EXERCISE 1

Assume that $y_1, \ldots, y_n$ are independent realizations of a positive continuous random variable Y with pdf

$$f(y) = \theta(1+\theta)y^{\theta-1}(1-y), \quad \theta > 0, \quad y \in (0,1).$$

(a) Show that $f(y)$ is a density function.

(b) Obtain the likelihood function, $L(\theta)$, and the log-likelihood function, $\ell(\theta)$.

(c) Write the score function for θ and verify that, with the data
`0.77, 0.95, 0.62, 0.85, 0.27, 0.01, 0.29, 0.67, 0.80, 0.38, 0.73, 0.18, 0.13`
$\hat{\theta} = 1.513$ is the maximum likelihood estimate of θ. (Hint: $\sum_{i=1}^{n} \log y_i = -13.765$.)

(d) Obtain an approximation for the distribution of $\hat{\theta}$. Build a confidence interval for θ of approximate level 0.95.

(e) Test $H_0 : \theta = \frac{1}{2}$ vs $H_1 : \theta > \frac{1}{2}$ at the significance level 0.05.

Sketch of the solution:

(a) It is clear that $f(y) > 0$ for every $y \in (0,1)$ and $\theta > 0$ and therefore it is sufficient to show that $\int_0^1 f(y,\theta)\, dy = 1$, as follows

$$\begin{aligned}\int_0^1 f(y)\, dy &= \theta(1+\theta) \textstyle\int_0^1 y^{\theta-1} - y^{\theta}\, dy \\ &= \theta(1+\theta)\left[\frac{y^{\theta}}{\theta} - \frac{y^{\theta+1}}{\theta+1}\right]_0^1 \\ &= \theta(1+\theta)\frac{1}{\theta(1+\theta)} = 1\end{aligned}$$

(b) Because the random sample is made up of independent and identically distributed observations the likelihood function can be obtained as follows.

$$\begin{aligned}L(\theta) &= \prod_{i=1}^{n} f(y_i) \\ &= \prod_{i=1}^{n} \theta(1+\theta)y_i^{\theta-1}(1-y_i) \\ &= \theta^n(1+\theta)^n \left(\prod_{i=1}^{n} y_i\right)^{\theta-1} \prod_{i=1}^{n}(1-y_i) \\ &\propto \theta^n(1+\theta)^n \left(\prod_{i=1}^{n} y_i\right)^{\theta}\end{aligned}$$

and therefore,

$$\ell(\theta) = \log L(\theta) = n\log\theta + n\log(1+\theta) + \theta\sum_{i=1}^{n}\log y_i.$$

(c) the score function is

$$\ell_*(\theta) = \frac{d}{d\theta}\ell(\theta) = \frac{n}{\theta} + \frac{n}{1+\theta} + \sum_{i=1}^{n} \log y_i,$$

while the likelihood equation is $\ell_*(\theta) = 0$, that is

$$\frac{n}{\theta} + \frac{n}{1+\theta} + \sum_{i=1}^{n} \log y_i = 0$$

and $\hat{\theta} = 1.513$ is the MLE of θ because

$$\frac{13}{1.513} + \frac{13}{1+1.513} - 13.765 = 0$$

and, furthermore,

$$\ell_{**}(\theta) = \frac{d}{d\theta}\ell_*(\theta) = -\frac{n}{\theta^2} - \frac{n}{(1+\theta)^2} < 0$$

(d) by the asymptotic theory of the likelihood, we know that

$$\hat{\theta} \dot{\sim} N(\theta, j(\hat{\theta})^{-1}),$$

where

$$\begin{aligned} j(\hat{\theta}) &= -\ell_{**}(\hat{\theta}) = \tfrac{n}{\hat{\theta}^2} + \tfrac{n}{(1+\hat{\theta})^2} \\ &= \tfrac{13}{1.513^2} + \tfrac{13}{(1+1.513)^2} = 7.737. \end{aligned}$$

Hence,

$$\hat{\theta}_n \dot{\sim} N(\theta, 1/7.737) \sim N(\theta, 0.129),$$

so that a confidence interval for θ of approximate level 0.95 can be computed as

$$\hat{\theta} \pm 1.96\sqrt{j(\hat{\theta})^{-1}} = (0.809, 2.217).$$

(e) By the asymptotic theory of the likelihood, we know that a pivotal quantity for $H_0 : \theta = \frac{1}{2}$ is

$$T = \frac{(\hat{\theta}_n - 1/2)}{\sqrt{j(\hat{\theta})^{-1}}} \dot{\sim} N(0,1). \tag{1}$$

Its observed value under H_0 is

$$t^{obs} = \frac{(\hat{\theta} - 1/2)}{\sqrt{j(\hat{\theta})^{-1}}} = 2.8176.$$

By hypotheses, we reject H_0 for large values of T.

More specifically, $t_{obs} = 2.8176$ is larger than the 0.95-quantile=1.64 of the standard normal distribution. Hence, the p-value is smaller than the significance level 0.05 and therefore we can reject H_0.

EXERCISE 2

Assume that $x_1, \ldots, x_n$ are independent realizations of a random variable X with probability density function

$$f(x) = C^\theta \, \theta \, x^{-\theta-1} \quad \text{with } x > C \text{ and } \theta > 1$$

where $C > 0$ is a known constant. Furthermore, it holds that $E(X) = \mu = C\theta/(\theta - 1)$.

(a) Show that $f(\cdot)$ is a proper density function.

(b) Obtain the likelihood function $L(\theta)$.

(c) Obtain the log-likelihood function $\ell(\theta)$.

(d) Obtain the score function $\ell_*(\theta)$.

(e) Compute the maximum likelihood estimate $\hat{\theta}$ of θ.

(f) Obtain an asymptotic approximation for the distribution of $\hat{\theta}$.

(g) **In the rest of this exercise assume that $C = 1$.**
Compute $\hat{\theta}$ with the data (see the R-script below),
`2.18 1.55 1.36 1.15 1.03 1.13 1.02 1.28 1.50 1.08`

(h) Use the asymptotic distribution of $\hat{\theta}$ to compute an approximate 95% confidence interval for θ.

(i) Explain if it is possible from the information provided to compute an approximate 95% confidence interval for μ and, in case, compute it.

This is some `R` code, potentially useful **for all the exercises**:

```
> x <- c(2.18, 1.55, 1.36, 1.15, 1.03, 1.13, 1.02, 1.28, 1.50, 1.08)
> length(x)
[1] 10
> mean(x)
[1] 1.328
> mean(log(x))
[1] 0.2565692
> mean(exp(x))
[1] 4.030509
> alpha <- c(0.001, 0.01, 0.025, 0.05, 0.95, 0.975, 0.99, 0.999)
> round(qnorm(alpha),2)
[1] -3.09 -2.33 -1.96 -1.64  1.64  1.96  2.33  3.09
> round(qchisq(alpha, 9), 2)
[1]  1.15  2.09  2.70  3.33 16.92 19.02 21.67 27.88
> round(qchisq(alpha, 10), 2)
[1]  1.48  2.56  3.25  3.94 18.31 20.48 23.21 29.59
> round(qt(alpha, 9), 2)
[1] -4.30 -2.82 -2.26 -1.83  1.83  2.26  2.82  4.30
> round(qt(alpha, 10), 2)
[1] -4.14 -2.76 -2.23 -1.81  1.81  2.23  2.76  4.14
```

Sketch of the solution:

(a)

$$\int_C^\infty f(x)\,dx = C^\theta \theta \int_C^\infty x^{-\theta-1}\,dx = \left[\frac{C^\theta \theta}{-\theta} x^{-\theta}\right]_C^\infty = C^\theta C^{-\theta} = 1$$

(b)

$$L(\theta) \propto C^{n\theta}\theta^n \left(\prod_{i=1}^n x_i\right)^{-\theta-1}$$

(c)

$$\ell(\theta) \quad = \quad n\theta \log(C) + n\log(\theta) - \theta \sum_{i=1}^n \log(x_i)$$

(d)

$$\ell_*(\theta) = n\log(C) + \frac{n}{\theta} - \sum_{i=1}^n \log(x_i)$$

(e) $\ell_*(\hat{\theta}) = 0$ gives

$$\hat{\theta} = \frac{n}{\sum_{i=1}^n \log(x_i) - n\log(C)}$$

and, furthermore,

$$\ell_{**}(\theta) = \frac{d\ell_*(\theta)}{d\theta} = -\frac{n}{\theta^2} < 0$$

(f)

$$j(\theta) = -\ell_{**}(\theta) = \frac{n}{\theta^2}$$

so that

$$j(\hat{\theta})^{-1} = \frac{\hat{\theta}^2}{n}.$$

(g)

$$\hat{\theta} = \frac{n}{\sum_{i=1}^n \log(x_i)} = \frac{1}{0.0.25657} = 3.898$$

(h) $\hat{\theta} \pm 1.96 \times \sqrt{j(\hat{\theta})^{-1}}$ gives $3.898 \pm 1.96 \times 1.233$ that is (1.48; 6.31).

(i) Because μ is a one-to-one (decreasing) transformation of θ, we can compute an approximate 95% confidence interval for μ as follows:

$$\left(\frac{6.31}{6.31-1},\ \frac{1.48}{1.48-1}\right) = (1.19;\ 3.08).$$

EXERCISE 3

Assume that $x_1, \dots, x_n$ are independent realizations of a random variable X with density function

$$f(x) = \frac{\theta\, x^{\theta-1}}{5^\theta} \quad \text{for } 0 \le x \le 5$$

with $\theta > 0$.

(a) Calculate the cumulative distribution function of X.

(b) Calculate the expected value $\mu = E(X)$ of X.

(c) There exists a value of the parameter θ such that the distribution of X is uniform over the interval $[0,\ 5]$?

(d) Obtain the likelihood function $L(\theta)$.

(e) Obtain the log-likelihood function $\ell(\theta)$.

(f) Obtain the score function $\ell_*(\theta)$.

(g) Compute the maximum likelihood estimate $\hat{\theta}$ for θ.

(h) Compute $\hat{\theta}$ with the data (see the R-script below),

`2.49  4.26  1.72  3.21  2.32  2.38  2.97  4.32  2.40  2.52`

(i) Obtain an approximation for the distribution of $\hat{\theta}$.

(j) Compute the maximum likelihood estimate $\hat{\mu}$ of μ.

(k) Obtain an approximate 95% confidence interval for θ

(l) Explain if you can use the asymptotic distribution of $\hat{\theta}$ to obtain an approximate 95% confidence interval for μ and, if possible, obtain such interval.

This is some potentially useful *R* code:

```
> x <- c(2.49, 4.26, 1.72, 3.21, 2.32, 2.38, 2.97, 4.32, 2.40, 2.52)
> length(x)
[1] 10
> sum(x)
[1] 28.59
> sum(log(x))
[1] 10.130
> sum(exp(x))
[1] 252.3478
> alpha <- c(0.5, 0.9, 0.95, 0.975, 0.99, 0.995)
> qt(alpha, 9)
[1] 0.000000 1.383029 1.833113 2.262157 2.821438 3.249836
> qnorm(alpha)
[1] 0.000000 1.281552 1.644854 1.959964 2.326348 2.575829
```

Sketch of the solution:

(a)

$$F(x) = \int_0^x f(t)\, dt = \frac{\theta}{5^\theta} \int_0^x t^{\theta-1}\, dt = \left[\left(\frac{t}{5} \right)^\theta \right]_0^x = \left(\frac{x}{5} \right)^\theta$$

(b)

$$E(X) = \int_0^5 x\, f(x)\; dx = \frac{\theta}{5^\theta}\int_0^5 x^\theta\; dx = \frac{\theta}{5^\theta}\left[\frac{x^{\theta+1}}{\theta+1}\right]_0^5 = \frac{5\theta}{\theta+1}$$

(c) For $\theta = 1$ the random variable X follows an uniform distributions in the interval $[0,\ 5]$.

(d) The likelihood function can be written as

$$L(\theta) = \prod_{i=1}^n \frac{\theta x_i^{\theta-1}}{5^\theta} = \frac{\theta^n \left(\prod_{i=1}^n x_i\right)^{\theta-1}}{5^{n\theta}}.$$

(e) The log-likelihood is

$$\ell(\theta) = n\log(\theta) - n\theta\log(5) + (\theta-1)\sum_{i=1}^n \log(x_i)$$

(f) The score function is

$$\ell_*(\theta) = \frac{n}{\theta} - n\log(5) + \sum_{i=1}^n \log(x_i)$$

(g) the score equation

$$\ell_*(\hat\theta) = \frac{n}{\hat\theta} - n\log(5) + \sum_{i=1}^n \log(x_i) = 0$$

gives

$$\hat\theta = \frac{n}{n\log(5) - \sum_{i=1}^n \log(x_i)} = \frac{1}{\log(5) - \sum_{i=1}^n \log(x_i)/n}$$

that is the MLE of θ because

$$\ell_{**}(\theta) = -\frac{n}{\theta^2} < 0.$$

(h) Hence, for the data provided,

$$\hat\theta = \frac{1}{\log(5) - 10.13/10} = 1.67662$$

(i)

$$j(\theta) = -\ell_{**}(\theta) = \frac{n}{\theta^2}.$$

so that

$$j(\hat\theta)^{-1} = \frac{\hat\theta^2}{n} = \frac{1.67662^2}{10} = 0.2812$$

and, thus,

$$\hat\theta \mathrel{\dot\sim} N(\theta, 0.2812)$$

(j) By the equivariance property of the MLE it holds that

$$\hat{\mu} = \frac{5\hat{\theta}}{\hat{\theta}+1} = 3.132$$

(k) an approximate 95% confidence interval for θ is

$$\hat{\theta} \pm 1.96 \times \sqrt{j(\hat{\theta})^{-1}} = 1.67662 \pm 1.96 \times \sqrt{0.2812} = (0.6373;\ 2.716)$$

(l) Because μ is a one-to-one (increasing) function of θ, an approximate 95% confidence interval for μ is give by

$$\left(\frac{5 \times 0.6373}{0.6373+1}; \frac{5 \times 2.716}{2.716+1}\right) = (1.946;\ 3.654).$$

EXERCISE 4

Assume that $x_1, \dots, x_n$ are independent realizations of a random variable X with cumulative distribution function

$$F(x) = 1 - \frac{e^{-x/\theta}(x+\theta)}{\theta}$$

for $x > 0$ and $\theta > 0$.

(a) Write the density function of X.

(b) Obtain the likelihood function $L(\theta)$.

(c) Obtain the log-likelihood function $\ell(\theta)$.

(d) Obtain the score function $\ell_*(\theta)$.

(e) Compute the maximum likelihood estimate $\hat{\theta}$ for θ (it is not required to check the sign of the second derivative).

(f) Compute and approximate distribution of $\hat{\theta}$.

Sketch of the solution:

(a)

$$f(x) = \frac{1}{\theta^2}\, x\, e^{-\frac{x}{\theta}}$$

(b)

$$L(\theta) = \frac{1}{\theta^{2n}} \exp\left(-\frac{n\bar{x}}{\theta}\right)$$

(c)

$$\ell(\theta) = -2n\log\theta - \frac{n\bar{x}}{\theta}$$

(d)

$$\ell_*(\theta) = -\frac{2n}{\theta} + \frac{n\bar{x}}{\theta^2}$$

(e)

$$\hat{\theta} = \frac{\bar{x}}{2}$$

(f)

$$\ell_{**} = \frac{2n}{\theta^2} - 2\frac{n\bar{x}}{\theta^3}$$

so that

$$\begin{aligned} j(\hat{\theta}) &= \frac{8n}{\bar{x}^2} - 16\frac{n\bar{x}}{\bar{x}^3} \\ &= -\frac{8n}{\bar{x}^2} \end{aligned}$$

and

$$\hat{\theta} \dot{\sim} N\left(\theta, \frac{\bar{x}^2}{8n}\right)$$

EXERCISE 5

Suppose that X is a discrete random variable with the following probability mass function, where $0 \leq \theta \leq 1$ is a parameter,

x	0	1	2	3
$P(X = x)$	$2\theta/3$	$\theta/3$	$2(1-\theta)/3$	$(1-\theta)/3$

(a) Compute the expected value of X.

(b) The following 10 independent observations were taken from such a distribution:

3, 0, 2, 1, 3, 2, 1, 0, 2, 1

Calculate the maximum likelihood estimate $\hat{\theta}$ of θ.

(c) Calculate the asymptotic distribution of $\hat{\theta}$.

(d) Give a 95% (approximate) confidence interval for θ.

Sketch of the solution:

(a)

$$\begin{aligned} E(X) &= 0 \times \frac{2\theta}{3} + 1 \times \frac{\theta}{3} + 2 \times \frac{2(1-\theta)}{3} + 3 \times \frac{(1-\theta)}{3} \\ &= \frac{\theta + 4 - 4\theta + 3 - 3\theta}{3} \\ &= \frac{7 - 6\theta}{3} \end{aligned}$$

(b) The likelihood function is,

$$\begin{aligned} L(\theta) &= \left(\frac{2\theta}{3}\right)^2 \left(\frac{\theta}{3}\right)^3 \left(\frac{2(1-\theta)}{3}\right)^3 \left(\frac{1-\theta}{3}\right)^2 \\ &\propto \theta^5(1-\theta)^5. \end{aligned}$$

The log-likelihood function therefore is,

$$\ell(\theta) = 5\log(\theta) + 5\log(1-\theta)$$

and

$$\ell_*(\theta) = \frac{5}{\theta} - \frac{5}{1-\theta}$$

and from the score equation $\ell_*(\hat{\theta}) = 0$ it follows that $\hat{\theta} = \frac{1}{2}$ that is the MLE because

$$\ell_{**}(\theta) = -\frac{5}{\theta^2} - \frac{5}{(1-\theta)^2} < 0$$

(c) We have

$$j(\hat{\theta})^{-1} = -\ell_{**}(1/2)^{-1} = \frac{1}{40}$$

So that $\hat{\theta} \dot{\sim} N(\theta, 1/40)$

(d) CI: $0.5 \pm 1.96 \times \frac{1}{\sqrt{40}}$ that is (0.19; 0.8).

EXERCISE 6

Assume that $x_1, \ldots, x_n$ are independent realizations of a random variable X with cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < 0; \\ x^{\frac{1}{\theta}} & \text{for } 0 \leq x \leq 1; \\ 1 & \text{for } x > 1; \end{cases}$$

with $\theta > 0$.

(a) Calculate the density function of X.

(b) Calculate the expected value $\mu = E(X)$ of X.

(c) Obtain the likelihood function $L(\theta)$.

(d) Obtain the log-likelihood function $\ell(\theta)$.

(e) Obtain the score function $\ell_*(\theta)$.

(f) Compute (analytically) the maximum likelihood estimate $\hat{\theta}$ for θ (it is not required to check the sign of the second derivative).

(g) Compute $\hat{\theta}$ with the data,

```
0.10, 0.12, 0.02, 0.11, 0.09, 0.01, 0.05, 0.07, 0.1, 0.21
```

(h) Obtain an approximation for the distribution of $\hat{\theta}$.

(i) Compute the maximum likelihood estimate $\hat{\mu}$ of μ.

(j) Use the asymptotic distribution of $\hat{\theta}$ to test the null hypothesis that the distribution of X is uniform over the interval [0, 1] against the alternative hypothesis that the distribution of X is not uniform (use a 5% significance level and provide the approximate p-value).

This is some possibly usefulR code,

```
> x <- c(0.10, 0.12, 0.02, 0.11, 0.09, 0.01, 0.05, 0.07, 0.1, 0.21)
> length(x)
[1] 10
> sum(log(x))
[1] -27.07349
> alpha <- c(0.5, 0.9, 0.95, 0.975, 0.99, 0.995)
> qnorm(alpha)
[1] 0.000000 1.281552 1.644854 1.959964 2.326348 2.575829
```

Sketch of the solution:

(a)

$$f_\theta(x) = \frac{dF(x)}{dx} = \frac{1}{\theta} x^{\frac{1}{\theta}-1} = \frac{1}{\theta} x^{\frac{1-\theta}{\theta}}$$

(b)

$$\begin{aligned} E(X) &= \int_0^1 x\, f(x)\, dx \\ &= \int_0^1 \frac{1}{\theta} x^{\frac{1}{\theta}}\, dx \\ &= \left[\frac{1}{\theta} \times \frac{1}{\frac{1+\theta}{\theta}} \times x^{\frac{1+\theta}{\theta}} \right]_0^1 \\ &= \left[\frac{1}{1+\theta} \times x^{\frac{1+\theta}{\theta}} \right]_0^1 \\ &= \frac{1}{1+\theta} \end{aligned}$$

(c)

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} x_i^{\frac{1-\theta}{\theta}} = \frac{1}{\theta^n} \left(\prod_{i=1}^{n} x_i \right)^{\frac{1-\theta}{\theta}}$$

(d)

$$\ell(\theta) = -n \log(\theta) + \frac{1-\theta}{\theta} \sum_{i=1}^{n} \log(x_i)$$

(e)

$$\begin{aligned}\ell_*(\theta) &= -\frac{n}{\theta} + \frac{-\theta-(1-\theta)}{\theta^2}\sum_{i=1}^n \log(x_i) \\ &= -\frac{n}{\theta} - \frac{1}{\theta^2}\sum_{i=1}^n \log(x_i)\end{aligned}$$

(f)

$$\hat{\theta} = -\frac{\sum_{i=1}^n \log(x_i)}{n}$$

(g)

$$\hat{\theta} = 2.71$$

(h)

$$\ell_{**}(\theta) = \frac{n}{\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^n \log(x_i)$$

so that

$$\begin{aligned}\ell_{**}(\hat{\theta}) &= \frac{n}{\hat{\theta}^2} + \frac{2}{\hat{\theta}^3}(-n\hat{\theta}) \\ &= -\frac{n}{\hat{\theta}^2}\end{aligned}$$

and

$$j(\hat{\theta})^{-1} = \frac{\hat{\theta}^2}{n} = 0.73$$

Finally

$$\hat{\theta} \dot{\sim} N(\theta, 0.73)$$

(i)

$$\hat{\mu} = \frac{1}{1+\hat{\mu}} = 0.27$$

(j)

$$\begin{cases} H_0: & \theta = 1 \\ H_1: & \theta \neq 1 \end{cases}$$

and

$$t^{obs} = \frac{\hat{\theta}-1}{\sqrt{0.73}} = 1.99$$

so thata $0.02 < p-$ value < 0.05 so that we can reject H_0 at the 5% significance level.

EXERCISE 7

The maximum height of the waves, in meters, at a certain beach is represented by a random variable X with density function:

$$f(x) = \frac{x}{\theta} \exp\left(-\frac{x^2}{2\theta}\right) \quad x \geq 0$$

where $\theta > 0$ is an unknown parameter. Furthermore, note that the expected value and the variance of X are

$$E(X) = \mu = \sqrt{\frac{\pi\theta}{2}} \qquad \text{and} \qquad \text{Var}(X) = \sigma^2 = \theta\left(\frac{4-\pi}{2}\right),$$

respectively.

(a) Obtain the likelihood function $L(\theta)$ for a random sample $x_1, \ldots, x_n$

(b) Obtain the log-likelihood function $\ell(\theta)$.

(c) Obtain the score function $\ell_*(\theta)$.

(d) Show that the maximum likelihood estimate for θ is $\hat{\theta} = \dfrac{\sum_{i=1}^n x_i^2}{2n}$ (it is not required to check the sign of the second derivative).

(e) Check whether $\hat{\theta}$ is a biased or an unbiased estimator of θ.

(f) Compute the observed information $j(\theta)$ for θ.

(g) Show that $j(\hat{\theta}) = n/\hat{\theta}^2$.

(h) Obtain an approximation for the distribution of $\hat{\theta}$.

(i) Compute $\hat{\theta}$ and its approximated standard error with the data,
`3.1, 2.4, 2.6, 2.2, 1.9, 2.8`

(j) Compute the maximum likelihood estimate of both μ and σ^2.

(k) Compute an approximate confidence interval for μ with confidence level 80%

This is some possibly useful`R` code,

```
> x <- c(3.1, 2.4, 2.6, 2.2, 1.9, 2.8)
> length(x)
[1] 6
> mean(log(x))
[1] 0.9037188
> mean(x)
[1] 2.5
> mean(x^2)
[1] 6.403333
> alpha <- c(0.5, 0.9, 0.95, 0.975, 0.99, 0.995)
> qnorm(alpha)
[1] 0.000000 1.281552 1.644854 1.959964 2.326348 2.575829
```

Sketch of the solution:

(a) The likelihood function is

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} f(x_i) \\ &= \left(\prod_{i=1}^{n} x_i\right) \theta^{-n} \exp\left(-\frac{\sum_{i=1}^{n} x_i^2}{2\theta}\right) \\ &\propto \theta^{-n} \exp\left(-\frac{\sum_{i=1}^{n} x_i^2}{2\theta}\right) \end{aligned}$$

(b) The log-likelihood function is

$$\ell(\theta) = -n\log(\theta) - \frac{1}{2\theta}\sum_{i=1}^{n} x_i^2$$

(c) The score function is

$$\ell_*(\theta) = -\frac{n}{\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2$$

(d) From the equation $\ell_*(\hat{\theta}) = 0$ one obtains $\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2}{2n}$.

(e) One can see that $E(X^2) = \text{Var}(X) + E(X)^2 = 2\theta$ so that

$$E(\hat{\theta}) = \frac{1}{2n}\sum_{i=1}^{n} E(X_i^2) = \frac{1}{2n} nE(X^2) = \frac{1}{2} 2\theta = \theta$$

and therefore $\hat{\theta}$ is unbiased.

(f) We have,

$$j(\theta) = -\ell_{**}(\theta) = -\frac{n}{\theta^2} + \frac{1}{\theta^3}\sum_{i=1}^{n} x_i^2 = -\frac{1}{\theta^2}\left(n - \frac{1}{\theta}\sum_{i=1}^{n} x_i^2\right)$$

(g) and therefore,

$$j(\hat{\theta}) = \frac{n}{\hat{\theta}^2}$$

(h)

$$\theta \dot{\sim} N(\theta, \hat{\theta}^2/n)$$

(i) $\hat{\theta} = 3.20$ and the approximate standard error is $\text{SE}(\hat{\theta}) = 3.20/\sqrt{6} = 1.31$

(j) $\hat{\mu} = \sqrt{\frac{\pi 3.2}{2}} = 2.24$ and $\hat{\sigma^2} = 3.2\left(\frac{4-\pi}{2}\right) = 1.37$.

(k) An 80% confidence interval for θ is $3.2 \pm 1.28 \times 1.31$ that is (1.52; 4.88) and therefore an interval for μ is (1.55; 2.77).

EXERCISE 8

Assume that X is a discrete random variable that takes values in $S = \{0, 1\}$ and has probability mass function

$$p(x) = \alpha e^{-\lambda x} \qquad \text{for } x = 0, 1$$

and let $x_1, \ldots, x_n$ be an i.i.d. sample from X such that r of the n sampling units take value 1.

(a) What constraints should one put on α and λ to ensure that $p(\cdot)$ is a valid probability mass function?

(b) Obtain the likelihood function $L(\lambda)$.

(c) Obtain the log-likelihood function $\ell(\lambda)$.

(d) Compute the maximum likelihood estimate of λ (hint: exploit the relationship with the Bernoulli random variable).

(e) Compute the maximum likelihood estimate of $p(1)$.

Sketch of the solution:

(a) We want (i) $p(x) > 0$ and therefore $\alpha > 0$ and (ii) $p(0) + p(1) = 1$ and therefore

$$\begin{aligned} 1 &= \alpha + \alpha e^{-\lambda} \\ &= \alpha(1 + e^{-\lambda}) \end{aligned}$$

so that

$$\alpha = \frac{1}{1 + e^{-\lambda}}$$

and we note that $p(0) = \alpha = (1 + e^{-\lambda})^{-1} \in (0, 1)$ for all $\lambda \in I\!R$. Furthermore

$$p(1) = \frac{e^{-\lambda}}{1 + e^{-\lambda}}$$

(b)

$$L(\lambda) = \frac{e^{-r\lambda}}{(1 + e^{-\lambda})^n}$$

(c)

$$\ell(\lambda) = -r\lambda - n \log(1 + e^{-\lambda})$$

(d) The score function is

$$\ell_*(\lambda) = -r + \frac{ne^{-\lambda}}{(1 + e^{-\lambda})}$$

so the score equation $\ell_*(\lambda) = 0$ gives

$$\frac{e^{-\lambda}}{1 + e^{-\lambda}} = \frac{r}{n}. \tag{2}$$

We note that (2) follows immediately from the fact that X follows a Bernoulli distribution where π is a bijective function of λ,

$$\pi(\lambda) = \frac{e^{-\lambda}}{1 + e^{-\lambda}}$$

and therefore one can remember that $\hat{\pi} = r/n$ and thus (2) follows from the equivariance property of MLEs.

Finally, from (2) one obtains that

$$\hat{\lambda} = \log\left(\frac{\hat{\pi}}{1 - \hat{\pi}}\right) = \log\left(\frac{n - r}{r}\right).$$

EXERCISE 9

Note that the solution of this exercise require the use of `R` Consider an experiment on antibiotic efficacy. A 1 litre culture of 5105 cells (this figure being known quite accurately) is set up and dosed with antibiotic. After 2 hours and every subsequent hour up to 14 hours after dosing, 0.1ml of the culture is removed and the live bacteria in this sample counted under a microscope. The data are:

Sample hour (t_i)	2	3	4	5	6	7	8	9	10	11	12	13	14
Live bacteria count (x_i)	35	33	33	39	24	25	18	20	23	13	14	20	18

A model for the sample counts, x_i, is that they are realizations of independent random variables X_i such that $E(X_i) = 50e^{-\delta t_i}$, where δ is an unknown 'death rate' parameter (per hour) and t_i is the sample time in hours, $i = 1, \ldots, 13$. Given the sampling protocol, it is reasonable to assume that the actual counts are observations of independent Poisson random variables with the above given mean.

(a) Explain why the Poisson model is a reasonable model.

(b) Obtain the likelihood, $L(\delta)$, and the log-likelihood, $\ell(\delta)$, functions.

(c) Use `R` to plot the log-likelihood function,

```
t <- 2:14
x <- c(35,33,33,39,24,25,18,20,23,13,14,20,18)
```

(d) Obtain, numerically, the maximum likelihood estimate, $\hat{\delta}$, of δ.

(e) Give an approximate distribution for the maximum likelihood estimator.

Sketch of the solution:

(a) In this experimental setting:

  (a) the variable is discrete with support $\{0, 1, \ldots\}$;
  (b) there is a domain of study, i.e., 0.1ml of the culture;
  (c) there is an underlying rate (changing with time) at which events arise;
  (d) it is reasonable to assume that alive(dead) bacteria arise at random in the domain.

(b) Let $\lambda_i = 50e^{-\delta t_i}$. The likelihood function is

$$L(\delta) = \exp(-\sum_{i=1}^{n} \lambda_i) \prod_{i=1}^{n} \lambda_i^{x_i};$$

the log-likelihood function is

$$\ell(\delta) = \sum_{i=1}^{n} [x_i \log(\lambda_i) - \lambda_i] = \sum_{i=1}^{n} x_i [\log(50) - \delta t_i] - \sum_{i=1}^{n} 50e^{-\delta t_i}.$$

This is equivalent to

$$\ell(\delta) = -\delta \sum_{i=1}^{n} x_i t_i - 50 \sum_{i=1}^{n} e^{-\delta t_i}.$$

(d) The score function is

$$l_*(\delta) = -\sum_{i=1}^{n} x_i t_i + 50 \sum_{i=1}^{n} t_i e^{-\delta t_i},$$

whereas

$$l_{**}(\delta) = -50 \sum_{i=1}^{n} t_i^2 e^{-\delta t_i}.$$