Rules of Inference

Arguments

An **argument** is a sequence of statements that starts with one or more **premises** (statements that are assumed to be true) and ends with a **conclusion**. We say that the argument is **valid** if the conclusion follows logically from the premises, i.e. if the conclusion must be true given that the premises are true.

In declaring an argument valid, we do NOT question the truth of the premises. Therefore, the following argument is valid, even though its premise is false.

Suppose I have ten million dollars. Then I can retire.

Argument Forms

To help us identify arguments as valid or invalid, we will identify valid **argument forms**. An argument form is an abstraction of an argument, where the particulars of the situation have been stripped away and only the logical structure remains. Basic valid argument forms are known as **rules of inference**.

Let us begin with the following example argument: *When the sun shines, I'm happy. The sun is shining. Therefore, I'm happy.*

Let's introduce the following propositional variables:

p = "the sun shines/is shining."   q = "I'm happy."

The argument above can than be rewritten as

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

It is customary to separate the conclusion with a horizontal bar. The triple dot ∴ symbol is read as "therefore". This particular argument form is called the *modus ponens*.

Proving a Rule of Inference

To formally demonstrate that a an argument form is valid, we must show that the conditional

if (conjunction of all premises) then conclusion

is true if all the premises are true. Since that conditional is true by default if the premises are not all true, it must always be true (be a tautology) if the argument form is to be valid.

Let us use this formal definition of validity to show that the modus ponens is valid. To do that, we must establish that

$$\big((p \to q) \wedge p\big) \to q$$

Is a tautology. We could use a truth table to do that, but it is more elegant to use logical equivalences:

$$\big((p \to q) \land p\big) \to q$$

$$\equiv \neg\big((\neg p \lor q) \land p\big) \lor q$$

$$\equiv (\neg(\neg p \lor q) \lor \neg p) \lor q$$

$$\equiv \big((p \land \neg q) \lor \neg p\big) \lor q$$

$$\equiv \big((p \lor \neg p) \land (\neg q \lor \neg p)\big) \lor q$$

$$\equiv (\neg q \lor \neg p) \lor q \equiv (\neg q \lor q\ ) \lor \neg p$$

$$\equiv T \lor \neg p \equiv T$$

Modus Tollens

We have learned that a conditional is equivalent to its contrapositive. Therefore, if we know $p \to q$, then we also know $\neg q \to \neg p$. If we then also know $\neg q$, we can conclude $\neg p$. This argument form is known as the **modus tollens**:

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

To prove the validity of the modus tollens, we have to show that the following statement is a tautology:

$$\big((p \to q) \land \neg q\big) \to \neg p$$

This is left as an exercise for the student.

Other Valid Rules of Inference

Argument Form	Name	Corresponding Tautology
$p \rightarrow q$ $q \rightarrow r$ $\therefore\ p \rightarrow r$	Hypothetical Syllogism	$(p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r)$
$p \lor q$ $\neg p$ $\therefore\ q$	Disjunctive Syllogism	$(p \lor q) \land \neg p \rightarrow q$
p $\therefore\ p \lor q$	Addition	$p \rightarrow p \lor q$
$p \land q$ $\therefore\ p$	Simplification	$p \land q \rightarrow p$
p q $\therefore p \land q$	Conjunction	$p \land q \rightarrow p \land q$
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	Resolution	$(p \lor q) \land (\neg p \lor r) \rightarrow (q \lor r)$

Some Real-Life Examples of the use of Rules of Inference

Hypothetical Syllogism: “If we had faster than light travel, we could travel to other star systems. If we could travel to other star systems, we would meet aliens. Therefore, if we had faster than light travel, we would meet aliens.”

Disjunctive Syllogism: “the accused is either innocent, or he is lying. He is not lying. Therefore, he is innocent.”

Addition: at a certain state park, you can get free admission if you are a student or a senior. You are a student. Therefore, you satisfy the condition of being a student or a senior.

Simplification: you apply for a scholarship. You mention on your resume that you have very good grades, and an outstanding collection of baseball cards. The selection committee makes a note that you have very good grades.

Fallacies

A **fallacy** is an invalid argument form. Two common fallacies are:

The fallacy of affirming the conclusion:

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore \ p \end{array}$$

This fallacy confuses the conditional premise with its converse. We have learned that a conditional and its converse are not logically equivalent.

The fallacy of denying the hypothesis:

$$\begin{array}{l} p \to q \\ \neg p \\ \hline \therefore \ \neg q \end{array}$$

This fallacy confuses the conditional premise with its inverse, which we know is not logically equivalent.

Analyzing an Argument (1)

Show that the following is a valid argument:

If it rains, I don't go to work. If I don't go to work, I get fired. I'm not getting fired today. Therefore, it is not raining.

To prove the validity of the argument, we will introduce assign appropriate propositional variables, rewrite the argument in terms of those variables and identify valid rules for inference that lead from the premises to the conclusion:

1. $p \rightarrow \neg q$ (premise)
2. $\neg q \rightarrow r$ (premise)
3. $\neg r$ (premise)
4. $p \rightarrow r$ (hypothetical syllogism from 1, 2)
5. $\therefore \neg p$ (modus tollens from 3, 4)

Needed Propositions
$p =$ "it rains"
$q =$ "I go to work"
$r =$ "I get fired"

Instantiation and Generalization

This slide discusses a set of four basic rules of inference involving the quantifiers.

1. If a statement is true about all objects, then it is true about any specific, given object. This is called **universal instantiation**.

$$\frac{\forall x P(x)}{\therefore\ P(c) \text{ for any arbitrary c}}$$

2. If a statement is true about every single object, then it is true about all objects. This is called **universal generalization**.

$$\frac{P(c) \text{ for any arbitrary c}}{\therefore \forall x P(x)}$$

3. If an object exists that makes a statement true, then it is possible to produce such an object. This is called **existential instantiation**.

$$\frac{\exists x P(x)}{\therefore\ P(c) \text{ for some particular c}}$$

4. If we have an object with a property, then we know that objects with this property exist. This is called **existential generalization**.

$$\frac{P(c) \text{ for some particular c}}{\therefore \exists x P(x)}$$

Analyzing an Argument (2)

Show that the following is a valid argument:

Every kid loves ice cream. Joey doesn't love ice cream. Therefore, Joey is not a kid.

To answer the question, we will introduce appropriate predicates, rewrite the argument using those predicates and identify the valid argument forms that lead from the premises to the conclusion.

Needed Predicates
$\mathrm{K}(x) =$ "x is a kid"
$\mathrm{L}(x) =$ "x loves ice cream"

1. $\forall x\big(K(x) \rightarrow L(x)\big)$ (premise)
2. $\neg L(\mathrm{Joey})$ (premise)
3. $K(\mathrm{Joey}) \rightarrow L(\mathrm{Joey})$ (universal instantiation of 1.)
4. $\therefore\ \neg K(\mathrm{Joey})$ (modus Tollens from 2 and 3) (conclusion)

Two Common Mistakes

A very common mistake students make in formally analyzing arguments is to forget the quantifiers. For example, they would give $K(x) \to L(x)$ as the first premise of the previous argument. But this “premise” is not even a statement because x is a free variable. Without quantification, the “statement” is meaningless.

A second common mistake is to *confuse conditional statements with conjunctions*. Let us consider the example on the previous slide.

Some would translate the statement *every kid loves ice cream* as $\forall x\big(K(x) \wedge L(x)\big)$. But this means that *everyone (every person) is a kid and loves ice cream,* quite a different statement. The correct symbolic representation of *every kid loves ice cream* is

$$\forall x\big(K(x) \to L(x)\big)$$

This is actually old news- in a previous presentation, we already learned that domain restricted universal quantification of a statement is equivalent to the unrestricted universal quantification of a conditional, where the membership in the restricted domain is the premise of the conditional, and the original statement is the conclusion. Therefore, *every kid loves ice cream* is equivalent to *it is true for every person that if the person is a kid, they love ice cream*.

Analyzing an Argument (3)

Show that the following is a valid argument:

When dogs play, they get dirty. Dogs that don't play sleep. Sleeping dogs do not chase cats. Billy is not a dirty dog. Therefore, Billy does not chase cats.

To answer the question, we will introduce appropriate predicates, rewrite the argument using those predicates and identify the valid argument forms that lead from the premises to the conclusion.

1. $\forall x(P(x) \rightarrow D(x))$ (premise)
2. $\forall x(\neg P(x) \rightarrow S(x))$ (premise)
3. $\forall x(S(x) \rightarrow \neg C(x))$ (premise)
4. $\neg D(\text{Billy})$ (premise)
5. $P(\text{Billy}) \rightarrow D(\text{Billy})$ (universal instantiation of 1.)
6. $\neg P(\text{Billy})$ (modus tollens using 4. and 5.)
7. $\neg P(\text{Billy}) \rightarrow S(\text{Billy})$ (universal instantiation of 2.)
8. $S(\text{Billy})$ (modus ponens using 6. and 7.)
9. $S(\text{Billy}) \rightarrow \neg C(\text{Billy})$ (universal instantiation of 3.)
10. $\therefore \neg C(\text{Billy})$ (modus ponens using 8. and 9.)

Needed Predicates
$P(x) =$ "x plays"
$\mathrm{S}(x) =$ "x sleeps"
$C(x) =$ "x chases cats"
$\mathrm{D}(x) =$ "x is dirty"

There is more than one correct solution. For example, we could have combined 7 and 9 using hypothetical syllogism to get . $\neg P(\text{Billy}) \rightarrow \neg C(\text{Billy})$ and then applied the modus ponens to that and to 6 to get the conclusion.

More Rules of Inference for Quantifiers

In the previous example, we repeatedly instantiated universally quantified conditionals and then used a modus ponens or tollens. It is convenient to adopt these argument patterns as new, named rules of inference:

Universal Modus Ponens:

$$\begin{array}{l} \forall x\big(P(x) \rightarrow Q(x)\big) \\ \underline{P(c) \qquad\qquad\qquad\qquad} \\ \therefore\ Q(c) \end{array}$$

Universal Modus Tollens:

$$\begin{array}{l} \forall x\big(P(x) \rightarrow Q(x)\big) \\ \underline{\neg Q(c) \qquad\qquad\qquad\qquad} \\ \therefore \neg P(c) \end{array}$$