Math 231 Second Midterm Solutions

Problem 1. Mark each of the following statements as true or false. Give a brief reason.

- If $\mathbf{u}$ is orthogonal to $\mathbf{v}$, then $2\mathbf{u}$ is orthogonal to $-\mathbf{v}$.

  True. If $\mathbf{u}$ is orthogonal to $\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = 0$. But then
  $$2\mathbf{u} \cdot -\mathbf{v} = (2)(-1)\mathbf{u} \cdot \mathbf{v} = (-2)(0) = 0$$
  which means $2\mathbf{u}$ is orthogonal to $-\mathbf{v}$.

- If the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span a vector space V, then $\dim V = 3$.

  False, because $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ may be linearly dependent. For example, the vectors $\mathbf{v}_1 = (1,0), \mathbf{v}_2 = (0,1), \mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2 = (1,1)$ span $\mathbb{R}^2$, but $\dim \mathbb{R}^2 = 2$.

- It is possible to find 7 linearly independent vectors in $\mathbb{R}^6$.

  False, because $\dim \mathbb{R}^6 = 6$ so any collection of more than 6 vectors in $\mathbb{R}^6$ must be linearly dependent.

- There is a 3×5 matrix A whose rank is 4.

  False, because if A is 3×5, then $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^3$ so $\mathrm{rank}(A) = \dim \mathrm{Col}(A) \leq 3$. Alternatively, $\mathrm{Row}(A)$ is spanned by 3 vectors in $\mathbb{R}^5$, so $\mathrm{rank}(A) = \dim \mathrm{Row}(A) \leq 3$.

Problem 2. Consider the vectors
$$\mathbf{u} = (-1, 0, 1) \qquad \text{and} \qquad \mathbf{v} = (0, 0, 2)$$
in $\mathbb{R}^3$. Find all scalars a, b such that the vector $a\,\mathbf{u} + b\,\mathbf{v}$ is orthogonal to $\mathbf{v}$ and has norm 1.

Write
$$\mathbf{w} = a\,\mathbf{u} + b\,\mathbf{v} = (-a, 0, a) + (0, 0, 2b) = (-a, 0, a + 2b).$$
We want $\mathbf{w}$ to be orthogonal to $\mathbf{v}$, so
$$\mathbf{w} \cdot \mathbf{v} = (-a, 0, a + 2b) \cdot (0, 0, 2) = 2(a + 2b) = 0 \Longrightarrow a + 2b = 0.$$
We also want $\mathbf{w}$ to have norm 1, so
$$\begin{aligned} \|\mathbf{w}\| = 1 &\Longrightarrow \sqrt{(-a)^2 + 0^2 + (a + 2b)^2} = 1 \\ &\Longrightarrow \sqrt{a^2} = 1 \quad (\text{because } a + 2b = 0) \\ &\Longrightarrow |a| = 1 \\ &\Longrightarrow a = 1 \text{ or } a = -1. \end{aligned}$$
Using $a + 2b = 0$, we obtain two sets of solutions:
$$a = 1, b = -\frac{1}{2} \qquad \text{or} \qquad a = -1, b = \frac{1}{2}.$$

Problem 3. Let W be the set of all 2×2 matrices A such that $\mathrm{tr}(A) = 0$. In other words, W consists of all matrices of the form
$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

where a, b, c are arbitrary real numbers.

(i) Verify that W is a subspace of $\mathcal{M}_{2,2}$.

We must verify that W is closed under addition and scalar multiplication:

- Suppose $A, B \in W$ so that $\operatorname{tr}(A) = \operatorname{tr}(B) = 0$. Then $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B) = 0 + 0 = 0$, so $A + B \in W$.

- Suppose λ is a scalar and $A \in W$ so that $\operatorname{tr}(A) = 0$. Then $\operatorname{tr}(\lambda A) = \lambda \operatorname{tr}(A) = \lambda \cdot 0 = 0$, so $\lambda A \in W$.

(ii) Find a basis and dimension of W.

Every element of W can be written as

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

It follows that the matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

span W. Moreover, A_1, A_2, A_3 are linearly independent because

$$aA_1 + bA_2 + cA_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Longrightarrow a = b = c = 0.$$

Thus $\{A_1, A_2, A_3\}$ forms a basis for W. In particular, $\dim W = 3$.

Problem 4. Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 3 & 1 & 5 \\ 1 & 3 & 1 & 2 & 1 \\ 3 & 9 & 4 & 5 & 2 \\ 4 & 12 & 8 & 8 & 8 \end{bmatrix}$$

(i) For what vectors $\mathbf{b} \in \mathbb{R}^4$ is the system $A\mathbf{x} = \mathbf{b}$ consistent?

We form the augmented matrix $[A \vdots \mathbf{b}]$ and reduce it to row-echelon form:

$$\begin{bmatrix} 0 & 0 & 3 & 1 & 5 & \vdots & a \\ 1 & 3 & 1 & 2 & 1 & \vdots & b \\ 3 & 9 & 4 & 5 & 2 & \vdots & c \\ 4 & 12 & 8 & 8 & 8 & \vdots & d \end{bmatrix} \xrightarrow{R_1 \leftrightarrows R_2} \begin{bmatrix} 1 & 3 & 1 & 2 & 1 & \vdots & b \\ 0 & 0 & 3 & 1 & 5 & \vdots & a \\ 3 & 9 & 4 & 5 & 2 & \vdots & c \\ 4 & 12 & 8 & 8 & 8 & \vdots & d \end{bmatrix}$$

$$\xrightarrow{-3R_1+R_3, -4R_1+R_4} \begin{bmatrix} 1 & 3 & 1 & 2 & 1 & \vdots & b \\ 0 & 0 & 3 & 1 & 5 & \vdots & a \\ 0 & 0 & 1 & -1 & -1 & \vdots & -3b+c \\ 0 & 0 & 4 & 0 & 4 & \vdots & -4b+d \end{bmatrix} \xrightarrow{R_2 \leftrightarrows R_3} \begin{bmatrix} 1 & 3 & 1 & 2 & 1 & \vdots & b \\ 0 & 0 & 1 & -1 & -1 & \vdots & -3b+c \\ 0 & 0 & 3 & 1 & 5 & \vdots & a \\ 0 & 0 & 4 & 0 & 4 & \vdots & -4b+d \end{bmatrix}$$

$$\xrightarrow{-3R_2+R_3,-4R_2+R_4}\left[\begin{array}{ccccc|c}1&3&1&2&1&b\\0&0&1&-1&-1&-3b+c\\0&0&0&4&8&a+9b-3c\\0&0&0&4&8&8b-4c+d\end{array}\right]\xrightarrow{-R_3+R_4}\left[\begin{array}{ccccc|c}1&3&1&2&1&b\\0&0&1&-1&-1&-3b+c\\0&0&0&4&8&a+9b-3c\\0&0&0&0&0&-a-b-c+d\end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_3}\left[\begin{array}{ccccc|c}1&3&1&2&1&b\\0&0&1&-1&-1&-3b+c\\0&0&0&1&2&\frac{1}{4}(a+9b-3c)\\0&0&0&0&0&-a-b-c+d\end{array}\right]$$

The consistency condition is therefore $-a-b-c+d=0$. In other words,

$$A\mathbf{x}=\mathbf{b}=\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\text{ is consistent}\iff a+b+c=d.$$

(ii) Find bases for Row(A), Col(A) and Null(A).

By part (i), a row-echelon form of A is

$$R=\begin{bmatrix}1&3&1&2&1\\0&0&1&-1&-1\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}.$$

Thus the vectors

$$\mathbf{u}_1=\begin{bmatrix}1&3&1&2&1\end{bmatrix}\quad \mathbf{u}_2=\begin{bmatrix}0&0&1&-1&-1\end{bmatrix}\quad \mathbf{u}_3=\begin{bmatrix}0&0&0&1&2\end{bmatrix}$$

form a basis for Row(A). The leading 1's of R occur along the first, third and fourth columns, so the vectors

$$\mathbf{c}_1=\begin{bmatrix}0\\1\\3\\4\end{bmatrix}\qquad \mathbf{c}_3=\begin{bmatrix}3\\1\\4\\8\end{bmatrix}\qquad \mathbf{c}_4=\begin{bmatrix}1\\2\\5\\8\end{bmatrix}$$

form a basis for Col(A). To find a basis for Null(A), we must solve the homogeneous system $A\mathbf{x}=\mathbf{0}$, or equivalently $R\mathbf{x}=\mathbf{0}$:

$$\begin{cases}x_1+3x_2+x_3+2x_4+x_5=0\\x_3-x_4-x_5=0\\x_4+2x_5=0\end{cases}$$

The leading variables are x_1,x_3,x_4 and the free variables are x_2,x_5. Hence the above system can be solved by assigning arbitrary values to x_2,x_5 and expressing x_1,x_3,x_4

in terms of them:

$$\begin{cases} x_1 = -3s + 4t \\ x_2 = s \\ x_3 = -t & \qquad\qquad s, t \in \mathbb{R} \\ x_4 = -2t \\ x_5 = t \end{cases}$$

Thus

$$\mathbf{x} \in \text{Null}(A) \iff \mathbf{x} = \begin{bmatrix} -3s + 4t \\ s \\ -t \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix} \qquad s, t \in \mathbb{R}$$

It follows that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

form a basis for $\text{Null}(A)$.

(iii) Find $\text{rank}(A)$ and $\text{nullity}(A)$ and verify that $\text{rank}(A) + \text{nullity}(A) = 5$.

We have

$$\text{rank}(A) = \dim \text{Row}(A) = \dim \text{Col}(A) = 3$$

and

$$\text{nullity}(A) = \dim \text{Null}(A) = 2.$$

Bonus Problem. Let A be an $n \times n$ matrix such that $\det(A) = 0$. Show that the rows of A, viewed as vectors in $\mathbb{R}^n$, must be linearly dependent.

Reduce A to an $n \times n$ row-echelon matrix R. Since $\det(A) = 0$, we have $\det(R) = 0$, so R must have an all-zero row. Hence $\text{rank}(A)$ (which is the number of non-zero rows of R) is less than n. This shows the rows of A must be dependent since otherwise $\text{rank}(A)$ would be n.