MATH 4305, Summer 2012,
Practice Exam 3, Solutions

Show all your work. Please give yourself 70 minutes.

Problm 1 Let $\mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $W = span\{\mathbf{u}, \mathbf{v}\}$.

a) Find the orthogonal projection of $\mathbf{y}$ onto W.

Solution Note: $\mathbf{u}$ is not orthogonal to $\mathbf{v}$. G-S process to an orthogonal basis of W.

Let $\mathbf{w}_1 = \mathbf{u}$, $\mathbf{w}_2 = \mathbf{v} - \frac{\mathbf{v} \bullet \mathbf{u}}{\mathbf{u} \bullet \mathbf{u}} \mathbf{u} = (0.5, -0.5, 1)^T$.

We choose $\{\mathbf{w}_1, \mathbf{w'}_2 = 2\mathbf{w}_2\}$ as the orthogonal basis for W.

$$Proj_W \mathbf{y} = \frac{\mathbf{y} \bullet \mathbf{w}_1}{\mathbf{w}_1 \bullet \mathbf{w}_1} \mathbf{w}_1 + \frac{\mathbf{y} \bullet \mathbf{w'}_2}{\mathbf{w'}_2 \bullet \mathbf{w'}_2} \mathbf{w'}_2 = (\frac{7}{3}, \frac{2}{3}, \frac{5}{3})^T.$$

b) Find the distance between $\mathbf{y}$ and W.

Solution Compute $\|\mathbf{y} - Proj_W \mathbf{y}\| = \frac{4}{3}\sqrt{3}$.

Problem 2 Find the trigonometric function of the form $f(t) = c_0 + c_1 sin(t) + c_2 cos(t)$ that best fits the data points $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 3)$, using least squares. Compute the least square error.

Solution We subject to solve the least square problem for $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & sin(1) & cos(1) \\ 1 & sin(2) & cos(2) \\ 1 & sin(3) & cos(3) \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}.$$

we thus solve the normal equation to obtain

$$\mathbf{c} = (A^T A)^{-1} A^T \mathbf{b}.$$

The least square error is $\|\mathbf{b} - A\mathbf{c}\|$.

Problem 3 Using determinant, find all possible values of a so that the columns of A given below are linearly dependent?

$$\begin{pmatrix} a & 2a & 0 & 0 \\ 0 & 0 & a-3 & 3(a-3) \\ 0 & -2a & 0 & 1 \\ 0 & 0 & a-2 & 2(a-2) \end{pmatrix}$$

Solution: Columns of A are linearly dependent if and only if $det(A) = 0$. Compute the determinant using row operations and co-factor expansion, one has $det(A) = -2a^2(a-2)(a-3)$. So, $det(A) = 0$ if $a = 0$, or $a = 2$ or $a = 3$.

Problem 4. Let A be the following matrix

$$\begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix}$$

a) Find the QR factorization of A.

Solution: First of all, it is easy to verify that A has linearly independent columns. So, QR factorization is possible. Let the columns of A are $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$. Then they form a basis for $Col(A)$. Q will be found by using G-S process with normalization on this basis. First of all, we find an orthogonal basis for $Col(A)$ using G-S process:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, 1, 1, 1)^T,$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \bullet \mathbf{v}_1}{\mathbf{v}_1 \bullet \mathbf{v}_1}\mathbf{v}_1 = (1, -1, -1, 1)^T,$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \bullet \mathbf{v}_1}{\mathbf{v}_1 \bullet \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \bullet \mathbf{v}_2}{\mathbf{v}_2 \bullet \mathbf{v}_2}\mathbf{v}_2 = (1, -1, 1, -1)^T.$$

Thus, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $Col(A)$. This basis can be normalized into an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ with

$$\mathbf{u}_1 = \frac{1}{2}\mathbf{v}_1,\ \ \mathbf{u}_2 = \frac{1}{2}\mathbf{v}_2,\ \ \mathbf{u}_3 = \frac{1}{2}\mathbf{v}_3.$$

The matrix Q is now given by $[\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]$, and R is given by $Q^T A$. The results are

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

$$R = \begin{pmatrix} 2 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

b) Find the orthoganal projection of $\mathbf{b} = (1, 2, 3, 4)^T$ onto $Col(A)$.

Solution: $proj_{Col(A)} = QQ^T\mathbf{b} = \frac{1}{4}\begin{pmatrix} 3 & -1 & 1 & 1 \\ -1 & 3 & 1 & 1 \\ 1 & 1 & 3 & -1 \\ 1 & 1 & -1 & 3 \end{pmatrix}\mathbf{b} = (2, 3, 2, 3)^T.$

Problem 5: If A is an $n \times n$ matrix, is it true that $det(AA^T) = det(A^TA)$? Why?
Solution: Yes, both are $(det(A))^2$.