MAT 243 Spring 2015
Review for Test 1

1. Which of the following sets are equal to the set of all integers that are even. There may be more than one or none.
   (a) $\{2n|n \in \mathbb{R}\}$
   (b) $\{2n|n \in \mathbb{Z}\}$
   (c) $\{n \in \mathbb{Z}|n = 2k \text{ and } k \in \mathbb{Z}\}$
   (d) $\{2n\}$
   (e) $\{0, 2, 4, 6, \ldots\}$

2. Suppose $A = \{a, b, c\}$ and $B = \{b, \{c\}, \{a, c\}\}$. True or false.
   (a) $B \subseteq A$
   (b) $\emptyset \in B$
   (c) $\{b, \{c\}\} \subseteq A \cap B$
   (d) $\{b, \{c\}\} \subset B$
   (e) $\{c\} \in B$
   (f) $|A \cup B| = 5$
   (g) $|A \cap B| = 3$
   (h) $\{\{c\}, \{a, c\}\} \subset B - A$
3. Suppose $A = \mathbb{N}$ and $B = \{x \in \mathbb{R}| -4 \leq x \leq 5\}$. True or false.
   (a) $(4, 6) \in B \times A$
   (b) $|A \cup B| = \infty$
   (c) $|A \cap B| = \infty$
4. Given $f : [0, \infty) \to [0, \infty)$, $f(x) = 2\sqrt{x}$, find
   (a) The image of $\{4, 9, 16\}$.
   (b) The preimage of $\{4, 9, 16\}$.
5. Let $g : \mathbb{R} \to [0, \infty)$ be defined by $g(x) = \lceil x^2 \rceil$. Let $A = \{x \in [0, \infty)|3.2 < x < 8.9\}$.
   (a) domain
   (b) codomain
   (c) range
   (d) Find $g(A)$.
   (e) Find $g^{-1}(A)$.
6. Let $g : \mathbb{N} \to \mathbb{R}$ be defined by $g(x) = \lfloor \frac{x-2}{3} \rfloor$. Let $A = \{x \in \mathbb{N}|4 \leq x \leq 10\}$.
   (a) domain
   (b) codomain
   (c) range
   (d) Find $g(A)$.
   (e) Find $g^{-1}(A)$.
7. Let $E = \{4n|n \in \mathbb{N}\}$ and consider the characteristic function $\chi_E : \mathbb{Z} \to \mathbb{Z}$. What is the ...
   (a) domain
   (b) codomain
   (c) range
   (d) $\chi_E(\{2n|n \in \mathbb{N}\})$
   (e) $\chi_E^{-1}(\{2n|n \in \mathbb{N}\})$

8. Circle all of the following statements that are equivalent to "If x is even, then y is odd"? There may be more than one or none.
   (a) y is odd only if x is even.
   (b) x is even is sufficient for y to be odd.
   (c) x is even is necessary for y to be odd.
   (d) If x is odd, then y is even.
   (e) x is even and y is even.
   (f) x is odd or y is odd.
9. Which of the following is the negation of the statement "If you go to the beach this weekend, then you should bring your books and study"?
   (a) If you do not go to the beach this weekend, then you should not bring your books and you should not study.
   (b) If you do not go to the beach this weekend, then you should not bring your books or you should not study.
   (c) If you do not go to the beach this weekend, then you should bring your books and study.
   (d) You will not go to the beach this weekend, and you should not bring your books and you should not study.
   (e) You will not go to the beach this weekend, and you should not bring your books or you should not study.
   (f) You will go to the beach this weekend, and you should not bring your books and you should not study.
   (g) You will go to the beach this weekend, and you should not bring your books or you should not study.
10. Which of the following is the negation of the statement "You will go to the beach this weekend or you will not go swimming"?
   (a) You will not go to the beach this weekend or you will go swimming.
   (b) You will not go to the beach this weekend or you will not go swimming.
   (c) You will not go to the beach this weekend and you will go swimming.
   (d) You will not go to the beach this weekend and you will not go swimming.
11. p is the statement "I will prove this by cases", q is the statement "There are more than 500 cases," and r is the statement "I can find another way."
   (a) State $(\neg r \vee \neg q) \rightarrow p$ in simple English.
   (b) State the *converse* of the statement in part (a) in simple English.
   (c) State the *inverse* of the statement in part (a) in simple English.
   (d) State the *contrapositive* of the statement in part (a) in simple English.
   (e) State the *negation* of the statement in part (a) in simple English. Do not use the expression "It is not the case."
12. Make a truth table for $(p \oplus \neg r) \vee (\neg q \rightarrow (p \vee r))$. Is this statement a tautology, contradiction, or neither of these?
13. Prove or disprove
   (a) $[(p \rightarrow q) \rightarrow r] \Leftrightarrow [p \rightarrow (q \rightarrow r)]$
   (b) $[(p \wedge q) \rightarrow r] \Leftrightarrow [p \rightarrow (q \rightarrow r)]$
14. Prove $[(p \rightarrow r) \vee (q \rightarrow r)] \Leftrightarrow [(p \wedge q) \rightarrow r]$ by using...
   (a) a truth table,
   (b) a verbal (cases) argument,

(c) propositional equivalences.

15. Circle all of the following that is equivalent to $\neg(p \to r) \to \neg q$? There may be more than one or none.
    (a) $\neg(p \to r) \vee q$
    (b) $(p \wedge \neg r) \vee q$
    (c) $(\neg p \to \neg r) \vee q$
    (d) $q \to (p \to r)$
    (e) $\neg q \to (\neg p \to \neg r)$
    (f) $\neg q \to (\neg p \vee r)$
    (g) $\neg q \to \neg(p \to r)$
16. Let $P(n, m)$ be the predicate $mn > 0$, where the domain for m and n is the set of integers. Which of the following statements are true? There may be more than one or none.
    (a) $P(-3, 2)$
    (b) $\forall m P(0, m)$
    (c) $\exists n P(n, -3)$
    (d) $\exists n \forall m P(n, m)$
    (e) $\forall n \exists m P(n, m)$
    (f) $\exists! m P(2, m)$
17. Let $P(x, y)$ be the predicate $2x + y = xy$, where the domain of discourse for x is $\{u \in \mathbb{Z} | u \neq 1\}$ and for y is $\{u \in \mathbb{Z} | u \neq 2\}$. Determine the truth value of each statement. Show work or briefly explain.
    (a) $P(-1, 1)$
    (b) $\exists x P(x, 0)$
    (c) $\exists y P(4, y)$
    (d) $\forall y P(2, y)$
    (e) $\forall x \exists y P(x, y)$
    (f) $\exists y \forall x P(x, y)$
    (g) $\forall x \forall y[((P(x, y)) \wedge (x > 0)) \to (y > 1)]$
18. True or false. Mark true if it is true for all possible predicates, false otherwise.
    (a) $\forall x \forall y P(x, y) \Leftrightarrow \forall y \forall x P(x, y)$
    (b) $\forall x \exists y P(x, y) \Rightarrow \exists y \forall x P(x, y)$
    (c) $\forall x \exists y P(x, y) \Leftrightarrow \forall y \exists x P(y, x)$
    (d) $\forall x[P(x) \wedge Q(x)] \Leftrightarrow [(\forall x P(x)) \wedge (\forall x Q(x))]$
    (e) $\exists x[P(x) \wedge Q(x)] \Rightarrow [(\exists x P(x)) \wedge (\exists x Q(x))]$
    (f) $\neg \exists x \forall y P(x, y) \Leftrightarrow \forall y \exists x \neg P(x, y)$
    (g) $\forall x \exists y[P(x, y) \to \neg Q(x, y)] \Rightarrow \neg \exists x \forall y[P(x, y) \wedge Q(x, y)]$
19. Suppose $S(x, y)$ is the predicate "x saw y," $L(x, y)$ is the predicate "x liked y," and $C(y)$ is the predicate "y is a comedy." The universe of discourse of x is the set of people and the universe of discourse for y is the set of movies. Write the following in proper English. Do not use variables in your answers.
    (a) $\forall y \neg S(\text{Margaret}, y)$
    (b) $\exists y \forall x L(x, y)$
    (c) $\exists x \forall y[C(y) \to S(x, y)]$
    (d) Give the negation for part 19c in symbolic form with the negation symbol to the right of all quantifiers.

(e) state the negation of part 19c in English without using the phrase "it is not the case."

20. Suppose the universe of discourse for x is the set of all ASU students, the universe of discourse for y is the set of courses offered at ASU, $A(y)$ is the predicate "y is an advanced course," $F(x)$ is "x is a freshman," $T(x, y)$ is "x is taking y," and $P(x, y)$ is "x passed y." Use quantifiers to express the statements
    (a) No student is taking every advanced course.
    (b) Every freshman passed calculus.
    (c) Some advanced course(s) is(are) being taken by no students.
    (d) Some freshmen are only taking advanced courses.
    (e) No freshman has taken and passed linear algebra.
21. Write using predicates and quantifiers.
    (a) For every $m, n \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that $m < p$ and $p < n$.
    (b) For all nonnegative real numbers a, b, and c, if $a^2 + b^2 = c^2$, then $a + b \geq c$.
    (c) There does not exist a positive real number a such that $a + \dfrac{1}{a} < 2$.
    (d) Every student in this class likes mathematics.
    (e) No student in this class likes mathematics.
    (f) All students in this class that are CS majors are going to take a 4000 level math course.
22. Give the negation of each statement in example 21 using predicates and quantifiers with the negation to the right of all quantifiers.
23. Give the negation of each statement in example 21 using an English sentence.

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Test 1 Review Solutions

1 (b) and (c) only,

2a) False, 2b) False, 2c) False, 2d) True, 2e) True, 2f) True, 2g) False, 2h) False (they are equal),

3a) True, 3b) True, 3c) False,

4a) $\{4, 6, 8\}$, 4b) $\{4, 81/4, 64\}$,

5a) $\mathbb{R}$, 5b) $[0, \infty)$, 5c) $\mathbb{N}$, 5d) $g(A) = \{11, 12, 13, \ldots, 80\}$, 5e) $g^{-1}(A) = [-\sqrt{8}, -\sqrt{3}) \cup (\sqrt{3}, \sqrt{8}]$,

6a) $\mathbb{N}$, 6b) $\mathbb{R}$, 6c) $\{-1, 0, 1, 2, 3, \ldots\}$, 6d) $g(\{x \in \mathbb{N} | 4 \leq x \leq 10\}) = \{0, 1, 2\}$, 6e) $g^{-1}(\{x \in \mathbb{N} | 4 \leq x \leq 10\}) = \{x \in \mathbb{N} | 14 \leq x < 35\}$,

7a) $\mathbb{Z}$, 7b) $\mathbb{Z}$, 7c) $\{0, 1\}$, 7d) $\{0, 1\}$, 7e) $\{k \in \mathbb{Z} | k$ is not a nonnegative multiple of $4\}$,

8) (b) and (f) are the only equivalent statements.

9) (g),

10) (c),

11a) If I cannot find another way or there are not more than 500 cases, then I will prove this by cases.

11b) If I prove this by cases, then I could not find another way or there are not more than 500 cases.

11c) If I can find another way and there are more than 500 cases, then I will not prove this by cases.

11d) If I cannot prove this by cases, then I can find another way and there are more than 500 cases.

11e) I cannot find another way or there are not more than 500 cases, but I will not prove this by cases.

12)

p	q	r	$\neg q$	$\neg r$	$p \oplus \neg r$	$p \vee r$	$\neg q \to (p \vee r)$	$(p \oplus \neg r) \vee (\neg q \to (p \vee r))$
T	T	T	F	F	T	T	T	T
T	T	F	F	T	F	T	T	T
T	F	T	T	F	T	T	T	T
T	F	F	T	T	F	T	T	T
F	T	T	F	F	F	T	T	T
F	T	F	F	T	T	F	T	T
F	F	T	T	F	F	T	T	T
F	F	F	T	T	T	F	F	T

This statement is a tautology because the statement will always be true as seen by the last column in the truth table.

13a) Consider the case where $p = F$, $q = F$, and $r = F$. Then
$[(p \to q) \to r)]$
$\Leftrightarrow [(F \to F) \to F]$
$\Leftrightarrow [T \to F] \Leftrightarrow F$.
However, $[p \to (q \to r)]$
$\Leftrightarrow [F \to (F \to F)]$
$\Leftrightarrow [F \to T] \Leftrightarrow T$.

This shows $[(p \to q) \to r)]$ and $[p \to (q \to t)]$ have different truth values in this case, so they are not equivalent.

The row in a truth table showing the statements have different truth values would also show the different truth values.

13b) $[(p \wedge q) \to r)] \Leftrightarrow [p \to (q \to r)]$

p	q	r	$p \wedge q$	$q \to r$	$(p \wedge q) \to r$	$p \to (q \to r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	F	T	T	T
F	T	F	F	F	T	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

Since the columns in the truth table for $(p \wedge q) \to r$ and $p \to (q \to r)$ are exactly the same, the statements are equivalent.

We could also use the other methods described in Propositional Equivalences.

14a) Truth table: The last 2 columns in the following truth table show the two statements are equivalent since their truth values are always the same.

p	q	r	$p \to r$	$q \to r$	$p \wedge q$	$(p \to r) \vee (q \to r)$	$(p \wedge q) \to r$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	T	T	F	T	T
T	F	F	F	T	F	T	T
F	T	T	T	T	F	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

14b)

Case 1 Suppose $(p \to r) \vee (q \to r)$ is true and $(p \wedge q) \to r$ is false. Since $(p \wedge q) \to r$ is false we must have r is false while $p \wedge q$ is true. Since $p \wedge q$ is true, both p and q are true. But in this case we would have both $p \to r$ and $q \to r$ is false. Thus we cannot have both $(p \to r) \vee (q \to r)$ true and $(p \wedge q) \to r$ false.

Case 2 Suppose $(p \to r) \vee (q \to r)$ is false and $(p \wedge q) \to r$ is true. Since $(p \to r) \vee (q \to r)$ is false both $(p \to r)$ and $(q \to r)$ are false. But then these imply p and q are true while r is false. In this case we would have $p \wedge q$ true and r false so $(p \wedge q) \to r$ would be false. Thus we cannot have both $(p \to r) \vee (q \to r)$ false and $(p \wedge q) \to r$ true.

The previous work shows the truth values of the two statements cannot be different and so they are equivalent.

14c)

$$\begin{aligned}
(p \to r) \vee (q \to r) &\Leftrightarrow (\neg p \vee r) \vee (\neg q \vee r) && \text{Implication Law}\\
&\Leftrightarrow \neg p \vee [r \vee (\neg q \vee r)] && \text{Associative Law}\\
&\Leftrightarrow \neg p \vee [(\neg q \vee r) \vee r] && \text{Commutative Law}\\
&\Leftrightarrow \neg p \vee [\neg q \vee (r \vee r)] && \text{Associative Law}\\
&\Leftrightarrow \neg p \vee [\neg q \vee r] && \text{Idempotent Law}\\
&\Leftrightarrow (\neg p \vee \neg q) \vee r && \text{Associative Law}\\
&\Leftrightarrow \neg(p \wedge q) \vee r && \text{DeMorgan's Law}\\
&\Leftrightarrow (p \wedge q) \to r && \text{Implication Law}
\end{aligned}$$

15) (d) only. The statement is equivalent to $(p \wedge \neg r) \to \neg q$ and $(p \to r) \vee \neg q$ as well.

16a) False, $P(-3, 2)$, 16b) False, $\forall m P(0, m)$, 16c) True, $\exists n P(n, -3)$, 16d) False, $\exists n \forall m P(n, m)$, 16e) False, $\forall n \exists m P(n, m)$. If 0 was excluded from the domain of discourse, then it would be true. 16f) False, $\exists! m P(2, m)$,

17a) True, $P(-1, 1)$ is the statement $2(-1) + (1) = (-1)(1)$, which is true. 17b) True, $\exists x P(x, 0)$. 17c) False, $\exists y P(4, y)$. Remember the universe is a subset of the integers. 17d) False, $\forall y P(2, y)$. 17e) False. However, if the domain of discourse for y were changed to $\{u \in \mathbb{R} | u \neq 2\}$, then it would be true. 17f) False. 17g) True. Notice that if the domain of discourse for x and y were changed to $\{u \in \mathbb{R} | u \neq 1\}$ and $\{u \in \mathbb{R} | u \neq 2\}$, respectively, then the statement would be false (consider $x = 1/2$ and $y = -2$).

18a) True, 18b) False (implication the other direction holds, though), 18c) True, 18d) True, 18e) True, 18f) False, 18g) True,

19a) Margaret has not seen any movies. 19b) There is a movie that everyone liked. 19c) There is a person that has seen every movie that is a comedy. 19d) The negation is $\forall x \exists y [C(y) \wedge \neg S(x, y)]$. 19e) in poor English the negation says, "For every person there is a movie that is a comedy and that person has not seen." To say this more clearly we can say "Noone has seen every movie that is a comedy."

20) The following answers are not unique.

20a) $\forall x \exists y (A(y) \wedge \neg T(x, y))$,

20b) $\forall x [F(x) \to P(x, calculus)]$,

20c) $\exists y \forall x [A(y) \wedge \neg T(x, y)]$,

20d) $\exists x \forall y [F(x) \wedge [T(x, y) \to A(y)]]$,

20e) $\forall x [F(x) \to \neg(T(x, Linear Algebra) \wedge P(x, Linear Algebra))]$,

21a) $\forall n \forall m \exists p [(m < p) \wedge (p < n)]$ with universe $\mathbb{N}$.

21b) $\forall a \forall b \forall c [((a \geq 0) \wedge (b \geq 0) \wedge (c \geq 0) \wedge (a^2 + b^2 = c^2)) \to (a + b \geq c)]$,with universe $\mathbb{R}$ for a, b and c.

21c) $\neg \exists a [(a > 0) \wedge (a + \frac{1}{a} < 2)] \Leftrightarrow \forall a [(a \leq 0) \vee (a + \frac{1}{a} \geq 2)$, with universe $\mathbb{R}$.

21d) Let $C(x)$ be "x is in this class" and $M(x)$ be "x likes math." The universe for x is the set of all people. $\forall x [C(x) \to M(x)]$.

21e) Let $C(x)$ be "x is in this class" and $M(x)$ be "x likes math." The universe for x is the set of all people. $\neg \exists x [C(x) \wedge M(x)] \Leftrightarrow \forall x [C(x) \to \neg M(x)]$.

21f) Let $C(x)$ be "x is in this class", $L(x, y)$ be "x's major is y", $T(x, z)$ is "x is taking z, $F(z)$ is "z is a 4000 level course", and $M(z)$ is "z is a math class". The universe for x is the set of all people, the universe for y is the set of possible majors, the universe for z is the set of all courses offered. $\forall x \exists z[(C(x) \wedge L(x, CS)) \rightarrow (T(x, z) \wedge F(z), \wedge M(z))]$.

22) 21a) $\exists n \exists n \forall p[(m \geq p) \vee (p \geq n)]$ with universe $\mathbb{N}$.

21b) $\exists a \exists b \exists c[((a \geq 0) \wedge (b \geq 0) \wedge (c \geq 0) \wedge (a^2+b^2=c^2)) \wedge (a+b<c)]$,with universe $\mathbb{R}$ for a, b and c.

21c) $\exists a[(a>0) \wedge (a+\frac{1}{a}<2)]$, with universe $\mathbb{R}$.

21d) Let $C(x)$ be "x is in this class" and $M(x)$ be "x likes math." The universe for x is the set of all people. $\exists x[C(x) \wedge \neg M(x)]$.

21e) Let $C(x)$ be "x is in this class" and $M(x)$ be "x likes math." The universe for x is the set of all people. $\exists x[C(x) \wedge M(x)]$.

21f) Let $C(x)$ be "x is in this class", $L(x, y)$ be "x's major is y", $T(x, z)$ is "x is taking z, $F(z)$ is "z is a 4000 level course", and $M(z)$ is "z is a math class". The universe for x is the set of all people, the universe for y is the set of possible majors, the universe for z is the set of all courses offered. $\exists x \forall z[(C(x) \wedge L(x, CS)) \wedge (\neg T(x, z) \vee \neg F(z), \vee \neg M(z))]$.

23) 21a) There are natural numbers m and n such that for all natural numbers p we have $m \geq p$ or $p \geq n$.

21b) There are nonnegative real numbers, a, b, and c, such that $a^2+b^2=c^2$ and $a+b<c$.

21c) There does exist a positive real number a such that $a+\frac{1}{a}<2$.

21d) There is at least one student in this class does not like mathematics.

21e) Some student in this class likes mathematics.

21f) There is at least one student in this class that is a CS major and will not take a 4000 level math course.