PROBLEMS - SET 2

Problem 1. Prove that the distribution μ_X of the random variable X is a probability.

Solution 1. Note that $\{X \in E\} = \Omega$, so

$$\mu_X(E) = \mathrm{P}(X \in E) = \mathrm{P}(\Omega) = 1.$$

Let now be $(A_n)_{n\geq 1}$ be disjoint elements of $\mathscr{E}$. Then the events $\{X \in A_n\}$ are disjoint, so by σ-additivity of P

$$\mu_X\left(\bigcup_n A_n\right) = P\left(X \in \bigcup_n A_n\right) = \mathrm{P}\left(\bigcup_n \{X \in A_n\}\right) = \sum_n \mathrm{P}(X \in A_n) = \sum_n \mu_X(A_n).$$

Problem 2. Prove that the distribution function F_X is

(i) non decreasing,

(ii) $\lim_{x\to -\infty} F_X(x) = 0$,

(iii) $\lim_{x\to +\infty} F_X(x) = 1$,

(iv) right continuous.

Solution 2. (i) Since $(-\infty, x] \subset (-\infty, y]$, then

$$F_X(x) = \mu_X(((-\infty, x]) \leq \mu_X(((-\infty, y]) = F_X(y).$$

(ii), iii) It is enough to show that

$$\lim_{n\to+\infty} F_X(-n) = 0, \quad \lim_{n\to+\infty} F_X(n) = 1.$$

We prove the first statement, the second is similar. Set $A_n := (-\infty, -n]$. Note that A_n is a decreasing sequence of events, and $\bigcap_n A_n = \emptyset$. So, using upper continuity of the probability

$$\lim_{n\to+\infty} F_X(-n) = \lim_{n\to+\infty} \mu_X(A_n) = \mu_X\left(\bigcap_n A_n\right) = \mu_X(\emptyset) = 0.$$

(iv) It is enough to show that

$$\lim_{n\to+\infty} F_X\left(x + \frac{1}{n}\right) = F_X(x).$$

Set $A_n := (-\infty, x + \frac{1}{n}]$. A_n is a decreasing sequence of events, and $\bigcap_n A_n = (-\infty, x]$. So, using upper continuity of the probability

$$\lim_{n\to+\infty} F_X\left(x + \frac{1}{n}\right) = \lim_{n\to+\infty} \mu_X(A_n) = \mu_X\left(\bigcap_n A_n\right) = \mu_X((-\infty, x]) = F_X(x).$$

Problem 3. An urn contains 8 white balls and 4 black balls. You toss a fair coin: if it shows *head* you make two draws *with* replacement, otherwise you make two draw *without* replacement. Let X be the number of white balls drawn. Compute mean and variance of X.

Solution 3. Clearly X can take the values $0,1,2$.

$$\begin{aligned} \mathrm{P}(X=1) &= \mathrm{P}(X=1|\text{head})\,\mathrm{P}(\text{head})+\mathrm{P}(X=1|\text{tail})\,\mathrm{P}(\text{tail}) \\ &= \frac{1}{2}\left[\mathrm{P}(X=1|\text{head})+\mathrm{P}(X=1|\text{tail})\right]. \end{aligned}$$

Moreover

$$\begin{aligned} \mathrm{P}(X=1|\text{head}) &= 2\cdot\frac{4}{12}\cdot\frac{8}{12}=\frac{4}{9} \\ \mathrm{P}(X=1|\text{tail}) &= \frac{8\cdot 4}{\binom{12}{2}}=\frac{16}{33}, \end{aligned}$$

so $\mathrm{P}(X=1)=\frac{46}{99}$. Similarly

$$P(X=2)=\frac{1}{2}\left[\mathrm{P}(X=2|\text{head})+\mathrm{P}(X=2|\text{tail})\right],$$

and

$$\begin{aligned} \mathrm{P}(X=2|\text{head}) &= \left(\frac{8}{12}\right)^2=\frac{4}{9} \\ \mathrm{P}(X=2|\text{tail}) &= \frac{\binom{8}{2}}{\binom{12}{2}}=\frac{14}{33}, \end{aligned}$$

so $\mathrm{P}(X=2)=\frac{43}{99}$. Finally:

$$\begin{aligned} \mathrm{E}(X) &= \frac{46}{99}+2\cdot\frac{43}{99}=\frac{132}{99}, \\ \mathrm{E}(X^2) &= \frac{46}{99}+4\cdot\frac{43}{99}=\frac{218}{99}. \end{aligned}$$

So

$$\mathrm{Var}(X)=\frac{218}{99}-\left(\frac{132}{99}\right)^2=\frac{14}{33}.$$

Problem 4. In the context of Problem 5 of set 1, let X denote the number of individuals, different from 1 and 2, which are friends with 1 but *not* friends with 2. Compute the density of X.

Solution 4. Each individual $i=3,4,\dots,n$ has probability $\frac{1}{4}$ of being friend with 1 but not with 2, independently of the others.

$$\mathrm{P}(X=k)=\binom{n-2}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{n-2-k}.$$

Problem 5. 120 students are divided into three groups, called A,B and C, containing respectively 36,40 and 44 students.

(i) Choose at random a group (I mean: each group has the same probability of being chosen), and let X be the number of students in the chosen group. Determine the density of X.

(ii) Choose a student at random, and let Y denote the number of students in his/her group. Determine the density of Y.

Solution 5. Note that both X and Y can take the values $\{36, 40, 44\}$. First

$$\mathrm{p}_X(36) = \mathrm{p}_X(40) = \mathrm{p}_X(44) = \frac{1}{3}.$$

Then

$$\mathrm{p}_Y(36) = \frac{36}{120} = \frac{3}{10}, \qquad \mathrm{p}_Y(40) = \frac{40}{120} = \frac{1}{3}, \qquad \mathrm{p}_Y(44) = \frac{44}{120} = \frac{11}{30}.$$

Problem 6. Let X be a E-valued random variable. Show that there is at most one $c \in E$ such that $\mathrm{P}(X = c) > \frac{1}{2}$.

Solution 6. If c_1 and c_2 have that property and $c_1 \neq c_2$,

$$\mathrm{P}(X \in \{c_1, c_2\}) = \mathrm{P}(X = c_1) + \mathrm{P}(X = c_2) > 1$$

which is absurd.