ANSWERS TO PRACTICE PROBLEMS CHAPTER 6

1.

(i) (a) $\lambda_1 = -1 \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 1 \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

(b) $\lambda_1 = i \quad \mathbf{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad \lambda_2 = -i \quad \mathbf{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}.$

(c) $\lambda_1 = \lambda_2 = 1 \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$

(d) $\lambda_1 = \lambda_2 = 1 \quad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

(ii) (a) The matrix is diagonalizable: $X = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) The matrix is diagonalizable: $X = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

(c) The matrix is diagonalizable: $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) The matrix is defective.

2.

(a) The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = 3$.

The eigenvalues of A^2 are 1, 0 9 and the eigenvalues of A^n are 1, 0 3^n.

(b) The associated eigenspaces for the matrix A have basis $\mathbf{x}_1 = [1, 0, 0]^T$, $\mathbf{x}_2 = [-2, 1, 0]^T$, $\mathbf{x}_3 = [5, 4, 2]^T$

These are also the basis for the eigenspaces for the matrix A^2 and A^n.

(c)

$$A^n = \begin{bmatrix} 1 & 2 & -\frac{13}{2} + \frac{5(3^n)}{2} \\ 0 & 0 & 2(3^n) \\ 0 & 0 & 3^n \end{bmatrix}$$

(d)

$$A^7 = \begin{bmatrix} 1 & 2 & -\frac{13}{2} + \frac{5(3^7)}{2} \\ 0 & 0 & 2(3^7) \\ 0 & 0 & 3^7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5461 \\ 0 & 0 & 4374 \\ 0 & 0 & 2187 \end{bmatrix}$$

3.

(a) $\lambda_1 = 2$ and $\lambda_2 = 3$.

(b) The eigenvalues of 5A are 10 and 15.

(b) $\lambda_1 = 2^2 + 2\alpha + \beta, \quad \lambda_2 = 3^2 + 3\alpha + \beta$

4. $k < 4$.

5. $\lambda_1 = -2, \quad \lambda_2 = 7$

6. **(a)** $\mathbf{v} = 4\mathbf{x}_1 + 5\mathbf{x}_2$ **(b)** $A\mathbf{v} = [37, 38, 1]^T$

7.

(a) Defective

(b) $X = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

8. $B = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ -4/3 & 4/3 & 1 \end{bmatrix}$

9. $\lambda_1 = 3 + 2i$, $\lambda_2 = 3 - 2i$, $\lambda_3 = 0$, $\lambda_4 = -2$.

10.

(a) If $\lambda = 0$ then $\det(A - \lambda I) = \det(A) = 0$ and the matrix is singular which contradicts the assumption, thus $\lambda \neq 0$.

(b) If $A\mathbf{x} = \lambda\mathbf{x}$ then $A^{-1}A\mathbf{x} = \lambda A^{-1}\mathbf{x}$ which gives $\mathbf{x} = \lambda A^{-1}\mathbf{x}$ or $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$ thus $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.with $\mathbf{x}$ the corresponding eigenvector.

11.

The matrix has the eigenvalue $\lambda = a$ with Algebraic Multiplicity 3.The basis of the corresponding eigenspace consists of two linearly independent eigenvectors: $[1, 0, 0]^T$ and $[0, 1, 0]^T$. Thus GM = 2 < AM = 3 and the matrix is defective.

12.

If A is diagonalizable, then $A = XDX^{-1}$ where D is a diagonal matrix. If B is similar to A, then there exists a nonsingular matrix S such that $B = S^{-1} A\, S$. It follows that

$$\begin{aligned} B &= S^{-1}(XDX^{-1})S \\ &= (S^{-1} X)D(S^{-1}X)^{-1} \end{aligned}$$

Therefore B is diagonalizable with diagonalizing matrix $S^{-1}X$.

13.

(a) rank(A) = 3.

(b) An orthonormal basis for R(A^T) is given by $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, where $\mathbf{v}_i$ is the ith column of V.

(c) An orthonormal basis is given by $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ where $\mathbf{u}_i$ is the ith column of U

(d) The rank-1 matrix B that is the closest matrix of rank-1 to A is given by $B = \begin{bmatrix} 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 \\ 30 & 30 & 30 & 30 \end{bmatrix}$

(e) $\|B - A\|_F = 10\sqrt{2}$.

(f) $C = \begin{bmatrix} 18 & 22 & 22 & 18 \\ 18 & 22 & 22 & 18 \\ 18 & 22 & 22 & 18 \\ 23 & 17 & 17 & 23 \\ 32 & 28 & 28 & 32 \end{bmatrix}$

(g) $\|C - A\|_F = 10$.

14. $\hat{\mathbf{x}} = \frac{(\mathbf{u}_1^T\mathbf{b})}{\sigma_1}\mathbf{v}_1 + \frac{(\mathbf{u}_2^T\mathbf{b})}{\sigma_2}\mathbf{v}_2 = -\frac{4.5}{20}\mathbf{v}_1 - \frac{1.5}{15}\mathbf{v}_2 = \begin{bmatrix} 0.12 \\ -.215 \end{bmatrix}$

15. $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$