DATA SCIENCE **Stochastic Methods**	Name:________________________
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Problem 1. [12] Let $X \sim Exp(\lambda)$ and $Y \sim Exp(\mu)$ and assume that they are independent.

(i) Compute $\mathrm{E}[XY^2]$;

(ii) Compute $\mathrm{P}[X \leq Y]$.

(i) since $X \perp\!\!\!\perp Y$, $\mathbb{E}[X \cdot Y^2] = \mathbb{E}[X] \cdot \mathbb{E}[Y^2]$

$$\mathbb{E}[X] = \int_0^{+\infty} x \lambda e^{-\lambda x} dx = 1/\lambda$$

$$\mathbb{E}[Y^2] = \int_0^{+\infty} y^2 \mu e^{-\mu x} = 2/\mu^2$$

$$\Rightarrow \mathbb{E}[XY^2] = \frac{2}{\lambda \cdot \mu^2}$$

(ii)

$$\mathbb{P}[X \leq Y] = \int_0^{+\infty} \lambda e^{-\lambda x} \left(\int_x^{+\infty} \mu e^{-\mu y} dy \right) dx$$

$$= \int_0^{+\infty} \lambda e^{-\lambda x} \left[-e^{-\mu y} \right]_x^{+\infty} dx$$

$$= \int_0^{+\infty} \lambda e^{-(\lambda + \mu)x} dx = \frac{\lambda}{\lambda + \mu}$$

Problem 2. [12] Let $(X_i)_{1\leq i\leq n}$ be a family of i.i.d. Exp(1) random variables and define $Z_i = \min\{X_i, 1\}$.

(i) (*) Compute the expectation of Z_1;

(ii) (**) Defined $\overline{Z}_n = \frac{1}{n}\sum_{i=1}^n Z_i$ and $\mu = E[Z_1]$, prove that

$$\mathrm{P}(\overline{Z}_n \geq \mu + \varepsilon) \leq e^{-n\frac{\varepsilon^2}{2}}, \qquad e^{-2n\varepsilon^2}$$

for $\varepsilon > 0$.

Hint:

(*) recall that for positive random variables $E[X] = \int_0^{+\infty} P[X > x]dx$

(**) use the Heoffding's inequality.

(i) $Z_1 = \min\{X_i, 1\}$

$$F_{Z_1}(z) = \begin{cases} 0 & z < 0 \\ 1 - e^{-\lambda z} & 0 \leq z < 1 \\ 1 & z \geq 1 \end{cases}$$

$$\mathbb{P}[Z_1 > z] = \begin{cases} e^{-\lambda z} & 0 \leq z < 1 \\ 0 & 1 \leq z \end{cases}$$

$$\Rightarrow \mathbb{E}[Z_1] = \int_0^1 e^{-\lambda z}\,dz = \left[-\frac{e^{-\lambda z}}{\lambda}\right]_0^1 = \frac{1 - e^{-\lambda}}{\lambda}$$

since $\lambda = 1$

$\mu = \frac{e-1}{e}$

(ii) $Z_1 \in [0,1]$; by the Heoffding's inequality

$$\mathbb{E}[Z_1 + \dots + Z_n - n\mu \geq t] \leq e^{-2t^2/n(b-a)^2}$$

$\Rightarrow a = 0,\ b = 1$

$$\mathbb{E}[\overline{Z}_n - \mu \geq \varepsilon] = \mathbb{E}[Z_1 + \dots + Z_n - n\mu \geq n\varepsilon]$$

$$\leq e^{-2\cdot\frac{n^2\varepsilon^2}{n\cdot 1}} = e^{-2n\varepsilon^2}$$

Problem 3. [12] Let $(X_n)_{n\geq 0}$ be a Markov Chain with state space $S = \{1, 2, \ldots, N\}$, such that, given $X_n = i$, the state X_{n+1} is determined as follows:

- Choose uniformly any state $j \in S$.
- If $j \leq i$, set $X_{n+1} = j$, otherwise set $X_{n+1} = j$ with probability $\frac{i}{j}$ and $X_{n+1} = i$ with probability $1 - \frac{i}{j}$.

For this Markov Chain:

(i) Determine the off-diagonal elements of the transition matrix;

(ii) Classify the states (communication, periodicity, recurrence);

(iii) Prove that this MC has a unique invariant distribution;

(iv) When $N = 4$, compute the invariant distribution.

(i) $$P_{ij} = \begin{cases} 1/N & j < i \\ 1/N \cdot \frac{i}{j} & i < j \end{cases}$$

(ii) $P_{ij} > 0 \ \forall i,j \Rightarrow$ the MC is regular ($\Rightarrow$ irreducible, aperiodic, recurrent).

(iii) By the Ergodic Theorem, this MC admits a unique invariant distribution

(iv) $N=4$

$$P = \begin{bmatrix} \cdot & 1/8 & \frac{1}{12} & \frac{1}{16} \\ 1/4 & \cdot & 2/12 & 2/16 \\ 1/4 & 1/4 & \cdot & 3/16 \\ 1/4 & 1/4 & 1/4 & \cdot \end{bmatrix}$$

By the detailed-balance equation:

$$\frac{\pi_1}{8} = \frac{\pi_2}{4} \Rightarrow \pi_1 = 2\pi_2$$

$$\frac{\pi_1}{12} = \frac{\pi_3}{4} \Rightarrow \pi_1 = 3\pi_3$$

$$\pi_1 = 4\pi_4 \Rightarrow \pi_1 = 4\pi_4$$

$$\pi_1\left(1 + 1/2 + 1/3 + 1/4\right) = 1 \Rightarrow \boxed{\pi = \left(\frac{12}{25}, \frac{6}{25}, \frac{4}{25}, \frac{3}{25}\right)}$$