Lecture 14

Moments generating function (Laplace trans.)
Characteristic function (Fourier trans.)

$m_X : \mathbb{R} \to [0, +\infty]$ $\qquad [0,+\infty) \cup \{\infty\}$

$$\boxed{m_X(t) := \mathbb{E}[e^{tX}]} \qquad X \text{ r.v.}$$

$m_X(t)$ is finite for $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$

$$\boxed{X \text{ admits mgf} \iff m_X(t) < +\infty \quad \forall\, |t| < \varepsilon,\ \varepsilon > 0}$$

Characteristic function of $X : \Omega \to \mathbb{R}$

$\varphi_X : \mathbb{R} \to \mathbb{C}$ complex numbers

$\mathbb{C} \approx \mathbb{R}^2$ $\qquad z = (a, b)$ $\qquad z = \underbrace{a}_{\text{real part}} + i \underbrace{b}_{\text{imaginary part}}$

$i \in \mathbb{C}$ $\qquad (i)^2 = -1$

i imaginary unit

$$\varphi_X(u) := \mathbb{E}\left[\underbrace{e^{iuX}}\right]$$

complex exponential

$x \in \mathbb{R}$

$$e^{ix} = \cos x + i \sin x \in \mathbb{C}$$

$$\underbrace{\varphi_X(u) = \mathbb{E}[\cos(uX)] + i\,\mathbb{E}[\sin(uX)]}$$

If $\mathbb{E}[|X|^n] < +\infty$ (X admits moment of order n finite) then

$$\frac{d^n}{du^n}\varphi_X(u)\Big|_{u=0} = \underbrace{i^n} \cdot \mathbb{E}[X^n]$$

Ex: $X \sim Bin(1,p)$ r.v.

$$\varphi_X(u) = \mathbb{E}[e^{iuX}] = \mathbb{E}[g(X)]$$

$$= g(0) \cdot \mathbb{P}[X=0] + g(1) \cdot \mathbb{P}[X=1]$$

$$= \overset{1}{\overset{\shortparallel}{\mathbb{E}[e^{iu0}]}} \cdot (1-p) + \mathbb{E}[e^{iu1}](p)$$

$$= (1-p) + e^{iu} \cdot p$$

$$= \underbrace{1-p} + p e^{iu} = \varphi_X(u)$$

$$\frac{d}{du} \varphi_X(u) = p \frac{d}{du} \left(e^{iu} \right) =$$

$$\hookrightarrow \cos(u) + i \sin(u)$$

$$= p \cdot e^{iu} \cdot i = \underbrace{p i e^{iu}}$$

$$\left. \frac{d}{du} \varphi_X(u) \right|_{u=0} = i \, \mathbb{E}[X]$$

$$\equiv$$

$$p \cdot i$$

$$\Rightarrow \mathbb{E}[X] = p$$

random variables $\longrightarrow$ random vectors

$$\mathbb{R} \rightsquigarrow \mathbb{R}^n \longleftarrow X = (X_1, \dots, X_n)$$

$$m_X(t) := \mathbb{E}\left[e^{\overbrace{tX}^{\in \mathbb{R}}}\right]$$

$$t = (t_1, \dots, t_n)$$

$$\boxed{t \cdot X}$$

$$(\mathbb{R}^n, \mathbb{R}^n) \longrightarrow \mathbb{R}$$

scalar $\boxed{\text{product}}$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = x_1 \cdot y_1 + x_2 \cdot y_2 + \dots + x_n y_n$$

$$\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

norm on $\mathbb{R}^n$

$$\|x\| = \sqrt{\langle x, x \rangle}$$

$$= \sqrt{\sum_{i=1}^{n} x_i^2}$$

$\mathbb{R}^2$ $\quad \|x\| = \sqrt{x_1^2 + x_2^2}$

$$\mathbb{R} \rightsquigarrow \mathbb{R}^n$$

$$m_{\underline{X}}(t) := \mathbb{E}\left[e^{\langle t, \underline{X} \rangle}\right]$$

$$t \in \mathbb{R}^n \qquad = \mathbb{E}\left[e^{t_1 X_1 + t_2 X_2 + \cdots + t_n X_n}\right]$$

$$\boxed{\text{mgf}}$$

$$u \in \mathbb{R}^n \qquad X = (X_1, \ldots, X_n)$$

$$\varphi_{\underline{X}}(u) := \mathbb{E}\left[e^{i\langle u, \underline{X} \rangle}\right]$$

$$= \mathbb{E}\left[e^{i(u_1 X_1 + u_2 X_2 + \cdots + u_n X_n)}\right]$$

$$\boxed{\text{cf}}$$

Proposition 1

The characteristic function φ_X characterizes the distribution of the random vector X.

This means:

if X and Y are random vectors

$$\underbrace{\varphi_X = \varphi_Y}_{} \quad \Longleftrightarrow \quad \underbrace{\mu_X = \mu_Y}_{}$$

$\Leftarrow$ easy $\qquad \Rightarrow$ not so easy

Proposition 2:

Let $(X, Y) \in \mathbb{R}^2$ two dim. random vector.

with $X \perp\!\!\!\perp Y$ (independent). Then

$$\varphi_{(X,Y)}(u, v) = \varphi_X(u) \cdot \varphi_Y(v)$$

$$m_{(X,Y)}(s, t) = m_X(s) \, m_Y(t)$$

Proof:

$$m_{(X,Y)}(s,t) = \mathbb{E}\left[e^{\langle (s,t),(X,Y)\rangle}\right]$$

$$= \mathbb{E}\left[e^{s\cdot X + t\cdot Y}\right]$$

$X \perp\!\!\!\perp Y$

$\Downarrow$

$e^{sX} \perp\!\!\!\perp e^{tY}$

$$= \mathbb{E}\left[e^{sX} \cdot e^{tY}\right]$$

$$\overset{X \perp\!\!\!\perp Y}{=} \mathbb{E}\left[e^{sX}\right] \cdot \mathbb{E}\left[e^{tY}\right]$$

$$= m_X(s) \cdot m_Y(t)$$

$\blacksquare$

Remark:

$X \perp\!\!\!\perp Y$, g_1, g_2 are meas. transformations,

then $g_1(X) \perp\!\!\!\perp g_2(Y)$.

Exercise Prove that the converse may not be true.

Corollary (of Prop. 2)

If $X \perp\!\!\!\perp Y$, then

$$\varphi_{X+Y}(\underset{\mathbb{R}}{u}) = \mathbb{E}\left[e^{iu(X+Y)}\right]$$

$$X+Y : \Omega \to \mathbb{R}$$

$$= \mathbb{E}\left[e^{iuX + iuY}\right]$$

$$= \mathbb{E}\left[e^{iuX} \cdot e^{iuY}\right]$$

$$\overset{X \perp\!\!\!\perp Y}{=} \mathbb{E}\left[e^{iuX}\right] \cdot \mathbb{E}\left[e^{iuY}\right]$$

$$= \varphi_X(u) \cdot \varphi_Y(u)$$

$$u$$

$$X \perp\!\!\!\perp Y \implies \varphi_{X+Y}(u) = \varphi_X(u) \cdot \varphi_Y(u)$$

$$\boxed{X+Y}$$

$$(x, y) \qquad f_X, \; f_Y$$

$X \perp\!\!\!\perp Y$

$X+Y$ is abs. cont

$f_X * f_Y$ convolution

Remark: from Prop. 1 and Prop. 2

$$\underbrace{X \perp\!\!\!\perp Y}_{\text{diff.}} \iff \underbrace{\varphi_{(X,Y)}(u,v)}_{} = \underbrace{\varphi_X(u)}_{} \cdot \underbrace{\varphi_Y(v)}_{}$$

$\Rightarrow$ easy

Example 1 Let $X_1, \dots, X_n$ be indep. $\mathrm{Bin}(1,p)$ r.v.'s.

$$\Rightarrow \underbrace{m_{X_i}(t) = (1-p) + p e^t}_{} \quad i = 1, \dots, n$$

Let us prove that $\underbrace{Z = X_1 + \dots + X_n \sim \mathrm{Bin}(n,p)}$

$$m_Z(t) = \underbrace{\mathbb{E}[e^{tZ}]}_{} = \qquad Z \sim \mathrm{Bin}(n,p)$$

$$= \mathbb{E}[g(Z)]$$

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, \dots, n$$

$$\mathbb{E}[e^{tz}] = \sum_{k=0}^{n} g(k) \cdot p(k)$$

$$g(z) = e^{tz}$$

$$= \sum_{k=0}^{n} e^{tk} \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$(e^t)^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} \underbrace{(pe^t)^k}_{a^k} \cdot \underset{b^{n-k}}{(1-p)^{n-k}}$$

$$= (a+b)^n = (1-p+pe^t)^n$$

$$= m_{X_1}(t) \cdot m_{X_2}(t) \cdots m_{X_n}(t)$$

$$m_{X_1}(t) = 1-p+pe^t$$

$$\Longrightarrow \quad X_1 + \dots + X_n \sim \mathrm{Bin}(n,p)$$

r.v.

$\blacksquare$

Example : $X \sim \text{Poisson}(\lambda)$, $\lambda > 0$

$$P_X(n) = e^{-\lambda} \cdot \frac{\lambda^n}{n!}$$

$$\boxed{\sum_{n \in \mathbb{N}} P_X(n) = 1}$$

$n \in \mathbb{N}$

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^{\lambda}$$

$$\mathbb{E}[g(X)] = \sum_{n \in \mathbb{N}} g(n) p(n)$$

$$\boxed{m_X(t)} = \mathbb{E}[e^{tX}]$$

$$= \sum_{n=0}^{+\infty} e^{tn} \cdot e^{-\lambda} \frac{\lambda^n}{n!}$$

$$= \sum_{n=0}^{+\infty} e^{-\lambda} \frac{(\lambda e^t)^n}{n!}$$

$$= e^{-\lambda} \underbrace{\sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!}}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t} = \boxed{e^{\lambda(e^t - 1)}}$$

A similar comp. gives $\boxed{\varphi_X(u) = e^{\lambda(e^{iu} - 1)}}$

Ex. 3 $\quad X \sim \text{Poisson}(\lambda)$, $\quad Y \sim \text{Poisson}(\mu)$, $\lambda, \mu \in \mathbb{R}$

$$X \perp\!\!\!\perp Y$$

$$X + Y$$

$$\varphi_{X+Y}(u) = \varphi_X(u) \cdot \varphi_Y(u)$$

$u \in \mathbb{R}$

$$\stackrel{\text{L}}{=} e^{\lambda(e^{iu}-1)} \cdot e^{\mu(e^{iu}-1)}$$

$$= e^{\lambda(e^{iu}-1) + \mu(e^{iu}-1)}$$

$$= e^{\boxed{(\lambda+\mu)}(e^{iu}-1)}$$

$$\Rightarrow \boxed{X + Y \sim \text{Poisson}(\lambda+\mu)}$$

Exemple 4

$$X \sim \text{Geo}(p), \quad p \in (0,1) \subseteq \mathbb{R}$$

$$P_X(n) = (1-p)^{n-1} \cdot p \qquad n \in \mathbb{N} \setminus \{0\}$$

$$\boxed{n \geq 1}$$

$$m_X(t) := \mathbb{E}[e^{tX}]$$

$$= \sum_{n=1}^{\infty} e^{tn} \cdot p(1-p)^{n-1}$$

$$= \frac{p}{1-p} \underbrace{\sum_{n=\boxed{1}}^{\infty} \left((1-p)e^t\right)^n}_{\text{Power series}}$$

$$\boxed{\sum_{n=0}^{\infty} p^n = \frac{1}{1-p}} \quad \text{if} \quad \boxed{|(1-p)e^t| < 1}$$

$$\sum_{n=0}^{\infty} \left((1-p)e^t\right)^n = \frac{1}{} \qquad \frac{1}{\underbrace{1-(1-p)e^t}} - 1$$

$$m_X(t) = \underbrace{\frac{p}{1-p} \cdot \frac{(1-p)e^t}{1-(1-p)e^t}} =$$

$$= \frac{p}{e^{-t}-(1-p)}$$

$X \sim Geo(p)$

$$\underbrace{m_X(t)} = \begin{cases} \dfrac{p}{e^{-t} - (1-p)} & t < -\lg(1-p) \\ +\infty & \text{otherwise} \end{cases}$$

$$|\underbrace{e^t(1-p)}_{>0}| < 1 \iff e^t(1-p) < 1$$

$$\iff e^t < \frac{1}{1-p}$$

$$\iff t < \lg\left(\frac{1}{1-p}\right) =$$

$$= -\lg(1-p) > 0$$

$$\boxed{m_X(t) < +\infty}$$

X admits mgf !