PROBLEMS - SET 6

Problem 1. For $q, r \in (0,1)$ consider the Markov Chain $(X_n)_{n\geq 0}$ with state space $\{1,2,3\}$ and transition matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1-q & q \\ r & 0 & 1-r \end{pmatrix}.$$

Show that the chain is irreducible, and find the unique stationary distribution.

Solution 1. Note that the path $1 \to 2 \to 3 \to 1$ so the chain is irreducible. To find the stationary distribution we have to find π such that $\pi P = \pi$, $\pi_1 + \pi_2 + \pi_3 = 1$ and $\pi_i > 0$. So we find

$$\begin{cases} \pi_1 - r\pi_3 \\ \pi_2 = \pi_1 + (1-q)\pi_2 \\ \pi_3 q\pi_2 + (1-r)\pi_3 \\ \pi + 1 + \pi_2 + \pi_3 = 1 \end{cases} \qquad \text{so} \begin{cases} \pi_1 = r\pi_3 \\ \pi_1 = q\pi_2 \\ \pi_1 + \pi_2 + \pi_3 = 1. \end{cases}$$

This gives

$$\pi = \left(\frac{qr}{qr+q+r}, \frac{r}{qr+q+r}, \frac{q}{qr+q+r} \right).$$

Problem 2. A Markov Chain $(X_n)_{n\geq 0}$ with state space $\{0,1,\dots,N\}$, $N \geq 2$, evolves as follows. If $X_n = 0$ (resp. $X_n = N$), then X_{n+1} is chosen with uniform probability in $\{1,\dots,N\}$ (resp. $\{0,1,\dots,N-1\}$). If $X_n = x \in \{1,\dots,N-1\}$, then $X_{n+1} = 0$ with probability $1/2$ and $X_{n+1} = N$ with probability $1/2$.

Show that the chain is irreducible and find the unique stationary distribution.

Solution 2. The nonzero elements of the transition matrix are

$$\begin{aligned} P_{0,i} &= \frac{1}{N} \quad i = 1,\dots,N \\ P_{N,i} &= \frac{1}{N} \quad i = 0,\dots,N-1 \\ P_{i,0} = P_{i,N} &= \frac{1}{2} \quad i = 1,\dots,N-1 \end{aligned}$$

Irreducibility comes from the fact that for every $i, j \in \{1,\dots,N-1\}$ the path $0 \to i \to N \to j \to 0$ has positive probability. Finally, we find

$$\begin{cases} \sum_{i=1}^{N-1} \frac{\pi_i}{2} + \frac{\pi_N}{N} = \pi_0 \\ \frac{\pi_0}{N} + \frac{\pi_N}{N} = \pi_i \\ \frac{\pi_0}{N} + \sum_{i=1}^{N-1} \frac{\pi_i}{2} = \pi_N \\ \sum_i \pi_i = 1 \end{cases} \qquad \begin{cases} \pi_i = \frac{\pi_0 + \pi_N}{N} \\ \sum_i \pi_i + \pi_0 + \pi_N = (\pi_0 + \pi_N)(\frac{N-1}{N} + 1) = 1 \\ \pi_0 - \frac{\pi_N}{N} = \pi_N - \frac{\pi_0}{N} \end{cases}$$

So the stationary distribution $\pi = (\pi_0, \pi_1, \dots, \pi_N)$ with $\pi_0 = \pi_N = \frac{N}{2(2N-1)}$ and $\pi_i = \frac{1}{2N-1}$ for $i = 1, \dots, N-1$.

Problem 3. The urn A contains two white balls, the urn B three red balls. Each step of the dynamics consists in drawing a ball from each urn, and then replace them exchanging the urn. Let X_n be the number of red balls in A after n draws.

Find the transition matrix of X_n and determine the stationary distributions.

Solution 3. Observe that $(X_n)_n$ has state space $\{0,1,2\}$. In A there are always 2 balls and in B 3 balls, and there are in total 2 white balls and 3 red.

If $X_n = 0$, in A there are 2 white balls and in B 3 red balls. Therefore $X_{n+1} = 1$ with probability 1: that is $P_{0,0} = 0 = P_{0,2}$ and $P_{0,1} = 1$.

If $X_n = 1$, then in A there are 1 red ball and 1 white and in B 2 red balls and 1 white. Therefore $P_{1,0} = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$, and $P_{1,2} = \frac{1}{2}\frac{2}{3} = \frac{1}{3}$. Therefore $P_{1,1} = 1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2}$.

If $X_n = 2$, then in A there are 2 red balls and in B 1 red ball and 2 white. Therefore $P_{2,0} = 0$ and $P_{2,2} = \frac{1}{3}$. This implies $P_{2,1} = \frac{2}{3}$. The transition matrix is given by

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

It is irreducible since $0 \to 1 \to 2 \to 2 \to 1 \to 0$ has positive probability.

To find the unique stationary distribution we solve $\pi P = \pi$ and we get

$$\begin{cases} \pi_1 = \frac{1}{6}\pi_2 \\ \pi_2 = \pi_1 + \frac{1}{2}\pi_2 + \frac{2}{3}\pi_3 \\ \pi_3 = \frac{1}{3}\pi_2 + \frac{1}{3}\pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1. \end{cases} \qquad \begin{cases} \pi_1 = \frac{1}{6}\pi_2 \\ \pi_3 = \frac{1}{2}\pi_2 \\ \pi_2 = \frac{3}{5}. \end{cases} \qquad \pi = \left(\frac{1}{10}, \frac{3}{5}, \frac{3}{10}\right)$$

Problem 4. N persons sit at a round table. Each one wear a bracelet, either on the right or left hand. At any time $n \geq 0$ we do what follows:

- choose at random (with uniform distribution) one of the N persons;
- the person chosen look at one of his two neighbors at random, and imitates him for the position, right or left, of the bracelet.

(a) Represent this dynamics with a transition matrix of a Markov Chain.

(b) Is this Markov Chain irreducible?

Solution 4. (a) The state space is $S = \{r, l\}^N$. For $x \in S$ denote by $x^{i,-}$ the element of S obtained from x by replacing x_i with x_{i-1} (note that $x^{i,-} = x$ if $x_i = x_{i-1}$); similarly, $x^{i,+}$ is obtained from x by replacing x_i with x_{i+1} (sums and differences are meant mod. N: that is $x_{0-1} = x_N$ and $x_{N+1} = x_1$). The only nonzero and non-diagonal elements of the transition matrix P are those of the form $P_{x,x^{i,\pm}}$, in the case $x \neq x^{i,\pm}$ and

$$P_{x,x^{i,\pm}} = \frac{1}{2N}.$$

(b) Denote by $\mathbf{r}$ and $\mathbf{l}$ the elements of S in which everyone wears the bracelet on the same side. Clearly, they are "traps", i.e. $P_{\mathbf{r},\mathbf{r}} = P_{\mathbf{l},\mathbf{l}} = 1$. (Note also that or any $x \neq \mathbf{r}, \mathbf{l}$ there is a positive probability of reaching these traps in a finite number of steps). So the chain cannot be irreducible.

Problem 5. Consider a Markov chain on the vertices of the graph in Fig. 0.1, evolving with the following rules:

- if the walk is in 0 then it moves to one of its neighbors each with probability $\frac{1}{4}$;
- if the walk is in $i = 1,2,3,4$ then with probability $\frac{1}{2}$ does not move, and with probability $\frac{1}{2}$ moves to 0.

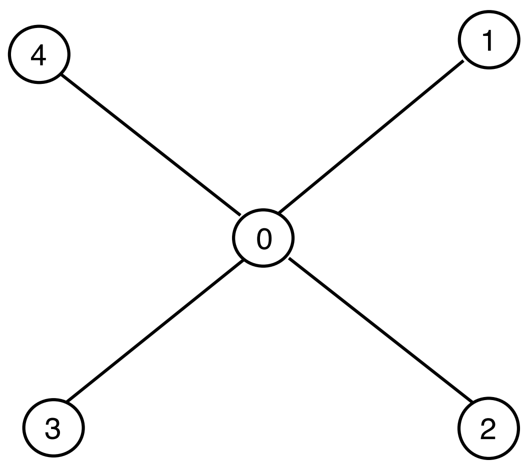

Figura 0.1

(a) Write the transition matrix P.

(b) Find the unique stationary distribution π.
Hint: it is useful to observe that, for symmetry reasons, $\pi_1 = \pi_2 = \pi_3 = \pi_4$.

Solution 5. (a)

$$P = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix}$$

(b) Solving $\pi P = \pi$ gives

$$\begin{cases} \sum_{i>0} \frac{1}{2}\pi_i = \pi_0 \\ \frac{1}{4}\pi_0 + \frac{1}{2}\pi_i = \pi_i \ i = 1,2,3,4 \\ \sum_i \pi_i = 1. \end{cases}$$

Therefore $\pi_1 = \pi_2 = \pi_3 = \pi_4 = \frac{\pi_0}{2}$ and $\pi_0 + 4\frac{\pi_0}{2} = 1$.
Then $\pi = (\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$.

Problem 6. Urn A and urn B contain n balls each. n of these balls are red and n are green. We draw one ball from each urn, then we exchange them and replace (i.e. the ball drawn form urn A goes to urn B and viceversa). We then iterate this procedure.

(a) Denote by X_n the number of red ball in urn A after n iterations. Determine the transition matrix of the Markov Chain X_n.

(b) For $n = 3$, determine the stationary distribution of this Markov Chain.

(c) Determine the stationary distribution for arbitrary n.
Hint: the combinatorial identity

$$\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}$$

is useful.

Solution 6. (a) X_n has state space $\{0, \ldots, n\}$.
Observe that $P_{0,1} = 1$ (so $P_{0,i} = 0$ for all other i), and $P_{n,n-1} = 1$ (so $P_{n,i} = 0$ for all other i), and finally for $i = 1, \ldots, n-1$ the only nonzero elements are $P_{i,i}$ $P_{i,i+1}$ and $P_{i,i-1}$.
Let $X_n = i$, then in A there are i red balls and $n-i$ green balls. So in B there are $n-i$ red balls and i green balls.

$$P_{i,i+1} = \frac{n-i}{n}\frac{n-i}{n} = \frac{(n-i)^2}{n^2}, \qquad P_{i,i-1} = \frac{i}{n}\frac{i}{n} = \frac{i^2}{n^2}.$$

and so

$$P_{ii} = 1 - P_{i,i+1} - P_{i,i-1} = 1 - \frac{(n-i)^2}{n^2} - \frac{i^2}{n^2} = \frac{2ni - 2i^2}{n^2} = \frac{2i(n-i)}{n^2}.$$

Note that this chain is irreducible, since $0 \to 1 \to 2 \to \ldots \to n-1 \to n \to n-1 \to \cdots \to 2 \to 1 \to 0$ has positive probability.

(b) It is a special case of (c). In any case the transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Solving $\pi P = \pi$ we get

$$\begin{cases} \pi_0 = \frac{1}{9}\pi_1 \\ \pi_1 = \pi_0 + \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2 \\ \pi_2 = \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2 + \pi_3 \\ \pi_3 = \frac{1}{9}\pi_2 \\ \sum_i \pi_i = 1 \end{cases} \qquad \begin{cases} \pi_0 = \frac{1}{9}\pi_1 \\ \pi_1 = \pi_2 \\ \pi_3 = \frac{1}{9}\pi_2 \\ 2\pi_1 + \frac{2}{9}\pi_1 = 1. \end{cases}$$

So $\pi = \left(\frac{1}{20}, \frac{9}{20}, \frac{9}{20}, \frac{1}{20}\right)$.

(c) We know that the chain is irreducible, so it admits a unique stationary distribution. Since reversible distributions are stationary, we look for a reversible distribution. So it must be

$$\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i}$$

i.e.

$$\pi_0 = \frac{1}{n^2}\pi_1 \quad \text{and for } i = 1, \ldots, n-1 \; \frac{(n-i)^2}{n^2}\pi_i = \frac{(i+1)^2}{n^2}\pi_{i+1}.$$

and so for all i there holds

$$\pi_{i+1} = \pi_i \frac{(n-i)^2}{(i+1)^2}.$$

Solving this recursion we get

$$\pi_1 = n^2\pi_0, \quad \pi_2 = \frac{(n-1)^2}{(2)^2} n^2 \pi_0 = \binom{n}{2}^2 \pi_0, \quad \pi_3 = \frac{(n-2)^2}{(3)^2}\binom{n}{2}^2 \pi_0 = \binom{n}{3}^2 \pi_0 \ldots.$$

and so

$$\pi_i = \binom{n}{i}^2 \pi_0.$$

Imposing $\sum_i \pi_i = 1$ we get

$$\pi_i = \frac{\binom{n}{i}^2}{\sum_{j=0}^n \binom{n}{j}^2} = \frac{\binom{n}{i}^2}{\binom{2n}{n}}.$$