ANSWERS TO CHAPTER TEST B – page 164-165.

1. The vectors are linearly dependent, since we can take $c_1 = 0$, $c_2 = 0$ and c_3 any nonzero number and
$0\,\mathbf{x}_1 + 0\,\mathbf{x}_2 + c_3\,\mathbf{x}_3 = \mathbf{0}$

2. (a) S_1 is a subspace (it is a line through the origin in $\mathbb{R}^2$)
(b) S_2 is not a subspace since it is not closed under addition. Counterexample: Let $\mathbf{x} = [1, 0]^T$ and $\mathbf{y} = [0, 1]^T$, then both $\mathbf{x}$ and $\mathbf{y}$ are in S_2, however the sum $\mathbf{x} + \mathbf{y} = [1,1]^T$ is not in S_2.

3 (a) The RREF of the given matrix is $U=\begin{bmatrix}1 & 3 & 0 & 2 & 3\\ 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$.

The free variable are x_2, x_4 and x_5 and a basis for N(A) is

$\{ [-3, 1, 0, 0, 0]^T,\ [-2, 0, -1, 1, 0]^T,\ [-3, 0, -1, 0, 1]^T \}$.

Since there are 3 vectors in the basis, we have that dim(N(A)) = 3.

(b) The pivots are in columns 1 and 3, thus

$\{ \mathbf{a}_1 = [1,0,0,0]^T,\ \mathbf{a}_3 = [1, 1, 2, 3]^T\}$

form a basis for the column space of A. Also, rank(A) = dimension of column space = 2.
(note: rank(A) + nullity = 2 + 3 = 5 = number of columns of A)

4. The columns of the matrix that correspond to the pivots are linearly independent and span the column space of the matrix. So the dimension of the column space is equal to the number of lead variables (pivots) in the Row Echelon Form of the matrix.
If there are r lead variables then there are $n - r$ free variables. By the Rank-Nullity Theorem the dimension of the nullspace is $n - r$. So the dimension of the nullspace is equal to the number of free variables in the echelon form of the matrix.

5. (a) Yes, it is possible: One-dimensional subspaces in $\mathbb{R}^3$ are lines through the origin. If $U_1 = \text{span}(\mathbf{u}_1)$ and $U_2 = \text{span}(\mathbf{u}_2)$ with $\mathbf{u}_1$ and $\mathbf{u}_2$ linearly independent (not collinear) then the two lines U_1 and U_2 will intersect only at the origin and $U_1 \cap U_2 = \{\mathbf{0}\}$
(b) No, it is not possible: Two dimensional subspaces are planes through the origin in $\mathbb{R}^3$. Any two distinct planes through the origin will intersect in a line. So $V_1 \cap V_2$ will contain infinitely many vectors.

6 (a) S is closed under addition and scalar multiplication (if we multiply a symmetric matrix by a scalar we still get a symmetric matrix; if we add two symmetric matrices, the sum is a symmetric matrix) and therefore it is a subspace.
(b) The symmetric matrices $\mathbf{E}_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ $\mathbf{E}_2=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ $\mathbf{E}_3=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ are linearly independent. An arbitrary symmetric matrix has the form $A=\begin{bmatrix}a & b\\ b & c\end{bmatrix}$ and can be written as a linear combination of $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$:
$A = a\,\mathbf{E}_1 + b\,\mathbf{E}_2 + c\,\mathbf{E}_3$. Thus $\{ \mathbf{E}_1 , \mathbf{E}_2, \mathbf{E}_3\}$ span the set and they form a basis for S.

7. (a) The dimension of the column space is 4 (the rank of A). By the rank- nullity Theorem dim(N(A)) = 0 and consequently N(A) = $\{\mathbf{0}\}$ (the Nullspace consists only of the zero vector).
(b) The column vectors of A are linearly independent since the rank of **A** is 4. However, they do not span $\mathbb{R}^6$ since you need at least **six** linearly independent vectors to span $\mathbb{R}^6$.
(c) By the consistency theorem, if **b** is in the column space of A then the system is consistent. The condition that the column vectors of A are linearly independent implies that there cannot be more than one solution. Therefore there must be exactly one solution.

8. (a) Since the dimension of $\mathbb{R}^3$ is 3, any collection of more than 3 vectors must be linearly dependent, thus $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and $\mathbf{x}_4$ are linearly dependent.

(b) Since $\dim(\mathbb{R}^3)=3$, it takes at least three vectors to span $\mathbb{R}^3$ so $\mathbf{x}_1$ and $\mathbf{x}_2$ do not span $\mathbb{R}^3$.

(c) The RREF of the matrix X= [$\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$] is $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, thus the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are linearly dependent ($\mathbf{x}_3 = -2\mathbf{x}_1 + 3\mathbf{x}_2$) and they only span a two- dimensional subspace of $\mathbb{R}^3$ (a plane).
The vectors do not form a basis for $\mathbb{R}^3$.

(d) The RREF of the matrix X= [$\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_4$] is the identity matrix, thus $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_4$ are linearly independent.
Since $\dim(\mathbb{R}^3)=3$, the three vectors form a basis for $\mathbb{R}^3$.

9. We have

$$c_1\mathbf{y}_1+c_2\mathbf{y}_2+c_3\mathbf{y}_3=c_1(\mathrm{A}\,\mathbf{x}_1)+c_2(\mathrm{A}\,\mathbf{x}_2)+c_3(\mathrm{A}\,\mathbf{x}_3)=\mathrm{A}(c_1\mathbf{x}_1+c_2\mathbf{x}_2+c_3\mathbf{x}_3)$$

Thus $c_1\mathbf{y}_1+c_2\mathbf{y}_2+c_3\mathbf{y}_3=\mathbf{0}$ if and only if $\mathrm{A}(c_1\mathbf{x}_1+c_2\mathbf{x}_2+c_3\mathbf{x}_3)=\mathbf{0}$.
Since A is nonsingular, this implies $c_1\mathbf{x}_1+c_2\mathbf{x}_2+c_3\mathbf{x}_3=\mathbf{0}$ and, from the independence of $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$, it follows that $c_1=c_2=c_3=0$. Thus $\mathbf{y}_1, \mathbf{y}_2$ and $\mathbf{y}_3$ are linearly independent.

10. (a) Since there are 3 linearly independent columns, we have that rank(A) = 3. By the Rank-Nullity theorem, dim(N(A)) = 5 – 3 = 2.

(b) Since $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are linearly independent, the first 3 columns of the reduced row echelon form of **U** will be $\mathbf{u}_1 = \mathbf{e}_1$, $\mathbf{u}_2 = \mathbf{e}_2$, $\mathbf{u}_3 = \mathbf{e}_3$.
The remaining columns of **U** satisfy the same dependency relations that the column vectors of **A** satisfy. Therefore
$\mathbf{u}_4 = \mathbf{u}_1 + 3\,\mathbf{u}_2 + \mathbf{u}_3 = \mathbf{e}_1 + 3\,\mathbf{e}_2 + \mathbf{e}_3$
$\mathbf{u}_5 = 2\,\mathbf{u}_1 - \mathbf{u}_3 = 2\,\mathbf{e}_1 - \mathbf{e}_3$
and it follows that

$$\mathbf{U}=\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

11. (a) Using the basic formula $\mathbf{x} = \mathrm{U}[\mathbf{x}]_{\mathrm{U}}$ and solving for $[\mathbf{x}]_{\mathrm{U}}$, gives $[\mathbf{x}]_{\mathrm{U}} = \mathrm{U}^{-1}\mathbf{x}$, thus the transition matrix from the standard basis to the basis U is $\mathrm{U}^{-1}=\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$

(b) Using the formula $\mathrm{U}[\mathbf{x}]_{\mathrm{U}} = \mathrm{V}[\mathbf{x}]_{\mathrm{V}}$ and solving for $[\mathbf{x}]_{\mathrm{U}}$, gives $[\mathbf{x}]_{\mathrm{U}} = \mathrm{U}^{-1}\mathrm{V}\,[\mathbf{x}]_{\mathrm{V}}$ and the transition matrix is
$\mathrm{U}^{-1}\mathrm{V}=\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 2 & 9 \end{bmatrix}=\begin{bmatrix} 31 & 10 \\ -13 & -3 \end{bmatrix}$.
If $\mathbf{z} = 2\,\mathbf{v}_1 + 3\mathbf{v}_2$, then $[\mathbf{z}]_{\mathrm{V}} = [2, 3]^T$ and $[\mathbf{z}]_{\mathrm{U}} = \begin{bmatrix} 31 & 10 \\ -13 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}=\begin{bmatrix} 92 \\ -35 \end{bmatrix}$
<u>Note:</u> we can easily check that $92\,\mathbf{u}_1 - 35\mathbf{u}_2 = 2\mathbf{v}_1 + 3\mathbf{v}_2 = (22, 31)^T$