Lecture 8 Stoch. Meth. Oct. 24th, 2024

- Expectation of a R.V.
- Abs. Continuous R.V.
- Variance
- Random Vectors

X discrete r.v.

$$\Rightarrow \quad \underbrace{\mathbb{E}[X]} = \underbrace{\sum_{x\in\mathbb{R}} x \cdot p(x)} \qquad p(x) = \mathbb{P}[X=x]$$

whenever this makes sense

$$S^+ = \sum_{x\geq 0} x \cdot p(x), \qquad S^- = \sum_{x<0} (-x)\, p(x)$$

$$S^+, S^- \in [0, +\infty]$$

If at least one of S^+ and S^- is finite, then we can define

$$\sum_{x\in\mathbb{R}} x\cdot p(x) = S^+ - S^- = \begin{cases} \boxed{< +\infty} \\ +\infty \\ -\infty \end{cases}$$

If both S^+ and S^- are finite, then

$$\mathbb{E}[X] = S^+ - S^- \in \mathbb{R}$$

$$\Leftrightarrow \underbrace{\sum_{x \in \mathbb{R}} |x| \, p(x) < +\infty}_{} \Leftrightarrow S^+ + S^- < +\infty$$

In this case we say that X is <u>integrable</u>

$$X \in \underbrace{L^1(\Omega)}_{}$$

<u>Remark:</u> If $X \geq 0$, $\mathbb{E}[X]$ is always well defined ($S^- = 0$), $\mathbb{E}[X] \in [0, +\infty]$

X discrete $\rightsquigarrow$ X r.v. general
$\mathbb{E}$

We start with the case $X \geq 0$

we approximate <u>from below</u> X by a sequence of non-negative, discrete <u>increasing</u> r.v.'s.

$X_n(\omega) \leq X(\omega)$

$x_n \leq x_{n+1}$

$X_n \uparrow \underbrace{X}\ \ge 0$

$E[X_n]$

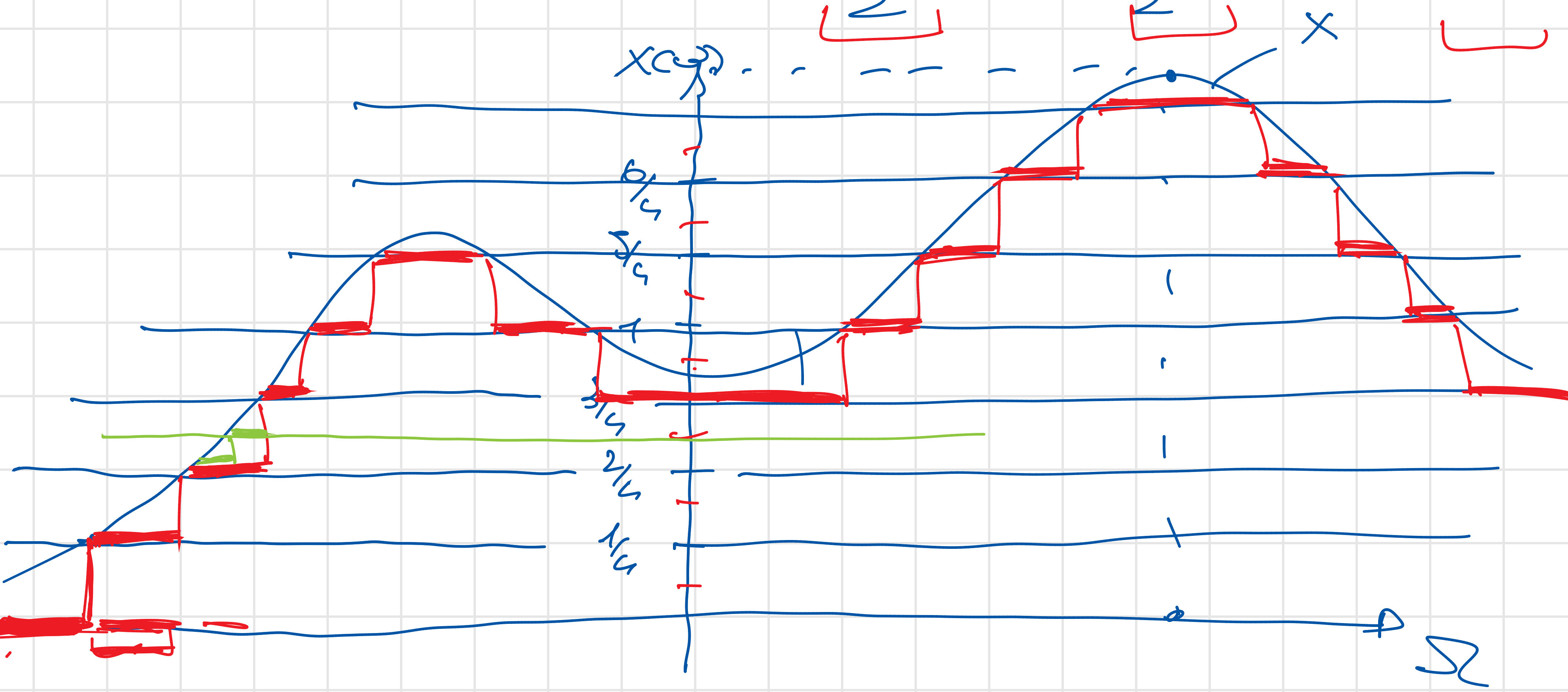
$\cap$

$\mathbb{R}$

$$\boxed{E[X] = \underbrace{\lim_{n\to\infty} E[X_n]}}$$

$n \ge 1, \quad k \in \mathbb{N}$

$X(\omega) \qquad X_n(\omega) = \frac{k}{2^n} \quad \text{if} \quad \frac{k}{2^n} \le X(\omega) < \frac{k+1}{2^n}$



X_2

$\frac{k}{2^2} = \frac{k}{4} \qquad n=2$

$X_2 \in \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{5}{4}, \frac{6}{4}, \frac{7}{4}\} \qquad E[X_2]$

$E[X_2] \le E[X_3] \le$

$X : \Omega \longrightarrow \mathbb{R}$

$$X^+ = \max(X, 0) \geq 0$$

$$X^- = -\min(X, 0) \geq 0$$

$X = X^+ - X^-$ and $|X| = X^+ + X^-$

<u>Def</u>: We say that X, a general r.v., admits <u>expectation</u> if at least one of $\mathbb{E}[X^+]$ and $\mathbb{E}[X^-]$ is finite.

If $\mathbb{E}[X^+]$ and $\mathbb{E}[X^-]$ are both finite, then we say that X is integrable, $\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-] < +\infty$ and we write $X \in L^1(\Omega)$.

Abs. cont. r.v. $\quad X \geq 0 \quad X_n \uparrow X$

f

$$\boxed{\mathbb{E}[X]} = \boxed{\lim_{n\to\infty} \mathbb{E}[X_n]}$$

$$= \lim_{n\to\infty} \sum_k \frac{k}{2^n} \mathbb{P}\left[\frac{k}{2^n} \leq X < \frac{k+1}{2^n}\right]$$

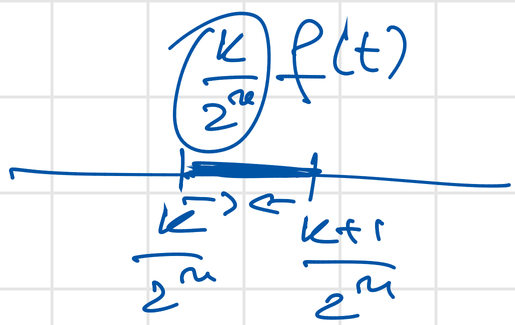
abs. cont.

$$X_n = \frac{k}{2^n} \quad \text{if} \quad \frac{k}{2^n} \leq X < \frac{k+1}{2^n}$$

$$\mathbb{P}\left[\frac{k}{2^n} \leq X < \frac{k+1}{2^n}\right]$$

$$= \lim_{n\to+\infty} \sum_{k\in\mathbb{N}} \frac{k}{2^n} \int_{k/2^n}^{\frac{k+1}{2^n}} f(t)\,dt$$

$$= \lim_{n\to\infty} \sum_{k\in\mathbb{N}} \int_{k/2^n}^{\frac{k+1}{2^n}} t\, f(t)\,dt$$



$$= \int_0^{+\infty} t\, f(t)\,dt$$

X discrete $\longrightarrow$ $\mathbb{E}[X] = \sum_{k\in N} k\, p(k)$

X abs. cont $\longrightarrow$ $\mathbb{E}[X] = \int_{\mathbb{R}} t\, f(t)\, dt$

$$\Omega \xrightarrow{X} E \xrightarrow{g} \mathbb{R}$$

$$g \circ X = g(X) = Y : \Omega \to \mathbb{R}$$

$X^2 = g(X)$

$g(x) = x^2$

- If X is discrete

$$\mathbb{E}[Y] = \sum_{x\in E} g(x) \cdot P_X(x)$$

$$\mathbb{E}[g(X)]$$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\, dx$$

- If X is abs. cont.

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x)\, f(x)\, dx$$

$g(x) \in \mathbb{R}$

Properties:

1. Monotonicity

if $\boxed{X \leq Y}$ then $\boxed{\mathbb{E}[X] \leq \mathbb{E}[Y]}$

$\mathbb{P}[X(\omega) \leq Y(\omega)] = 1$ $\quad X : \Omega \to \mathbb{R}$, $\quad Y : \Omega \to \mathbb{R}$

almost surely

2. If $\mathbb{P}[X = c] = 1$, then $\mathbb{E}[X] = c$

3. Linearity $\quad \alpha, \beta \in \mathbb{R}$, X, Y r.v.

$$\mathbb{E}[\alpha X + \beta Y] = \alpha \cdot \mathbb{E}[X] + \beta\, \mathbb{E}[Y]$$

if $X, Y \in L^1(\Omega)$, then $\alpha X + \beta Y \in L^1(\Omega)$

$X \in \underline{\mathbb{N}}$ $\boxed{p_X(k) = \frac{c}{k^2}}$ is this a density?

$k \in \mathbb{N} \setminus \{0\}$ $X \sim \{1, 2, 3, \dots, n, \dots\}$

(1) $p \geq 0$

(2) $\underbrace{\sum_n p_X(n) \cancel{< +\infty} = 1}$ $(< +\infty)$

If $\sum_n p_X(n) = \alpha \in \mathbb{R}$ $\sum \frac{p_X(n)}{\alpha} = 1$

(1) $\frac{c}{k^2} \geq 0$ YES $c \geq 0$

(2) $\sum_k p_X(k) = \sum_{k=1}^{+\infty} \frac{c}{k^2} < +\infty$ YES $= 1$

$\boxed{c = 1 / \sum_k 1/k^2}$ $k \in \mathbb{Z} \setminus \{0\}$

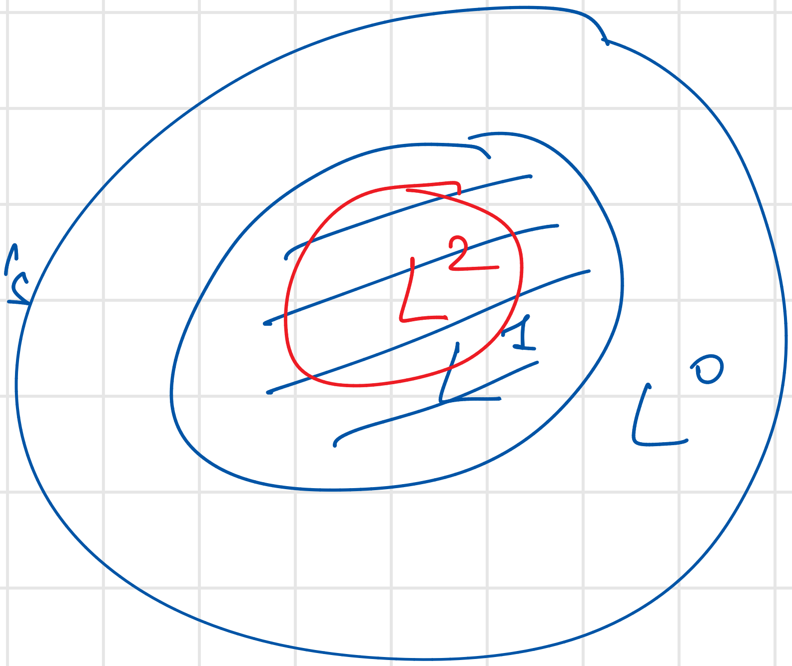
$X \in L^1(\Omega)$? $\Longleftrightarrow$ $\mathbb{E}[X] < +\infty$?

$\Longleftrightarrow \sum_{k=1}^{\infty} k \cdot \frac{c}{k^2} < +\infty$ $\sum_{k=1}^{\infty} \frac{c}{k} = +\infty$

$L^0(\Omega)$ = set of all poss. R.V.'s

$L^1(\Omega)$ = integrable R.V.'s

L ▷ Square integrable R.V.'s



$\iff \mathbb{E}[X^2] < +\infty$

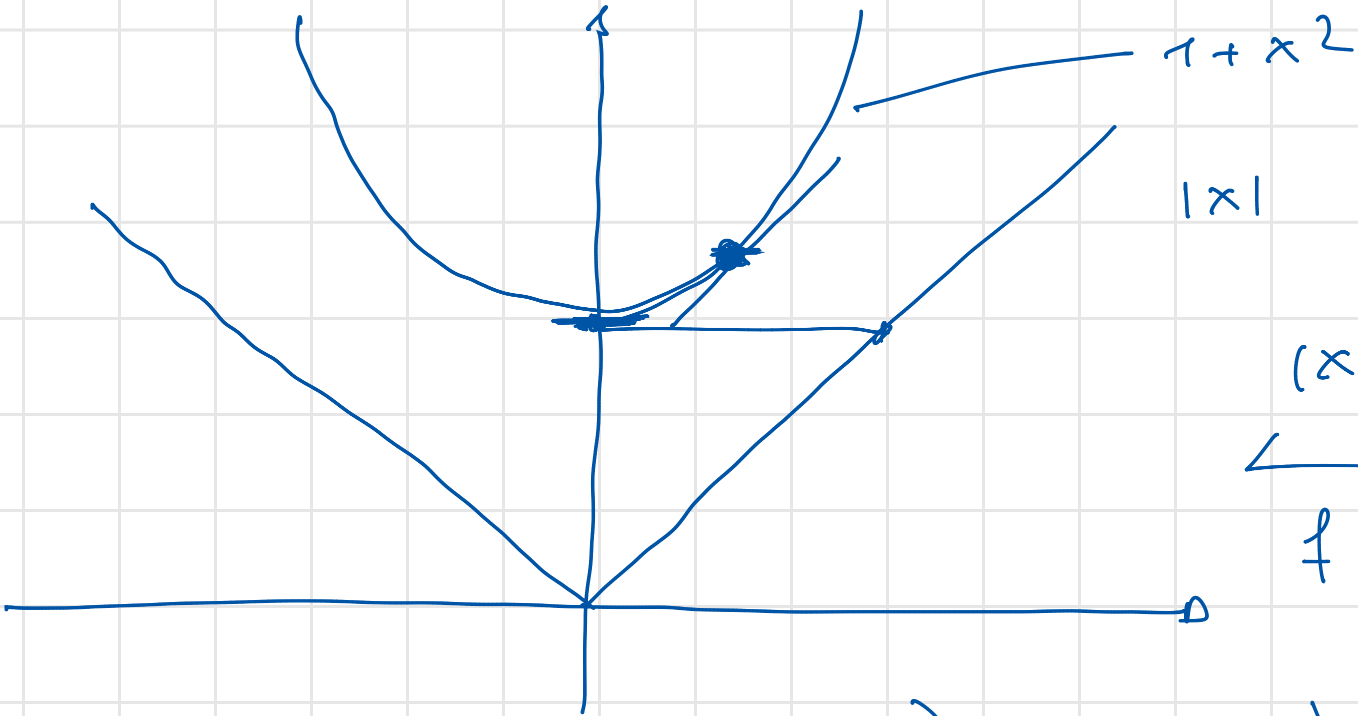
If X is abs. cont., X is square integrable $\iff \int_{\mathbb{R}} x^2 f(x)\, dx < +\infty$

We call $L^2(\Omega)$ the set of all square integrable r.v.'s

(1) $\boxed{L^2(\Omega) \subseteq L^1(\Omega)}$

Monotonicity of expectation:

$\boxed{f(x) = |x| \leq 1 + x^2 = g(x)}$ $\forall x \in \mathbb{R}$



$$\underbrace{|x|}_{f} < \underbrace{1 + x^2}_{g} \quad \forall x$$

$$f(x) \le g(x) \Rightarrow f(X) \le g(X)$$

$$\Rightarrow \mathbb{E}[f(X)] \le \mathbb{E}[g(X)]$$

$$\parallel$$

$$\mathbb{E}[|X|] \le \mathbb{E}[1 + X^2] =$$

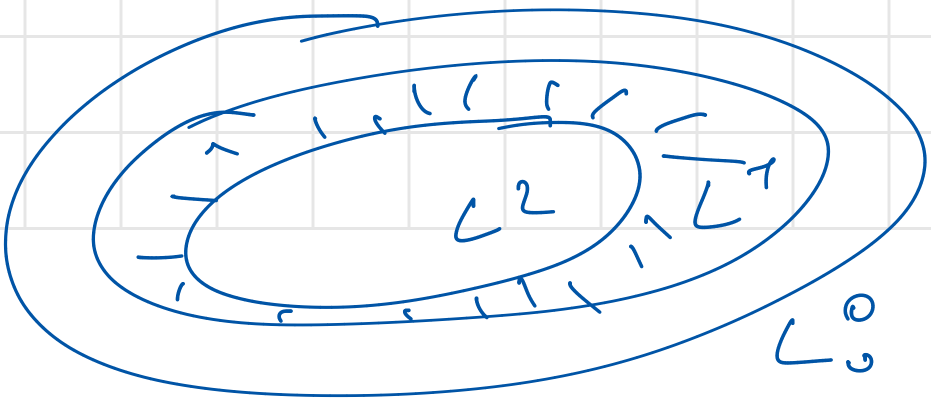
$$= \mathbb{E}[1] + \mathbb{E}[X^2]$$

$$= 1 + \mathbb{E}[X^2]$$

If $X \in L^2 \Rightarrow \mathbb{E}[X^2] < +\infty \Rightarrow \mathbb{E}[|X|] < +\infty$

$$\Rightarrow S^+ + S^- < +\infty \Rightarrow S^+ - S^- < +\infty \Rightarrow$$

$$\mathbb{E}[X] < +\infty \Rightarrow X \in L^1$$

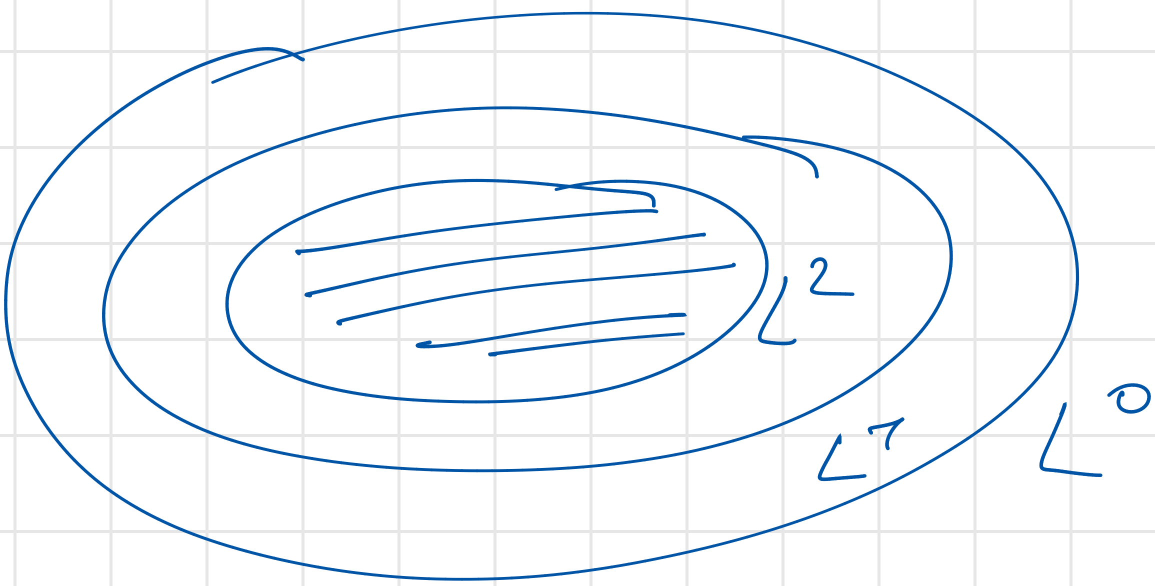


Ex: $X \in L^1$, but $X \notin L^2$

$$X > 0, \quad X \in \mathbb{N}, \quad P_X(k) = \frac{c}{k^3}$$

- $\sum 1/k^3 < +\infty \Rightarrow X \in L^0$
- $\sum k \cdot 1/k^3 = \sum 1/k^2 < +\infty \Rightarrow X \in L^1$
- $\sum g(k) \cdot P_X(k) = \sum k^2 \cdot \frac{1}{k^3} = \sum \frac{1}{k} = +\infty$

  $\Rightarrow X \notin L^2$



$L^n \subseteq L^{n-1}$

$L^3, L^4, L^5, \dots, L^n,$

$X \in L^n \iff |X|^n \in L^1$

$\mathbb{E}[X^2]$

$0 \le (X - \mathbb{E}[X])^2$

Random.

D

Variance

$$\mathrm{Var}[X] := \mathbb{E}\left[\underbrace{(X - \mathbb{E}[X])^2} \right]$$

$\mathbb{E}[X]$ $X(\omega)$

$$\mathrm{Var}[X] \geq 0$$

$$\mathrm{Var}[X] = \mathbb{E}\left[X^2 + (\mathbb{E}[X])^2 - 2X\,\mathbb{E}[X] \right]$$

$$= \mathbb{E}[X^2] + \mathbb{E}\left[(\mathbb{E}[X])^2\right] + \mathbb{E}\left[\boxed{-2}\,X\,\boxed{\mathbb{E}[X]}\right]$$

$$= \mathbb{E}[X^2] + (\mathbb{E}[X])^2 \underbrace{- 2\,\mathbb{E}[X]} \cdot \mathbb{E}[X]$$

$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$