“For all x, there exists y.. “ Proof Writing

The basic logical structure of a “for all $x \in X$, there exists $y \in Y$ such that $P(x, y)$ ” proof is like this:

“Suppose $x \in X$ is arbitrary. Let $y =$ (some function of x that comes seemingly out of nowhere). Then (show how with this choice of y, the property $P(x, y)$ is satisfied.)”

So you’re the magician and the y is a rabbit you pull out of your hat. You don’t let the audience know how it got in there. How it got there, that’s your scratchwork, and it’s not part of your published proof. The scratchwork consists of writing down what you want, i.e. $P(x, y)$, and then figuring out how to select the y, given x, to make $P(x, y)$ happen.

The technical reason why starting with what you want $- P(x, y) -$ and then finding the y to make it happen is not acceptable proof of existence is that it reverses the flow of the logic. It assumes the conclusion that y exists, and then derives necessary conditions for it. Perhaps those necessary conditions narrow the candidate pool down to one candidate. Even then, that doesn’t prove that such a y actually exists, because it is not clear that the logical chain is reversible.

Here is an example of how assuming existence and deriving conditions does not prove existence. Someone wants to prove that a real number exists whose square is -9 (no such number exists of course because all squares of real numbers are greater than or equal to zero.) He starts by assuming existence.

Suppose $x \in \mathbb{R}$ such that $x^2 = -9$. Then by squaring both sides, we get $x^4 = 81$. This is true for $x = 3$ and $x = -3$. Therefore, I have proved the existence of real numbers whose square is -9.

It is undeniable that if such real numbers existed, their fourth powers would be 81, and it is also undeniable that because of that, the only possible candidate real numbers are $x = \pm 3$. So the conditional “if a real number has a square of 9, the number must be 3 or -3” is true. But it’s only true because its premise is false.

Assuming that unicorns don’t exist, the following is also a true statement, for the same reason: *“If unicorns exist, they have 4 legs, a tail and a horn, and if you grind their horn into dust, that dust is a room temperature superconductor and allows us to build faster than light engines.”*

This is why your actual existence proof must go in the other logical direction. It must begin by identifying the object, and then show that it does satisfy the claimed properties.

When our imagined proof writer attempts to do that for the "$x^2 = -9$" proof, he would like to write the following, and immediately realize that it is wrong:

Suppose $x = 3$. Then $x^2 = -9$.

Oops. The candidate x=3 doesn't work after all. Its square is 9, not -9.

[Based on this insight, we can in fact now give a correct proof of non-existence:

Suppose, to get a contradiction, that there is $x \in \mathbb{R}$ such that $x^2 = -9$. Then by squaring both sides, we get $x^4 = 81$. This is only true for $x = 3$ and $x = -3$. But the squares of those numbers are 9, not -9. That contradicts the assumption $x^2 = -9$. Therefore, real numbers whose squares are -9 do not exist.]

So let's repeat for the record one more time: when you write a proof that "for all $x \in X$, there exists $y \in Y$ such that $P(x, y)$", you can and should assume existence for your scratch work, to find the candidate y that will make P(x,y) true. But in the existence proof you publish, you cannot assume existence. You introduce the candidate y and show that it satisfies P(x,y). The chain of logic goes in the opposite direction here.

<u>A proof is not a teaching device.</u> You don't share the thought process that went into designing the proof. You don't reveal how you arranged for the rabbit to come out of the hat. You share only what is logically necessary to convince the reader that for each x, a suitable y exists.

Let's take a simple example theorem:

Theorem: for any real number x, there is a real number y so that their sum is 10.

Formally, we could write this statement as follows: $\forall x \exists y (x + y = 10)$.

This is a challenge-response game. Given an x, can I find the y so that $x + y = 10$?

Here's the <u>scratch work that convinces me that I can always win this game.</u> I write down what I want: $x + y = 10$ and then solve for y: $y = 10 - x$. Aha. Given x, I need to select 10-x for the y.

This scratch work has no place in my proof, because I assumed existence of what I want and used that to narrow down the pool of infinitely many real numbers y that could conceivably add up to 10 when we add them to x, to a single candidate. Technically, therefore, I haven't proved existence. All that this proves is that IF y exists, it must be 10-x. It doesn't prove that this y actually exists. Just like assuming that IF shape-shifting aliens exist and showing that then, your neighbor Joe must be one of them, combined with the fact that Joe undeniably exists, doesn't prove that shape-shifting aliens exist.

So the existence proof must reverse the logical direction. Here's the correct proof:

Suppose x is an arbitrary real number. Let $y = 10 - x$. Subtraction is defined for any pair of real numbers, so this y always exists. Then, $x + y = 10$. We have shown that for any x, a y exists with $x + y = 10$.* *(* let, pick, select, choose, they all convey the same meaning.)*

Other semi-common mistakes

"Proof by example": you can't prove statements that are true about infinitely many values of a variable by citing example. Appealing to the reader's intuition or pattern recognition ability doesn't make it any better. A proof has to convince a hostile jury, not a friendly audience. So don't do this:

Let's take $x = 2$ as an example. Then if we select $y = 8$, $x + y = 10$.

The tragedy of this incorrect proof lies in the fact that the writer was so close to a correct proof. She only needed to make that small abstraction step of realizing that how she found the y for the special case x=2, namely by subtracting the x from 10, works for any x, and writing it down accordingly.

The following is another attempted "proof by example", but it is worse than the previous.

Let's take $x = 2$ and $y = 8$. Then $x + y = 10$.

This proof writer shows no awareness of the fact that the statement to be proved is not symmetric between x and y. We are not proving that numbers exist whose sum is 10. We are proving something more subtle: that for any given number, a second number can be picked so that their sum is 10.

"Proof by affirmation or declaration": this is the attempt to prove existence by merely claiming it.

Suppose x is an arbitrary real number. Then let y be a real number such that $x + y = 10$. Therefore, we have proved that for every real number x, a real number y exists so that their sum is 10.

Exercise:

Prove that for any integer $n \geq 2$, you can always find an odd number k strictly between $2n$ and $3n$, i.e. you can always find an odd number k such that $2n < k < 3n$.