Solutions to Assignment 7

Math 217, Fall 2002

4.3.10 Find a basis for the null space of the following matrix: $A = \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix}$.

We need to find a basis for the solutions to the equation $A\mathbf{x} = \mathbf{0}$. To do this we first put A in row reduced echelon form. The result (according to the computer) is: $\begin{bmatrix} 1 & 0 & -5 & 0 & 7 \\ 0 & 1 & -4 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$.

From this we can read the general solution, $\mathbf{x} = \begin{bmatrix} 5x_3 - 7x_5 \\ 4x_3 - 6x_5 \\ x_3 \\ 3x_5 \\ x_5 \end{bmatrix}$. We can also write this as $\mathbf{x} = x_3 \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix}$, or $\mathrm{Span}\left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$. Because these two vectors are clearly not multiples of one another, they also give a basis. So a basis for null(A) is $\left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$.

3.3.28 Let $\mathbf{v}_1 = \begin{bmatrix} 7 \\ 4 \\ -9 \\ -5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ -7 \\ 2 \\ 5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -5 \\ 3 \\ 4 \end{bmatrix}$. It can be verified that $\mathbf{v}_1 - 3\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$. Use this information to find a basis for $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

By the Spanning Set Theorem, some subset of the $\mathbf{v}_i$ is a basis for H. In class we showed how to find this subset. We simply remove any of the vectors involved in a non-trivial linear relation. So I choose to remove $\mathbf{v}_3$ (I could have removed any of the $\mathbf{v}_i$ because they each occur with a non-zero coefficient in the dependency relation $\mathbf{v}_1 - 3\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$). The remaining vectors then give a basis for H. We know they span by the Spanning Set Theorem. They are also linearly independent, because they are not multiples of one another.

4.3.30 Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$, with $k > n$. Use a theorem from Chapter 1 to explain why S cannot be a basis for $\mathbb{R}^n$.

Theorem 8, page 69, claims that any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$. That is exactly the situation we find ourselves in (expect that we have used the letter k instead of p). Thus $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly dependent, and cannot be a basis.

4.3.32 Suppose that T is a one-to-one transformation. Show that if the set of images $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is linearly dependent, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent.

This problem should say that T is a linear transformation (the book has a typo).

If the set $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is linearly dependent, then there are $c_1, \ldots, c_p \in \mathbb{R}$ not all zero such that $c_1 T(\mathbf{v}_1) + \cdots + c_p T(\mathbf{v}_p) = \mathbf{0}$. Because T is a linear transformation, we can rewrite this as $T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = \mathbf{0}$. We know that $T(\mathbf{0}) = 0$, so because T is one-to-one it must be the case that $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$. This gives a dependency relation on the $\mathbf{v}_i$ (recall that not all the c_i are zero), and thus $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent.

4.4.14 The set $\mathcal{B} = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 3 + t - 6t^2$ relative to $\mathcal{B}$.

We need to write $\mathbf{p}$ in terms of the basis $\mathcal{B}$, that is, find $x_1, x_2, x_3 \in \mathbb{R}$ such that $x_1(1 - t^2) + x_2(t - t^2) + x_3(2 - 2t + t^2) = 3 + t - 6t^2$. Multiplying things out, we get $(x_1 + 2x_3) + (x_2 - 2x_3)t + (-x_1 - x_2 + x_3)t^2 = 3 + t - 6t^2$. Thus we have to solve the three linear equations:

$$\begin{aligned} x_1 + 2x_3 &= 3 \\ x_2 - 2x_3 &= 1 \\ -x_1 - x_2 + x_3 &= -6 \end{aligned}$$

We form the augmented matrix for this system,

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ -1 & -1 & 1 & -6 \end{bmatrix}.$$

In row reduced echelon form, this is the matrix $\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$. So we see that $x_1 = 7$, $x_2 = -3$, and $x_3 = -2$.

This implies that $7(1-t^2) + (-3)(t-t^2) + (-2)(2-2t+t^2) = 3 + t - 6t^2$, so

$$[3 + t - 6t^2]_{\mathcal{B}} = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix} \in \mathbb{R}^3.$$

4.4.20 Suppose that $\{\mathbf{v}_1, \dots, \mathbf{v}_4\}$ is a linearly dependent spanning set for a vector space V. Show that each $\mathbf{w}$ in V can be expressed in more than one way as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_4$.

Because the $\mathbf{v}_i$ are linearly dependent, there are $c_1, \dots, c_4 \in \mathbb{R}$ not all zero such that $c_1\mathbf{v}_1 + \cdots + c_4\mathbf{v}_4 = \mathbf{0}$. Given $\mathbf{w}$ there are $d_1, \dots, d_4 \in \mathbb{R}$ such that $d_1\mathbf{v}_1 + \cdots + d_4\mathbf{v}_4 = \mathbf{w}$ (because the $\mathbf{v}_i$ are a spanning set). Consider the linear combination $(c_1+d_1)\mathbf{v}_1 + \cdots + (c_4+d_4)\mathbf{v}_4$. We have that $(c_1+d_1)\mathbf{v}_1 + \cdots + (c_4+d_4)\mathbf{v}_4 = (c_1\mathbf{v}_1 + \cdots + c_4\mathbf{v}_4) + (d_1\mathbf{v}_1 + \cdots + d_4\mathbf{v}_4) = \mathbf{0} + \mathbf{w} = \mathbf{w}$. This constitutes a different linear combination than $d_1\mathbf{v}_1 + \cdots + d_4\mathbf{v}_4$ because not all of the c_i are zero, and hence for some i between 1 and 4, we have that $c_i + d_i \neq d_i$.

4.4.32 Let $\mathbf{p}_1(t) = 1 + t^2$, $\mathbf{p}_2(t) = 2 - t + 3t^2$, $\mathbf{p}_3(t) = 1 + 2t - 4t^2$.

(a) Use coordinate vectors to show that these polynomials form a basis for $\mathbb{P}_2$.

We know that a basis for $\mathbb{P}_2$ is $\mathcal{B} = \{1, t, t^2\}$. It is not difficult to see that if the $\mathbf{p}_i$ are dependent, then so are the images $[\mathbf{p}_1]_{\mathcal{B}}, [\mathbf{p}_2]_{\mathcal{B}}, [\mathbf{p}_3]_{\mathcal{B}}$. That is, if $\mathbf{0} = c_1\mathbf{p}_2 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3$ is a nontrivial dependency (i.e., not all c_i are zero), then taking the coordinate mapping of both sides of the equation yields the dependency relation $\mathbf{0} = [\mathbf{0}]_{\mathcal{B}} = [c_1\mathbf{p}_2 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3]_{\mathcal{B}} = c_1[\mathbf{p}_2]_{\mathcal{B}} + c_2[\mathbf{p}_2]_{\mathcal{B}} + c_3[\mathbf{p}_3]_{\mathcal{B}}$.

We also know that if $[\mathbf{p}_1]_{\mathcal{B}}, [\mathbf{p}_2]_{\mathcal{B}}, [\mathbf{p}_3]_{\mathcal{B}}$ spans $\mathbb{R}^3$, then the $\mathbf{p}_i$ span $\mathbb{P}_2$. That is, assuming the $[\mathbf{p}_i]_{\mathcal{B}}$ span, we know that for each $f \in \mathbb{P}_2$ there are d_i such that $[f]_{\mathcal{B}} = d_1[\mathbf{p}_1]_{\mathcal{B}} + d_2[\mathbf{p}_2]_{\mathcal{B}} + d_3[\mathbf{p}_3]_{\mathcal{B}} = [d_1\mathbf{p}_1 + d_2\mathbf{p}_2 + d_3\mathbf{p}_3]_{\mathcal{B}}$, and because the coordinate mapping is one-to-one, this implies that $f = d_1\mathbf{p}_1 + d_2\mathbf{p}_2 + d_3\mathbf{p}_3$.

So to show the $\mathbf{p}_i$ are a basis of $\mathbb{P}_2$, it is enough to show that the $[\mathbf{p}_i]_{\mathcal{B}}$ are a basis of $\mathbb{R}^3$. By the Invertible Matrix Theorem, this set will be a basis if and only if the matrix $\big[[\mathbf{p}_1]_{\mathcal{B}}[\mathbf{p}_2]_{\mathcal{B}}[\mathbf{p}_3]_{\mathcal{B}}\big] = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}$ has a pivot in every row (because then by IMT the columns will span and will be linearly independent). So we put $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}$ in row reduced echelon form and obtain: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We conclude that the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ does form a basis of $\mathbb{P}_2$.

(b) Consider the basis $\mathcal{B} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for $\mathbb{P}_2$. Find $\mathbf{q}$ in $\mathbb{P}_2$ given that $[\mathbf{q}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$.

Because $[\mathbf{q}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$, we know that

$$\mathbf{q} = -3\mathbf{p}_1 + \mathbf{p}_2 + 2\mathbf{p}_3 = -3(1+t^2) + (2-t+3t^2) + 2(1+2t-4t^2) = 1+3t-8t^2.$$

4.5.12 Find the dimension of the vector space spanned by $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$.

To find the dimension, we need to find the number of elements in a basis. So we form the matrix $\begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix}$ and count the number of pivot columns (this because we know that $\dim(\mathrm{Col}(A))$ for a matrix A is precisely the number of pivot columns of A). So we put $\begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix}$ in row reduced echelon form, obtaining the matrix: $\begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. This has 3 pivot columns, so the dimension of the subspace spanned by the vectors given is 3.

4.5.22 The first four Laguerre polynomials are 1, $1-t$, $2-4t+t^2$, and $6-18t+9t^3-t^3$. Show that these polynomials form a basis of $\mathbb{P}_3$.

Consider the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ of $\mathbb{P}_3$. Utilizing the same arguments as we did in question 4.4.32, we know that it is enough to show that the images of these polynomials form a basis of $\mathbb{R}^4$ under to coordinate mapping. Their images under this mapping are: $[1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $[1-t]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $[2-4t+t^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \end{bmatrix}$, and $[6 - 18t + 9t^3 - t^3]_{\mathcal{B}} = \begin{bmatrix} 6 \\ -18 \\ 9 \\ -1 \end{bmatrix}$.

By IMT, these will form a basis if and only if the matrix $\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ is invertible. This matrix will be invertible if and only if its determinate is non-zero (theorem 4 page 194). The computer tells me that the determinate of this matrix is equal to 1. So we conclude that the given polynomials do form a basis.

4.5.32 Let H be a nonzero subspace of V, and suppose T is a one-to-one (linear) mapping of V into W. Prove that dim $T(H)$ = dim H. If T happens to be a one-to-one mapping of V onto W, then dim V = dim W. Isomorphic finite dimensional vector spaces have the same dimension.

We know that if $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is linearly independent, then because T is one-to-one, $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)$ is linearly independent (that was an earlier homework problem, 4.3.32). We showed in class that if $\mathbf{v}_1, \ldots, \mathbf{v}_p$ spans H, then $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)$ spans $T(H)$. This means that because the $\mathbf{v}_1, \ldots, \mathbf{v}_p$ form a basis of H, the $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)$ give a basis of $T(H)$. They both have the same size, so both spaces have the same dimension.

When T is also onto, then taking $H = V$ we have that $T(V) = W$, and the statement follows from what we have already done.