PROBLEMS - SET 7

Problem 1. A Markov Chain with a finite number of states is said to be **regular** if there exists a non negative integer $\overline{n}$ such that for any $i,j \in S$, $P^n_{i,j} > 0$ for any $n \geq \overline{n}$.

(i) Prove that a regular Markov Chain is irreducible.
(ii) Prove that if a Markov Chain is irreducible and there exists $k \in S$ such that $P_{k,k} > 0$, then it is regular.
(iii) Find an example of an irreducible Markov chain which is not regular.

Solution 1.

(i) The fact that $P^n_{i,j} > 0$ for any i,j at some time n, implies that all the states communicate.
(ii) By irreducibility, we have that for any pair $i,j \in S$, there exists $m(i,j) \in \mathbb{N}$ such that $P^{m(i,j)}_{i,j} > 0$. Let $m = \max_{i,j\in S}\{m_{i,j}\}$; since S is finite, m is finite. For any pair of states i,j and every $n \geq 2m+1$, we get

$$P^n_{i,j} = \sum_{r\in S} P^{m(i,r)}_{i,r} P^{n-m(i,r)}_{r,j} = \sum_{r\in S, q\in S} P^{m(i,r)}_{i,r} P^{n-m(i,r)-m(q,j)}_{r,q} P^{m(q,j)}_{q,j}$$
$$\geq P^{m(i,k)}_{i,r} P^{n-m(i,k)-m(k,j)}_{k,k} P^{m(k,j)}_{k,j} > 0$$

since $n - m(i,k) - m(k,j) \geq 2m+1 - m(i,k) - m(k,j) 2m+1-m-m = 1$ and $P^l_{k,k} \geq (P_{k.k})^l > 0$ for every $l \geq 1$. So the MC is regular.
(iii) The two states MC with transition matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is irreducible, but $P^{2n+1}_{0,0} = 0$ for any $n \in \mathbb{N}$.

Problem 2. Let $\{S_n, n \geq 0\}$ be a symmetric Random Walk on $\mathbb{Z}$. Defined $T_0 = \inf\{n \geq 1 : S_n = 0\}$ the time of first passage to state 0, prove that

$$\mathbb{P}[T_0 = 2n | S_0 = 0] = \frac{1}{2n-1} \binom{2n}{n} \left(\frac{1}{2}\right)^{2n}$$

for any $n \geq 1$.

Solution 2. Any path that goes from 0 to 0 in exactly $2n$ steps has probability $\left(\frac{1}{2}\right)^{2n}$. So, in order to evaluate $\mathbb{P}[T_0 = 2n|S_0 = 0]$ we have to determine how many of these paths do not reach 0 before the step $2n$. This number is equal to twice the number of paths that go from 0 to 1 in $2n-1$ steps without returning to 0.

Let A be the set of all the paths that go from 0 to 1 in $2n-1$ steps. To reach 1 from 0 you should go up (+1) n times and down (-1) $n-1$ times. So in total the cardinality of A is

$$\binom{2n-1}{n}$$

and is equal to the disjoint union of the sets B, C and D, where

$$B = \{\text{paths from 0 to 1 in } 2n-1 \text{ steps not returning to } 0\}$$

$$C = \{\text{paths not in } B \text{ such that } S_1 = 1\}$$

$$D = \{\text{paths not in } B \text{ such that } S_1 = -1\}.$$

Note that both D and C contains only paths which pass through 0.

By the reflection principle, the sets C and D have the same cardinality. Indeed, if a fix a path in C this path satisfies $S_0 = 0, S_1 = 1$ and $S_{2n-1} = 1$ and moreover $S_k = 0$ for some $k \in \{2, \dots, 2n-2\}$. Let $\bar{k}$ the first number between 2 and $2n-2$ such that $S_{\bar{k}} = 0$. So, I have that $S_i > 0$ for every $i \in \{1, \dots, \bar{k}-1\}$. I consider the path obtained by reflecting this over the first axis: therefore a path with has $\tilde{S}_0 = 0$, $\tilde{S}_1 = -1$, $\tilde{S}_{\bar{k}} = 0$ and $\tilde{S}_i < 0$ for all $i \in \{1, \dots, \bar{k}-1\}$. So the path which $(\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_{\bar{k}}, S_{\bar{k}+1}, \dots, S_{2n-1})$ is a path in D, since $S_{2n-1} = 1$ and $\tilde{S}_1 = -1$. If we fix a path on D we proceed in the same way: we fix the first time of passage through 0, and we reflect this first part of the path, so we attach the other part of the path and we end up with a path in C.

Note that in D the constraint of passing through 0 is always satisfied, so D coincides with the set of all the paths that goes from -1 to 1 in $2n-2$ steps (so $n-2$ times down and n times up), its cardinality is equal to

$$\binom{2n-2}{n-2} \quad .$$

Therefore the cardinality of the set B will be equal to

$$\binom{2n-1}{n} - 2 \cdot \binom{2n-2}{n-2} = \frac{(2n-2)!}{n!\,(n-1)!}$$

Then

$$\mathbb{P}[T_0 = 2n | S_0 = 0] = 2\, \frac{(2n-2)!}{n!\,(n-1)!}\, \left(\frac{1}{2}\right)^{2n} = \frac{1}{2n-1} \binom{2n}{n} \left(\frac{1}{2}\right)^{2n}.$$

Problem 3. Let $(X_n)_{n \geq 0}$ be a MC with state space S and unique limiting distribution π. Define the process $(Y_n)_{n \geq 1} = (X_n, X_{n-1})$, which has state space $S \times S$. Prove that Y_n is a MC, determine its transition matrix, and find the limiting probabilities

$$\pi_{(i,j)} = \lim_{n \to \infty} \mathbb{P}(Y_n = (i,j)).$$

Solution 3. To show the Markov property for Y_n, we compute

$$\begin{aligned}&\mathbb{P}\big(Y_{n+1} = (i', j') | Y_n = (i_n, j_n), \dots, Y_1 = (i_1, j_1)\big) \\ &\quad = \mathbb{P}\big(X_{n+1} = i', X_n = j' | X_n = i, X_{n-1} = j, X_{n-1} = i_{n-1}, X_{n-2} = j_{n-1}, X_{n-2} = i_{n-2}, \dots, X_0 = j_1\big)\end{aligned}$$

This expression is zero unless

$$j' = i_n, j_n = i_{n-1}, j_{n-1} = i_{n-2}, \dots, j_2 = i_1,$$

and if these conditions are satisfies, by the Markov property for X_n, it equals

$$\mathbb{P}\big(X_{n+1} = i' | X_n = i_n\big) = P_{i_n,i'},$$

which is independent of $(i_{n-1}, j_{n-1}), \dots, (i_1, j_1)$. This proves the Markov property. It also shows that

$$\begin{aligned} P_{(i,j),(i',j')} &= \mathbb{P}\big(Y_{n+1} = (i',j') | Y_n = (i,j)\big) \\ &= \begin{cases} P_{i,i'} & j' = i \\ 0 & \text{else} \end{cases} \end{aligned}$$

which is independent of the time n. Thus Y_n is a MC with the above transition matrix. For the limiting distribution, since if it exists it is the stationary distribution, we need to solve

$$\begin{aligned} \pi_{(i,j)} &= \sum_{(i',j')} \pi_{(i',j')} P_{(i',j'),(i,j)} = \\ &= \sum_{j'} \pi_{(j,j')} P_{ji}. \end{aligned}$$

We make the intuitive ansatz $\pi_{(i,j)} = \pi_j P_{ji}$, where π_i is the limiting distribution of X_n. Then

$$\sum_{j'} \pi_{(j,j')} P_{ji} = \sum_{j'} \pi_{j'} P_{j'j} P_{ji} = \pi_j P_{ji} = \pi_{(i,j)}$$

does indeed solve the eigenvector equation for the limiting distribution, and since

$$\sum_{(i,j)} \pi_j P_{ji} = \sum_j \pi_j \sum_i P_{ji} = \sum_j \pi_j = 1,$$

it is also correctly normalized. We have thus found the limiting distribution.

Problem 4. A Queen moves randomly on an otherwise empty chessboard, with each step being uniformly selected from all possible moves (a Queen is allowed to move to any position in the same row, same column, or on one of the diagonals that intersect her current position). What is the long term proportion of time the Queen spends on the 28 “border squares” of the chessboard?

Solution 4. This is an example of a random walk on a connected finite graph: the vertices V are the squares of the chessboard and the edges are the moves that the Queen can take. So the long term proportion of time the Queen spends on one square of the chessboard is equal to the value of the invariant distribution π at that square, since $\lim_n \frac{1}{n}\mathbb{E}\sum_{k=0}^{n} f(X_k) = \sum_{v\in V} f(v)\pi_v$ for every $f : V \to \mathbb{R}$ (so taking f to be the function which is 1 on a fixed vertex and 0 elsewhere we get exactly this property)

The value of the invariant distribution at one square is equal to the valency of that square, i.e the number of squares that the Queen can reach in one move starting from that square, divided by the sum of the valencies of all the squares in the chessboard (which is equal to twice the number of edges). Indeed let $\deg(v)$ to be the valency of a square v. Then the distribution given by π where $\pi_v = \deg(v)/\sum_{w\in V}\deg w$ is the invariant distribution. We check it. First of all $\sum_{v\in V}\pi_v = 1$ by definition and moreover if P is the transition matrix note that $P_{v,w} = \frac{1}{\deg(v)}$ if there is a edge from v to w (so if the Queen can move in one step from v to w- the probability for the Queen to move from v to w is 1 over all possible moves of the Queen from v) and 0 otherwise. Therefore, since the graph is symmetric, $P_{v,w} > 0$ implies $P_{w,v} > 0$, and we get, for all $w \in V$ fixed

$$(\pi P)_w = \sum_{v\in V} \pi_v P_{v,w} = \frac{\sum_{v\in V, P_{v,w}>0} \deg(v)\frac{1}{\deg(v)}}{\sum_{z\in V}\deg z} = \frac{\sum_{v\in V, P_{w,v}>0} 1}{\sum_{z\in V}\deg z} = \frac{\deg(w)}{\sum_z \deg z} = \pi_w.$$

Let us denote by (i, j) the squares of the chessboard, with $i, j \in \{1,\dots,8\}$. A simple computation gives that:

the valency of each of the 28 squares of the first ring R_1, where $R_1 = \{(i, j) \in \{1,\dots,8\}^2 : i \in \{1,8\}$ or $j \in \{1,8\}\}$, is 21;

that of each of the 20 squares in the second ring R_2,where $R_2 = \{(i, j) \in \{2,\dots,7\}^2 : i \in \{2,7\}$ or $j \in \{2,7\}\}$, is 23;

that of each of the 12 squares in the third ring R_3, where $R_3 = \{(i, j) \in \{3,\dots,6\}^2 : i \in \{3,6\}$ or $j \in \{3,6\}\}$, is 25;

that of each of the 4 squares in the inner ring R_4, where $R_4 = \{4,5\}^2$, is 27.

So the sum of all the valencies is

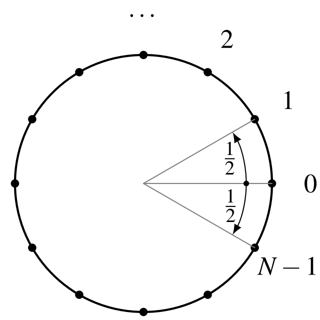
$$28\cdot 21 + 20\cdot 23 + 12\cdot 25 + 4\cdot 27 = 1456$$

and the long term proportion of time the Queen spends on the 28 squares of R_1 is equal to

$$\sum_{(i,j)\in R_1} \frac{21}{1456} = \frac{21}{1456}\cdot 28 = \frac{588}{1456} = 0.4038$$

(being the same long term proportion for the second, third and fourth rings equal to 0.3159, 0.2020 and 0.0741, respectively).

Problem 5. Divide the circle into N points, and consider a particle randomly moving between these points, in such a way that the position of the particle is a Markov Chain, and at each step the particle jumps to a neighboring point with equal probabilities:



Compute $\mathbb{P}(X_n = 0|X_0 = 0)$ for all $n \geq 0$. *Hint:* Use the discrete Fourier transform to diagonalize the transition matrix. That is, express the transition matrix in the orthonormal basis $\hat{e}^{(0)}, \hat{e}^{(1)}, \ldots, \hat{e}^{(N-1)}$ of $\mathbb{C}^N$ which has the following components in the standard basis: $\hat{e}^{(k)}_j = \frac{1}{\sqrt{N}} e^{\frac{2\pi i}{N} jk}$, ($j = 0, \ldots, N-1$, $i = \sqrt{-1}$).

Solution 5. Note that $\mathbb{P}(X_n = 0|X_0 = 0) = P^n_{0,0}$.

The transition matrix is

$$P_{ij} = \begin{cases} \frac{1}{2} & j = i \pm 1 \bmod N \\ 0 & \text{else} \end{cases}$$

Note that this is a symmetric matrix. We compute

$$\begin{aligned} \left(P \cdot \hat{e}^{(k)}\right)_j &= \sum_i P_{j,i} \hat{e}^{(k)}_i = \frac{1}{2}\hat{e}^{(k)}_{j+1} + \frac{1}{2}\hat{e}^{(k)}_{j-1} \\ &= \frac{1}{2\sqrt{N}} \cdot \left[e^{\frac{2\pi i}{N}(j+1)k} + e^{\frac{2\pi i}{N}(j-1)k} \right] = \left[\frac{e^{\frac{2\pi i}{N}k} + e^{-\frac{2\pi i}{N}k}}{2} \right] \frac{e^{\frac{2\pi i}{N}jk}}{\sqrt{N}}. \end{aligned}$$

Above, we don't have to write $(j+1) \bmod N$, because $e^{\frac{2\pi i}{N}jk}$ is N periodic in j whenever $k \in \mathbb{Z}$. Thus

$$P \cdot \hat{e}^{(k)} = \cos \tfrac{2\pi k}{N} \cdot \hat{e}^{(k)}$$

is an eigenvector of P. Since the $\hat{e}^{(k)}$ are an orthonormal basis, we have that the matrix O given by $O_{ij} = \hat{e}^{(j)}_i$ satisfies $O^T O = id$ and $O^T P O = \text{diag}\ (\cos \frac{2\pi i}{N})$, where $\text{diag}\ (\cos \frac{2\pi i}{N})$ is the diagonal matrix D with $D_{i,i} = \cos \frac{2\pi i}{N}$. Therefore using both these properties we get $O^T P^n O = \text{diag}\ (\cos^n \frac{2\pi i}{N})$ which implies

$$P^n = O \text{diag}\ (\cos^n \tfrac{2\pi i}{N}) O^T$$

from which we compute

$$P^n_{00} = \sum_{k=0}^{N-1} \left(\hat{e}^{(k)}_0\right)^2 \cdot \left[\cos \tfrac{2\pi k}{N}\right]^n = \frac{1}{N} \sum_{k=0}^{N-1} \left[\cos \tfrac{2\pi k}{N}\right]^n.$$

Problem 6. When I listen to music on my smartphone, the app decides to repeat a song with probability $\alpha \in (0,1)$, or, if it doesn't repeat, to randomly select one of the other $N-1$ songs and play that instead. Find the limiting probability

$$\lim_{n\to\infty} \mathbb{P}(n\text{th song is the same as the first song}).$$

Hint: Define a suitable two state Markov Chain with states $\{\text{initial song, other song}\}$.

Solution 6. Given the two states $\{\text{initial song, other song}\}$, the transition probabilities of the two states Markov chain are given by the 2×2 matrix

$$P = \begin{bmatrix} \alpha & 1-\alpha \\ \frac{1-\alpha}{N-1} & 1-\frac{1-\alpha}{N-1} \end{bmatrix}$$

Note that $\mathbb{P}(n\text{th song is the same as the first song}) = P^n_{1,1}$.

All the elements in the transition matrix are positive, so the Markov chain is irreducible and aperiodic: this implies that $\lim_n \nu_0 P^n = \pi$, for every initial distribution ν_0. In particular this implies that, given $\nu_0 = (1,0)$

$$\lim \mathbb{P}(n\text{th song is the same as the first song}) = \lim_n P^n_{1,1} = \pi_1$$

where $\pi = (\pi_1, \pi_2)$ is the invariant distribution. We look for the invariant distribution, so we have to solve $\pi P = \pi$ along with $\pi_1 + \pi_2 = 1$.

$$\begin{cases} \alpha\pi_1 + \frac{(1-\alpha)}{N-1}\pi_2 = \pi_1 \\ \pi_1 + \pi_2 = 1 \end{cases} \qquad \begin{cases} \pi_1 = \frac{1}{N-1}\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases} \qquad \begin{cases} \pi_1 = \frac{1}{N} \\ \pi_2 = \frac{N-1}{N}. \end{cases}$$

Therefore

$$\lim \mathbb{P}(n\text{th song is the same as the first song}) = \lim_n P^n_{1,1} = \frac{1}{N}.$$

We could also compute explicitly the n–th transition probabilities for the two states Markov Chain with a recursive type argument.

Since $\mathbb{P}(n\text{th song is the same as the first song}) = \mathbb{P}(X_n = 1)$, we have

$$\begin{aligned} \mathbb{P}(X_n = 1) &= \mathbb{P}(X_n = 1|X_{n-1} = 1)\mathbb{P}(X_{n-1} = 1) + \mathbb{P}(X_n = 1|X_{n-1} = 2)\mathbb{P}(X_{n-1} = 2) \\ &= \mathbb{P}(X_n = 1|X_{n-1} = 1)\mathbb{P}(X_{n-1} = 1) + \mathbb{P}(X_n = 1|X_{n-1} = 2)(1-\mathbb{P}(X_{n-1} = 2)) \\ &= \alpha\mathbb{P}(X_{n-1} = 1) + \frac{1-\alpha}{N-1}(1-\mathbb{P}(X_{n-1} = 1)), \end{aligned}$$

Let denote $a_n = \mathbb{P}(X_n = 1)$, so the previous formula gives

$$a_{n+1} = \alpha a_n + \frac{1-\alpha}{N-1}(1-a_n) = a_n\left(\alpha - \frac{1-\alpha}{N-1}\right) + \frac{1-\alpha}{N-1} = \frac{N\alpha - 1}{N-1}a_n + \frac{1-\alpha}{N-1}$$

coupled with the initial condition $a_0 = \mathbb{P}(X_0 = 1) = 1$. If w solve recursively, we find

$$\begin{aligned}\mathbb{P}(X_n = 1) = a_n &= \left(\frac{N\alpha - 1}{N-1}\right)^n + \frac{1-\alpha}{N-1}\sum_{k=0}^{n-1}\left(\frac{N\alpha-1}{N-1}\right)^k \\ &= \left(\frac{N\alpha - 1}{N-1}\right)^n + \frac{1-\alpha}{N-1}\frac{1-\left(\frac{N\alpha-1}{N-1}\right)^n}{1-\frac{N\alpha-1}{N-1}} \\ &= \left(\frac{N\alpha - 1}{N-1}\right)^n + \frac{1}{N}\left(1-\left(\frac{N\alpha-1}{N-1}\right)^n\right) \\ &= \frac{1}{N} + \frac{N-1}{N}\left(\frac{\alpha N-1}{N-1}\right)^n\end{aligned}$$

Since $\left|\frac{\alpha N-1}{N-1}\right| < 1$, we get

$$\lim_{n\to\infty} \mathbb{P}(n\text{th song is the same as the first song}) = 1/N\ .$$