DATA SCIENCE **Stochastic Methods**	Name:__________________________
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Problem 1. [8] Let $X_1, \ldots, X_n$ be a family of i.i.d. random variables with uniform distribution on the interval $[0,1]$.

(i) Prove that

$$\mathrm{P}[\max(X_1, \ldots, X_n) \leq x] = \begin{cases} 0 & x < 0 \\ x^n & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

(ii) Compute

$$\mathrm{E}[\max(X_1, \ldots, X_n)]$$

Solution

(i) Let $Y = \max(X_1, \ldots, X_n)$. By the independence

$$\mathrm{F}_Y(y) = \mathrm{F}_{X_1}(y) \cdot \mathrm{F}_{X_2}(y) \cdot \ldots \cdot \mathrm{F}_{X_n}(y) = \Big[\mathrm{F}_{X_1}(y)\Big]^n .$$

(ii) Since $Y \geq 0$:

$$\begin{aligned} E[Y] &= \int_0^1 (1 - F_Y(y))\, dy = \big[y\big]_0^1 - \int_0^1 \Big(y\Big)^n dy \\ &= 1 - \left[\frac{y^{n+1}}{n+1}\right]_0^1 = 1 - \frac{1}{n+1} = \frac{n}{n+1} . \end{aligned}$$

Problem 2. [8] A discrete time Markov chain $\{X_n, n \geq 0\}$ with state space $S = \{0,1,2,3\}$ has transition probability matrix

$$P = \begin{bmatrix} 1/4 & 1/4 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(i) Determine the communicating classes.

(ii) Find the stationary distribution of the Markov chain.

(iii) Is this distribution reversible?

(iv) If $X_0 = 3$, compute the expected number of steps until the first time the Markov chain will return to state 3?

Solution

(i) The Markov chain is irreducible and regular ($p_{i,j}^{(4)} > 0$ for any $i, j \in S$).

(ii) The unique distribution solution of the system $\pi = \pi P$ is

$$\pi = (\pi_0, \pi_1, \pi_2, \pi_3) = \left(\frac{4}{25}, \frac{6}{25}, \frac{10}{25}, \frac{5}{25}\right)$$

(iii) No, since $p_{0,2} > 0$ and $p_{2,0} = 0$.

(iv) Since the Markov Chain is ergodic, the expected number m_3 of steps starting from 3 until the first time the Markov chain will return to state 3 is equal to $\pi_3^{-1} = 25/5 = 5$.

Problem 3. [10] Let $S = \{M, M \subset \{1,\dots,N\}\}$ be the set of subsets of $\{1,\dots,N\}$. Define $(X_n)_{n\geq 0}$ to be a Markov Chain with state space S, such that, given $X_n = M$, the state X_{n+1} is determined as follows:

- Choose uniformly a number $i \in \{1,\dots,N\}$.
- If $i \notin M$, set $X_{n+1} = M \cup \{i\}$. Otherwise, set $X_{n+1} = M$ or $X_{n+1} = M \setminus \{i\}$, both with probability $\frac{1}{2}$.

(i) Prove that this MC has a unique invariant distribution.

(ii) Determine the off-diagonal elements of the transition matrix.

(iii) Determine the invariant distribution.

Solution

(i) The MC is irreducible, since any set can be reached with positive probability from any other set (by first selecting all vertices which belong to the second, but not the first set, and then selecting all vertices that belong to the first, but not the second set, and flipping Tail in step 2.). It is also aperiodic, since $X_{n+1} = X_n$ if a vertex from X_n is selected, and Head is flipped in step 2. Since the state space is finite, this implies existence and uniqueness of the invariant measure.

(ii) The off-diagonal elements are

$$P_{M,M'} = \begin{cases} \frac{1}{N} & M' = M \cup \{i\} \text{ with } i \notin M \\ \frac{1}{2N} & M = M' \cup \{i\} \text{ with } i \notin M' \\ 0 & \text{else} \end{cases}$$

(iii) The invariant measure is

$$\pi_M = \text{const } 2^{|M|},$$

where $|M|$ is the number of elements of the subset M. Indeed, if $M = M' \cup \{i\}$,

$$\pi_M P_{M,M'} = \text{const } 2^{|M|} \frac{1}{2N} = \text{const } 2^{|M|-1} \frac{1}{N} = \pi_{M'} P_{M',M},$$

and so this choice satisfies the detailed balance equations, and is therefore the invariant measure. The constant must be determined from the normalization condition:

$$\text{const} = \Big[\sum_{M\in S} 2^{|M|}\Big]^{-1}.$$

(Advanced remark: the sum equals 3^N, and so const $= 3^{-N}$.).
Here is a suggestion about how to guess π_M: The algorithm described in steps 1 and 2 is just the Hastings algorithm, which adopts a proposed set M' (either $M' = M \cup \{i\}$ or $M' = M \setminus \{i\}$) with probability $\min\{1, \frac{\pi_{M'}}{\pi_M}\}$. Thus, looking at step 2, we know that $\frac{\pi_{M'}}{\pi_M} = \frac{1}{2}$ if $M' = M \setminus \{i\}$. In other words, π_M decreases by a factor $\frac{1}{2}$ if we delete a point from M, and so $\pi_M = \text{const } 2^{|M|}$

Problem 4. [10] Let $(X_i)_{1\leq i\leq n}$ be a family of i.i.d. Bernoulli random variables.

(i) Defined $X = X_1 + \ldots + X_n$ and $\mu = \mathrm{E}[X]$, prove:

$$\mathrm{P}(X \geq (1+\delta)\mu) \leq e^{-\mu\frac{\delta^2}{2+\delta}},$$

for $0 < \delta < 1$.

(ii) We toss a fair coin 200 times. Find un upper bound for the probability to see at least 120 heads.

Solution

(i) Note that $\mu = np$. So the required formula is simply the upper tail estimate of the Chernoff bounds proved in class in the case of i.i.d. Bernoulli r.v.'s.

(ii) In this special case $p = 1/2$ and $n = 200$; so $\mu = 100$ and the previous estimate reads

$$\mathrm{P}(X \geq 120) = \mathrm{P}(X \geq (1+0.2)100) \leq e^{-100\frac{(0.2)^2}{2.2}} \approx 0.1623$$