Lecture 15 Stoch. Methods Nov. 18th, 2024

mgf

cf

Abs. cont. r.v.

$$X \sim \mathcal{U}([a,b]) \qquad f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

$$m_X(t) = \begin{cases} \boxed{1} & t=0 \\ \frac{e^{tb}-e^{ta}}{t(b-a)} & t \neq 0 \end{cases} \qquad \mathbb{E}[e^{tX}]$$

$m_X(t)$ is continuous in 0? $\boxed{\text{YES}}$

$$\lim_{t\to 0} m_X(t) = \lim_{t\to 0} \frac{e^{tb}-e^{ta}}{t(b-a)} = \frac{0}{0}$$

$$\overset{(H)}{=} \lim_{t\to 0} \frac{D(e^{tb}-e^{ta})}{D(t(b-a))} = \lim_{t\to 0} \frac{be^{tb}-ae^{ta}}{b-a}$$

$$= \frac{b-a}{b-a} = 1$$

$$E[X] = \frac{d}{dt} m_X(t) \Big|_{t=0}$$

$m_X(t)$ is cont. everywhere

$$m_X'(t) = \frac{(be^{tb} - ae^{ta}) \cdot t(b-a) - (e^{tb} - e^{ta}) \cdot (b-a)}{t^2 (b-a)^2}$$

$t \neq 0$

$$\frac{d}{dt}\left(\frac{e^{tb} - e^{ta}}{t(b-a)}\right)$$

$$= \boxed{\frac{(b e^{tb} - a e^{ta})}{t(b-a)}} - \boxed{\frac{e^{tb} - e^{ta}}{t^2 (b-a)}}$$

$$\boxed{\lim_{t \to 0} m_X'(t)} = \lim_{t \to 0} \left(\frac{be^{tb} - ae^{ta}}{t(b-a)} - \frac{e^{tb} - e^{ta}}{t^2(b-a)}\right) = \boxed{\infty - \infty}$$

$\to$ (indeterminate form)

$$E[X] = m'_X(t)\big|_{t=0} = m'_X(0)$$

Let us consider the definition of $m'_X(0)$:

Let us compute $m'_X(0) = \lim_{h\to 0} \dfrac{m_X(h) - m_X(0)}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{e^{hb}-e^{ha}}{h(b-a)} - 1}{h} = \lim_{h\to 0} \frac{e^{hb}-e^{he}-h(b-a)}{(b-a)h^2}$$

$$= \frac{1}{b-e} \lim_{h\to 0} \frac{e^{hb}-e^{he}-h(b-a)}{h^2} = \frac{0}{0}$$

$$\overset{(H)}{=} \frac{1}{b-a} \lim_{h\to 0} \frac{be^{hb}-ae^{he}-(b-e)}{2h} = \frac{0}{0}$$

$$\overset{(H)}{=} \frac{1}{b-e} \cdot \lim_{h\to 0} \frac{b^2e^{hb}-a^2e^{ha}}{2} = \frac{b^2-a^2}{2(b-a)} =$$

$$= \frac{b+a}{2}$$

$$\lim_{t\to 0} \frac{e^{tb}-e^{ta}}{t^2(b-a)} = \frac{0}{0} \overset{(H)}{=} \lim_{t\to 0} \frac{be^{tb}-ae^{ta}}{2t(b-a)}$$

$$\boxed{m_X'(t)\Big|_{t=0} = \frac{a+b}{2}}$$

$$m_X'(t)\Big|_{t=0} = \lim_{h\to 0} \frac{m_X''(t+h) - m_X''(t)}{h}$$

$X \sim Exp(\lambda)$ r.v.

Normal distribution

$X \sim N(\mu, \sigma^2)$, $Z \sim N(0,1)$

$\mu = 0,\ \sigma^2 = 1$ standard Normal

$$f_Z(z) = \underbrace{\frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2}} \qquad z \in \mathbb{R}$$

$$m_X(t) = \mathbb{E}\left[e^{tZ}\right] = \int_{-\infty}^{+\infty} e^{tz} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2}\, dz$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\, e^{tz - z^2/2}\, dz = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(t-z)^2}{2}} \cdot e^{t^2/2}\, dz$$

$$e^{(tz - z^2/2)} = e^{-\frac{(t-z)^2}{2} + t^2/2}$$

$$-z^2/2 + \frac{\cancel{2}tz}{\cancel{2}} - t^2/2$$

$$\boxed{m_X(t)} = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-(t-z)^2/2} \cdot \underbrace{e^{t^2/2}}_{(z-t)^2} \, dz$$

$$= e^{t^2/2} \cdot \int_{-\infty}^{+\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{(t-z)^2}{2}}}_{\text{density of a } N(t,1) \text{ r.v.}} \, dz$$

$$= \boxed{e^{t^2/2}}$$

$$i^2 = -1$$

$$\boxed{\varphi_Z(u) = e^{(iu)^2/2} = e^{-u^2/2}} \quad \forall u \in \mathbb{R}$$

$$Z \sim N(0,1) \rightsquigarrow \varphi_Z(u) = e^{-u^2/2} \in \mathbb{R}$$

<u>Remark</u>: If X is a symmetric r.v., i.e. X and $-X$ have the same distribution, then $\varphi_X(u) \in \mathbb{R} \quad \forall u \in \mathbb{R}$.

$\varphi_X(u) \in \mathbb{R}$

$$\underbrace{\varphi_{-X}(u)} = \mathbb{E}\left[e^{iu(-X)}\right] =$$

$$= \mathbb{E}\left[e^{-iuX}\right] = \underbrace{\varphi_X(-u)}$$

$$\underbrace{\varphi_X(-u)} = \mathbb{E}[\cos(-uX)] + i\,\mathbb{E}[\sin(-uX)]$$

$$= \mathbb{E}[\cos(uX)] + i\,\mathbb{E}[-\sin(uX)]$$

$$= \underbrace{\mathbb{E}[\cos(uX)] - i\,\mathbb{E}[\sin(uX)]}$$

$z = a + ib$ complex number

$\bar{z} = a - ib$ complex conjugate

$$\boxed{\varphi_{-X}(u) = \overline{\varphi_X(u)}} \quad \forall u \in \mathbb{R}$$

If X is symmetric $\Rightarrow$ $\boxed{\varphi_X = \varphi_{-X}}$

$\Rightarrow$ $\boxed{\varphi_X = \overline{\varphi_X}}$

$$z = \bar{z} \iff a + ib = a - ib$$

$$\iff \boxed{b = 0}$$

$$\boxed{\varphi_X(u) \in \mathbb{R} \quad \forall u \in \mathbb{R}}$$

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

$$Z \sim \mathcal{N}(0,1) \qquad \boxed{X = \mu + \sigma Z}$$

$$m_X(t) = \mathbb{E}[e^{tX}] = \mathbb{E}[e^{t(\mu + \sigma Z)}]$$

$$= \mathbb{E}[e^{t\mu + t\sigma Z}] = \mathbb{E}[e^{t\mu} \cdot e^{t\sigma Z}]$$

$$= e^{t\mu} \cdot \underbrace{\mathbb{E}[e^{(t\sigma)\cdot Z}]}_{} =$$

$$= \underbrace{e^{t\mu}}_{} \cdot m_Z(t\sigma) = \boxed{e^{t\mu} \cdot e^{\frac{t^2\sigma^2}{2}}}$$

$$\boxed{\varphi_X(u)} = e^{u\mu} \cdot \varphi_Z(u\sigma) = \boxed{e^{u\mu - \frac{u^2\sigma^2}{2}}}$$

Proposition: If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ and $X \perp\!\!\!\perp Y$, then $X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

Proof:

$$\varphi_{X+Y}(u) = \varphi_X(u) \cdot \varphi_Y(u)$$

$$= e^{u\mu_1} e^{-\frac{u^2\sigma_1^2}{2}} \cdot e^{u\mu_2} e^{-\frac{u^2\sigma_2^2}{2}}$$

$$= e^{u(\mu_1+\mu_2)} \cdot e^{-\frac{u^2}{2}(\sigma_1^2+\sigma_2^2)}$$

this is the cf $N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$ r.v. $\blacksquare$

If X is a d-dimensional r. vector, $A \in M(n \times d)$ (real valued $n \times d$ matrix) and $b \in \mathbb{R}^n$

$$Y := \underset{(n\times d)\times(d\times 1) = n \times 1}{A \cdot X} + \underset{n \times 1}{b}$$

d-dim. r.v. $\rightsquigarrow$ n-dim. r.v.

$\underbrace{X}$ d-dim. r.v. $\rightsquigarrow$ Y $\underline{n\text{-dim. r.v.}}$

$$m_Y(t) = \mathbb{E}\left[e^{\langle t, Y\rangle}\right] =$$

$t \in \mathbb{R}^n$

$$= \mathbb{E}\left[e^{\langle t, AX+b\rangle}\right]$$

$$= \mathbb{E}\left[e^{\langle t, AX\rangle + \langle t, b\rangle}\right]$$

$$\underbrace{\langle t, v\rangle = \sum_{i=1}^{n} t_i v_i}$$

$$\leq \mathbb{E}\left[e^{\langle t, AX\rangle} \cdot \underbrace{e^{\langle t, b\rangle}}\right]$$

$$= e^{\langle t, b\rangle} \cdot \mathbb{E}\left[e^{\langle t, AX\rangle}\right]$$

$$\boxed{\langle t, A\bar{X}\rangle = \langle A^T t, \bar{X}\rangle}$$

$A \in M(n \times d) \rightsquigarrow A^T \in M(d \times n)$

$$(A^T_{ij}) = A_{ji}$$

$$m_Y(t) = e^{\langle t,b\rangle} \cdot \underbrace{\mathbb{E}\left[e^{\langle A^T t, \bar{X}\rangle}\right]}$$

$$= e^{\langle t,b\rangle} \cdot m_X(A^T t)$$

With a similar computation you get

$$\boxed{\varphi_Y(u) = e^{i\langle u,b\rangle} \cdot \varphi_{\bar{X}}(A^T \cdot u) \qquad u \in \mathbb{R}^n}$$

Exercise: If $X_1, X_2, X_3, \ldots, X_n$ are indep. $\mathcal{U}(0,1)$ r.v.'s and

$Z = X_1 + \cdots + X_n$ prove that

$$\mathbb{P}[Z \le x] = \frac{x^n}{n!} \quad \text{for } 0 \le x \le 1$$

$X_1 \sim \mathcal{U}([0,1])$

$Z = X_1 + X_2 \quad \cancel{\in \mathcal{U}([0,2])}$?? **NO !!**

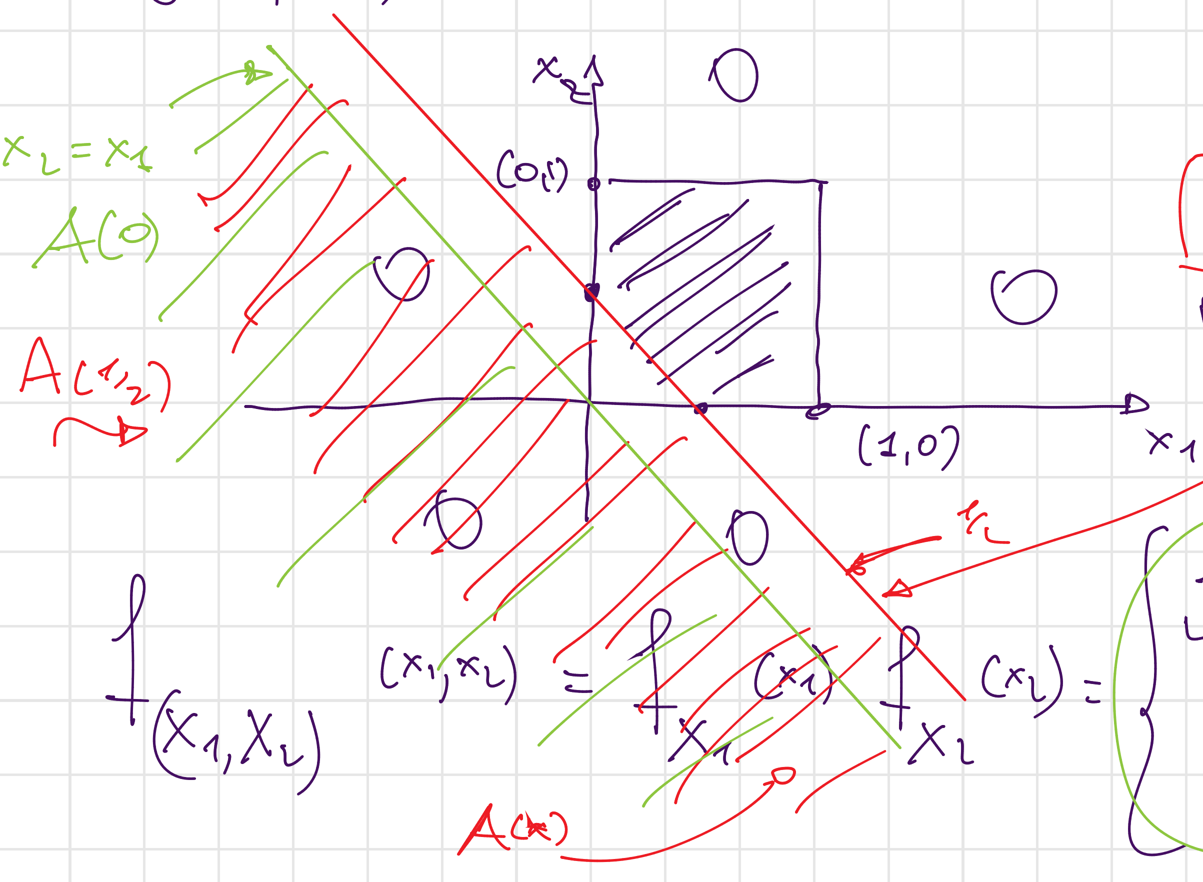
$z \in (0,2)$

$F_Z(x) = \mathbb{P}[Z \le x] = \mathbb{P}[(X_1,X_2) \in A(1/2)]$

$= \mathbb{P}[X_1 + X_2 \le x] = \iint_{A(1/2)} f_{X_1X_2}(x_1,x_2)\,dx_1\,dx_2$

$x = 1/2$

(X_1, X_2) is an abs. cont. r.v.



$x = 1/2$

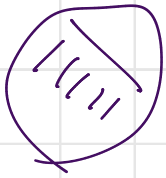
$\boxed{X_1 + X_2 \le \frac{1}{2}}$

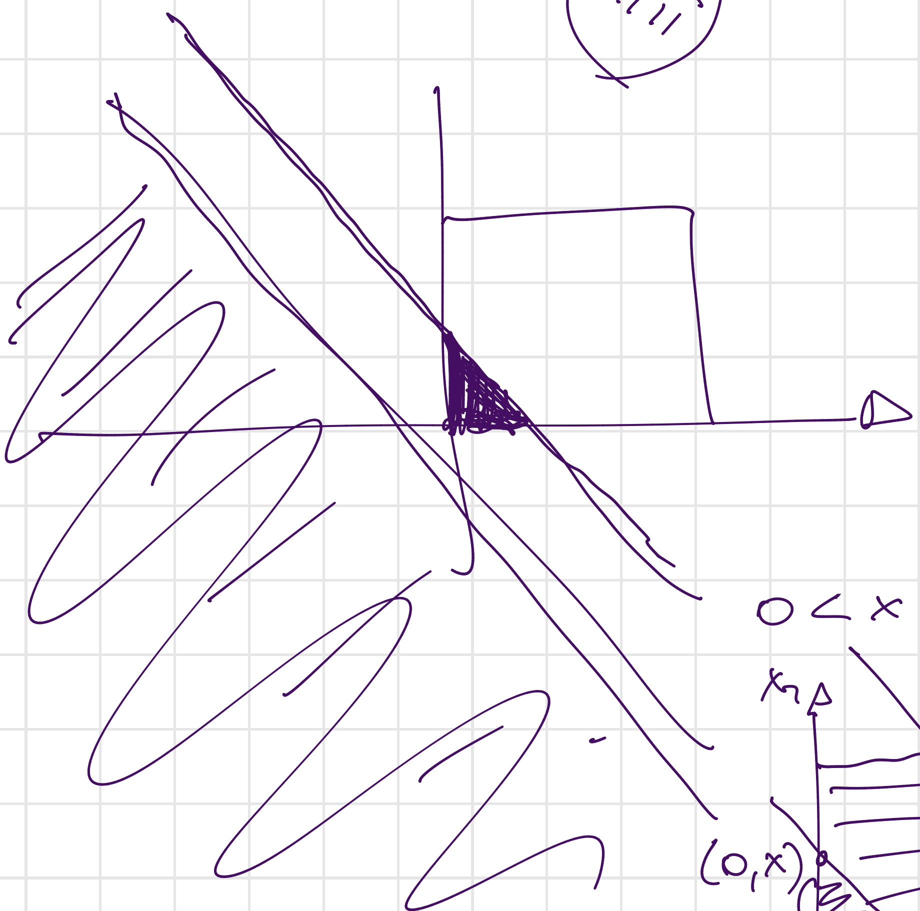
$\boxed{X_2 = \frac{1}{2} - X_1}$

$f_{(X_1,X_2)}(x_1,x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2) = \begin{cases} 1 & 0<x_1<1,\ 0<x_2<1 \\ 0 & \text{otherwise} \end{cases}$

$\mathbb{P}[X_1 + X_2 \le x] = \iint_{A(x)} \underbrace{f_{X_1}(x_1)\, f_{X_2}(x_2)}\,dx_1\,dx_2$

$$\mathbb{P}[X_1 + X_2 \le x] = \iint f_{X_1}(x_1) f_{X_2}(x_2)\, dx_1 dx_2$$

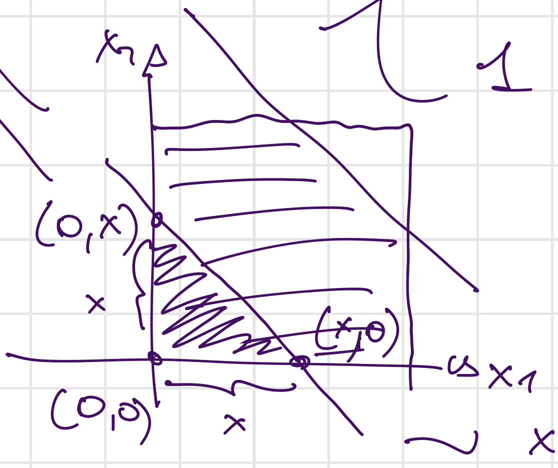




$x \le 0$

$$= \begin{cases} \iint 0 \, dx_1 dx_2 = 0 & x < 0 \\ x^2/2 & 0 < x < 1 \\ 0 & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

$0 < x < 1$



$\sim x_2 = x - x_1$

$$\iint 1 \, dx_1 dx_2 = x^2/2$$

$Z = X_1 + X_2$

$$\mathbb{P}[Z \le x] = F_Z(x) = \frac{x^2}{2} \qquad x \in [0,1]$$

$n = 1 \qquad Z = X_1 \qquad \mathbb{P}[Z \le x] = x \qquad 0 \le x \le 1$

$n = 2 \qquad Z = X_1 + X_2 \qquad \mathbb{P}[Z \le x] = \frac{x^2}{2} \qquad 0 \le x \le 1$

$\vdots$

$$n \in \mathbb{N} \qquad \underbrace{\underbrace{Z = X_1 + X_2 + \dots + X_n}_{[0,n]}, \quad \mathbb{P}[Z \le x] = \frac{x^n}{n!}} \qquad \underline{\underline{0 \le x \le 1}}$$

Prove by induction over n

Assume is true for n, and prove that is true for $n+1$.

$$Z = X_1 + \dots + X_{n+1} \qquad \mathbb{P}[Z \le x] = \frac{x^{n+1}}{(n+1)!}$$

$$\boxed{Z = \underbrace{W}_{X_1 + X_2 + \dots + X_n} + X_{n+1}} \qquad \underline{W \perp\!\!\!\perp X_{n+1}} \text{ are ind.}$$

$\mathbb{P}[X_{n+1} \le x] = x \qquad 0 \le x \le 1$

$\mathbb{P}[W \le x] = \frac{x^n}{n!} \qquad 0 \le x \le 1$

2) Determine the expected number of independent $\mathcal{U}(0,1)$ r.v.'s that need to be summed to exceed 1.