Lecture 13 — Stoch. Meth. — Nov. 8th, 2024

$X \sim \mathcal{U}(a,b)$

$X \sim \mathrm{Exp}(\lambda)$

$X \sim \mathcal{N}(\mu, \sigma^2)$

Exercise: $\underbrace{X \sim \mathcal{U}(0,1)}$, $\underbrace{Y \sim \mathrm{Bin}(1, 1/2)}$, $X \perp\!\!\!\perp Y$

$W := \max(X, Y)$

$$F_W(w) = \mathbb{P}[W \le w] = \mathbb{P}[\underbrace{\max(X,Y) \le w}]$$

$$= \mathbb{P}[X \le w, Y \le w] = \mathbb{P}[X \le w] \cdot \mathbb{P}[Y \le w]$$

$V := \min(X, Y)$ $\qquad F_V(v) = \mathbb{P}[\underbrace{\min(X,Y) \le v}]$

$$= 1 - \mathbb{P}[\underbrace{\min(X,Y)} > v] =$$

$$= \underbrace{1 - \mathbb{P}[X > v] \cdot \mathbb{P}[Y > v]}$$

$$F_V(v) = \begin{cases} \_\_\_\_ & \circ \\ \_\_\_\_ & \circ \\ \_\_\_\_ & \circ \end{cases}$$

Normal (Gaussian) distributions

A real valued r.v. Z is called **standard Normal** if it is absolutely continuous with density

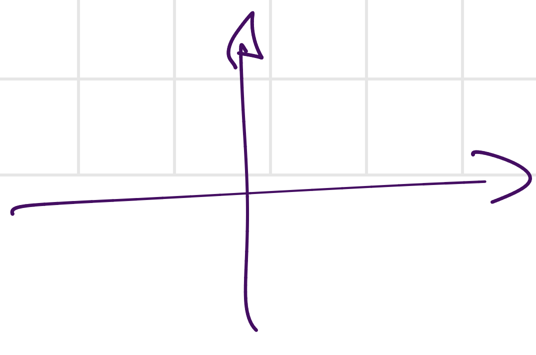
$$\boxed{f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}}$$

$F' = f$

$\Rightarrow \quad f_Z \geq 0$

$$\int_{\mathbb{R}} f_Z(z)\,dz = 1 \iff \boxed{\sqrt{2\pi} = \int_{-\infty}^{+\infty} e^{-z^2/2}\,dz}$$

$$\int_{\mathbb{R}^2} e^{-x^2/2} \cdot e^{-y^2/2}\,dx\,dy$$



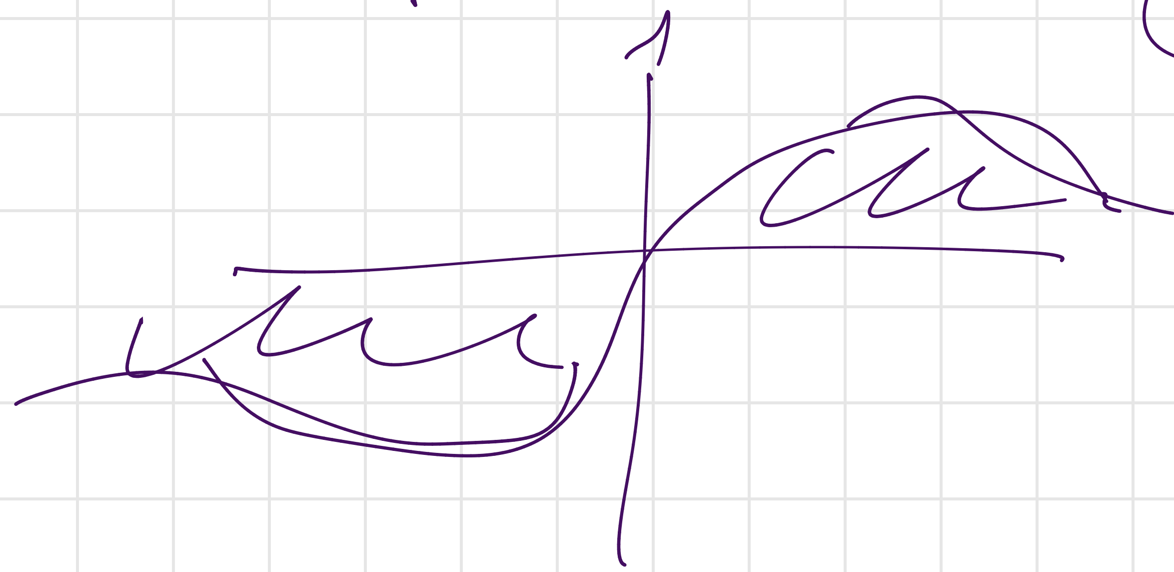
2π

$Z \in L^1(\Omega)$

$$\int_{\mathbb{R}} \underbrace{|z|}_{t} \frac{1}{\sqrt{2\pi}} \underbrace{e^{-z^2/2}}_{1} dz < +\infty$$

$$\mathbb{E}[Z] = \int_{\mathbb{R}} z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \int_{-\infty}^{0} + \int_{0}^{+\infty} = 0$$

$f(z) = z\, e^{-z^2/2}$, $f(-z) = -f(z)$ $(0,1)$



$$\mathbb{E}[Z^2] = \mathrm{Var}[Z] = \int_{-\infty}^{+\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \int_{-\infty}^{+\infty} z \cdot \frac{z}{\sqrt{2\pi}} e^{-z^2/2} dz =$$

$$= \frac{1}{\sqrt{2\pi}} \left[z \cdot \left(-\underbrace{e^{-z^2/2}}_{=0}\right) \right]_{-\infty}^{+\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} 1 \cdot \left(-e^{-z^2/2}\right) dz$$

$$\mathbb{E}[Z^2] = \mathrm{Var}[Z] = \int_{-\infty}^{+\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-z^2/2}}\, dz = 1$$

$$\underbrace{Z \quad \text{standard Normal}}, \quad \mathbb{E}[Z] = 0, \quad \mathrm{Var}[Z] = 1$$

$$Z \sim \mathcal{N}(0,1) \rightsquigarrow X \sim \mathcal{N}(\mu, \sigma^2)$$

$$\mu \in \mathbb{R}, \quad \underbrace{\sigma > 0}$$

$$\boxed{X = \mu + \sigma Z}$$

$$\underbrace{\Phi}'(t) = f_Z$$

$$\Phi(t) = \mathbb{P}[Z \le t]$$

$$F_X(x) = \mathbb{P}[X \le x] = \mathbb{P}[\mu + \sigma Z \le x]$$

$$= \mathbb{P}[\sigma Z \le x - \mu] = \mathbb{P}\left[Z \le \frac{x-\mu}{\sigma}\right]$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$\phi'(t) = \underbrace{\frac{1}{\sqrt{2\pi}} \cdot e^{-t^2/2}}$$

$$\frac{d}{dx} F_X(x) = \frac{d}{dx} \Phi\left(\frac{x-\mu}{\sigma}\right) = \Phi'\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}$$

$$\shortparallel$$

$$f_X(x)$$

$X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2 / 2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

μ, σ^2

X Normal (Gaussian) r.v.

$X = \mu + \sigma Z$

$Z \sim N(0,1)$

$$\boxed{\mathbb{E}[X]} = \mathbb{E}[\mu + \sigma Z]$$

$$= \mathbb{E}[\mu] + \mathbb{E}[\sigma \cdot Z]$$

$$= \mu + \sigma \underbrace{\mathbb{E}[Z]}_{= 0} = \boxed{\mu}$$

$$\boxed{Var[X]} = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$X = \mu + \sigma Z$

$$Var[\mu + \sigma Z] = \underbrace{Var[\mu]}_{= 0} + Var[\sigma Z] = \sigma^2 \underbrace{Var[Z]}_{= 1}$$

$$= \boxed{\sigma^2}$$

Remarks:

(a) If $X \sim N(\mu, \sigma^2)$, then

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1) \quad \text{standard Normal}$$

(b) If $X \sim N(\mu, \sigma^2)$, $a, b \in \mathbb{R}$, $b \neq 0$

then $a + bX \sim N(\underbrace{a + b\mu}, \underbrace{b^2\sigma^2})$

Exercise: $X, Y \sim Exp(\lambda)$ $\quad X \perp\!\!\!\perp Y$

$Z = \min(X, Y)$ $\quad \mathbb{E}[Z]$ $\quad \underline{Z \sim Exp(2\lambda)}$

$W = \max(X, Y)$ $\quad \mathbb{E}[W]$

$$F_Z(z) = \mathbb{P}[Z \le z] = \mathbb{P}[\min(X,Y) \le z] =$$

$$= 1 - \mathbb{P}[\underbrace{\min(X,Y)} > z] = 1 - \mathbb{P}[\underbrace{X > z, Y > z}]$$

$$\overset{IND.}{=} 1 - \mathbb{P}[X > z] \cdot \mathbb{P}[Y > z] = 1 - e^{-\lambda z} \cdot e^{-\lambda z}$$

$$\underline{= 1 - e^{-2\lambda z}} \qquad \mathbb{E}[Z] = \frac{1}{2\lambda}$$

<u>Remark</u> : $X \perp\!\!\!\perp Y$, $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$

$$\min(X,Y) \sim Exp(\lambda+\mu)$$

$W = \max(X,Y)$

$$F_W(w) = \mathbb{P}[W \le w] = \mathbb{P}[\max(X,Y) \le w]$$

$$= \mathbb{P}[X \le w, Y \le w] \overset{IND.}{=} \underbrace{\mathbb{P}[X \le w]} \cdot \underbrace{\mathbb{P}[Y \le w]}$$

$$= \begin{cases} 0 & w < 0 \\ (1-e^{-\lambda w})(1-e^{-\lambda w}) = (1-e^{-\lambda w})^2 & w \ge 0 \end{cases}$$

$$F_W' = f_W$$

$$\mathbb{E}[W] = \begin{cases} \int_{\mathbb{R}} w f_W(w)\,dw \\ \int_0^{+\infty} (1-\underbrace{F_W(w)})\,dw = \end{cases}$$

$$= \int_0^{+\infty} \Big(1 - \underbrace{(1-e^{-\lambda w})^2}\Big)\,dw = \int_0^{+\infty} (\cancel{1} - \cancel{1} - e^{-2\lambda w} + 2e^{-\lambda w})\,dw$$

$$= \int_0^{+\infty} (2\underbrace{e^{-\lambda w}} - e^{-2\lambda w})\,dw = 2\left[\frac{e^{-\lambda w}}{-\lambda}\right]_0^{+\infty} - \left[\frac{e^{-2\lambda w}}{-2\lambda}\right]_0^{+\infty}$$

$$= 2\left[\frac{e^{-\lambda\omega}}{-\lambda}\right]_0^{+\infty} - \left[\frac{e^{-2\lambda\omega}}{-2\lambda}\right]_0^{+\infty}$$

$\omega \to +\infty$

$\omega = 0$

$$= \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}$$

$$\mathbb{E}[\max(X,Y)] = \frac{3}{2}\cdot\frac{1}{\lambda}$$

■

$\boxed{\text{Two generating functions}}$

- Moment generating function
- Characteristic function

χ_A

$\boxed{\mathbb{1}_A}$

Scalar case $(\mathbb{R})$ $\quad X : \Omega \to \mathbb{R}$

Define

$$m_X : \mathbb{R} \to [0, +\infty] = [0,+\infty) \cup \{\infty\}$$

$$\underbrace{m_X(t)} := \mathbb{E}\left[e^{tX}\right]$$

$E[X]$

$E[X^2]$

$\vdots$

$E[X^n]$

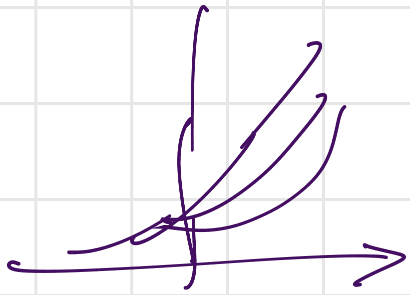
$$\mathbb{E}[g(X)] = \int_{-\infty}^{+\infty} g(x)\, f(x)\, dx$$

$g(x) = x$ $\quad f(x)$

$g(x) = x^2$

x^n

$$\mathbb{E}[e^X] = \int \underbrace{e^x f(x)}\, dx < +\infty$$



$$\mathbb{E}[e^{tX}]$$

$\boxed{m_X}$ is called the <u>moment generating function</u>

Fix ω

$$\underbrace{\frac{d^n}{dt^n}}\, e^{\underbrace{tX(\omega)}} = X^n e^{tX}$$

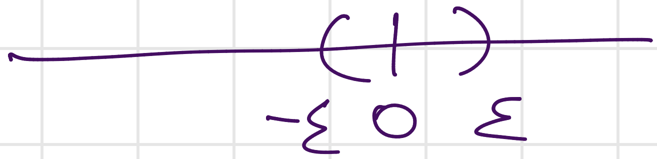
$$\boxed{\begin{aligned} \frac{d^n}{dt^n} m_X(t) &= \frac{d^n}{dt^n}\, \mathbb{E}[e^{tX}] = \\ &= \mathbb{E}\left[\frac{d^n}{dt^n} e^{tX}\right] = \mathbb{E}[X^n e^{tX}] \end{aligned}}$$

$$\boxed{\left.\frac{d^n}{dt^n} m_X(t)\right|_{t=0} = \mathbb{E}[X^n]}$$

$\forall n \in \mathbb{N}$

It can be proven that this last identity is true whenever $\exists\, \varepsilon > 0$ s.t.

$$\boxed{\underbrace{m_X(t) < +\infty}_{} \qquad \forall\, |t| < \varepsilon}$$



$$\frac{d^n}{dt^n} m_X(t)\Big|_{t=0} = \mathbb{E}[X^n]$$

Example

$X \sim Exp(\lambda)$

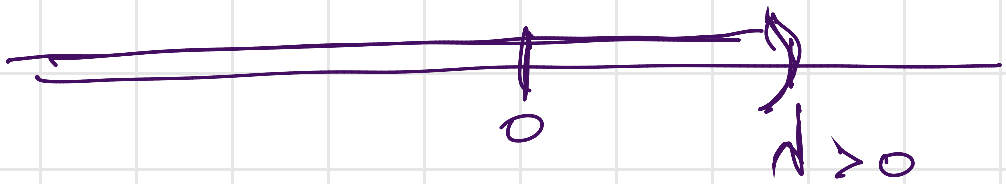
$$f_X = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases}$$

$$m_X(t) = \mathbb{E}\left[e^{tX}\right] = \int_{\mathbb{R}} e^{tx} \cdot f_X(x)\, dx$$

$$= \int_0^{+\infty} e^{tx} \cdot \lambda e^{-\lambda x}\, dx = \int_0^{+\infty} \lambda \underbrace{e^{-(\lambda - t)x}}_{}\, dx$$

$$= \begin{cases} +\infty & \lambda - t \leq 0 \\ \int_0^{+\infty} \lambda \underbrace{e^{-(\lambda - t)x}}_{}\, dt = \dfrac{\lambda}{\lambda - t} < +\infty & \lambda - t > 0 \end{cases}$$

$$m_X(t) = \begin{cases} +\infty & t \geq \lambda \\ \underbrace{\dfrac{\lambda}{\lambda - t}} & t < \lambda \end{cases}$$



$m_X(t) < +\infty$ in $(-\varepsilon, \varepsilon)$

$$\frac{d}{dt} m_X(t) \Big|_{\underline{t=0}} = \mathbb{E}[X]$$

$$\frac{d}{dt} m_X(t) = \frac{d}{dt}\left(\frac{\lambda}{\lambda - t}\right) = (-1)\frac{\lambda}{(\lambda-t)^2}(-1) = \frac{\lambda}{(\lambda-t)^2}$$

$$\frac{d}{dt} m_X(t) \Big|_{t=0} = \frac{\lambda}{\lambda^2} = 1/\lambda = \mathbb{E}[X]$$

<u>Exercise</u>: Compute $\mathbb{E}[X^2]$, $\mathbb{E}[X^3]$ and prove $\boxed{\mathbb{E}[X^n] = \frac{n!}{\lambda^n}}$

If $\underbrace{m_X(t) < +\infty \quad \forall t \in (-\varepsilon, \varepsilon)}$, then

$$\underbrace{\mathbb{E}[X^n] < +\infty}\ \forall n \quad \text{and} \quad \underbrace{\frac{d^n}{dt^n} m_X(t)\Big|_{t=0} = \mathbb{E}[X^n]}$$

<u>Exercise</u>

<u>Log normal distribution</u>

$$Y \sim \mathcal{N}(\mu, \sigma^2), \quad \boxed{X = e^Y}$$

$$\underbrace{\mathbb{E}[X^n] < +\infty} \quad \text{but} \quad m_X(t) \not< +\infty \quad \forall \cancel{t} \in (-\varepsilon, \varepsilon)$$

$$\mathbb{E}[X^n], \quad m_X(t)$$

$X \sim \mathrm{Bin}(1, p)$ $\qquad m_X(t) = \mathbb{E}[e^{tX}]$

$$X = \begin{cases} 0 & 1-p \\ 1 & p \end{cases}$$

$$m_X(t) = \underbrace{e^{t\cdot 0} \cdot \mathbb{P}[X=0]} + e^{t\cdot 1}\,\mathbb{P}[X=1]$$

$$= 1-p + e^t \cdot p < +\infty \quad \forall t \in \mathbb{R}$$

Exer. $X \sim Geo(p)$ $\quad m_X(t)$

$X \sim Poisson(\lambda)$ $\quad m_X(t)$