{y}^T$ = column times row.
The columns are multiples of $\mathbf{x}$ and the rows are multiples of $\mathbf{y}^T$.

Now assume $\mathbf{u}$ and $\mathbf{v}$ are two more vectors, then $B = \mathbf{x}\mathbf{y}^T + \mathbf{u}\mathbf{v}^T$ is a **rank 2** matrix, (provided $\mathbf{x}$ and $\mathbf{u}$ are linearly independent).

2. An $n \times n$ matrix $Q$ is called an **Orthogonal Matrix** if the columns of $Q$ are orthonormal vectors. Then $Q^TQ = I$, but since $Q$ is square, with linearly independent columns, it is invertible and $Q^T = Q^{-1}$.

**Examples of orthogonal matrices:**

- *Rotation matrices*
- *Permutation matrices* (obtained by permuting rows of $I$).
- *Reflection matrices:* they are of the form $2\mathbf{u}\mathbf{u}^T - I$ where $\mathbf{u}$ is the unit vector in the direction we are reflecting into. (This follows from the fact that the reflection of the vector $\mathbf{v}$ into the vector $\mathbf{u}$ is $2\mathbf{p} - \mathbf{v} = 2\frac{\mathbf{u}^T\mathbf{v}}{||\mathbf{u}||^2}\mathbf{u} - \mathbf{v} = 2(\mathbf{u}^T\mathbf{v})\mathbf{u} - \mathbf{v}$. Now, from the associative property of matrix multiplication, we have $(\mathbf{u}\mathbf{u}^T)\mathbf{v} = \mathbf{u}(\mathbf{u}^T\mathbf{v}) = (\mathbf{u}^T\mathbf{v})\mathbf{u}$ and it follows that the reflection can be written as $2(\mathbf{u}\mathbf{u}^T)\mathbf{v} - \mathbf{v} = (2\mathbf{u}\mathbf{u}^T - I)\mathbf{v}$).
  Reflection matrices are symmetric and also orthogonal. They also have the property $Q^2 = I$ (reflecting twice brings back the original).

**Properties of orthogonal matrices**

- orthogonal matrices leave lengths unchanged: $||Q\mathbf{x}||_2 = ||\mathbf{x}||_2$ for every vector $\mathbf{x}$.
- orthogonal matrices preserve dot products: $(Q\mathbf{x})^T(Q\mathbf{y}) = \mathbf{x}^TQ^TQ\mathbf{y} = \mathbf{x}^T\mathbf{y}$.
- Orthogonal matrices are excellent for computations - numbers can never grow too large when lengths of vectors are fixed.

3. Recall: Frobenius norm of $A$ is $||A||_F = \sqrt{\sum_{ij}(a_{ij})^2}$

## Singular Value Decomposition

Assume $A$ is $m \times n$ and $m \geq n$ (but all results hold also if $m < n$). We want to factor $A$ in the form

$$A = U\Sigma V^T$$

where $U$ is $m \times m$ orthogonal matrix
$V$ is $n \times n$ orthogonal matrix
$\Sigma$ is $m \times n$ with off diagonal entries all 0's and diagonal entries $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$.

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ldots & 0 \\ 0 & & \ddots & 0 \\ 0 & \ldots & & \sigma_n \\ 0 & & \ldots & 0 \\ 0 & & \vdots & 0 \end{bmatrix}$$

$\sigma_i$'s are called **SINGULAR VALUES of** $A$.
The factorization $U\Sigma V^T$ is called the Singular Value Decomposition (SVD) of $A$.

---

In many applications we need to determine the rank of a matrix. Computationally, reducing to RREF and counting the number of non zero rows is not efficient because of round off errors.
A better way to compute the rank is by using the following Theorem:

**THEOREM:**
rank(A) = number of non zero singular values $\sigma_i$.

*Proof:*

$$A = U\Sigma V^T = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_n & \mathbf{u}_{n+1} & \ldots & \mathbf{u}_m \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ldots & 0 \\ 0 & & \ddots & 0 \\ 0 & \ldots & & \sigma_n \\ 0 & & \ldots & 0 \\ 0 & & \vdots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} =$$

$$= \begin{bmatrix} \sigma_1\mathbf{u}_1 & \sigma_2\mathbf{u}_2 & \ldots & \sigma_n\mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} = \sigma_1(\mathbf{u}_1\mathbf{v}_1^T) + \sigma_2(\mathbf{u}_2\mathbf{v}_2^T) + \ldots + \sigma_n(\mathbf{u}_n\mathbf{v}_n^T)$$

By Preliminary 1., $\mathbf{u}_i\mathbf{v}_i^T$ are matrices of rank one, thus, if $\sigma_1, \sigma_2, \ldots, \sigma_k \neq 0$,
and $\sigma_{k+1} = \sigma_{k+2} = \ldots = \sigma_n = 0$, then the sum is a matrix of rank $k$. ■

---

The previous Theorem and, in particular, the expression for $A$:

$$A = \sigma_1(\mathbf{u}_1\mathbf{v}_1^T) + \sigma_2(\mathbf{u}_2\mathbf{v}_2^T) + \ldots + \sigma_n(\mathbf{u}_n\mathbf{v}_n^T)$$

are extremely important in applications.
From the above Theorem the following Remarks follow:

**IMPORTANT REMARKS:**

1. If $A$ has rank $n$,
$$B = \sigma_1(\mathbf{u}_1\mathbf{v}_1^T) + \sigma_2(\mathbf{u}_2\mathbf{v}_2^T) + \ldots + \sigma_k(\mathbf{u}_k\mathbf{v}_k^T), \quad k \leq n$$
is the matrix of rank $k$ that is closest to $A$ w.r.t the Frobenius norm, i.e.
$$||B - A||_F = \text{minimum among all matrices } B \text{ of rank } k$$

2. It can be shown that $||A-B||_F = \sqrt{\sigma_{k+1}^2 + \sigma_{k+2}^2 + \ldots + \sigma_n^2}$

3. In particular, if $A$ is non singular $n \times n$, then

$$A' = \sigma_1(\mathbf{u}_1\mathbf{v}_1^T) + \sigma_2(\mathbf{u}_2\mathbf{v}_2^T) + \ldots + \sigma_{n-1}(\mathbf{u}_{n-1}\mathbf{v}_{n-1}^T)$$

is singular and $||A-A'||_F = \sigma_n$.
**Thus $\sigma_n$ may be taken as a measure of how close a square matrix is to being singular.** Here "close to singular" means that if we perturb slightly $A$, then $A\mathbf{x} = \mathbf{0}$ could give solutions that are NOT close to zero.

4. In general, the determinant is not a good measure of how close a matrix is to being singular, (i.e. small determinant does not imply "close to singular"); however, small $\sigma_n$ does imply close to singular.

**EXAMPLE**. Given the SVD decomposition

$$A = U\Sigma V^T = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 30 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

(a) Determine rank(A).

*Solution:* The rank is three since there are three non zero singular values.

(b) Find the closest (w.r.t Frobenius norm) matrix of rank 1 to $A$.

*Solution:* By Remark 1: $B = \sigma_1(\mathbf{u}_1\mathbf{v}_1^T) = 30 \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 6 & 12 & 12 \\ 8 & 16 & 16 \\ 0 & 0 & 0 \end{bmatrix}$

(c) Determine $||A-B||_F$.

*Solution:* By Remark 2.: $||A-B||_F = \sqrt{\sigma_2^2+\sigma_3^2} = \sqrt{234} \approx 15.29$.

(d) Find the closest matrix of rank 2.

*Solution:* By Remark 1.:

$$A_2 = \sigma_1(\mathbf{u}_1\mathbf{v}_1^T) + \sigma_2(\mathbf{u}_2\mathbf{v}_2^T) = B + \sigma_2(\mathbf{u}_2\mathbf{v}_2^T) = B + 15 \begin{bmatrix} -4/5 \\ 3/5 \\ 0 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 & -2/3 \end{bmatrix} = \begin{bmatrix} -2 & 8 & 20 \\ 14 & 19 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

(Note that $\mathbf{a}_3 = -2\mathbf{a}_1 + 2\mathbf{a}_2$ which confirms rank$(A_2) = 2$).

(e) Determine $||A-A_2||_F$

*Solution:* By Remark 2: $||A-A_2||_F = \sqrt{\sigma_3^2} = 3$.

## How do we find the SVD?

**THEOREM:**
Every $m \times n$ matrix has an SVD.

*Idea of Proof.* (For simplicity we assume $A$ is $n \times n$ and has rank $n$).
Let $\lambda_i$ be an e-value of $A^TA$ and $\sigma_i = \sqrt{\lambda_i}$.
Let $V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$ with $\mathbf{v}_i$ orthonormal e-vectors of $A^TA$, then, by definition of e-vector, $A^TA\mathbf{v}_i = \lambda_i\mathbf{v}_i$ and

$$||A\mathbf{v}_i||^2 = (A\mathbf{v}_i)^T(A\mathbf{v}_i) = \mathbf{v}_i^TA^TA\mathbf{v}_i = \mathbf{v}_i^T\lambda_i\mathbf{v}_i = \lambda_i\mathbf{v}_i^T\mathbf{v}_i = \lambda_i$$

Thus $||A\mathbf{v}_i|| = \sqrt{\lambda_i} = \sigma_i$.

Let $\Sigma = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ 0 & & \ddots & 0 \\ 0 & \dots & & \sigma_n \end{bmatrix}$ (since rank$(A) = n$ we have that $\sigma_i \neq 0, \quad i = 1, \dots, n$) and let

$$U = AV\Sigma^{-1} = \begin{bmatrix} \frac{A\mathbf{v}_1}{\sigma_1} & \frac{A\mathbf{v}_2}{\sigma_2} & \dots & \frac{A\mathbf{v}_n}{\sigma_n} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}.$$

We need to verify that the columns of $U$ are orthonormal:
$||\mathbf{u}_i|| = \dfrac{||A\mathbf{v}_i||}{\sigma_i} = \dfrac{\sigma_i}{\sigma_i} = 1$.
Also, $\mathbf{u}_i^T\mathbf{u}_j = (A\mathbf{v}_i)^T A\mathbf{v}_j = \mathbf{v}_i^T \lambda_j \mathbf{v}_j = \lambda_j \mathbf{v}_i^T \mathbf{v}_i = 0$ (since $\mathbf{v}_i \perp \mathbf{v}_j$).
We then have

$$U = AV\Sigma^{-1} \Rightarrow U\Sigma = AV \Rightarrow A = U\Sigma V^{-1}$$

and since $V$ is an orthgonal matrix we have $V^T = V^{-1}$ which implies $A = U\Sigma V^T$. ∎

**STEPS FOR FINDING THE SVD:**

1. $\sigma_i = \sqrt{\lambda_i}$ with $\lambda_i$ e-values of $A^TA$.

2. Find the e-vectors of $A^TA$. Since $A^TA$ is symmetric, these e-vectors will be automatically orthogonal, however, you may need to normalize them. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be these orthonormal e-vectors, then $V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$

3. If $\sigma_1, \dots, \sigma_r \neq 0, \quad r < n$, then $\mathbf{u}_i = \dfrac{1}{\sigma_i} A\mathbf{v}_i, \quad i = 1, \dots, r$.
   The others $\mathbf{u}_{r+1}, \mathbf{u}_{r+2}, \dots, \mathbf{u}_m$ are an orthonormal basis of $N(A^T)$ . We can compute a basis for $N(A^T)$ in the usual way (by finding the RREF of $A^T$) and, if the vectors are not already orthogonal, we can use Gram-Schmidt to orthonormalize them.

**REMARK: The matrices $U$ and $V$ contain orthonormal bases for all four fundamental subspaces.**
Let rank$(A) = r$, i.e $\sigma_1, \sigma_2, \dots, \sigma_r \neq 0$, then

- $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\}$ is an orthonormal basis of $R(A)$ (by definition the $\mathbf{u}_i = \frac{A\mathbf{v}_i}{\sigma_i}$ are linear combinations of the columns of $A$ and they are linearly independent since they are orthogonal)

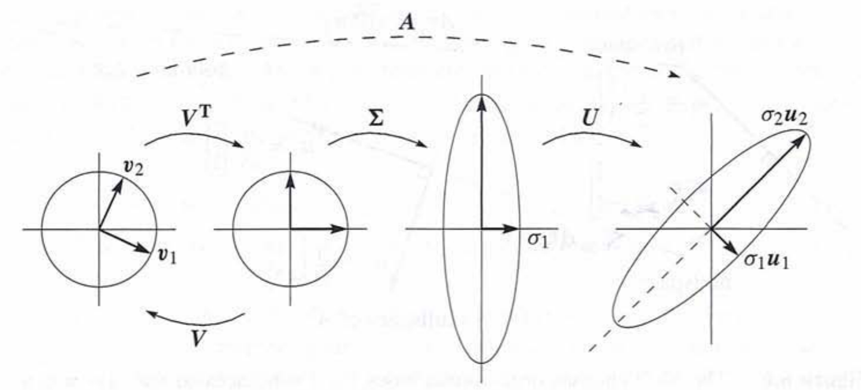
- $\{\mathbf{u}_{r+1}, \dots, \mathbf{u}_m\}$ is an orthonormal basis of the nullspace of $A^T$, $N(A^T)$.

- $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is an orthonormal basis of $R(A^T)$. (Since $A^T\mathbf{u}_j = \sigma_j\mathbf{v}_j$ we have that the $\mathbf{v}_j$ are linear combinations of the rows of $A$ and they are lin. independent since they are orthogonal).

- $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ is an orthonormal bases for $N(A)$.

---

In the picture below we can see a geometric representation of the SVD for a 2 by 2 invertible matrix. This matrix **transforms the unit circle into an ellipse**.
$U$ and $V$ are rotations and reflections. $\Sigma$ is a stretching matrix.
The orthonormal columns of $V$, $\mathbf{v}_1$ and $\mathbf{v}_2$, are mapped by $V^T$ into $\mathbf{e}_1$ and $\mathbf{e}_2$. $\Sigma$ then stretches these vectors into $\sigma_1\mathbf{e}_1$ and $\sigma_2\mathbf{e}_2$. Since $U(\sigma_1\mathbf{e}_1) = \sigma_1\mathbf{u}_1$ and $U(\sigma_2\mathbf{e}_2) = \sigma_2\mathbf{u}_2$, the vectors $\sigma_1\mathbf{e}_1$ and $\sigma_2\mathbf{e}_2$ are mapped by $U$ into $\sigma_1\mathbf{u}_1$ and $\sigma_2\mathbf{u}_2$ which are the axis of the ellipse.




**EXAMPLE**

Let $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(a) Find the SVD decomposition of $A$

*Solution:*

$A^TA = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$, $|A^TA - \lambda I| = (10-\lambda)^2 - 36$. The e-values of $A^TA$ are $\lambda_1 = 16$ and $\lambda_2 = 4$, thus the singular values are $\sigma_1 = \sqrt{16} = 4$ and $\sigma_2 = \sqrt{4} = 2$.

The rref of $A^TA - 16I$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, thus an e-vector associated to $\lambda_1 = 16$ is $(1,1)^T$.

The rref of $A^TA - 4I$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, thus an e-vector associated to $\lambda_2 = 4$ is $(1,-1)^T$.

As expected the two e-vectors are orthogonal, but not orthonormal. To find the columns of $V$ we need to normalize the e-vectors:

$\mathbf{v}_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$ and $\mathbf{v}_2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})^T$; thus the matrix

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

We have $\mathbf{u}_1 = \dfrac{A\mathbf{v}_1}{\sigma_1} = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, 0, 0\right)^T$ and $\mathbf{u}_2 = \dfrac{A\mathbf{v}_2}{\sigma_2} = \left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, 0, 0\right)^T$

The matrix $U$ is $4 \times 4$, thus we need two more columns: $\mathbf{u}_3$ and $\mathbf{u}_4$. These two columns are an orthonormal basis of $N(A^T)$.

The rref of $A^T$ is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and a basis of $N(A^T)$ is given by $\{(0,0,1,0)^T, (0,0,0,1)^T\}$.

$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We can verify that $A = U\Sigma V^T$.

(b) Determine the rank of $A$.

*Solution:* Since we have two nonzero singular values, the rank of the matrix is two.

(c) Find an orthonormal basis for $R(A)$.

*Solution:* $\mathbf{u}_1$ and $\mathbf{u}_2$ are a basis for $R(A)$.

(d) Find an orthonormal basis for $R(A^T)$.

*Solution:* $\mathbf{v}_1$ and $\mathbf{v}_2$ are a basis for $R(A^T)$.

## The SVD and Least Squares

**THEOREM** Let $A$ be an $m \times n$ matrix of rank $n$, then the least squares solution of $A\mathbf{x} = \mathbf{b}$ is given by

$$\hat{\mathbf{x}} = \frac{(\mathbf{u}_1^T\mathbf{b})}{\sigma_1}\mathbf{v}_1 + \ldots + \frac{(\mathbf{u}_n^T\mathbf{b})}{\sigma_n}\mathbf{v}_n$$

Proof: We start with the normal equations:

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

Replace $A$ with its SVD decomposition $A = U\Sigma V^T$:

$$(U\Sigma V^T)^T(U\Sigma V^T)\mathbf{x} = (U\Sigma V^T)^T\mathbf{b}$$

Distributing the transpose gives:

$$V\Sigma^T(U^TU)\Sigma V^T\mathbf{x} = V\Sigma^T U^T\mathbf{b}$$

$U$ is orthogonal and therefore $U^TU = I$. Also, since $V$ is nonsingular we can left multiply by its inverse:

$$(\Sigma^T\Sigma)V^T\mathbf{x} = \Sigma^T U^T\mathbf{b}$$

$$V^T\mathbf{x} = (\Sigma^T\Sigma)^{-1}\Sigma^T U^T\mathbf{b}$$

Since $V^T = V^{-1}$, left multiplying both sides by $V$ gives:

$$\mathbf{x} = V(\Sigma^T\Sigma)^{-1}\Sigma^T U^T\mathbf{b}$$

$$(\Sigma^T\Sigma)^{-1}\Sigma^T = \begin{bmatrix} 1/\sigma_1^2 & \ldots & 0 \\ & \ddots & 0 \\ 0 & \ldots & 1/\sigma_n^2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \ldots & 0 & 0 & 0 \\ & \ddots & 0 & \vdots & \vdots \\ 0 & \ldots & \sigma_n & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sigma_1 & \ldots & 0 & 0 & 0 \\ & \ddots & 0 & \vdots & \vdots \\ 0 & \ldots & 1/\sigma_n & 0 & 0 \end{bmatrix}, \text{ thus}$$

$$\mathbf{x} = V\begin{bmatrix} 1/\sigma_1 & \ldots & 0 & 0 & 0 \\ & \ddots & 0 & \vdots & \vdots \\ 0 & \ldots & 1/\sigma_n & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{u}_1^T\mathbf{b} \\ \vdots \\ \mathbf{u}_n^T\mathbf{b} \end{bmatrix} = V\begin{bmatrix} \frac{\mathbf{u}_1^T\mathbf{b}}{\sigma_1} \\ \vdots \\ \frac{\mathbf{u}_n^T\mathbf{b}}{\sigma_n} \end{bmatrix} = \frac{(\mathbf{u}_1^T\mathbf{b})}{\sigma_1}\mathbf{v}_1 + \ldots + \frac{(\mathbf{u}_n^T\mathbf{b})}{\sigma_n}\mathbf{v}_n \qquad \blacksquare$$

**EXAMPLE:**
An SVD of a matrix $A$ is given by :

$$A = \begin{bmatrix} .5 & .5 & .5 & .5 \\ .5 & -.5 & -.5 & .5 \\ .5 & .5 & -.5 & -.5 \\ .5 & -.5 & .5 & -.5 \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} .6 & .8 \\ -.8 & .6 \end{bmatrix}$$

Use the SVD to find the least squares solution of $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = (4, 4, 1, -4)^T$.

*Solution:*
$$\hat{\mathbf{x}} = \frac{\mathbf{u}_1^T\mathbf{b}}{\sigma_1}\mathbf{v}_1 + \frac{\mathbf{u}_2^T\mathbf{b}}{\sigma_2}\mathbf{v}_2 = \frac{2.5}{10}\begin{bmatrix} .6 \\ .8 \end{bmatrix} + \frac{2.5}{5}\begin{bmatrix} -.8 \\ .6 \end{bmatrix} = \begin{bmatrix} -.25 \\ .5 \end{bmatrix}.$$