Structural Induction

Let S be the set of binary strings defined recursively as follows:

Basis step: $\lambda \in S, \text{where } \lambda$ denotes the empty string.

Recursive step: If $x \in S$ then $0x \in S$ and $x1 \in S$.

a. **List the elements of S produced by the first 3 applications of the recursive definition.**

$$S_0 = \lambda \ S_1 = \{0, 1\},\ S_2 = \{00, 01, 11\},\ S_3 = \{000, 001, 011, 111\}$$

The elements of S produced by the first 3 applications of the recursive definition is

$$S_0 \cup S_1 \ \cup S_2 \cup S_3 = \{\lambda, 0, 1, 00, 01, 11, 000, 001, 011, 111\}.$$

b. **Use structural induction to prove that S does not contain any string x in which a 0 occurs to the right of a 1 in x.**

When we use structural induction to show that the elements of a recursively defined set S have a certain property, then we need to do the following procedure:

1. **Basis step**: show all the elements defined in the basis step have the desired property.
2. **Inductive step**: assume that an arbitrary element of the set S has the desired property. This is your inductive hypothesis. Using the inductive hypothesis, prove that, when you create more elements in S by using the recursive definition, these newly created elements of S have the same property.
3. **Conclusion:** state that by the principle of structural induction all the elements in S have the same property.

Basis step: $\lambda \in S$ and the empty string λ does not contain a 0 which occurs to the right of a 1.

Recursive Step: Assume $x \in S$ and x has the property that it does not contain a 0 which occurs to the right of a 1. Then we need to prove that the strings constructed by using the recursive definition, $0x \in S$ and $x1 \in S$, have the same property.

Proof:
Case1: The string $0x$ can not contain a 0 to the right of a 1, since that 1 and 0 would be the part of $\text{string } x$. According to our inductive hypothesis x does not have that property.

Case2: The string $x1$ can not contain a 0 to the right of a 1, since that 1 and 0 would be the part of $\text{the string } x$. According to our inductive hypothesis x does not have that property.

By **structural induction** we have proved that S does not contain any string x in which a 0 occurs to the right of a 1 in x.