Mathematical Induction

Use induction to prove that

$$\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\dots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

for all positive integers $\boldsymbol{n \geq 2}$.

Let P(n) denote the proposition $\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$,
where n is a positive integer, $n \geq 2$.

BASIS STEP: P(2) is true since $\prod_{i=2}^{2}\left(1-\frac{1}{i^2}\right)=1-\frac{1}{2^2}=\frac{3}{4}$ and $\frac{2+1}{2\cdot 2}=\frac{3}{4}$

Inductive Step:

Let us assume P(n), that is $\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$
is true for an arbitrary positive integer n$\geq$ 2 (This is the inductive hypothesis).

We have to show the statement P(n+1),

$\prod_{i=2}^{n+1}\left(1-\frac{1}{i^2}\right)=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\ldots\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right)=\frac{(n+1)+1}{2(n+1)}=\frac{n+2}{2n+2}$ is
true assuming the inductive hypothesis P(n).

Proof:

$\prod_{i=2}^{n+1}\left(1-\frac{1}{i^2}\right) = \prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)\cdot\left(1-\frac{1}{(n+1)^2}\right) = \left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\ldots\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right) = \frac{n+1}{2n}\cdot(1-\frac{1}{(n+1)^2})$ by the inductive hypothesis.

Now, we have to use algebra to prove the inductive step.

$$\frac{n+1}{2n}\cdot\left(1-\frac{1}{(n+1)^2}\right) = \frac{n+1}{2n}\cdot 1-\frac{n+1}{2n}\frac{1}{(n+1)^2} = \frac{n+1}{2n}-\frac{1}{2n(n+1)} = \frac{(n+1)^2-1}{2n(n+1)} = \frac{n(n+2)}{2n(n+1)} = \frac{\boldsymbol{n+2}}{\boldsymbol{2n+2}}.$$

By the Principle of Mathematical Induction (Basis Step and Inductive Step together) we have proved the for all positive integers $n \geq 2$,

$$\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right) = \left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\ldots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$$