Lecture 11 Stoch. Meth. 4th Nov, 2024

Random Vectors ...

Discrete R.V.'s

Binomials r.v. $Bin(n, p)$

$n = 1$ $\underbrace{Bin(1,p)}_{}$, $\underbrace{Bernoulli(p)}_{}$

win

$\Omega = \underbrace{\{b_1, b_2\}}$ $\mathcal{A} = 2^{\Omega}$

$\mathbb{P}[\{b_1\}] = p \in [0,1]$, $\mathbb{P}[\{b_2\}] = 1-p$

$X : \Omega \to \boxed{\mathbb{R}}$

$X(b_1) = \underline{1}$, $X(b_2) = \underline{0}$

$X \sim Bin(1,p)$ random variable μ_X:

$N = \{0,1\}$ $p_X(0) = 1-p$, $p_X(1) = p$

$y \in \{1, -1\}$ $\qquad$ $y(b_1) = \underbrace{1}$
$y(b_2) = \underbrace{-1}$

$y \not\sim Bin(1, p)$

$X \sim Bin(1, p) \rightsquigarrow y = f(X)$

$$\boxed{X(b_1) = \underbrace{1}, \quad X(b_2) = \underbrace{0}}$$

$y = ax + b$

linear trans.

b_1
$$\begin{cases} 1 = a \cdot 1 + b \\ -1 = a \cdot 0 + b \end{cases}$$
b_2

$$\begin{cases} a + b = 1 \\ b = -1 \end{cases} \Rightarrow \begin{cases} a - 1 = 1 \\ b = -1 \end{cases} \Rightarrow \begin{cases} a = 2 \\ b = -1 \end{cases}$$

$$X \rightsquigarrow \boxed{y = 2X - 1}$$

$X \sim \mathrm{Bin}(n,p)$

$$\Omega = \{ \omega = (\omega_1, \dots, \omega_n) : \omega_i \in \{b_1, b_2\},\ i=1,\dots,n \}$$

$$= \{b_1, b_2\}^n, \quad \mathcal{A} = 2^{\Omega}$$

Defini $A_i = \{ \omega \in \Omega : \omega_i = b_1 \}$

$$\mathbb{P}[A_i] = p \in [0,1], \quad \mathbb{P}[A_i^c] = 1-p$$

Note that any $\omega \in \Omega$ can be written as

$$\underbrace{A_1^* \cap A_2^* \cap \dots \cap A_n^*}$$

$$A_i^* = \begin{cases} A_i \\ A_i^c \end{cases}$$

For example

$$\boxed{\omega = (b_1, b_1, \dots, b_1)} \iff \boxed{\underbrace{A_1} \cap \underbrace{A_2} \cap \dots \cap \underbrace{A_n}}$$

$$(b_1, b_2, \dots, b_2) \iff A_1 \cap A_2^c \cap A_3^c \cap \dots \cap A_n^c$$

$$\boxed{\mathbb{P}[\{\omega\}] = p^k (1-p)^{n-k}}$$

$k = \# \{ i : \underbrace{A_i^* = A_i} \}, \quad n-k := \# \{ i : A_i^* = A_i^c \}$

$X \sim \mathrm{Bin}(n,p)$ $\qquad\qquad$ $\omega \in \Omega$

$X(\underbrace{\omega}) :=$ the number of outcomes equal to b_1

$= \# \{ i : \omega_i = b_1 \}$

$X : \Omega \to \mathbb{R}$ $\qquad$ $\underbrace{N = \{0, 1, 2, \dots, n\}}$

$X((b_2, \dots, b_2)) = 0$

$X((b_1, b_2, \dots, b_2)) = 1$ $\qquad$ $X \sim \mathrm{Bin}(n,p)$

$X((b_1, b_1, b_2, \dots, b_2)) = 2$

$\vdots$

$X((b_1, b_1, \dots, b_1)) = n$ $\qquad$ $\forall k \in N$

$$\boxed{\mathbb{P}[X = k] = \binom{n}{k} \underbrace{p^k (1-p)^{n-k}}}$$

$(b_1, \dots, b_1, b_2, \dots, b_2)$ —

$b_1, \dots, b_1$ $\qquad$ $\binom{n}{k}$

2 $\quad$ 3 $\quad$ 50

1 $\quad$ 2 $\quad$ 3 $\quad$ . . . $\quad$ n

$$\mathbb{E}[X] = \boxed{n \cdot p}$$

$$n^2 p^2$$

$$\mathrm{Var}[X] = \underbrace{\mathbb{E}[X^2]}_{\displaystyle \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}} - \left(\mathbb{E}[X]\right)^2$$

$X \sim \mathrm{Bin}(n,p)$, $\exists\, X_1, \ldots, X_n \sim \mathrm{Bin}(1,p)$ independent s.t.

$$\boxed{X = X_1 + X_2 + \cdots + X_n}$$

linearity of the $\mathbb{E}[\cdot]$

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \underbrace{\mathbb{E}[X_i]}_{\displaystyle \parallel\!\!\!\,\, p}$$

$$= p\left(\sum_{i=1}^{n} 1\right) = n \cdot p$$

indep.

$$\boxed{\mathrm{Var}[X]} = \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}[X_i] = \boxed{= n \cdot p(1-p)}$$

$$\mathrm{Var}[X_i] = \mathbb{E}[X_i^2] - \underbrace{\left(\mathbb{E}[X_i]\right)^2}_{= p^2} = p - p^2 = p(1-p)$$

$$\mathbb{E}[X_i^2] = 1^2 \cdot p + 0^2 (1-p) = p$$

Geometric r.v. with parameter $p \in [0,1]$

$$\Omega = \{ \omega = (\omega_1, \dots, \omega_n, \dots) \,,\ \omega_i \in \{b_1, b_2\} \}$$

Ω is more than countable $\quad \Omega \approx \underline{(0,1]}$

$\underline{\mathcal{A} = \cancel{2^{\Omega}}}$ = minimal σ-field that contains all the cylindric sets

$$X : \Omega \longrightarrow \mathbb{N}_+ = \{1, 2, \dots, n, \dots\}$$

$$X(\omega) := \underline{\inf} \{ n \geq 1 : \omega_n = b_1 \}$$

$X = 5 \Rightarrow$ 4 unsuccesses before the first success.

$$\omega = (b_2, b_2, b_2, b_2, b_1, \underline{\cdot \longrightarrow})$$

Remark: $\omega = (b_2, b_2, \dots, b_2, \dots)$

$\{ n \geq 1 : \omega_n = b_1 \} = \emptyset \qquad \boxed{\inf \emptyset = +\infty}$

$k \in \mathbb{N}_+$

$$\boxed{\mathbb{P}[X=k] = (1-p)^{k-1} \cdot p}$$

discrete density

$$\omega = (\underbrace{b_2, \dots, b_2}_{\substack{k-1 \\ \text{positions}}}, \underset{\substack{k\text{th} \\ \text{position}}}{b_1}, \_\_\_\_\_\_)$$

Geo(p) random variable

$$\mathbb{E}[X] = 1/p, \quad Var[X] = \frac{1-p}{p^2}$$

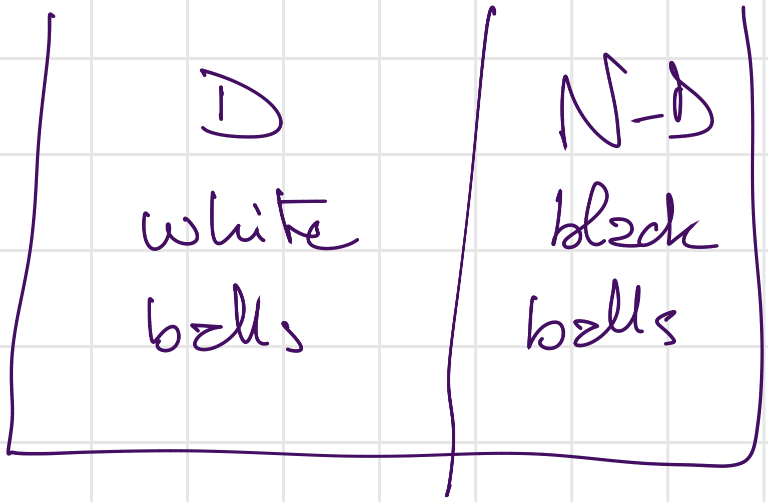
$Y \sim Geo(p)$ $\quad Y \in \mathbb{N}$

$Y = 0 \iff$ the first b_1 in the first position

$X = Y + 1$

$$\mathbb{E}[Y] = \mathbb{E}[X] - 1$$
$$= \frac{1}{p} - 1$$

$Y \sim$ Hypergeometric r.v.



draw n balls without replacement

and Y = # of white balls

$$N \in \mathbb{N}, \quad D \in \mathbb{N}, \quad D \le N$$

$$n \le N \qquad Y \in \underbrace{\{0, 1, \ldots, \underline{\underline{n}}\}}$$

$$\mathcal{D} = \{ k \in \mathbb{N} : \underbrace{0 \vee (n-N+D)}_{\max(0,\, n-N+D)} \le k \le \underbrace{n \wedge D}_{\min\{n,D\}} \}$$

$$\underbrace{0 \le k} \le \begin{matrix} n \\ D \end{matrix}$$

$$0 \le n-k \le N-D$$

$$n-k \le N-D \Rightarrow \boxed{k \ge n-N+D}$$

.... $\mathbb{P}[Y=k]$, $\mathbb{E}[Y] = ?$