MAT 275 TEST 2 **PRACTICE PROBLEMS**

I. Existence and uniqueness. Fundamental sets.

1. Determine the longest interval on which the given initial value problem is certain to have a unique twice differentiable solution.

(a) $(x-3)y''+\dfrac{x}{x-3}y'+\sqrt{x-1}\,y=0, \quad y(2)=0, y'(2)=1$

(b) $(t-1)y''+ty'+y=\sec(t), \quad y(0)=1, \ y'(0)=3$

(c) $t(t-4)y''+3y'+\ln(t)y=\sin(t), \quad y(1)=1, y'(1)=1$

2. Which of the following is a true statement?

I. Two functions defined on an open interval I are said to be linearly independent on I provided that one is a constant multiple of the other on I.
II. Two functions defined on an open interval I are said to be linearly dependent on I provided that one is a constant multiple of the other on I.
III. Two functions defined on an open interval I are said to be linearly independent on I provided that neither is a constant multiple of the other on I.
IV. Two functions defined on an open interval I are said to be linearly dependent on I provided that neither is a constant multiple of the other on I.

3. Which of the following pairs of functions is linearly independent on the entire real line?

A. $\{\sin x, \cos x\}$ B. $\{e^x, xe^x\}$ C. $\left\{x, \left(\dfrac{e}{\pi}\right)^3 x\right\}$ D. $\{x, 3x\}$

E. $\{1, \ e^{-t}\}$ F. $\{\cos t, \ \sin(t+\pi/2)\}$ G. $\{e^{-2t}\cos 2\text{t}, \ e^{-2\text{t}}\sin 2\text{t}\}$

H. $\{2e^{-t}, \ 4e^{-t+3}\}$ I. $\{e^{2t}, e^{2t}-6\}$ J. $\{x, |x|\}$

4. Which of the following is NOT a fundamental set of solutions for $y''-y=0$?

A. $\{e^t, e^{-t}\}$ B. $\{2e^t, 2e^{-t}\}$ C. $\{te^t, e^{-t}\}$ D. $\left\{(e^t+e^{-t}), \dfrac{1}{2}(e^t+e^{-t})\right\}$ E. $\left\{\dfrac{1}{2}(e^t+e^{-t}), \dfrac{1}{2}(e^t-e^{-t})\right\}$

F. $\{(e^t+e^{-t}), e^t\}$

5. Suppose $y_1(t)=t$ and $y_2(t)=t^2$ are both solutions of the second order linear equation $y''+p(t)y'+q(t)y=0.$ Which of the functions below are guaranteed to also be solutions of the same equation?

A. $y=t^2-1$ B. $y=5t^2$ C. $y=-9t^2+17t$ D. $y=0$

6. Consider the ODE $t^2y''+3ty'+y=0$ with the initial conditions $y(1) = 1$, $y'(1) = 1$.

(i) What is the maximum interval of validity, I, of the solution?

(ii) Verify that the functions $y_1(t)=t^{-1}$ and $y_2(t)=t^{-1}\ln t$ satisfy the ODE for t in the interval I.

(iii) Use the Wronskian to show that the functions y_1 and y_2 from ii. form a fundamental set of solutions.

(iv) Solve the initial value problem.

II. HODEs/IVP with constant coefficients.

1. Find a real valued solution to the following initial value problems. Sketch a graph of the solution.

   a. $y''-6y'+13y=0$ with $y(0)=1,\ y'(0)=1$.

   b. $y''+4y'+4y=0,$ with $y(0)=1,\ y'(0)=-4$.

   c. $y''+3y'+2y=0,$ with $y(0)=3,\ y'(0)=0$.

2. For which values of α (if any) are all solutions of $y''-(2\alpha-1)y'+\alpha(\alpha-1)y=0$ unbounded as $t\to\infty$?

3. The characteristic equation of a homogeneous 9th order linear Differential Equation with constant coefficients has roots $r=0$ with multiplicity three, $r=-2$ with multiplicity two, $r=-3\pm 2i$ with multiplicity two.
   Write the general solution of the Differential Equation.

4. One solution of the DE $6y^{(4)}+5y^{(3)}+25y''+20y'+4y=0$ is $y=\cos(2x)$. Find the general solution.

III. Reduction of order:

1. The ODE $t^2y''+3ty'+y=0$ has a solution $y_1(t)=t^{-1}$ for $t>0$. Find the general solution.

2. The ODE $t^2y''-t(t+2)y'+(t+2)y=0$ has a solution $y_1(t)=t$ for $t>0$. Find the general solution.

IV. Undetermined coefficients

1. Find the general solution of the ODE $y''+2y'+y=e^{-t}$

2. Solve the IVP: $y''-y'-2y=6x+6e^{-x},\quad y(0)=1, y'(0)=0$

3. Solve the IVP: $y''-y'-2y=6te^{2t}, y(0)=0, y'(0)=1$

4. Determine a suitable form for the particular solution Y(t) if the method of undetermined coefficients is to be used.
   You do not need to determine the values of the coefficients.

   (i) $y''+3y'=2t^2+t^2e^{-3t}+\sin 3t$

   (ii) $y''+y=t(1+\sin t)$

   (iii) $y''-5y'+6\mathrm{y}=e^t\cos 2\mathrm{t}+e^{2t}(3\mathrm{t}+4)\sin t$

   (iv) $y''+2y'+2y=3e^{-t}+2e^{-t}\cos t+4e^{-t}t^2\sin t$

   (v) $y''-4y'+4y=2t^2+4te^{2t}+t\sin(2t)$

V. Mass-Spring system

1. Consider the IVP: $y''+4y=0$ with $y(0)=-3$ and $y'(0)=6.$. Write the solution as $y(t)=R\cos(\omega_0 t-\delta)$.

2. A mass of 2 kilograms stretches a spring 0.5 meters. If the mass is set in motion from its equilibrium with a downward velocity of 10 cm/s, and there is no damping, write an IVP for the position u (in meters) of the mass at any time t (in seconds). Use g=9.8 m/s^2 for the acceleration due to gravity.

3. For the following, choose the best description of the system from the following:
   Simple Harmonic Motion (SHM) Overdamped (OD) Underdamped (UD) Critically Damped (CD)
   Beating (B) Resonant (R) Steady-State plus Transient (SST)

   a. $y''+4y=0$

   b. $y''+(1.8)^2y=\cos(2t)$

   c. $y''+4y=\cos(2t)$

d. $y''+y'+y=0$

e. $y''+y'+y=\cos(t)$

f. $y''+2y'+y=0$

4. The motion of a force mass-spring system is described by the following IVP:

$$u''+9u=\cos(3t), \quad u(0)=0, \quad u'(0)=0$$

(a) Explain why you expect resonance to occur.

(b) solve this IVP and sketch the graph of the solution.

5. The motion of a force mass-spring system is described by the following IVP:

$$u''+(2.8)^2 u=\cos(3t), \quad u(0)=0, u'(0)=0.$$

(a) Explain why you expect the beats phenomenon to occur.

(b) Solve the IVP and write your solution in the form $A\sin(\alpha t)\sin(\beta t)$

(c) Determine the length of the beats and the period of the oscillation.

6. A mass *m* =1 is attached to a spring with constant *k* = 2 and damping constant γ. Determine the value of γ so that the motion is critically damped.

7. The position function of a mass-spring system satisfies the differential equation

$$mx''+\gamma x'+kx=\cos(\omega t), \qquad x(0)=x'(0)=0 \quad .$$

Assume *m* = 1 and *k* = 9.

If $\gamma\neq 0$, the amplitude of the forced oscillation is given by $C=\dfrac{1}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}$.

Assume $\gamma=1$.

Differentiate *C* to find the value of ω at which practical resonance occurs. Determine the corresponding value of *C*.

VI. Introduction to systems:

1. Transform the given IVP into an initial value problem for two first order equations.

$$u''+4u'+5t\,u=7-\sin(2t), \quad u(0)=-2, \; u'(0)=1$$

2. (a) Write the following IVP for a system of 2 linear ODEs as an IVP for a single second-order linear ODE

$$\begin{aligned} x'&=-y, & x(0)&=1 \\ y'&=10x-7y, & y(0)&=-7 \end{aligned} \qquad (1)$$

(b) Find the solution of the IVP (1)

3. Match the description of the phase portrait with the corresponding system (one description will not match)

I $x'=y,\ y'=-x$ II $x'=y,\ y'=x$ III $x'=-2y,\ y'=x$

A. circles B. Ellipses C. hyperbolas D. parallel lines

TEST 2 **ANSWERS TO PRACTICE PROBLEMS**

I.

1. (a) $1 < x < 3$ (b) $-\frac{\pi}{2} < t < 1$ (c) $0 < t < 4$.

2. II and III

3. A, B, E, G, I, J

4. C: te^t is not a solution; D: the two functions are not linearly independent.

5. By the principle of superposition: B, C, D

6. (i) $t > 0$ (iii) $W(y_{1,} y_2) = t^{-3}$ Since the Wronskian is nonzero on I, y_1 and y_2 form a fundamental set of solutions.

(iv) $y = \frac{(1+2\ln t)}{t}$

II.

1. (a) $y = -e^{3t}\sin(2t) + e^{3t}\cos(2t)$ (b) $y(t) = (1-2t)e^{-2t}$ (c) $y(t) = -3e^{-2t} + 6e^{-t}$

2. The general solution is $y = c_1 e^{\alpha t} + c_2 e^{(\alpha-1)t}$. Thus all solutions are unbounded if $\alpha > 1$

3. $c_1 + c_2 t + c_3 t^2 + c_4 e^{-2t} + c_5 t\, e^{-2t} + e^{-3t}(c_6\cos(2t) + c_7\sin(2t)) + te^{-3t}(c_8\cos(2t) + c_9\sin(2t))$

4. Since $y = \cos(2x)$ is a solution, $2i$ and $-2i$ must be roots of the characteristic equation and $r^2 + 4$ must be a factor. Using long division, another factor is $6r^2 + 5r + 1$. Thus the characteristic equation can be written as $(r^2+4)(3r+1)(2r+1)$ and the general solution is $y = c_1\cos(2x) + c_2\sin(2x) + c_3 e^{-x/3} + c_4 e^{-x/2}$

III.

1. $y(t) = c_1 t^{-1} + c_2 t^{-1}\ln t$ **2.** $y(t) = c_1 t + c_2 t\, e^t$

IV.

1. $y = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{2} t^2 e^{-t}$ **2.** $y = \frac{3}{2} e^{2x} - 2e^{-x} + \left(\frac{3}{2} - 3x\right) - 2xe^{-x}$ **3**. $y = \frac{5}{9} e^{2t} - \frac{5}{9} e^{-t} + (t^2 - \frac{2}{3} t) e^{2t}$

4.

(i) $Y(t) = t(A_0 t^2 + A_1 t + A_2) + t(B_0 t^2 + B_1 t + B_2)e^{-3t} + E\sin(3t) + F\cos(3t)$

(ii) $Y(t) = A_0 t + A_1 + t(B_0 t + B_1)\sin t + t(C_0 t + C_1)\cos t$

(iii) $Y(t) = A_0 e^t\cos(2t) + A_1 e^t\sin(2t) + e^{2t}(B_0 t + B_1)\sin t + e^{2t}(C_0 t + C_1)\cos t$

(iv) $Y(t) = A_0 e^{-t} + t\, e^{-t}(B_0 t^2 + B_1 t + B_2)\sin t + t\, e^{-t}(C_0 t^2 + C_1 t + C_2)\cos t$

(v) $Y(t) = A_0 t^2 + A_1 t + A_2 + t^2(B_0 t + B_1)e^{2t} + (C_0 t + C_1)\sin(2t) + (D_0 t + D_1)\cos(2t)$

V.

1. $y(t) = -3\cos(2t) + 3\sin(2t) = 3\sqrt{2}\cos\left(2t - \frac{3\pi}{4}\right)$ **2.** $2u'' + 39.2u = 0, \quad u(0) = 0, u'(0) = 0.1$

3. a. undamped free motion: Simple Harmonic Motion
b. undamped motion with $\omega_0 = 1.8 \approx 2 = \omega$: Beats
c. undamped motion with $\omega_0 = 2 = \omega$: Resonance
d. free damped motion; roots of the characteristic equation are complex: Under Damped
e. damped motion with forcing term: Steady State plus Transient
f. free damped motion; roots of the characteristic equation are repeated: Critically Damped

4. (a) $\omega_0 = 3 = \omega$ (b) $u(t) = \frac{t}{6}\sin(3t)$

5. (a) $\omega_0 = 2.8 \approx 3 = \omega$ (b) $\frac{1}{1.16}(\cos(2.8t) - \cos(3t)) = \frac{2}{1.16}\sin(0.1t)\sin(2.9t)$

(c) Length of beats = $\frac{2\pi}{2(0.1)} = 10\pi$ Period of oscillation = $\frac{2\pi}{2.9}$

6. $2\sqrt{2}$ 7**.** $\omega = \frac{\sqrt{34}}{2} \approx 2.91$ The corresponding maximum value of the amplitude is $C\left(\frac{\sqrt{34}}{2}\right) \approx 0.338$

VI.

1. $x_1' = x_2, \quad x_2' = -5t x_1 - 4x_2 + 7 - \sin(2t), \quad x_1(0) = -2 \quad x_2(0) = 1$

2. (a) $y'' + 7y' + 10y = 0, \quad y(0) = -7, y'(0) = 59$ (b) $x(t) = 4e^{-2t} - 3e^{-5t}, \quad y(t) = 8e^{-2t} - 15e^{-5t}$

3. I: Solving $\frac{dy}{dx} = -\frac{x}{y}$ yields $y^2 + x^2 = C,$ hence the trajectories are circles and I matches A

II: Solving $\frac{dy}{dx} = \frac{x}{y}$ yields $y^2 - x^2 = C,$ hence the trajectories are hyperbolas and II matches C

III: Solving $\frac{dy}{dx} = -\frac{x}{2y}$ yields $y^2 + \frac{x^2}{2} = C,$ hence the trajectories are ellipses and III matches B