Inverse Trig Functions

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Introduction

Just as trig functions arise in many applications, so do the inverse trig functions. What may be most surprising is that they are useful not only in the calculation of angles given the lengths of the sides of a right triangle, but they also give us solutions to some common integrals. For example, suppose you need to evaluate the following integral:

$$\int_a^b \frac{1}{\sqrt{1-x^2}} \, dx$$

for some appropriate values of a and b. You can use the inverse sine function to solve it! In this capsule we do not attempt to derive the formulas that we will use; you should look at your textbook for derivations and complete explanations. This material will simply summarize the key results and go through some examples of how to use them. As usual, all angles used here are in radians.

Restrictions on the Domains of the Trig Functions

A function must be one-to-one for it to have an inverse. As we are sure you know, the trig functions are not one-to-one and in fact they are periodic (i.e. their values repeat themselves periodically). So in order to define inverse functions we need to restrict the domain of each trig function to a region in which it is one-to-one but also attains all of its values. We do this by selecting a specific period for each function and using this as a domain on which an inverse can be defined. Clearly there are an infinite number of different restrictions we could chose but the following are choices that are normally used.

Standard Restricted Domains		
Function	**Domain**	**Range**
$sin(x)$	$[\frac{-\pi}{2}, \frac{\pi}{2}]$	$[-1, 1]$
$cos(x)$	$[0, \pi]$	$[-1, 1]$
$tan(x)$	$(\frac{-\pi}{2}, \frac{\pi}{2})$	$(-\infty, \infty)$
$cot(x)$	$(0, \pi)$	$(-\infty, \infty)$
$sec(x)$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$	$(-\infty, -1] \cup [1, \infty)$
$csc(x)$	$[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$	$(-\infty, -1] \cup [1, \infty)$

Definitions of the Inverse Functions

When the trig functions are restricted to the domains above they become one-to-one functions, so we can define the inverse functions. For the sine function we use the notation $sin^{-1}(x)$ or $arcsin(x)$. Both are read "arc sine" . Look carefully at where we have placed the -1. Written this way it indicates the *inverse of the sine function*. If, instead, we write $(sin(x))^{-1}$ we mean the fraction $\frac{1}{sin(x)}$. The other functions are similar.

The following table summarizes the domains and ranges of the inverse trig functions. Note that for each inverse trig function we have simply swapped the domain and range for the corresponding trig function.

Standard Restricted Domains		
Function	**Domain**	**Range**
$sin^{-1}(x)$	$[-1, 1]$	$[\frac{-\pi}{2}, \frac{\pi}{2}]$
$cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$tan^{-1}(x)$	$(-\infty, \infty)$	$(\frac{-\pi}{2}, \frac{\pi}{2})$
$cot^{-1}(x)$	$(-\infty, \infty)$	$(0, \pi)$
$sec^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
$csc^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

We can now define the inverse functions more clearly. For the *arcsin* function we define

$$y = sin^{-1}(x) \text{ if } -1 \le x \le 1, \ y \text{ is in } [\frac{-\pi}{2}, \frac{\pi}{2}], \text{ and } sin(y) = x$$

Note that this is only defined when x is in the interval $[-1, 1]$. The other inverse functions are similarly defined using the corresponding trig functions.

Some Useful Identities

Here are a few identities that you may find helpful.
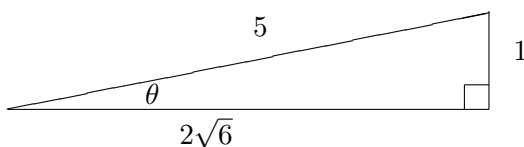
$$cos^{-1}(x) + cos^{-1}(-x) = \pi$$

$$sin^{-1}(x) + cos^{-1}(x) = \frac{\pi}{2}$$

$$tan^{-1}(-x) = -tan^{-1}(x)$$

Practicing with the Inverse Functions

**Example 1:** Find the value of $tan(sin^{-1}(\frac{1}{5}))$.
Solution: The best way to solve this sort of problem is to draw a triangle for yourself using the Pythagorian Theorem.

Here we use θ for the value of $sin^{-1}(\frac{1}{5})$. Notice that we labeled the hypotenuse and the side opposite θ by using the value of the sin of the angle. We then used the Pythagorian Theorem to get the remaining side. We now have the information that is needed to find $tan(\theta)$. Since $tan(\theta) = \frac{opposite}{adjacent}$, the answer is $\frac{1}{\sqrt{24}} = \frac{1}{2\sqrt{6}}$

Example 2: Find the value of $sin(cos^{-1}(-\frac{3}{5}))$.
Solution: Look at the following picture:
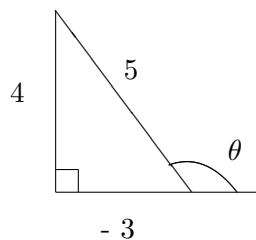
In this picture we let $\theta = cos^{-1}(-\frac{3}{5})$. Then $0 \leq \theta \leq \pi$ and $cos\theta = -\frac{3}{5}$. Because $cos(\theta)$ is negative, θ must be in the second quadrant, i.e. $\frac{\pi}{2} \leq \theta \leq \pi$. Using the Pythagorean

Theorem and the fact that θ is in the second quadrant we get that $sin(\theta) = \frac{\sqrt{5^2-3^2}}{5} = \frac{\sqrt{25-9}}{5} = \frac{4}{5}$. Note that although θ does not lie in the restricted domain we used to define the *arcsin* function, the unrestricted *sin* function is defined in the second quadrant and so we are free to use this fact.

Derivatives of Inverse Trig Functions

The derivatives of the inverse trig functions are shown in the following table.

Derivatives	
Function	**Derivative**
$sin^{-1}(x)$	$\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}, \quad |x| < 1$
$cos^{-1}(x)$	$\frac{d}{dx}(cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}, \quad |x| < 1$
$tan^{-1}(x)$	$\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}$
$cot^{-1}(x)$	$\frac{d}{dx}(cot^{-1}x) = \frac{-1}{1+x^2}$
$sec^{-1}(x)$	$\frac{d}{dx}(sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$
$csc^{-1}(x)$	$\frac{d}{dx}(csc^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$

In practice we often are interested in calculating the derivatives when the variable x is replaced by a function $u(x)$. This requires the use of the *c*hain rule. For example,

$$\frac{d}{dx}(sin^{-1}u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}, \quad |u| < 1$$

The other functions are handled in a similar way.

Example 1: Find the derivative of $y = cos^{-1}(x^3)$ for $|x^3| < 1$

Solution: Note that $|x^3| < 1$ if and only if $|x| < 1$, so the derivative is defined whenever $|x| < 1$.

$$\frac{d}{dx}(cos^{-1}(x^3)) = -\frac{1}{\sqrt{1-(x^3)^2}} \cdot \frac{d}{dx}(x^3)$$

$$= -\frac{1}{\sqrt{1-(x^3)^2}} \cdot (3x^2)$$

$$= -\frac{3x^2}{\sqrt{1-x^6}}$$

Example 2: Find the derivative of $y = tan^{-1}(\sqrt{3x})$.

Solution:

$$\frac{d}{dx}(tan^{-1}(\sqrt{3x})) = \frac{1}{1+(\sqrt{3x})^2} \cdot \frac{d}{dx}(\sqrt{3x})$$

$$= \frac{1}{1+(\sqrt{3x})^2} \cdot \frac{1}{2\sqrt{3x}} \cdot 3$$

$$= \frac{3}{2\sqrt{3x}\,(1+3x)}$$

Exercise 1: For each of the following, find the derivative of the given function with respect to the independent variable.

(a) $y = tan^{-1} t^4$

(b) $z = t\ cot^{-1}(1+t^2)$

(c) $x = sin^{-1}\sqrt{1-t^4}$

(d) $s = \frac{t}{\sqrt{1-t^2}} + cos^{-1}t$

(e) $y = sin^{-1}\sqrt{x}$

(f) $z = cot^{-1}(\frac{y}{1-y^2})$

Solutions:

(a) $y = tan^{-1}\ t^4$

$$\frac{dy}{dt} = \frac{d}{dt} tan^{-1}\ (t^4)$$

$$= \frac{1}{1 + (t^4)^2} \cdot \frac{d}{dt}\ (t^4)$$

$$= \frac{4t^3}{1 + t^8}$$

(b) $z = t\ cot^{-1}(1 + t^2)$

$$\frac{dz}{dt} = \frac{d}{dt}\ t\ cot^{-1}(1 + t^2)$$

$$= cot^{-1}(1 + t^2) + t \cdot \frac{-1}{1 + (1 + t^2)^2} \cdot (2t)$$

$$= cot^{-1}(1 + t^2) - \frac{2t^2}{t^4 + 2t^2 + 2}$$

(c) $x = sin^{-1}\sqrt{1 - t^4}$

$$\frac{dx}{dt} = \frac{d}{dt}\ sin^{-1}\sqrt{1 - t^4}$$

$$= \frac{1}{\sqrt{1 - (\sqrt{1 - t^4})^2}} \cdot \frac{d}{dt}(\sqrt{1 - t^4})$$

$$= \frac{1}{\sqrt{1 - (1 - t^4)}} \cdot \frac{1}{2} \cdot (1 - t^4)^{\frac{-1}{2}} \cdot (-4t^3)$$

$$= \frac{1}{\sqrt{1 - 1 + t^4}} \cdot \frac{1}{\sqrt{1 - t^4}} \cdot (-2t^3)$$

$$= \frac{1}{t^2} \cdot \frac{1}{\sqrt{1 - t^4}} \cdot (-2t^3)$$

$$= \frac{-2t}{\sqrt{1 - t^4}}$$

6

(d) $s = \frac{t}{\sqrt{1-t^2}} + cos^{-1}t$

$$\frac{ds}{dt} = \frac{d}{dt}\frac{t}{\sqrt{1-t^2}} + \frac{d}{dt}cos^{-1}t$$

$$= \frac{(\sqrt{1-t^2}) \cdot 1 - t \cdot \frac{1}{2}(1-t^2)^{\frac{-1}{2}} \cdot (-2t)}{(\sqrt{1-t^2})^2} + \frac{-1}{\sqrt{1-t^2}}$$

$$= \frac{\sqrt{1-t^2} + \frac{t^2}{\sqrt{1-t^2}}}{(1-t^2)} - \frac{1}{\sqrt{1-t^2}}$$

$$= \frac{(\sqrt{1-t^2})(\sqrt{1-t^2}) + t^2}{(\sqrt{1-t^2})(1-t^2)} - \frac{(1-t^2)}{(1-t^2)} \cdot \frac{1}{(\sqrt{1-t^2})}$$

$$= \frac{(1-t^2) + t^2 - (1-t^2)}{(\sqrt{1-t^2})(1-t^2)}$$

$$= \frac{t^2}{(1-t^2)^{\frac{3}{2}}}$$

(e) $y = sin^{-1}\sqrt{x}$

$$\frac{dy}{dx} = \frac{d}{dx}sin^{-1}\sqrt{x}$$

$$= \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}\sqrt{x}$$

$$= \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2}x^{\frac{-1}{2}}$$

$$= \frac{1}{2\sqrt{x(1-x)}}$$

(f) $z = cot^{-1}(\frac{y}{1-y^2})$

$$\frac{dz}{dy} = \frac{d}{dy}cot^{-1}(\frac{y}{1-y^2})$$

$$= \frac{-1}{1+(\frac{y}{1-y^2})^2} \cdot \frac{d}{dy}(\frac{y}{1-y^2})$$

$$= \frac{-1}{\frac{(1-y^2)^2+y^2}{(1-y^2)^2}} \cdot \frac{d}{dy}(\frac{y}{1-y^2})$$

$$= \frac{-(1-y^2)^2}{(1-y^2)^2+y^2} \cdot \frac{(1-y^2) \cdot 1 - y \cdot (-2y)}{(1-y^2)^2}$$

$$= \frac{-1}{(1-y^2)^2+y^2} \cdot \frac{(1-y^2) \cdot 1 - y \cdot (-2y)}{1}$$

$$= \frac{-1(1-y^2+2y^2)}{1-2y^2+y^4+y^2}$$

$$= \frac{-(1+y^2)}{1-y^2+y^4}$$

1 Solving Integrals

The formulas listed above for the derivatives lead us to some nice ways to solve some common integrals. The following is a list of useful ones. These formulas hold for any constant $a \neq 0$

$$\int \frac{du}{\sqrt{a^2-u^2}} = sin^{-1}(\frac{u}{a}) + C \qquad \text{for } u^2 < a^2$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}tan^{-1}(\frac{u}{a}) + C \qquad \text{for all } u$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}sec^{-1}|\frac{u}{a}| + C \qquad \text{for } |u| > a > 0$$

Exercise 2: Verify each of the equations above by taking the derivative of the right hand side.

We now want to use these formulas to solve some common integrals.

Example 1: Evaluate the integral $\int \frac{dx}{\sqrt{9-16x^2}}$

Solution: Let $a = 3$ and $u = 4x$. Then $16x^2 = (4x)^2 = u^2$ and $du = 4dx$. We get the following for $16x^2 < 9$:

$$\int \frac{dx}{\sqrt{9-16x^2}} \quad = \frac{1}{4} \int \frac{du}{\sqrt{a^2-u^2}}$$

$$= \frac{1}{4} sin^{-1}\left(\frac{u}{a}\right) + C$$

$$= \frac{1}{4} sin^{-1}\left(\frac{4x}{3}\right) + C$$

$$= \frac{1}{4} sin^{-1}\left(\frac{4}{3}x\right) + C$$

Exercise 3: Evaluate the following integrals.

(a) $\int \frac{dx}{\sqrt{25-4x^2}}$

(b) $\int \frac{dy}{36+4y^2}$

(c) $\int \frac{z\, dz}{5+2z^4}$

(d) $\int \frac{sin\, x\, dx}{\sqrt{10-cos^2 x}}$

(e) $\int \frac{dx}{\sqrt{5+4x-x^2}}$

(f) $\int \frac{7\, dx}{25-12x+4x^2}$

Solutions:

(a) $\int \frac{dx}{\sqrt{25-4x^2}}$. For this problem use the formula $\int \frac{du}{\sqrt{a^2-u^2}} = sin^{-1}\frac{u}{a} + C$ with $a = 5$, $u = 2x$ and $du = 2\, dx$, giving you $\int \frac{dx}{\sqrt{25-4x^2}} = \frac{1}{2} \int \frac{du}{\sqrt{a^2-u^2}} = \frac{1}{2} sin^{-1}\left(\frac{2x}{5}\right) + C$

(b) $\int \frac{dy}{36+4y^2}$. Use the formula $\int \frac{du}{a^2+u^2} = \frac{1}{a} tan^{-1}\left(\frac{u}{a}\right) + C$ with $a = 6$, $u = 2y$ and $du = 2\, dy$.

This gives us $\int \frac{dy}{36+4y^2} = \frac{1}{2} \int \frac{du}{a^2+u^2} = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right) tan^{-1}\left(\frac{2y}{6}\right) + C = \frac{1}{12} tan^{-1}\left(\frac{y}{3}\right) + C$

(c) $\int \frac{z\, dz}{5+2z^4}$. In order to make the calculations a bit simpler, it is useful to multiply the numerator and denominator by 2 in order to get the term $4z^4$ instead of $2z^4$ in the denominator. This gives us $\int \frac{z\, dz}{5+2z^4} = \int \frac{2z\, dz}{10+4z^4}$.
Now let $u = 2z^2$, $du = 4z\, dz$ and $a = \sqrt{10}$ and we have

$$\int \frac{z\, dz}{5+2z^4} = \frac{1}{2} \int \frac{4z\, dz}{10+4z^4} = \frac{1}{2} \int \frac{du}{\left(\sqrt{10}\right)^2+u^2} = \frac{1}{2\sqrt{10}} tan^{-1}\left(\frac{2z^2}{\sqrt{10}}\right) + C$$

(d) $\int \frac{sin\, x\, dx}{\sqrt{10-cos^2 x}}$. Let $u = cos\, x$, $du = -sin\, x\, dx$ and $a = \sqrt{10}$. Then

$$\int \frac{sin\, x\, dx}{\sqrt{10-cos^2 x}} = (-1) \int \frac{-sin\, x\, dx}{\sqrt{10-cos^2 x}} = -sin^{-1}\left(\frac{cos\, x}{\sqrt{10}}\right) + C$$

9

(e) $\int \frac{dx}{\sqrt{5+4x-x^2}}$. We want to transform this expression into something with the

form $\int \frac{du}{\sqrt{a^2-u^2}}$. To do this we need to complete the square of the expression in the

denominator as follows:

$$
\begin{aligned}
5 + 4x - x^2 \quad &= \quad 5 + 4 - 4 + 4x - x^2 \\
&= \quad 9 - 4 + 4x - x^2 \\
&= \quad 9 - (x^2 - 4x + 4) \\
&= \quad (3)^2 - (x-2)^2
\end{aligned}
$$

This gives us

$$
\int \frac{dx}{\sqrt{5+4x-x^2}} \quad = \quad \int \frac{dx}{\sqrt{(3)^2-(x-2)^2}} \quad = \quad sin^{-1}(\tfrac{x-2}{3}) \; + \; C
$$

(f) $\int \frac{7\,dx}{25-12x+4x^2}$. Again we need to complete the square. This time we want to transform
the expression into something with the form $\int \frac{du}{a^2+u^2}$. We rewrite the denominator as
follows:

$$
\begin{aligned}
25 - 12x + 4x^2 \quad &= \quad 16 + 9 - 12x + 4x^2 \\
&= \quad (4)^2 + (2x-3)^2
\end{aligned}
$$

Now, letting $u \; = \; 2x - 3$ and $du \; = \; 2\,dx$ we get

$$
\int \frac{7\,dx}{25-12x+4x^2} \quad = \quad \tfrac{7}{2} \int \frac{2\,dx}{(4)^2+(2x-3)^2} = \tfrac{7}{8}\, tan^{-1}(\tfrac{2x-3}{4}) \; + \; C
$$