PROBLEMS - SET 4

Problem 1. Let $X \sim Bin(n,p)$. Compute

$$\mathrm{P}[X \text{ is odd}]$$

Solution 1. X can be written as the sum of n independent $Bin(1,p)$ r.v.'s. Then $Z_i = 2X_i - 1$ for $i = 1,\dots,n$ form a sequence of i.i.d. r.v.'s with $\mathrm{P}[Z_1 = 1] = p = 1 - \mathrm{P}[Z_1 = -1]$. In the previous SET 3 we have proved that if $Z := Z_1 \cdot Z_2 \cdots Z_n$, we have

$$\mathrm{P}(Z = \pm 1) = \frac{1 \pm (2p-1)^n}{2}.$$

Now, it is easy to see that if n is odd, than $[X \text{ is odd}] = [Z = 1]$, while if n is even, than $[X \text{ is odd}] = [Z = 1]$. Then if n is odd

$$\mathrm{P}[X \text{ is odd}] = \frac{1 + (2p-1)^n}{2}$$

while if n is even

$$\mathrm{P}[X \text{ is odd}] = \frac{1 - (2p-1)^n}{2}.$$

This is equal in both the cases to

$$\mathrm{P}[X \text{ is odd}] = \frac{1 - (1-2p)^n}{2}$$

Problem 2. A box contains k balls numbered from 1 to k. We extract n balls from the box and let X denote the maximum number that we obtain. In both the cases with or without replacement, compute the distribution F_X.

Solution 2. Let us start by the case with replacement. In this case $X \in \{1,2,\dots,k\}$ and we get

$$\mathrm{P}[X \le x] = \begin{cases} 0 & x < 1 \\ \mathrm{P}[\{(1,1,\dots,1)\}] = \left(\frac{1}{k}\right)^n & 1 \le x < 2 \\ \mathrm{P}[\{(\omega_1,\omega_2,\dots,\omega_n) : \omega_i \in \{1,2\}\}] = \left(\frac{2}{k}\right)^n & 2 \le x < 3 \\ \dots & \\ \mathrm{P}[\{(\omega_1,\omega_2,\dots,\omega_n) : \omega_i \in \{1,2,\dots,l\}\}] = \left(\frac{l}{k}\right)^n & l \le x < l+1 \\ \dots & \\ 1 & x \ge k \end{cases}$$

In the case without replacement, where $n \le k$, $X \in \{n, n+1, \dots, k\}$ and

$$
\mathrm{P}[X \le x] = \begin{cases} 0 & x < n \\ \frac{\binom{n}{n}}{\binom{k}{n}} = \frac{1}{\binom{k}{n}} & n \le x < n+1 \\ \frac{\binom{n+1}{n}}{\binom{k}{n}} & n+1 \le x < n+2 \\ \dots & \\ \frac{\binom{n+l}{n}}{\binom{k}{n}} & n+l \le x < n+l+1 \\ \dots & \\ \frac{\binom{k}{n}}{\binom{k}{n}} = 1 & x \ge k \end{cases}
$$

Problem 3. Compute the characteristic function and the moment generating function of the absolutely continuous random variables with densities

i.
$$
f(x) = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{2}|x|\right) & \text{if } |x| \le 2 \\ 0 & \text{otherwise} \end{cases}
$$

ii.
$$
f(x) = \frac{1}{b-a} 1_{[a,b]}(x)
$$

iii.
$$
f(x) = \frac{1}{2} \exp[-|x-a|].
$$

Solution 3. Only the characteristic function is computed, the moment generating function is obtained replacing u by $-iu$ (that is $m_X(t) = \varphi(-it)$).

i.
$$
\begin{aligned}
\varphi(u) = \frac{1}{2}\int_{-2}^{2} e^{iux}\left(1 - \frac{1}{2}|x|\right)dx = \int_0^2 \cos(ux)\left(1 - \frac{1}{2}x\right)dx \\
= \frac{1}{u}\sin(ux)\Big|_0^2 - \frac{1}{2}\int_0^2 x\cos(ux)dx \\
= \frac{\sin(2u)}{u} - \frac{1}{2u}x\sin(ux)\Big|_0^2 + \frac{1}{2u}\int_0^2 \sin(ux)dx = \frac{\cos(1-2u)}{2u^2}.
\end{aligned}
$$

ii.
$$
\varphi(u) = \frac{1}{b-a}\int_a^b e^{iux}dx = \frac{1}{iu}\frac{e^{iub} - e^{iua}}{b-a}.
$$

iii.
$$
\begin{aligned}
\varphi(u) = \frac{1}{2}\int e^{iux}e^{-|x-a|}dx = e^{iua}\frac{1}{2}\int e^{iuy}e^{-|y|}dy \\
= \frac{1}{2}e^{iua}\left[\int_{-\infty}^{0} e^{iuy}e^{y}dy + \int_0^{+\infty} e^{iuy}e^{-y}dy\right] = \frac{e^{iua}}{1+u^2}.
\end{aligned}
$$

Problem 4. We know that for a real, discrete random X variable with density p, the characteristic function is given by

$$\varphi_X(u) = \sum_{n \in \mathbb{Z}} e^{iun} p(n).$$

Determine the densities of the discrete random variables having the following characteristic functions:

i.
$$\varphi(u) = \frac{1}{4}\left(1 + e^{iu}\right)^2$$

ii.
$$\varphi(u) = \frac{1}{2 - e^{iu}}$$

iii.
$$\varphi(u) = cos(u)$$

iv.
$$\varphi(u) = cos^2(u)$$

v.
$$\varphi(u) = \sum_{k=0}^{+\infty} a_k cos(kt)$$

where $a_k > 0$ and $\sum_k a_k = 1$.

Solution 4. i.

$$\varphi(u) = \frac{1}{4} + \frac{1}{2}e^{iu} + \frac{1}{4}e^{2iu} \;\Rightarrow\; p(0) = \frac{1}{4} = p(2),\; p(1) = \frac{1}{2}.$$

ii.
$$\varphi(u) = \frac{1}{2}\frac{1}{1 - \frac{e^{iu}}{2}} = \frac{1}{2}\sum_{n=0}^{+\infty} \frac{e^{inu}}{2^n} \;\Rightarrow\; p(n) = \frac{1}{2^{n+1}}.$$

iii.
$$\varphi(u) = \frac{e^{iu} + e^{-iu}}{2} \;\Rightarrow\; p(1) = p(-1) = \frac{1}{2}.$$

iv.
$$\varphi(u) = \left(\frac{e^{iu} + e^{-iu}}{2}\right)^2 = \frac{1}{4}e^{2iu} + \frac{1}{4}e^{-2iu} + \frac{1}{2} \Rightarrow p(0) = \frac{1}{2}, p(2) = p(-2) = \frac{1}{4}.$$

v.
$$\varphi(u) = \sum_k a_k \left(\frac{e^{iuk} + e^{-iuk}}{2}\right) = \sum_{k=0}^{\infty} \frac{a_k}{2} e^{iuk} + \sum_{k=0}^{+\infty} \frac{a_k}{2} e^{-iuk} \Rightarrow p(0) = a_0, p(k) = p(-k) = \frac{a_k}{2}.$$

Problem 5. Let $X \sim \mathrm{N}(1,4)$ and $Y \sim \mathrm{N}(2,1)$ be independent random variables.

(a) Set $U := X+Y$ and $V := X-2Y$. Find μ and Σ such that $(U,V) \sim \mathrm{N}(\mu,\Sigma)$.

(b) Find a 2×2 matrix A such that defining

$$\begin{pmatrix} Z \\ W \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix},$$

Z and W are independent and have both variance 1.

Solution 5.
(a) Note that $(X,Y) \sim \mathrm{N}\left((1,2), \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\right)$ and $\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$. Let us denote $B = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ Therefore $(U,V) \sim \mathrm{N}\left(B\begin{pmatrix} 1 \\ 2 \end{pmatrix}, B\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}B^t\right)$. So, $(U,V) \sim \mathrm{N}\left(\begin{pmatrix} 3 \\ -3 \end{pmatrix}, \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}\right)$.
(b) We have that $\begin{pmatrix} Z \\ W \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathrm{N}\left(A\begin{pmatrix} 1 \\ 2 \end{pmatrix}, A\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}A^t\right)$. So we need to find a matrix A such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}A^t = \begin{pmatrix} 4a_{11}^2 + a_{12}^2 & 4a_{11}a_{21} + a_{22}a_{12} \\ 4a_{11}a_{21} + a_{22}a_{12} & 4a_{21}^2 + a_{22}^2 \end{pmatrix}.$$

A solution is given by $a_{11} = \frac{1}{2}, a_{12} = a_{21} = 0, a_{22} = 1$, and so $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$.

Problem 6. Set $V := \{1,2,\dots,N\}$. To each *unordered* pair $\{i,j\}$ with $i,j \in V$ and $i \neq j$ we assign a random variable $X_{ij} \sim N(0,1)$ and we assume that they are all independent. You can interpret it as follows: if the elements of V are geographical locations, $e^{-X_{ij}}$ is the time needed to travel from i to j of viceversa. We say that i and j are *far* if $X_{ij} < 0$. For $i \in V$, set

$$X_i := \sum_{j:j\neq i} X_{ij};$$

$$N_i := \text{ number of locations far from } i \ = |\{j : X_{ij} < 0\}|.$$

(a) Find the distribution of N_i.

(b) What is the joint distribution of X_1 and X_2? Are they independent?

(c) We say that i is *isolated* if it is far from all other elements of V. Let p_N be the probability that there exists at least one isolated point. Show that

$$\lim_{N\to+\infty} p_N = 0.$$

Solution 6. (a) Note that $P(X_{ij} < 0) = \pi^{-1/2}\int_{-\infty}^{0} e^{-x^2/2}dx = \frac{1}{2}$.
So $N_i \sim \mathrm{Bin}(N-1, 1/2)$.

(b) Note that the vector (X_1, X_2) is a linear transform of $(X_{ij} : \{i, j\}$ unordered pair), so there exists a matrix $A \in M\left(2 \times \binom{N}{2}\right)$ such that $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = A \begin{pmatrix} X_{11} \\ X_{12} \\ \dots \\ X_{N-1,N} \end{pmatrix}$.

This implies that $(X_1, X_2) \sim \mathrm{N}(0, AA^t)$.

To obtain its joint distribution it is therefore enough to compute the covariance matrix:

$$Var(X_1) = \sum_{k:k\neq 1} Var(X_{1k}) = N - 1 = Var(X_2).$$

$$Cov(X_1, X_2) = \sum_{k:k\neq 1} \sum_{h:h\neq 2} Cov(X_{1k}, X_{2h}) = Var(X_{12}) = 1.$$

In particular, X_1 and X_2 are not independent.

(c) For a given i

$$P(i \text{ is isolated}) = P(N_i = N - 1) = \frac{1}{2^{N-1}}.$$

So

$$p_N = P(\cup_i \{N_i = N - 1\}) \leq \sum_i P(N_i = N - 1) = \frac{N}{2^{N-1}} \to 0$$

as $N \to +\infty$.