Structural Induction

Let S be the set of ordered pairs of integers defined recursively as follows:

Basis step: $(0, 0) \in S$

Recursive step: If $(a, b) \in S$ then $(a + 2, b) \in S$ $(a + 1, b + 1) \in S$.

a. List the elements of S produced by the first 2 applications of the recursive definition.

$$S_0 = \{(0,0)\},\ S_1 = \{(2,0),(1,1)\},\ S_2 = \{(4,0),(3,1),(2,2)\}.$$

The elements of S produced by the first 2 applications of the recursive definition are

$$S_0 \cup S_1 \cup S_2 = \{(0,0),(2,0),(1,1),(4,0),(3,1),(2,2)\}.$$

b. Use structural induction to prove that if $(\boldsymbol{a},\boldsymbol{b}) \in \boldsymbol{S}$ then $\boldsymbol{a}+\boldsymbol{b}$ is even.

When we use structural induction to show that the elements of a recursively defined set S have a certain property, then we need to do the following procedure:

1. **Basis step**: show that all the elements defined in the basis step have the desired property.
2. **Inductive step**: assume that an arbitrary element of the set S has the desired property. This is your inductive hypothesis. Using the inductive hypothesis, prove that, when you create more elements in S by using the recursive definition, these newly created elements of S have the same property.
3. **Conclusion:** state that by the principle of structural induction all the elements in S have the same property.

Recall: $\boldsymbol{a}$ is even integer if there exists an integer $\boldsymbol{k}$ such that $\boldsymbol{a = 2 \cdot k}$

Basis step: $(0,0) \in S$ and $0 + 0 = 0$ is even by the definition of even integers, $2 \cdot 0 = 0$.

Recursive Step: Assume $(a, b) \in S$ with the property that $a + b$ is even, that is $a + b = 2k$ for some integer k. We need to prove that the following elements of S, created by using the recursive definition, $(a + 2, b), (a + 1, b + 1)$ has the same property. That is, the sum of the first and the second coordinate is even.

Proof: Using the inductive hypothesis,

$Case$ 1: $(a + 2) + b = (a + b) + 2 = 2 \cdot k + 2 = 2 \cdot (k + 1)$, where k+1 is an integer.

Case 2: $(a + 1) + (b + 1) = (a + b) + 2 = 2 \cdot k + 2 = 2 \cdot (k + 1)$, where $k + 1$ is an integer.

Thus, in both cases the sum of the first and the second coordinate is even by the definition of even integers.

By **structural induction** we proved that if $(a, b) \in S$ then $a + b$ is even.