Lecture 23 Stoch. Meth. Dec. 9th, 2026

Theorem: For an irreducible, ergodic MC $(X_n)_{n\geq 0}$ exists

$$\boxed{\pi_j := \lim_{n\to+\infty} P^n_{ij}} \quad \forall j \in S$$

(indep. of $i$)

$$\boxed{(\pi_1, \dots, \pi_N) \qquad |S| = N}$$

In addition

(a) $\pi = (\pi_1, \dots, \pi_N)$ is the unique sol.

$$\begin{cases} \underline{\pi = \pi P} \\ \sum_{i\in S} \pi_i = 1 \end{cases}$$

$\pi$ is a distribution, stationary or invariant distribution

(b) $\pi_j = \dfrac{1}{m_j} > 0 \qquad m_j = \mathbb{E}[T_j \mid X_0 = j]$

(c) $\pi_j$ = long run proportion of time the chain spends in state $j$

# Reversible Markov Chains

**Def:** Let $P$ be a transition matrix.

A distribution $\pi$ is called reversible for $P$ if

(*) $$\boxed{\pi_x P_{xy} = \pi_y \cdot P_{yx}} \qquad \forall x,y \in S$$

Remark 1: (*) is trivial for $x=y$

We have to check (*) only for $x \neq y$

If $\pi = (\pi_1, \pi_2, \dots)$ and $\pi_i = \pi_j \quad \forall i,j$

$|S| = N < +\infty \qquad (\pi_1, \dots, \pi_N) = \left(\frac{1}{N}, \frac{1}{N}, \dots, \frac{1}{N}\right)$

$$\boxed{\sum_i \pi_i = +\infty}$$

$$\boxed{(\alpha\pi) = (\alpha\pi) P}$$

$\pi = (\pi_i)_{i \in S}$

$\pi_i \geq 0 \qquad \sum_i \pi_i < +\infty$

(*) is satisfied by $\pi = (\frac{1}{N}, \dots, \frac{1}{N})$ $\iff$

$$P_{xy} = P_{yx} \quad \forall x \neq y$$

$P$ is a symmetric matrix.

$\pi$ and $P$ are in det. bal.

$$\boxed{\pi_x P_{xy} = \pi_y P_{yx}}$$

detailed balance condition (equation)

<u>Proposition</u>: If $\pi$ is <u>reversible</u>, then $\pi$ is invariant for $P$, i.e. $\boxed{\pi = \pi P}$.

<u>Proof</u>:

$$\underbrace{(\pi P)_x}_{} = \sum_{y \in S} \pi_y \cdot P_{yx} = \underbrace{\sum_{y \in S}}_{} \underbrace{\pi_x}_{} \cdot P_{xy}$$

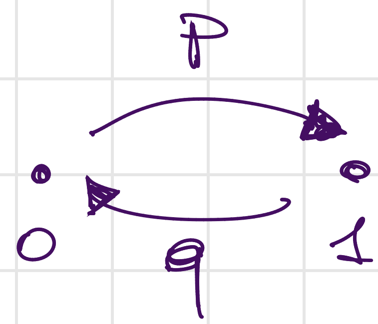
$$= \pi_x \cdot \overbrace{\sum_{y \in S} P_{xy}}^{1} = \underbrace{\pi_x}_{} \qquad \forall x \in S$$

but $P$ is a stoch. matrix $\Rightarrow \forall x \quad \sum_{y} P_{xy} = 1$

∎

$|S| = 2$ $\qquad S = \{0, 1\}$

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \qquad \underbrace{p, q}_{} \in [0,1] \; ,$$




$P$ is irreducible $\iff$ $p \neq 0$ and $q \neq 0$

If $p = q = 0$ $\qquad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\{0\}$ and $\{1\}$ are absorbing states.

$$\boxed{\pi = \pi P} \qquad \pi = (\pi_0, \pi_1)$$

infinite family

$$(\underbrace{\pi_0}_{}, \underbrace{\pi_1}_{}) = (\underbrace{\pi_0, \pi_1}_{}) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\pi_0 \in [0,1]$

$$\begin{cases} \pi_0 = \pi_0 \\ \pi_1 = \pi_1 \\ \pi_0 + \pi_1 = 1 \end{cases}$$

$\pi_1 = 1 - \pi_0$

$\pi = (\pi_0, 1 - \pi_0)$ is invariant distrib.

If at least one of p and q is different from zero.

$$P = \begin{bmatrix} 1-p & \boxed{p} \\ \boxed{q} & 1-q \end{bmatrix}$$

Detailed balance equation

$$\pi_x P_{xy} = \pi_y P_{yx}$$

$x=0$
$y=1$

$$\pi_0 P_{01} = \pi_1 P_{10}$$

$$\pi_0 p = \pi_1 \cdot q$$

Let $q \neq 0$

$$\boxed{\pi_1 = p/q \, \pi_0}$$

$\to (\pi_0, p/q \, \pi_0)$ are all the solut.

$$\boxed{\pi = P\pi}$$

$$\boxed{\pi_0 + \pi_1 = 1}$$

$$\pi_0 + \frac{p}{q}\pi_0 = 1$$

$$\Rightarrow \frac{p+q}{q}\pi_0 = 1 \Rightarrow \begin{cases} \pi_0 = \dfrac{q}{p+q} \\ \pi_1 = \dfrac{p}{p+q} \end{cases}$$

$(q>0 \Rightarrow p+q>0)$

Ex. 2 Sentinel

$p = 1/2$

$$P = \begin{array}{c} \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$|S| < +\infty$

This MC is irreducible, reversible

$\Rightarrow \exists!$ invariant distribution

Detailed balance equation:

$$\pi_x \underset{1/2}{\underset{\parallel}{P_{xy}}} = \pi_y \underset{1/2}{\underset{\parallel}{P_{yx}}} \qquad \forall x \neq y$$

$$\Leftrightarrow \quad \pi_1 = \pi_2 = \pi_3 = \pi_4 = \frac{1}{4}$$

The invariant distribution $\pi = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

A double stochastic matrix is:

$$\sum_x P_{xy} = 1 \qquad \forall y \in S$$

If $\underbrace{|S|}_{N} < +\infty$, $\pi = \left(\frac{1}{N}, \dots, \frac{1}{N}\right)$ for these matrices is always an invariant distrib.

$$(\pi P)_x = \sum_y \underbrace{\pi_y}_{=\pi} P_{yx} = \pi \cdot \underbrace{\boxed{\sum_y P_{yx}}}_{=1} = \pi$$

$(\pi, \dots, \pi)$

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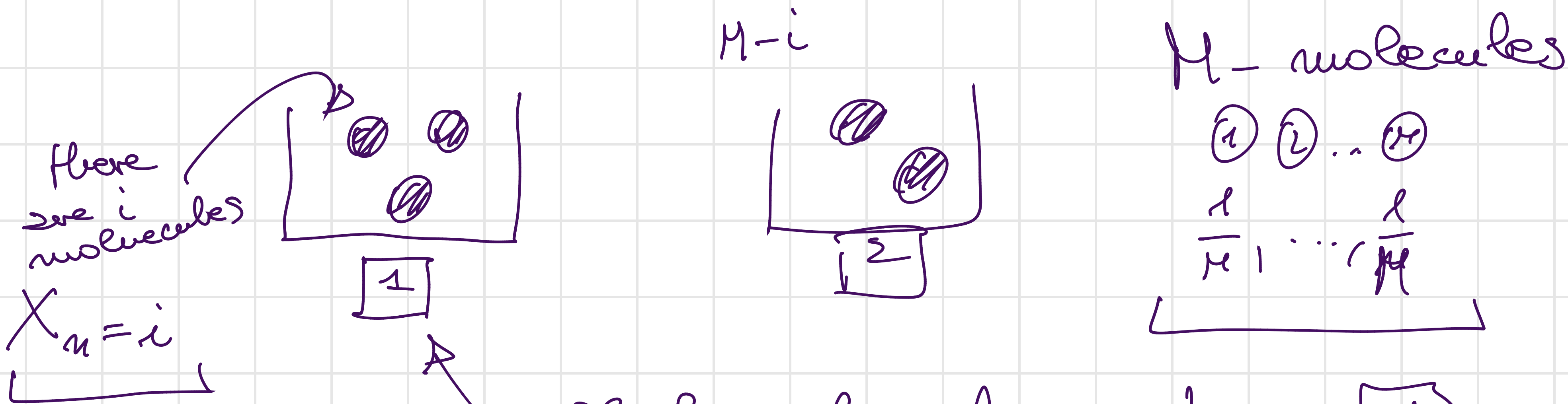
Example 1 Ehrenfest Urns model



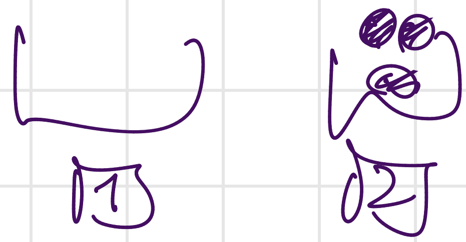

N° of molecules in box $\boxed{1}$

$X_0, X_1, \dots, X_n, \dots \qquad S = \{0, 1, \dots, M\}$

$S = \{0, 1, \dots, M\}$


$X_n = 0$

$$i = 0 \xrightarrow{1} j = 1$$




$$i = M \xrightarrow{1} j = M-1$$

$i \in \{1, \dots, M-1\}$

$$\frac{1}{M} + \dots + \frac{1}{M}$$

$M-i$ times

$$\begin{cases} P_{i,i+1} = \dfrac{M-i}{M} \\[2ex] P_{i,i-1} = \dfrac{i}{M} \end{cases}$$

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & & & 0 \\ * & 0 & * & & & & \\ & \ddots & 0 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & * & 0 & * \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

Irreducible MC

$$\boxed{\pi = \pi P}$$

if $|S| < +\infty$ and $(X_n)$ is irreducib.
it exists always a unique $\pi$

Exercise: compute the invariant distribution

Hint: consider the detailed balance equation

. . . $\pi_i = \binom{M}{i} \frac{1}{2^M} \Rightarrow \boxed{\pi \sim Bin(M, 1/2)}$

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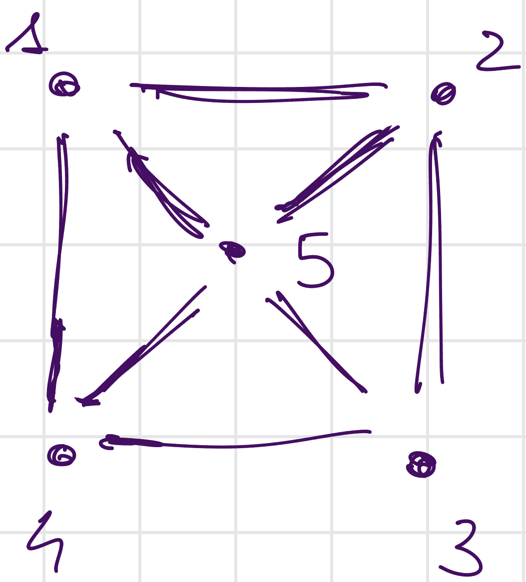
Example: Random Walk on indirect connected graphs.




$$P = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & \frac{1}{3} & 0 & 1/3 \\ 0 & 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{bmatrix} \end{array}$$

Irreducible $\pi = \pi P$

dbe $\underbrace{\pi_x P_{xy} = \pi_y P_{yx}}$ $\forall x \neq y$

$x_i$ = # of neighbours of the vertex $i$

$i = 1, 2, 3, 4$ $\qquad x_i = 3$

$x_5 = 4$

$$x = \sum_{i \in S} x_i = 3 + 3 + 3 + 3 + 4 = 16$$

$$\pi_i = \frac{x_i}{x}$$

$\pi = (\pi_1, \dots, \pi_5)$ is a distribution

$$\boxed{\pi_i}\, P_{ij} = \underbrace{\pi_j}\, P_{ji}$$

$$\frac{1}{x} = \frac{x_i}{x} \cdot {}^{1}/_{x_i} = \frac{x_j}{x}\, {}^{1}/_{x_j} = {}^{1}/_{x}$$

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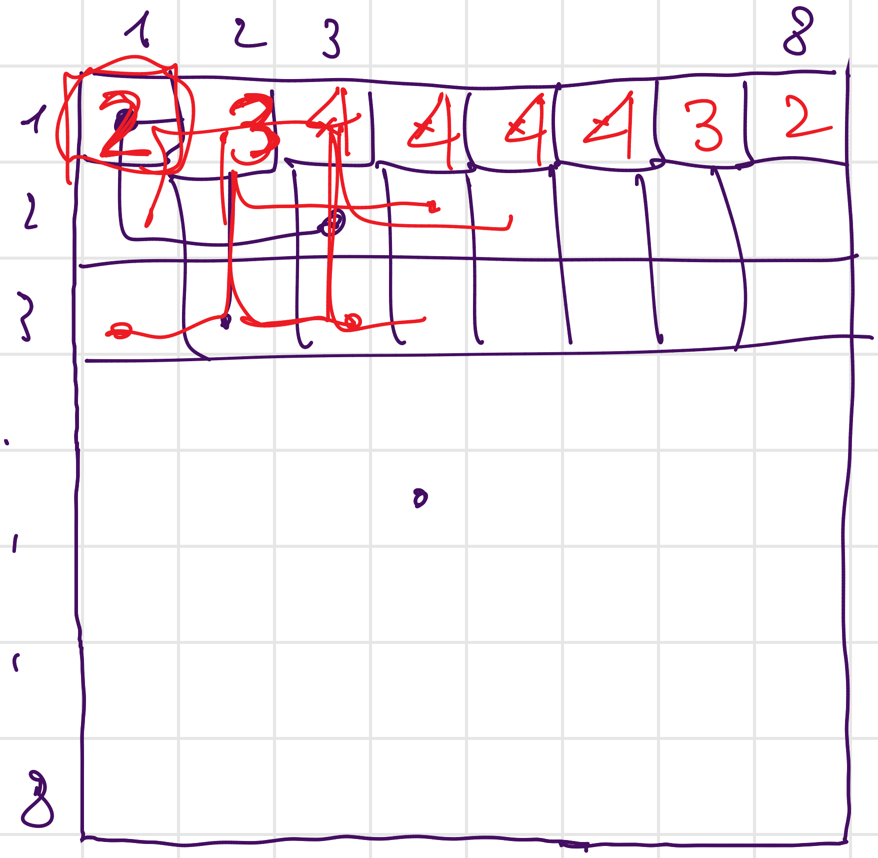
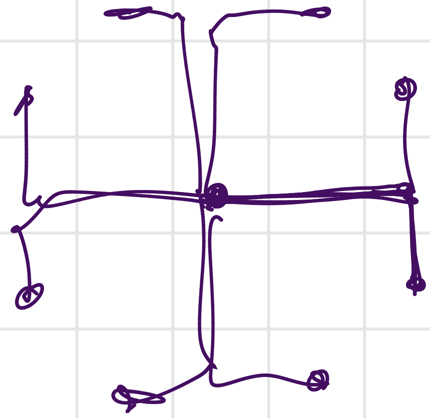
<u>Example</u>: A "random knight" (Chess)

(1,1)



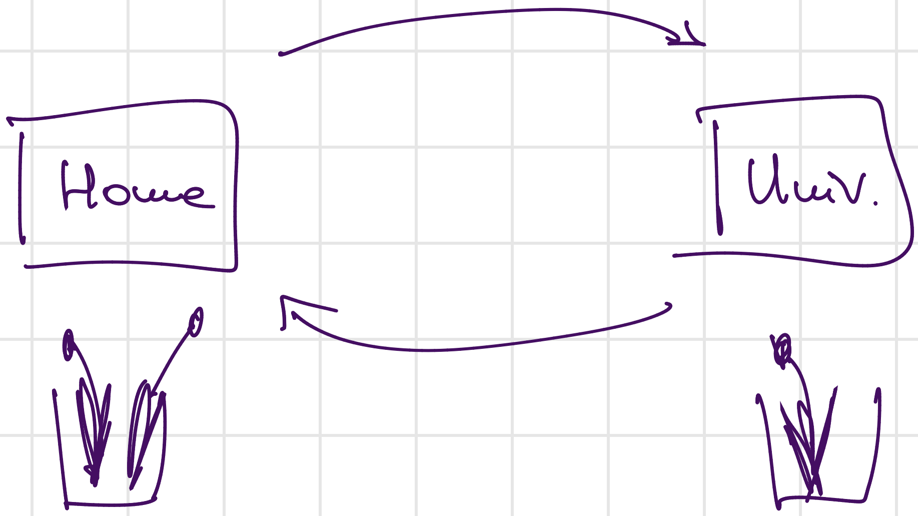
64 squares

If the knight is in a corner, how long it takes on average to return to the same corner?

$$\mathbb{E}\left[T_{(1,1)} \mid X_0 = (1,1)\right] =$$

$$= \underbrace{m_{(1,1)}} = \frac{1}{\underbrace{\pi_{(1,1)}}}$$

Example: Professor with M umbrellas.




$0 < p < 1$

prob. that it rains.

What is the long run proportion of journeys on which the professor gets wet?

Hint: $X_n$ = # umbrellas at his current location

$$S = \{0, 1, \dots, M\}$$

$$P_{i,j} \quad \forall i j$$

Compute the invariant dist $\pi$.

$$\pi_0 \cdot p$$

$p = 1/2$, $M = 10$ $\qquad \pi_0 \cdot p \approx 0.026$ $\boxed{2\%}$