Lecture 18 Stoch. Meth. Nov. 28th, 2024

WLLN $(X_n)_{n\in\mathbb{N}}$ iid r.v. s.t. $\mu = \mathbb{E}[X_1] < +\infty$

$\sigma^2 = \mathrm{Var}[X_1] < +\infty$

$$\overline{X}_n \xrightarrow[n\to\infty]{\mathbb{P}} \mu$$

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \approx 1/n$$

$$\forall \varepsilon > 0 \qquad \boxed{\mathbb{P}\left[\,|\overline{X}_n - \mu| \geq \varepsilon\right] \leq \frac{\sigma^2}{n\varepsilon^2}} \qquad \forall n \quad \forall \varepsilon > 0$$

Chernoff's bounds

Let $X_1, \dots, X_n$ be iid r.v's and set

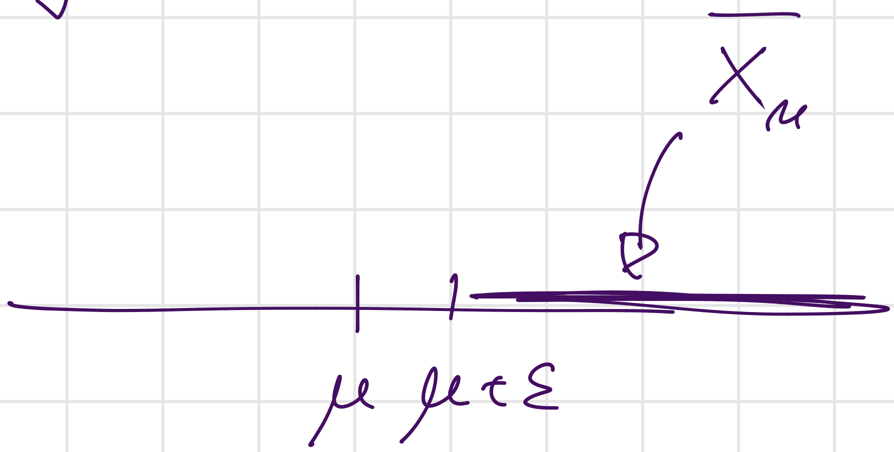
$$\underline{m(t)} := \mathbb{E}\left[e^{t\cdot X_1}\right] \quad \text{mgf of } X_1 \; (X_1, \dots, X_n)$$

We assume that $m(t) < +\infty$ $\forall t \in (-\varepsilon, \varepsilon)$

for a suitable $\varepsilon > 0$.

$$\mathbb{P}[\underbrace{|\bar{X}_n - \mu|} \geq \varepsilon] = \mathbb{P}[\underbrace{\bar{X}_n - \mu \geq \varepsilon}]^{①} + \mathbb{P}[\underbrace{\bar{X}_n - \mu \leq -\varepsilon}]^{②}$$

$$\mathbb{P}[\bar{X}_n - \mu \geq \varepsilon] =$$



$$= \mathbb{P}[\bar{X}_n \geq \mu + \varepsilon] \qquad \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$= \mathbb{P}[\underbrace{X_1 + X_2 + \dots + X_n \geq n(\mu+\varepsilon)}] \qquad \boxed{t>0}$$

$$= \mathbb{P}[\underbrace{e^{t(X_1+\dots+X_n)}} \geq \underbrace{e^{tn(\mu+\varepsilon)}}] \qquad t \geq s \iff e^t \geq e^s$$

positive r.v.

Markov Inequality

$$\leq \frac{\mathbb{E}[e^{t(X_1+X_2+\dots+X_n)}]}{e^{tn(\mu+\varepsilon)}}$$

X_i are indep

$$= \frac{\mathbb{E}[e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n}]}{e^{tn(\mu+\varepsilon)}} \overset{IND}{=} \frac{(m(t))^n}{e^{tn(\mu+\varepsilon)}}$$

$m(t)$

$$\mathbb{E}[e^{tX_1}] \cdot \mathbb{E}[e^{tX_2}] \dots$$

$$\mathbb{P}[\overline{X}_n \geq \mu+\varepsilon] \leq \exp\left(-nt(\mu+\varepsilon) + n\log(m(t))\right)$$

$$\boxed{\frac{(m(t))^n}{e^{nt(\mu+\varepsilon)}}} = \exp($$

$$(m(t))^n = e^{\log (m(t))^n} = e^{n\log(m(t))}$$

$$\frac{1}{e^{nt(\mu+\varepsilon)}} = \underbrace{e^{-nt(\mu+\varepsilon)}}$$

$$= \exp(-n \cdot g(t)) \qquad t \geq 0$$

where $\boxed{g(t) = t(\mu+\varepsilon) - \log(m(t))}$

$$\mathbb{P}[\overline{X}_n \geq \mu+\varepsilon] \leq \exp(-n\, g(t)) \qquad \boxed{\frac{g(t)>0}{t>0}}$$

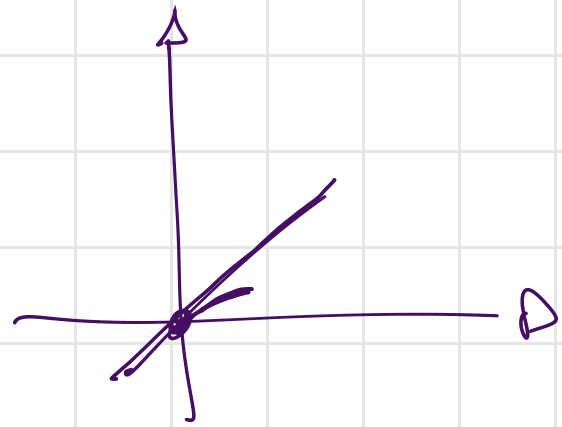
Look for $\underbrace{t^*}$ s.t. $g(t^*) \geq g(t) \quad \forall t \geq 0$

$$\Downarrow$$

$$\Rightarrow \underbrace{e^{-n\,g(t^*)} \leq e^{-n\,g(t)}}_{\forall t\geq 0} \qquad -n\,g(t^*) \leq -n\,g(t)$$

$\exists\, \bar{t} > 0 \quad \text{s.t.} \quad g(\bar{t}) > 0$

$t \longmapsto g(t)$



$t \geq 0$

$$\underbrace{g(t) = t(\mu+\varepsilon) - \log(\underbrace{m(t)})}$$

$$g(0) = 0 \cdot (\mu+\varepsilon) - \log(m(0))$$

$$= 0 - \log(1) = 0$$

$$\underbrace{m(0) = \mathbb{E}[e^{0 \cdot X}] = \mathbb{E}[1] = 1}$$

if we are able to prove that $\underbrace{g'(0) > 0}$

$$\Rightarrow \boxed{\exists\, \bar{t} > 0 \quad \text{s.t.} \quad g(\bar{t}) > 0}$$

$$\underbrace{m'(t)\big|_{t=0} = \mathbb{E}[X]} = \mu$$

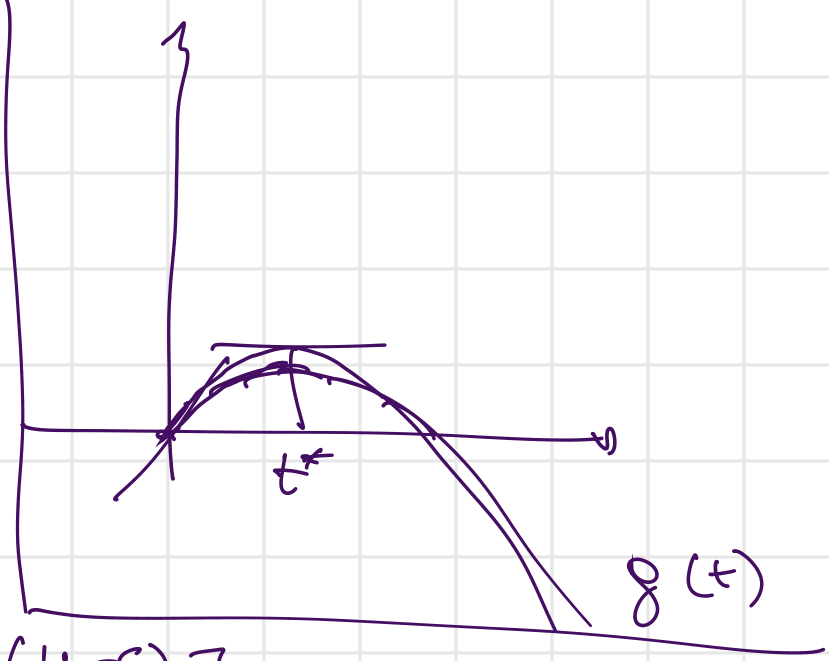
$$g'(t) = \mu + \varepsilon - \frac{1}{m(t)} \cdot m'(t)$$

$$g'(t)\big|_{t=0} = \mu + \varepsilon - \frac{1}{1} \cdot \mu = \varepsilon > 0$$

<u>upper tail estimate</u>

$$\mathbb{P}[\bar{X}_n \geq \mu+\varepsilon] \leq e^{-n\, g(\bar{t})} \quad \forall n$$

② $\mathbb{P}[\overline{X}_n \le \mu - \varepsilon]$



$$= \mathbb{P}[\underbrace{X_1 + X_2 + \dots + X_n} \le \underbrace{n(\mu - \varepsilon)}]$$

$$= \mathbb{P}[\underbrace{e^{-t(X_1 + \dots + X_n)}}_{Y} \ge e^{-tn(\mu-\varepsilon)}]$$

$t \ge 0$

$$\overset{M.I}{\le} \frac{\mathbb{E}[e^{-t(X_1+\dots+X_n)}]}{e^{\underbrace{-tn(\mu-\varepsilon)}}} =$$

$$= \exp\left(\underbrace{-n}\left(-t(\mu-\varepsilon) - \log(m(-t))\right)\right)$$

$$= \exp(-n\, h(t))$$

$$h(t) = \underbrace{-t(\mu-\varepsilon)} - \underbrace{\log(m(-t))}$$

$$h(0) = 0 - \log(m(0)) = 0$$

$$h'(t) = -(\mu-\varepsilon) - \frac{1}{m(-t)} \cdot m'(-t)\,(-1)$$

$$= \varepsilon - \mu + \frac{\overbrace{m'(-t)}}{\underbrace{m(-t)}}$$

$t > 0$

$$h'(0) = \varepsilon - \mu + \mu = \varepsilon > 0$$

$\exists\ \bar{t} > 0$ s.t. $h(\bar{t}) > 0$

$$\boxed{\begin{array}{l}\text{Lower tail estimate} \\ \mathbb{P}[\bar{X}_n \leq \mu - \varepsilon] \leq e^{-n h(\bar{t})}\end{array}}$$

$\underbrace{\bar{t} > 0}$

ε

$\bar{t}(\varepsilon)$

Example 1: Normal r.v.'s

$X_1, \dots, X_n \sim N(0,1)$ independent $\underbrace{\mu = 0}$

$\underbrace{m(t) = e^{t^2/2}}$ $t \in \mathbb{R}$

Upper tail estimate

$\mathbb{P}[\bar{X}_n \geq \mu + \varepsilon] \leq e^{-n \underbrace{g(t)}}$, $t > 0$

$g(t) = t(\mu + \varepsilon) - \log(m(t))$

$g(t) = t \cdot \varepsilon - t^2/2$ $\boxed{t \geq 0}$

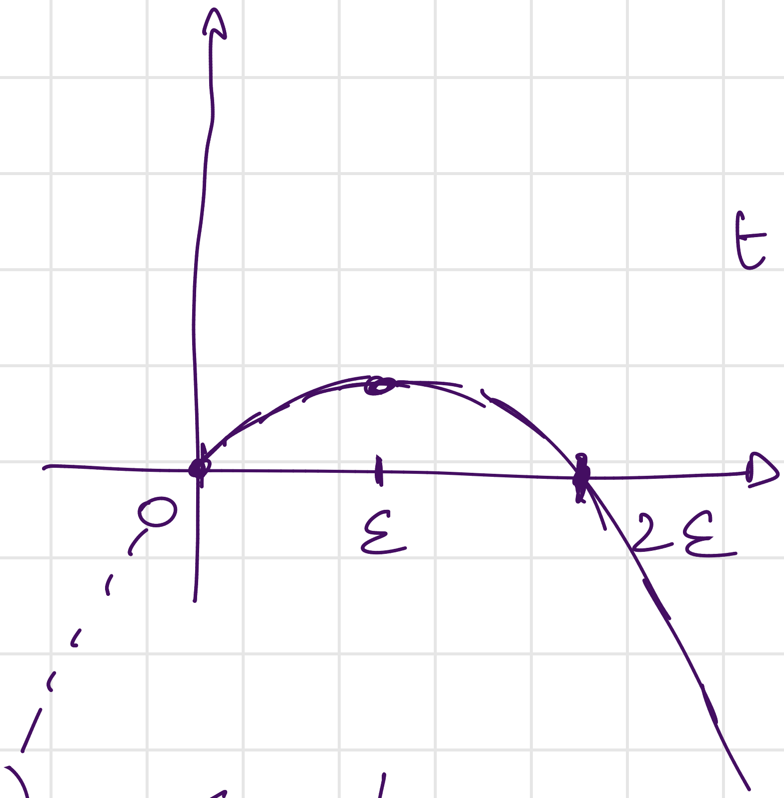
$\underbrace{g(t) = t \cdot \varepsilon - t^2/2}$

$g(t) = 0$

$t = 0$

$t/2 - \varepsilon = 0$

$t = 2 \cdot \varepsilon$



$t \geq 0$

t^*

$g'(t) = \varepsilon - t$

$g'(t) = 0 \iff \underbrace{t = \varepsilon}$

$t^*: \quad g(t^*) \geq g(t) \quad \forall t \geq 0 \quad \boxed{t^* = \varepsilon}$

$g(t^*) = g(\varepsilon) = \varepsilon^2/2$

$X_1 \sim N(0,1)$

$$\boxed{\mathbb{P}[\overline{X}_n \geq \varepsilon] \leq \underbrace{e^{-n\,\varepsilon^2/2}}}$$

Standard Normal

$X \sim N(0,1)$

$$\mathbb{P}[X > x] \leq \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}} \, \underline{\underline{e^{-x^2/2}}}, \quad x \geq 0$$

Example 2: $Z_1, \dots, Z_n$ indep. $N(0,1)$, $\forall i = 1, \dots, n$

$$\underbrace{X_i = Z_i^2}_{} \sim \text{Chi-square distribution with 1 degree of freedom}$$

$$X_i \sim \Gamma\left(1/2, 1\right) \quad \text{(Gamma)}$$

$$\mathbb{E}[X_i] = \mathbb{E}[Z_i^2] = 1 \qquad g(z) = e^{t \cdot z^2}$$

$$m(t) = \mathbb{E}[e^{t \cdot X_1}] = \mathbb{E}[e^{t \cdot Z_1^2}] =$$

$$= \int_{-\infty}^{+\infty} g(z) f_{Z_1}(z)\, dz$$

$$= \int_{-\infty}^{+\infty} e^{t \cdot z^2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2}\, dz$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \underbrace{e^{\boxed{-\frac{z^2}{2}\left(\boxed{1-2t}\right)}}}_{e^{-s^2/2}}\, dz$$

$$\boxed{1-2t>0}$$

$$s = \underbrace{\sqrt{1-2t}}_{\neq 0} \cdot z \qquad s^2 = (1-2t) z^2$$

$$1-2t=0$$

$$dz = \underbrace{\frac{1}{\sqrt{1-2t}}}_{} \cdot ds \qquad 1-2t \neq 0$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-s^2/2} \cdot \left(\frac{1}{\sqrt{1-2t}}\right) ds$$

$$m(t) = \begin{cases} +\infty & 1-2t \leq 0, \quad t \geq 1/2 \\ \dfrac{1}{\sqrt{1-2t}} & 1-2t > 0, \quad \boxed{t < 1/2} \end{cases}$$

$$\mathbb{P}\left[\overline{X}_n \geq \mu + \varepsilon\right] \leq e^{-n\,g(t)} \qquad (-\varepsilon, \varepsilon)$$

$$\boxed{g(t)} = t(\mu+\varepsilon) - \lg(m(t)) \qquad \mu = \mathbb{E}[X_1] = 1$$

$$= t(1+\varepsilon) - \lg\left(\frac{1}{\sqrt{1-2t}}\right) \qquad \boxed{t < 1/2}$$

$$= t(1+\varepsilon) - \lg\left((1-2t)^{-1/2}\right)$$

$$= \boxed{t(1+\varepsilon) + \frac{1}{2}\lg(1-2t)}$$

t^*

$$g'(t) = 0 \iff (1+\varepsilon) + \frac{1}{2}\,\frac{1}{1-2t}\cdot(-2) = 0$$

$$1+\varepsilon - \frac{1}{1-2t} = 0 \iff 1+\varepsilon = \frac{1}{1-2t}$$

$$\iff 1-2t = \frac{1}{1+\varepsilon}$$

$$\iff 2t = 1 - \frac{1}{1+\varepsilon} = \frac{1+\varepsilon-1}{1+\varepsilon} = \frac{\varepsilon}{1+\varepsilon}$$

$$t^* = \frac{\varepsilon}{2(1+\varepsilon)} \quad < 1/2 \; ? \quad \text{YES}$$

$$\mathbb{P}[\bar{X}_n \geq 1+\varepsilon] \leq e^{-n\, g(t^*)}$$

$$g(t^*) \qquad\qquad g(t) = t(1+\varepsilon) + \frac{1}{2}\log(1-2t)$$

$$\| $$

$$\frac{\varepsilon}{2\cancel{(1+\varepsilon)}} \cdot \cancel{(1+\varepsilon)} + \frac{1}{2}\log\left(\frac{1}{1+\varepsilon}\right) =$$

$$= \varepsilon/2 - 1/2 \log(1+\varepsilon)$$

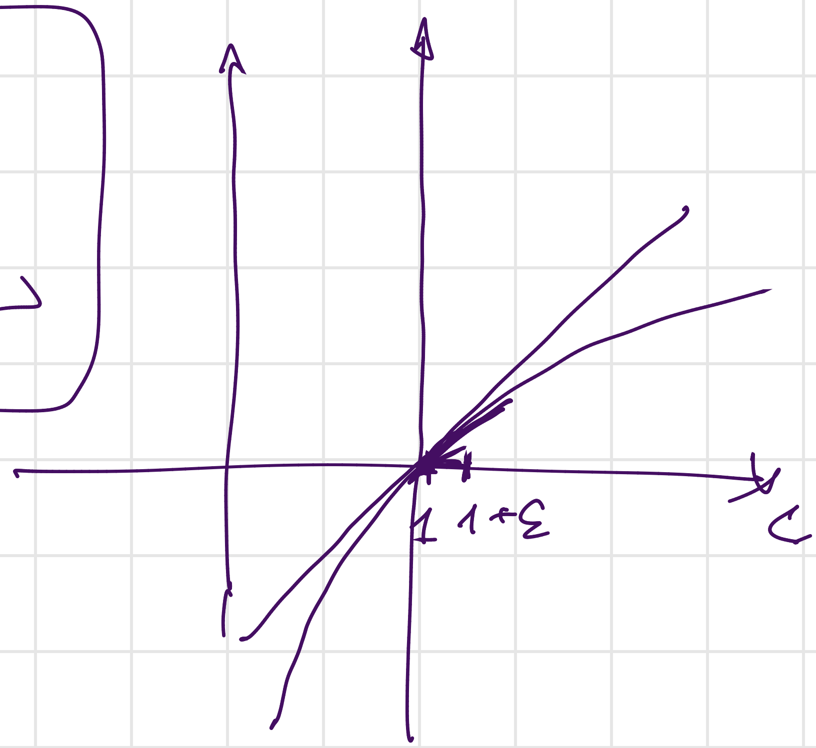
$$1-2t^* = 1 - \cancel{2} \cdot \frac{\varepsilon}{\cancel{2}(1+\varepsilon)} = \frac{1+\varepsilon-\varepsilon}{1+\varepsilon} =$$

$$= \frac{1}{1+\varepsilon}$$

$$\mathbb{P}[\overline{X}_n \geq 1+\varepsilon] \leq e^{-n\, g\left(\frac{\varepsilon}{2(1+\varepsilon)}\right)}$$

$$g\left(\frac{\varepsilon}{2(1+\varepsilon)}\right) = \frac{1}{2}\cdot\left(\varepsilon - \underbrace{\lg(1+\varepsilon)}\right)$$

$$\boxed{\lg(1+\varepsilon) \leq \varepsilon - \underbrace{\frac{\varepsilon^2}{2(1+\varepsilon)}}}$$



$$a(\varepsilon) = \lg(1+\varepsilon)$$

$$b(\varepsilon) = \varepsilon - \frac{\varepsilon^2}{2(1+\varepsilon)}$$

$$\boxed{a(\varepsilon) \leq b(\varepsilon)}$$

$\underline{\text{small } \varepsilon}$

$\varepsilon = 0 \quad a', b'$