Lecture 24 Stoch. Meth. Dec. 13th, 2024

Martingales (discrete - time)

Markov Chains

$$\boxed{\mathbb{P}[X_{n+1} = x \mid X_n = y, X_{n-1} = i_{n-1}, \dots, X_0 = i_0] = \mathbb{P}[X_{n+1} = x \mid \boxed{X_n = y}]}$$

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}, \quad \mathbb{P}[B] > 0$$

Example: $\mathbb{P}[X_{n+1} = x \mid \boxed{X_n \geq 0}, \boxed{\cancel{X_{n-1} = i}}]$

$$\omega \longmapsto \mathbb{E}[X \mid Y(\omega)] \qquad \mathbb{P}[X_{n+1} = x \mid X_n \geq 0]$$

Martingales $\to$ $\boxed{\mathbb{E}[X \mid Y]}$ conditional expectation

$$\underbrace{X : \Omega \to \mathbb{R}} \leadsto \underbrace{\mathbb{E}[X] \in \mathbb{R}}$$

X

$=$

$E[X|Y]$

$=$

$E[X]$

$$\boxed{\mathbb{E}[X \mid Y] := h(Y)}$$

X a discrete r.v.

$$\Omega \xrightarrow{Y} \mathbb{R} \xrightarrow{h} \mathbb{R}$$

$$\mathbb{E}[X \mid Y](\omega)$$

$$\mathbb{E}[X] = \sum_x x \cdot p_X(x)$$

$$\underbrace{\mathbb{E}[X \mid Y = y]} = \sum_x x \; \underbrace{p_{X|Y}(x \mid y)} = h(y)$$

(X, Y) is a discrete r vector $\quad \underbrace{p_{X,Y}}(x,y)$ joint density

$p_{X,Y} \longrightarrow$ marginal density of Y $\quad p_Y(y) = \sum_x p_{X,Y}(x,y)$

$$\underbrace{p_{X|Y}(x \mid y)} = \mathbb{P}[\underbrace{X = x}_{A} \mid \underbrace{Y = y}_{B}] =$$

conditional density of X given $Y = y$

$$= \frac{\mathbb{P}[X = x, Y = y]}{\mathbb{P}[Y = y]}$$

$$= \frac{p_{X,Y}(x,y)}{\underbrace{p_Y(y)}} \qquad \forall y : p_Y(y) \neq 0$$

Fix y

$$\sum_x \underbrace{p_{X|Y}(x \mid y)} = 1 \, ,$$

Ex 1: X, Y $\perp\!\!\!\perp$ Binomial (n,p) r.v.'s

$$h(m) = \mathbb{E}[X \mid X+Y = m]$$

Sol:

$$\mathbb{E}[X \mid X+Y = m] = \sum_x x \cdot p_{X|X+Y}(x|m)$$

$$\mathbb{P}[X=k \mid X+Y=m]$$

$X, Y \sim Bin(n,p)$

$\to \boxed{0 \le m \le 2n}$

$$= \frac{\mathbb{P}[X=k, X+Y=m]}{\mathbb{P}[X+Y=m]}$$

$\boxed{0 \le k \le n}$

$\boxed{0 \le k \le m}$

$$= \frac{\mathbb{P}[X=k, Y=m-k]}{\mathbb{P}[X+Y=m]}$$

$\to \boxed{0 \le k \le \min(n,m)}$

$X, Y \sim Bin(n,p)$

$X+Y \sim Bin(2n,p)$

Remark:

$X \sim Bin(n,p)$

$Y \sim Bin(m,p)$

indep. $\Rightarrow X+Y \sim Bin(n+m,p)$

(prove with mgf)

$$= \frac{\binom{n}{k} \cancel{p^k} \cancel{(1-p)^{n-k}} \cdot \binom{n}{m-k} \cancel{p^{m-k}} \cancel{(1-p)^{n-m+k}}}{\binom{2n}{m} \cdot \cancel{p^m} \cancel{(1-p)^{2n-m}}}$$

$$\frac{P[X=k] \, \underbrace{P[Y=m-k]}}{P[X+Y=m]}$$

$k \leq m$

$$= \underbrace{\frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}} = P[X=k \mid X+Y=m]$$

$$\boxed{n \cdot \frac{D}{N}}$$

$N \quad D \quad n$

$$X \mid X+Y=m \sim Hyp(2n, n, m)$$

$$h(m) = E[X \mid X+Y=\underbrace{m}] = m \cdot \frac{n}{2n} = \frac{m}{2}$$

$$\underbrace{E[X \mid \underbrace{X+Y}]} = h(X+Y) = \underbrace{\frac{X+Y}{2}}$$

(X, Y) is abs. continuous r.v.

$$f_{X,Y}(x,y) \rightsquigarrow f_Y(y)$$

$$\underbrace{f_{X|Y}(x|y)} := \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_Y(y)} & f_Y(y) > 0 \\ 0 & f_Y(y) = 0 \end{cases}$$

$$E[X] = \int_{\mathbb{R}} x \cdot \underbrace{f_X(x)} \, dx$$

$$\underbrace{E[X \mid Y = y]} = \int_{\mathbb{R}} x \cdot f_{X|Y}(x|y) \, dx = h(y)$$

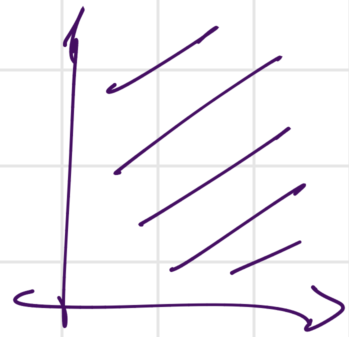
$$\boxed{E[X \mid Y](\omega)} := h(Y)$$

r.v.

$$y \rightsquigarrow h(y)$$

Ex:

$$f_{X,Y}(x,y) = \frac{e^{-x/y}\, e^{-y}}{y} \qquad x>0,\ y>0$$



$$h(y) = \mathbb{E}[X \mid Y = y]$$

Sol:

$$h(y) = \int_{\mathbb{R}} x \cdot f_{X|Y}(x|y)\, dx$$

$$\boxed{\frac{f_{X,Y}(x,y)}{f_Y(y)}}$$

$$\boxed{f_Y(y)} = \int_{\mathbb{R}} f_{X,Y}(x,y)\, dx \qquad x>0,\ y>0$$

$$\| \quad e^{-y}$$

$$= \int_0^{+\infty} \frac{1}{y} e^{-x/y} e^{-y}\, dx \qquad f_{X,Y} = \frac{e^{-x/y} e^{-y}}{y}$$

$$= \frac{e^{-y}}{y} \cdot \int_0^{+\infty} e^{-x/y}\, dx \qquad y>0$$

$$= \frac{e^{-y}}{y} \Big[(-y) e^{-x/y} \Big]_0^{+\infty} = \frac{e^{-y}}{y} \cdot y = \boxed{e^{-y}}$$

$Y \sim \mathrm{Exp}(1)$ r.v.

fixed

$$f_{X|Y}(x|y) = \frac{\frac{e^{-x/y}\, e^{-y}}{y}}{e^{-y}} = \qquad x>0,\ y>0$$

$$= \frac{e^{-x/y}}{y} = \left(\frac{1}{y}\right) e^{-\frac{1}{y}\cdot x}$$

$$\underbrace{X}\ \Big|\ \underbrace{Y=y}_{y>0} \sim \mathrm{Exp}(1/y)$$

$$\mathbb{E}[X \mid Y=y] = y$$

$$\boxed{\mathbb{E}[X \mid Y] = Y}$$

■

$\underbrace{\mathbb{E}[X \mid Y]}$ in the general setting

$X \in L^1(\Omega) \iff \underbrace{\mathbb{E}[|X|] < +\infty}$

$(X \in L^2)$

$$\boxed{\begin{gathered}\mathbb{E}[X \mid Y = y] = h(y) \\ h(Y) = \mathbb{E}[X \mid Y]\end{gathered}}$$

$\sigma(Y)$ σ-field generated by Y

$\shortparallel$ $\qquad Y : \Omega \longrightarrow \mathbb{R}$

$\{A : \exists B \in \mathcal{B}(\mathbb{R}^?) : Y^{-1}(B) = A\}$ $\qquad B \in \mathcal{B}(\mathbb{R})$

$\mathbb{E}[c \cdot X] = c \cdot \mathbb{E}[X]$

$\mathbb{E}[X \cdot \underbrace{\mathbb{1}_A}] = \underbrace{\mathbb{E}[\mathbb{E}[X \cdot \mathbb{1}_A \mid Y]]}$

$A \in \sigma(Y)$ TOWER PROPERTY

Remark: If (X,Y) is discrete or ab. cont.

$$\mathbb{E}[X \mid Y] = h(Y) \qquad h(y) = \mathbb{E}[X \mid Y = y]$$

Tower property: $\underbrace{\mathbb{E}(\boxed{h(Y)})}_{\mathbb{E}[\mathbb{E}[X|Y]]} = \underbrace{\mathbb{E}[X]}$

$$\boxed{\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]}$$

$$\boxed{\mathbb{E}[X \mathbb{1}_A]} = \mathbb{E}[\overbrace{\mathbb{E}[X \cdot \mathbb{1}_A \mid Y]}] = \boxed{\mathbb{E}[\mathbb{1}_A \, \mathbb{E}[X|Y]]}$$

$\boxed{A \in \sigma(Y)}$

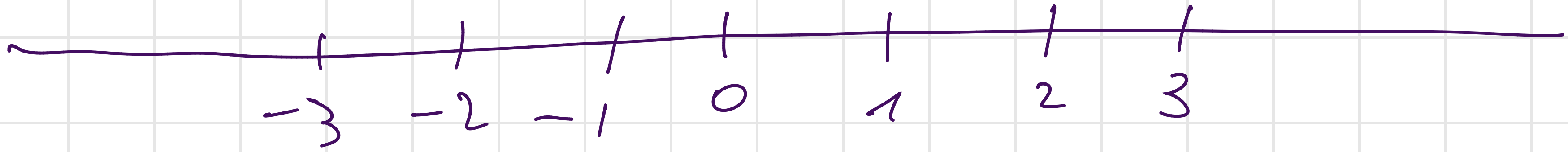
$$\mathbb{E}[cX] = c\,\mathbb{E}[X]$$

Def: For r.v.'s X, Y let $\mathbb{E}[X|Y]$ which is called the conditional expectation of X given Y, denote that function $h(Y)$ having the property that for any $\underline{\underline{A \in \sigma(Y)}}$

$$\boxed{\mathbb{E}[X \cdot \boxed{\mathbb{1}_A}] = \mathbb{E}[\underline{h(Y)} \cdot \boxed{\mathbb{1}_A}]}$$

$$\boxed{h(Y) = \mathbb{E}[X|Y]}$$

Symmetric Random Walk on $\mathbb{Z}$



$$\begin{cases} S_0 , \\ S_1 = S_0 + X_1 \\ S_2 = S_0 + X_1 + X_2 \end{cases}$$

$X_1, X_2, \dots$ iid

$$X_i = \begin{cases} 1 & 1/2 \\ -1 & 1/2 \end{cases}$$

$\vdots$

$$S_n = S_0 + X_1 + \dots + X_n = S_{n-1} + X_n$$

$\vdots$

$E[S_0]$

$$\begin{aligned} E[S_1] &= E[S_0 + X_1] \\ &= E[S_0] + \underbrace{E[X_1]}_{=0} \\ &= E[S_0] \end{aligned}$$

$\dots$ $E[S_n] = E[S_0]$

$n \geq m$

$$\mathbb{E}[\underbrace{S_n}_{} - \underbrace{S_m}_{} \mid \underbrace{S_0 = i_0, S_1 = i_1, \dots,}_{} \underbrace{S_m = i_m}_{}] = 0$$

$$\overset{\shortparallel}{i_n - i_m}$$

$$S_n = S_m + \underbrace{X_{m+1}}_{} + \dots + \underbrace{X_n}_{}$$

$$\shortparallel$$

$$i_m$$

$$\boxed{\mathbb{E}[S_n \mid S_m] = \underbrace{S_m}_{}}$$

$n \geq m$

This property says that S_n is a Martingale

$$S_0,\ S_1,\ \boxed{S_m},\ S_n$$

$$\mathbb{E}[S_n \mid S_m] = S_m$$