Name: ______________________

Math 243, Spring 2006, Professor Callahan
Final Exam, Tue., May 9

Note 1: This test is closed book. You may use one $8\frac{1}{2} \times 11$ sheet of notes (both sides).

Note 2: Show your work. Clarity counts. If I can't follow your reasoning I can't give credit.

Problem 1: Of a classroom of 25 students, 12 are men and 13 are women. 15 are engineering majors and 10 are sophomores. 6 are male engineering majors, 6 are sophomore engineering majors and 5 are sophomore men. 2 are male sophomore engineering majors. How many women are there in the class who are neither sophomores nor engineering majors?

Answer: Let M be the set of men in the class, E be the set of engineering majors and S be the set of sophomores. By the inclusion–exclusion principle,

$$|M \cup E \cup S| = |M| + |E| + |S| - |M \cap E| - |M \cap S| - |E \cap S| + |M \cap E \cap S| = 12 + 15 + 10 - 6 - 6 - 5 + 2 = 22.$$

Thus 22 of the 25 students are men or sophomores or engineering majors. The number of people who are none of these things is therefore 3.

Problem 2: Solve

$$277 \cdot x \equiv 1 \pmod{343}$$

for x.

Answer: We use the Euclidean algorithm:

$$\begin{aligned} 343 &= 1 \cdot 277 + 66 \\ 277 &= 4 \cdot 66 + 13 \\ 66 &= 5 \cdot 13 + 1. \end{aligned}$$

Thus we have

$$\begin{aligned} 1 &= 66 - 5 \cdot 13 = 66 - 5 \cdot (277 - 4 \cdot 66) \\ &= 21 \cdot 66 - 5 \cdot 277 = 21 \cdot (343 - 277) - 5 \cdot 277 \\ &= 21 \cdot 343 - 26 \cdot 277. \end{aligned}$$

Thus, modulo 343, we have

$$1 = (-26) \cdot 277,$$

so that $x = (-26) \bmod 343 = 317$. Thus $x = 343n + 317$ for any integer n.

Problem 3: Evaluate

$$3^{(5^{81})} \bmod 34.$$

Answer: We first find $\phi(34) = 16$. One way to see this is to observe that the numbers < 34 that are relatively prime to 34 are the odd numbers, not counting 17. There are 16 of these. The other way is to factor 34 into powers of primes:

$$\phi(34) = \phi(2)\phi(17) = 1 \cdot 16 = 16.$$

Now

$$3^{(5^{81})} \bmod 34 = 3^{(5^{81} \bmod 16)} \bmod 34,$$

so we need to calculate $5^{81} \bmod 16$. To do this we note that $\phi(16) = 8$ (all the odd numbers), so that $5^8 \bmod 16 = 1$. Thus

$$5^{81} \bmod 16 = 5^{81 \bmod 8} \bmod 16 = 5^1 \bmod 16 = 5.$$

Thus

$$3^{(5^{81})} \bmod 34 = 3^{(5^{81} \bmod 16)} \bmod 34 = 3^5 \bmod 34 = 243 \bmod 34 = 5.$$

Problem 4: How many ways are there to distribute 12 identical cookies among 4 (distinguishable) children?

Answer:

$$\binom{12+4-1}{12} = \frac{15!}{12!3!} = \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} = 455.$$

Problem 5: We flip a fair coin 5 times. What is the probability of getting a run of at least three heads in a row?

Answer: The possibilities with three heads in a row are

HHHHH
HHHHT
HHHTT
HHHTH
THHHH
THHHT
TTHHH
HTHHH

There are eight of these, out of $2^5 = 32$ total possibilities, so the probability is $8/32 = 1/4$.

Problem 6: Solve the recurrence relation

$$\begin{aligned} a_0 &= 2 \\ a_1 &= 1 \\ a_n &= a_{n-1} + 6a_{n-2}, \qquad n \geq 2. \end{aligned}$$

The characteristic equation is

$$r^2 = r + 6 \quad \Rightarrow \quad r^2 - r - 6 = (r+2)(r-3) = 0 \quad \Rightarrow \quad r = -2, 3.$$

Thus the solution is of the form

$$a_n = A(-2)^n + B3^n.$$

For $n = 0$ we get

$$2 = A + B,$$

while for $n = 1$ we get

$$1 = -2A + 3B.$$

The solution to these two equations in two unknowns is $A = B = 1$, so that the solution to the recurrence relations is

$$a_n = (-2)^n + 3^n.$$