PROBLEMS - SET 8

Problem 1. Define a time reversible Markov Chain on the positive integers $\{1,2,\dots\}$ with stationary distribution $\pi(i) \propto i^{-5/4}$ for $i \in \{1,2,\dots\}$.

Solution 1. Let $P_{i,j}$ denote the transition probabilities of a random walk on the positive integers. We assume that

$$\begin{array}{l} P_{i,i+1} = p\ ,\ P_{i+1,i} = 1-p \quad \text{for any } i \geq 1 \\ P_{1,1} = 1-p \end{array}$$

with $p \in (0,1)$.

For $i \neq j$, define $\alpha(i,j) = 0$ if $j \neq i+1, i-1$,

$$\alpha(i,i+1) = \min\left\{\frac{\pi_{i+1}P_{i+1,i}}{\pi_i P_{i,i+1}}, 1\right\} = \min\left\{\frac{1-p}{p}\frac{\pi_{i+1}}{\pi_i}, 1\right\} = \min\left\{\frac{(i+1)^{-5/4}(1-p)}{i^{-5/4}p}, 1\right\}$$

$$\alpha(i,i-1) = \min\left\{\frac{\pi_{i-1}P_{i+1,i}}{\pi_i P_{i,i-1}}, 1\right\} = \min\left\{\frac{p}{1-p}\frac{\pi_{i-1}}{\pi_i}, 1\right\} = \min\left\{\frac{(i-1)^{-5/4}p}{i^{-5/4}(1-p)}, 1\right\}$$

Note that

$$\pi_i P_{i,i+1}\alpha(i,i+1) = \pi_{i+1}P_{i+1,i}\alpha(i+1,i).$$

The transition probabilities for the desired Markov Chain are $Q_{i,j} = P_{i,j}\alpha(i,j)$ for $j \neq i$, by construction indeed $\pi_i P_{i,j}\alpha(i,j) = \pi_j P_{j,i}\alpha(j,i)$. When $i = 1$

$$Q_{1,j} = \begin{cases} p\alpha(1,2) = p\ \min\{1, 2^{-5/4}\frac{1-p}{p}\} & \text{if } j = 2 \\ 1 - Q_{1,2} = 1 - p\ \min\{1, 2^{-5/4}\frac{1-p}{p}\} & \text{if } j = 1 \\ 0 & \text{if } j > 2 \end{cases}$$

When $i > 1$

$$Q_{i,j} = \begin{cases} (1-p)\alpha(i,i-1) = (1-p)\min\{1, \left(\frac{i-1}{i}\right)^{-5/4}\frac{p}{1-p}\} = \min\{1-p, p\left(\frac{i}{i-1}\right)^{5/4}\} & \text{if } j = i-1 \\ p\alpha(i,i+1) = p\min\{1, \left(\frac{i+1}{i}\right)^{-5/4}\frac{1-p}{p}\} & \text{if } j = i+1 \\ 1 - Q_{i,i-1} - Q_{i,i+1} & \text{if } j = i \\ 0 & \text{if } |i-j| > 1 \end{cases}$$

Note that by choosing $p = 1/2$, all the previous expressions can be simplified.

Problem 2. Let (V,E) be a graph. Define a reversible Markov Chain X_n with state space $S = \{X, X \subset V\} = 2^V$ the set of subsets of V, such that its invariant measure is the measure in which the measure of a subset $X \subset V$ is proportional to $2^{E|_X}$, where $E|_X$ is the number of edges in E that have both endpoints in X.

Solution 2. First of all we have to fix a reference transition matrix. To do this, we define a rule to pass from one state X to another state X' of the chain. Note that the states of the chain are subsets of V.

We define the following rule to update a set $X_n \subset V$.
Metropolis Hastings:

(i) Randomly select a vertex $v \in V$.
(ii) If $v \notin X_n$, set $X_{n+1} = X_n \cup \{v\}$.
(iii) If $v \in X_n$ set $X_{n+1} = X_n \setminus \{v\}$ with probability p and $X_{n+1} = X_n$ with probability $1-p$.

So, we get that the transition matrix is given by

$$P_{X,X'} = \begin{cases} 0 & \text{if } X, X' \text{ differ on more than 1 vertex} \\ \frac{1}{|V|} & \text{if } X' \setminus X = \{v\} \text{ for exactly one } v \\ \frac{1}{|V|} p & \text{if } X \setminus X' = \{v\} \text{ for exactly one } v \\ \text{generally incomputable} & \text{if } X = X'. \end{cases}$$

As usual, the diagonal elements of the transition matrix are not known, but the algorithm can be simulated without this knowledge. Note that the invariant distribution is given by $\pi = (\pi_X)_{X \in 2^V}$, defined as

$$\pi_X = c 2^{E|_X}$$

for some normalization constant $c > 0$. So, in order for the chain to be reversible, we need that

$$\pi_X P_{X,X'} = \pi_{X'} P_{X',X}.$$

Obviously these equations are trivial if X, X' differ for more than one vertex. Assume now that $X' \setminus X = \{v\}$, then the previous equation reads

$$c 2^{E|_X} \frac{1}{|V|} = c 2^{E|_{X'}} \frac{p}{|V|}.$$

Note that

$$2^{E|_{X'}} = 2^{E|_{X \cup v}} = 2^{E|_X} \cdot 2^{|\text{neighbors of } v \text{ in } X|}$$

because $E|_{X'} = E|_X + |\text{neighbors of } v \text{ in } X|$. Therefore, substituting in the balance equation we get

$$p = 2^{-|\text{neighbors of } v \text{ in } X|}.$$

With this choice of p, we get that $P_{X,X'}$ is the transition matrix of the required MC. This is just the Metropolis-Hastings algorithm.

Problem 3. Let $X_1, X_2, \ldots$ be a sequence of independent exponential random variables with parameter λ and N be a Geometric random variable of parameter α. Assume that $X_1, X_2, \ldots$ and N are also independent. Prove that the random variable

$$T = \sum_{i=1}^{N} X_i$$

is an exponential distribution of parameter $\alpha\lambda$.

Solution 3. It is sufficient to show that the moment generating function of T coincides with the moment generating function of an exponential random variable of parameter $\alpha\lambda$.

We compute the moment generation function of T. We fix $t \in \mathbb{R}$ and we define the function $f : \mathbb{N} \to \mathbb{R}$ as $f(n) = \mathbb{E}[e^{t(X_1+\ldots+X_n)}]$. Observe that $\varphi_T(t) = \mathbb{E}[e^{t(X_1+\ldots+X_N)}] = \mathbb{E}(f(N))$. Therefore

$$\varphi_T(t) = \sum_{n=1}^{\infty} \mathbb{E}[e^{t(X_1+\ldots+X_n)}]\, \mathbb{P}[N = n].$$

Since the random variables X_i are iid, and the moment generation function of X_1 is defined for $t < \lambda$ as

$$\varphi_{X_1}(t) = \mathbb{E}[e^{tX_1}] = \frac{\lambda}{\lambda - t}$$

we get

$$\mathbb{E}[e^{t(X_1+\ldots+X_n)}] = \prod_{i=1}^{n} \varphi_{X_1}(t) = \left(\frac{\lambda}{\lambda - t}\right)^n.$$

Then

$$\varphi_T(t) = \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda - t}\right)^n \mathbb{P}[N = n] = \frac{\lambda\alpha}{\lambda - t} \sum_{n=1}^{\infty} \left(\frac{\lambda(1-\alpha)}{\lambda - t}\right)^{n-1}$$
$$= \frac{\lambda\alpha}{\lambda - t} \frac{1}{1 - \frac{\lambda(1-\alpha)}{\lambda - t}} = \frac{\lambda\alpha}{\lambda\alpha - t}$$

which is the moment generating function of an exponential random variables of parameter $\alpha\lambda$.