Lecture 9 — Stoch. Meth. — Oct. 25th, 2024

Variance

$$\underbrace{\mathbb{E}\left[\left(X-\mathbb{E}(X)\right)^2\right]}_{\shortparallel} = Var[X]$$

$$\underbrace{\mathbb{E}[X^2]}_{\substack{\wedge \\ +\infty}} - \left(\mathbb{E}[X]\right)^2 \qquad (X-\mathbb{E}[X])(X-\mathbb{E}[X])$$

$$X \in L^2$$

discrete case $\quad \sum x^2 p_X(x) < +\infty$

$\boxed{X, Y \in L^2}$, we can define their covariance

$$Cov(X,Y) := \mathbb{E}\left[\left(X-\mathbb{E}[X]\right)\cdot\left(Y-\mathbb{E}[Y]\right)\right]$$

$$= \mathbb{E}[X\cdot Y] - \mathbb{E}[X]\cdot\mathbb{E}[Y]$$

$X, Y \in L^2 \Rightarrow \boxed{X\cdot Y \in L^1}$

$\forall x, y \in \mathbb{R}$

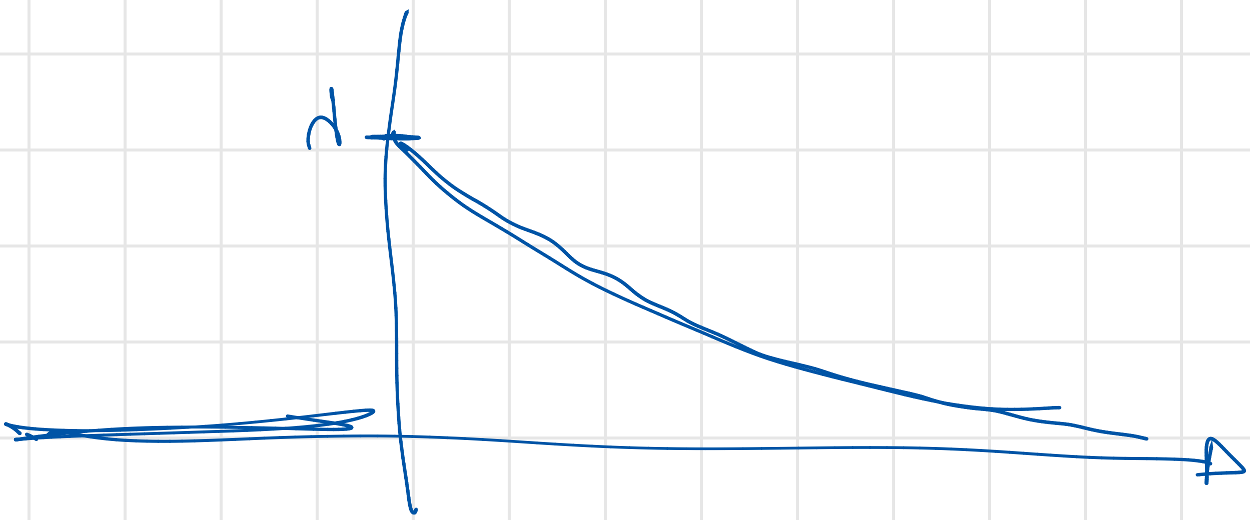
$$\boxed{|xy| \le x^2 + y^2} \Rightarrow |XY| \le X^2 + Y^2$$

$\boxed{XY} = Z \qquad Z, \quad \mathbb{E}[Z]$

Ex: $X \sim Exp(\lambda)$, $\lambda > 0$

X abs. continuous r.v.

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases}$$



$$\int_{\mathbb{R}} f_X = 1$$

$X \geq 0$

$$\boxed{\mathbb{E}[X]} = \int_{-\infty}^{+\infty} x \, f_X(x) \, dx$$

$$= \int_0^{+\infty} \underbrace{x}_{f} \cdot \underbrace{\lambda e^{-\lambda x}}_{g = G'} \, dx$$

$+\infty$

$< +\infty$

$$= f \cdot G - \int f' G$$

$$= \Big[x \left(-e^{-\lambda x} \right) \Big]_0^{+\infty} + \int_0^{+\infty} 1 \left(+ e^{-\lambda x} \right) dx$$

$$= 0 + \int_0^{+\infty} e^{-\lambda x} \, dx = \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^{+\infty} =$$

$$= 0 - \left(\frac{e^0}{-\lambda} \right) = \boxed{1/\lambda}$$

$$\boxed{\mathrm{Var}[X]} = \underbrace{E[X^2]}_{2/\lambda^2} - \underbrace{(E[X])^2}_{\left(\frac{1}{\lambda}\right)^2} = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

$X \in L^2$?

$$E[g(X)] = \int \underbrace{g(x) f(x)}\, dx$$

$$\boxed{E[X^2]} = \int_0^{+\infty} \underbrace{x^2}_{f} \underbrace{\lambda e^{-\lambda x}}_{g} \, dx$$

$$= \underbrace{\left[x^2 \cdot \underbrace{(-e^{-\lambda x})}_{G} \right]_0^{+\infty}}_{=0} - \int_0^{+\infty} 2x \cdot (-e^{-\lambda x}) \, dx$$

$$= \int_0^{+\infty} \underbrace{2x}_{f} \underbrace{e^{-\lambda x}}_{g} \, dx = \underbrace{\left[2x \cdot \underbrace{\frac{e^{-\lambda x}}{-\lambda}}_{G} \right]_0^{+\infty}}_{=0}$$

$$- \int_0^{+\infty} 2 \cdot \frac{e^{-\lambda x}}{-\lambda} \, dx = \int_0^{+\infty} \frac{2}{\lambda} \underbrace{e^{-\lambda x}} \, dx$$

$$= \frac{2}{\lambda} \underbrace{\left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^{+\infty}}_{\lim_{x \to +\infty} \frac{e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda 0}}{-\lambda}} = \boxed{\frac{2}{\lambda^2}}$$

Ex: $X \sim \text{Exp}(\lambda)$

$X \geq 0$

$1 \geq 0$

$Y = \min(X, 1) \geq 0$

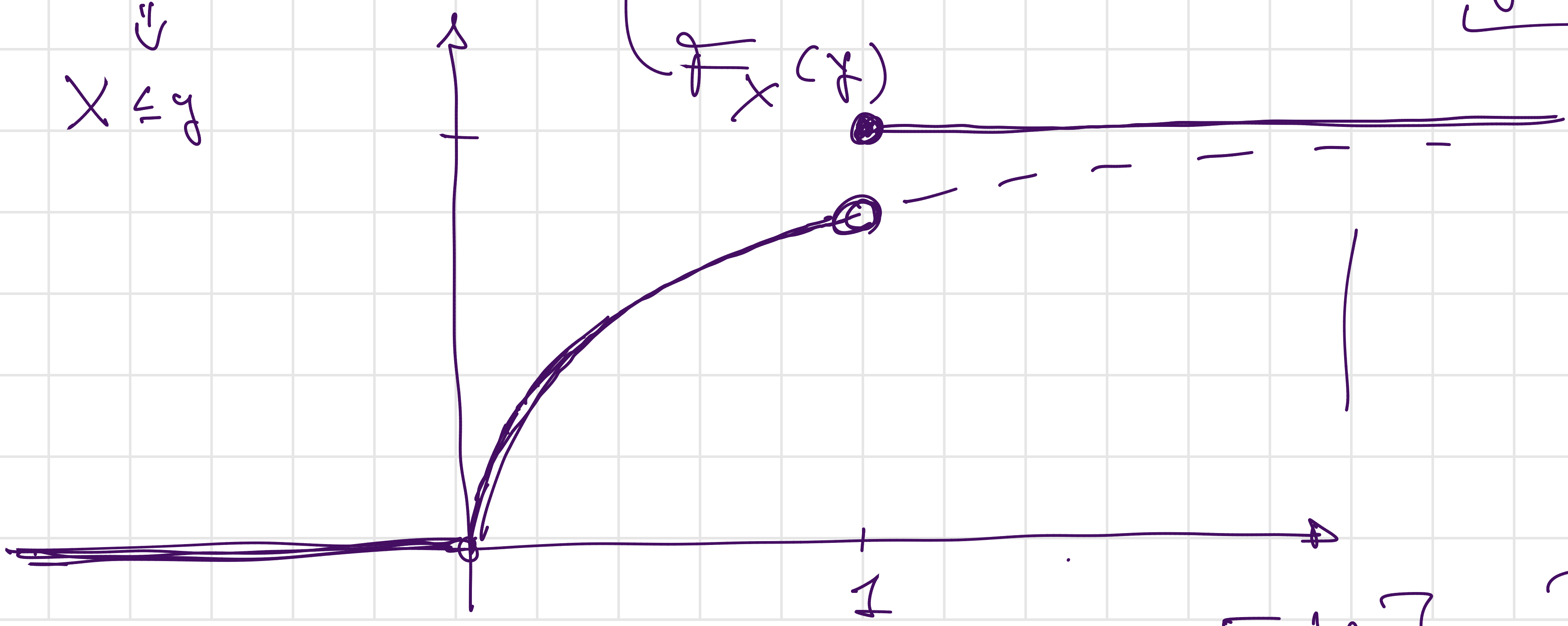
$F_Y(y) = \mathbb{P}[Y \leq y] \quad \forall y \in \mathbb{R}$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \mathbb{P}[Y \leq y] = \mathbb{P}[\min(X,1) \leq y] = \mathbb{P}[X \leq y] = 1 - e^{-\lambda y} = F_X(y) & 0 \leq y < 1 \\ \mathbb{P}[\min(X,1) \leq y] = 1 & y \geq 1 \end{cases}$$

$\min(X,1) \leq y$

$\Leftrightarrow$

$X \leq y$



$\mathbb{E}[Y] = ??$

Proposition:

① If $X \geq 0$ r.v. then

$$\underbrace{\mathbb{E}[X] = \underbrace{\int_0^{+\infty} (1 - F_X(x))\,dx}_{} = \int_0^{+\infty} \mathbb{P}[X > x]\,dx}_{}$$

② if $X \in L^1(\Omega)$ then

$$\mathbb{E}[X] = -\int_{-\infty}^{0} F_X(x)\,dx + \int_0^{+\infty} (1 - F_X(x))\,dx$$

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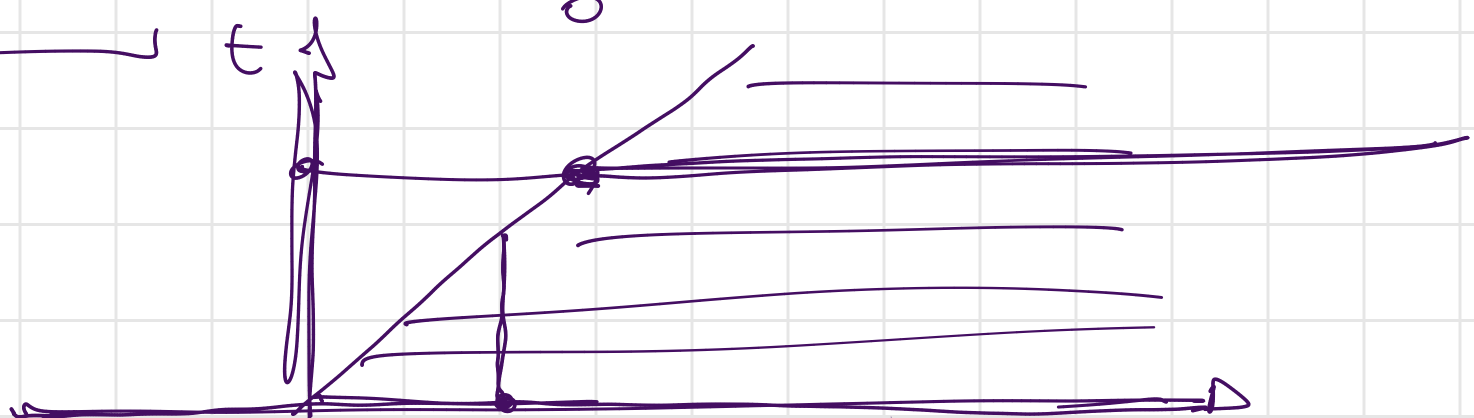
① in the case of abs. cont. r.v. f

$X \geq 0$ $\qquad \mathbb{E}[X] = \int_0^{+\infty} x \cdot f(x)\,dx$

$\underbrace{x = \int_0^x 1\,dt}_{} \qquad = \int_0^{+\infty} \left(\int_0^x dt \right) f(x)\,dx$



$$= \int_0^{+\infty} dt \left(\int_t^{+\infty} f(x)\,dx \right) = \int_0^{+\infty} (1 - F(t))\,dt$$

$Y = \min(X, 1)$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \underbrace{1 - e^{-\lambda y}} & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases}$$

$$E[Y] = \int_0^{+\infty} (1 - F_Y(y))\, dy =$$

$$1 - F_Y(y) = \begin{cases} e^{-\lambda y} & 0 \le y < 1 \\ 1 - 1 = 0 & y \ge 1 \end{cases}$$

$$= \int_0^1 e^{-\lambda y}\, dy = \left[\frac{e^{-\lambda y}}{-\lambda} \right]_0^1$$

$$= -\frac{e^{-\lambda}}{\lambda} - \frac{e^{-\lambda 0}}{-\lambda}$$

$$\boxed{E[Y]} = -\frac{e^{-\lambda}}{\lambda} + \frac{1}{\lambda} = \boxed{\underbrace{\frac{1}{\lambda}} - \underbrace{\frac{e^{-\lambda}}{\lambda}}}$$

Random Variables $\longrightarrow$ Random Vectors

X $\qquad\qquad$ $(X_1, X_2, \ldots, X_d)$

ω

$1, 2, 3, 4, \ldots$

time

Definition: A random vector is a random variable taking values in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ where $\mathcal{B}(\mathbb{R}^d)$ is the minimum σ-field generated by the open sets of $\mathbb{R}^d$.

$$\underline{X} : \Omega \longrightarrow \mathbb{R}^d$$

$$\omega \longmapsto (X_1(\omega), X_2(\omega), \ldots, X_d(\omega))$$

where $X_1, X_2, \ldots, X_d$ are real-random variab.

The law of $\underline{X}$

$$\mu_{\underline{X}}(B) = \mathbb{P}[\underline{X}^{-1}(B)]$$

$B \in \mathcal{B}(\mathbb{R}^d)$

$\mu_{\underline{X}}$ is a probability on $\mathcal{B}(\mathbb{R}^d)$ and it is called the joint distribution of $X_1, X_2, \dots, X_d$.

$$\boxed{\mu_{\underline{X}}} \rightsquigarrow \boxed{\mu_{\underline{X}_1}, \dots, \mu_{\underline{X}_d}}$$

$\mu_{\underline{X}_k}$ is the marginal distr.

$A \in \mathcal{B}(\mathbb{R})$ $\boxed{\mu_{\underline{X}_1}(A)} = \mathbb{P}[X_1 \in A] =$

$$= \mathbb{P}[\underbrace{X_1 \in A}, \underbrace{X_2 \in \mathbb{R}}_{= \Omega}, X_3 \in \mathbb{R}, \dots, X_d \in \mathbb{R}]$$

$$= \mathbb{P}[(X_1, \dots, X_d) \in A \times \mathbb{R} \times \dots \times \mathbb{R}]$$

$$= \mu_{\underline{X}}(A \times \underbrace{\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}})$$

and sim. for $X_2, \dots, X_d$

Independent r.v.'s

Def: $X_1, X_2, \dots, X_d$ r.v.'s are independent iff for any choice of $\underbrace{A_1, A_2, \dots, A_d \in \mathcal{B}(\mathbb{R})}$

$$\mathbb{P}[X_1 \in A_1, X_2 \in A_2, \dots, X_d \in A_d] = \prod_{i=1}^{d} \mathbb{P}[X_i = A_i]$$

$$\|$$

$$\rightarrow \quad \underbrace{\mu_{\underline{X}}(A_1 \times \dots \times A_d)} = \underbrace{\prod_{i=1}^{d} \mu_{X_i}(A_i)}$$

Remark: It can be proven that the values of $\mu_{\underline{X}}$ on $A_1 \times \dots \times A_d$ uniquely determine $\mu_{\underline{X}}$ everywhere!

In the case of indep. r.v's

$$X_1, \dots, X_d \rightsquigarrow X = (\underbrace{X_1, X_2,}_{\mu_{\underline{X}}} \dots, X_d)$$