Lecture 7 Stoch. Math. Oct. 21, 2024

Random Variables

Discrete R.V.'s

Absolutely continuous R.V

$X : \Omega \to \mathbb{R}$ s.t. $\underbrace{f_X}$ $: \underbrace{\mathbb{R} \to \mathbb{R}_+}_{x}$ s.t.

$$\underbrace{F_X}_{\mathcal{C}}(x) = \underbrace{\mathbb{P}[X \le x]} = \underbrace{\int_{-\infty}^{x} f_X(t)\,dt} \quad \forall x \in \mathbb{R}$$

f_X is called density of X

① $\displaystyle\int_{-\infty}^{+\infty} f_X(t)\,dt = \int_{\underbrace{\mathbb{R}}} f_X(t)\,dt = 1$

$$\mu_X(\mathbb{R}) = \mathbb{P}[X \in \mathbb{R}] = 1$$

$a < b$

$(X \in (a,b]) \cup (X \le a) = (X \le b)$

② $\underbrace{\mathbb{P}[X \in (a,b]]} = F_X(b) - F_X(a) = \underbrace{\mathbb{P}[X \le b]} - \underbrace{\mathbb{P}[X \le a]}$

$$= \int_{-\infty}^{b} f_X(t)\,dt - \int_{-\infty}^{a} f_X(t)\,dx =$$

$$= \underline{\int_a^b f_X(t)\,dt} = P[X \in (a,b]]$$

$$\int_{(a,b)} \qquad \int_{(a,b]}$$

$$P[X \in (a,b)] = P[X \in [a,b)]$$

$$= P[X \in [a,b]] = \int_a^b f_X(t)\,dt$$

More generally, $\forall B \in \mathcal{B}(\mathbb{R})$

$$\boxed{P[X \in B] = \int_B f_X(t)\,dt}$$

$$\boxed{P[X = a] = 0 \qquad \forall a \in \mathbb{R}}$$

$$\| \quad \int_a^a f_X(t)\,dt = 0$$

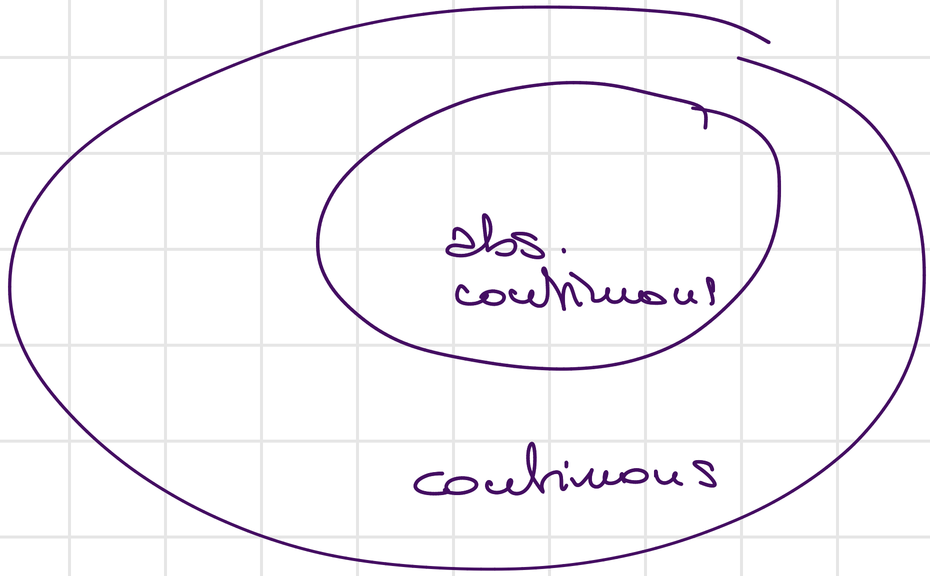
disjoint

$$1 = P[X \in \mathbb{R}] = P\Big[\bigcup_{x \in \mathbb{R}} \{X = x\}\Big] \quad \underline{\text{countable}}$$

$$\overset{?}{\neq} \sum_{x \in \mathbb{R}} P[X = x] = \sum_{x \in \mathbb{R}} 0 = 0$$

Def: $X : \Omega \to \mathbb{R}$ is continuous $\iff$

F_X is continuous

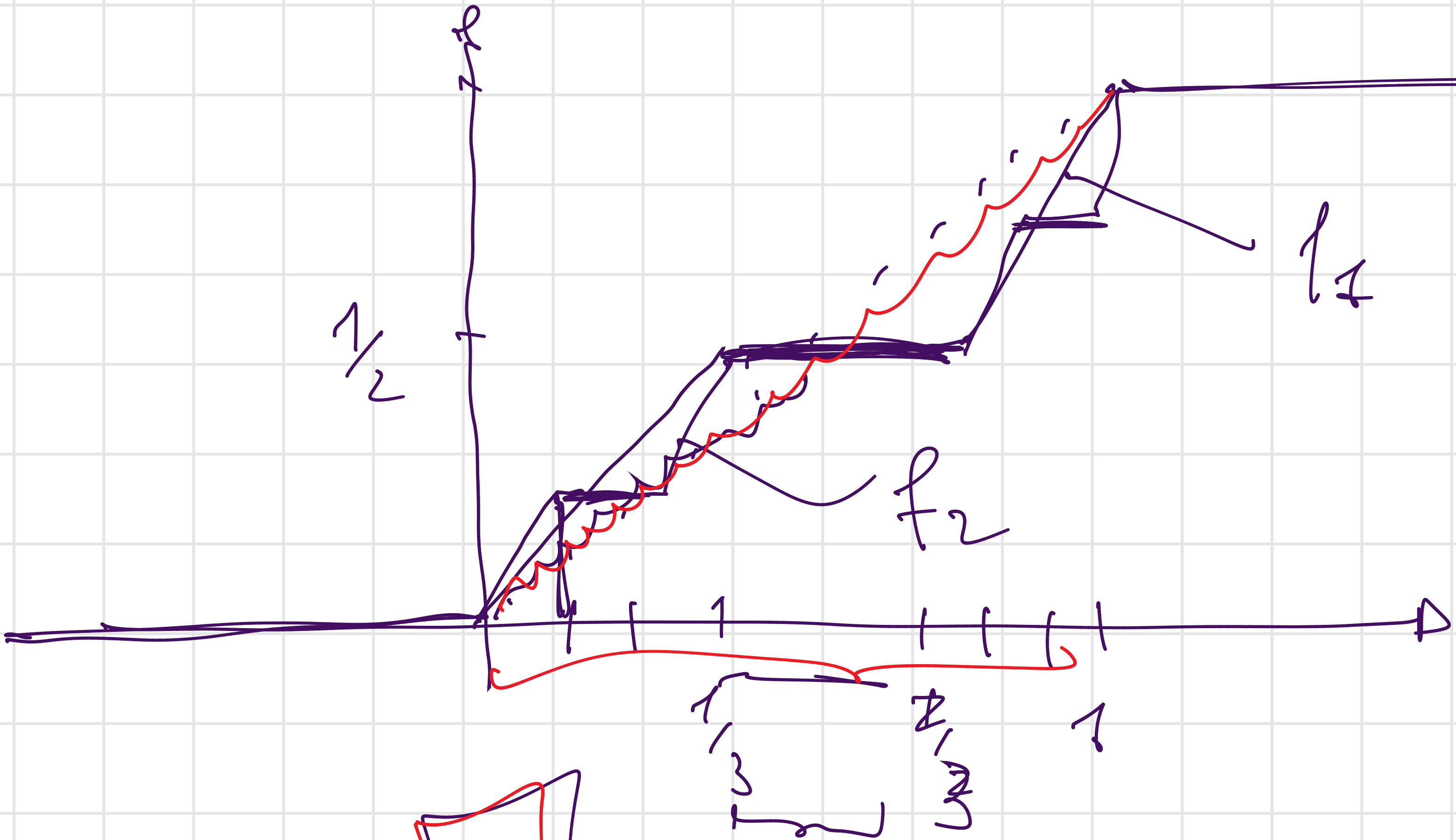


F_X is continuous, but it is impossible to write $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

$$f_X = F_X'$$

F_X s.t. $F_X' = 0$ almost everywhere

but $F_X \neq \int F_X' = 0$



$f_n \longrightarrow \boxed{f}$ $\quad 1/9 \quad 1/3 \quad 1/9$

$1/3 + 2/9 + \dots$

Ex: abs. cont. r.v.

$\lambda > 0$, X is an exponential random variable with parameter λ

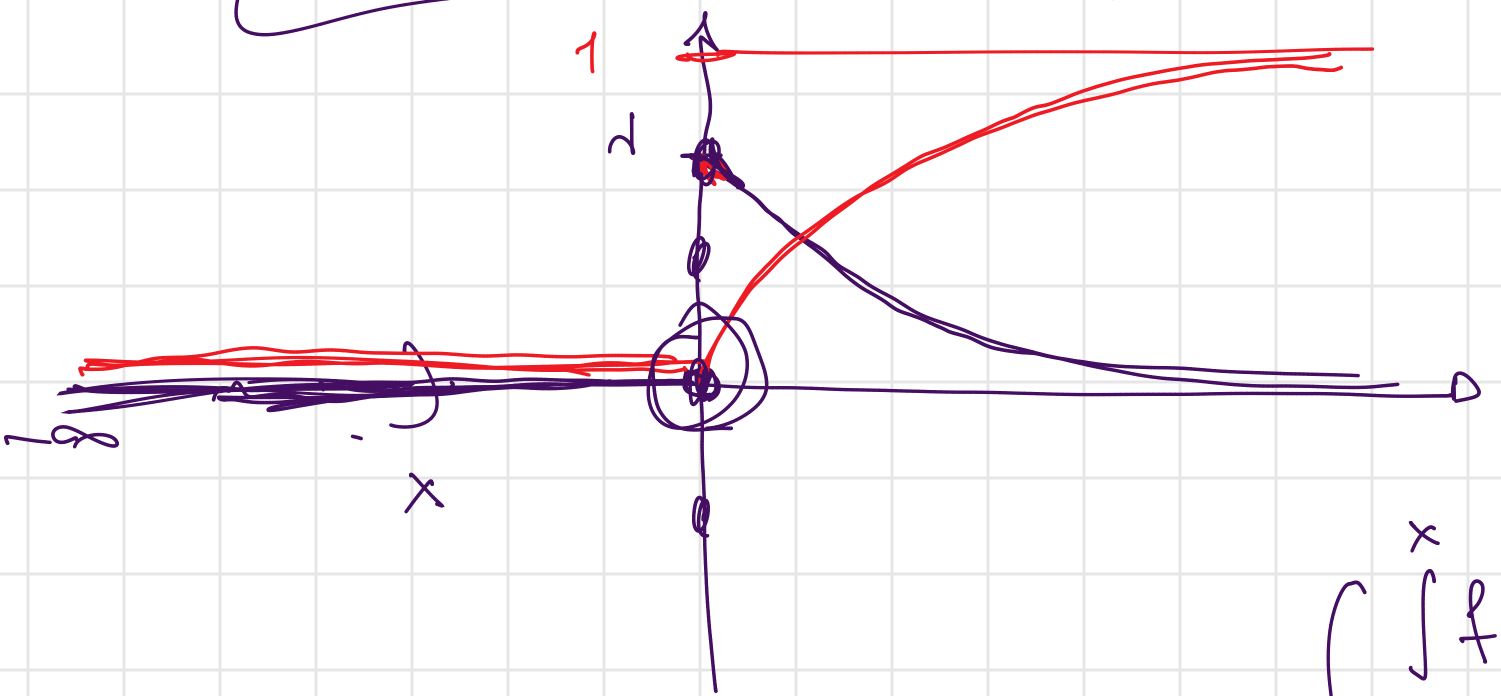
$$f_X(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

$$f(x) = \begin{cases} \boxed{0} & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

$>$

$x = 0$

$$f(0) = \lambda e^{-\lambda \cdot 0} = \lambda$$



$$F_X(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} \int_{-\infty}^{x} f(t)\,dt = & x < 0 \\ \quad = \int_{-\infty}^{x} 0\,dt = 0 & \\ \int_{-\infty}^{x} f(t)\,dt = & x \geq 0 \\ \quad = \cancel{\underbrace{\int_{-\infty}^{0} f(t)\,dt}_{=0}} + \int_{0}^{x} f(t)\,dt & \\ \quad = \int_{0}^{x} f(t)\,dt & \\ \quad = \int_{0}^{x} \lambda e^{-\lambda t}\,dt & \end{cases}$$

$$= \left[-e^{-\lambda t}\right]_0^x = -e^{-\lambda x} + e^{-\lambda 0}$$

$$= 1 - e^{-\lambda x}$$

$$\boxed{F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}}$$

Mean, $\boxed{\text{Expectation}}$ of a R.V.

$X \sim Bin(1, p)$ $\qquad\qquad$ $X : \Omega \to \mathbb{R}$

$$\mathbb{P}[X = 0] = 1 - p$$

$$\mathbb{P}[X = 1] = p$$

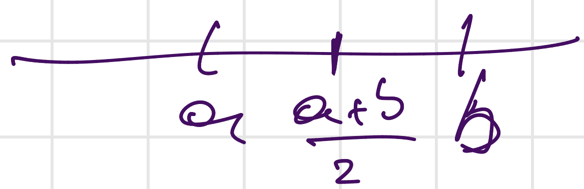
$$\mathbb{E}[X] = 0 \cdot \underbrace{\mathbb{P}[X=0]} + 1 \cdot \underbrace{\mathbb{P}[X=1]}_{p}$$

$$= \boxed{p}$$

$p = 1/2$

$\mathbb{P}[X = a] = p$, $\mathbb{P}[X = b] = 1 - p$

$a, b \in \mathbb{R}, a \neq b$



$$\mathbb{E}[X] = a \cdot \underbrace{\mathbb{P}[X=a]}_{p} + b\, \underbrace{\mathbb{P}[X=b]}_{1-p}$$

$$= a \cdot p + b(1-p)$$

$p = 1/2$ $\qquad$ $= \frac{1}{2} \cdot a + \frac{1}{2} b = \frac{a+b}{2}$

X be a discrete r.v. with density p_X.

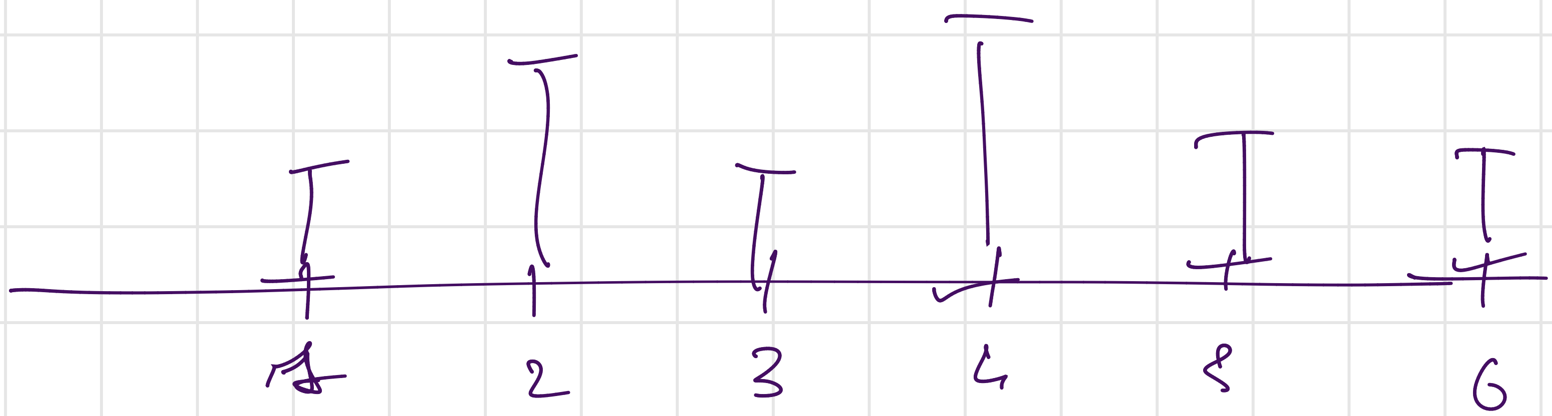
We define the expectation

$$\boxed{\mathbb{E}[X] := \sum_{\substack{x \in \mathbb{R} \\ (x \in \mathbb{N})}} x \cdot p_X(x)}$$

$1, 2, 3, 4, 5, 6$ $\qquad$ $1/6$

$$\mathbb{E}[X] = 3.5 = \left(\underbrace{1+2+3+4+5}_{p}+6\right)\frac{1}{6}$$

$p \qquad 1-5p$



Ex. 1

$X \sim Bin(1,p)$

$$\mathbb{E}[X] = \sum_{x\in\mathbb{R}} x \cdot p_X(x)$$

$$= 0 \cdot p_X(0) + 1\, p_X(1) = p$$

$X \sim Bin(n,p)$

$k = 0, 1, \dots, n$

$$\mathbb{E}[X] = \sum_{x\in\mathbb{R}} x \cdot p_X(x) =$$

$$= \sum_{k=0}^{n} k \cdot p_X(k)$$

$$= \sum_{\underline{k=\boxed{0}}}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{\underline{k=1}}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \cancel{k} \frac{n!}{\underbrace{\cancel{k!}}_{(k-1)!}(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \boxed{\frac{n!}{(k-1)!\,(n-k)!}} p^k (1-p)^{n-k}$$

$$(n-k)! = [(n-1)-(k-1)]!$$

$$\frac{n!}{(k-1)!\,(n-k)!} = n \cdot \frac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!}$$

$$= n \cdot \binom{n-1}{k-1}$$

$$\sum_{k=1}^{n} n \cdot \binom{n-1}{k-1} \cdot p^{k} (1-p)^{n-k}$$

$$\underbrace{p^{k-1} \cdot p}$$

$$= \underbrace{n \cdot p}\ \underbrace{\sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} \cdot (1-p)^{(n-1)-(k-1)}}_{= 1}$$

$$l = k-1$$

$$\underbrace{(p+(1-p))^{n-1}} = \underbrace{\sum_{l=0}^{n-1} \binom{n-1}{l} p^{l} (1-p)^{(n-1)-l}}_{= 1}$$

$$\boxed{\mathbb{E}[X] = n p}$$

X is discrete

$$\mathbb{E}[X] = \sum_{x \in \mathbb{R}} x \cdot p_X(x)$$

$x > 0$

$$S^+ = \sum_{x \geq 0} x \cdot p_X(x), \qquad S^- = \sum_{x < 0} (-x)\, p_X(x)$$

$$S^+ \in [0, +\infty] = [0, +\infty) \cup \{+\infty\}$$

$$S^- \in [0, +\infty]$$

S^- can be zero

$$\mathbb{E}[X] = \sum_{x \in \mathbb{R}} x\, p_X(x) = S^+ - S^-$$

If at least one of S^+ and S^- is finite then we can define

$$\mathbb{E}[X] = S^+ - S^-$$

- $< +\infty$ — $S^+ < +\infty$, $S^- < +\infty$
- $+\infty$ — $S^+ = +\infty$, $S^- < +\infty$
- $-\infty$ — $S^+ < +\infty$, $S^- = +\infty$

If both S^+ and S^- are finite, we say that X is <u>integrable</u> and we have

$$\underbrace{\mathbb{E}[X] = \underbrace{S^+}_{} - \underbrace{S^-}_{} < +\infty}_{}$$

<u>Remark</u>: $S^+ - S^- < +\infty \iff S^+ + S^- < +\infty$

$$\iff \underbrace{\sum_{x \in \mathbb{R}} |x| \, p_X(x) < +\infty}_{}$$

$L^1(\Omega)$ = space of all integrable r.v.'s on Ω

$$X \in L^1(\Omega) \iff \boxed{\sum_{x \in \mathbb{R}} |x| \, p_X(x) < +\infty}$$

<u>Remark</u>: $Y \geq 0$, discrete

$\mathbb{E}[Y]$ is always well-defined (since $S^- = 0$)

$\mathbb{E}[Y] \in [0, +\infty]$

$X = k$, $\mathbb{E}[X] = k$? $\quad k \in \mathbb{R}$ constant

$\llcorner k \cdot \underset{\substack{\shortparallel \\ 1}}{\mathbb{P}[X=k]}$

<u>Ex</u> : $X \sim \text{Geo}(p) \qquad p \in [0,1]$

$\underbrace{X \geq 0}$, $\boxed{\mathbb{P}[X=k] = (1-p)^{k-1} \cdot p}$

$k \in \mathbb{N} \setminus \{0\}$
$= \{1, 2, 3, \dots\}$

$\mathbb{E}[X] = ?$ YES $< \begin{matrix} \in \mathbb{R} \\ +\infty \end{matrix}$

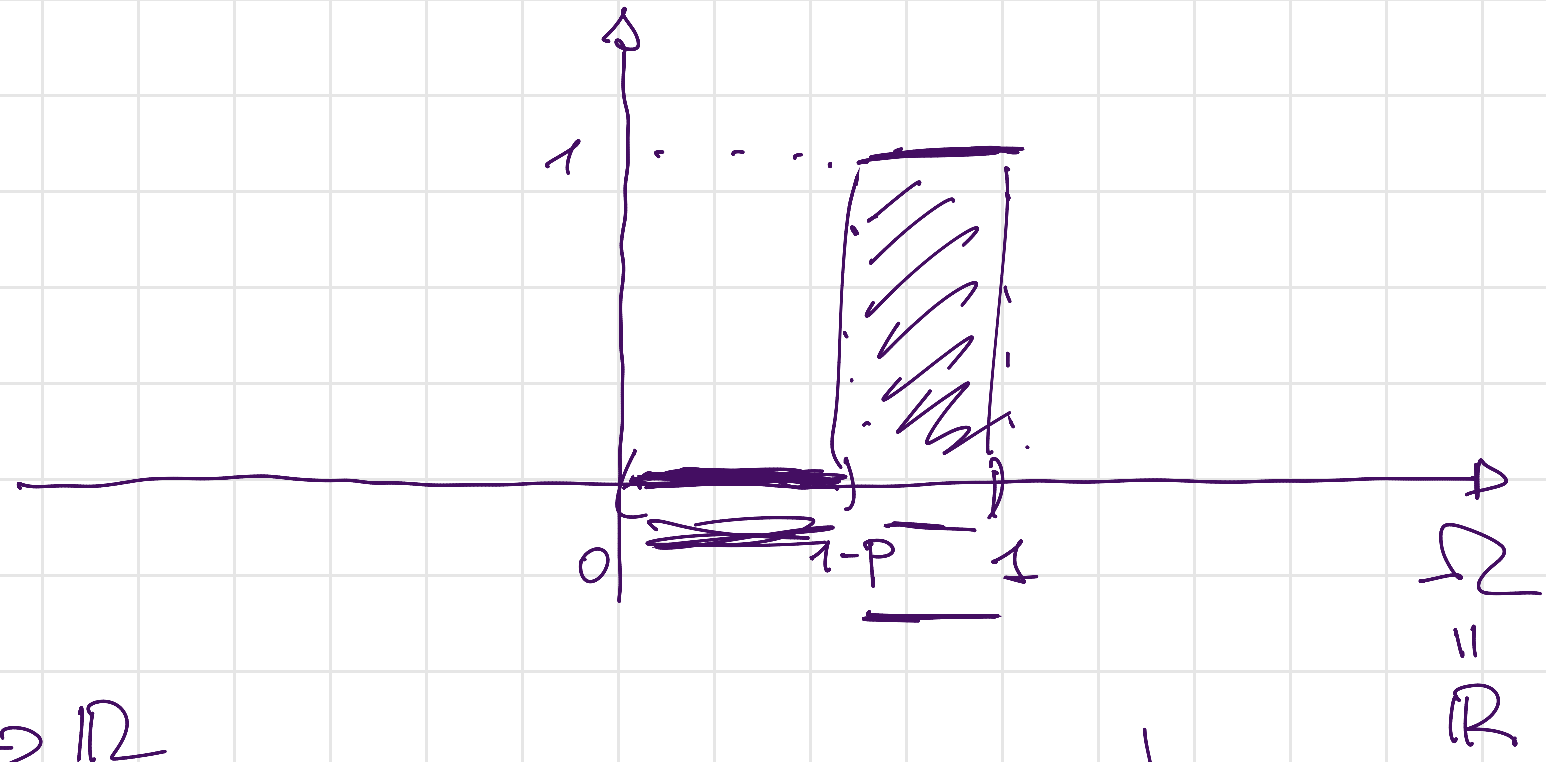
$\underbrace{S^+ < +\infty}$, $\underbrace{S^- < +\infty}$

$S^- = 0$

$$\mathbb{E}[X] = \underbrace{\sum_{k=1}^{+\infty} k\,(1-p)^{k-1} \cdot p} = \frac{1}{p}$$

$X \in L^1(\Omega) \iff S^+ < +\infty$
$S^- < +\infty$

$X \geq 0$



$$X : \Omega \to \mathbb{R}, \quad \Omega \subseteq \mathbb{R}$$

$$X \sim \mathrm{Bin}(1, p)$$

$$\mathbb{P}[X=0] = (1-p-0) = 1-p$$

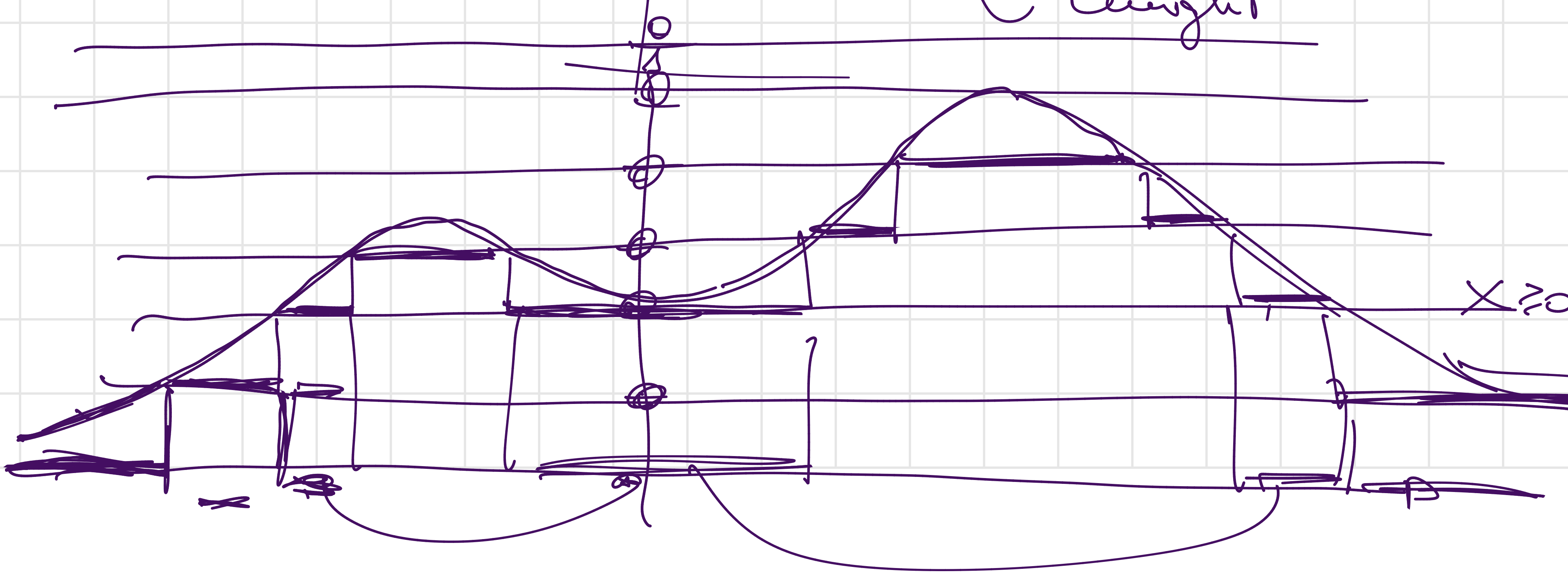
$$\mathbb{P}[X=1] = 1-(1-p) = p$$

$$X = \begin{cases} 0 & 1-p \\ 1 & p \end{cases}$$

$$X : (0,1) \to \mathbb{R}$$

$$\big((0,1), \mathcal{L}\big)$$

lenght



$$f(k) = \frac{c}{k^2}$$

$$\sum \frac{1}{k^2} < +\infty$$

$$\underbrace{k \in \{1, 2, 3, \dots\}}$$

$$1 + \frac{1}{4} + \frac{1}{9} + \dots$$

$$\mathbb{E}[X] = k \cdot \frac{c}{k^2}$$

$$c = \frac{1}{\sum 1/k^2}$$

$$= \frac{c}{k}$$

$$\sum_{n=1}^{\infty} \frac{1}{k} = +\infty$$