PROBLEM - SET 0

Problem 1. Consider the random experiment of rolling two balanced dice with six faces and sum the two numbers that appears.

(a) Describe the probability space for this random experiment.

(b) Compute the probability to obtain an even number.

Solution 1.

(a) Let us start by defining the probability space of all the possible outcomes rolling two dice, i.e. $\Omega = \{(i,j) : i,j \in \{1,2,3,4,5,6\}\}$ with $\mathscr{A} = 2^{\Omega}$. It is reasonable to define on this space the uniform probability P, i.e. $\mathrm{P}[\{(i,j)\}] = \frac{1}{36}$. The probability space that describe the sum of two dice is given by $\Omega_1 = \{\mathbf{2},\mathbf{3},\mathbf{4},\ldots,\mathbf{10},\mathbf{11},\mathbf{12}\}$ with $\mathscr{A} = 2^{\Omega_1}$. The probability P_1 will NOT be uniform and we will get $\mathrm{P}_1[\{\mathbf{2}\}] = \mathrm{P}[\{(1,1)\}] = \frac{1}{36}, \mathrm{P}_1[\{\mathbf{3}\}] = \mathrm{P}[\{(1,2),(2,1)\}] = \frac{2}{36}, \ldots, \mathrm{P}_1[\{\mathbf{12}\}] = \mathrm{P}[\{(6,6)\}] = \frac{1}{36}$.

(b) $\mathrm{P}_1[\{\text{even number}\}] = \mathrm{P}_1[\{\mathbf{2},\mathbf{4},\mathbf{6},\mathbf{8},\mathbf{10},\mathbf{12}\}] = \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{3}{36} + \frac{1}{36} = \frac{18}{36} = \frac{1}{2}$.

Problem 2. Instead of rolling two dice, assume now that we extract at random two balls without replacement from a box that contains six balls numbered from 1 to 6.

(a) Describe the probability space for this random experiment.

(b) Compute the probability to obtain two balls with consecutive numbers.

Solution 2.

(a) $\Omega = \{(i,j) : i,j \in \{1,2,3,4,5,6\}, \mathbf{i} \neq \mathbf{j}\}$, $\mathscr{A} = 2^{\Omega}$ and P is the uniform probability on Ω.

(b) $\mathrm{P}[\{\text{consecutive numbers}\}] = \mathrm{P}[\{(i,j) : i,j \in \{1,2,3,4,5,6\}, |\mathbf{i}-\mathbf{j}| = \mathbf{1}\}] = \frac{\mathbf{10}}{\mathbf{30}}$.

Problem 3. Let $\Omega = \mathbb{R}$ and define the following subset of 2^{Ω}

$$\mathscr{A} = \{A \subset \mathbb{R} : A \text{ is countable}\} \cup \{A \subset \mathbb{R} : A^c \text{ is countable}\}$$

(a) Prove that $\mathscr{A}$ is a σ–field (it is called the countable/co-countable σ–field)

(b) Prove that $A = (-\infty, 0]$ does not belong to $\mathscr{A}$.

Solution 3.

(a) $\mathscr{A}$ is a σ-field if the following three conditions are satisfied:

(i) $\Omega \in \mathscr{A}$? Yes since the complement of Ω is the empty set, which is countable.

(ii) $A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$? Yes

(iii) $(A_n)_{n\in\mathbb{N}} \subset \mathscr{A} \Rightarrow \cap_{n\in\mathbb{N}} A_n \in \mathscr{A}$? Yes: if at least one among the A_n is countable, the intersection is countable. Otherwise, when all the A_n^c are countable, we have $(\cap_{n\in\mathbb{N}} A_n)^c = \cup_{n\in\mathbb{N}} A_n^c$ is countable, which implies that $\cap_{n\in\mathbb{N}} A_n \in \mathscr{A}$.

(b) Since $A = (-\infty, 0]$ and its complement $(0, +\infty)$ are both not countable, we have that $a \notin \mathscr{A}$.

Problem 4. Let $\Omega = \mathbb{N}$ and define

$$\mathscr{A} = \{A \subset \mathbb{N} : A \text{ or } A^c \text{ is finite}\}$$

Show that $\mathscr{A}$ is a field, but not a σ–field.

Solution 4.

$\mathscr{A}$ is a field if the following three conditions are satisfied:

(i) $\mathbb{N} \in \mathscr{A}$? Yes since the complement of $\mathbb{N}$ is the empty set, which is finite.

(ii) $A \in \mathscr{A} \;\Rightarrow\; A^c \in \mathscr{A}$? Yes

(iii) $A, B \in \mathscr{A} \;\Rightarrow\; A \cap B \in \mathscr{A}$? Yes: if at least one among A and B is finite, the intersection is finite. Otherwise, when both A and B have the complement finite, then $(A \cap B)^c = A^c \cup B^c$ is finite, which implies that $A \cap B \in \mathscr{A}$.

$\mathscr{A}$ is NOT a σ-field: define $A_n = \{2n\}$ for any $n \in \mathbb{N}$. A_n belongs to $\mathscr{A}$, but the (countable) union $\cup_{n \in \mathbb{N}} A_n = \{\text{even numbers}\} \notin \mathscr{A}$, since is not countable as its complement which is the set of the odd numbers.

Problem 5. (a) Prove that the intersections of σ–fields is a σ–field.

(b) Given $\Omega = \{1, 2, 3, 4, 5, 6\}$, define the minimal σ–field containing the sets $\{1\}$ and $\{2, 4\}$.
Recall that given a collection of subsets $\mathscr{C}$ of Ω, the σ–field generated by $\mathscr{C}$, denoted $\sigma(\mathscr{C})$, is the σ–field satisfying:

(i) $\sigma(\mathscr{C}) \supset \mathscr{C}$

(ii) If $\mathscr{B}$ is a σ–field containing $\mathscr{C}$, then $\mathscr{B} \supset \sigma(\mathscr{C})$.

Solution 5.

(a) Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two σ–fields. Their intersection will be the set

$$\mathscr{A}_1 \cap \mathscr{A}_2 = \{A \subset \Omega : A \in \mathscr{A}_1 \text{ and } A \in \mathscr{A}_2\}$$

It is easy to prove that this set is a σ–field: for example $\Omega \in \mathscr{A}_1 \cap \mathscr{A}_2$, since $\Omega \in \mathscr{A}_1$ and $\Omega \in \mathscr{A}_2$.

(b) Let $\mathscr{C} = \{\{1\}, \{2, 4\}\}$. The minimal σ–field containing $\mathscr{C}$ will be

$$\sigma(\mathscr{C}) = \{\emptyset, \Omega, \{1\}, \{2,4\}, \{3,5,6\}, \{2,3,4,5,6\}, \{1,3,5,6\}, \{1,2,4\}\}$$