PROBLEM - SET 1

Problem 1. Consider the random experiment of rolling twice a balanced die with six faces.

(a) Describe the probability space for this random experiment.

(b) Compute the probability the number 6 appears *exactly* once in the outcomes.

(c) Compute the probability the number 6 appears *at least* once in the outcomes *knowing* that the total score is 9.

Solution 1. (a) $\Omega = \{1,2,3,4,5,6\}^2$, P uniform probability on Ω.

(b) $10/36$.

(c) Set $A =$ " the total score is 9", $B =$ "6 appears at least once".

$$A = \{(3,6),(4,5),(5,4),(6,3)\}, \ \ A \cap B = \{(3,6),(6,3)\}.$$

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} = \frac{2/36}{4/36} = \frac{1}{2}.$$

Problem 2. A deck of 52 cards is accurately shuffled.

(a) Describe the probability space for this random experiment.

(b) What is the probability that the ace of spades is found in its original (before shuffling) position?

(c) What is the probability that the four aces occupy their four original positions, but not necessarily in the same order?

(d) What is the probability that the ace of spades and the ace of clubs are found in neighboring positions?

Solution 2. (a) $\Omega = S_{52} =$ set of permutation of 52 objects. P uniform probability on Ω. $|\Omega| = 52!$.

(b) There are $51!$ permutations leaving the ace of spades in the same position, so the required probability is $51!/52! = \frac{1}{52}$.

(c) The 48 cards different from the aces can be permuted arbitrarily, there are $48!$ different possibilities. The four aces can exchange their positions in $4! = 24$ ways. So the number of permutations leaving the aces in their original positions, but not necessarily in the same ordes, is $24 \cdot 48!$. Thus the required probability is

$$\frac{24 \cdot 48!}{52!} = \frac{24}{52 \cdot 51 \cdot 50 \cdot 49}.$$

(d) Let us count the permutations for which the ace of spades immediately precedes the ace of clubs. We have 51 possible positions of the ace of spades, with the ace of clubs following it. The other cards can be rearranged in $50!$ ways. The probability of this event is then $\frac{51 \cdot 50!}{52!}$. The probability other case, so the case in which the ace of clubs immediately precedes the ace of spades, can be computed in a completely analogous way. So the required probability is

$$2\frac{51\cdot 50!}{52!}=\frac{2}{52}=\frac{1}{26}.$$

Problem 3. An urn contains 10 red balls and 20 black balls. We make five successive draws *without replacement* (drawn balls are *not* re-inserted in the urn).

(a) Describe the probability space for this random experiment.

(b) Compute the probability of drawing *at least* a black ball.

(c) Knowing that at least one red ball was drawn, compute the probability of drawing *at least* a black ball.

Solution 3. (a) Ω is the family of all subsets of five balls chosen among the 30. In particular $|\Omega|=\binom{30}{5}$. P is the uniform probability on Ω.

(b) Let A be the event whose probability is required. Then A^c contains all subsets of five balls chosen among the red ones, so $|A^c|=\binom{10}{5}$. Finally

$$\mathrm{P}(A)=1-\mathrm{P}(A^c)=1-\frac{\binom{10}{5}}{\binom{30}{5}}.$$

(c) Let $B=$ "at least one red ball was drawn" and $C=$ "at least one black ball was drawn". As above

$$\mathrm{P}(B)=1-\frac{\binom{20}{5}}{\binom{30}{5}}.$$

Note that $(B\cap C)^c=B^c\cup C^c$ is the family of subsets of five balls which are either all red of all black. Thus

$$|(B\cap C)^c|=\binom{10}{5}+\binom{20}{5}.$$

So

$$\mathrm{P}(B\cap C)=1-\frac{\binom{10}{5}+\binom{20}{5}}{\binom{30}{5}}.$$

The required conditional probability $\mathrm{P}(C|B)=\frac{\mathrm{P}(B\cap C)}{\mathrm{P}(B)}$ is therefore easily computed.

Problem 4. Consider a set of n individuals, identified with the numbers $\{1,2,\dots .n\}$. To each individual i we assign a random binary label $\sigma_i\in\{0,1\}$, in such a way that all assignments are equally likely.

(a) Describe the probability space for this random experiment.

(b) What is the probability that exactly k individuals have label 1 ($0\le k\le n$)? For which values of k this probability is maximized?

Solution 4. (a) $\Omega=\{0,1\}^n$, with the uniform probability. $|\Omega|=2^n$.

(b) There are $\binom{n}{k}$ such assignments, since we just need to chose the k individuals with label 1. Thus the required probability is

$$p_k := \frac{1}{2^n}\binom{n}{k}.$$

Note that
$$\frac{p_{k+1}}{p_k} = \frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{k!(n-k)!}{(k+1)!(n-k-1)!} = \frac{n-k}{k+1}.$$

Thus
$$\frac{p_{k+1}}{p_k} \geq 1 \iff n-k \geq k+1 \iff k \leq \frac{n-1}{2}.$$

This implies that if n is odd the maximum is attained at $k = \frac{n-1}{2}$.
If n is even p_k are increasing for $k \leq \frac{n}{2} - 1$ and decreasing for $k \geq \frac{n}{2}$. We have to compare $p_{\frac{n}{2}-1}$ and $p_{\frac{n}{2}}$. Note that

$$\binom{n}{\frac{n}{2}} - \binom{n}{\frac{n}{2}-1} = \frac{n!}{(n/2-1)!(n/2)!}\left[\frac{1}{n/2} - \frac{1}{n/2+1}\right] > 0$$

and so $p_{\frac{n}{2}-1} < p_{\frac{n}{2}}$ and the maximum is attained at $k = \frac{n}{2}$.

Problem 5. Consider again a set of n individuals. To each *unordered pair* $\{i, j\}$, with $i \neq j$, we assign a random binary label $\sigma_{ij} \in \{0,1\}$, in such a way that all assignments are equally likely. If $\sigma_{ij} = 1$ we say i and j are *friends*.

(a) Describe the probability space for this random experiment.
(b) What is the probability that individual 1 has exactly k friends?
(c) What is the probability that 1 is friend of 2, 2 is friend of 3 but 1 is *not* friend of 3?

Solution 5. (a) There are $\frac{n(n-1)}{2}$ unordered pair $\{i, j\}$, with $i \neq j$. Thus $\Omega = \{0,1\}^{\frac{n(n-1)}{2}}$, with the uniform probability.

(b) There are $n-1$ unordered pairs involving individual 1. We can chose in $\binom{n-1}{k}$ different ways his friends. The other $\frac{n(n-1)}{2} - (n-1) = \frac{(n-1)(n-2)}{2}$ pairs can be abitrarily labeled. So the required probability is

$$\frac{\binom{n-1}{k}2^{\frac{(n-1)(n-2)}{2}}}{2^{\frac{n(n-1)}{2}}} = \frac{\binom{n-1}{k}}{2^{n-1}}$$

(not surprising, why?).

(c) We are just assigning label 1 to $\{1,2\}$ and $\{2,3\}$ and label 0 to $\{1,3\}$. The other $\frac{n(n-1}{2} - 3$ pairs are arbitrarily labeled. Thus the required probability is

$$\frac{2^{\frac{n(n-1)}{2}-3}}{2^{\frac{n(n-1)}{2}}} = \frac{1}{8}.$$

Problem 6. A set A of n elements is randomly partitioned into two *nonempty* subsets B and B^c. All such partitions are equally likely.

(a) Describe the probability space for this random experiment.

(b) Two distinct elements x and y of A are said to be connected if they belong to the same element of the partition (i.e. to either B or B^c). Given x and y, what is the probability that they are connected?

(c) An element $x \in A$ is said to be isolated if it is not connected to any other element of A. Find the probability that there exists an isolated element.

Solution 6. (a) We can take $\Omega := \{B \subseteq A : B \neq \emptyset, B \neq A\}$, with the uniform probability. Note that $|\Omega| = 2^n - 2$. Note that with this choice we are considering *ordered* partitions, i.e. $(B, B^c) \neq (B^c, B)$. It would be equally correct to take unordered partitions: cardinalities would be divided by two.

(b) The event "x and y are connected" is the disjoint union of

$$E := \{B \in \Omega : x, y \in B\}$$

$$F := \{B \in \Omega : x, y \in B^c\}.$$

By symmetry, E and F have the same number of elements. Observing that choosing an element of E is the same as choosing a subset of $A \setminus \{x, y\}$ except $A \setminus \{x, y\}$ itself, we have

$$|E| = 2^{n-2} - 1,$$

so that

$$P(E \cup F) = \frac{2(2^{n-2} - 1)}{2^n - 2} = \frac{2^{n-2} - 1}{2^{n-1} - 1}.$$

(c) An isolated element exists if and only if $|B| = 1$ or $|B^c| = 1$. Thus the required probability is

$$\frac{2n}{2^n - 2}.$$