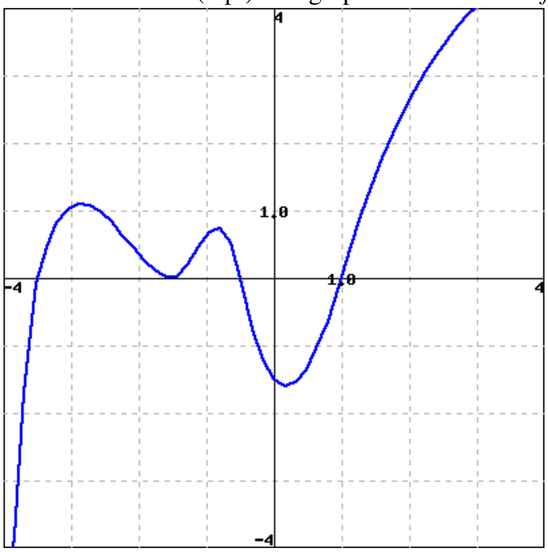
Problem 1. 1. (1 pt) The graph of the function $f(x)$ is



(the horizontal axis is x.)
Given the differential equation $x'(t) = f(x(t))$.
List the constant (or equilibrium) solutions to this differential equation in increasing order and indicate whether or not these equations are stable, semi-stable, or unstable.

_____ [?]
_____ [?]
_____ [?]
_____ [?]

Answer(s) submitted:

- `-3.5`
- `unstable`
- `-1.5`
- `semi-stable`
- `-0.5`
- `stable`
- `1.0`
- `unstable`

(correct)

Problem 2. 2. (1 pt) A function $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = -y^4 + 8y^3 + 9y^2.$$

(a) What are the constant solutions of this equation?
Separate your answers by commas.

______________________.

(b) For what values of y is y increasing?
_____ $< y <$ _____.

Answer(s) submitted:

- `0, -1, 9`
- `-1`
- `1`

(score 0.666670024394989)

Problem 3. 3. (1 pt) Which of the following functions are solutions of the differential equation $y'' - 12y' + 36y = 0$?

- A. $y(x) = 6x$
- B. $y(x) = x^2e^{6x}$
- C. $y(x) = xe^{6x}$
- D. $y(x) = xe^{-6x}$
- E. $y(x) = e^{-6x}$
- F. $y(x) = 0$
- G. $y(x) = e^{6x}$

Answer(s) submitted:

- `( C, G )`

(incorrect)

Problem 4. 4. (1 pt) It can be helpful to classify a differential equation, so that we can predict the techniques that might help us to find a function which solves the equation. Two classifications are the **order of the equation** – (what is the highest number of derivatives involved) and whether or not the equation is **linear** .
Linearity is important because the structure of the the family of solutions to a linear equation is fairly simple. Linear equations can usually be solved completely and explicitly.

Determine whether or not each equation is linear:

[?] 1. $\frac{dy}{dt} + ty^2 = 0$
[?] 2. $y'' - y + t^2 = 0$
[?] 3. $\frac{d^3y}{dt^3} + t\frac{dy}{dt} + (cos^2(t))y = t^3$
[?] 4. $\frac{d^4y}{dt^4} + \frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} = 1$

Answer(s) submitted:

- `1Nonlinear`
- `2Linear`
- `3Linear`
- `4Linear`

(correct)

Problem 5. 5. (1 pt) Solve the following initial value problem:

$$t\frac{dy}{dt}+3y=6t$$

with $y(1)=2$.
Put the problem in standard form.
Then find the integrating factor, $\rho(t)=$ __________,
and finally find $y(t)=$ ____________________.

Answer(s) submitted:

- `e^((3/2)t)`
- `((3t)/2) + (1/(2t^3)))`

(incorrect)

Problem 6. 6. (1 pt) Find the function $y(t)$ that satisfies the differential equation

$$\frac{dy}{dt}-2ty=3t^2e^{t^2}$$

and the condition $y(0)=3$.
$y(t)=$ _________________________.

Answer(s) submitted:

- `((t^3) + 3)(e^(t^2))`

(correct)

Problem 7. 7. (1 pt) Find the solution to the differential equation

$$6\frac{du}{dt}=u^2,$$

subject to the initial conditions $u(0)=7$.
$u=$ __________________

Solution:
SOLUTION
Separating variables gives

$$\int\frac{1}{u^2}du=\int\frac{1}{6}dt$$

or

$$-\frac{1}{u}=\frac{1}{6}t+C.$$

The initial condition gives $C=\frac{-1}{7}$ and so, after rearranging slightly,

$$u=\frac{42}{6-7t}.$$

Answer(s) submitted:

- `(t^2)((t/6)-(1/7))`

(incorrect)

Problem 8. 8. (1 pt) Find the particular solution of the differential equation

$$\frac{x^2}{y^2-7}\frac{dy}{dx}=\frac{1}{2y}$$

satisfying the initial condition $y(1)=\sqrt{8}$.
Answer: $y=$ ________________________.
Your answer should be a function of x.

Answer(s) submitted:

- `sqrt((e^(1-(1/x))) + 7)`

(correct)

Problem 9. 9. (1 pt) Water leaks from a vertical cylindrical tank through a small hole in its base at a rate proportional to the square root of the volume of water remaining. The tank initially contains 200 liters and 15 liters leak out during the first day.

A. When will the tank be half empty? $t=$ ____________ days

B. How much water will remain in the tank after 2 days? volume = ____________ Liters

Solution:
SOLUTION
Let $V(t)$ be the volume of water in the tank at time t, then

$$\frac{dV}{dt}=k\sqrt{V}.$$

This is a separable equation which has the solution

$$V(t)=(\frac{kt}{2}+C)^2.$$

Since $V(0)=200$ this gives $200=C^2$ so

$$V(t)=(\frac{kt}{2}+\sqrt{200})^2.$$

However, $V(1)=185$, and so

$$185=(\frac{k}{2}+\sqrt{200})^2,$$

so that $k=2(\sqrt{185}-\sqrt{200})=-1.0814$. Therefore,

$$V(t)=(-0.5407t+\sqrt{200})^2.$$

The tank will be half-empty when $V(t)=100$, so we solve

$$100=(-0.5407t+\sqrt{200})^2$$

to obtain $t=7.661$ days. The tank will be half empty in 7.661 days.

The volume after 2 days is $V(2)$ which is approximately 170.583 liters.

Answer(s) submitted:

-
-

(incorrect)

Problem 10. 10. (1 pt) A bacteria culture starts with 420 bacteria and grows at a rate proportional to its size. After 2 hours there will be 840 bacteria.

(a) Express the population after t hours as a function of t.
population: ________________ (function of t)

(b) What will be the population after 8 hours?

____

(c) How long will it take for the population to reach 1770 ?

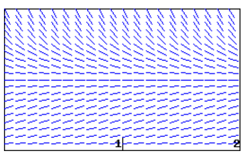
____

Answer(s) submitted:

-
-
-

(incorrect)

Problem 11. 11. (1 pt) The slope field for a population P modeled by $dP/dt = 1.5P - 3P^2$ is shown in the figure below.



(a) On a print-out of the slope field, sketch three non-zero solution curves showing different types of behavior for the population P. Give an initial condition that will produce each:
$P(0) =$ ______,
$P(0) =$ ______, and
$P(0) =$ ______.

(b) Is there a stable value of the population? If so, give the value; if not, enter **none**:
Stable value = ______

(c) Considering the shape of solutions for the population, give any intervals for which the following are true. If no such interval exists, enter **none**, and if there are multiple intervals, give them as a list. *(Thus, if solutions are increasing when P is between 1 and 3, enter (1,3) for that answer; if they are decreasing when P is between 1 and 2 or between 3 and 4, enter (1,2),(3,4). Note that your answers may reflect the fact that P is a population.)*
P is increasing when P is in __________
P is decreasing when P is in __________

Think about what these conditions mean for the population, and be sure that you are able to explain that.

In the long-run, what is the most likely outcome for the population?
$P \to$ __________
(Enter **infinity** *if the population grows without bound.)*

Are there any inflection points in the solutions for the population? If so, give them as a comma-separated list (e.g., 1,3); if not, enter **none**.

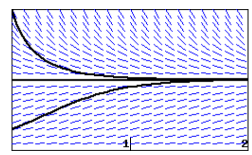
Inflection points are at $P =$ __________

Be sure you can explain what the meaning of the inflection points is for the population.

(d) Sketch a graph of dP/dt against P. Use your graph to answer the following questions.
When is dP/dt positive?
When P is in __________
When is dP/dt negative?
When P is in __________
(Give your answers as intervals or a list of intervals.)
When is dP/dt zero?
When $P =$ __________
(If there is more than one answer, give a list of answers, e.g., **1,2***.)*
When is dP/dt at a maximum?
When $P =$ __________

Be sure that you can see how the shape of your graph of dP/dt explains the shape of solution curves to the differential equation.

Solution:
SOLUTION
A graph of the slope field with some solutions added is

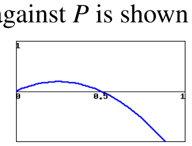


Thus we get different solution behaviors for the initial conditions $P(0) = 0.5$, $P(0) > 0.5$, and $0 < P(0) < 0.5$. The initial condition $P(0) = 0$ gives the zero solution, which is an equilibrium and has the same behavior as the initial condition $P(0) = 0.5$. Note that we can also say that the behavior for initial conditions $0 < P(0) < 0.25$ is different from that for initial conditions $0.25 \leq P(0) < 0.5$, because the former will be sigmoidal while the latter are strictly exponential.

From the solutions graphed above, or by looking at the slope field, we can see that the equilibrium solution $P = 0.5$ is stable. The other equilibrium solution, $P = 0$ is unstable.

Also by looking at the solution curves, we can see that P is increasing when $0 < P < 0.5$ and decreasing when $P > 0.5$. In the long run, we're most likely to start with a non-zero population and thus $P \to 0.5$. The solution curves with initial populations of less than $P = 0.25$ have inflection points at $P = 0.25$. (This will be demonstrated algebraically below.) At the inflection point, the population is growing fastest.

A graph of dP/dt against P is shown below.



Since $\frac{dP}{dt} = 1.5P - 3P^2 = 3P(0.5 - P)$, the graph of $\frac{dP}{dt}$ against P is a parabola, opening downward with P intercepts at 0 and 0.5. The quantity $\frac{dP}{dt}$ is positive for $0 < P < 0.5$, negative for $P > 0.5$ (and $P < 0$). The quantity $\frac{dP}{dt}$ is 0 at $P = 0$ and $P =$, and maximum at $P = 0.25$. The fact that $\frac{dP}{dt} = 0$ at $P = 0$ and $P = 0.5$ tells us that these are equilibria. Further, since $\frac{dP}{dt} > 0$ for $0 < P < 0.5$, we see that solution curves starting here will increase toward $P = 0.5$.

If the population starts at a value $P < 0.25$, it increases at an increasing rate up to $P = 0.25$. After this, P continues to increase, but at a decreasing rate. The fact that $\frac{dP}{dt}$ has a maximum at $P = 0.25$ tells us that there is a point of inflection when $P = 0.25$. Similarly, since $\frac{dP}{dt} < 0$ for $P > 0.5$, solution curves starting with $P > 1$ will decrease to $P = 0.5$. Thus, $P = 0.5$ is a stable equilibrium.

Answer(s) submitted:

-
-
-
-
-
-
-
-
-
-
-
-

(incorrect)

Problem 12. 12. (1 pt) Use Euler's method with step size 0.5 to compute the approximate y-values $y_1 \approx y(1.5)$, $y_2 \approx y(2)$, $y_3 \approx y(2.5)$, and $y_4 \approx y(3)$ of the solution of the initial-value problem

$$y' = -2 + 4x - 4y, \quad y(1) = 4.$$

$y_1 =$ _____,
$y_2 =$ _____,
$y_3 =$ _____,
$y_4 =$ _____.

Answer(s) submitted:

- `-3`
- `5`
- `-2`
- `6`

(correct)

Problem 13. 13. (1 pt) Find the general solution to the homogeneous differential equation

$$\frac{d^2y}{dx^2} - 24\frac{dy}{dx} + 140y = 0$$

The solution has the form

$$y = C_1 f_1(x) + C_2 f_2(x)$$

with

$f_1(x) =$ ____________
$f_2(x) =$ ____________

Left to your own devices, you will probably write down the correct answers, but in case you want to quibble, enter your answers so that f_1, f_2 are normalized with their value at $x = 0$ equal to 1.

Answer(s) submitted:

- `e^(14x)`
- `e^(10x)`

(correct)