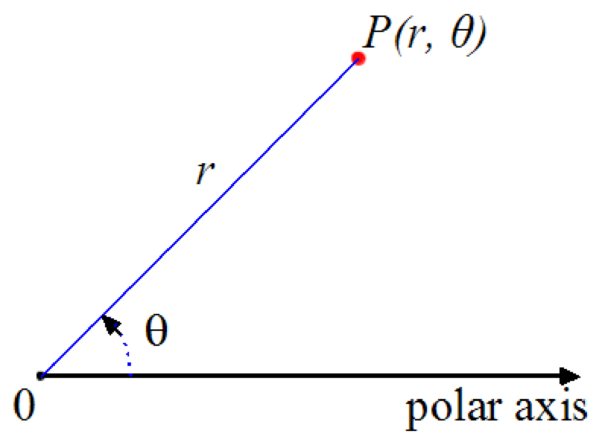
Polar Coordinates

Polar Coordinates

The **Polar Coordinate System** consists of a Pole (or origin), denoted by 0 and a ray starting at 0 called the polar axis.

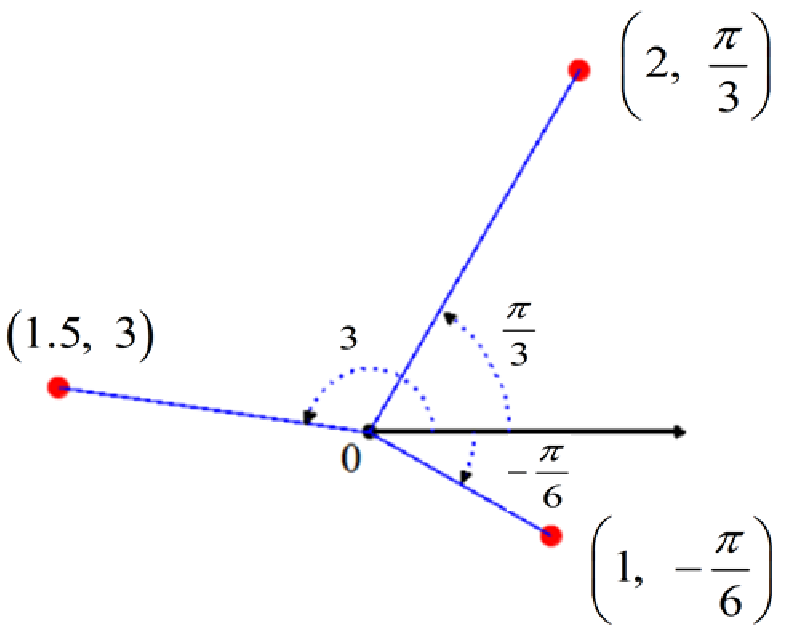


r = distance from O to P

θ = angle between polar axis and the line segment OP

(r, θ) are the **polar coordinates** of the point P

The Pole has coordinates $(0, \theta)$ with θ any angle.



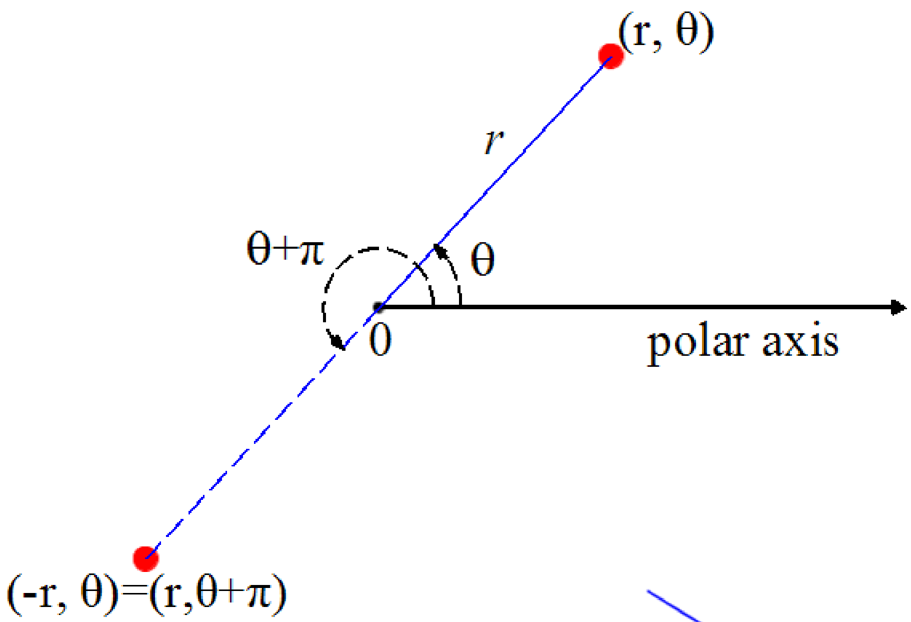
A point P has infinitely many polar coordinates representations:

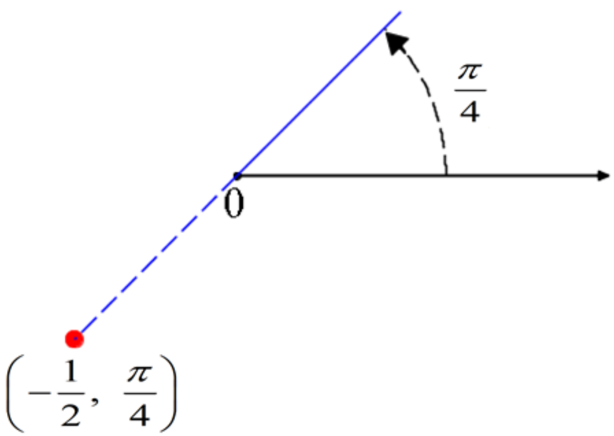
$$(r, \theta) = (r, \theta + 2n\pi)$$

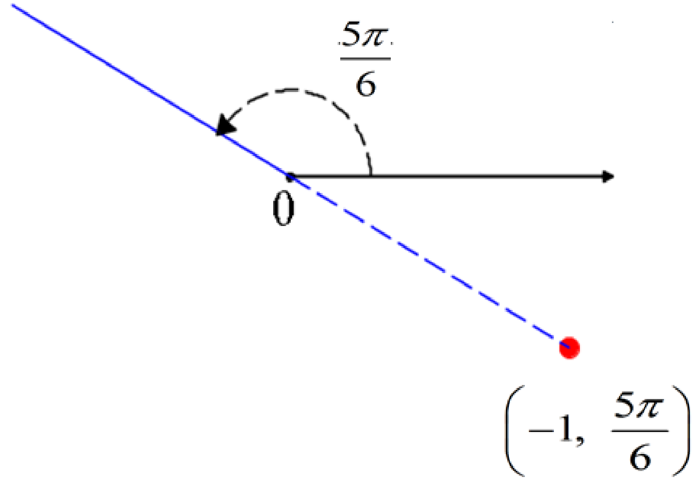
with n any integer

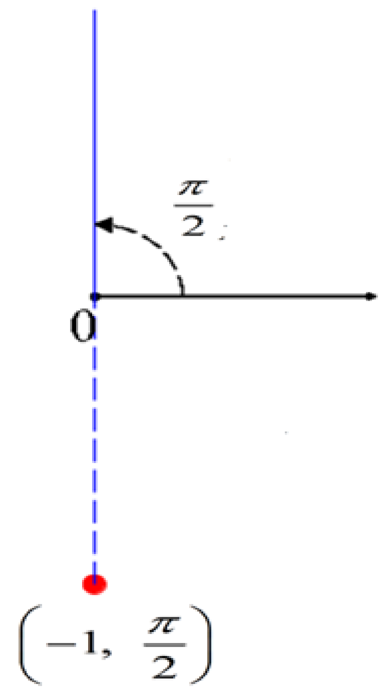
Polar Coordinates

r can be negative







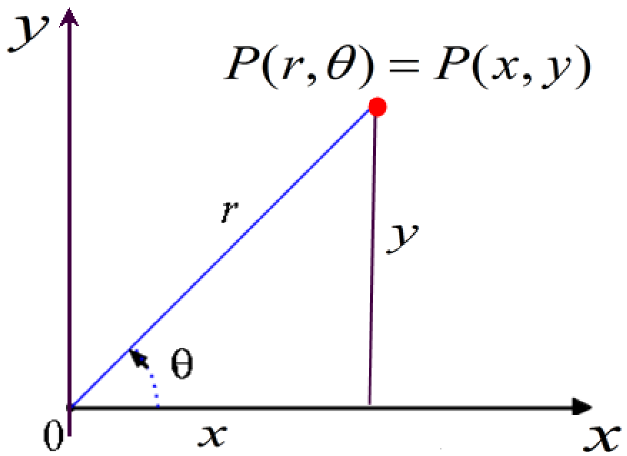


Polar Coordinates

Connection between polar and Cartesian coordinates:

$$x = r\cos\theta \qquad x = r\sin\theta$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

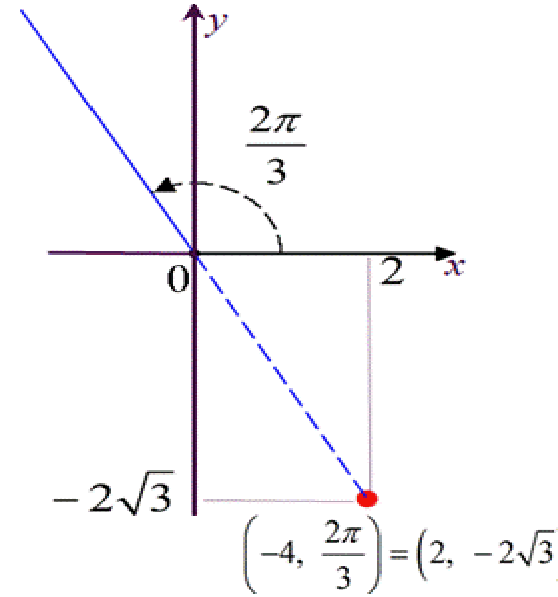


$\tan\theta = \dfrac{y}{x}$ DOES NOT necessarily imply $\theta = \arctan\left(\dfrac{y}{x}\right)$.

Always choose θ so that the point (r, θ) lies in the correct quadrant.

Polar coordinates Example 1

i. Convert $\left(-4,\ \dfrac{2\pi}{3}\right)$ into Cartesian coordinates.

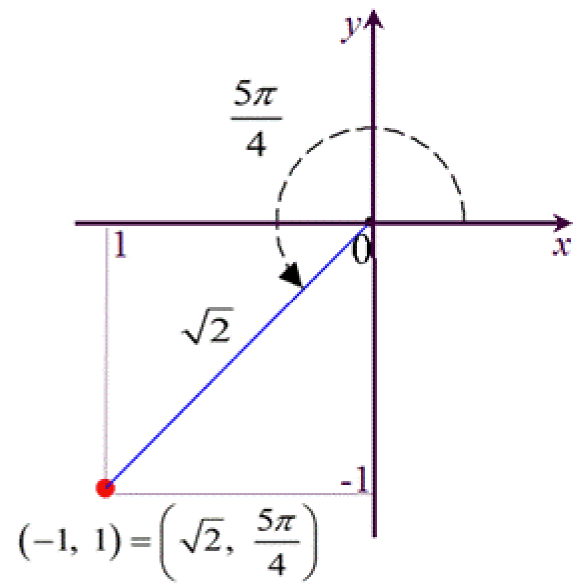


$$x = r\cos\theta = -4\cos\left(\frac{2\pi}{3}\right) = -4\left(-\frac{1}{2}\right) = 2$$

$$y = r\sin\theta = -4\sin\left(\frac{2\pi}{3}\right) = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

In Cartesian coordinates this point is $\left(2,\ -2\sqrt{3}\right)$

ii. Convert $(-1,\ -1)$ into polar coordinates.



If we choose a positive *r* :

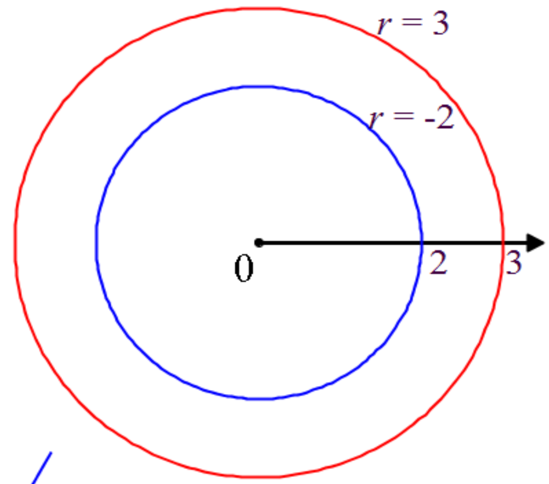
$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{-1} = 1$$

The point is in Quadrant III $\Rightarrow$ $\theta = \tan^{-1}(1) + \pi = \dfrac{\pi}{4} + \pi = \dfrac{5\pi}{4}$

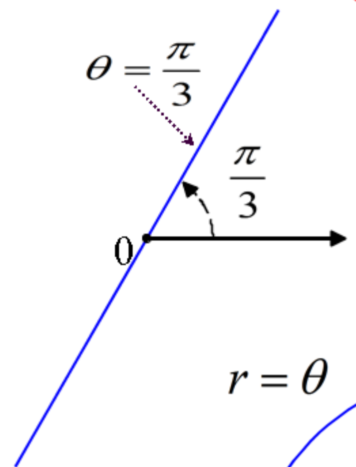
Possible polar representation of the point: $\left(\sqrt{2},\ \dfrac{5\pi}{4}\right)$

Polar coordinates Basic Polar Graphs

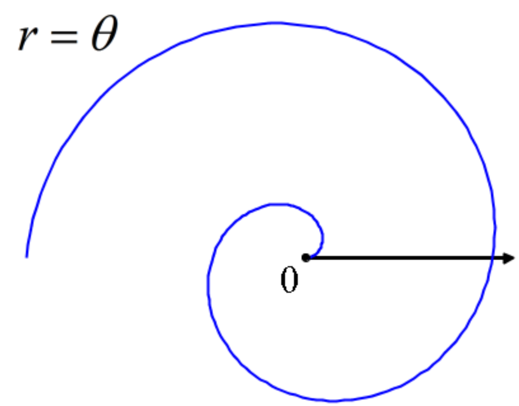


- The polar equation $r = 3$ represents a circle centered at the origin with radius 3.

  In general, $r = a$, represents a circle centered at the origin with radius $|a|$.



- The polar equation $\theta = \frac{\pi}{3}$ represents a radial line through the origin that makes an angle of $\frac{\pi}{3}$ with the polar axis.



- The polar equation $r = \theta$ is a spiral.

Polar Coordinates Example 2

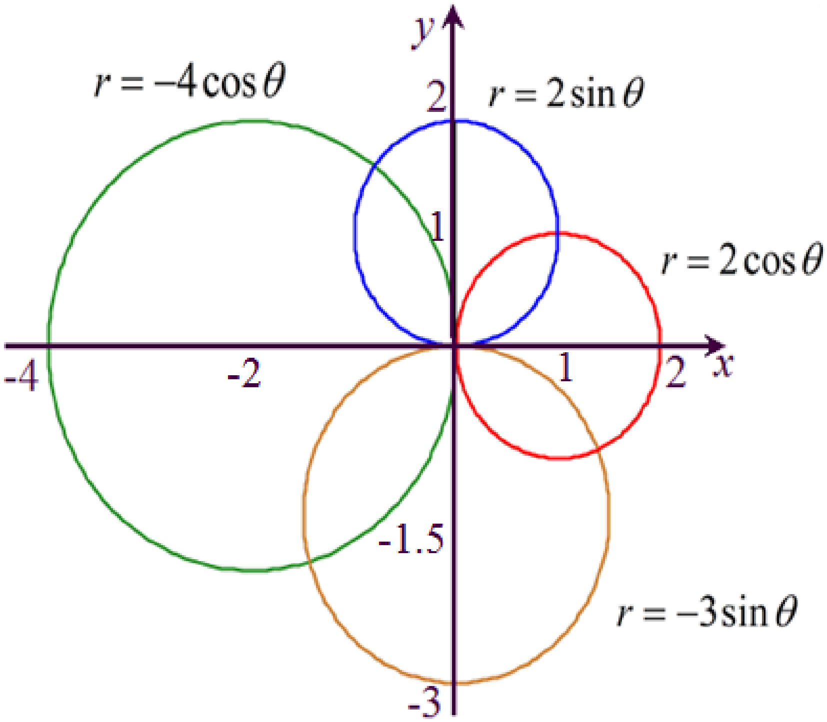
Convert $r = -4\cos\theta$ into Cartesian coordinates.

Multiply both sides of the equation by r:

$$x = r\cos\theta$$

$$r^2 = -4r\cos\theta \quad \Rightarrow \quad x^2 + y^2 = -4x$$

$$r^2 = x^2 + y^2$$



Complete the squares: $x^2 + 4x + 4 + y^2 = 4$

$$(x+2)^2 + y^2 = 4$$

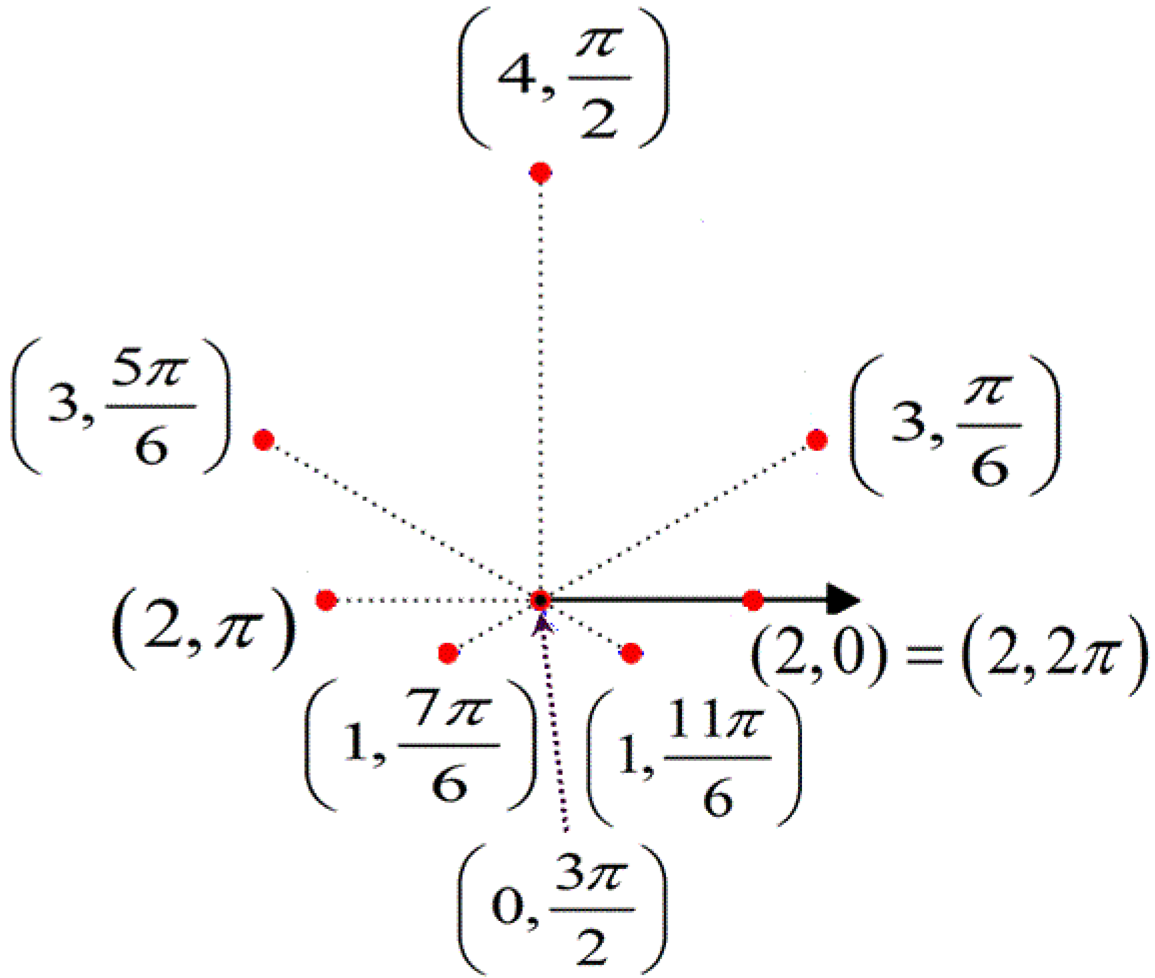
Circle of radius 2 centered at (-2,0)

- In general: $r = a\cos\theta$ is a circle of radius $\frac{|a|}{2}$ centered at $\left(\frac{a}{2}, 0\right)$
- Similarly: $r = a\sin\theta$ is a circle of radius $\frac{|a|}{2}$ centered at $\left(0, \frac{a}{2}\right)$

Polar Coordinates Example 3

Graph the CARDIOID $r = 2 + 2\sin\theta$ by plotting points.

θ	$r = 2 + 2\sin\theta$
0	2
$\frac{\pi}{6}$	3
$\frac{\pi}{2}$	4
$\frac{5\pi}{6}$	3
π	2
$\frac{7\pi}{6}$	1
$\frac{3\pi}{2}$	0
$\frac{11\pi}{6}$	1
2π	2

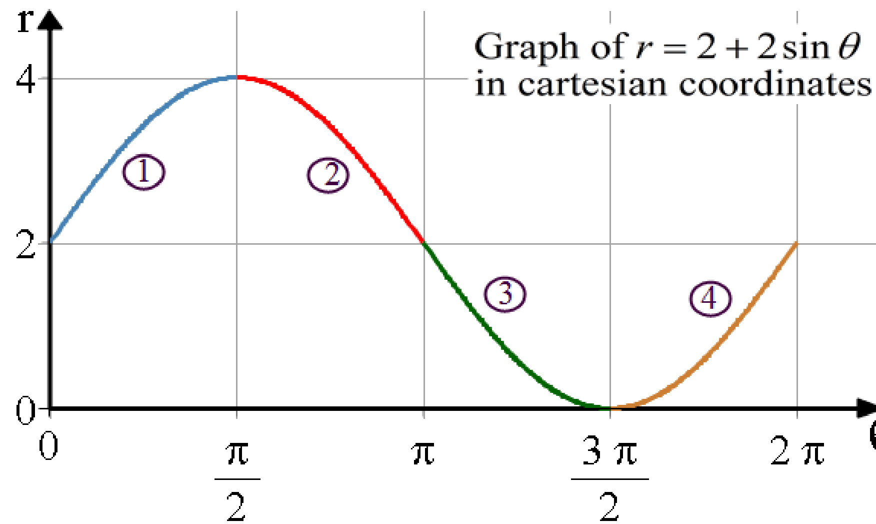


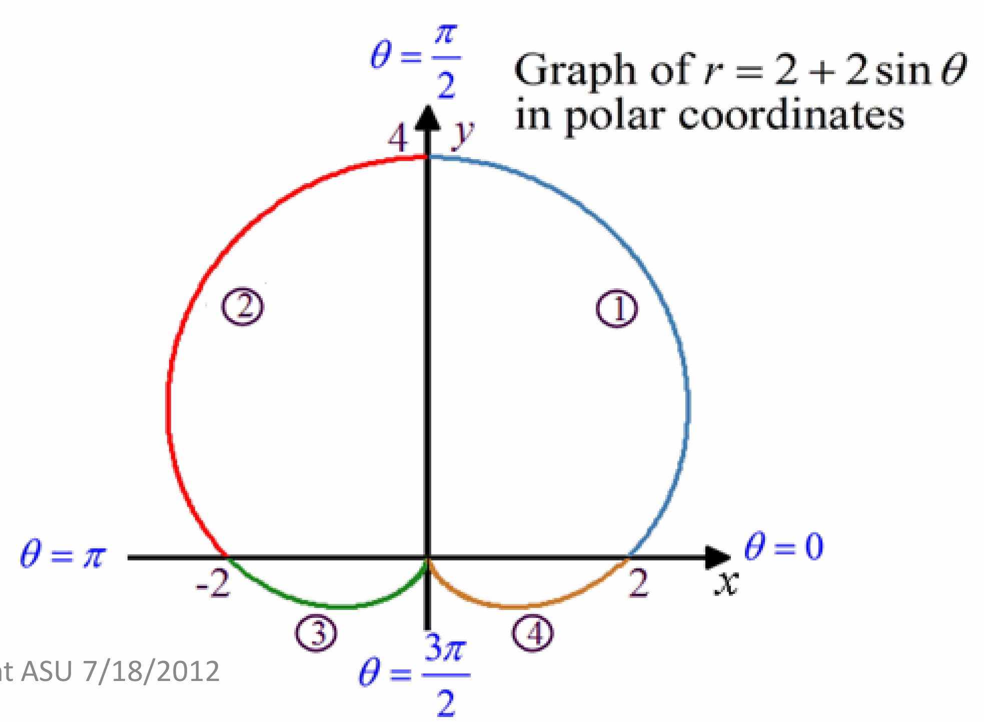
Polar Coordinates Example 3 continued

Faster way to sketch the graph:

First graph $r = 2 + 2\sin\theta$ as if *r* and θ were Cartesian coordinates with θ on the horizontal axis and *r* on the vertical axis.

Then use the Cartesian graph as a guide to sketch the $(r,\ \theta)$ points on the polar curve.





1 $0 \le \theta \le \frac{\pi}{2}$: r increases from 2 to 4

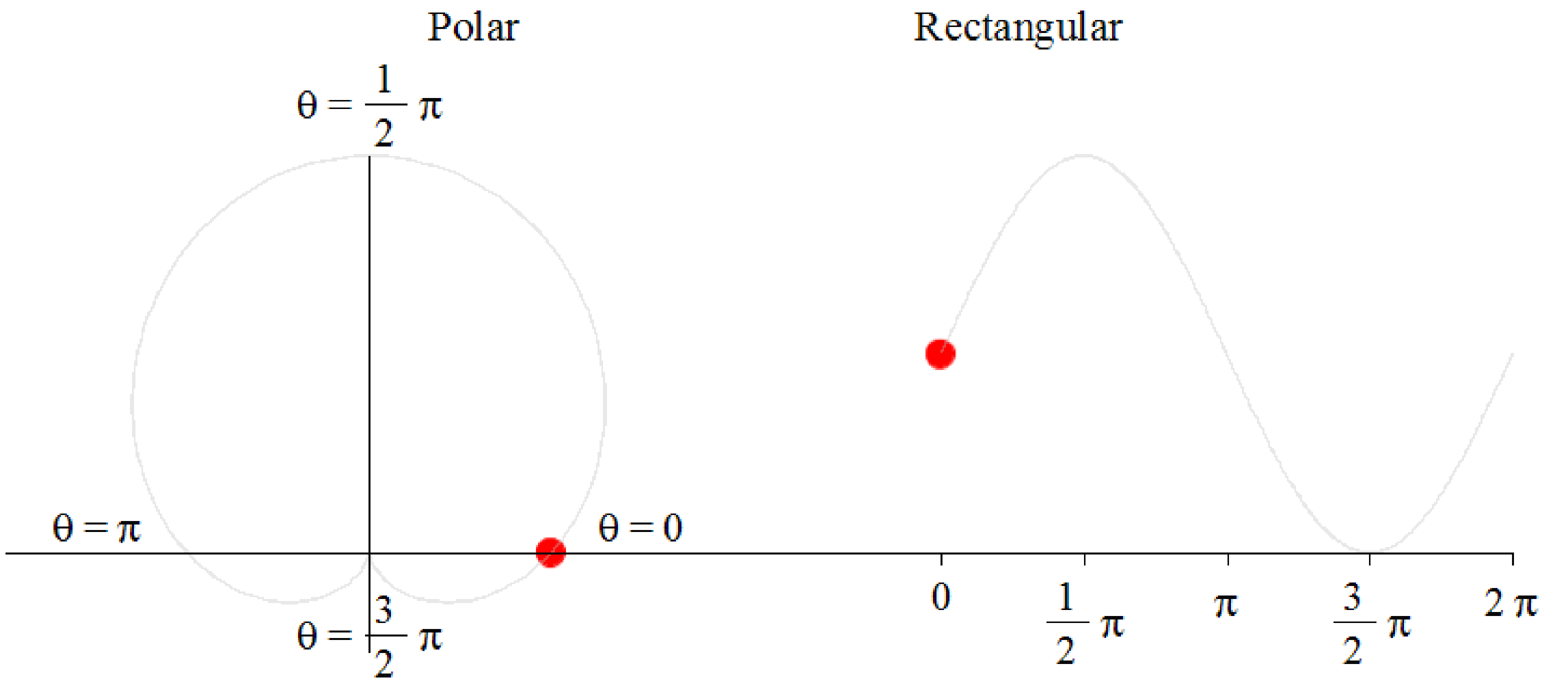
2 $\frac{\pi}{2} \le \theta \le \pi$: r decreases from 4 to 2

3 $\pi \le \theta \le \frac{3\pi}{2}$: r decreases from 2 to 0

4 $\frac{3\pi}{2} \le \theta \le 2\pi$: r increases from 0 to 2

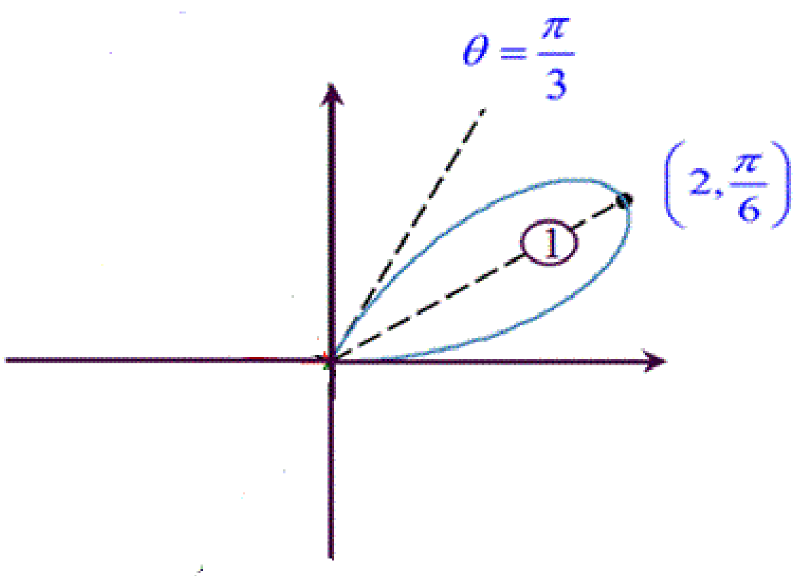
Polar Coordinates Example 3 continued

Animated graph of $r = 2 + 2\sin(\theta)$ in Cartesian and polar coordinates.

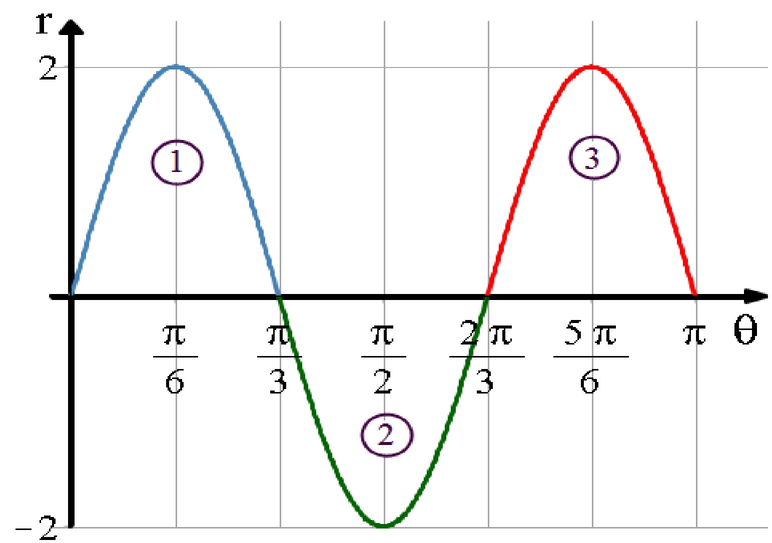


Polar Coordinates Example 4

Sketch the graph of the 3-petal rose: $r = 2\sin(3\theta)$



Graph in Cartesian Coordinates



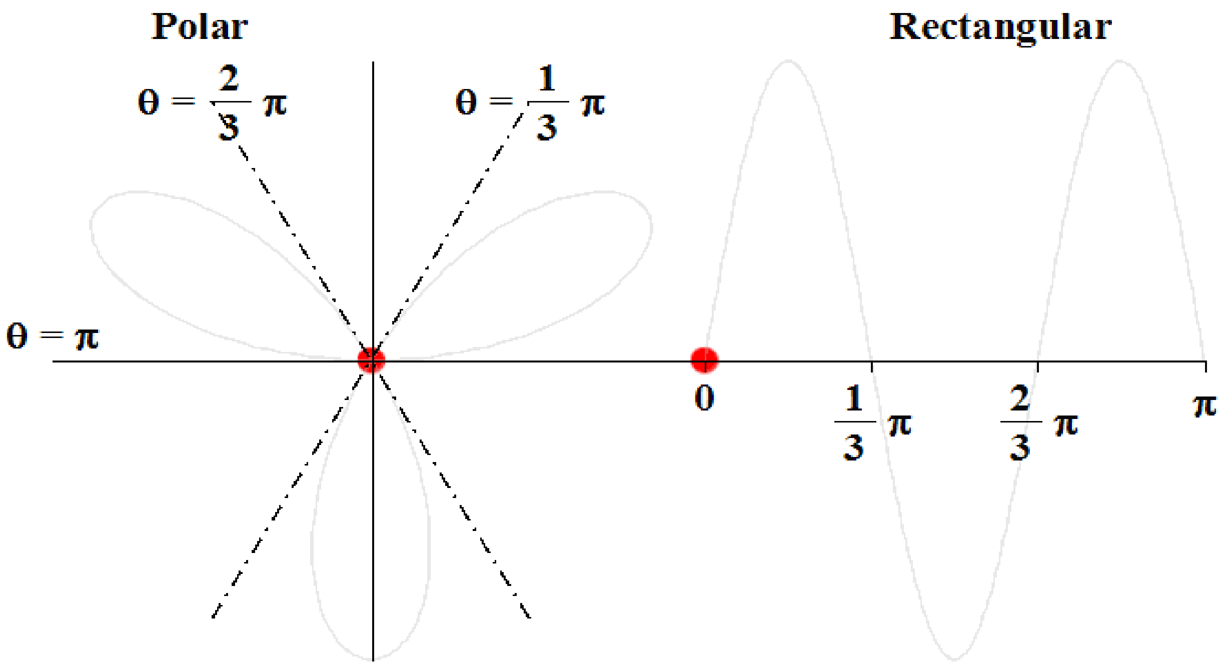
1. $0 \le \theta \le \dfrac{\pi}{3}$: r goes from 0 back to 0
2. $\dfrac{\pi}{3} \le \theta \le \dfrac{2\pi}{3}$: r goes from 0 back to 0. Note that r is negative
3. $\dfrac{2\pi}{3} \le \theta \le \pi$: r goes from 0 back to 0

The polar curve is traced for $0 \le \theta \le \pi$

Each Cartesian arc corresponds to a polar loop

Polar Coordinates Example 4 continued

Animated graph of $r = 2\sin(3\theta)$ in Cartesian and polar coordinates.



Polar Coordinates Tangents to Polar Curves

Consider the curve with polar equation $r = f(\theta)$

Write the curve in parametric form (θ is the parameter)

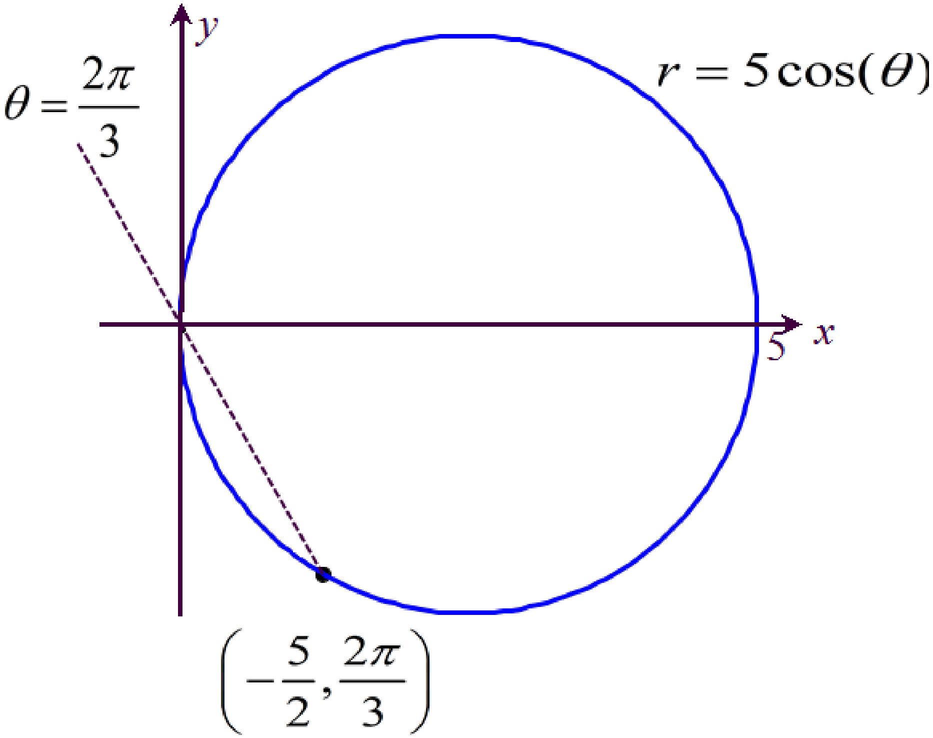
$$x = r\cos(\theta) = f(\theta)\cos(\theta), \quad y = r\sin(\theta) = f(\theta)\sin(\theta)$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos(\theta) - r\sin(\theta), \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)$$

$$\boxed{\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

Polar Coordinates Example 5

Find the slope of the tangent line to the polar curve $r = 5\cos(\theta)$ at $\theta = \dfrac{2\pi}{3}$



$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$= \frac{(-5\sin\theta)\sin\theta + (5\cos\theta)\cos\theta}{(-5\sin\theta)\cos\theta - (5\cos\theta)\sin\theta}$$

$$= \frac{-\sin^2\theta + \cos^2\theta}{-2\sin\theta\cos\theta}$$

$$= \frac{\cos(2\theta)}{-\sin(2\theta)}$$

$$= -\cot(2\theta)$$

Slope: $$\left.\frac{dy}{dx}\right|_{\theta = \frac{2\pi}{3}} = -\cot\left(\frac{4\pi}{3}\right) = -\frac{1}{\sqrt{3}}$$