# §1.5
# Elementary Matrices

## Elementary Matrices

If we start with the identity matrix $I$ and perform exactly one elementary row operation, the resulting matrix is called an **Elementary Matrix**.
There are 3 types of elementary matrices corresponding to the three types of elementary row operations:

**Type I:** obtained by interchanging two rows of $I$ (*Permutation matrix*).
**Type II:** obtained by multiplying a row of $I$ by a nonzero constant.
**Type III:** obtained by adding a multiple of one row to another row. (*Elimination Matrix*).

**PROPERTY:**
**Left multiplication** by $E$ performs the same row operation on $A$ as the one done on the Identity matrix to obtain $E$.

**EXAMPLE:** Consider the elementary matrix $E$ obtained by interchanging row2 and row3 of the Identity matrix:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \leftrightarrow R_3 \rightarrow E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Now left multiply an arbitrary $3 \times 3$ matrix $A$ by the matrix $E$:

$$E \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$EA$ interchanges row2 and row3 of $A$.

**EXAMPLE:** Let $A$ be a $3 \times 3$ matrix.
What $3 \times 3$ matrix $F$ adds 5 times row2 to row3 and then adds 2 times row1 to row2 when it multiplies A?

*Solution*: First find an elementary matrix $E_1$ that adds $5 \cdot row2$ to $row3$. We can obtain this matrix by performing this row operation on the identity:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 = r_3 + 5r_2 \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = E_1$$

Then find $E_2$ that performs $R_2 = r_2 + 2r_1$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 = r_2 + 2r_1 \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2$$

To perform the two operations in sequence we need 1st to multiply by $E_1$ and then by $E_2$.

$$\Rightarrow \boxed{F = E_2 \cdot E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}}$$

$F$ is the $3 \times 3$ matrix that performs the two operations in sequence.

CHECK:

$$FA = E_2E_1A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{11} + a_{21} & 2a_{12} + a_{22} & 2a_{13} + a_{23} \\ 5a_{21} + a_{31} & 5a_{22} + a_{32} & 5a_{23} + a_{33} \end{bmatrix}$$

**THEOREM:** If $E$ is an elementary matrix, then $E$ is nonsingular and $E^{-1}$ is an elementary matrix of the same type.

*Proof.*

- **Type I**
  $E$ is its own inverse, that is, $E = E^{-1}$.
  Recall that Type I elementary matrices are obtained by interchanging two rows of the Identity, if we interchange the two rows again we get back the Identity.

- **Type II**

$$E = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \ddots & & & \\ 0 & 0 & \cdots & \alpha & \cdots & 0 \\ \vdots & & & & \ddots & \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \quad \to E^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \ddots & & & \\ 0 & 0 & \cdots & \frac{1}{\alpha} & \cdots & 0 \\ \vdots & & & & \ddots & \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

- **Type III**

$$E = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \ddots & & & \\ 0 & m & \cdots & 1 & \cdots & 0 \\ \vdots & & & & \ddots & \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

(obtained by performing the row operation $row_j = row_j + m \cdot row_i$ on the identity)
To "undo" the row operation we need to subtract $m \cdot row_i$ from $row_j$:

$$E^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \ddots & & & \\ 0 & -m & \cdots & 1 & \cdots & 0 \\ \vdots & & & & \ddots & \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

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**Definition**: A matrix $B$ is **row equivalent** to $A$ if there exists a finite sequence $E_1, E_2, \ldots, E_k$ of elementary matrices such that

$$B = E_k \cdot E_{k-1} \cdot \ldots \cdot E_1 \cdot A$$

(that is, $B$ can be obtained from $A$ by performing a finite sequence of elementary row operations — recall that left multiplication by an elementary matrix corresponds to performing a row operation on $A$).

## Theorem: Equivalent conditions for non singularity

Let $A$ be an $n \times n$ matrix. The following conditions are equivalent:

(a) $A$ is non singular (i.e. $A^{-1}$ exists).

(b) The homogeneous system $A\mathbf{x} = \mathbf{0}$ has <u>only</u> the trivial solution $\mathbf{x} = (0, 0, \ldots, 0)^T$.

(c) $A$ is row equivalent to $I$( i.e. we can obtain the Identity by performing a finite sequence of row operations on $A$, that is, $I = E_k E_{k-1} \cdots E_1 A$ for some elementary matrices $E_j$, i.e. **the RREF of $A$ is the Identity matrix**)

*Proof.*

(a)$\Rightarrow$ (b) $A\mathbf{x} = \mathbf{0} \Rightarrow A^{-1}A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}$.

(b)$\Rightarrow$ (c) If $A$ is not row equivalent to $I$, then rref(A) has a row of zeros and the system $A\mathbf{x} = \mathbf{0}$ would have infinitely many solutions which contradicts (b).

(c)$\Rightarrow$ (a) If $I = E_k E_{k-1} \cdots E_1 A$ then $A^{-1} = (E_k E_{k-1} \cdots E_1)$, thus $A^{-1}$ exists and the matrix is nonsingular.

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**Corollary:** Let $A$ be $n \times n$, then
$A$ is nonsingular, iff the system $A\mathbf{x} = \mathbf{b}$ has <u>a unique solution</u> for <u>every vector $\mathbf{b}$</u>.

*Proof:* If $A$ is nonsingular, then $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.
Conversely, if $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$, then, in particular, $A\mathbf{x} = \mathbf{0}$ has a unique solution and therefore $A$ is nonsingular by the Theorem on equivalent condition for nonsingularity.

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**EXAMPLE**
Let $A = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix}$

(i) Show that $A$ is row equivalent to $I$.

(ii) Write $A^{-1}$ as a product of elementary matrices.

(iii) Solve the system $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ using $A^{-1}$.

*Soln.*

(i) We need to find elementary matrices s.t. $I = E_k E_{k-1} \cdots E_2 E_1 A$. The elementary matrices correspond to the row operations needed to put $A$ in RREF. $A = \begin{bmatrix} 1 & 0 \\ \boxed{5} & 2 \end{bmatrix}$ To zero out $\boxed{5}$ we need to perform the row operation $R_2 = r_2 - 5r_1$. If we perform this same row operation on the identity matrix we obtain the elementary matrix $\boxed{E_1 = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}}$

We have $E_1 A = \begin{bmatrix} 1 & 0 \\ 0 & \boxed{2} \end{bmatrix}$
To make the $\boxed{2}$ a 1, we need to perform the row operation $R_2 = r_2/2$. Performing the same row operation on the identity matrix gives the elementary matrix $\boxed{E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}}$

Now we can easily check that $I = E_2 E_1 A$.

(ii) From $I = E_2E_1A$ we have $I = (E_2E_1)A$. Let $B = E_2E_1$, then $B \cdot A = I$ and $B = A^{-1}$ by definition of inverse. Thus

$$A^{-1} = E_2E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 \\ -5/2 & 1/2 \end{bmatrix}}.$$

(iii) From the Corollary:

$$\mathbf{x} = A^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -5/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -3/2 \end{bmatrix}} \leftarrow \text{Solution to the system.}$$

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From (c) in Theorem (Equivalent conditions for nonsingularity), we have that the same row operations that transform $A$ into $I$ will transform $I$ into $A^{-1}$(in the Example, $I = E_2E_1A \Rightarrow A^{-1} = E_2E_1 \Rightarrow A^{-1} = E_2E_1I$).
This gives us a method for computing $A^{-1}$:

## Computing the Inverse

1. Create the augmented matrix $[A|I]$.
2. Perform row operations that transform $A$ into $I$ on the augmented matrix $[A|I]$.
3. The RREF of $[A|I]$ will be $[I|A^{-1}]$.

**EXAMPLE**

Find the inverse of $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{bmatrix}$.

*Solution*

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & \\ -1 & 1 & 1 & 0 & 1 & 0 \\ -1 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 = r_2 + r_1 \\ R_3 = r_3 + r_1 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -2 & -2 & 1 & 0 & 1 \end{array}\right]$$

$$R_3 = r_3 + 2r_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 3 & 2 & 1 \end{array}\right] R_3 = r_3/2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3/2 & 1 & 1/2 \end{array}\right]$$

$$\begin{array}{l} R_1 = r_1 - r_3 \\ R_2 = r_2 - 2r_3 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & -1 & -1/2 \\ 0 & 1 & 0 & -2 & -1 & -1 \\ 0 & 0 & 1 & 3/2 & 1 & 1/2 \end{array}\right] = [I|A^{-1}] \boxed{\Rightarrow A^{-1} = \begin{bmatrix} -1/2 & -1 & -1/2 \\ -2 & -1 & -1 \\ 3/2 & 1 & 1/2 \end{bmatrix}}$$

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**NOTE:** A diagonal matrix has an inverse provided no diagonal entries are zero.

If $A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} \frac{1}{d_1} & & \\ & \ddots & \\ & & \frac{1}{d_n} \end{bmatrix}$

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> **IMPORTANT REMARK**
> In practice we <u>never</u> use the inverse to solve a system b/c it is computationally inefficient and it introduces round off errors. Solving the system using RREF is better.
> In MATLAB, use $A\backslash\mathbf{b}$ to solve the system $A\mathbf{x} = \mathbf{b}$

**Definition:** A matrix is **upper triangular** if all elements below the diagonal are zero, it is **lower triangular** if all elements above the diagonal are zero.

## LU Factorization

The goal is to factor the $m \times n$ matrix $A$ into the product $A = LU$, where $L$ is $m \times m$ lower triangular and $U$ is $m \times n$ upper triangular.

**EXAMPLE** Given $A = \begin{bmatrix} -1 & -3 & -3 \\ 3 & 8 & 3 \\ 2 & 4 & 1 \end{bmatrix}$,

(a) Determine elementary matrices of $\boxed{\text{TYPE III}}$ such that $E_3E_2E_1A = U$ where $U$ is upper triangular.

*Solution*
In order to zero out the entries 3 and 2 in the first column of the matrix $A$ we need to perform the row operations $R_2 = r_2 + 3r_1$ and $R_3 = r_3 + 2r_1$. Performing these same row operations on the identity matrix we obtain the first two elementary matrices.

$$I \to R_2 = r_2 + 3r_1 \to \boxed{E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \to E_1A = \begin{bmatrix} -1 & -3 & -3 \\ 0 & -1 & -6 \\ 2 & 4 & 1 \end{bmatrix}$$

$$I \to R_3 = r_3 + 2r_1 \to \boxed{E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}} \to E_2(E_1A) = \begin{bmatrix} -1 & -3 & -3 \\ 0 & -1 & -6 \\ 0 & -2 & -5 \end{bmatrix}$$

In order to zero out the entry -2 in $E_2(E_1A)$ we need to perform the row operation $R_3 = r_3 - 2r_2$. Performing the same row operation on the identity matrix will give us the third elementary matrix:

$$I \to R_3 = r_3 - 2r_2 \to \boxed{E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \to E_3(E_2E_1A) = \boxed{\begin{bmatrix} -1 & -3 & -3 \\ 0 & -1 & -6 \\ 0 & 0 & 7 \end{bmatrix} = U}$$

(b) Determine the inverses of $E_1, E_2$ and $E_3$ and let $L = E_1^{-1}E_2^{-1}E_3^{-1}$. What type of matrix is L? Verify that $A = LU$.

*Solution*

$$E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} = L}$$

$L$ is a lower triangular matrix and it is easy to check that $A = LU$.

$\boxed{\text{The matrices } L \text{ and } U \text{ are called the } \underline{\text{LU factorization}} \text{ or the triangular factorization of the matrix } A}$

**Shortcut for determining the matrix L**:

1st row operation: $R_2 = r_{\boxed{2}} + \ 3 \ r_{\boxed{1}} \quad \Rightarrow \quad l_{\boxed{21}} = - \ 3$ (note the negative)

2nd row operation: $R_3 = r_{\boxed{3}} + \ 2 \ r_{\boxed{1}} \quad \Rightarrow \quad l_{\boxed{31}} = - \ 2$ (note the negative)

3rd row operation: $R_3 = r_{\boxed{3}} - \ 2 \ r_{\boxed{2}} \quad \Rightarrow \quad l_{\boxed{32}} = 2$ (note the positive)

Diagonal elements of $L$ are all <u>ones</u> $\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$.

$\boxed{U \textbf{ is determined with the usual row reduction but } \underline{\textbf{using only row operations of TYPE III}}}$.

**EXAMPLE**

Find the $LU$ factorization of $A = \begin{bmatrix} -1 & 4 & 1 \\ 4 & -17 & -2 \\ 2 & -12 & 5 \end{bmatrix}$.

*Solution*

$$\begin{array}{lll} R_2 = r_2 + 4r_1 & \Rightarrow & l_{21} = -4 \\ R_3 = r_3 + 2r_1 & \Rightarrow & l_{31} = -2 \end{array} \Rightarrow \begin{bmatrix} -1 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & -4 & 7 \end{bmatrix}$$

$$R_3 = r_3 - 4r_2 \quad \Rightarrow \quad l_{32} = 4 \Rightarrow \boxed{\begin{bmatrix} -1 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = U} \qquad \boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix}}$$

WE CAN EASILY CHECK THAT $A = LU$.

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**NOTE:** Not all matrices have an $LU$ factorization b/c not all matrices can be reduced to upper triangular form using only row operation III, but if we allow the rows to be interchanged, then all matrices have an $LU$ factorization (but we have to keep track of the rows we permute).

**EXAMPLE**

$A = \begin{bmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{bmatrix}$ does not have an $LU$ (when we try to zero out the entries in the first column we obtain $\begin{bmatrix} -1 & -3 & -3 \\ 0 & 0 & -5 \\ 0 & -1 & -3 \end{bmatrix}$, and it is impossible to put this matrix in triangular form without permuting the rows). However, the matrix $\begin{bmatrix} -1 & -3 & -3 \\ 3 & 8 & 3 \\ 2 & 6 & 1 \end{bmatrix}$ does have an $LU$ factorization.

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## Solving systems using LU Factorization

The $LU$ factorization is very useful when studying computer methods for solving linear systems.
We can use the $LU$ to solve systems in the following manner:

To solve $A\mathbf{x} = \mathbf{b}$ or, equivalently: $LU\mathbf{x} = \mathbf{b}$,
Let $\mathbf{y} = U\mathbf{x}$
First solve $L\mathbf{y} = \mathbf{b}$ (USING SUBSTITUTION!) and then $U\mathbf{x} = \mathbf{y}$ (USING SUBSTITUTION!!!!)

**EXAMPLE**

Solve the system using the $LU$ factorization: $\begin{bmatrix} -1 & 4 & 1 \\ 4 & -17 & -2 \\ 2 & -12 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -22 \\ 88 \\ 46 \end{bmatrix}$

*Solution:* We already found before the $LU$ decomposition of the coefficient matrix.

First solve $L\mathbf{y} = \mathbf{b}$, i.e. $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -22 \\ 88 \\ 46 \end{bmatrix}$

This system reduces to
$y_1 = -22$
$-4y_1 + y_2 = 88$
$-2y_1 + 4y_2 + y_3 = 46$
and using forward substitution we find the solution $y_1 = -22,\ y_2 = 0$ and $y_3 = 2$.
We now solve $U\mathbf{x} = \mathbf{y}$, i.e. $\begin{bmatrix} -1 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -22 \\ 0 \\ 2 \end{bmatrix}$
The system reduces to
$-x_1 + 4x_2 + x_3 = -22$
$-x_2 + 2x_3 = 0$
$-x_3 = 2$
and using back substitution we find the solution

$$\mathbf{x} = (4, -4, -2)^T$$