MAT 275 **TEST 2 - LIST OF CONCEPTS**

Existence, Uniqueness, Fundamental sets for second order linear differential equations (Section 3.2)

- Know how to determine whether two functions of one variable are linearly independent or dependent on a given interval by determining whether one of them is a constant multiple of the other or not.
- A second order linear differential equation in standard form has the form: $y''+p(t)y'+q(t)y=g(t)$. If $g(t)=0$, it is called homogeneous (HODE); otherwise non-homogeneous.
- Know Theorem 3.2.1: Existence and Uniqueness. Consider the Initial Value Problem
  $$y''+p(t)y'+q(t)y=g(t), \quad y(t_0)=y_{0,} \;\; y'(t_0)=y'_0$$
  where p, q and g are continuous on an open interval *I* that contains $t_{0.}$ Then there is exactly one solution of this problem, and the solution exists throughout the interval *I*.
- Know Theorem 3.2.2: **Principle of Superposition**. If y_1 and y_2 are solutions of the **homogeneous** differential equation $y''+p(t)y'+q(t)y=0$, so is $c_1y_1+c_2y_2$ for any constant c_1 and c_2.
- Know Theorem 3.2.4: **Wronskian of Solutions**: Given two solutions y_1 and y_2 of a homogeneous ODE, always check whether the Wronskian of the two solutions is **not** everywhere zero. In this is case the two solutions are linearly independent and we say they form a Fudamental Set of Solutions for the ODE. The general solution is given by $c_1y_1+c_2y_2$ with c_1 and c_2 arbitrary constants.

Linear Homogeneous DEs with constant coefficients (Sections 3.1, 3.3, 3.4)
Second order, linear, homogeneous, constant coefficient ODE:
Given the homogeneous ODE $ay''+by'+cy=0$, with *a*, *b* and *c* constant:
Let r_1 and r_2 be the roots of the characteristic equation, $ar^2+br+c=0$

- If r_1 and r_2 are real and distinct, then the general solution of the homogeneous ODE is $y=c_1e^{r_1t}+c2\,e^{r_2t}$
- If $r_1=r_2=r$, then the general solution is $y=(c_1+c_2t)e^{rt}$.
- If the roots of the characteristic equation, r_1 and r_2 are complex conjugate $a\pm bi$, then the general solution is $y=c_1e^{at}\cos(bt)+c_2e^{at}\sin(bt)$

This can be generalized to a higher order differential equations:
If a root *r* of the characteristic equation has multiplicity *k*, then $e^{rt}, te^{rt}, t^2e^{rt}, \ldots, t^{k-1}e^{rt}$ are *k* linearly independent solutions.
If the complex roots $a\pm bi$ are repeated *k* times, then the general solution must contain the terms:
$e^{rt}\cos(bt), e^{rt}\sin(bt), t\,e^{rt}\cos(bt), t\,e^{rt}\sin(bt), t^2e^{rt}\cos(bt), t^2e^{rt}\sin(bt), \ldots, t^{k-1}e^{rt}\cos(bt), t^{k-1}e^{rt}\sin(bt)$.

Reduction of order (Section 3.4)
Suppose we know one solution $y_1(t)$, not everywhere zero of $y''+p(t)y'+q(t)=0$. To find another solution, let $y_2(t)=v(t)y_1(t)$. To find the unknown function *v*(*t*), substitute $y_2(t)$ into the Differential equation. The result will be a DE involving v' and v'' . This DE can be reduced to first order by letting $u=v'$ and solved using techniques from chapters 1 and 2.

Non-Homogeneous Equations, Undetermined Coefficients (Section 3.5)

- Know Theorem 3.5.2: **Solutions of Nonhomogeneous Equations**: The general solution of a Non-Homogeneous ODE is given by $y=y_c+y_p$ where y_c is the complementary solution (the general solution of the associated HODE) and y_p is a particular solution of the Non-Homogeneous ODE.

- Method of **undetermined coefficients** for linear DEs with constant coefficients: This method works only when the function *g*(*t*) is a polynomial, an exponential function, a sine or cosine and or a sum/product of these functions. The method consists of taking as a trial solution for y_p a linear combination of linearly independent terms appearing in *g*(*t*) and in all their derivatives *g*'(*t*), *g*''(*t*), … If any of these terms duplicates a solution of the associated homogeneous ODE, then we need to multiply the term by t^s where *s* is the smallest non negative integer such that no term in y_p duplicates a term in the complementary function y_c. The undetermined coefficients *A*, *B*, *C*, … are then determined by substituting y_p and the appropriate derivatives of it into the original DE. Once y_p is determined, a general solution is given by $y=y_c+y_p$ where y_c is the complementary function.
  - You must be able to write the correct expression for y_p with the minimal number of undetermined coefficients.
  - You must be able to solve for the undetermined coefficients and thus write explicitly a solution of the given non-homogeneous ODE.
  - You must be able to find a general solution for the given ODE.
  - You must be able to find the unique solution for the given ODE with initial conditions (the Initial Value Problem).

Mechanical vibrations (Section 3.7)

- Assume a mass *m* is attached to a spring with constant *k*, and assume there is a dashpot producing damping proportional to the velocity, with constant γ. When no external force is applied, the ODE governing the motion of the mass is given by $mu''+\gamma u'+ku=0$.
- *Undamped Free Vibration*: if $\gamma=0$, the equation reduces to $mu''+ku=0$ with solution $u(t)=A\cos(\omega_0 t)+B\sin(\omega_0 t)$, (Simple Harmonic Motion) where $\omega_0=\sqrt{k/m}$ is called the circular frequency. The period of the motion is given by $T=2\pi/\omega_0$. Be able to rewrite the solution as $u(t)=R\cos(\omega_0 t-\delta)$ where *R* is the amplitude and δ is the phase angle.
- *Damped Free Vibration*: if $\gamma\neq 0$ we can have three different kinds of solutions depending on the roots of the characteristic equation. If the roots are both real, they must be negative and the motion is **overdamped**; if the roots are repeated (necessarily real and negative) the motion is **critically damped**; if the roots are complex the motion is **underdamped**. For underdamped motion, know how to determine the quasi frequency and quasi period. Know to graph solutions in the three different cases of damping.

Forced Vibrations (Section 3.8)

We consider the case of a mass m attached to a spring with constant *k*, a dashpot with constant γ and also acted upon by an external force *F*(*t*). The governing ODE is given by: $mu''+\gamma u'+ku=F(t)$.
We consider the particular case where $F(t)=F_0\cos(\omega t)$.

- *Forced Vibrations Without Damping*: if $\gamma=0$, the equation reduces to $u''+ku=F_0\cos(\omega t)$.
  - If $\omega\neq\omega_0$ the solution is $u=c_1\cos(\omega_0 t)+c_2\sin(\omega_0 t)+C\cos(\omega t)$ where $u_c=c_1\cos(\omega_0 t)+c_2\sin(\omega_0 t)$ is the complementary solution and $U_p=C\cos(\omega t)$ is the particular solution (determined using the method of undetermined coefficients). Note that the constants c_1 and c_2 depend on the initial conditions. So we see that the resulting motion is the superposition of two oscillations, one with natural frequency ω_0, the other with frequency ω of the external force.

    - **BEATS**: If $\omega \approx \omega_0$ and the initial conditions are set to $u(0)=u'(0)=0,$ we have the phenomenon of beasts (a rapid oscillation with a (comparatively) slowly varying periodic amplitude).
    - **RESONANCE**: If $\omega = \omega_0$ we have the phenomenon of pure resonance, the increase without bound in the amplitude of the oscillations
- *Forced Vibrations with Damping*: if $\gamma \neq 0,$ we can have three different kind of solutions depending on the roots of the characteristic equation. In any case, the complementary solutions $u_c(t) \to 0$ as $t \to \infty.$ So, $u_c(t)$ is a *transient* solution (i.e. dying out in time) leaving only the particular solution $u_p(t)=C\cos(\omega t-\alpha)$ (*steady state solution* or *forced response*) with $C=\frac{F_0}{\sqrt{(k-m\omega^2)^2+(\gamma\omega)^2}}.$ Note that, if $\gamma > 0,$ then the "forced amplitude" *C* always remains finite, in contrast with the case of resonance in the undamped case, but *C* may attain a maximum for some value of $\omega,$ in which case we speak of *practical resonance*. To determine the practical resonance value ω we need to differentiate *C* as a function of ω and solve $\frac{dC}{d\omega}=0$.

NOTE: It is not a good idea to memorize the formulas above. Just remember the assumptions for Beats and Pure Resonance and be able to solve the Differential Equations using the method of Undetermined Coefficients from Section 3.5.

Introduction to linear systems (Section 7.1)

- Know how to transform a linear ODE of any order *n* into a system of *n* first-order ODEs.
- Know how to write a system of first-order linear ODEs in matrix form.
- Know how to solve simple two-dimensional systems by rewriting them as a single higher order ODE.
- Know how to determine the trajectories of the phase portraits for simple 2-dimensional systems