Sequences and Summation

Sequences

Informally, a sequence is an infinite progression of objects (usually numbers), consisting of a first, a second, a third, and so on. The members of a sequence are called elements or terms.

Example sequence: 2,4,6,8,10, …

It is customary to denote sequences with the letters a, b, c and to use subscript notation to refer to individual terms: a_n is the nth term of the sequence a. The notation $\{a_n\}$ refers to the entire sequence, not to the set of the terms. A sequence is an ordered list, whereas a set is an unordered collection of objects.

Example: if $\{a_n\} = 2,4,6,8,10, \dots$ then $a_0 = 2$, $a_1 = 4$.

(To simplify some of the formulas, the index n will always start with $n = 0$ in this presentation. This will not always be the case when you encounter sequence problems – the first element may well correspond to $n = 1$.)

How not to define a sequence

On the previous slide, we defined a sequence by giving the first 5 terms and expected that a reasonable reader would understand that we mean the sequence of positive even numbers.

This way of defining a sequence, by giving finitely many terms of it and expecting the reader to recognize a pattern in them is mathematically indefensible because there is always more than one conceivable pattern to continue a sequence, and, more importantly, a sequence does not have to fit any pattern in the first place. Each term is independent from all other terms and can assume any value.

Given $\{a_n\} = 1,2,3, \ldots$ $\{a_n\}$ could be the sequence that repeats these 3 numbers in perpetuity: $\{a_n\} = 1,2,3,1,2,3, \ldots$ or $\{a_n\}$ could be constant after the third term: $\{a_n\} = 1,2,3,7,7,7,7, \ldots$

If you think that these examples are far-fetched and exaggerate the issue of misunderstanding, consider the following example: $\{a_n\} = 1,2,4, \ldots$ could represent $\{a_n\} = 1,2,4, 8, 16, 32, 64, \ldots$ (each term is double the previous) but also $\{a_n\} = 1,2,4, 7,11,16, \ldots$ where the nth term plus n produces the next term, for all n.

Defining a sequence properly

A proper definition of a sequence requires us to define **all** terms, not just finitely many of them. This can be done in two ways, directly and recursively. (We will discuss recursive definition later in this presentation).

A direct definition gives each a_n as a function of n. We often just give an equation for a_n without bothering to quantify the "for all $n \in \mathbb{N}_0$".

Examples:

$a_n = 2n$ is the sequence of nonnegative even numbers.

$a_n = n^2$ is the sequences of squares.

$a_n = 2^n$ is the sequence of powers of 2 that are integers.

Sequences as Functions

Technically, a sequence is a special kind of function, namely a function whose domain is $\mathbb{N}_0$. Therefore, we could use standard function notation to represent sequences and write $f(n)$ instead a_n, but we use the latter for reasons of tradition.

Arithmetic sequences

A sequence that has a constant difference between successive terms is called **arithmetic**. An arithmetic sequence has the form

$$a, a + d, a + 2d, a + 3d, a + 4d, \ldots$$

Where a is the first term and d is the common difference between successive terms. The general formula is

$$a_n = a + nd.$$

An arithmetic sequence is just a linear function with a domain restricted to the natural numbers.

Example: $a_n = 1 + 2n$ is arithmetic with $a = 1$ and $d = 2$.

Geometric Sequences

A sequence that has a constant quotient between successive terms is called **geometric**. A geometric sequence has the form

$$a, aq, aq^2, aq^3, \ldots$$

Where a is the first term and q is the common quotient between successive terms. The general formula is $a_n = aq^n$.

A geometric sequence is just an exponential function with a domain restricted to the natural numbers.

Example: $a_n = 3 \cdot 2^n$ is arithmetic with $a = 3$ and $q = 2$.

Recursive Definition

A recursive definition gives each term of a sequence as a function of previous sequence terms:

$$a_n = f(a_{n-1}, a_{n-2}, \dots, a_{n-k})$$

This equation is called a **recurrence relation**, or more precisely, a k-step recurrence relation. A recursive definition involving a k-step recurrence relation requires the values of the first k terms: $a_0, a_1, \dots, a_k$. These values are called the initial conditions.

For example, $a_n = a_{n-1} + 2$ and $a_0 = 0$ defines the sequence of non-negative even numbers recursively. The equation $a_n = a_{n-1} + 2$ is the recurrence relation.

Each arithmetic sequence $a_n = a + nd$ has the recursive definition $a_n = a_{n-1} + d, a_0 = a$.

Each geometric sequence $a_n = aq^n$ has the recursive definition $a_n = qa_{n-1}$, $a_0 = a$.

These three recursive definitions all involve 1-step relations.

An Example of a Multi-Step Recurrence Relations

The **Fibonacci Sequence** is the sequence $\{f_n\}$ defined by the initial conditions $f_0 = 1$, $f_1 = 1$ and the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n = 2,3,4, \ldots$:

$$\{f_n\} = 1,1,2,3,5,8,13,21,34,55, \ldots$$

$f_n = f_{n-1} + f_{n-2}$ is a two-step recurrence.

Summation

The **sigma notation** is a convenient way to express lengthy sums that follow a pattern:

$$\sum_{k=n}^{N} f(k) = f(n) + f(n+1) + \cdots + f(N)$$

The index variable k always runs from the integer n to the integer N in steps of 1. Examples:

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \qquad \sum_{k=4}^{8} 2^k = 16 + 32 + 64 + 128 + 256$$

$$\sum_{k=1}^{5} \frac{k}{2} = \frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \frac{4}{2} + \frac{5}{2}$$

Since the terms in a sum can be arbitrarily rearranged, and common multiplicative constants can be factored out, we have the general laws

$$\sum_{k=n}^{N} (f(k) + g(k)) = \sum_{k=n}^{N} f(k) + \sum_{k=n}^{N} g(k)$$

$$\sum_{k=n}^{N} c f(k) = c \sum_{k=n}^{N} f(k)$$

Sums of Consecutive Integers

There is a convenient summation formula available for the sum of the first n positive integers:

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

Such a formula for a sigma sum is known as a "closed form" formula.

For even n, this formula has a simple explanation. The first and the last term have a sum of $n+1$. The second and the second-to-last term also have a sum of $n+1$, and so on. Since there are $\frac{n}{2}$ such pairs, the sum is $\frac{n(n+1)}{2}$. The formula is also valid for odd n. Think about how this explanation needs to be adjusted for that case.

Examples:

$$1 + 2 + 3 + \cdots + 100 = \frac{100 \cdot 101}{2} = 50 \cdot 101 = 50050$$

$$\sum_{k=50}^{100} k = \sum_{k=1}^{100} k - \sum_{k=1}^{49} k = \frac{100 \cdot 101}{2} - \frac{49 \cdot 50}{2}$$

Application to Arithmetic Sums

Using the summation formula we just learned, we can evaluate all arithmetic sums, i.e. all sums of the form

$$\sum_{k=1}^{n}(a+kd)=\sum_{k=1}^{n}a+d\sum_{k=1}^{n}k=na+d\frac{n(n+1)}{2}$$

Example:

$$\sum_{k=1}^{100}(2+3k)=200+3\cdot\frac{100\ \cdot\ 101}{2}$$

Sums of Consecutive Squares

Summation formulas are also available for the sum of the squares and cubes of the first n positive integers:

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 = \frac{n^2(n+1)^2}{4}$$

Such formulas for $\sum_{k=1}^{n} k^p$ exist, in fact, for all positive integers p.

Example: $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 = \frac{10 \cdot 11 \cdot 21}{6}$

Index Shifting

Let us consider another summation example:

$$\sum_{k=1}^{50}(k+1)^2 = 2^2 + 3^2 + \cdots + 51^2$$

Since we have a summation formula for the k^2, we could evaluate this sum by expanding $(k+1)^2$ into $k^2 + 2k + 1$:

$$\sum_{k=1}^{50}(k+1)^2 = \sum_{k=1}^{50} k^2 + 2\sum_{k=1}^{50} k + \sum_{k=1}^{50} 1 = \frac{50 \cdot 51 \cdot 101}{6} + 2\frac{50\ \cdot 51}{2} + 50 = 45525$$

There is a better way though, which is to perform an **index shift**. Index shifting means to increase the limits of the index variable k by some integer constant c and simultaneously substitute $k - c$ for k in the expression being summed:

$$\sum_{k=n}^{N} f(k) = \sum_{k=n+c}^{N+c} f(k-c)$$

If we apply an index shift with $c = -1$ to our example sum, we get

$$\sum_{k=1}^{50}(k+1)^2 = \sum_{k=2}^{51} k^2 = \sum_{k=1}^{51} k^2 - 1 = \frac{51 \cdot 52 \cdot 103}{6} - 1 = 45525$$

Geometric Sums

We shall determine a summation formula that helps us evaluate geometric sums, i.e. sums of the form

$$\sum_{k=0}^{n} aq^k$$

Since the constant multiplier a can be factored out, we only require a formula for

$$\sum_{k=0}^{n} q^k = 1 + q + q^2 + \cdots + q^n$$

Let us multiply that expression by $(q-1)$ and distribute:

$$(1 + q + q^2 + \cdots + q^n)(q-1) = q + q^2 + \cdots + q^{n+1} - 1 - q - \cdots - q^n$$

Every "positive" term here is canceled by a "negative term", except for two terms that remain:

$$(1 + q + q^2 + \cdots + q^n)(q-1) = q^{n+1} - 1$$

If $q \neq 1$, we can divide by $(q-1)$ and obtain

$$\sum_{k=0}^{n} q^k = \frac{q^{n+1} - 1}{q - 1}$$

For $q = 1$, $1 + q + q^2 + \cdots + q^n = n$.

Geometric Sums II

To evaluate a geometric sum where the exponent does not start at zero, we could use the difference approach we have already encountered earlier:

$$\sum_{k=n}^{N} q^k = \sum_{k=0}^{N} q^k - \sum_{k=0}^{n-1} q^k$$

There is a better way, however. We factor out the common highest factor of q, which is q^n and then perform an index shift:

$$\sum_{k=n}^{N} q^k = q^n \sum_{k=n}^{N} q^{k-n} = q^n \sum_{k=0}^{N-n} q^k = q^n \frac{q^{N-n+1} - 1}{q - 1} = \frac{q^{N+1} - q^n}{q - 1}$$

Geometric Sums III

Let us work a more complex example of a summation involving the geometric sum.

$$\sum_{k=5}^{20} \frac{3^{2k+1}}{5^{3k-1}} = \sum_{k=5}^{20} \frac{3^{2k} \cdot 3^1}{5^{3k} \cdot 5^{-1}} = 15 \sum_{k=5}^{20} \frac{9^k}{125^k} = 15 \sum_{k=5}^{20} \left(\frac{9}{125}\right)^k$$

We now use the formula we just discovered to evaluate:

$$15 \sum_{k=5}^{20} \left(\frac{9}{125}\right)^k = 15 \left(\frac{\left(\frac{9}{125}\right)^{21} - \left(\frac{9}{125}\right)^{5}}{\frac{9}{125} - 1} \right)$$

The geometric series

If we let $n \to \infty$ in the geometric sum $\sum_{k=0}^{n} q^k$, we obtain the **geometric series**:

$$\sum_{k=0}^{\infty} q^k$$

Technically, this quantity is the limit of the geometric sums as $n \to \infty$. It is a calculus fact that we shall not explain that this limit only exists when $|q| < 1$. In that case, the limit of q^{n+1} as $n \to \infty$ is zero. Therefore,

$$\sum_{k=0}^{\infty} q^k = \lim_{n\to\infty} \frac{q^{n+1} - 1}{q - 1} = \frac{1}{1-q} \text{ for } |q| < 1.$$

Example: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{1-\frac{1}{2}} = 2.$

Telescoping Sums

Let us consider

$$\sum_{k=1}^{n}\frac{1}{k(k+1)}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}$$

The partial fraction decomposition of $\frac{1}{k(k+1)}$ is

$$\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$$

By substituting this identity into the sum we get

$$\sum_{k=1}^{n}\frac{1}{k(k+1)}=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

We can see that all terms in this sum cancel except $\frac{1}{1}$ and $-\frac{1}{n+1}$. The sum collapses like an old-style telescope and is therefore named a **telescoping sum**. Therefore,

$$\sum_{k=1}^{n}\frac{1}{k(k+1)}=1-\frac{1}{n+1}$$