Solutions

## Assignment four

**Problem 1.** Compute the coefficient of $x^7$ in $(1 + x^4 + x^{12})^{15}$.

Answer: 0 (in $(1 + x^4 + x^{12})^{15}$ every power of $x$ is divisible by 4.)

**Problem 2.** Compute the coefficient of $x^{10}$ in $(1 + x)/(1 - x)^2$.

$(1 + x)/(1 - x)^2 = (1 + x)(1 + 2x + 3x^2 + 4x^3 + \dots)$ and so the answer is $11 + 10 = 21$.

**Problem 3.** Compute the coefficient of $x^3$ in $(1 + x^2)(1 + x)^{100}$.

Use the binomial theorem. $(1+x)^{100} = 1+\binom{100}{1}x+\binom{100}{2}x^2+\binom{100}{3}x^3+\dots$. thus the answer is $\binom{100}{3} + \binom{100}{1}$.

**Problem 4.** Compute the coefficient of $x^{10}$ in $(1 + x)^{10}(1 - x)^{10}$.

We have that $(1 - x)^{10}(1 + x)^{10} = (1 - x^2)^{10}$. So the answer is $-\binom{10}{5}$.

**Problem 5.** Compute the coefficient of $x^n$ in $(2 + x)/(2 - x)$.

Observe that $1/(2 - x) = \frac{1}{2(1-x/2)} = \sum_{i=0}^{\infty} 2^{-(i+1)}x^i$. It follows that the answer is $2/2^{n+1} + 1/2^n = 1/2^{n-1}$ if $n > 0$ and is equal to 1 if $n = 1$.

## Assignment five

**Problem 1.** In how many ways can we put 31 people in 3 rooms such that each room has an odd number of people?

The solution is the coefficiont of $x^{31}$ in $(x+x^3/3!+x^5/5!+\dots)^3$ multiplied by 31!. Using $x + x^3/3! + x^5/5! + \dots = (e^x - e^{-x})/2$ we get $(x + x^3/3! + x^5/5! + \dots)^3 = (e^x - e^{-x})^3/8$. This is equal to $(e^{3x} - 3e^x + 3e^{-x} - e^{-3x})/8$. So the answer is: $(3^{31} - 3 - 3 + 3^{31})/8 = (3^{31} - 3)/4$.

**Problem 2.** Compute the coeeficient of $x^{10}$ in $\left(\sum_{i=1}^{\infty} x^i/i!\right)^3$. (Warning: the summation goes from 1 and not from 0.)

We need to compute the coefficient of $x^{10}$ in $(e^x - 1)^3 = e^{3x} - 3e^{2x} + 3e^x - 1)$. This is $(3^{10} - 3 \times 2^{10} + 3)/10!$.

**Problem 3.** Check that the exponential generating function of the sequence $\{i^2\}_{i=0}^{\infty}$ is equal to $x(x+1)e^x$. (*Hint:* use that $i^2 = i(i-1) + i$)

We have that $\sum_{i=0}^{\infty} i^2/i!x^i = \sum_{i=0}^{\infty}(i(i-1)+i)/i!x^i = \sum_{i=0}^{\infty} i(i+1)/i!x^i + \sum_{i=0}^{\infty} i/i!x^i = x^2e^x + xe^x$.

**Problem 4.** Six people draw their names from a hat. What is the probability that nobody draws his/her own name? (*Hint:* use the inclusion-exclusion formula in the set of all possible draws.)

$(6! - 6 \times 5! + \binom{6}{2}4! - \binom{6}{3}3! + \binom{6}{4}2! - \binom{6}{5} + 1)/6! = (1 - 1 + 1/2! - 1/3! + 1/4! - 1/5! + 1/6!)$

**Problem 5.** Let $V = \{1, 2, \ldots, 100\}$ and let $G_1$ be the graph on the vertex set $V$ in which $a, b$ are connected if $a - b$ is odd. Let $G_2$ be the graph on $V$ in which $a, b$ are connected if $a \neq b$ and $a - b$ is even. Are the graphs $G_1$ and $G_2$ isomorphic?

Solution: The degree of every point in $G_1$ is 50 and the degree of every point in $G_2$ is 49.