Chapter 5

Inner product spaces

5.1 Length and Dot product in $\mathbb{R}^n$

Homework: [Textbook, §5.1 Ex. 9, 11, 15, 19, 23, 27, 31, 39, 41, 59, 67, 75, 77, 85, 99, 103; p. 290].

In this section *we discuss*

1. *Length of vectors in* $\mathbb{R}^n$.
2. *Dot product of vectors in* $\mathbb{R}^n$.
3. *Cauchy Swartz Inequality in* $\mathbb{R}^n$.
4. *Triangular Inequality in* $\mathbb{R}^n$.

Definition 5.1.1 We give the main definitions in this section as follows. Let

$$\mathbf{u} = (u_1, u_2, \ldots, u_n), \quad \mathbf{v} = (v_1, v_2, \ldots, v_n)$$

be two vectors in $\mathbb{R}^n$.

1. The **length** or **magnitude** of vector $\mathbf{v}$ is defined as

$$\| \mathbf{v} \| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

   (a) The length of $\mathbf{v}$ is also called the **norm** of $\mathbf{v}$.

   (b) Also, if $\| \mathbf{v} \| = 1$, then we say $\mathbf{v}$ is a **unit vector.**

   (c) The definition shows that $\| \mathbf{v} \| \geq 0$ and

$$\| \mathbf{v} \| = 0 \quad if\ and\ only\ if \quad \mathbf{v} = \mathbf{0}.$$

2. The **distance** between $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$d(\mathbf{u}, \mathbf{v}) = \| \mathbf{u} - \mathbf{v} \| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}.$$

3. The **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

4. The **angle** θ between $\mathbf{u}$ and $\mathbf{v}$ is defined by the formula

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} \qquad 0 \leq \theta \leq \pi.$$

   Remark. For this definition to make sense, we need to assert that

$$-1 \leq \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} \leq 1.$$

   We will prove this later.

Remarks. Here are some obvious comments:

1. The standard basis vectors $\mathbf{e_i} \in \mathbb{R}^n$ are unit vectors.
2. For a nonzero vector $\mathbf{v}$ and a nozero scalar, $c\mathbf{v}$ and $-c\mathbf{v}$ point to opposite directions.

Theorem 5.1.2 Suppose $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ is a vector and c is a scalar. Then

$$\| c\mathbf{v} \| = |c| \, \| \mathbf{v} \|,$$

where $|c|$ denotes the absolute value of c.

Proof. We have $c\mathbf{v} = (cv_1, cv_2, \ldots, cv_n)$. Therefore, $\| c\mathbf{v} \| =$

$$\sqrt{(cv_1)^2 + (cv_2)^2 + \cdots + (cv_n)^2} = \sqrt{c^2 \left(v_1^2 + v_2^2 + \cdots + v_n^2\right)} = |c| \, \| \mathbf{v} \| .$$

The proof is complete. ∎

Theorem 5.1.3 Suppose $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ is a nonzero vector. Then,

$$\mathbf{u} = \frac{\mathbf{v}}{\| \mathbf{v} \|}$$

has lenght 1. We say, $\mathbf{u}$ is the **unit vector in the direction of v.**

Proof.(*First, note that the statement of the theorem would not make sense. unless* $\mathbf{v}$ *is nonzero.*) Now,

$$\| \mathbf{u} \| = \left\| \frac{1}{\| \mathbf{v} \|} \mathbf{v} \right\| = \left| \frac{1}{\| \mathbf{v} \|} \right| \, \| \mathbf{v} \| = 1.$$

The proof is complete. ∎

Reading assignment: Read [Textbook, Example 1,2, p. 279-].

5.1.1 On Distance

The distance between $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ was defined in the main definition 5.1.1(2) as

$$d(\mathbf{u}, \mathbf{v}) = \parallel \mathbf{u} - \mathbf{v} \parallel$$

We have the following proposition

Proposition 5.1.4 Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are two vectors. Then

1. $d(\mathbf{u}, \mathbf{v}) \geq 0$.
2. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$.
3. $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$.

Proof. The proofs follow directly from the definition of distance. I will only prove the last statement. We have

$$d(\mathbf{u}, \mathbf{v}) = 0 \iff \parallel \mathbf{u} - \mathbf{v} \parallel \iff \mathbf{u} - \mathbf{v} = \mathbf{0} \iff \mathbf{u} = \mathbf{v}.$$

The proof is complete. ■

5.1.2 On Dot product

The following theorem describes some of the properties of dot product.

Theorem 5.1.5 Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are three vectors and c is a scalar. Then

1. (*Commutativity*):
$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}.$$
2. (*Distributivity*):
$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.$$

3. (*Associativity*):

$$c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u}) \cdot (c\mathbf{v}).$$

4. (*dot product and Norm*):

$$\mathbf{v} \cdot \mathbf{v} = \| \mathbf{v} \|^2 .$$

5. We have $\mathbf{v} \cdot \mathbf{v} \geq 0$ and

$$\mathbf{v} \cdot \mathbf{v} \Longleftrightarrow \mathbf{v} = \mathbf{0}.$$

Proof. Follows easily from the definition 5.1.1.

Definition 5.1.6 The vector space $\mathbb{R}^n$ together with (1) length, (2) dot product is called the **Euclidian** $n-$**Space.**

Reading assignment: Read [Textbook, Example 3-6, p. 282-].

5.1.3 Two Inequalities

Theorem 5.1.7 (Cauchy-Schwartz Inequality) Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are two vectors. Then,

$$| \mathbf{u} \cdot \mathbf{v} | \quad \leq \quad \| \mathbf{u} \| \| \mathbf{v} \| .$$

Proof. (*Case 1.*): Assume $\mathbf{u} = \mathbf{0}$. So,

$$| \mathbf{u} \cdot \mathbf{v} | = | \mathbf{0} \cdot \mathbf{v} | = 0 \quad \textit{and} \; \| \mathbf{u} \| \| \mathbf{v} \| = 0 \, \| \mathbf{v} \| = 0.$$

So, the inequality is valid if $\mathbf{u} = \mathbf{0}$.

(*Case 2.*): Assume $\mathbf{u} \neq \mathbf{0}$. So, $a = \mathbf{u} \cdot \mathbf{u} = \parallel \mathbf{u} \parallel^2 > 0$. Let t be any real number. We have

$$(t\mathbf{u} + \mathbf{v}) \cdot (t\mathbf{u} + \mathbf{v}) = \parallel (t\mathbf{u} + \mathbf{v}) \parallel^2 \geq 0.$$

Expanding it, we have

$$t^2(\mathbf{u} \cdot \mathbf{u}) + 2t(\mathbf{u} \cdot \mathbf{v}) + (\mathbf{v} \cdot \mathbf{v}) \geq 0.$$

We have $a = \mathbf{u} \cdot \mathbf{u} = \parallel \mathbf{u} \parallel^2 > 0$, and write $b = 2(\mathbf{u} \cdot \mathbf{v})$ abd $c = (\mathbf{v} \cdot \mathbf{v})$. So, the polynomial $f(t) = at^2 + bt + c \geq 0$ for all t. So, $f(t)$ either has no real root or has a single repeated root. By the Quadratic formula, we have

$$b^2 - 4ac \leq 0 \qquad \textit{or} \qquad b^2 \leq 4ac.$$

This means

$$4(\mathbf{u} \cdot \mathbf{v})^2 \leq 4(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) = 4 \parallel \mathbf{u} \parallel^2 \parallel \mathbf{v} \parallel^2 .$$

Taking square root, we have

$$\mid \mathbf{u} \cdot \mathbf{v} \mid \quad \leq \quad \parallel \mathbf{u} \parallel \parallel \mathbf{v} \parallel .$$

The proof is complete. ■

Theorem 5.1.8 (Triangule Inequality) Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are two vectors. Then,

$$\parallel \mathbf{u} + \mathbf{v} \parallel \quad \leq \quad \parallel \mathbf{u} \parallel + \parallel \mathbf{v} \parallel .$$

Proof. We have

$$\parallel \mathbf{u} + \mathbf{v} \parallel^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v}$$

$$= \parallel \mathbf{u} \parallel^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \parallel \mathbf{v} \parallel^2 \leq \parallel \mathbf{u} \parallel^2 + 2 \mid \mathbf{u} \cdot \mathbf{v} \mid + \parallel \mathbf{v} \parallel^2 .$$

By Cauchy-Schwartz Inequality 5.1.7 $\mid \mathbf{u} \cdot \mathbf{v} \mid \quad \leq \quad \parallel \mathbf{u} \parallel \parallel \mathbf{v} \parallel$. So, we get

$$\parallel \mathbf{u} + \mathbf{v} \parallel^2 \leq \parallel \mathbf{u} \parallel^2 + 2 \parallel \mathbf{u} \parallel \parallel \mathbf{v} \parallel + \parallel \mathbf{v} \parallel^2 = (\parallel \mathbf{u} \parallel + \parallel \mathbf{v} \parallel)^2$$

The theorem is established by taking square root. ■

Definition 5.1.9 Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are two vectors. We say that they are **orthogonal**, if $\mathbf{u} \cdot \mathbf{v} = 0$.

Theorem 5.1.10 (Pythagorian) Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are two orthogoanl vectors. Then

$$\| \mathbf{u} + \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2 .$$

Proof. From the proof or triangular inequality 5.1.8

$$\| \mathbf{u} + \mathbf{v} \|^2 = \mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v} = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2 .$$

The proof is complete. ■

Reading assignment: Read [Textbook, Example 7-10, p. 285-].

Exercise 5.1.11 (Ex. 10, p. 290) Let

$$\mathbf{u} = (1, 2, 1), \quad \mathbf{v} = (0, 2, -2).$$

1. Compute $\| \mathbf{u} \|$.

   Solution: We have

   $$\| \mathbf{u} \| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}.$$

2. Compute $\| \mathbf{v} \|$.

   Solution: We have

   $$\| \mathbf{v} \| = \sqrt{0^2 + 2^2 + (-2)^2} = \sqrt{8}.$$

3. Compute $\| \mathbf{u} + \mathbf{v} \|$.

   Solution: We have

   $$\| \mathbf{u} + \mathbf{v} \| = \| \mathbf{u} + \mathbf{v} \| = \| (1, 4, -1) \| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}.$$

Exercise 5.1.12 (Ex. 16, p. 290) Let

$$\mathbf{u} = (-1, 3, 4).$$

1. Compute the unit vector in the direction of $\mathbf{u}$

   Solution: First,

   $$\| \mathbf{u} \| = \sqrt{(-1)^2 + 3^2 + 4^2} = \sqrt{26}.$$

   The unit vector in the direction of $\mathbf{u}$ is

   $$\mathbf{e} = \frac{\mathbf{u}}{\| \mathbf{u} \|} = \frac{(-1, 3, 4)}{\sqrt{26}} = \left(-\frac{1}{\sqrt{26}}, \frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}} \right).$$

2. Compute the unit vector in the direction opposite of $\mathbf{u}$.

   Solution: Answer is

   $$-\mathbf{e} = \left(\frac{1}{\sqrt{26}}, -\frac{3}{\sqrt{26}}, -\frac{4}{\sqrt{26}} \right).$$

Exercise 5.1.13 (Ex. 24, p. 290) Let $\mathbf{v}$ be a vector in the

$$\textit{same direction as} \quad \mathbf{u} = (-1, 2, 1) \quad \textit{and} \quad \| \mathbf{v} \| = 4.$$

Compute $\mathbf{v}$.

Solution: We have $\mathbf{v} = c\mathbf{u}$ with $c > 0$. So,

$$4 = \| \mathbf{v} \| = \| c\mathbf{u} \| = |c| \, \| \mathbf{u} \| = |c|\sqrt{(-1)^2 + 2^2 + 1^2} = |c|\sqrt{6}.$$

Since $c > 0$, we have $c = \frac{4}{\sqrt{6}}$ and $\mathbf{v} = c\mathbf{u} = \frac{4}{\sqrt{6}} (-1, 2, 1)$.

Exercise 5.1.14 (Ex. 28, p. 290) Let $\mathbf{v} = (-1, 3, 0, 4)$.

1. Find **u** such that **u** has same direction as **v** and one-half its length.

   Solution: In general,

   $$\parallel c\mathbf{v} \parallel = |c| \parallel \mathbf{v} \parallel .$$

   So, in this case,

   $$\mathbf{u} = \frac{1}{2}\mathbf{v} = \frac{1}{2}(-1,3,0,4) = \left(-\frac{1}{2}, \frac{3}{2}, 0, 2\right).$$

2. Find **u** such that **u** has opposite direction as **v** and one-fourth its length.

   Solution: Since it has opposite direction

   $$\mathbf{u} = -\frac{1}{4}\mathbf{v} = -\frac{1}{4}(-1,3,0,4) = \left(\frac{1}{4}, -\frac{3}{4}, 0, -1\right)$$

3. Find **u** such that **u** has opposite direction as **v** and twice its length.

   Solution: Since it has opposite direction

   $$\mathbf{u} = -2\mathbf{v} = -2(-1,3,0,4) = (2,-6,0,-8).$$

Exercise 5.1.15 (Ex. 32, p. 290) Find the distance between

$$\mathbf{u} = (1,2,0) \qquad \text{and} \qquad \mathbf{v} = (-1,4,1).$$

Solution: Distance

$$d(\mathbf{u},\mathbf{v}) = \parallel \mathbf{u} - \mathbf{v} \parallel = \parallel (2,-2,-1) \parallel = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3.$$

Exercise 5.1.16 (Ex. 40, p. 290) Let

$$\mathbf{u} = (0,4,3,4,4) \qquad \text{and} \qquad \mathbf{v} = (6,8,-3,3,-5).$$

1. Find $\mathbf{u} \cdot \mathbf{v}$.

   Solution: We have

$$\mathbf{u} \cdot \mathbf{v} = (0,4,3,4,4) \cdot (6,8,-3,3,-5) =$$

$$0*6+4*8+3*(-3)+4*3+4*(-5) = 15.$$

2. Compute $\mathbf{u} \cdot \mathbf{u}$.

   Solution: We have

$$\mathbf{u} \cdot \mathbf{u} = (0,4,3,4,4) \cdot (0,4,3,4,4) = 0+16+9+16+16 = 57.$$

3. Compute $\| \mathbf{u} \|^2$.

   Solution: From (2), we have

$$\| \mathbf{u} \|^2 = \mathbf{u} \cdot \mathbf{u} = 57.$$

4. Compute $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$.

   Solution: From (1), we have

$$(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 15\mathbf{v} = 15(0,4,3,4,4) = (0,60,45,60,60).$$

Exercise 5.1.17 (Ex. 42, p. 290) Let $\mathbf{u}, \mathbf{v}$ be two vectors in $\mathbb{R}^n$. It is given,

$$\mathbf{u} \cdot \mathbf{u} = 8, \quad \mathbf{u} \cdot \mathbf{v} = 7, \quad \mathbf{v} \cdot \mathbf{v} = 6.$$

Find $(3\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - 3\mathbf{v})$.

Solution: We have

$$(3\mathbf{u}-\mathbf{v}) \cdot (\mathbf{u}-3\mathbf{v}) = 3\mathbf{u} \cdot \mathbf{u} - 10\mathbf{u} \cdot \mathbf{v} + 3\mathbf{v} \cdot \mathbf{v} = 3*8-10*7+3*6 = -28.$$

Exercise 5.1.18 (Ex. 62, p. 291) Let

$$\mathbf{u} = (1, -1, 0) \qquad and \qquad \mathbf{v} = (0, 1, -1).$$

Verify Cauchy-Schwartz inequality (see 5.1.7).

Solution: We have

$$\| \mathbf{u} \| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2} \quad and \quad \| \mathbf{v} \| = \sqrt{0^2 + 1^2 + (-1)^2} = \sqrt{2}.$$

Also,

$$\mathbf{u} \cdot \mathbf{v} = 1 * 0 + (-1) * 1 + 0 * (-1) = -1.$$

Therefore, it is verified that

$$|\mathbf{u} \cdot \mathbf{v}| = 1 \leq 2 = \| \mathbf{u} \| \| \mathbf{v} \| .$$

Exercise 5.1.19 (Ex. 68, p. 291) Let

$$\mathbf{u} = (2, 3, 1) \qquad and \qquad \mathbf{v} = (-3, 2, 0).$$

Find the angle θ between them.

Solution: The **angle** θ between **u** and **v** is defined (see 5.1.1), by the formula

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} \qquad 0 \leq \theta \leq \pi.$$

We have

$$\| \mathbf{u} \| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}, \quad \| \mathbf{v} \| = \sqrt{(-3)^2 + 2^2 + 0^2} = \sqrt{13}$$

and

$$\mathbf{u} \cdot \mathbf{v} = 2 * (-3) + 3 * 2 + 1 * 0 = 0.$$

So,

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} = 0 \qquad and \quad \theta = \pi/2.$$

Exercise 5.1.20 (Ex. 78, p. 291) Let $\mathbf{u} = (2, -1, 1)$. Find all vectors that are orthogonal to $\mathbf{u}$.

Solution: Suppose $\mathbf{v} = (x, y, z)$ be orthogonal to $\mathbf{u}$. By definition, it means,

$$\mathbf{u} \cdot \mathbf{v} = 2x - y + z = 0.$$

A parametric solution to this system is

$$x = t, y = s, z = s - 2t.$$

So, the set of vectors orthogonal to $\mathbf{u}$, is given by

$$\{\mathbf{v} = (t, s, s - 2t) : \quad t, s \in \mathbb{R}\}$$

Exercise 5.1.21 (Ex. 82, p. 291) Let

$$\mathbf{u} = (4, 3) \quad \mathbf{v} = \left(\frac{1}{2}, -\frac{2}{3}\right).$$

Determine if are $\mathbf{u}, \mathbf{v}$ orthogonal to each other or not?

Solution: We need to check, if $\mathbf{u} \cdot \mathbf{v} = 0$ or not. We have

$$\mathbf{u} \cdot \mathbf{v} = 4 * \frac{1}{2} + 3 * \left(-\frac{2}{3}\right) = 0.$$

So, $\mathbf{u}, \mathbf{v}$ are orthogonal to each other.

Exercise 5.1.22 (Ex. 86, p. 291) Let

$$\mathbf{u} = (0, 1, 6) \quad \mathbf{v} = (1, -2, -1).$$

Determine if are $\mathbf{u}, \mathbf{v}$ orthogonal to each other or not?

Solution: We need to check, if $\mathbf{u} \cdot \mathbf{v} = 0$ or not. We have

$$\mathbf{u} \cdot \mathbf{v} = 0 * 1 + 1 * (-2) + 6 * (-1) = -7 \neq 0.$$

So, $\mathbf{u}, \mathbf{v}$ are not orthogonal to each other.

Exercise 5.1.23 (Ex. 100, p. 292) Let

$$\mathbf{u} = (1, 1, 1) \quad \mathbf{v} = (0, 1, -2).$$

Verify, triangle Inequality (see 5.1.8).

Solution: We have

$$\| \mathbf{u} \| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}, \quad \| \mathbf{v} \| = \sqrt{0^2 + 1^2 + (-2)^2} = \sqrt{5}$$

and

$$\| \mathbf{u} + \mathbf{v} \| = \| (1, 2, -1) \| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}.$$

We need to check,

$$\| \mathbf{u} + \mathbf{v} \|^2 = 6 \quad \leq \quad \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2 = 15.$$

So, the triangle inequality is verified.

Exercise 5.1.24 (Ex. 104, p. 292) Let

$$\mathbf{u} = (3, -2) \quad \mathbf{v} = (4, 6).$$

Verify Pythagorian Theorem (see 5.1.10).

Solution: We have $\mathbf{u} \cdot \mathbf{v} = 3 * 4 - 2 * 6 = 0$. So, $\mathbf{u}, \mathbf{v}$ are orthogonal to each other and Pythagorian Theorem (see 5.1.10) must hold.

$$\| \mathbf{u} \| = \sqrt{3^2 + (-2)^2} = \sqrt{13}, \quad \| \mathbf{v} \| = \sqrt{4^2 + 6^2} = \sqrt{52}$$

and

$$\| \mathbf{u} + \mathbf{v} \| = \| (7, 4) \| = \sqrt{7^2 + 4^2} = \sqrt{65}.$$

We need to check,

$$\| \mathbf{u} + \mathbf{v} \|^2 = 65 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2 = 13 + 52.$$

So, the Pythagorian Theorem is verified.

5.2 Inner product spaces

Homework: [Textbook, Ex. 3, 5, 7, 11, 13, 15, 59, 61; p. 303-].

In this section *we define abstract inner product spaces. The concepts of length and dot product on the Euclidean spaces* $\mathbb{R}^n$ *is extended to vector spaces with innner products as follows.*

Definition 5.2.1 Suppose V is a vector space. An **inner product** on V is a function

$$< *, * >: V \times V \to \mathbb{R}$$

that associates each pair $(\mathbf{u}, \mathbf{v})$ of elements in V to a real number $< \mathbf{u}, \mathbf{v} >$ such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and scalar c, we have

1. $< \mathbf{u}, \mathbf{v} >=< \mathbf{v}, \mathbf{u} >$.
2. $< \mathbf{u}, \mathbf{v} + \mathbf{w} >=< \mathbf{u}, \mathbf{v} > + < \mathbf{u}, \mathbf{w} >$.
3. $c < \mathbf{u}, \mathbf{v} >=< c\mathbf{u}, \mathbf{v} >$.
4. $< \mathbf{v}, \mathbf{v} >\geq 0$ and $v = 0 \Longleftrightarrow < \mathbf{v}, \mathbf{v} >= 0$.

A vector space V together with an inner product $< *, * >$ is called an **inner product space**. For such an inner product space,

1. The **length** of a vector $\mathbf{v} \in V$ is defined as

$$\| \mathbf{v} \| = \sqrt{(< \mathbf{v}, \mathbf{v} >)}.$$

   The length $\| \mathbf{v} \|$, is also called the **norm** of v.

2. The distance between two vectors $\mathbf{u}, \mathbf{v} \in V$ is defined as

$$d(\mathbf{u}, \mathbf{v}) = \parallel \mathbf{u} - \mathbf{v} \parallel$$

3. The angle θ between two vectors $\mathbf{u}, \mathbf{v} \in V$ is defined by the formula:

$$\cos\theta = \frac{< \mathbf{u}, \mathbf{v} >}{\parallel \mathbf{u} \parallel \parallel \mathbf{v} \parallel} \qquad 0 \leq \theta \leq \pi.$$

Example 5.2.2 For $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^n$, define

$$< \mathbf{u}, \mathbf{v} >= \mathbf{u} \cdot \mathbf{v}.$$

This is an inner product on $\mathbb{R}^n$. So, $\mathbb{R}^n$, together with dot product is an inner product space.

A better and nontrivial example is [Textbook, Example 5], which discuss as follows.

Example 5.2.3 Let $V = C[a, b]$ be the vector space of all continuous functions $f : [a, b] \to \mathbb{R}$. For $f, g \in C[a, b]$, define inner product

$$< f, g >= \int_a^b f(x)g(x)dx.$$

It is easy to check that $< f, g >$ satisfies the properties of definition 5.2.1 of inner product space. Namely, we have

1. $< f, g >=< g, f >$, for all $f, g \in C[a, b]$.

2. $< f, g + h >=< f, g > + < f, h >$, for all $f, g, h \in C[a, b]$.

3. $c < f, g >=< cf, g >$, for all $f, g \in C[a, b]$ and $c \in \mathbb{R}$.

4. $< f, f >\geq 0$ for all $f \in C[a, b]$ and $f = 0 \Leftrightarrow < f, f >= 0$.

Accordingly, for $f \in C[a,b]$, we can define length (or norm)

$$\| f \| = \sqrt{< f, f >} = \sqrt{\int_a^b f(x)^2 dx}.$$

This 'length' will have all the properties that you expect length to have.

The following are some properties of inner product:

Theorem 5.2.4 Let V be an inner product space and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c \in \mathbb{R}$, be a scalar. Then,

1. $< \mathbf{v}, \mathbf{0} > = < \mathbf{0}, \mathbf{u} > = 0$.
2. $< \mathbf{u} + \mathbf{v}, \mathbf{w} > = < \mathbf{u}, \mathbf{w} > + < \mathbf{v}, \mathbf{w} >$.
3. $< \mathbf{u}, c\mathbf{v} > = c < \mathbf{u}, \mathbf{v} >$.

Proof. All these three statements follows from cummutativity, (proerty (1) of definition 5.2.1).

First, the first equaity of (1) follows from cummutativity, (proerty (1) of definition 5.2.1). Then, we have

$$< \mathbf{v}, \mathbf{0} > = < \mathbf{v}, \mathbf{0} + \mathbf{0} > = < \mathbf{v}, \mathbf{0} > + < \mathbf{v}, \mathbf{0} > .$$

Now, subtracting $< \mathbf{v}, \mathbf{0} >$ from both sides, we get $< \mathbf{v}, \mathbf{0} > = 0$.

To prove (2), we have

$$< \mathbf{u}+\mathbf{v}, \mathbf{w} > = < \mathbf{w}, \mathbf{u}+\mathbf{v} > = < \mathbf{w}, \mathbf{u} > + < \mathbf{w}, \mathbf{v} > = < \mathbf{u}, \mathbf{w} > + < \mathbf{v}, \mathbf{w} > .$$

To prove (3), we have

$$< \mathbf{u}, c\mathbf{v} > = < c\mathbf{v}, \mathbf{v} > = c < \mathbf{v}, \mathbf{u} > = c < \mathbf{u}, \mathbf{v} > .$$

The proof is complete. ■

Theorem 5.2.5 Let V be an inner product space and $\mathbf{u}, \mathbf{v} \in V$. Then,

1. Cauchy-Schwartz Inequality:
$$| < \mathbf{u}, \mathbf{v} > | \leq \| \mathbf{u} \| \| \mathbf{v} \| .$$

2. Triangle Inequality:
$$\| \mathbf{u} + \mathbf{v} \| \leq \| \mathbf{u} \| + \| \mathbf{v} \| .$$

3. (Definition) We say $\mathbf{u}, \mathbf{v}$ are (mutually) orthogonal if
$$< \mathbf{u}, \mathbf{v} >= 0.$$
In this case, we write $\mathbf{u} \perp \mathbf{v}$, and say they are perpendicular to each other.

4. If $\mathbf{u}, \mathbf{v}$ are orthogonal, then
$$\| \mathbf{u} + \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2 .$$
This is called **Pythagorean Theorem.**

Proofs. The proofs are exactly, line for line, similar to that of the corresponding theorems in section 5.1.

1. To prove (1) Cauchy-Schwartz Inequality, repeat the proof of theorem 5.1.7.

2. To prove (2) the Triangle Inequality, repeat the proof of theorem 5.1.8.

3. To prove the Pythagorean Theorem, repeat the proof of theorem 5.1.10.

So, the proofs are complete. ■

5.2.1 Orthogonal Projections

Definition 5.2.6 Let V be an inner product space. Suppose $\mathbf{v} \in V$ is a vector. Then,

$$for \quad \mathbf{u} \in V \quad define \qquad proj_{\mathbf{v}}(\mathbf{u}) = \frac{< \mathbf{v}, \mathbf{u} >}{\| \mathbf{v} \|^2} \mathbf{v}$$

It is easy to check that $(\mathbf{u} - proj_{\mathbf{v}}(\mathbf{u})) \perp proj_{\mathbf{v}}(\mathbf{u})$.

Reading assignment: Read [Textbook, Example 1-8, p. 293-].

Exercise 5.2.7 (Ex. 4, p.303) In $\mathbb{R}^2$, define an inner product

$$for \quad \mathbf{u} = (u_1, u_2), \ \mathbf{v} = (v_1, v_2) \quad define \quad < \mathbf{u}, \mathbf{v} >= u_1 v_1 + 2u_2 v_2.$$

(*It is easy to check that it defines an inner product, as defined in (5.2.1).*) Now let

$$\mathbf{u} = (0, -6), \quad \mathbf{v} = (-1, 1).$$

1. Compute $< \mathbf{u}, \mathbf{v} >$.

   Solution: We have

   $$< \mathbf{u}, \mathbf{v} >= u_1 v_1 + 2u_2 v_2 = 0 * (-1) + 2(-6) * 1 = -12.$$

2. Compute $\| \mathbf{u} \|$.

   Solution: We have

   $$\| \mathbf{u} \| = \sqrt{< \mathbf{u}, \mathbf{u} >} = \sqrt{u_1 u_1 + 2u_2 u_2} = \sqrt{0 * 0 + 2(-6) * (-6)} = \sqrt{72}.$$

3. Compute $\| \mathbf{v} \|$.

   Solution: We have

   $$\| \mathbf{v} \| = \sqrt{< \mathbf{v}, \mathbf{v} >} = \sqrt{(-1)(-1) + 2(1) * (1)} = \sqrt{3}.$$

4. Compute $d(\mathbf{u}, \mathbf{v})$.

   Solution: We have

$$d(\mathbf{u}, \mathbf{v}) = \parallel \mathbf{u} - \mathbf{v} \parallel = \parallel (1, -7) \parallel = \sqrt{1 * 1 + 2(-7)(-7)} = \sqrt{99}.$$

Exercise 5.2.8 (Ex. 12, p. 303) Let $V = C[-1, 1]$ with inner product

$$< f, g > = \int_{-1}^{1} f(x)g(x)dx \qquad for \qquad f, g, \in V.$$

Let $f(x) = -x$ and $g(x) = x^2 - x + 2$.

1. Compute $< f, g >$.

   Solution: We have

$$< f, g > = \int_{-1}^{1} f(x)g(x)dx = \int_{-1}^{1} \left(-x^3 + x^2 - 2x\right) dx = \left[\frac{-x^4}{4} + \frac{x^3}{3} - 2\frac{x^2}{2}\right]_{x=-1}^{1}$$

$$= \left[\frac{-1}{4} + \frac{1}{3} - 2\frac{1}{2}\right] - \left[\frac{-1}{4} + \frac{-1}{3} - 2\frac{1}{2}\right] = \frac{2}{3}.$$

2. Compute norm $\parallel f \parallel$.

   Solution: We have

$$\parallel f \parallel = \sqrt{< f, f >} = \sqrt{\int_{-1}^{1} f(x)^2 dx} = \sqrt{\int_{-1}^{1} x^2 dx}$$

$$= \sqrt{\left[\frac{x^3}{3}\right]_{x=-1}^{1}} = \sqrt{\frac{1}{3} - \left(-\frac{1}{3}\right)} = \sqrt{\frac{2}{3}}$$

3. Compute norm $\parallel g \parallel$.

   Solution: We have

$$\parallel g \parallel = \sqrt{< g, g >} = \sqrt{\int_{-1}^{1} g(x)^2 dx} = \sqrt{\int_{-1}^{1} (x^2 - x + 2)^2 dx}$$

$$= \sqrt{\int_{-1}^{1} (x^4 - 2x^3 + 5x^2 - 4x + 4)\, dx}$$

$$= \sqrt{\left[\frac{x^5}{5} - 2\frac{x^4}{4} + 5\frac{x^3}{3} - 4\frac{x^2}{2} + 4x\right]_{-1}^{1}} = \sqrt{\frac{2}{5} + \frac{10}{3} + 8}.$$

4. Compute $d(f, g)$.

   Solution: We have

$$d(f, g) = \parallel f - g \parallel = \sqrt{< f, f >} = \sqrt{\int_{-1}^{1} (-x^2 - 2)^2 dx}$$

$$= \sqrt{\int_{-1}^{1} (x^4 + 4x^2 + 4)\, dx} = \sqrt{\left[\frac{x^5}{5} + 4\frac{x^3}{3} + 4x\right]_{-1}^{1}}$$

$$= \sqrt{\frac{2}{5} + \frac{8}{3} + 8}.$$

Exercise 5.2.9 (Ex. 60, p. 305) Let $V = C[-1, 1]$ with inner product

$$< f, g >= \int_{-1}^{1} f(x)g(x)dx \qquad for \qquad f, g, \in V.$$

Let $f(x) = x$ and $g(x) = \frac{3x^2-1}{2}$. Show that f and g are orthogonal.

Solution: We have to show that $< f, g >= 0$. We have

$$< f, g >= \int_{-1}^{1} f(x)g(x)dx = \int_{-1}^{1} x\frac{3x^2 - 1}{2}dx = \int_{-1}^{1} \frac{1}{2}(3x^3 - x)dx$$

$$= \left[\frac{1}{2}\left(3\frac{x^4}{4} - \frac{x^2}{2}\right)\right]_{-1}^{1} = \frac{1}{2}\left(3\frac{1}{4} - \frac{1}{2}\right) - \frac{1}{2}\left(3\frac{1}{4} - \frac{1}{2}\right) = 0.$$

So, $f \perp g$.