DATA SCIENCE **Stochastic Methods**

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Solutions

Problem 1. [10] Let $X \sim Exp(\lambda)$ and define $Y = \min\{X, 3\}$ and $Z = \max\{Y, 1\}$.

(i) Compute $P[Z = 1]$;

(ii) Compute $P[Z \le z]$ and $P[Z^2 \le z]$ for any $z \in \mathbb{R}$;

(iii) Compute $E[Z]$ and $Var[Z]$.

(i) $\mathbb{P}[Z=1] = \mathbb{P}[X \le 1] = 1 - e^{-\lambda}$

(ii)
$$\mathbb{P}[Z \le z] = \begin{cases} 0 & z < 1 \\ 1 - e^{-\lambda z} & 1 \le z < 3 \\ 1 & z \ge 3 \end{cases}, \quad \mathbb{P}[Z^2 \le z] = \begin{cases} 0 & z < 1 \\ 1 - e^{-\lambda \sqrt{z}} & 1 \le z < 9 \\ 1 & z \ge 9 \end{cases}$$

(iii)
$$\mathbb{E}[Z] = \int_0^{+\infty} \mathbb{P}[Z > z]\,dz = \int_0^1 1\,dz + \int_1^3 e^{-\lambda z}\,dz$$
$$= 1 + \left[\frac{e^{-\lambda z}}{-\lambda}\right]_1^3 = 1 + \frac{e^{-\lambda} - e^{-3\lambda}}{\lambda}$$
$$\mathbb{E}[Z^2] = \int_0^{+\infty} \mathbb{P}[Z^2 > z]\,dz = \int_0^1 1\,dz + \int_1^9 e^{-\lambda\sqrt{z}}\,dz = 1 + 2\int_1^3 t\, e^{-\lambda t}\,dt =$$
$$= 1 + 2\left[t\,\frac{e^{-\lambda t}}{-\lambda}\right]_1^3 - 2\int_1^3 \frac{e^{-\lambda t}}{-\lambda}\,dt = 1 + \frac{2}{\lambda}e^{-\lambda} - \frac{6}{\lambda}e^{-\lambda} + \frac{2}{\lambda}\int_1^3 e^{-\lambda t}\,dt$$
$$= 1 + \frac{2}{\lambda}e^{-\lambda} - \frac{6}{\lambda}e^{-3\lambda} + \frac{2}{\lambda^2}e^{-\lambda} - \frac{2}{\lambda^2}e^{-3\lambda}$$
$$Var[Z] = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2$$

Problem 2. [10] Let $(Y_i)_{1\le i\le n}$ be a family of i.i.d. Standard Normal random variables and define $Z_i = Y_i^2$.

(i) Compute the expectation of Z_1;

(ii) Compute the mgf of $Z_1 + Z_2$;

(iii) Defined $\overline{Z}_n = \frac{1}{n}\sum_{i=1}^{n} Z_i$, prove that

$$\mathrm{P}(\overline{Z}_n - 1 \le -\varepsilon) \le e^{-n\frac{\varepsilon^2}{8}},$$

for $0 < \varepsilon < 1$.

(i) $\mathbb{E}[Z_1] = \mathbb{E}[Y_1^2] = 1$

(ii) $\mathbb{E}[e^{u(Z_1+Z_2)}] = \mathbb{E}[e^{uZ_1}]\cdot\mathbb{E}[e^{uZ_2}] = \left(\frac{1}{\sqrt{1-2u}}\right)^2$

$= \frac{1}{1-2u}$

(iii) $\mathbb{P}[\overline{Z}_n \le 1-\varepsilon] \le e^{-n\varepsilon^2/4} \le e^{-n\varepsilon^2/8}$

(the proof is on the Lecture Notes)

Problem 3. [12] Let $(X_n)_{n\geq 0}$ be a Markov chain on $\mathbb{N}\setminus\{0\} = \{1,2,3,\dots\}$ with transition probabilities given by

$$p_{i,1} = \frac{i}{i+1}\ , \quad p_{i,i+1} = \frac{1}{i+1}\ , \quad i \geq 1$$

(i) Is the Markov chain irreducible?

(ii) Is the Markov chain aperiodic?

(iii) Compute $E[X_3|X_0 = 1]$;

(iv) Determine the invariant distribution.

(i) YES : $P_{n,1} = \frac{n}{n+1} > 0$ and $P_{1,n} \geq \frac{1}{2}\cdot\frac{1}{3}\cdot\dots\cdot\frac{1}{n} > 0$

(ii) YES : $P_{11}^{(2)} = \frac{1}{2}\cdot\frac{2}{3} > 0$, $P_{11}^{(3)} = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{3}{4} > 0$

(iii) $\mathbb{P}[X_3 = 4 | X_0 = 1] = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4} = \frac{1}{24}$, $\mathbb{P}[X_3 = 3 | X_0 = 1] = \frac{1}{12}$

$\mathbb{P}[X_3 = 2 | X_0 = 1] = \frac{7}{24}$, $\mathbb{P}[X_3 = 1 | X_0 = 1] = \frac{14}{24}$

$$E[X_3 | X_0 = 1] = \frac{14}{24} + \frac{14}{24} + \frac{6}{24} + \frac{4}{24} = \frac{19}{12}$$

(iv)
$$\begin{cases} \pi_2 = \frac{1}{2}\pi_1 \\ \pi_3 = \frac{1}{3}\pi_2 = \frac{1}{2}\cdot\frac{1}{3}\pi_1 = \frac{1}{3!}\pi_1 \\ \pi_4 = \frac{1}{4}\pi_3 = \frac{1}{4!}\pi_1 \\ \vdots \\ \pi_n = \frac{1}{n!}\pi_1 \\ \vdots \end{cases}$$

$$\Rightarrow \pi_1\left(1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \dots\right) = 1$$

$$\Rightarrow \pi_1 = \frac{1}{e-1}$$

$$\pi = \frac{1}{e-1}\left(1, \frac{1}{2!}, \frac{1}{3!}, \dots, \frac{1}{n!}, \dots\right)$$