Give yourself 1 hour and fifty minutes to work this practice exam. Do as much as you can. Email Dr. Taylor when you have done so

MAT 243

Practice Final Exam

1. (*0 points*) Prove that this statement is true: *For all integers $a, b,$ and c if $a|b$ and $b|c$ then $a|c$*

$a|b$ iff there exists an integer k such that $b = ka$, and $b|c$ iff there is an integer ℓ such that $c = \ell b$. Thus it follows that $c = \ell ka$, hence that $a|c$.

2. (*0 points*) Let A be the set $\{a, b, c, d, e\}$. Consider the relation on A,
$R = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (b, c), (a, c), (b, a), (c, b), (c, a), (d, e), (e, d)\}$.

a) Is R reflexive? Prove it.
Yes. R is reflexive iff $\forall x \in A \quad (x, x) \in R$. But $\{(a, a), (b, b), (c, c), (d, d), (e, e)\} \subseteq R$.

b) Is R symmetric? Prove it. Yes. For the elements in $\{(a, a), (b, b), (c, c), (d, d), (e, e)\} \subseteq R$, this is certainly true, since these elements don't change when I switch the first and second component. The other part of R is $\{(a, b), (b, c), (a, c), (b, a), (c, b), (c, a), (d, e), (e, d)\}$. With a little rearranging we can break this down to $\{(a, b)--(b, a), \quad (b, c)--(c, b), \quad (a, c)--(c, a), \quad (d, e)--(e, d)\}$

c) Is R transitive? Prove it. Yes. R is transitive iff $(x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R$. When $x = y$ or $y = z$ this is obviously true. The other cases to consider are $(a, b) \wedge (b, c) \Rightarrow (a, c)$–which is true, $(a, c) \wedge (c, b) \Rightarrow (a, b)$–which is true, along with all of those cases like $(a, b) \wedge (b, a) \Rightarrow (a, a)$–which take one back to the case that $x = y$, as well as all of those cases we can get from the preceding by switching the first and second components of each ordered pair.

d) Is R an equivalence relation? Prove it. Yes, R is an equivalence relation iff R is reflexive, symmetric and transitive, which we have now shown it is.

3. (*0 points*) Consider 'words' of length n in the alphabet $\{a, b, c\}$

a) Use induction to prove that the number of such words is 3^n Base step: the number of strings of length 0 is $1 = 3^0$. Induction step: suppose the number of strings of length n is 3^n. For every word w of length n, there are three words of length $n + 1$; wa, wb, wc. Since every word of length $n + 1$ can be uniquely specified by a word of length n and one of three letters, it follows that there are three times as many words of length $n + 1$ as there are words of length n, i.e. $3 \cdot 3^n = 3^{n+1}$.

b) Now consider the restricted class of all words of length n of the form $w = \ell_1\ell_2 \cdots \ell_n$, with the property that each letter ℓ_i cannot be followed by the same letter: $\ell_i \neq \ell_{i+1}$. Prove that the number of such words is $3 \times 2^{n-1}$. Base step: the number of strings of length 1 is $1 = 3^1$ since there are three possible letters. Induction step: suppose the number of strings of length $n > 1$ is $3 \times 2^{n-1}$. For every word $w = \ell_1\ell_2 \cdots \ell_n$ of length n, there are two words of length $n+1$: the two words of the form $w\ell_{n+1}$ for which $\ell_n \neq \ell n+1$ and $\ell_{n+1} \in \{a,b,c\}$. Since every word of length $n+1$ can be uniquely specified by a word of length n and one of two letters, it follows that there are twice times as many words of length $n+1$ as there are words of length n, i.e. $2 \times 3 \times 2^{n-1} = 3^{n+1}$.

4. (*0 points*) Suppose that a class of 25 students are all assigned grades in the range$\{A, B, C, D, E, I\}$. Show that there are at least 5 students who got the same grade. Since there are 25 students and six possible grades it follows from the pigeonhole principle that there are at least

$$\left\lceil \frac{25}{6} \right\rceil = 5$$

students who got the same grade.

5. (*0 points*) Find the solution of the recurrence relation $a_n = -2a_{n-1} - a_{n-2}$ with the initial conditions $a_0 = 0, a_1 = 1$ The characteristic equation of this homogeneous linear recurrence relation is $r^2 - 2r + 1 = 0$, so that there is just one root $r = 1$, with multiplicity 2. From this it follows that every solution can be written as $a_n = \alpha 1^n + \beta n \times 1^n = \alpha + \beta n$. Plugging in the values $n = 0$ and $n = 1$ gives us the two equations $0 = a_0 = \alpha + \beta \times 0$, $1 = a_1 = \alpha + \beta \times 1$. The first of these equations tells that $\alpha = 0$ while the second then tells us that $\beta = 1$. From this it follows that $a_n = n$.

6. (*0 points*) Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in A_1,1000 elements in A_2, and 10,000 elements in A_3 if there are two elements common to each pair of sets, and one element in all three sets. The inclusion-exclusion identity is $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| = 100 + 1000 + 10,000 - 2 - 2 - 2 + 1$ or $11,100 - 6 + 1 = 11,095$.

7. (*0 points*) Prove that

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^k$$

.

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = (1+2)^n,$$

from the binomial theorem.

8. (*0 points*) Use induction to prove that $n^2 - 7n + 12 \geq 0$ for every $n \geq 3$. When $n = 3$ we get $3^2 - 7 * 3 + 12 = -12 + 12 = 0 \geq 0$. Suppose that $n^2 - 7n + 12 \geq 0$. Then

$$\begin{aligned} ccc(n+1)^2 - 7(n+1) + 12 &= (n^2 + 2n + 1) - (7n + 7) + 12 \\ &= n^2 - 7n + 12 + (2n + 1 - 7) \\ &= (n^2 - 7n + 12) + (2n - 6) \end{aligned}$$

Since $2n - 6 \geq 0$ for $n \geq 0$, and by assumption $(n^2 - 7n + 12) \geq 0$ when $n \geq 0$.

9. (*0 points*) Find the greatest common divisor of the numbers 966 and 4788 Implementing the Euclidean algorithm, 966 divides the 4788 with a remainder of 924. 924 divides 966 with a remainder of 42. 42 divides 924 exactly so the GCD is 42.