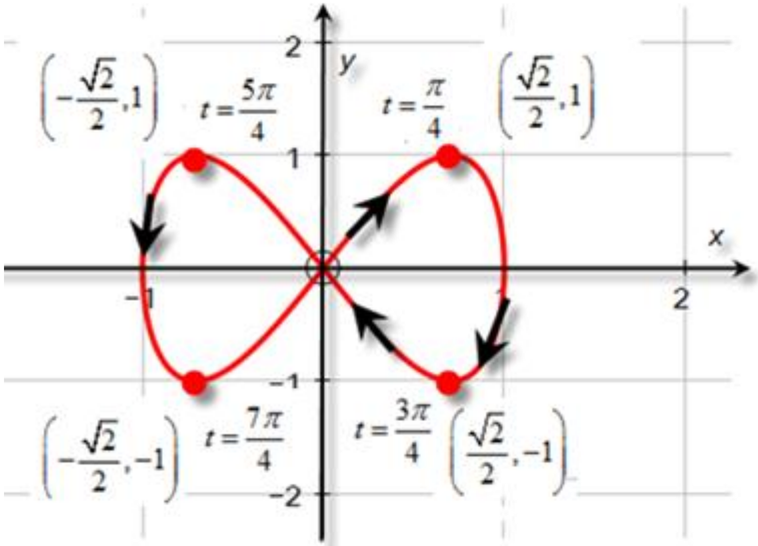
Calculus With Parametric Curves

Calculus with Parametric Curves

Suppose *f* and *g* are both differentiable functions where

$$x = f(t)$$
$$y = g(t)$$

Recall, these are parametric equations, and *t* is called the parameter.



$$x = \sin(t)$$
$$y = \sin(2t)$$

By the chain rule, we have:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \qquad if \ \frac{dx}{dt} \neq 0$$

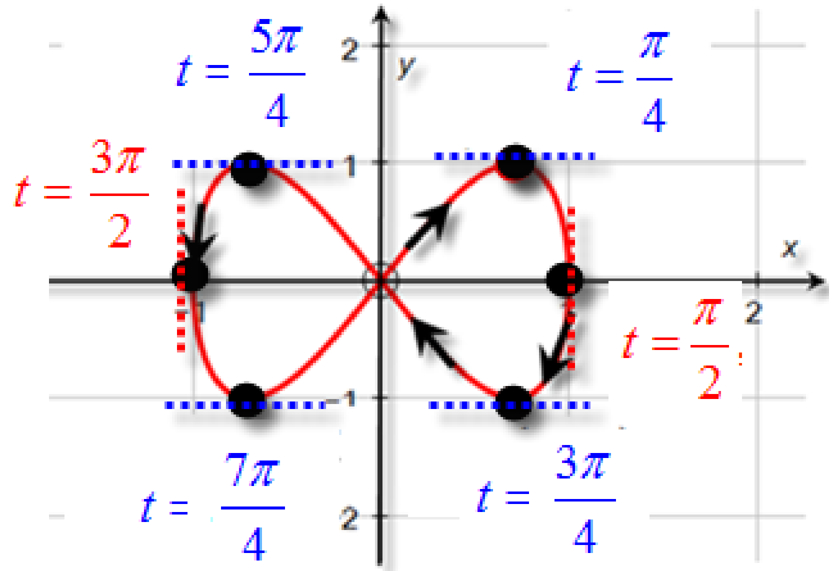
Calculus with Parametric Curves

Find where there exits a

a) Horizontal tangent

b) Vertical Tangent for

$$x = \sin(t)$$
$$y = \sin(2t)$$



$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad dx/dt = \cos(t)$$
$$dy/dt = 2\cos(2t)$$

$$\frac{dy}{dx} = \frac{2\cos(2t)}{\cos(t)} \qquad \frac{dy}{dx} = \frac{2\cos(2t)}{\cos(t)}$$

HT : Set numerator equal to 0 and solve for t

VT: Set denominator equal to 0 and solve for t

Horizontal Tangent:

$$2\cos(2t) = 0 \qquad t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\cos(2t) = 0$$

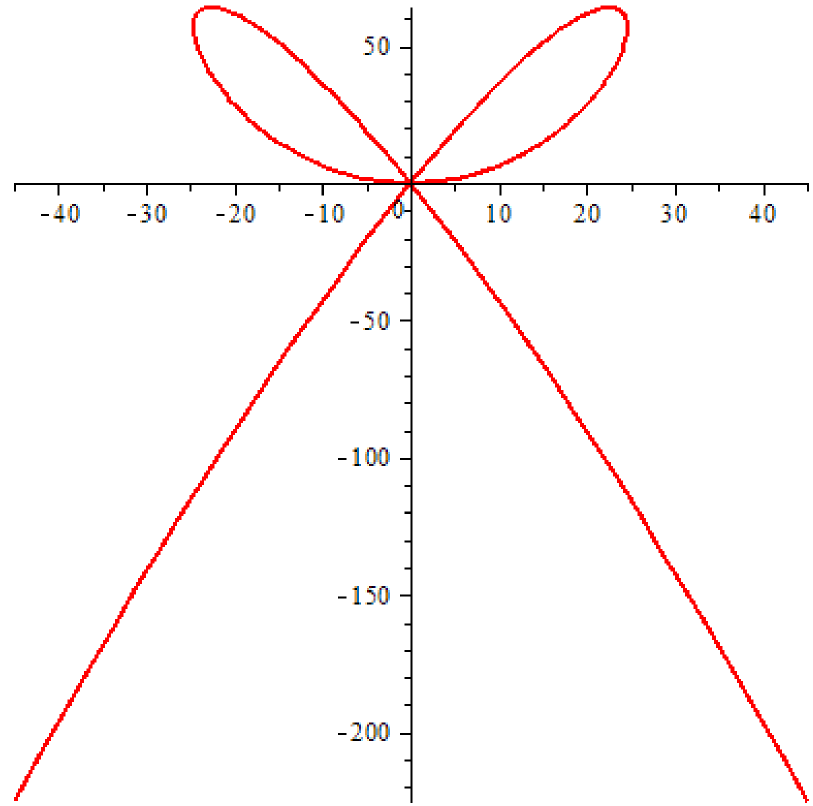
$$2t = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{7\pi}{2}$$

Vertical Tangent:

$$\cos(t) = 0 \qquad t = \frac{\pi}{2}, \frac{3\pi}{2}$$

Calculus with Parametric Curves

Find the equation of the line tangent to the parametric curve at $t = 4$:



$$x = t^3 - 16t$$

$$y = 16t^2 - t^4$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad dx/dt = 3t^2 - 16$$

$$dy/dt = 32t - 4t^3$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{32t - 4t^3}{3t^2 - 16}$$

Find slope:

$$\left.\frac{dy}{dx}\right|_{t=4} = \frac{32(4) - 4(4)^3}{3(4)^2 - 16} = -4$$

Find point:

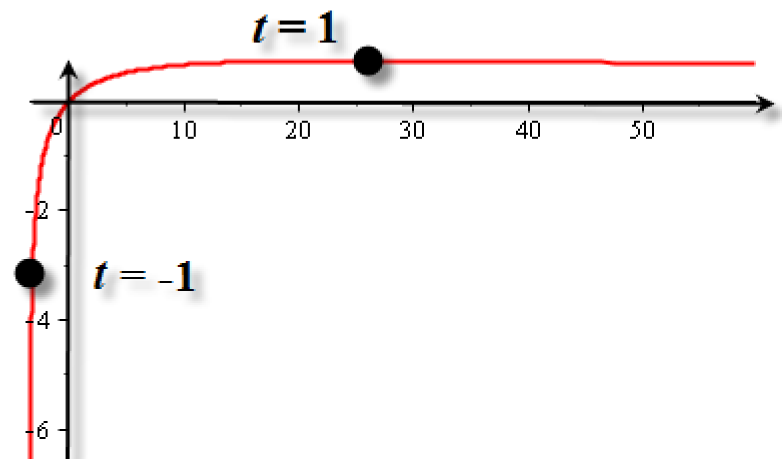
$$x(4) = 4^3 - 16(4) = 0$$

$$y(4) = 16t^2 - t^4 = 0$$

Find line:

$$y = mx + b \Rightarrow y = -4x$$

Calculus with Parametric Curves



Find the highest and left most point on the parametric curve:

$$x = 9te^{t}$$

$$y = 2te^{-t}$$

$$dx/dt = 9e^{t} + 9te^{t}$$

$$dy/dt = 2e^{-t} - 2te^{-t}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{-t} - 2te^{-t}}{9e^{t} + 9te^{t}} = \frac{2e^{-t}(1-t)}{9e^{t}(1+t)}$$

$HT : at \ t = 1$

$VT : at \ t = -1$

Highest Point:

$$x(1) = 9(1)e^{1} = 9e$$

$$y(1) = 2(1)e^{-1} = \frac{2}{e}$$

Left Most Point:

$$x(-1) = 9(-1)e^{-1} = -\frac{9}{e}$$

$$y(-1) = 2(-1)e^{1} = -2e$$

Calculus with Parametric Curves

We already know how to find the length L
of a curve C in the form *y* = *F*(*x*) on [*a*, *b*] .

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Suppose C can also be described with parametric equations *x* = *f*(*t*) and *y* = *g*(*t*)

Suppose C is traversed once on $\alpha \le t \le \beta$

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_\alpha^\beta \sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}\,\frac{dx}{dt}\,dt$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2\left(\frac{dy/dt}{dx/dt}\right)^2}\,dt$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Calculus with Parametric Curves

Find the length of the curve on [0, 5]:

$$x = 1 + 18t^2$$

$$y = 6 + 12t^3$$

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$dx/dt = 36t$$

$$dy/dt = 36t^2$$

We have:

$$L = \int_0^5 \sqrt{(36t)^2 + \left(36t^2\right)^2}\, dt = \int_0^5 \sqrt{36^2 t^2 + 36^2 t^4}\, dt$$

$$= \int_0^5 \sqrt{36^2 t^2 \left(1 + t^2\right)}\, dt$$

$$= \int_0^5 36t\sqrt{1 + t^2}\, dt$$

$$= 18\int_1^{26} u^{1/2} du \quad = 18 \cdot \frac{2}{3} u^{3/2} \Big|_1^{26} = 12\left(26^{3/2} - 1\right)$$

$$u = 1 + t^2 \qquad t = 0,\ u = 1$$

$$du = 2tdt \qquad t = 0,\ u = 26$$

$$18du = 36tdt$$

Calculus with Parametric Curves

Find the length of the curve:

$$x = 12(\cos\theta + \theta\sin\theta) \qquad 0 \le \theta \le \frac{9\pi}{10}$$
$$y = 12(\sin\theta - \theta\cos\theta)$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$dx/dt = 12(-\sin\theta + \sin\theta + \theta\cos\theta) = 12\theta\cos\theta$$
$$dy/dt = 12(\cos\theta - (\cos\theta - \theta\sin\theta)) = 12\theta\sin\theta$$

We have:

$$L = \int_0^{9\pi/10} \sqrt{(12\theta\cos\theta)^2 + (12\theta\sin\theta)^2}\, d\theta$$

$$L = \int_0^{9\pi/10} \sqrt{12^2\theta^2(\sin^2\theta + \cos^2\theta)}\, d\theta$$

$$L = \int_0^{9\pi/10} 12\theta\, d\theta = 6\theta^2\Big|_0^{9\pi/10} = 6\left(\frac{81\pi^2}{100} - 0\right) = \frac{243}{50}\pi^2$$