PROBLEMS - SET 3

Problem 1. A telecommunication channel sends sequences of binary digits (0 or 1). Due to transmission noise the receiver may get some digit wrong.

(a) Suppose first each digit is changed in the transmission with probability 0.0002, independently of the other digits. Let X denote the number of changed digits in a transmission of 10000 digits. Compute (possibly with approximations) $\mathrm{E}(X)$, $\mathrm{Var}(X)$ and $\mathrm{P}(X \geq 3)$.

(b) Suppose instead that the noise in the channel increases with time, so that the i-th digit is received wrong with probability $0.0003 \cdot \left(1 - \exp\left[-\frac{i}{1000}\right]\right)$. Let X denote the number of changed digits in a transmission of 10000 digits. Compute (possibly with approximations) $\mathrm{E}(X)$, $\mathrm{Var}(X)$ and $\mathrm{P}(X \geq 3)$.
Hint: use the identity

$$\sum_{i=1}^{N} a^i = a\frac{a^N - 1}{a - 1}.$$

Solution 1. (a) Note that $X \sim \mathrm{Bin}(10000, 0.0002)$, so

$$\mathrm{E}(X) = 10000 \cdot 0.0002 = 2, \quad \mathrm{Var}(X) = 10000 \cdot 0.0002 \cdot 0.9998 \simeq 2.$$

The probability $\mathrm{P}(X \geq 3)$ can be computed using the Poisson approximation. Since $np^2 = 10000 \cdot (0.0002)^2 = 0.0004$ the approximation is quite good. So, setting $Y \sim \mathrm{Pois}(2)$,

$$\mathrm{P}(X \geq 3) \simeq \mathrm{P}(Y \geq 3) = 1 - \sum_{k=0}^{2} \mathrm{P}(Y = k) = 1 - e^{-2} - 2e^{-2} - 2e^{-2} = 1 - 5e^{-2}.$$

(b) In this case

$$X = \sum_{i=1}^{10000} X_i$$

where $X_i \sim \mathrm{Be}(0.0003 \cdot \left(1 - \exp\left[-\frac{i}{1000}\right]\right))$. Thus using the identity

$$\sum_{i=0}^{N} a^i = \frac{1 - a^{N+1}}{1 - a}$$

and so

$$\sum_{i=1}^{N} a^i = \left[\sum_{i=1}^{N} a^i\right] + 1 - 1 = \sum_{i=0}^{N} a^i - 1 = \frac{1 - a^{N+1}}{1 - a} = \frac{a - a^{N+1}}{1 - a} = a\frac{1 - a^N}{1 - a}$$

we get

$$\mathrm{E}(X) = \sum_{i=1}^{10000} 0.0003 \cdot \left(1 - \exp\left[-\frac{i}{1000}\right]\right) = 3 - 0.0003 \sum_{i=1}^{10000} \exp\left[-\frac{i}{1000}\right]$$
$$= 3 - 0.0003 \cdot \exp\left[-\frac{1}{1000}\right] \frac{1-e^{-10}}{1-\exp\left[-\frac{1}{1000}\right]} \simeq 2.7.$$

In the approximation we use the fact that $1-\exp\left[-\frac{1}{1000}\right] \simeq 1000$. Indeed for $t \simeq 0$ $\exp t \simeq 1+t$ and we apply this for $t = -1/1000$.

$$\mathrm{Var}(X) = \sum_{i=1}^{10000} 0.0003 \cdot \left(1 - \exp\left[-\frac{i}{1000}\right]\right)\left[1 - 0.0003 \cdot \left(1 - \exp\left[-\frac{i}{1000}\right]\right)\right]$$
$$\simeq \sum_{i=1}^{10000} 0.0003 \cdot \left(1 - \exp\left[-\frac{i}{1000}\right]\right) \simeq 2.7.$$

To compute $\mathrm{P}(X \geq 3)$ we approximate X with $Y \sim \mathrm{Pois}(2.7)$, obtaining

$$\mathrm{P}(X \geq 3) \simeq \mathrm{P}(Y \geq 3) = 1 - \sum_{k=0}^{2} \mathrm{P}(Y=k) = 1 - e^{-2.7} - 2.7e^{-2.7} - \frac{(2.7)^2}{2} e^{-2.7}$$
$$= 1 - 5e^{-2} \simeq 0.506.$$

Problem 2. Let X, Z e W be independent random variables with $X \sim \mathrm{Be}(p)$ and $Z, W \sim \mathrm{Pois}(\lambda)$. Define $Y := XZ + W$.

(i) Determine the discrete densities of (X,Y) and Y.

(ii) Using p_Y obtained above, compute $\mathrm{E}(Y)$ e $\mathrm{Var}(Y)$.

(iii) Compute $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$ *without* using p_Y.

Solution 2. (i)

(ii) $p_{X,Y}(0,n) = P(X=0, Y=n) = P(X=0, W=n) = (1-p)e^{-\lambda}\frac{\lambda^n}{n!}$, $p_{X,Y}(1,n) = P(X=1, Y=n) = P(X=1, Z+W=n) = pe^{-2\lambda}\frac{(2\lambda)^n}{n!}$, since $Z+W = \mathrm{Pois}(2\lambda)$.

Therefore $p_Y(n) = p_{X,Y}(0,n) + p_{X,Y}(1,n) = pe^{-2\lambda}\frac{(2\lambda)^n}{n!} + (1-p)e^{-\lambda}\frac{\lambda^n}{n!}$.

(iii)

$$\mathrm{E}(Y) = \sum_{n=0}^{+\infty} n p_Y(n) = \sum_{n=0}^{+\infty} npe^{-2\lambda}\frac{(2\lambda)^n}{n!} + \sum_{n=0}^{+\infty} n(1-p)e^{-\lambda}\frac{\lambda^n}{n!}$$
$$= \sum_{k=0}^{+\infty} pe^{-2\lambda}\frac{(2\lambda)^{k+1}}{k!} + \sum_{k=0}^{+\infty}(1-p)e^{-\lambda}\frac{\lambda^{k+1}}{k!} =$$
$$= pe^{-2\lambda}(2\lambda)\sum_{k=0}^{+\infty} pe^{-2\lambda}\frac{(2\lambda)^k}{k!} + (1-p)e^{-\lambda}\lambda\sum_{k=0}^{+\infty}\frac{\lambda^k}{k!} = p \cdot 2\lambda + (1-p)\cdot\lambda = \lambda(p+1).$$

$$\mathrm{E}(Y^2) = \sum_{n=0}^{+\infty} n^2 p_Y(n) = \sum_{n=0}^{+\infty} n^2 pe^{-2\lambda}\frac{(2\lambda)^n}{n!} + \sum_{n=0}^{+\infty} n^2(1-p)e^{-\lambda}\frac{\lambda^n}{n!} = p\cdot[(2\lambda)^2 + (2\lambda)] + (1-p)\cdot(\lambda^2+\lambda)$$

so $\mathrm{Var}(Y) = \lambda^2(p-p^2) - \lambda(1+p)$.

(iv) $\mathrm{E}(Y) = \mathrm{E}(X)\,\mathrm{E}(Z) + \mathrm{E}(W) = p\lambda + \lambda$, $\mathrm{E}(Y^2) = \mathrm{E}(X^2)\,\mathrm{E}(Z^2) + \mathrm{E}(W^2) + 2\,\mathrm{E}(x)\,\mathrm{E}(W)\,\mathrm{E}(Z) = p(\lambda^2+\lambda) + (\lambda^2+\lambda) + 2p\lambda^2$, so $\mathrm{Var}(Y) = \mathrm{E}(Y^2) - (\mathrm{E}(Y))^2$.

Problem 3. For given $p \in (0,1)$ and $n \geq 2$, let $Z_1, \ldots, Z_n$ be independent random variables with values in $\{-1,1\}$, with $\mathrm{P}(Z_i = 1) = p$ for all $i = 1, \ldots, n$. Define

$$X := \prod_{i=1}^{n} Z_i = Z_1 \cdot Z_2 \cdots Z_n \,.$$

(i) Determine the distribution of X.

(ii) Show that X is independent of the random vector $(Z_2, \ldots, Z_n)$ if and only if $p = \frac{1}{2}$.

Solution 3. (i) Note that

$$\mathrm{E}(X) = \prod_{i=1}^{n} \mathrm{E}(Z_i) = \mathrm{E}(Z_1)^n = (p \cdot 1 + (1-p)\cdot(-1))^n = (2p-1)^n.$$

Since $\mathrm{E}(X) = \mathrm{P}(X=1) - \mathrm{P}(X=-1) = 2\,\mathrm{P}(X=1) - 1$, we have $\mathrm{P}(X = \pm 1) = \frac{1 \pm \mathrm{E}(X)}{2}$ so

$$\mathrm{P}(X = \pm 1) = \frac{1 \pm (2p-1)^n}{2}.$$

(ii) If $p = \frac{1}{2}$ we have

$$\mathrm{P}(X = \pm 1 | Z_2 = t_2, \ldots, Z_{n-1} = t_{n-1}) = \mathrm{P}(Z_1 = \pm\,\mathrm{sign}(t_2 \cdot t_3 \cdots t_{n-1})) = \frac{1}{2} = \mathrm{P}(X = \pm 1)\,,$$

so X e $(Z_2, \ldots, Z_n)$ are independent. If instead $p \neq \frac{1}{2}$

$$\mathrm{P}(X = 1 | Z_2 = 1, \ldots, Z_n = 1) = \mathrm{P}(Z_1 = 1) = p \neq \frac{1}{2}(1 + (2p-1)^n) = \mathrm{P}(X = 1)\,.$$

Problem 4. Let X be a point uniformly chosen in the interval $[0,2]$. What is the probability that the area of the equilateral triangle of side X is greater than 1?

Solution 4. The area of the equilateral triangle of side X is $A := \frac{\sqrt{3}}{4}X^2$, so

$$\begin{aligned}
\mathrm{P}(A > 1) &= \mathrm{P}\left(X^2 > \frac{4}{\sqrt{3}}\right) = \mathrm{P}\left(X \in \left(-\infty, -\frac{2}{(3)^{1/4}}\right) \cup \left(\frac{2}{(3)^{1/4}}, +\infty\right)\right) \\
&= \int_{(-\infty, -\frac{2}{(3)^{1/4}}) \cup (\frac{2}{(3)^{1/4}}, +\infty)} f_X(x)\,\mathrm{d}x \\
&= \int_{(-\infty, -\frac{2}{(3)^{1/4}}) \cup (\frac{2}{(3)^{1/4}}, +\infty)} \frac{1}{2}\,\mathbb{1}_{(0,2)}(x)\,\mathrm{d}x \\
&= \frac{1}{2}\left(2 - \frac{2}{(3)^{1/4}}\right) = 1 - 3^{-1/4} \simeq 0.24\,.
\end{aligned}$$

Problem 5. Let $X \sim \mathrm{U}(0,1)$ and $Y := 4X(1-X)$. Compute the distribution function F_Y of Y, show that Y is absolutely continuous and compute its density

Solution 5. Clearly $F_Y(y) = 0$ if $y < 0$ while $F_Y(y) = 1$ if $y > 1$, since, for all $X \in [0,1]$, $4X(1-X) \in [0,1]$. Thus it is enough to consider the case $0 \leq y \leq 1$. In this case, the inequality $4x(1-x) \leq y$ has solution $x \leq \frac{1}{2}(1-\sqrt{1-y})$ or $x \geq \frac{1}{2}(1+\sqrt{1-y})$, thus

$$F_Y(y) \;=\; \mathrm{P}(Y \leq y) = \mathrm{P}\left(X \leq \frac{1}{2}(1-\sqrt{1-y})\right) + \mathrm{P}\left(X \geq \frac{1}{2}(1+\sqrt{1-y})\right).$$

Note that $\mathrm{P}(X \leq x) = x$ and $\mathrm{P}(X \geq x) = 1-x$ if $x \in [0,1]$. Since the values $\frac{1}{2}(1 \pm \sqrt{1-y})$ are in $[0,1]$ for $y \in [0,1]$, we have

$$F_Y(y) \;=\; 1 - \frac{1}{2}(1+\sqrt{1-y}) + \frac{1}{2}(1-\sqrt{1-y}) \;=\; 1-\sqrt{1-y}.$$

Finally,

$$F_Y(y) \;=\; \begin{cases} 0 & \text{se } y < 0 \\ 1-\sqrt{1-y} & \text{se } y \in [0,1] \\ 1 & \text{se } y > 1 \end{cases}.$$

Taking the derivative we get

$$f_Y(y) \;=\; F_Y'(y) \;=\; \frac{1}{2}\,\frac{1}{\sqrt{1-y}}\,1_{(0,1)}(y),$$

and it is easily checked that

$$F_Y(y) = \int_{-\infty}^{y} f_Y(u)du.$$

Problem 6. Let X be a point uniformly chosen in the interval $[0,4]$. Moreover let Q be the square centered in the origin whose side has length X. Compute the probability that Q is contained in the unit circle, i.e. the circle centered in the origin and with radius 1.

Solution 6. Q is contained in the unit circle if and only if the lenght of its half diagonal, $X/\sqrt{2}$ is less than 1. So the required probability is

$$P(X < \sqrt{2}) = \frac{\sqrt{2}}{4}.$$

Problem 7. Consider the random variables $X \sim \mathrm{Be}(p)$, $Y \sim \mathrm{Exp}(\lambda)$, and assume they are independent. Set $Z := X+Y$. Compute the distribution function F_Z of Z. Is Z an absolutely continuous random variable?

Solution 7.

$$\begin{aligned}
F_Z(z) &= \mathrm{P}(X+Y \le z) = \mathrm{P}(Y \le z, X=0) + \mathrm{P}(Y \le z-1, X=1) \\
&= \mathrm{P}(Y \le z)\,\mathrm{P}(X=0) + \mathrm{P}(Y \le z-1)\,\mathrm{P}(X=1) = \\
&= (1-p)\left(1-e^{-\lambda z}\right)\mathbb{1}_{[0,+\infty)}(z) + p\left(1-e^{-\lambda(z-1)}\right)\mathbb{1}_{[1,+\infty)}(z) \\
&= \begin{cases} (1-p)\left(1-e^{-\lambda z}\right) + p\left(1-e^{-\lambda(z-1)}\right) & \text{for } z \ge 1 \\ (1-p)\left(1-e^{-\lambda z}\right) & \text{for } 0 \le z < 1 \\ 0 & \text{for } z < 0 \end{cases}
\end{aligned}$$

Note that F_Z is continuous, and it is continuously differentiable except at $z = 0, 1$, so $F_Z(z) = \int_{-\infty}^{z} F_Z'(x)dx$, i.e. Z is absolutely continuous with density F_Z'.