DATA SCIENCE **Stochastic Methods**	Name: Solution (sketch)
February 25, 2020 Prof. Marco Ferrante	Student number:

Problem 1. [10 marks]

Define a Markov Chain on $S = \{1, 2, 3, \ldots\}$ with transition probabilities

$$p_{i,1} = \frac{i}{i+1},$$

$$p_{i,i+1} = \frac{1}{i+1},$$

for any $i \geq 1$.

(i) Is the MC irreducible?

(ii) Find the invariant distribution;

(iii) Is this distribution reversible?

$$P = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \cdots \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} & \left[\begin{array}{cccccc} 1/2 & 1/2 & 0 & 0 & 0 & \\ 2/3 & 0 & 1/3 & 0 & 0 & \cdots \\ 3/4 & 0 & 0 & 1/4 & 0 & \cdots \\ 4/5 & 0 & 0 & 0 & 1/5 & \cdots \\ \vdots & & & & & \end{array}\right] \end{array}$$

(i) X_n is irreducible

$\forall\, i, j \in S$

(ii) $\pi = \pi P \implies \begin{cases} \dfrac{\pi_1}{2} = \pi_2, \quad \dfrac{\pi_2}{3} = \pi_3, \ \ldots, \ \dfrac{\pi_N}{N+1} = \pi_N, \ \ldots \\ \displaystyle\sum_{i=1}^{\infty} \pi_i = 1 \end{cases}$

$\implies \pi_2 = \dfrac{\pi_1}{2}, \quad \pi_3 = \dfrac{\pi_1}{2 \cdot 3} = \dfrac{\pi_1}{3!}, \quad \pi_4 = \dfrac{\pi_1}{4!}, \ \ldots, \ \pi_N = \dfrac{\pi_1}{N!}$

$\implies \pi_1 \cdot \displaystyle\sum_{i=1}^{\infty} 1/i! = 1 \implies \pi_1 = 1/_{e-1} = (e-1)^{-1}$

$$\Rightarrow \quad \pi_N = \frac{(e-1)^{-1}}{N!} \qquad \forall N \geq 1$$

(iii) is π reversible? NO

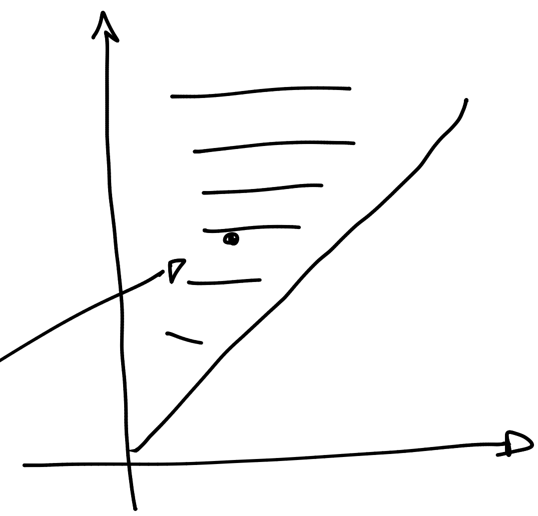
$$0 = \pi_1 P_{13} \neq \pi_3 P_{31} = \frac{(e-1)^{-1}}{3!} \cdot \frac{3}{4}$$

Problem 2. [14 marks] Let X and Y be two independent Exponential random variables with parameters, respectively, λ and μ.

(i) Compute $\mathbb{P}[X < Y]$;

(ii) Compute $\mathbb{E}[X \cdot Y^2]$;

(iii) Compute $\mathbb{E}[X|X < a]$, where $a > 0$. (Hint: compute first $\mathbb{P}[X > x|X < a]$ for any $x > 0$).



(i) $$\mathbb{P}[X<Y] = \iint_B \lambda e^{-\lambda x} \mu e^{-\mu y}\, dx\, dy$$

$$B = \{(x,y) \in (0,+\infty)^2 : x < y\}$$

$$= \int_0^{+\infty} dy \left(\int_0^y \lambda e^{-\lambda x} dx \right) \mu e^{-\mu y}$$

$$= \int_0^{+\infty} \mu e^{-\mu y} \left[-e^{-\lambda x} \right]_0^y dy$$

$$= \int_0^{+\infty} \mu \cdot \left[e^{-\mu y} - e^{-(\mu+\lambda) y} \right] dy$$

$$= \left[-e^{-\mu y} \right]_0^{+\infty} - \left[-\frac{\mu}{\mu+\lambda} e^{-(\mu+\lambda) y} \right]_0^{+\infty}$$

$$= 1 - \frac{\mu}{\mu+\lambda} = \frac{\mu+\lambda-\mu}{\mu+\lambda} = \frac{\lambda}{\mu+\lambda}$$

(ii) Since $X \perp\!\!\!\perp Y$, then

$$\mathbb{E}[X \cdot Y^2] = \mathbb{E}[X] \cdot \mathbb{E}[Y^2]$$

$$= \frac{1}{\lambda} \cdot \frac{2}{\mu^2} = \frac{2}{\lambda \cdot \mu^2}$$

$$\left(\mathbb{E}[Y^2] = \int_0^{+\infty} y^2 \mu e^{-\mu y} \, dy \underset{\text{integ. by parts}}{=} \int_0^{+\infty} 2y \, e^{-\mu y} \, dy \right.$$

$$= \frac{2}{\mu} \cdot \mathbb{E}[Y] = \frac{2}{\mu} \cdot \frac{1}{\mu} = \frac{2}{\mu^2}$$

(iii) $\mathbb{E}[X \mid X < a]$, where $a > 0$.

$$\mathbb{P}[X > x \mid X < a] = \frac{\mathbb{P}[x < X, X < a]}{\mathbb{P}[X < a]} =$$

$$= \begin{cases} \dfrac{\mathbb{P}[x < X < a]}{\mathbb{P}[X < a]} & \text{if } x < a \\ 0 & \text{if } a < x \end{cases}$$

So $\mathbb{E}[X \mid X < a] = \displaystyle\int_0^{+\infty} \mathbb{P}[X > x \mid X < a] \, dx = \int_0^a \frac{\mathbb{P}[x < X < a]}{\mathbb{P}[X < a]} \, dx$

$$= \int_0^a \frac{e^{-\lambda x} - e^{-\lambda a}}{1-e^{-\lambda a}}\,dx = \circledast$$

$$\mathbb{P}[X < a] = F_X(a) = 1 - e^{-\lambda a}$$

$$\mathbb{P}[x < X < a] = F_X(a) - F_X(x) = 1 - e^{-\lambda a} - 1 + e^{-\lambda x} = e^{-\lambda x} - e^{-\lambda a}$$

$$\circledast = \frac{1}{1-e^{-\lambda a}} \cdot \left[\left[-\frac{e^{-\lambda x}}{\lambda} \right]_0^a - a e^{-\lambda a} \right]$$

$$= \frac{1}{1-e^{-\lambda a}} \cdot \left[-\frac{e^{-\lambda a}}{\lambda} + \frac{1}{\lambda} - a e^{-\lambda a} \right]$$

$$= \frac{1}{1-e^{-\lambda a}} \left[\frac{1-e^{-\lambda a}}{\lambda} - a \cdot e^{-\lambda a} \right]$$

$$= \frac{1}{\lambda} - \frac{a e^{-\lambda a}}{1-e^{-\lambda a}}$$

Problem 3. [12] Let $(Z_i)_{1\leq i\leq n}$ be a family of i.i.d. Standard Normal random variables and define $X_i = Z_i^2$.

(i) Compute the expectation of X_1;

(ii) Defined $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$, prove that

$$\mathrm{P}(\overline{X_n} - 1 \leq -\varepsilon) \leq e^{-n\frac{\varepsilon^2}{8}},$$

for $0 < \varepsilon < 1$.

(See Lecture 12.1)