DATA SCIENCE **Stochastic Methods**

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Problem 1. [10] Let X_1, X_2, X_3 be independent, binomial $(2, p)$ random variables. Define $Y_1 = \min\{X_1, X_2, X_3\}$, $Y_3 = \max\{X_1, X_2, X_3\}$ and $Y_2 = X_1 + X_2 + X_3 - Y_1 - Y_3$.

(i) Compute $P[Y_1 = 2]$ and $P[Y_3 = 0]$;

(ii) Compute $P[Y_1 = Y_2 = Y_3]$;

(iii) Compute $P[Y_2 > n]$ for $n = 0, 1$;

(iv) Compute $E[Y_2]$.

(i) $\mathbb{P}[Y_1 = 2] = \mathbb{P}[X_1 = X_2 = X_3 = 2] = p^2 \cdot p^2 \cdot p^2 = p^6$

$\mathbb{P}[Y_3 = 0] = \mathbb{P}[X_1 = X_2 = X_3 = 0] = (1-p)^2 \cdot (1-p)^2 \cdot (1-p)^2 = (1-p)^6$

(ii) $\mathbb{P}[Y_1 = Y_2 = Y_3] = \sum_{k=0}^{2} \mathbb{P}[X_1 = X_2 = X_3 = k] = (1-p)^6 + 8p^3(1-p)^3 + p^6$

(iii) $\mathbb{P}[Y_2 > 0] = 1 - \mathbb{P}[Y_2 = 0] = 1 - \mathbb{P}[X_1 = X_2 = X_3 = 0]$

$- 3\,\mathbb{P}[X_1 = 0, X_2 = 0, X_3 > 0]$

$= 1 - (1-p)^6 - 3(1-p)^4 \cdot (1 - (1-p)^2) =$

$= 1 - (1-p)^6 - 3(1-p)^4 (2p - p^2)$

$\mathbb{P}[Y_2 > 1] = 3 \cdot \mathbb{P}[X_1 \leq 1, X_2 = X_3 = 2] + \mathbb{P}[X_1 = X_2 = X_3 = 2]$

$= 3 \cdot (1 - p^2)\, p^4 + p^6$

(iv) $\mathbb{E}[Y_2] = \mathbb{P}[Y_2 > 0] + \mathbb{P}[Y_2 > 1] = 1 - (1-p)^6 +$

$- 3(1-p)^4(2p - p^2) + 3(1-p^2) \cdot p^4 + p^6$

$\mathbb{E}[Y_2] = 0 \cdot \mathbb{P}[Y = 0] + 1\,\mathbb{P}[Y = 1] + 2\,\mathbb{P}[Y = 2]$

$= (\mathbb{P}[Y = 1] + \mathbb{P}[Y = 2]) + \mathbb{P}[Y = 2]$

$= \mathbb{P}[Y > 0] + \mathbb{P}[Y > 1]$

Problem 2. [10] Let $(X_n)_{1\le i\le n}$ be a family of i.i.d. uniform random variables on $[-1,1]$. Define $S_n=\sum_{i=1}^n X_i$ and $Z_n=S_n^2-n/3$.

(i) Compute the expectation and the variance of S_n and the expectation of Z_n;

(ii) Compute the conditional expectation of S_{n+1}^2 given $X_1,X_2,\dots,X_n$;

(iii) Is Z_n a martingale?

(i) $\mathbb{E}[S_n]=\sum_{i=1}^{n}\mathbb{E}[X_i]=0$

$$Var[S_n]=Var\Big[\sum_{i=1}^{n}X_i\Big]\overset{IND.}{=}\sum_{i=1}^{n}Var[X_i]=n\cdot Var[X_1]$$

$$=n\cdot\mathbb{E}[X_1^2]=n\cdot\int_{-1}^{1}\frac{x^2}{2}\,dx=n\cdot\frac{1}{6}\big[x^3\big]_{-1}^{1}=$$

$$=n\cdot\frac{1+1}{6}=n/3$$

$$\mathbb{E}[Z_n]=\mathbb{E}[S_n^2]-n/3=Var[S_n]-n/3=n/3-n/3=0$$

(ii) $$\mathbb{E}[S_{n+1}^2\mid X_1,\dots,X_n]=\mathbb{E}[(S_n+X_{n+1})^2\mid X_1,\dots,X_n]$$

$$=\mathbb{E}[S_n^2\mid X_1,\dots,X_n]+2\,\mathbb{E}[S_n\cdot X_{n+1}\mid X_1,\dots,X_n]$$

$$+\mathbb{E}[X_{n+1}^2\mid X_1,\dots,X_n]=S_n^2+2\cdot S_n\underbrace{\mathbb{E}[X_{n+1}\mid X_1,\dots,X_n]}_{=0}$$

$$+\mathbb{E}[X_{n+1}^2]=S_n^2+1/3$$

(iii) YES! INDEED

$$\mathbb{E}[Z_{n+1}\mid X_1,\dots,X_n]=\mathbb{E}\Big[S_{n+1}^2-\frac{n+1}{3}\,\Big|\,X_1,\dots,X_n\Big]$$

$$=\mathbb{E}[S_{n+1}^2\mid X_1,\dots,X_n]-\frac{n+1}{3}=S_n^2+1/3-\frac{n+1}{3}=S_n^2+\frac{n}{3}=Z_n$$

Problem 3. [12] Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{1,2,3,4\}$ with transition probabilities given by

$$p_{i,1}=\frac{i}{i+1} \quad , \quad p_{i,i+1}=\frac{1}{i+1} \quad , \quad 1\leq i\leq 3 \quad \text{and} \quad p_{4,2}=1$$

(i) Is the Markov chain irreducible?

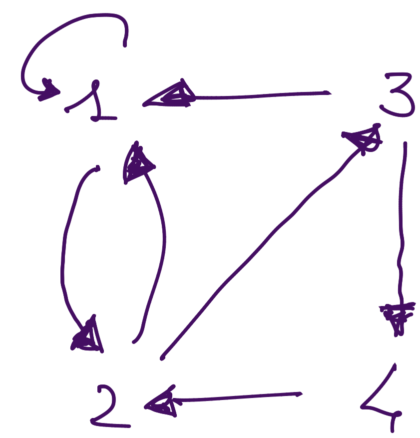
(ii) Is the Markov chain aperiodic?

(iii) Compute $E[X_3|X_0=1]$;

(iv) Determine the invariant distribution.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 \\ 3/4 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

(i)



The MC is irreducible

(ii) $P^2_{11}>0$, $P^3_{11}>0 \Rightarrow$ the MC is aperiodic

(iii) $\mathbb{E}[X_3 | X_0=1] = \sum_{k=1}^{4} k \cdot \mathbb{P}[X_3=k \,|\, X_0=1]$

$= 1\cdot\left[\left(\frac{1}{2}\right)^3 + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{3}{4}\right]$

$+ 2\left[\left(\frac{1}{2}\right)^2\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{2}\right] + 3\left[\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\right] + 4\left[\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\right]$

(iv) $\pi P = \pi \Rightarrow \begin{cases} \frac{1}{2}\pi_1 + \frac{2}{3}\pi_2 + \frac{3}{4}\pi_3 = \pi_1 \\ \frac{1}{2}\pi_1 + \pi_4 = \pi_2 \\ \frac{1}{3}\pi_2 = \pi_3 \\ \frac{1}{4}\pi_3 = \pi_4 \\ \pi_1 + \dots + \pi_4 = 1 \end{cases} \Rightarrow \begin{cases} \pi_1 = 22/39 \\ \pi_2 = 12/39 \\ \pi_3 = 4/39 \\ \pi_4 = 1/39 \end{cases}$