1. (20) How many ways are there to choose 6 objects from 10 **distinct** ones when

(a) the order in which the objects are chosen **matters** and repetition is **not** allowed?

Answer. $P(10, 6) = 10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151,200$

(b) the order in which the objects are chosen **matters** and repetition **is** allowed?

Answer. 10^6.

(c) the order in which the objects are chosen does **not** matter and repetition is **not** allowed?

Answer. $\binom{10}{6} = \frac{10!}{6!4!} = 210$.

(d) the order in which the objects are chosen does **not** matter and repetition **is** allowed?

Answer. $\binom{6+10-1}{10-1} = \binom{10+6-1}{6} = \binom{15}{9} = \frac{15!}{9!6!} = 5005$.

2. (15) How many strings of length 10 made from the letters $\{a, b, c\}$ have either **exactly 3** a, or **exactly 4** b, but **not** both.

Answer. $2^7\binom{10}{3} + 2^6\binom{10}{4} - \binom{10}{3}\binom{7}{4} = 24,600.$

3. (10) What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x - 3y)^{200}$?

Answer. $-2^{101}3^{99}\binom{200}{99}$.

4. (15) A fortune cookie company makes 213 different fortunes. A student eats at a restaurant that uses fortunes from this company. What is the largest possible number of times that the student can eat at the restaurant without getting the same fortune 4 times?

Answer. 639.

5. (10) The sequence of Lucas numbers is defined by $l_0 = 2,\ l_1 = 1$, and $l_n = l_{n-1} + l_{n-2}$, for $n = 2, 3, 4, \ldots$. Show that $l_0^2 + l_1^2 + \cdots + l_n^2 = l_n l_{n+1} + 2$ whenever n is a nonnegative integer.

Proof. Base step: $l_0^2 = 2^2 = 4$. On the other hand, $l_0 l_1 + 2 = 2 \times 1 + 2 = 4$. Thus the equation is true for $n = 0$.

Assume that $l_0^2 + l_1^2 + \cdots + l_k^2 = l_k l_{k+1} + 2$

We want to show that $l_0^2 + l_1^2 + \cdots + l_k^2 + l_{k+1}^2 = l_{k+1} l_{k+2} + 2$

$$\begin{aligned} l_0^2 + l_1^2 + \cdots + l_k^2 + l_{k+1}^2 &= l_k l_{k+1} + 2 + l_{k+1}^2 \qquad \text{(IH)} \\ &= l_{k+1}(l_k + l_{k+1}) + 2 \\ &= l_{k+1} l_{k+2} + 2 \end{aligned}$$

6. (10) Show that given any set of 10 **distinct** positive integers not exceeding 50 there exist at least 2 different five-element subsets of this set that have the same sum. (Hint: what are the maximum and minimum possible sum of 5 distinct integers not exceeding 50.?)

Solution. The maximum possible sum is 240, and the minimum possible sum is 15. So the number of possible sums is 226. Because there are $\binom{10}{5} = 252$ possible subsets with 5 elements of a set with 10 elements, by the pigeonhole principle it follows that at least two have the same sum. //

7. (20) Show that if n and k are integers with $1 \leq k \leq n$, then $k\binom{n}{k} = n\binom{n-1}{k-1}$.

(a) using a combinatorial argument

Solution. Claim: both sides count the number of ways to choose from a set of n people a subset of k people, and to put a hat on one them. For the left hand side, you first choose the k people, in $\binom{n}{k}$ ways, and then determine whom should wear the hat, in k ways. For the right hand-side, you first choose the person who will wear the hat, in n ways, then you choose the remaining $k-1$ people from the remaining $n-1$ people, in $\binom{n-1}{k-1}$ ways.

//

(b) by algebraic manipulation.

Proof. $k\binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}$.

$\square$

NAME: ____________________

MAT 243

EXAM 3

November 15, 2007

INSTRUCTIONS. You have until 11:55am to complete this exam. This exam consists of 7 numbered questions totaling 100 points. The test is worth 100 points. Make sure your exam is complete before you begin.

Be sure to read and follow the instructions for each problem. Be sure to understand each problem carefully before starting work on it. Be sure to clearly indicate your final answers, and to **justify** all your answers.
Failing to do so will earn you 0 points for this problem. Write your answers on the blank space left between each question. If you need more space please use the back of the page. You can obtain scratch paper from me.

No notes or books are allowed. Calculators are allowed, as well as handwritten notes on 1 sheet letter size paper (both sides).

Relax and good luck!

Problem	Points	Score
1	20	
2	15	
3	10	
4	15	
5	10	
6	10	
7	20	
Total	100	

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