Lecture 12 Stoch. Meth. Nov. 7th, 2024

$X \sim Bin(1,p)$, $\underbrace{Y \sim Exp(\lambda)}$ $p \in (0,1)$, $\lambda > 0$

$Z = X \cdot Y$, $X \perp\!\!\!\perp Y$

$\mathbb{E}[Z] = \mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y] = p \cdot \frac{1}{\lambda}$

$\boxed{\mathbb{P}[Z=0]}$, $\mathbb{P}[Z>0] = p$

$Var[Z] = Var[X \cdot Y] = \mathbb{E}[X^2 Y^2] - \underbrace{(\mathbb{E}[X \cdot Y])^2}$

$\mathbb{E}[X^2] \cdot \mathbb{E}[Y^2]$

$X, Y$ are indep. and $g_1, g_2$ are two meas. funct.

$\Longrightarrow$ $g_1(X)$ and $g_2(Y)$ are still indep.

$\Longleftarrow$

$\mathbb{P}[Z=0] = \mathbb{P}[X \cdot Y = 0] = \boxed{\mathbb{P}[X=0]} = 1-p$

$= \mathbb{P}[X \cdot Y = 0 \mid X=0] \boxed{\mathbb{P}[X=0]} +$

$\underbrace{\mathbb{P}[X \cdot Y = 0 \mid X=1]} \cdot \mathbb{P}[X=1]$

$\mathbb{P}[Y=0 \mid X=1] \cdot \mathbb{P}[X=1]$

$\mathbb{P}[Y=0] = 0$

$\forall z \in \mathbb{R}$ $\quad \boxed{F_Z(z) = \mathbb{P}[X \cdot Y \leq z] =}$

$$= \mathbb{P}[X \cdot Y \leq z \mid X=0] \cdot \mathbb{P}[X=0] +$$

$$+ \mathbb{P}[X \cdot Y \leq z \mid X=1] \cdot \mathbb{P}[X=1]$$

$\mathbb{E}[X] = \int_0^{+\infty} (1-F(x))\, dx$

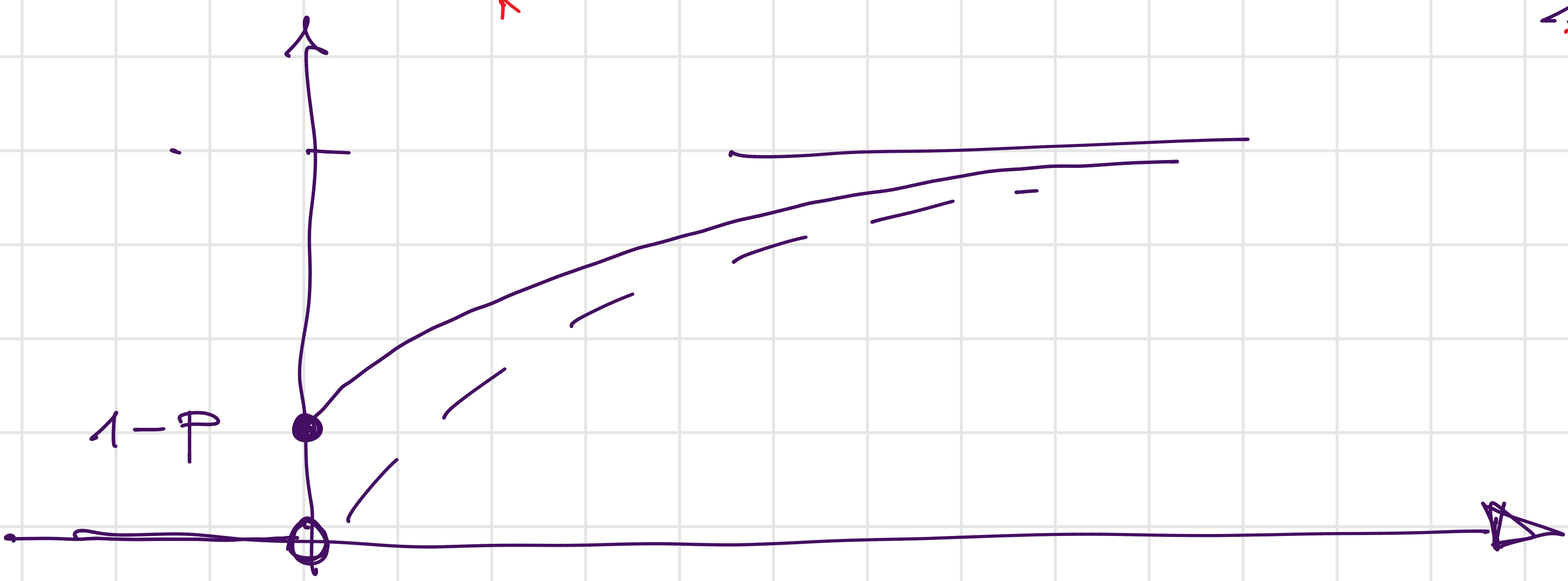
$$= \boxed{\underbrace{\mathbb{P}[0 \leq z \mid \cancel{X=0}] \cdot \underbrace{\mathbb{P}[X=0]}}} + \underbrace{\mathbb{P}[Y \leq z \mid \cancel{X=1}]} \cdot \mathbb{P}[X=1]$$

$X \perp\!\!\!\perp Y$

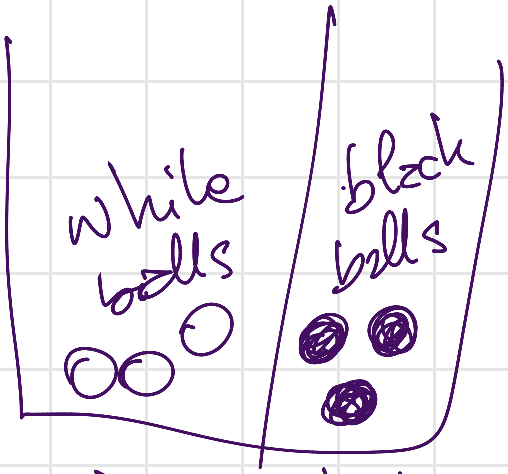
$$= \begin{cases} 0 & z<0 \\ 1-p & z=0 \\ 1-p + \underbrace{\mathbb{P}[Y \leq z]} \cdot \underbrace{\mathbb{P}[X=1]} & 0<z \\ \underbrace{1-p} + p \cdot \underbrace{(1-e^{-\lambda t})} & \end{cases}$$

$1-p+p-pe^{-\lambda t}$

$1-\underbrace{pe^{-\lambda t}}$



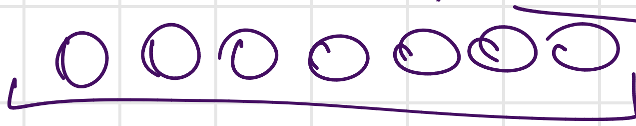
Hypergeom. r.v.




$\underline{D}$ $\quad N-D$

$n \le N$

$\boxed{n}$



$\boxed{\text{without replacement}}$

$$\boxed{Y = \#\ \text{white balls}}$$

$$Y \sim Bin\left(n, \frac{D}{N}\right) \qquad \boxed{n \le N}$$

$$L = \{k \in \mathbb{N} : \underbrace{0 \overset{\max}{\vee} (n-N+D)} \le k \le \underbrace{D \overset{\min}{\wedge} n}\}$$

$$\mathbb{P}[Y=k] = \frac{\binom{D}{k}\binom{N-D}{n-k}}{\binom{N}{n}}$$

$$\mathbb{E}[Y] = \sum_{k \in L} k \cdot \frac{\binom{D}{k}\binom{N-D}{n-k}}{\binom{N}{n}}$$

$$Y = \overbrace{Y_1 + Y_2 + \dots + Y_n}^{\text{are not indep.}} \qquad Y_i = \begin{cases} 1 & \text{white} \\ 0 & \text{black} \end{cases}$$

$Y = Y_1 + \dots + Y_n$, $\quad Y_1, \dots, Y_n$ are not i.i.d.

but $Y_i \sim \text{Bin}\left(1, \frac{D}{N}\right)$ r.v. $\forall i$

$Y_1 \sim \text{Bin}\left(1, \frac{D}{N}\right)$

$Y_2 \sim \text{Bin}\left(1, \frac{D}{N}\right)$

$$\mathbb{E}[Y] = \mathbb{E}[Y_1 + \dots + Y_n]$$
$$= \sum_{i=1}^{n} \mathbb{E}[Y_i] = n \cdot \frac{D}{N}$$

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1 2 ... D

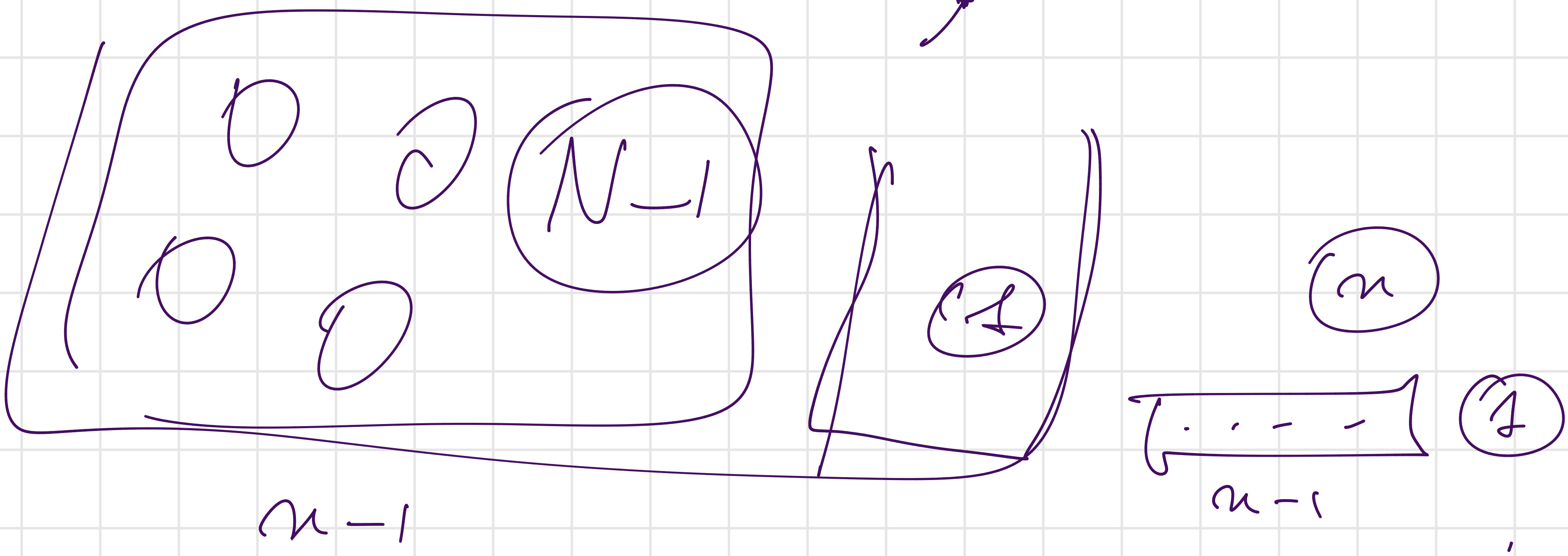
$$Y = Z_1 + \dots + Z_D$$

binomial $\quad Z_1 = \begin{cases} 1 & \text{if the white ball 1 has been withdrawn} \\ 0 & \text{, otherwise} \end{cases}$

$$\mathbb{E}[Y] = D \cdot \mathbb{E}[Z_1] = D \cdot \mathbb{P}[Z_1 = 1]$$

$\mathbb{P}[Z_1 = 1] =$


$$= \frac{\binom{N-1}{n-1}\binom{1}{1}}{\binom{N}{n}}$$




$$= \frac{\dfrac{(N-1)!}{(n-1)!\,(N-1-\cancel{(n-1)})!}}{\dfrac{N!}{n!\,(\cancel{N-n})!}} = \frac{1}{N}(n) = \frac{n}{N}$$

$$\mathbb{E}[Z] \boxed{= D \cdot \mathbb{P}[Z_1 = 1] = D \cdot \frac{n}{N}} = n \cdot \frac{D}{N}$$

Exercise: Compute $Var[Z]$.

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Poisson Distribution $\lambda$, $\lambda > 0$

$X \sim Poisson(\lambda)$ will be a possible approximation for $Bin(n,p)$ when $n$ is big, but $n \cdot p$ is bounded.

$$\boxed{\lambda = np}$$

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$$X : \Omega \to \underbrace{\mathbb{N}}$$

$$P_X(n) = e^{-\lambda} \frac{\lambda^n}{n!}$$

$$\underbrace{e^{\lambda}} = \underbrace{\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}}$$

$E[X] = ?$

$Var[X] = ?$

**Theorem:** Let $X_1, X_2, \dots, X_n$ be independent, Binomial $(1, p_i)$, $\underbrace{p_i}$ $\in [0,1]$.

and set $\underbrace{S_n := X_1 + X_2 + \dots + X_n}$.

Moreover, let $W_n \sim \text{Poisson}(p_1 + \dots + p_n)$.

Then, for any $A \subseteq \mathbb{N}$

$$\left| \underbrace{\mathbb{P}[S_n \in A]} - \underbrace{\mathbb{P}[W_n \in A]} \right| \leq \sum_{i=1}^{n} p_i^2$$

**Corollary:** Choose $\underbrace{p_i = d/n}$

we get $S_n \sim \text{Bin}(n, \frac{d}{n})$. Since

$$\sum_i p_i^2 = \sum_{i=1}^{n} \underbrace{d^2/n^2} = \frac{d^2}{n^2} \cdot n = \underbrace{d^2/n}$$

$$n \to \infty \quad \downarrow \quad 0$$

**Proof:**

Discrete $\rightsquigarrow$ Abs. cont.

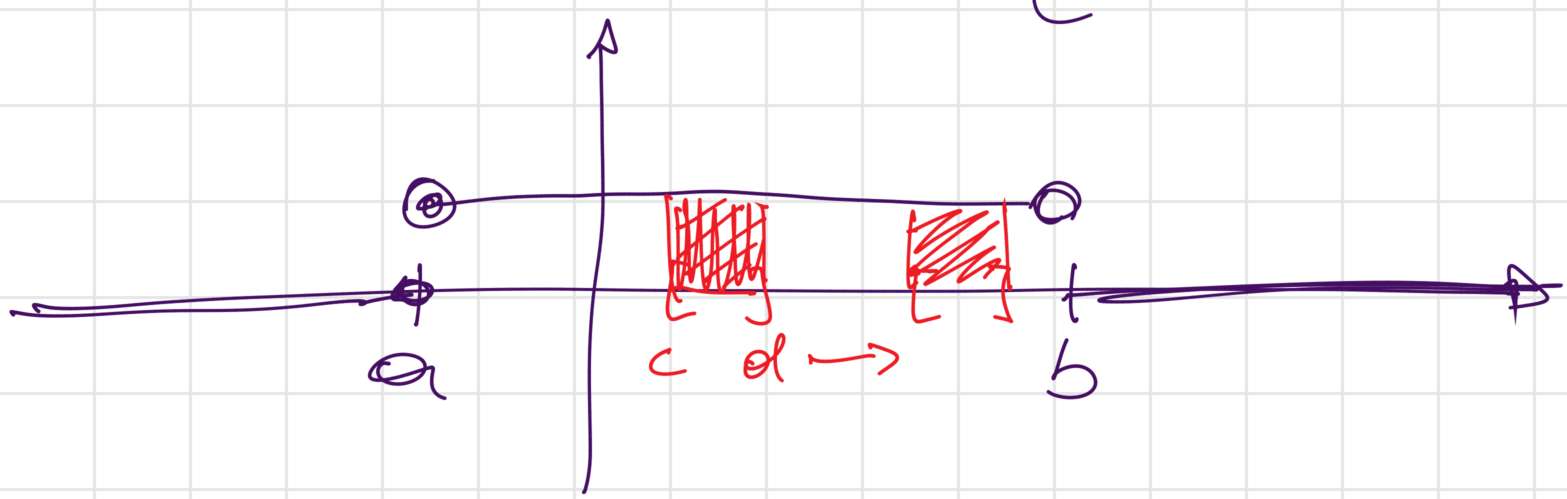
## Uniform distribution

$X \sim \mathcal{U}(a,b)$

$-\infty < a < b < +\infty$

$(a,b)$

$$\iff \quad f_U(x) = \begin{cases} \dfrac{1}{b-a} & x \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$




$k = ?$ (i) $f_U \geq 0 \Rightarrow k \geq 0$

(ii) $\displaystyle\int_{\mathbb{R}} f_U(t)\,dt = 1$ $\qquad k = \dfrac{1}{b-a}$ $\Updownarrow$

$$1 = \int_{\mathbb{R}} f_U(t)\,dt = \cancel{\int_{-\infty}^{a} 0\,dt} + \int_a^b k\,dt + \cancel{\int_b^{+\infty} 0\,dt} = $$

$= k(b-a)$

$\boxed{\text{Uniform}}$

$\underbrace{[c,d]}\subseteq\underbrace{(a,b)}$

$$\mathbb{P}[X\in[c,d]]=\int_c^d \underbrace{\frac{1}{b-a}}\,dt=\frac{d-c}{b-a}$$

$[c+\varepsilon,\, d+\varepsilon]\subseteq \underline{(a,b)}$

$$\mathbb{P}[X\in[c+\varepsilon,\, d+\varepsilon]]=\frac{d-c}{b-a}$$

$\boxed{\varepsilon\in\mathbb{R}}$

$$\boxed{-\infty<a<b<+\infty}$$

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$X\sim \mathrm{Exp}(\lambda)\qquad \lambda>0$

$$F_X(x)=\begin{cases}1-e^{-\lambda x} & x\geq 0\\ 0 & x<0\end{cases}$$

$\mathbb{E}[X]=\underbrace{\frac{1}{\lambda}}$ , $\mathrm{Var}[X]=\underbrace{1/\lambda^2}$

**Theorem**: A random variable $T:\Omega \to (0,+\infty)$ has an exponential distribution if and only if it has the following memoryless property:

$$\underbrace{P[T>s+t \mid T>s]}_{e^{-\lambda t}} = \underbrace{P(T>t)}_{e^{-\lambda t}} \qquad \forall s,t \geq 0$$

**Proof**:

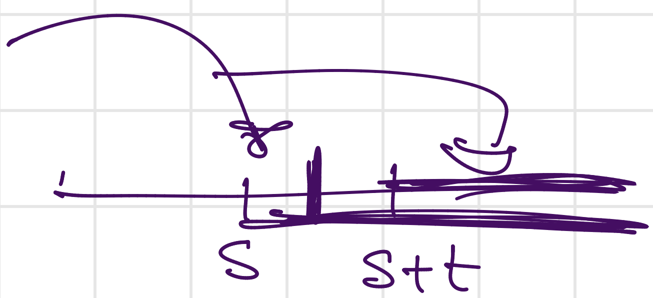
$\Rightarrow$ $\qquad T \sim Exp(\lambda)$

$$P[T \leq t] = 1-e^{-\lambda t} \qquad t \geq 0$$

$$\boxed{\begin{gathered} P[T>t] = e^{-\lambda t} \qquad t \geq 0 \\ P[T>s] = e^{-\lambda s}, \quad P[T>s+t] = e^{-\lambda(s+t)} \end{gathered}}$$

$$P[T>s+t \mid T>s] = \qquad s,t \geq 0$$

$$= \frac{P[(T>s+t),(T>s)]}{P[T>s]}$$




$$(T>s+t) \subseteq (T>s)$$

$$\frac{P[T>s+t]}{P[T>s]} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda s-\lambda t+\lambda s} = \boxed{e^{-\lambda t}}$$

$\Leftarrow$

$$\mathbb{P}[T > s+t \mid T > s] = \mathbb{P}[T > t]$$

$\forall t, s \geq 0$

then $T \sim Exp(\lambda)$ ??

$$\Rightarrow \quad \frac{\mathbb{P}[T > s+t)}{\mathbb{P}[T > s]} = \mathbb{P}[T > t)$$

$$\Leftrightarrow \quad \mathbb{P}[T > s+t] = \mathbb{P}[T > t] \cdot \mathbb{P}[T > s]$$

$g(t) := \mathbb{P}[T > t]$

$$\boxed{g(s+t) = g(t) \cdot g(s)} \quad \forall s, t \geq 0$$

Cauchy's functional equation

If $g$ is continuous, the unique sol. of this equation are the exponential.

$\square$