SOLUTION

STOCHASTIC METHODS - June 17, 2021

Problem 1. [12]
Let X_1, X_2 and X_3 be three absolutely continuous, independent, Uniform $(0,1)$ random variables. Define $X_{(1)} = \min\{X_1, X_2, X_3\}$ and $X_{(3)} = \max\{X_1, X_2, X_3\}$.

(i) Compute the density and the expectation of $X_{(1)}$;

(ii) Compute the density and the expectation of $X_{(3)}$;

(iii) Compute the joint density of $(X_{(1)}, X_{(3)})$.

Hint: evaluate first $P[X_{(1)} > x, X_{(3)} \le y]$ and use this to compute the joint distribution of $(X_{(1)}, X_{(3)})$, i.e. $P[X_{(1)} \le x, X_{(3)} \le y]$. Then differentiate the distribution with respect to x and y.

(i) $$\underbrace{\mathbb{P}[X_{(1)} > x]}_{= 1 - F_{(1)}(x)} = \mathbb{P}[X_1 > x, X_2 > x, X_3 > x] \overset{\text{IND}}{=} \prod_{i=1}^{3} \mathbb{P}[X_i > x] = (1 - F(x))^3$$

$$\Rightarrow F_{(1)}(x) = 1 - (1-F(x))^3 = \begin{cases} 0 & x < 0 \\ 1-(1-x)^3 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$f_{(1)}(x) = \frac{d}{dx} F_{(1)}(x) = \begin{cases} 3(1-x)^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\mathbb{E}[X_{(1)}]} = \int_0^1 (1 - F_{(1)}(x))\,dx = \int_0^1 (1-x)^3\,dx = \left[-\frac{(1-x)^4}{4}\right]_0^1 = \boxed{1/4}$$

(ii) Similarly, $$F_{(3)}(x) = \begin{cases} 0 & x \le 0 \\ x^3 & 0 < x \le 1 \\ 1 & x \ge 1 \end{cases}, \quad f_3(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\mathbb{E}[X_{(3)}]} = \int_0^1 (1 - x^3)\,dx = \left[x - \frac{x^4}{4}\right]_0^1 = 1 - 1/4 = \boxed{3/4}$$

(iii) $$\mathbb{P}[X_{(1)} > x, X_{(3)} \le y] = \prod_{i=1}^{3} \mathbb{P}[x < X_i \le y] = (y-x)^3 \qquad 0 \le x \le y \le 1$$

$$F_{X_{(1)}, X_{(3)}}(x,y) = \mathbb{P}[X_{(3)} \le y] - \mathbb{P}[X_{(1)} > x, X_{(3)} \le y] = y^3 - (y-x)^3 \qquad 0 \le x \le y \le 1$$

$$\Rightarrow f_{(X_{(1)}, X_{(3)})} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{X_{(1)}, X_{(3)}} = 6(y-x) \qquad 0 < x < y < 1$$

$$= 0 \qquad \text{otherwise}$$

Problem 2. [12] Let $(X_i)_{1\le i\le n}$ be a family of i.i.d. Standard Normal random variables and define $Y_i = X_i^2$.

(i) Compute the probability that $Y_1 < Y_2$;

(ii) Compute the expectation of Y_1;

(iii) Defined $\overline{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i$, prove that

$$\mathrm{P}(\overline{Y}_n - 1 \ge \varepsilon) \le e^{-n\frac{\varepsilon^2}{4(1+\varepsilon)}},$$

for $\varepsilon > 0$.

(i) $\mathbb{P}[Y_1 < Y_2] = \mathbb{P}[Y_2 < Y_1]$ (since Y_1 and Y_2 are iid)

since are continuous, $\mathbb{P}[Y_1 = Y_2] = 0 \Rightarrow \mathbb{P}[Y_1 < Y_2] = 1/2$

(ii) $\mathbb{E}[Y_1] = \mathbb{E}[X_1^2] = \mathrm{Var}[X_1] + (\mathbb{E}[X_1])^2 = 1 + 0 = 1$

since $X_1 \sim N(0,1)$

(iii) (see the Lecture Notes)

Problem 3. [12] Two servers are used to support the e-mail of a department, only one of which is in operation at any given time. A server may break down on any given day with probability p. It takes 2 days to restore the server to normal and only one server at a time can be repaired.

(i) Define a **four states** $\{(2,0),(1,0),(1,1),(0,1)\}$ suitable Markov Chain, where the first number indicates how many servers are in operating condition at the end of a day and the second is equal to 1 if a day's labor has been expended on a server not yet repaired and 0 otherwise;

(ii) Classify the states of this Markov Chain;

(iii) Compute the invariant distribution;

(iv) Determine the long run probability that both the servers are inoperative.

Denote $q = 1-p$

(i)

$$P = \begin{array}{c} \\ (2,0) \\ (1,0) \\ (1,1) \\ (0,1) \end{array} \begin{array}{c} \begin{array}{cccc} (2,0) & (1,0) & (1,1) & (0,1) \end{array} \\ \begin{bmatrix} q & p & 0 & 0 \\ 0 & 0 & q & p \\ q & p & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

(ii)

The Chain is irreducible and aperiodic.

(iii)

$\pi P = \pi$

$$\begin{cases} q\pi_1 + q\pi_3 = \pi_1 \\ p\pi_1 + p\pi_3 + \pi_4 = \pi_2 \\ q\pi_2 = \pi_3 \\ p\pi_2 = \pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases} \Rightarrow \boxed{\begin{cases} \pi_1 = \dfrac{q^2}{1+p^2} \\ \pi_2 = p/1+p^2 \\ \pi_3 = pq/1+p^2 \\ \pi_4 = \dfrac{p^2}{1+p^2} \end{cases}}$$

(iv)

$$\pi_4 = \frac{p^2}{1+p+p^2}$$