ANSWERS TO RECOMMENDED PROBLEMS CHAPTER 4 TEST B (Problems 1- 6, pg. 197)

1. (a)

(i) $$L(\mathbf{x}+\mathbf{y})=\begin{bmatrix}(x_1+y_1)+(x_2+y_2)\\ x_1+y_1\end{bmatrix}$$

$$L(\mathbf{x})+L(\mathbf{y})=\begin{bmatrix}x_1+x_2\\ x_1\end{bmatrix}+\begin{bmatrix}y_1+y_2\\ y_1\end{bmatrix}=\begin{bmatrix}x_1+y_1+x_2+y_2\\ x_1+y_1\end{bmatrix}$$

Thus $L(\mathbf{x})+L(\mathbf{y})=L(\mathbf{x}+\mathbf{y})$ and the first condition for linearity is satisfied.

(ii) $$L(\alpha\mathbf{x})=\begin{bmatrix}\alpha x_1+\alpha x_2\\ \alpha x_1\end{bmatrix}$$

$$\alpha L(\mathbf{x})=\alpha\begin{bmatrix}x_1+x_2\\ x_1\end{bmatrix}=\begin{bmatrix}\alpha x_1+\alpha x_2\\ \alpha x_1\end{bmatrix}$$

Thus the second condition $\alpha L(\mathbf{x})=L(\alpha\mathbf{x})$ is also satisfied and L is linear.

(b)

(i) $$L(\mathbf{x}+\mathbf{y})=\begin{bmatrix}(x_1+y_1)(x_2+y_2)\\ x_1+y_1\end{bmatrix}=\begin{bmatrix}(x_1x_2+x_1y_2+y_1x_2+y_1y_2)\\ x_1+y_1\end{bmatrix}$$

$$L(\mathbf{x})+L(\mathbf{y})=\begin{bmatrix}x_1x_2\\ x_1\end{bmatrix}+\begin{bmatrix}y_1y_2\\ y_1\end{bmatrix}=\begin{bmatrix}x_1x_2+y_1y_2\\ x_1+y_1\end{bmatrix}$$

Obviously $L(\mathbf{x}+\mathbf{y})\neq L(\mathbf{x})+L(\mathbf{y})$ and the operator is NOT linear.

(ii) We can easily check that also this property fails.

2. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are two linearly independent vectors in $\mathbb{R}^2$ and therefore they form a basis for $\mathbb{R}^2$. Any other vector in $\mathbb{R}^2$ can then be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. In particular, $\mathbf{v}_3 = 3\,\mathbf{v}_1 + 2\,\mathbf{v}_2$.

Thus $$L(\mathbf{v_3})=L(3\,\mathbf{v_1}+2\,\mathbf{v_2})=3\,L(\mathbf{v_1})+2\,L(\mathbf{v_2})=3\begin{bmatrix}2\\5\end{bmatrix}+2\begin{bmatrix}-3\\1\end{bmatrix}=\begin{bmatrix}0\\17\end{bmatrix}$$

3. (a) The kernel of L is the set of vectors $\mathbf{x}$ such that $L(\mathbf{x}) = \mathbf{0}$. By definition of L, this is the set of vectors $\mathbf{x}$ solutions to the system

$$x_2-x_1=0$$
$$x_3-x_2=0$$
$$x_3-x_1=0$$

i.e., the set of vectors $\mathbf{x}$ in the nullspace of the matrix $A=\begin{bmatrix}-1&1&0\\0&-1&1\\-1&0&1\end{bmatrix}$

The rref of this matrix is $\begin{bmatrix}1&0&-1\\0&1&-1\\0&0&0\end{bmatrix}$. Setting the free variable $x_3=t$ gives the solution

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=t\begin{bmatrix}1\\1\\1\end{bmatrix}.\quad \text{Thus}\quad \ker(L)=\text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right).$$

(b) The vector $[1, 0, 1]^T$ is mapped by L into the vector $[-1, 1, 0]^T$, thus $L(S) = \text{span}([-1, 1, 0]^T)$

4. Range of $L = \left\{ \mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix} \right\} = \left\{ \mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = x_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$

5. From Theorem 4.2.1, the first column of **A** is given by $\mathbf{a}_1 = L(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ and the second column is given by $\mathbf{a}_2 = L(\mathbf{e}_2) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Thus the matrix representation of L is $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}$

6. If we rotate the vector $\mathbf{e}_1$ counterclockwise by 30°, we obtain the vector

$\left[\cos(30^o) \quad \sin(30^o)\right]^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}^T$ and reflecting this vector in the y–axis gives $\begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}^T$.

Since $\mathbf{a}_1 = L(\mathbf{e}_1)$ we have that $\begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}^T$ is the first column of the matrix **A**.

Rotating the vector $\mathbf{e}_2$ counterclockwise by 30°, gives the vector

$\left[\cos(30^o+90^o) \quad \sin(30^o+90^o)\right]^T = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}^T$

Reflecting this vector in the y–axis gives $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}^T$.

Since $\mathbf{a}_2 = L(\mathbf{e}_2)$ we have that $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}^T$ is the second column of the matrix **A**, thus $\mathbf{A} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

Note that the matrix **A** could have also been obtained by taking the product of the rotation matrix

$\mathbf{R} = \begin{bmatrix} \cos(30^o) & -\sin(30^o) \\ \sin(30^o) & \cos(30^o) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ and the reflection matrix in the y-axis: $\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathbf{A} = \mathbf{BR} = \begin{bmatrix} \frac{-\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ (Note that the order in which we multiply the matrices **B** and **R** is important since we first rotate and then reflect.)