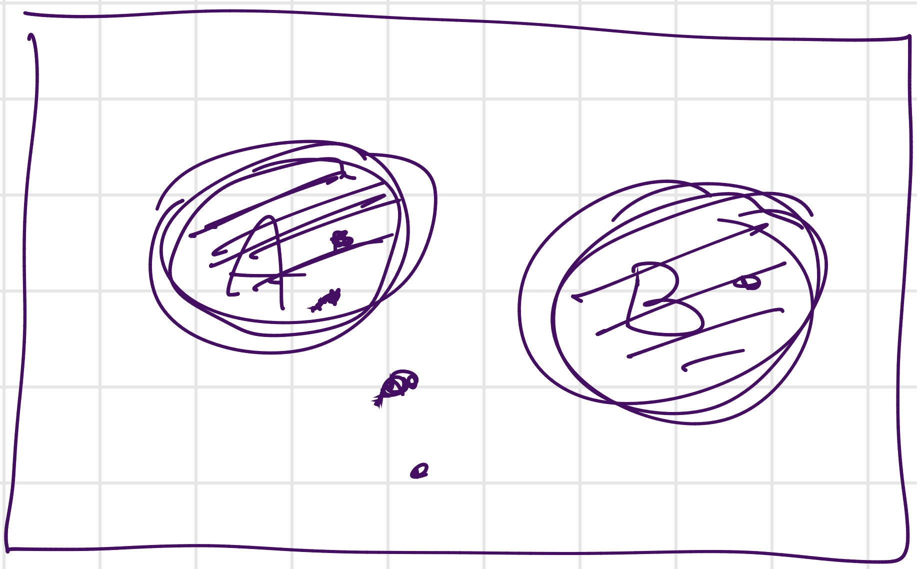
Lecture 5 Stoc. Math. Oct. 10, 2024

$(A_n)_{n \in \mathbb{N}} \qquad A_n \in \mathcal{A}$

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \left(\bigcup_{m=n}^{\infty} A_m \right)$$

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \left(\bigcap_{m=n}^{\infty} A_m \right)$$



$A_1 = A$

$A_2 = B$

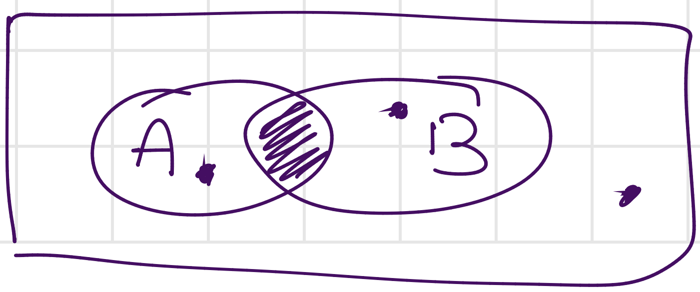
$A_3 = A$

$A_4 = B = A_6 = A_8 = \dots = A_{2k}$

$A_5 = A = A_{2k+1}$

$\limsup A_n = \boxed{\{A_n \text{ i.o.}\}} = A \cup B$

$\liminf A_n = \emptyset$



$\limsup A_n = A \cup B$

$\liminf A_n = A \cap B$

Ex: If $(A_n)_{n\in\mathbb{N}}$

$$\underbrace{\limsup_{n\to\infty} \mathbb{1}_{A_n}} - \underbrace{\liminf_{n\to\infty} \mathbb{1}_{A_n}} = \boxed{\mathbb{1}_{\left(\limsup\limits_{n\to\infty} A_n \setminus \liminf\limits_{n\to\infty} A_n\right)}}$$

where $\mathbb{1}_A(x) := \begin{cases} 1 & x\in A \\ 0 & x \notin A \end{cases}$

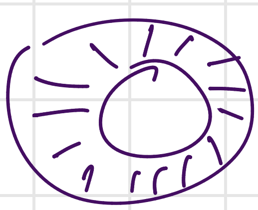
Remark

$f = g \iff \forall x \quad f(x) = g(x)$

$$\underbrace{\mathbb{1}_{(\limsup A_n \setminus \liminf A_n)}(\omega)} = \begin{cases} 1 & \omega\in \limsup A_n \setminus \liminf A_n \\ 0 & \text{otherwise} \end{cases}$$

$$\|$$

$$1 \iff \omega \in \underbrace{\limsup_{n\to\infty} A_n \setminus \liminf_{n\to\infty} A_n}$$



$\underbrace{\omega\in\limsup A_n}$ and $\omega\notin \liminf A_n$

$\exists$ infinitely many n s.t. $\underbrace{\omega \in A_n}$

$\Rightarrow \mathbb{1}_{A_n}(\omega) = 1 \qquad f_n(\omega) = 1$ i.o.

$\omega \in \limsup_n A_n \implies \boxed{\limsup_{n\to\infty} \mathbb{1}_{A_n} = 1}$

$\omega \notin \liminf_{n\to\infty} A_n$

$\omega \in \liminf A_n \implies \exists \bar{n} : \forall n \geq \bar{n} \quad \omega \in A_n$

$\omega \notin \liminf A_n \implies \forall \bar{n}, \exists n \geq \bar{n} \quad \omega \notin A_n$

$\implies \mathbb{1}_{A_n}(\omega) = 0$

$\implies \boxed{\liminf_{n\to\infty} \mathbb{1}_{A_n} = 0}$

YES

$$\underset{\substack{\shortparallel \\ 1}}{\limsup \mathbb{1}_{A_n}} - \underset{\substack{\shortparallel \\ 0}}{\liminf \mathbb{1}_A} = \mathbb{1}_{(\limsup A_n \setminus \liminf)}$$

$= 1$

$(\ldots)$

First - Second Borel Cantelli Lemma

Let $(A_n)_{n \in \mathbb{N}}$ be a family of events

(i) if $\boxed{\sum_{n=1}^{\infty} \underbrace{\mathbb{P}[A_n]}_{\geq 0} < +\infty}$ then

$$\mathbb{P}[A_n \text{ i.o.}] = 0$$

Remark $\sum \mathbb{P}[A_n] < +\infty \Rightarrow \underbrace{\lim_{n \to \infty} \mathbb{P}[A_n] = 0}$

$\sum 1/n = +\infty$

(ii) If the events A_n are independent

and $\sum_{n=1}^{\infty} \mathbb{P}[A_n] = +\infty$ then

$$\mathbb{P}[A_n \text{ i.o.}] = 1$$

<u>Example</u> Consider a sequence of trials in which the probability of success at the n-th trial is $p_n \in [0,1]$.

Set A_n = "the n-th trial is a success"

$$\mathbb{P}[A_n] = p_n$$

The events A_n are independent

$$\{A_n \text{ i.o.}\} = \{\text{infinitely many successes occur}\}$$

- if $\sum_{n=1}^{\infty} p_n < +\infty \Rightarrow \mathbb{P}\{A_n \text{ i.o.}\} = 0$

- if $\sum_{n=1}^{\infty} p_n = +\infty \Rightarrow \mathbb{P}\{A_n \text{ i.o.}\} = 1$

$\mathbb{P}[A_n] = p \in (0,1)$ $\Rightarrow$ almost surely

Flip a coin infinitely often

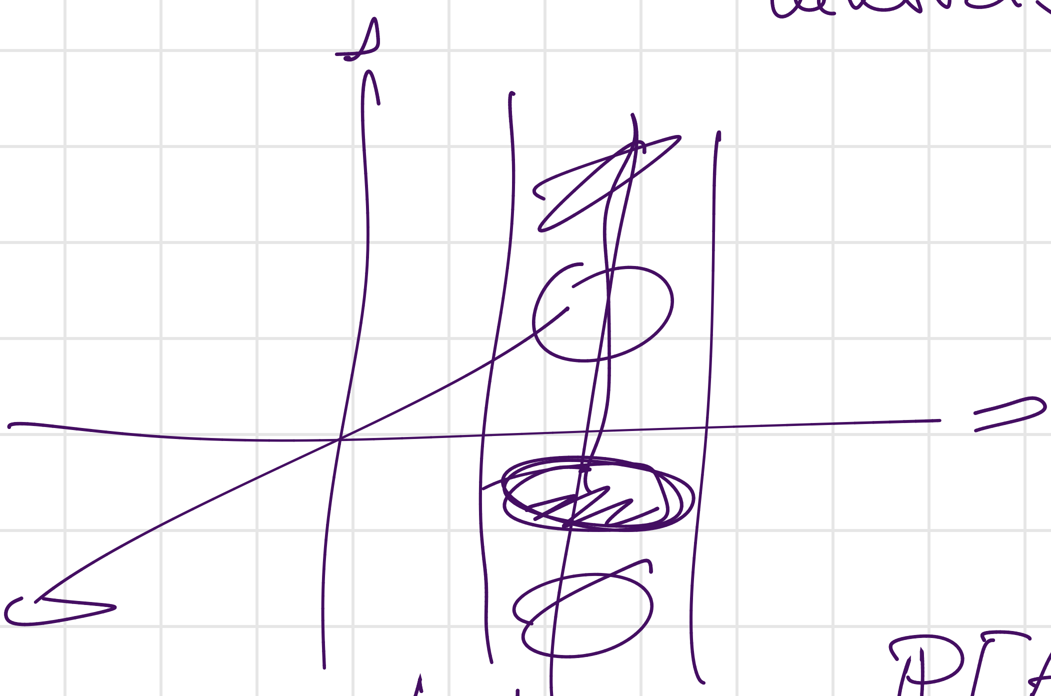
$$\Omega = \{ (\omega_1, \dots, \omega_n, \dots), \ \omega_i \in \{H,T\} \}$$

$0, i_1 i_2 i_3 \dots$ $\qquad \Omega \simeq [0,1]$

$$\mathcal{A} \neq 2^{\Omega}$$

$\Omega = \mathbb{R}$, $\quad \mathcal{A} = \mathcal{B}(\mathbb{R})$ = Borel sets

= minimal σ-field containing all the $\underbrace{\text{open sets}}_{\text{intervals}}$ of $\mathbb{R}$.



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$(x,y) \in D$

$D \times \mathbb{R}$

$$P[A] = p \cdot p \cdot (1-p)$$

$$A = \{ (H, H, T, \omega_4, \omega_5, \dots), \ \omega_n \in \{H,T\} \ \forall n \geq 4 \}$$

Random Variables

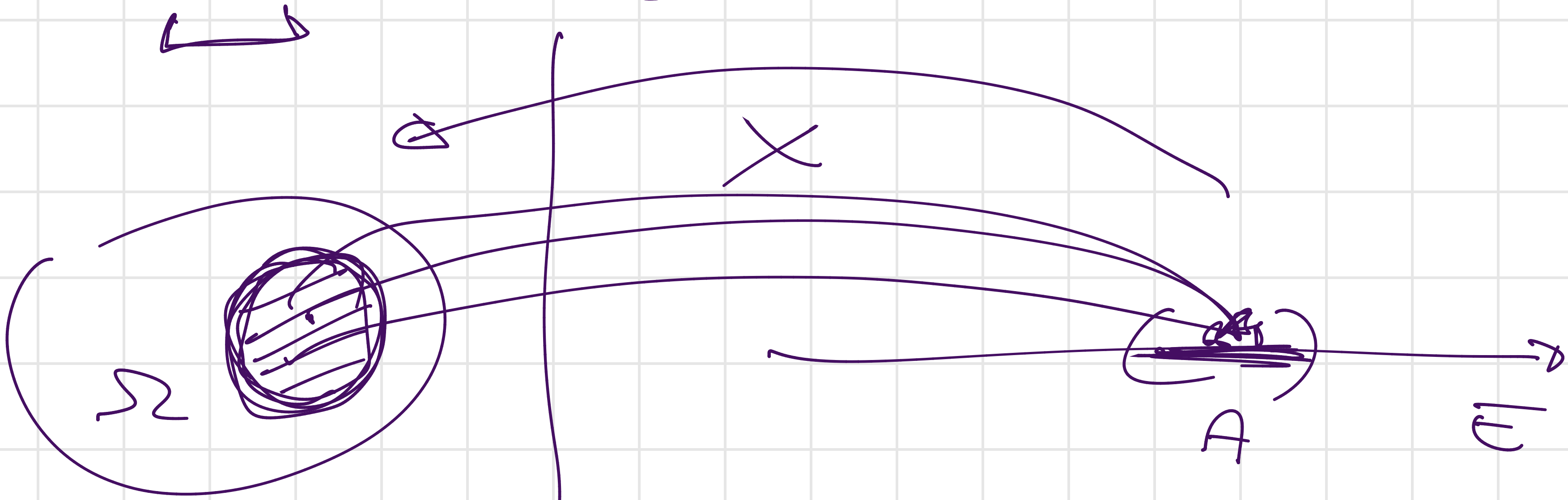
Let E be a set ; a E-valued random variable, defined on the prob. space $(\Omega, \mathcal{A}, \mathbb{P})$ is a function

$$\boxed{X : \Omega \longrightarrow E}$$

s.t. . . . . $A \longmapsto \underbrace{X^{-1}(A)}$

$\boxed{A \subseteq E}$, we can define the counter image of A through X $E = \mathbb{R}$

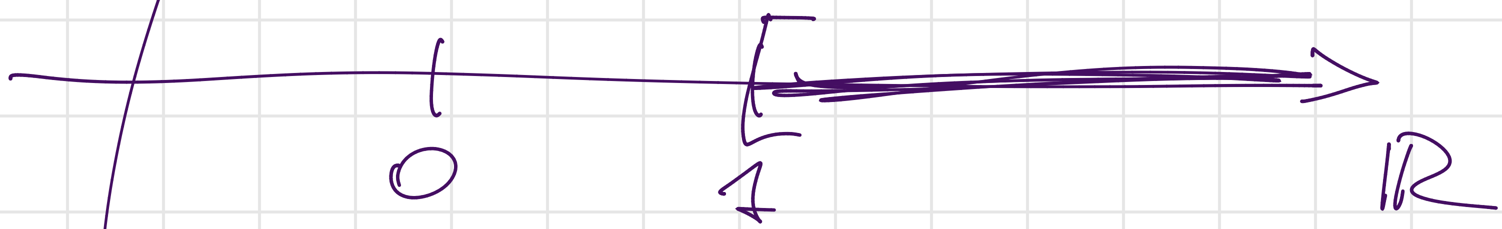
$$\Omega \supseteq \underbrace{X^{-1}}(A) = \{\omega \in \Omega : X(\omega) \in A\}$$



$$= \{X \in A\} \subseteq \Omega$$

$X : \Omega \longrightarrow \mathbb{R}$

$P\left(\boxed{\underbrace{\{X \text{ takes values larger than } 1\}}}\right)$



$\{\omega : X(\omega) \geq 1\}$

$= \{\omega : X(\omega) \in [1, +\infty)\}$

$= \{X \in [1, +\infty)\}$

$= \underbrace{X^{-1}([1, +\infty))}_{\text{I can compute the probability}} \in \mathcal{A}$

$X : \Omega \longrightarrow E,$

$X^{-1}(A) \longleftarrow A$

$(E, \mathcal{E})$

$\mathcal{E}$ a σ-field on E

$$(\Omega, \mathcal{A}, \mathbb{P}) \xrightarrow{X} (E, \mathcal{E})$$

$$\omega \longmapsto X(\omega) \in E$$

$$\forall \underbrace{A \in \mathcal{E}}, \quad \underbrace{X^{-1}(A)} \in \mathcal{A}$$

X will be a random variable.

$$\mu_X : \underbrace{\mathcal{E}} \longrightarrow [0,1]$$

$$\overset{\in}{A} \longmapsto \mu_X(A) := \underline{\mathbb{P}}[X^{-1}(A)]$$

μ_X <u>is called</u> <u>"the distribution" or</u>

<u>the law</u> of X

Thm: If $X : \underbrace{(\Omega, \mathcal{A}, \mathbb{P})}$ $\longrightarrow (E, \mathcal{E})$ is a random variable

$(E, \mathcal{E}, \mu_X)$ is a prob. space

$(\Omega, \mathcal{A}, \mathbb{P}) \longrightarrow (E, \mathcal{E}, \mu_X)$

$E = \mathbb{R}$, $\mathcal{E} = \mathcal{B}(\mathbb{R})$ real random variables

$E = \mathbb{R}^d$, $\mathcal{E} = \mathcal{B}(\mathbb{R}^d)$ random vectors

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X $(\Omega, \mathcal{A}, \mathbb{P}) \longrightarrow \boxed{(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_X)}$

$\underline{X(\omega)}$

Discrete Random Variables

Def: We say that a r.v. $X : \Omega \to E$ is discrete if there exists $\underbrace{N}_{} \in \mathcal{E}$, at most countable, such that

$$\mathbb{P}[X \in N] = \mu_X[N] = 1$$

N is called "a support of μ_X"

$$\underbrace{N_1}_{} = N \cup \{e\} \in \mathcal{E}$$

If X is discrete, we can define the map

$$p_X : E \longrightarrow [0,1]$$
$$\underbrace{x}_{\in} \longmapsto \mu_X(\{x\}) = P[X=x]$$

p_X is called the (discrete) $\boxed{\text{density}}$ of the r.v. X.

Thm: If X is a E-valued, discrete r.v., then $\forall A \in \mathcal{E}$

$$\mathbb{P}[X \in A] = \mu_X(A) = \sum_{x \in A} p_X(x)$$

Remark:

Any function $p : \boxed{E} \to [0,1]$ s.t. there exists $\underbrace{N}\subseteq E$ finite or countable s.t.

1. $\underbrace{p(x) = 0}$ $\quad \forall x \notin N$
2. $\underbrace{p(x) \geq 0}$ $\quad \forall x \in N$
3. $\displaystyle\sum_{x \in E} p(x) = \underbrace{\sum_{x \in N} p(x)} = 1$

is a possible density for a discrete r.v. whose support is N.

$$p(x) = \mathbb{P}[X = x]$$

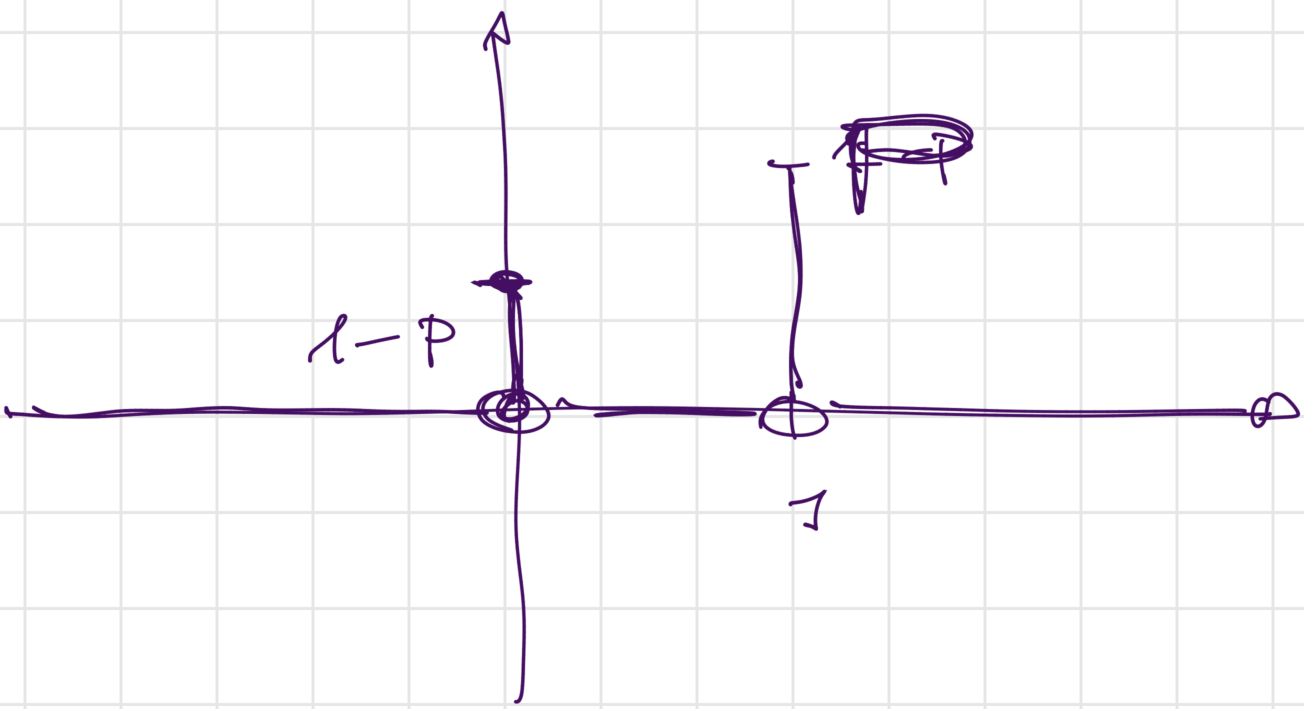
$$X \in \{0,1\} \subseteq \mathbb{R}$$

$$\mathbb{P}[X=1] = p \in [0,1], \quad \mathbb{P}[X=0] = 1-p$$

$$N = \{0,1\}$$

$$p_1 = \mathbb{P}[X=1] = p$$

$$p_0 = \mathbb{P}[X=0] = 1-p = q$$



$\mathrm{Ber}(1,p)$