Structural Induction

Let S be the set of ordered pairs of integers defined recursively as follows:

Basis step: $(0,0) \in S$

Recursive step: If $(a, b) \in S$ then $(a, b + 1) \in S$ $(a + 1, b + 1) \in S$ and $(a + 2, b + 1) \in S$.

a. List the elements of S produced by the first 2 applications of the recursive definition.

$$S_0 = \{(0,0)\},\ S_1 = \{(0,1),(1,1),(2,1)\},\ S_2 = \{(0,2),(1,2),(2,2),(3,2),(2,1),(4,2)\}.$$

The elements of S produced by the first 2 applications of the recursive definition are

$S_0 \cup S_1 \cup S_2 = \{(0,0),(0,1),(1,1),(2,1),(0,2),(1,2),(2,2),(3,2),(2,1),(4,2)\}.$

b. Use structural induction to prove that $\boldsymbol{a \leq 2b}$, when $\boldsymbol{(a,b) \in S}$.

When we use structural induction to show that the elements of a recursively defined set S have a certain property, then we need to follow the following procedure:

1. **Basis step**: show that all the elements defined in the basis step have the desired property.
2. **Inductive step**: assume that an arbitrary element of the set S has the desired property. This is your inductive hypothesis. Using the inductive hypothesis, prove that, when you create more elements in S by using the recursive definition, these newly created elements of S have the same property.
3. **Conclusion:** state that by the principle of structural induction all the elements in S have the same property.

Basis step: $(0,0) \in S$ and $0 \leq 2 \cdot 0$.

Recursive Step: Assume $(a, b) \in S$ with the property that $a \leq 2b$. We need to prove that the following elements of S, created by using the recursive definition, $(a, b+1)$, $(a+1, b+1)$ and $(a+2, b+1)$ have the same property. That is, the first coordinate of each of these elements is less than or equal to twice as much as the second coordinate.

Proof: Using the inductive hypothesis,
$Case$ 1: $a \leq 2b < 2b + 2 = 2(b+1)$.
Case 2: $a + 1 \leq 2b + 2 = 2(b+1)$.
Case 3: $a + 2 \leq 2b + 2 = 2(b+1)$.

By **structural induction** we have proved that if $(a, b) \in S$ then $a \leq 2b$.