DATA SCIENCE **Stochastic Methods**	Name: SOLUTION
December 16, 2020 Prof. Marco Ferrante	Student number:

Problem 1. [12] Let $X \sim Exp(\lambda)$ and $Y \sim Bin(n, p)$ and assume that they are independent.

3 (i) Compute $\mathrm{E}[X(Y+1)]$;

5 (ii) Compute $\mathrm{P}[X > Y]$;

4 (iii) When $n = 2$, compute $\mathrm{E}[X^{Y+1}]$

(i) $X \perp\!\!\!\perp Y \Rightarrow \mathbb{E}[X(Y+1)] = \mathbb{E}[X] \cdot (\mathbb{E}[Y+1]) =$

$= \mathbb{E}[X] \cdot (\mathbb{E}[Y] + 1) = \frac{1}{\lambda}(np+1)$

(ii) $\mathbb{P}[X > Y] = \sum_{k=0}^{n} \mathbb{P}[X > Y \mid Y = k] \cdot \mathbb{P}[Y = k] =$

$= \sum_{k=0}^{n} \mathbb{P}[X > k] \cdot \mathbb{P}[Y = k] = \sum_{k=0}^{n} e^{-\lambda k} \cdot \binom{n}{k} p^k (1-p)^{n-k}$

$= (1 - p + e^{-\lambda} p)^n$

(iii) $\mathbb{E}[X^{Y+1}] = \sum_{k=0}^{2} \mathbb{E}[X^{Y+1} \mid Y = k) \cdot \mathbb{P}[Y = k]$

$= \mathbb{E}[X] \cdot (1-p)^2 + \mathbb{E}[X^2]\, 2p(1-p) +$

$+ \mathbb{E}[X^3] \cdot p^3 = \frac{1}{\lambda}(1-p)^2 + \frac{4}{\lambda^2} p(1-p) + \frac{6}{\lambda^3} p^2$

Problem 2. [12] If $Z_1, \ldots, Z_n$ are independent, bounded random variables, with $a \le Z_i \le b$ for all i, and denoting $S_n = Z_1 + \ldots + Z_n$, the Heoffding's inequality states that

$$\mathrm{P}[S_n - \mathrm{E}[S_n] \ge t] \le e^{\frac{-2t^2}{n(b-a)^2}}$$

Let $X_1, \ldots, X_n$ be a family of i.i.d. Uniform $[-1,1]$ random variables.

3 (i) Compute $\mathrm{P}[X_1 \ge X_2]$;

4 (ii) Compute $\mu = \mathrm{E}[X_1^2]$;

5 (iii) Prove a Chernoff Bound Upper tail estimate for $\widehat{X}_n = \frac{1}{n}\sum_{i=1}^n X_i^2$, i.e. for any $\delta > 0$ an upper bound for the probability

$$\mathrm{P}[\widehat{X}_n \ge (1+\delta)\mu].$$

(i) Since $X_1 \perp\!\!\!\perp X_2$ and (X_1, X_2) is absolutely continuous $\mathbb{P}[X_1 = X_2] = 0$. Moreover, by symmetry, $\mathbb{P}[X_1 > X_2] = \mathbb{P}[X_2 > X_1] = 1/2$.

(ii) $\boxed{\mu} = \mathbb{E}[X_1^2] = \int_{-1}^{1} \frac{1}{2} x^2 \, dx = \left[\frac{x^3}{6}\right]_{-1}^{1} = \frac{1}{6} + \frac{1}{6} = \boxed{1/3}$

(iii) $\mathbb{P}[\widehat{X}_n \ge (1+\delta)\mu] = \mathbb{P}[X_1 + \cdots + X_n \ge (1+\delta)n\mu]$

$= \mathbb{P}[X_1 + \cdots + X_n - \mathbb{E}[X_1 + \cdots + X_n] \ge \delta n \mu]$ $\quad a=0, b=1 \quad \mu = 1/3 \quad (0 \le X_1^2 \le 1)$

$\le e^{-\frac{2\delta^2 n^2 \mu^2}{n}} = e^{-2\delta^2\mu^2 n}$

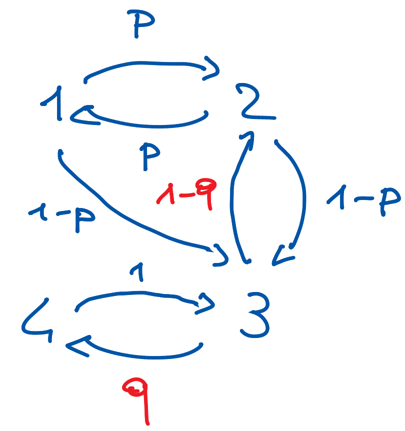
by the Hoeffding's ineq.

$= e^{-\frac{2\delta^2}{9} n}$

Problem 3. [12] A discrete time Markov chain $\{X_n, n \geq 0\}$ with state space $S = \{1,2,3,4\}$ has transition probability matrix

$$P = \begin{bmatrix} 0 & p & 1-p & 0 \\ p & 0 & 1-p & 0 \\ 0 & 1-q & 0 & q \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where $p, q \in [0,1]$.



3 (i) Prove that the MC is irreducible iff $p, q \in (0,1)$.

3 (ii) When the chain is irreducible, prove that it is also aperiodic.

4 (iii) When the chain is irreducible, find the stationary distribution.

2 (iv) Is this distribution reversible?

(i) if $p=0$ $\{1\}$ is a single class; if $p=1$ $\{1,2\}$ is a class, if $q=0$ $\{4\}$ is a class, if $q=1$ $\{3,4\}$ is a class.

if $0<p<1$ and $0<q<1$, $\{1,2,3,4\}$ is a class $\Rightarrow$ MC is irred.

(ii) $P^2_{11} > 0$ $\left(1 \xrightarrow[p]{} 2 \xrightarrow[p]{} 1 \right)$ and $P^3_{11} > 0$ $\left(1 \xrightarrow[1-p]{} 3 \xrightarrow[1-q]{} 2 \xrightarrow[p]{} 1 \right)$

$\Rightarrow$ 1 is aperiodic $\Rightarrow$ the MC is aperiodic

(iii) $\begin{cases} \pi = \pi P \\ \sum \pi_i = 1 \end{cases} \Rightarrow \begin{cases} p\pi_2 = \pi_1 \\ p\pi_1 + (1-q)\pi_3 = \pi_2 \\ (1-p)\pi_1 + (1-p)\pi_2 + \pi_4 = \pi_3 \\ q\pi_3 = \pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases} \Rightarrow \begin{cases} \pi_2 = \frac{1}{p}\pi_1 \\ \pi_3 = \frac{1}{1-q}\left(\frac{1}{p} - p\right)\pi_1 \\ \pi_4 = \frac{q}{1-q}\left(\frac{1-p^2}{p}\right)\pi_1 \\ \pi_1 + \dots + \pi_4 = 1 \end{cases}$

$$\Rightarrow \quad \pi_1 + \frac{1}{p}\pi_1 + \frac{1}{1-q}\,\frac{1-p^2}{p}\pi_1 + \frac{q}{1-q}\,\frac{(1-p^2)}{p}\pi_1 = 1$$

$$\pi_1 = \frac{p(1-q)}{p(1-q) + (1-q) + (1-p^2) + q(1-p^2)}, \quad \pi_2 = \frac{1-q}{p(1-q) + \dots + q(1-p^2)}$$

$$\pi_3 = \frac{1-p^2}{p(1-q) + \dots + q(1-p^2)}, \quad \pi_4 = \frac{q(1-p^2)}{p(1-q) + \dots + q(1-p^2)}$$

(iv) NO!

$$\underbrace{\pi_1 P_{13}}_{\neq 0} \neq \underbrace{\pi_3 P_{31}}_{= 0}$$