Math 415 Spring 2011 Homework 7 Solutions

Sec 5.1, No 1: Find the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$ in each of the following:

(a) $\mathbf{v} = (2, 1, 3)^T, \mathbf{w} = (6, 3, 9)^T$

Solution: We use the result

$$\cos\theta = \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} = \frac{2(6) + 1(3) + 3(9)}{\sqrt{2^2 + 1^2 + 3^2}\sqrt{6^2 + 3^2 + 9^2}} = \frac{42}{\sqrt{14}(3\sqrt{14})} = 1$$

and so the angle is zero. Indeed, we now observe that $\mathbf{w} = 3\mathbf{v}$, so the two vectors are parallel to each other.

(c) $\mathbf{v} = (4, 1)^T, \mathbf{w} = (3, 2)^T$

Solution: We use the result

$$\cos\theta = \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} = \frac{4(3) + 1(2)}{\sqrt{4^2 + 1^2}\sqrt{3^2 + 2^2}} = \frac{14}{\sqrt{17}\sqrt{13}} = .94174$$

and so the angle is $\arccos(.94174) = .34303$ radians, or $.34303\frac{180}{\pi} = 19.654$ degrees.

(d) $\mathbf{v} = (-2, 3, 1)^T, \mathbf{w} = (1, 2, 4)^T$

Solution: We use the result

$$\cos\theta = \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} = \frac{-2(1) + 3(2) + 1(4)}{\sqrt{(-2)^2 + 3^2 + 1^2}\sqrt{1^2 + 2^2 + 4^2}} = \frac{8}{\sqrt{14}\sqrt{21}} = .46657$$

and so the angle is $\arccos(.46657) = 1.0854$ radians, or $1.0854\frac{180}{\pi} = 62.189$ degrees.

Sec 5.1, No 3: For each of the following pairs of vectors $\mathbf{x}$ and $\mathbf{y}$, find the projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$ and verify that $\mathbf{p}$ and $\mathbf{x} - \mathbf{p}$ are orthogonal:

(a) $\mathbf{x} = (3, 4)^T, \mathbf{y} = (1, 0)^T$

Solution: In this question the projection is

$$\mathbf{p} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y} = \frac{3(1) + 4(0)}{1^2 + 0^2}(1, 0)^T = (3, 0)^T$$

Then

$$\mathbf{p}^T(\mathbf{x} - \mathbf{p}) = (3, 0)\begin{pmatrix} 3 - 3 \\ 4 - 0 \end{pmatrix} = (3, 0)\begin{pmatrix} 0 \\ 4 \end{pmatrix} = 0$$

and so $\mathbf{p}$ and $\mathbf{x} - \mathbf{p}$ are orthogonal.

(d) $\mathbf{x} = (2, -5, 4)^T, \mathbf{y} = (1, 2, -1)^T$

Solution: In this question the projection is

$$\mathbf{p} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y} = \frac{2(1) - 5(2) + 4(-1)}{1^2 + 2^2 + (-1)^2}(1, 2, -1)^T = \frac{-12}{6}(1, 2, -1)^T = (-2, -4, 2)^T$$

Then

$$\mathbf{p}^T(\mathbf{x} - \mathbf{p}) = (-2, -4, 2)\begin{pmatrix} 2 - (-2) \\ -5 - (-4) \\ 4 - 2 \end{pmatrix} = (-2, -4, 2)\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = 0$$

and so $\mathbf{p}$ and $\mathbf{x}-\mathbf{p}$ are orthogonal.

Sec 5.2, No 1: For each of the following matrices A, determine a basis for each of the subspaces $R(A^T), N(A), R(A)$, and $N(A^T)$:

(b)

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \end{pmatrix}$$

Solution: First we will find the null space of A: using Gaussian Elimination

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

so vectors $\mathbf{z}$ in the null space of A satisfy $z_1 - 2t = 0, z_2 + t = 0, z_3 = t$, that is $\mathbf{z} = t(2,-1,1)^T$ and therefore a basis of $N(A)$ is $\left\{(2,-1,1)^T\right\}$. The echelon form above also shows that all rows of A can be written as linear combos of $(1,0,-2)$ and $(0,1,1)$, and therefore a basis of $R(A^T)$ is $\{(1,0,-2)^T,(0,1,1)^T\}$. As we should expect, $(2,-1,1)^T$ is orthogonal to both $(1,0,-2)^T$ and $(0,1,1)^T$.

Now consider in the same way the null space of A^T :

$$A^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 0 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and so $\mathbf{z} = \mathbf{0}$ is the only vector in the null space of A^T. Therefore the null space has no basis in this case. $R(A)$ is the orthogonal complement of this null space and so is all of R^2. A basis would be $\{\mathbf{e}_1, \mathbf{e}_2\} = \{(1,0)^T,(0,1)^T\}$

(c)

$$\begin{pmatrix} 4 & -2 \\ 1 & 3 \\ 2 & 1 \\ 3 & 4 \end{pmatrix}$$

Solution: First we will find the null space of A:

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 3 \\ 2 & 1 \\ 3 & 4 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and so $\mathbf{z} = \mathbf{0}$ is the only vector in the null space of A. Therefore the null space has no basis in this case. $R(A^T)$ is the orthogonal complement of this null space and so is all of R^2. A basis would be $\{\mathbf{e}_1, \mathbf{e}_2\} = \{(1,0)^T,(0,1)^T\}$.

Now consider in the same way the null space of A^T :

$$A^T = \begin{pmatrix} 4 & 1 & 2 & 3 \\ -2 & 3 & 1 & 4 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 0 & \frac{5}{14} & \frac{5}{14} \\ 0 & 1 & \frac{4}{7} & \frac{11}{7} \end{pmatrix}$$

so vectors $\mathbf{z}$ in the null space of A^T satisfy $z_1 + \frac{5}{14}t + \frac{5}{14}s = 0, z_2 + \frac{4}{7}t + \frac{11}{7}s = 0, z_3 = t, z_4 = s$, that is,

$$\mathbf{z} = \begin{pmatrix} -\frac{5}{14}t - \frac{5}{14}s \\ -\frac{4}{7}t - \frac{11}{7}s \\ t \\ s \end{pmatrix} = t\begin{pmatrix} -\frac{5}{14} \\ -\frac{4}{7} \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -\frac{5}{14} \\ -\frac{11}{7} \\ 0 \\ 1 \end{pmatrix}$$

Therefore a basis of $N(A^T)$ is $\left\{(-\frac{5}{14}, -\frac{4}{7}, 1, 0)^T, (-\frac{5}{14}, -\frac{11}{7}, 0, 1)^T\right\}$. Similarly the non-zero rows of the echelon form are a basis of the row space of A^T, that is, $R(A)$ has as a basis the set $\{(1, 0, \frac{5}{14}, \frac{5}{14})^T, (0, 1, \frac{4}{7}, \frac{11}{7})^T\}$. Note that each vector in our basis of $N(A^T)$ is orthogonal to each vector in our basis of $R(A)$.

Sec 5.2, No 3 (a): Let S be the subspace of $\mathbb{R}^3$ spanned by the vectors $\mathbf{x} = (x_1, x_2, x_3)^T$ and $\mathbf{y} = (y_1, y_2, y_3)^T$. Let

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

Show that $S^\perp = N(A)$.

Solution: A vector $\mathbf{z}$ is in $S^\perp$, i.e. is orthogonal to every linear combo of $\mathbf{x}$ and $\mathbf{y}$, if $\mathbf{x}^T\mathbf{z} = 0$ and $\mathbf{y}^T\mathbf{z} = 0$. Now let us express the null space $N(A)$ in terms of $\mathbf{x}$ and $\mathbf{y}$. $\mathbf{z}$ is in this null space if

$$A\mathbf{z} = \begin{pmatrix} - & \mathbf{x}^T & - \\ - & \mathbf{y}^T & - \end{pmatrix}\mathbf{z} = \begin{pmatrix} \mathbf{x}^T\mathbf{z} \\ \mathbf{y}^T\mathbf{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

that is, if $\mathbf{x}^T\mathbf{z} = 0$ and $\mathbf{y}^T\mathbf{z} = 0$. Therefore $S^\perp = N(A)$.

Sec 5.2, No 3 (b): Find the orthogonal complement of the subspace of $\mathbb{R}^3$ spanned by $(1, 2, 1)^T$ and $(1, -1, 2)^T$.

Solution: Here we apply the result in part (a): by Gaussian Elimination

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & -\frac{1}{3} \end{pmatrix}$$

and so $\mathbf{z}$ is in $S^\perp$ when $z_1 + \frac{5}{3}t = 0, z_2 - \frac{1}{3}t = 0, z_3 = t$. Thus $\mathbf{z}$ has the form $\mathbf{z} = t(-\frac{5}{3}, \frac{1}{3}, 1)^T$, and therefore

$$S^\perp = N(A) = \text{ Span}\left\{(-\frac{5}{3}, \frac{1}{3}, 1)^T\right\}$$

Sec 5.3, No 1 (c): Find the least squares solution of the following system:

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ -x_1 + x_2 + x_3 &= 0 \\ -x_2 + x_3 &= 1 \\ x_1 \qquad + x_3 &= 2 \end{aligned}$$

Solution: The vector $\hat{\mathbf{x}}$ that we want in this case is $\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$ where A is the coefficient matrix and $\mathbf{b}$ is the vector of right-hand-sides. Thus

$$A^TA = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

$$(A^TA)^{-1} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{11}{30} & \frac{1}{30} & -\frac{1}{10} \\ \frac{1}{30} & \frac{11}{30} & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{1}{10} & \frac{3}{10} \end{pmatrix}$$

$$\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{pmatrix} \frac{11}{30} & \frac{1}{30} & -\frac{1}{10} \\ \frac{1}{30} & \frac{11}{30} & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{1}{10} & \frac{3}{10} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} 4 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{8}{5} \\ \frac{3}{5} \\ \frac{6}{5} \end{pmatrix}$$

Sec 5.3, No 2: For your solution $\hat{\mathbf{x}}$ in Exercise 1 (c):

(a) determine the projection $\mathbf{p} = A\hat{\mathbf{x}}$

Solution: Here is the calculation:

$$\mathbf{p} = A\hat{\mathbf{x}} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{8}{5} \\ \frac{3}{5} \\ \frac{6}{5} \end{pmatrix} = \begin{pmatrix} \frac{17}{5} \\ \frac{1}{5} \\ \frac{3}{5} \\ \frac{14}{5} \end{pmatrix}$$

(b) calculate the residual $r(\hat{\mathbf{x}})$

Solution: The residual $r(\hat{\mathbf{x}}) = \mathbf{b} - \mathbf{p}$ is

$$r(\hat{\mathbf{x}}) = \mathbf{b} - \mathbf{p} = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} \frac{17}{5} \\ \frac{1}{5} \\ \frac{3}{5} \\ \frac{14}{5} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ \frac{2}{5} \\ -\frac{4}{5} \end{pmatrix}$$

(c) verify that $r(\hat{\mathbf{x}}) \in N(A^T)$

Solution: We simply need to show that $A^Tr(\hat{\mathbf{x}}) = \mathbf{0}$

$$A^Tr(\hat{\mathbf{x}}) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ \frac{2}{5} \\ -\frac{4}{5} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ \frac{2}{5} \\ -\frac{4}{5} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Sec 5.3, No 5 (a): Find the best least squares fit by a linear function to the data

x	-1	0	1	2
y	0	1	3	9

Solution: For a linear fit $y = c_1 + c_2 x$ we need to solve $A\mathbf{c} = \mathbf{y}$ where

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 9 \end{pmatrix}$$

Of course, this system does not (generally) have a solution and so we use the least squares solution. The normal equations are

$$A^T A\mathbf{c} = A^T \mathbf{y}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}^T \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}^T \begin{pmatrix} 0 \\ 1 \\ 3 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 13 \\ 21 \end{pmatrix}$$

$$\mathbf{c} = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 13 \\ 21 \end{pmatrix} = \begin{pmatrix} \frac{9}{5} \\ \frac{29}{10} \end{pmatrix}$$

Therefore the best fit line is $y = \frac{9}{5} + \frac{29}{10} x$.