Stochastic Methods

30/9/2024

Lecture 1

A probabilistic Model

Coin toss a coin

$$\text{coin} \to \begin{cases} T \\ H \end{cases}$$

1. Ω a set and it will contains all the possible outcomes of a random experiment

Ω = sample space

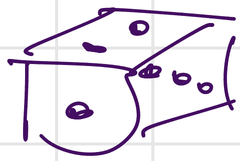
Ex. 1 Flip a coin $\quad \Omega = \{H, T\}$

$\Omega, \quad \cup, \cap, \setminus, {}^c, \Delta$

Ex. 2 Flip two coins

$$\Omega = \{(H,H), (T,H), (H,T), (T,T)\}$$

Ex. 3 Toss a die



$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Toss two dice

$$\Omega = \{(1,1), (1,2), \dots, (6,6)\}$$
$$= \{(i,j) : i, j \in \{1,2,3,4,5,6\}\}$$

$$2 \longleftrightarrow P_2$$

Ω

② $\boxed{\text{Events}}$

if Ω is finite, the set of the events will be $2^{\Omega} = P(\Omega) = \{$set of all the subset of $\Omega\}$

empty set

$\Omega = \{H, T\}$, $2^{\Omega} = \{\emptyset, \{H\}, \{T\}, \{H,T\}\}$

$$\Omega = \{1,2,3,4,5,6\}$$

$$2^{\Omega} \overset{?}{=} \{ \emptyset, \{1\}, \{2\}, \ldots, \{6\},$$
$$\{1,2\}, \{1,3\}, \ldots, \{5,6\},$$
$$\{1,2,3\}, \ldots, \{4,5,6\}$$
$$\vdots$$
$$\{1,2,3,4,5,6\} \}$$

$$\{1,2\} = \{2,1\}$$

$|\Omega| = 6$ cardinality of Ω, number of elements in Ω

$$|\underbrace{2^{\Omega}}| = 2^6 = \underbrace{2}^{|\Omega|}$$

$$\Omega \supseteq \underbrace{A} = \{1, \cancel{2}, 3, \cancel{4}, \cancel{5}, 6\}$$

$$2 \times 2 \times 2 \times 2 \times 2 \times 2$$

Toss 2 dice $\quad |\overline{\Omega}| = |\{(i,j) : i,j \in \{1,\ldots,6\}\}|$

$$= 6 \times 6 = 6^2 = 36$$

$$|2^{\overline{\Omega}}| = 2^{|\overline{\Omega}|} = 2^{36} = \underbrace{68.719.476}$$

Ω = sample space

set of events 2^{Ω}

$\mathcal{A}$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\emptyset$

$$\boxed{\mathcal{A}} \subseteq \underbrace{2^{\Omega}} = \{\underbrace{A} : A \subseteq \Omega\}$$

$\underbrace{\mathcal{A} \text{ a } \sigma\text{-field}}$

$\boxed{\text{(i) } \Omega \in \mathcal{A}}$

(ii) $\underbrace{A \in \mathcal{A}} \Rightarrow \underbrace{A^c \in \mathcal{A}}$

(iii) if $(A_n)_{n \in \mathbb{N}}$ is an <u>infinite</u> sequence of events ($A_n \in \mathcal{A}$, $\forall n$) then

$$\bigcup_{\substack{n=1 \\ n \in \mathbb{N}}}^{\infty} A_n \in \mathcal{A}$$

$$\mathcal{A} \subsetneq 2^{\Omega}$$

$\Omega = \{H, T\}$

$2^{\Omega} = \underbrace{\{\emptyset, \{H\}, \{T\}, \{H,T\}\}}$

$\mathcal{A} \subseteq 2^{\Omega}$ $\quad \underbrace{\mathcal{A}} = \{\emptyset, \{H\}\}$

$\mathcal{A}$ is a σ-field? $\quad \underbrace{\text{(i) } \cancel{\Omega \in \mathcal{A}}}$

no, $\mathcal{A}$ is not a σ-field

$\mathcal{A}_1 = \{\underline{\emptyset, \Omega}\}$ is a σ-field? $\quad$ trivial σ-field

(i) $\Omega \in \mathcal{A}_1$? $\quad$ YES

$\underbrace{\text{(ii) } A \in \mathcal{A}_1 \Rightarrow A^c \in \mathcal{A}_1}$ YES $\quad A = \begin{cases} \emptyset \\ \Omega \end{cases}$

$\underbrace{A_n} \in \mathcal{A}_1 \Rightarrow \bigcup_n A_n \in \mathcal{A}_1$ $\quad \emptyset^c = \Omega, \ \underbrace{\Omega^c = \emptyset}$

YES

$\Omega = \{H, T\}$, $\quad \{\emptyset, \Omega\}$, $\underbrace{2^{\Omega}}$

① Ω

② events will be all the elements of a σ-field $\mathcal{A} \subseteq 2^{\Omega}$

$A \in \mathcal{A}$

③ $\underbrace{\mathbb{P}[A]}$ $\in [0,1]$ probability

Ex:

$\underbrace{\Omega = \{1,2,3,4,5,6\}}$ $\quad \mathcal{A} = 2^{\Omega}$

$A, B \in \mathcal{A}$

$\boxed{A \cup B} = \{1,2,4,6\}$

$\boxed{A \cap B} = \{2\}$

$\boxed{(A \cup B)^c = A^c \cap B^c}$ $\quad \underbrace{A^c} = \{1,3,5\}$ odd number

$\underline{A = \{2,4,6\}}$ even number

$B = \{1,2\}$

① Ω = sample space (outcomes)

② $\mathcal{A} \subseteq 2^{\Omega}$ σ-field (events)

③ A function $\mathbb{P}$ is called "probability"

$$\mathbb{P} : \mathcal{A} \longmapsto [0,1] \subseteq \mathbb{R}$$

(i) $\mathbb{P}[\Omega] = 1$

(ii) if $(A_n)_{n\in\mathbb{N}}$ is a family of pairwise disjoint events, i.e.

$A_n \in \mathcal{A}$ $\forall n$, $\underbrace{A_n \cap A_m = \emptyset}$

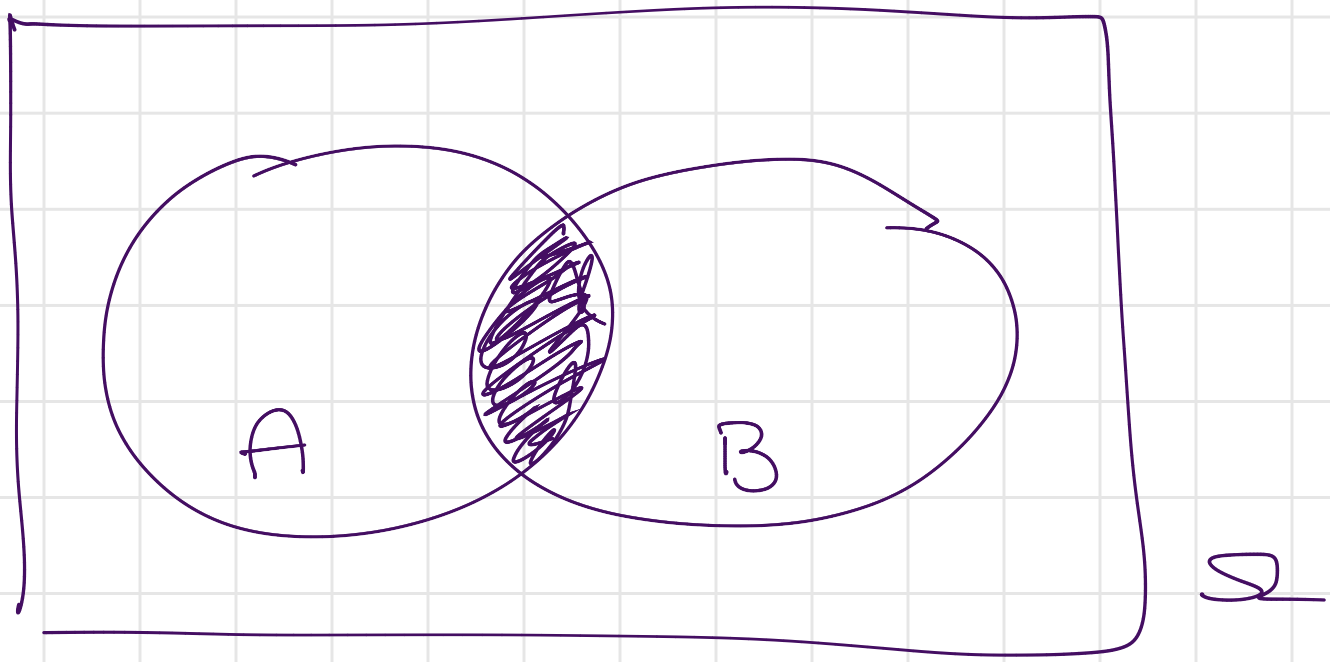
$\forall n \neq m$, then

$$\mathbb{P}\Big[\underbrace{\bigcup_{n\in\mathbb{N}} A_n}_{\substack{\cap \\ \mathcal{A}}}\Big] = \sum_{n\in\mathbb{N}} \mathbb{P}[A_n]$$

Remark: (ii) $\Rightarrow$ $A, B \in \mathcal{A}$, $A \cap B = \emptyset$

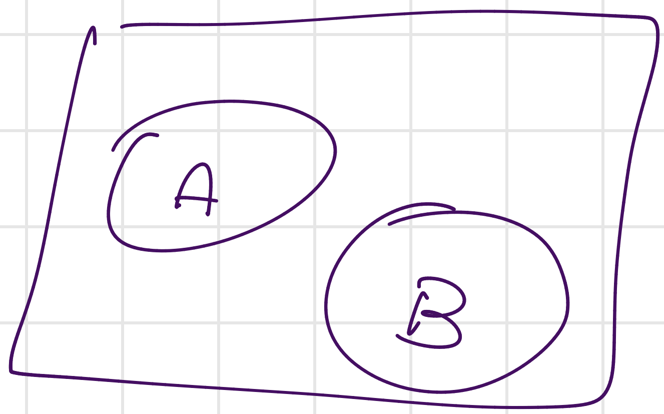
$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]$

Venn - diagrams



$A \cap B$

A and B are disjoint



Definition: The triple $(\Omega, \mathcal{A}, \mathbb{P})$ is called a probability space

Ex 1 : $\Omega = \{H, T\}$ $\quad p \in [0,1]$

$$\mathcal{A} = 2^{\Omega} = \{\emptyset, \Omega, \{H\}, \{T\}\}$$

$\quad 0 \quad 1 \quad p \quad 1-p$

$$\mathbb{P} : \mathcal{A} \longrightarrow [0,1]$$

$$\mathbb{P}[\{H\}] = p \in [0,1]$$

$$\mathbb{P}[\Omega] = 1$$

$\emptyset = \Omega^c$ $\quad \boxed{\mathbb{P}[\emptyset] = 0}$ $\quad \left(\mathbb{P}[A^c] = 1 - \mathbb{P}[A]\right)$

$$\Omega = A \cup A^c$$

$$\underset{(i)}{1} = \mathbb{P}[\Omega] = \mathbb{P}[A \cup A^c] \underset{(ii)}{=} \mathbb{P}[A] + \mathbb{P}[A^c]$$

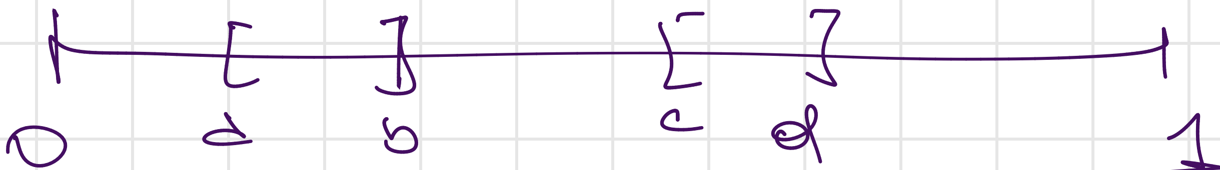
$\boxed{\mathbb{P}[\{H\}] = p}$ $\Rightarrow \mathbb{P}[\{T\}] = 1 - p$

$$\{T\} = \{H\}^c$$

$$\Omega = [0,1] \subseteq \mathbb{R}, \qquad [a,b] \subseteq [0,1]$$

$$\mathbb{P}[[a,b]] = b - a$$

$$\mathbb{P}[\Omega] = 1 - 0 = 1$$



$$\boxed{\mathbb{P}\big[[a,b] \cup [c,d]\big] = \mathbb{P}\big[[a,b]\big] + \mathbb{P}\big[[c,d]\big]}$$

$$\mathbb{P} : \underbrace{A} \longmapsto [0,1]$$

$$2^{\Omega} \neq 2^{[0,1]}$$

$$\mathbb{P} : \underbrace{\mathcal{B}([0,1])}_{\text{Borel } \sigma\text{-field}} \longrightarrow [0,1]$$

YES