Example: Find the area of the region enclosed by each loop of the curve
$x = \sin t - \cos t$, $y = \cos^2 t$, $0 \le t < 2\pi$

Solution:

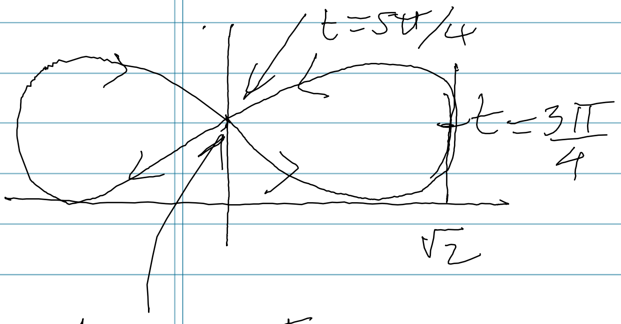

FIG. 1

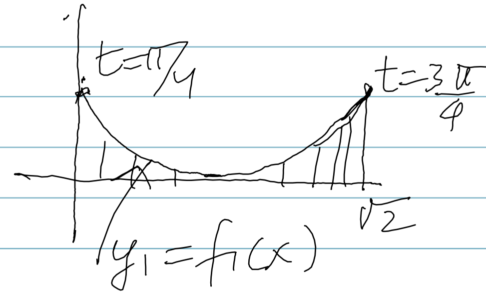

FIG. 2

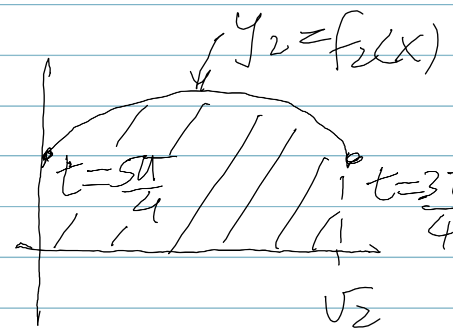

FIG. 3.

$x = \sin t - \cos t = 0, \implies \tan t = 1$

$t = \frac{\pi}{4}$ or $\frac{5\pi}{4}$ on $[0, 2\pi)$.

$\frac{dx}{dt} = \cos t + \sin t = 0$; $\tan t = -1$

$t = \frac{3\pi}{4}$ for the right loop.

$t = \frac{3\pi}{4}$, $x = \sin\left(\frac{3\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right)$
$= \sqrt{2}$

FIG 2

The area under this curve is given by

$$A_1 = \int_0^{\sqrt{2}} y_1\,dx = \int_{\pi/4}^{3\pi/4} \cos^2 t(\cos t + \sin t)\,dt$$

$$= \left(\sin t - \frac{1}{3}\sin^3 t - \frac{1}{3}\cos^3 t\right)\Big|_{\pi/4}^{3\pi/4}$$

$$= \frac{\sqrt{2}}{6}$$

HG 3.

The area under this curve is given by

$$A_2 = \int_0^{\sqrt{2}} y_2\,dx = \int_{5\pi/4}^{3\pi/4} \cos^2 t\,(\cos t + \sin t)\,dt$$

$$= \left(\sin t - \frac{1}{3}\sin^3 t - \frac{1}{3}\cos^3 t\right)\Big|_{5\pi/4}^{3\pi/4}$$

$$= \frac{5\sqrt{2}}{6}$$

Area enclosed in the right loop $= A_2 - A_1$

$$= \frac{5\sqrt{2}}{6} - \frac{\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$$

By symmetry,

area enclosed in the left loop $= \dfrac{2\sqrt{2}}{3}$.