Lecture 20 Stoch. Meth. Dec. 2^{nd}, 2024

Stochastic Processes

Conditional prob.

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

$$\underbrace{\mathbb{P}[A \cap B]} = \mathbb{P}[B] \cdot \mathbb{P}[A|B]$$

$$A \perp\!\!\!\perp B \quad \mathbb{P}[A \cap B] = \mathbb{P}[B] \cdot \mathbb{P}[A]$$

$$\Updownarrow$$

$$\mathbb{P}[A|B] = \mathbb{P}[A]$$

r.v.

$$X : \Omega \longrightarrow \mathbb{R}$$

$$\omega \longmapsto X(\omega) \in \mathbb{R}$$

$$\boxed{T \times \Omega \longrightarrow X(t,\omega) \in \mathbb{R}}$$

Time — Prob. Space

$$t \longmapsto X_t(\omega)$$

$$\in [0,+\infty)$$

$$\underbrace{(t,\omega) \longmapsto X(t,\omega)}$$

time

$T = \mathbb{N}$ discrete time stochastic processes

$$(X_1(\omega), X_2(\omega), X_3(\omega), \dots, X_n(\omega), \dots)$$

$T = [0,+\infty)$ continuous time stoch. process.

$$t \longmapsto X_t(\omega)$$

$(t,\omega) \longmapsto X_t(\omega) \qquad (= X(t,\omega))$

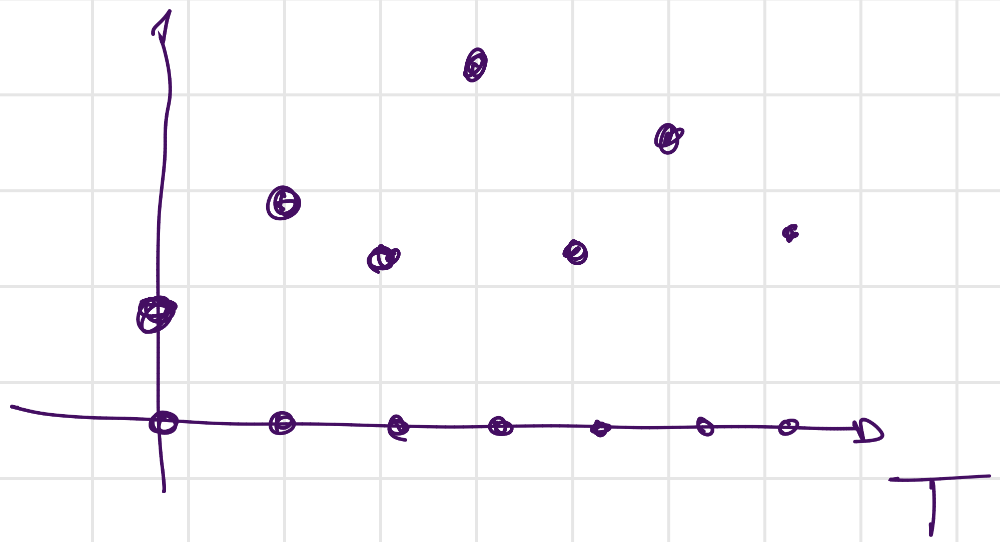
fix $t \in T$ $\qquad \omega \longmapsto X_t(\omega)$ random variable

fix $\omega \in \Omega$ $\qquad$ path $\qquad t \longmapsto X_t(\underset{\text{fix}}{\omega})$

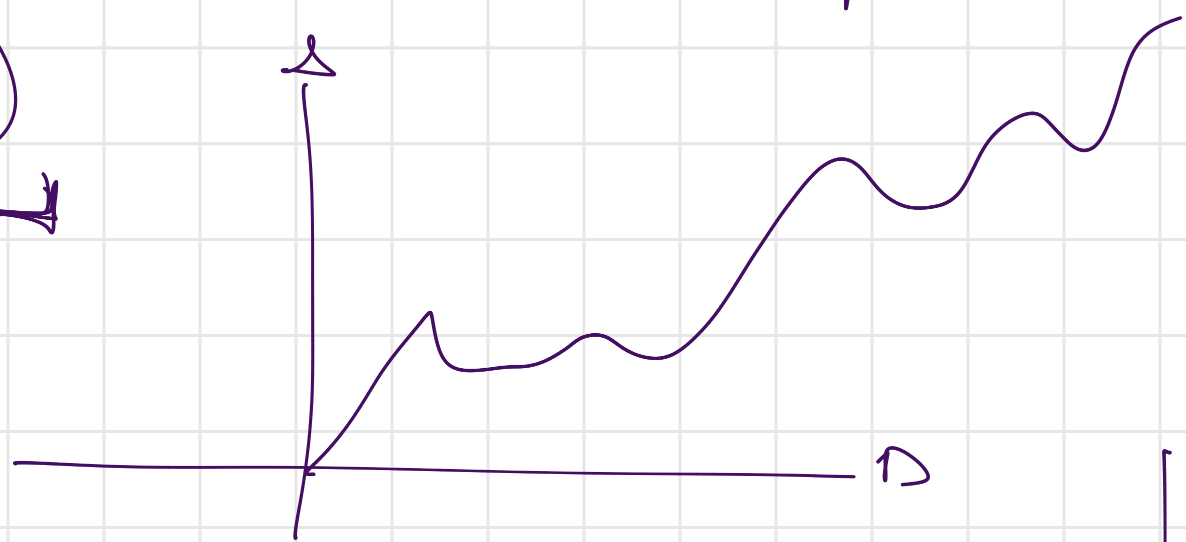
$=$

$f : \mathbb{N} \to \mathbb{R}$

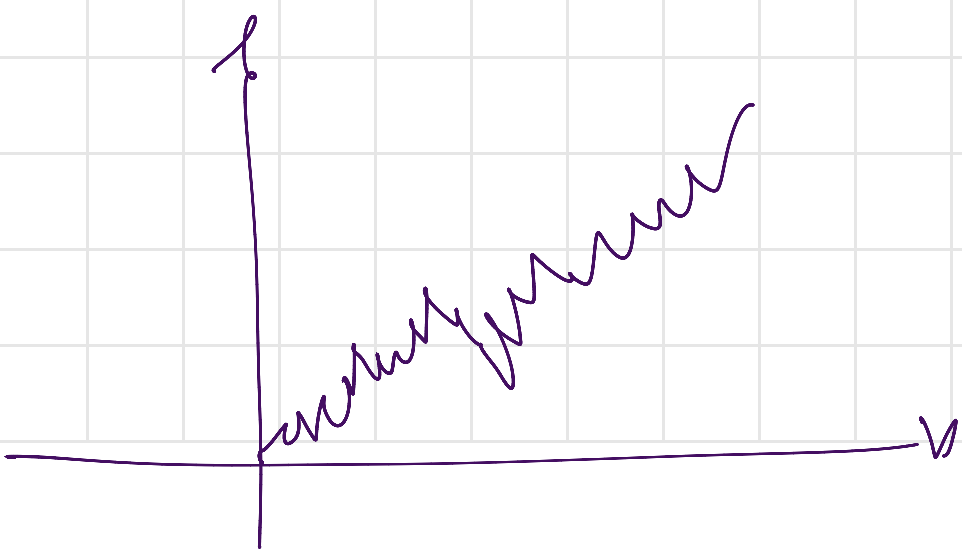
fix ω

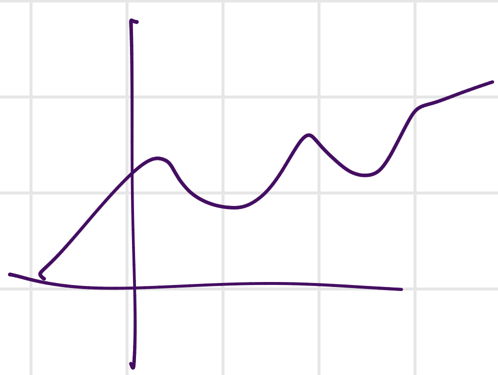


$T = [0,+\infty)$



$t \mapsto X_t(\omega)$





$T = \mathbb{N}$ discrete time stoch. $\boxed{\text{process}}$

$$\mathbb{N} \times \Omega \longrightarrow \underbrace{\mathbb{R}}$$

$\underbrace{\text{Discrete time}}_{T=\mathbb{N}}$ $\underbrace{\boxed{\text{Markov}}}_{\text{dependence}}$ $\boxed{\text{chains}}$

dependence

$X_1, X_2, X_3, \ldots$

$X_1 : \Omega \longrightarrow \underbrace{S}$

X_2

S is finite or at most countable

$\{0, \ldots, n\}$

$\mathbb{N}, \mathbb{Z}$

$$\mathbb{N} \times \Omega \longrightarrow \underbrace{S}$$

discrete set

$\underbrace{X_0, X_1, X_2, \ldots, X_n}, \ldots$

$\underbrace{\mathbb{P}[X_i = k]}$ $\forall k \in S$ $\forall i \in \mathbb{N}$

$(X_0, \ldots, X_n) \in S^{n+1}$

$(X_0, \ldots, X_n)$

$\mathbb{P}[X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n] =$

$\cap \quad \cap \quad \cap$

$$\mathbb{P}[X_0=x_0, X_1=x_1, \dots, X_n=x_n] = \underbrace{\mathbb{P}[X_0=x_0]} \cdot \mathbb{P}[\underbrace{X_1=x_1} | X_0=x_0]$$

$$\cdot \mathbb{P}[\underbrace{X_2=x_2} \,|\, \underbrace{\cancel{X_0=x_0}, X_1=x_1}] \cdot \mathbb{P}[X_3=x_3 \,|\, \cancel{X_0=x_0}, \cancel{X_1=x_1}, X_2=x_2]$$

$$\dots \mathbb{P}[\underbrace{X_n=x_n} \,|\, \underbrace{\cancel{X_0=x_0}, \cancel{X_1=x_1}, \cancel{\dots}, \underbrace{X_{n-1}=x_{n-1}}}]$$

$$= \mathbb{P}[X_0=x_0] \cdot \underbrace{\mathbb{P}[X_1=x_1 | X_0=x_0]} \cdot \underbrace{\mathbb{P}[X_2=x_2 \,|\, X_1=x_1]} \dots \underbrace{\mathbb{P}[X_n=x_n | X_{n-1}=x_{n-1}]}$$

Markov property

Fix the notation: $x_m^n = (x_m, x_{m+1}, \dots, x_{n-1}, x_n)$

$m < n$

<u>Definition</u>: A Markov Chain is a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables, taking values in a finite or countable set S s.t.

$\forall\ x_0^{n+1} \in S^{n+2}$ $\quad \left(x_0^{n+1} = (x_0, x_1, \dots, x_{n+1}) \right)$

and $n \geq 1$

$$\mathbb{P}[X_{n+1} = x_{n+1} \,|\, \underbrace{X_0}^{n} = \underbrace{x_0^n}] = \mathbb{P}[\underbrace{X_{n+1}=x_{n+1}} \,|\, X_n = x_n]$$

This last property is called Markov property.

S is called the state space of the MC.

The conditional probabilities

$$\mathbb{P}[X_{n+1} = x_{n+1} \mid X_n = x_n]$$

are called the (1-step) transition probabilities.

The complete law of the Markov Chain can be obtained if we know the transition probabilities and the initial distribution:

$$\mathbb{P}[X_0^n = x_0^n] = \mathbb{P}[X_n = x \mid X_{n-1} = x_{n-1}] \cdot$$
$$\times\ \mathbb{P}[X_{n-1} = x_{n-1} \mid X_{n-2} = x_{n-2}]$$
$$\vdots$$
$$\times\ \mathbb{P}[X_1 = x_1 \mid X_0 = x_0]$$
$$\times\ \mathbb{P}[X_0 = x_0]$$

We say that a MC is homogeneous (in time) if the transition probabilities $\mathbb{P}[X_n = x_n \mid X_{n-1} = x_{n-1}]$ do not depend on n.

x_n, x_{n-1}

$$y \in S \longrightarrow x \in S$$

$y, x \in S$

$$\boxed{P_{yx}} := \mathbb{P}[X_n = x \mid X_{n-1} = y]$$

homog. MC.

$$\mathbb{P}[X_0^n = x_0^n] = P_{x_{n-1},x_n} \cdot P_{x_{n-2}x_{n-1}} \cdots P_{x_0 x_1} \cdot \underline{\lambda_{x_0}}$$

$$\lambda_{x_0} = \mathbb{P}[X_0 = x_0]$$

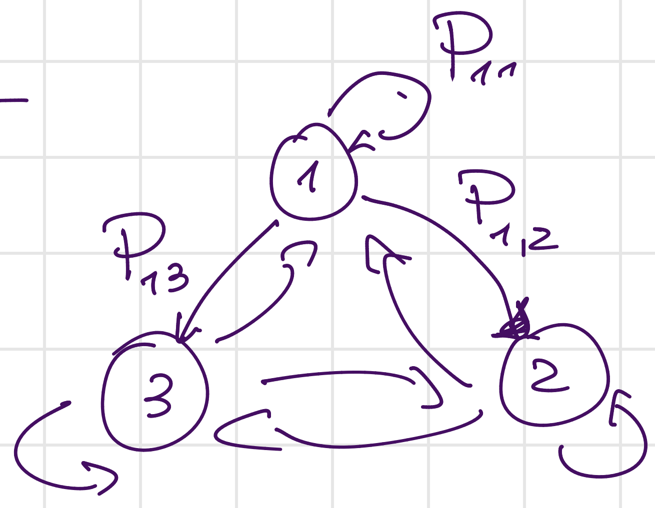
$P := (P_{yx})_{y,x \in S}$ is called the transition matrix

→ if $|S| < +\infty$, this is a square matrix $|S| \times |S|$

→ if $|S| = +\infty$, P is a generalized matrix

Example

$$S = \{1, 2, 3\}$$



$$\underbrace{P = (P_{yx})_{y,x \in S}}$$

$P_{1,x}$ $\quad x = 1, 2, 3$

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1/2 & 1/2 & 0 \\ 2 & \boxed{0} & \boxed{1/3} & \boxed{2/3} \\ 3 & 1/3 & 1/2 & 1/6 \end{array}$$

$P_{2,1}$

$P_{2,2}$

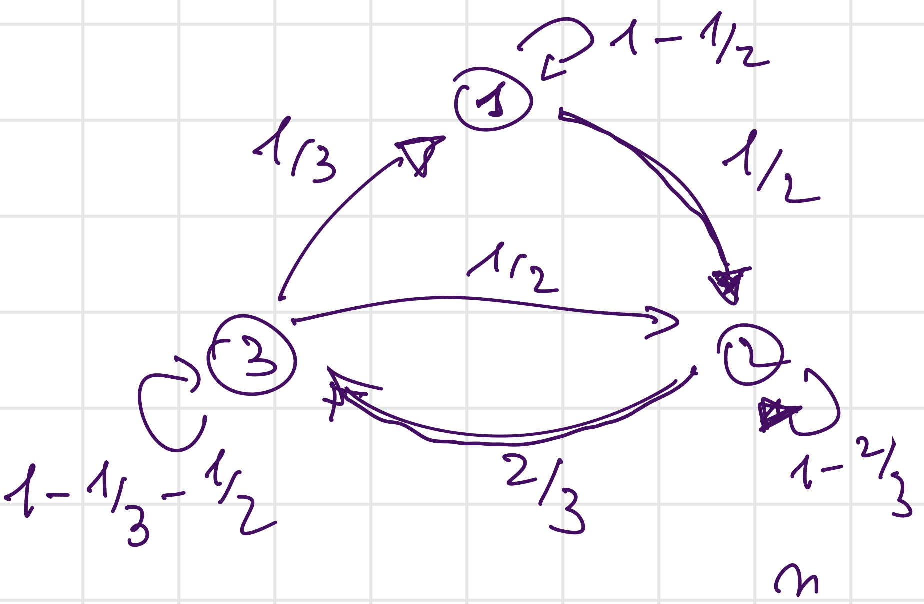
$P_{2,3}$

P_{yx}

$y \bullet \longrightarrow \bullet x$

$2 \bullet \longrightarrow \bullet 2$

$2 \bullet \longrightarrow \bullet 3$



$$P_{11} + \underbrace{P_{12}} + \underbrace{P_{13}} = 1$$

$$\mathbb{P}[X_n = 1 | X_{n-1} = 1] + \mathbb{P}[X_n = 2 | X_{n-1} = 1] + \mathbb{P}[X_n = 3 | X_{n-1} = 1] = 1$$

$\boxed{B}$

$A \longmapsto \mathbb{P}[A | B]$

is a prob.

$$P = \begin{pmatrix} \boxed{1/2} & 1/2 & 0 \\ 0 & \boxed{1/3} & 2/3 \\ 1/3 & 1/2 & \boxed{1/6} \end{pmatrix}$$

$$P = (P_{yx})_{y,x\in S}$$ is a <u>Stochastic Matrix</u>

$$\Leftrightarrow \begin{cases} P_{yx} \geq 0 & \forall y,x \\ \sum_{x\in S} P_{yx} = 1 & \forall y \in S \end{cases}$$

<u>Example</u> : X_0 – S valued r.v. indep of a sequence of iid $(U_1, U_2, \dots)$ r.v.'s

$\underline{f : S \times \mathbb{R} \longrightarrow S}$ be a map

$$\begin{cases} X_1 = f(X_0, U_1) \\ X_2 = f(X_1, U_2) \\ \vdots \\ X_{n+1} = f(X_n, U_{n+1}) \\ \vdots \end{cases}$$

$(X_0, X_1, \dots, X_n, \dots)$

this is an S-valued Markov Chain !!

$$\mathbb{P}[\underbrace{X_{n+1}}=x_{n+1} \mid X_n=x_n, \underbrace{X_{n-1}=x_{n-1}, \dots, X_0=x_0}]$$

$$X_{n+1}=f(X_n, U_{n+1})$$

$$=\mathbb{P}[f(X_n, U_{n+1})=x_{n+1} \mid \underbrace{X_n=x_n}, \underbrace{X_{n-1}=x_{n-1}, \dots}]$$

$$=\mathbb{P}[\underbrace{f(x_n, U_{n+1})=x_{n+1}}_{U_{n+1}} \mid \underbrace{\cancel{X_n=x_n, X_{n-1}=x_{n-1}, \dots}}_{U_n, U_{n-1}, \dots, U_1, X_0}]$$

$$\underbrace{U_{n+1}} \perp\!\!\!\perp \underbrace{U_n, U_{n-1}, \dots, U_1, \underline{X_0}}$$

$$=\mathbb{P}[f(x_n, U_{n+1})=x_{n+1}]$$

$$=\mathbb{P}[X_{n+1}=x_{n+1} \mid X_n=x_n]$$

YES, this is a M.C.

$\boxed{\text{Random Walk}}$

<u>Example</u> <u>Gambler's Ruin</u> model

$$S=\{0, 1, \dots, n\} \text{ €}$$

$\boxed{1}$ k €

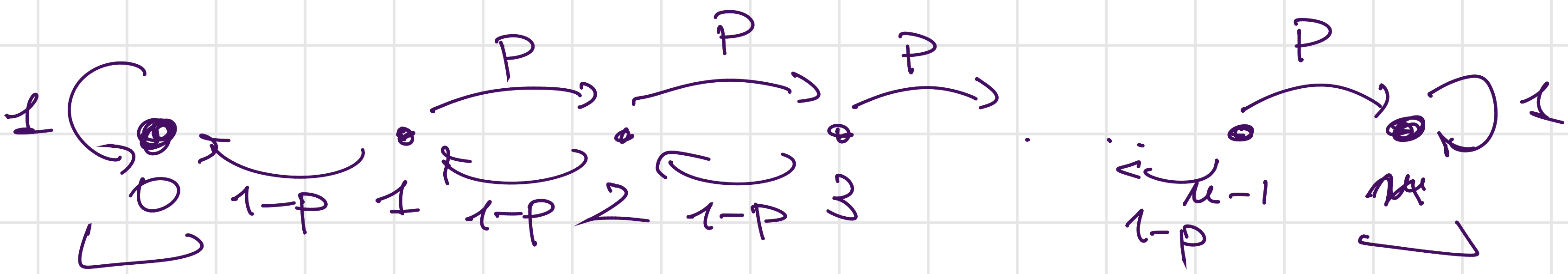
$\boxed{2}$ $n-k$ €

$$U_i = \begin{cases} +1 & p & \text{win } 1€ \to \boxed{1} \\ -1 & (1-p) & \text{lose } 1€ \end{cases}$$

$\boxed{1}$

$$X_0 = k \,, \quad X_1 = X_0 + U_1 \,, \quad X_2 = X_0 + U_1 + U_2$$
$$= X_1 + U_2$$

$$\dots \quad \underline{X_{n+1} = X_n + U_{n+1}}$$



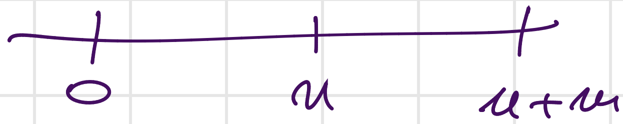
$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & & & & 0 \\ 1-p & 0 & p & 0 & \cdots & & 0 \\ 0 & 1-p & 0 & p & & & \vdots \\ \vdots & & 1-p & 0 & \ddots & & 0 \\ & & & 1-p & \ddots & \ddots & \\ \vdots & & & & 1-p & 0 & p \\ 0 & \cdots & & & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \leftarrow 1 \\ \leftarrow 1 \\ \leftarrow 1 \\ \\ \\ \\ \\ \end{array}$$

Chapman – Kolmogorov equation

$$P^{n+m}_{yx} = \mathbb{P}[X_{n+m} = x \mid X_0 = y]$$

$$= \sum_{z \in S} \mathbb{P}[X_{n+m} = x, X_n = z \mid X_0 = y]$$

$$\mathbb{P}[A \cap B] = \mathbb{P}[A \mid B] \cdot \mathbb{P}[B]$$



$$= \sum_{z \in S} \mathbb{P}[X_{n+m} = x \mid X_n = z, \cancel{X_0 = y}] \cdot \mathbb{P}[X_n = z \mid X_0 = y]$$

Markov property

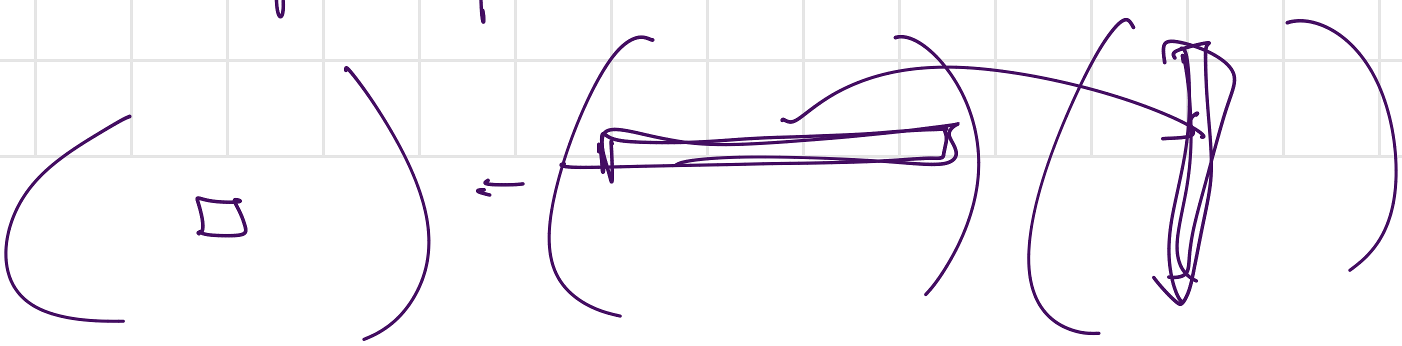
$$P^{n+m}_{yx} = \sum_{z \in S} P^{m}_{zx} \cdot P^{n}_{yz} =$$

$$= \sum_{z \in S} P^{n}_{yz} \cdot P^{m}_{zx} \qquad |S| < +\infty$$

$$\boxed{P^{n+m} = P^{n} \cdot P^{m}}$$

C-K equation

matrix product



Remark

$$\underbrace{P^2_{yx}} = \mathbb{P}[X_{n+2} = x \mid X_n = y]$$

$$= (\underbrace{P \cdot P})_{yx} \qquad (P^2)_{yx}$$

$$\left(P^2{}_{yx}\right) = \begin{pmatrix} & & \\ & P & \\ & & \end{pmatrix} \begin{pmatrix} & & \\ & P & \\ & & \end{pmatrix}$$

$$(P^2)_{yx} \neq (P_{yx})^2$$

the entry yx of the matrix P^2

the square of the entry yx of P

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/3 & 2/3 \\ 1/3 & 1/2 & 1/6 \end{bmatrix} \qquad (P_{yx})$$

P^2_{32} .

$$P^2 = P \cdot P$$

$$= \begin{bmatrix} 1/4 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \boxed{\cdot} & \cdot \end{bmatrix}$$

$$\shortparallel$$

$$\left(1/2\right)^2 = \left(P_{32}\right)^2$$