Lecture 17 Stoch. Meth. 25 Nov. 2024

$X_n \longrightarrow X \qquad \lim_{n\to\infty} X_n$

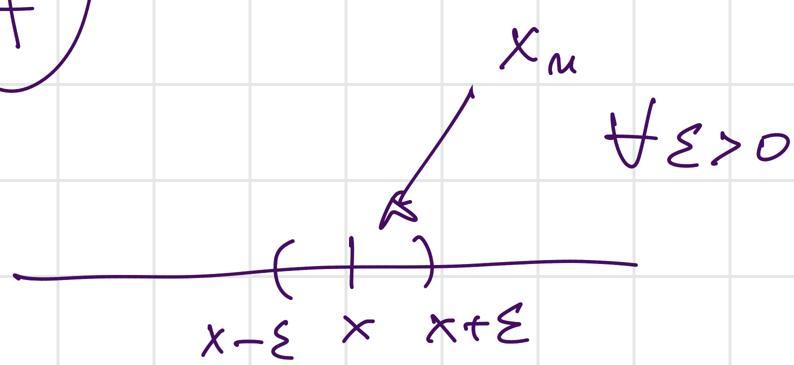
$X : \Omega \longrightarrow \mathbb{R} \quad , \quad X_n : \Omega \to \mathbb{R} \qquad n=1,2,\dots$

$X_n \xrightarrow[n\to\infty]{} X$

f_n sequence of functions

$f_n \xrightarrow[n\to\infty]{} f$??

$\boxed{x_n \longrightarrow x}$



$x_n \to x \Rightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}$

s.t. $\underbrace{\forall n > N}_{|x - x_n| < \varepsilon}$

$f_n \xrightarrow[n\to\infty]{} f \iff \forall x \in D$

$\lim_{n\to\infty} f_n(x) = f(x)$

$\boxed{\text{pointwise convergence}}$

$X_n : \Omega \to \mathbb{R}$, $n = 1, 2, \dots$, $X : \Omega \to \mathbb{R}$

$$\underbrace{X_n \xrightarrow[n\to\infty]{} X}_{} \iff \underbrace{\forall \omega \in \Omega : \underbrace{\overset{\mathbb{R} \atop \cup}{X_n(\omega)} \xrightarrow[n\to\infty]{} \overset{\mathbb{R} \atop \cup}{X(\omega)}}_{}}_{}$$

$$\boxed{X \overset{a.e.}{=} Y} \iff \mathbb{P}\left[\{\omega \in \Omega : X(\omega) = Y(\omega)\}\right] = 1$$

almost everywhere

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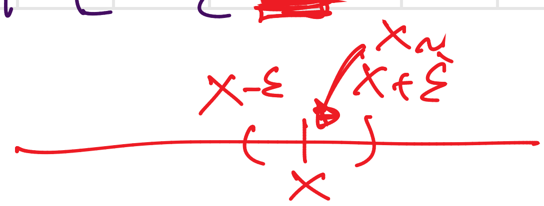
1 Almost sure convergence

$(X_n)_{n \geq 1}$ sequence of r.v.'s $X_n : \Omega \to \mathbb{R}$

we say that X_n converges almost surely to X, where $X : \Omega \to \mathbb{R}$, and we write

$$\boxed{X_n \xrightarrow[n\to\infty]{a.s.} X}$$

$$\iff \mathbb{P}\left[\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}\right] = 1$$



(2) Convergence in Probability

$(X_n)_{n\geq 1}$, X $\qquad X_n : \Omega \to \mathbb{R}$, $X : \Omega \to \mathbb{R}$

$$\boxed{X_n \xrightarrow[n\to\infty]{\mathbb{P}} X \iff \forall\, \varepsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}\big[\underbrace{|X_n - X| > \varepsilon}\big] = 0}$$

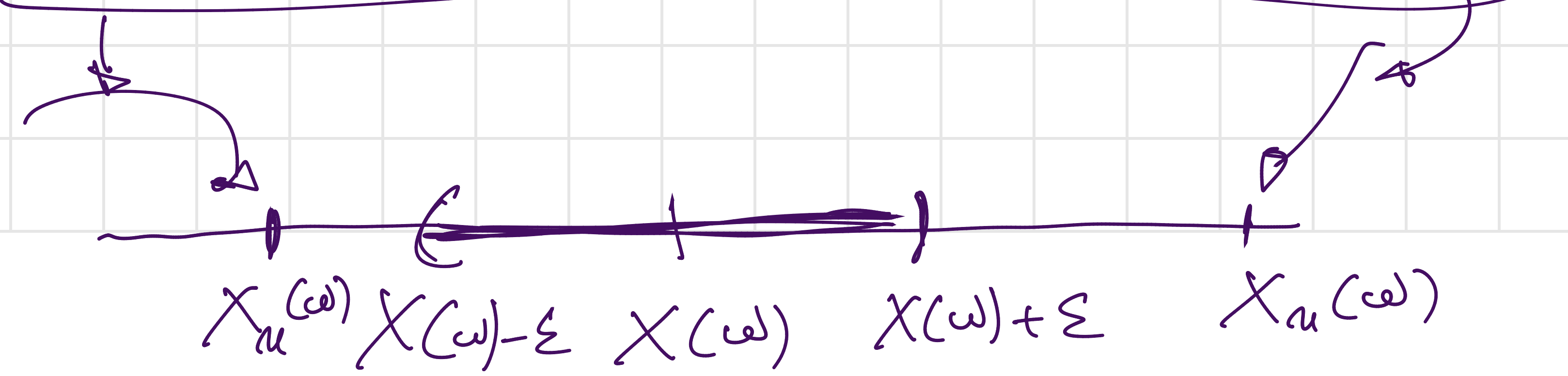
$\{|X_n - X| > \varepsilon\} \subseteq \Omega$

$\{\omega \in \Omega : \ |X_n(\omega) - X(\omega)| > \varepsilon\}$

$\omega \in \underbrace{\{|X_n - X > \varepsilon\}}$ $\iff$ $|X_n(\omega) - X(\omega)| > \varepsilon$

$\iff \quad X_n(\omega) - X(\omega) < -\varepsilon \quad$ or $\quad X_n(\omega) - X(\omega) > \varepsilon$

$\iff \quad \underbrace{X_n(\omega) < X(\omega) - \varepsilon} \quad$ or $\quad \underbrace{X_n(\omega) > X(\omega) + \varepsilon}$



3 Convergence in L^p-norm

$X \in L^2 \iff \mathbb{E}[X^2] < +\infty$

$X \in L^1 \iff \mathbb{E}[|X|] < +\infty$

$X \in L^3 \iff \mathbb{E}[|X|^3] < +\infty$

$\boxed{p \geq 1}$ $\quad p \in \mathbb{R}$ $\quad \boxed{p = 1+\varepsilon}$

$X \in L^p \iff \mathbb{E}[|X|^p] < +\infty$

$\|X\|_p^p$ norm

$(X_n)_{n \geq 1}, X \quad , \quad X_n : \Omega \to \mathbb{R}, \quad X : \Omega \to \mathbb{R}$

$\boxed{X_n \in L^p \ \forall n, \quad X \in L^p}$

$$X_n \xrightarrow[n\to\infty]{L^p} X \iff \lim_{n\to\infty} \mathbb{E}[|X_n - X|^p] = 0$$

converges in L^p-norm

(4) Weak convergence : Convergence in distribution

$(X_n)_{n\geq 1}$, X r.v.'s $\qquad \cancel{\Omega}$

$$X_n \xrightarrow[n\to\infty]{\overset{w}{\mathcal{D}}} X \iff \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

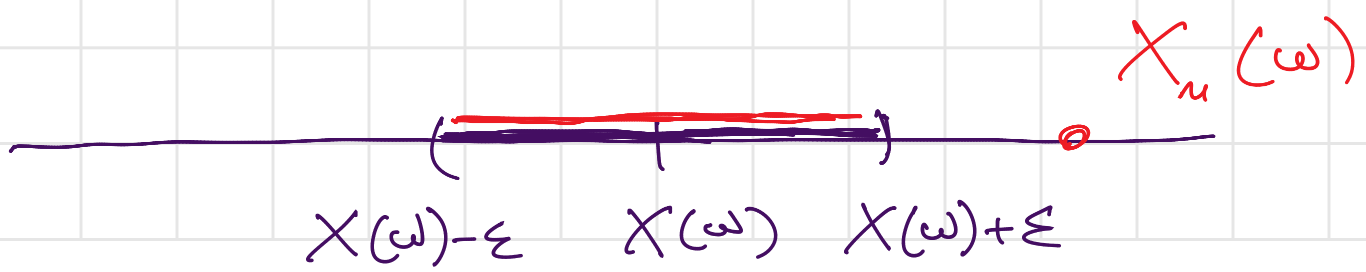
$\forall x \in \mathbb{R}$ where F_X is <u>continuous</u>

<u>Theorem</u>: $X_n \xrightarrow[n\to\infty]{a.s.} X \iff$

$$\boxed{\forall \varepsilon > 0 \quad \mathbb{P}\left[\{|X_n - X| > \varepsilon\} \text{ i.o.}\right] = 0}$$

$$= \mathbb{P}\left[\limsup_{n\to\infty} \{|X_n - X| > \varepsilon\}\right] = 0$$

$\ni \omega$



$\omega \in \limsup \{|X_n - X| > \varepsilon\} \iff$ i.o. $|X_n(\omega) - X(\omega)| > \varepsilon$

<u>Corollary</u> : By the Borel - Cantelli Lemma (first)

if $\boxed{\underbrace{\sum_{n=1}^{\infty} \mathbb{P}\left[\{ |X_n - X| > \varepsilon \} \right]} < +\infty}$

then $\mathbb{P}\left[\{ |X_n - X| > \varepsilon \} \text{ i.o.} \right] = 0 \Rightarrow X_n \xrightarrow{a.s.} X$

<u>Remark</u> : Conv. in prob.

$X_n \xrightarrow[n\to\infty]{P} X \iff \lim_{n\to\infty} \underbrace{\mathbb{P}\left[\{ |X_n - X| > \varepsilon \} \right] = 0}$

<u>Proposition</u> :

$$X_n \xrightarrow{a.s} X \implies X_n \xrightarrow{P} X$$

$$\cancel{\Leftarrow}$$

<u>Remark</u>: $X_n \xrightarrow{P} X$, but $X_n \not\rightarrow$ any r.v.

$X_n \sim \text{Bin}(1, p_n)$ $\quad X_n \xrightarrow{P} 0$

$X_n \not\xrightarrow{a.s.}$

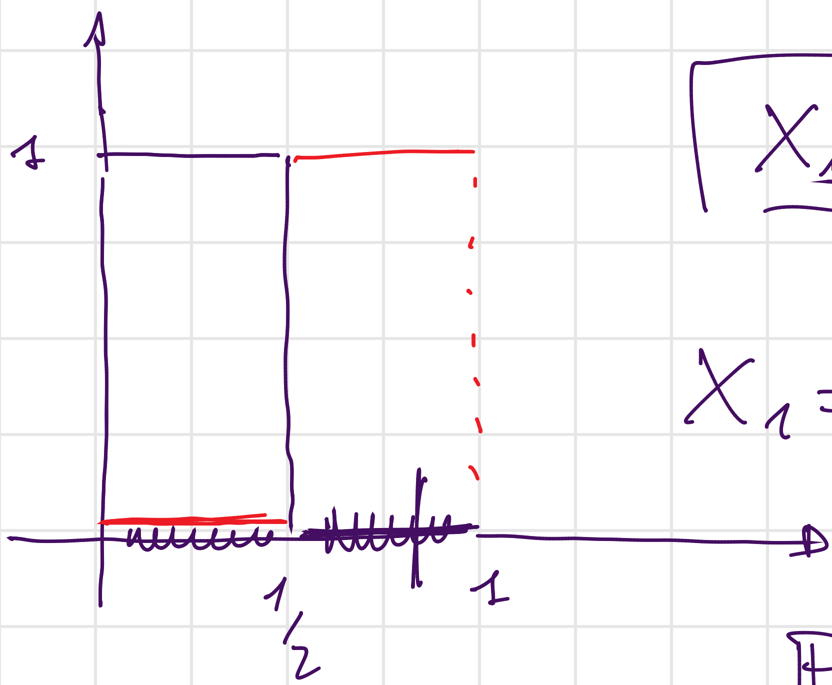
$\Omega = [0,1)$, P = Lebesgue measure (length)

Lebesgue

$X_n : \underline{\underline{\Omega}} \to \mathbb{R}$ $\qquad$ $X_n : [0,1) \to \mathbb{R}$

[0,1)

Leb



$$\boxed{X_1 \sim Bin(1, 1/2)}$$

$$X_1 = \begin{cases} 1 & x \in [0, 1/2) \\ 0 & x \in [1/2, 1) \end{cases}$$

$$\mathbb{P}[X_1 = 1] = \mathcal{L}(0, 1/2) = 1/2 - 0 = \frac{1}{2}$$

$X_2 \sim Bin(1, 1/2)$

$$X_2 = \begin{cases} 0 & x \in [0, 1/2) \\ \boxed{1} & x \in [1/2, 1) \end{cases}$$

<u>Remark</u> : $\forall\, \omega \in \Omega = [0,1)$ $\quad X_1(\omega) \neq X_2(\omega)$

X_3, X_4, X_5

$\sim Bin(1, 1/3)$

ω : $X_1(\omega), X_2(\omega), X_3(\omega), X_4(\omega), X_5(\omega)$

$\mathrm{Bin}(1, 1/2)$, $\mathrm{Bin}(1, 1/2)$,

$\mathrm{Bin}(1, 1/3)$, $\mathrm{Bin}(1, 1/3)$, $\mathrm{Bin}(1, 1/3)$

n $\quad \mathrm{Bin}(1, 1/n), \dots, \mathrm{Bin}(1, 1/n)$

$$X_n \xrightarrow{\mathbb{P}} \underline{0} \quad ?? \quad n \to \infty$$

$\boxed{\text{YES}}$

$$\mathbb{P}\left[\{|X_n - 0| > \varepsilon\}\right] \xrightarrow[n\to\infty]{} 0 \quad \forall \varepsilon$$

$$\parallel$$

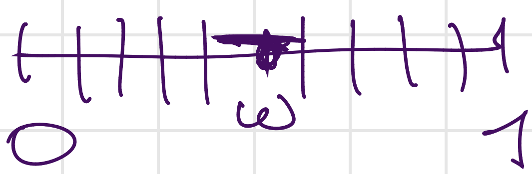
$$\mathbb{P}[X_n = 1]$$

$$\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots \frac{1}{n}, \dots, \frac{1}{n} \dots$$

$$\xrightarrow[n\to\infty]{} 0$$

$$X_n \xrightarrow{a.s} 0 \quad ??$$

NO



$$\lim_{n\to\infty} X_n(\omega) \nexists$$

$$X_n(\omega) = \underbrace{0\ 1}\ \underbrace{0\ 0\ 1}\ \underbrace{0\ 0\ 1\ 0}\ \underbrace{0\ 0\ 0\ 0\ 1}$$

Law of Large Numbers LLN

$\boxed{\text{WLLN}}$ ← weak ← conv. in Prob

SLLN ← strong ← conv. a.s.

<u>Theorem</u>: Assume that $(X_n)_{n\in\mathbb{N}}$ is a sequence of iid r.v's

i.i.d. = <u>independent</u>, <u>identically distributed</u>

and let $\mu = \mathbb{E}[X_i] < +\infty$ and $\mathrm{Var}[X_i] = \sigma^2 < +\infty$.

(i) $\forall \varepsilon > 0 \quad \mathbb{P}\left[\,\left|\overline{X}_n - \mu\right| \geq \varepsilon\right] \leq \underbrace{\dfrac{\sigma^2}{n\cdot\varepsilon^2}}$ the

where $\overline{X}_n = \dfrac{1}{n}\sum_{i=1}^{n} X_i$.

(ii) $\overline{X}_n \xrightarrow{P} \mu$

Proof:

(i) $\Longrightarrow$ (ii)

$$\bar{X}_n \xrightarrow{P} \mu \iff \underbrace{\lim_{n\to\infty} \mathbb{P}[|\bar{X}_n - \mu| > \varepsilon] = 0}$$

$$\text{(i)} \quad \underbrace{\mathbb{P}[|\bar{X}_n - \mu| \geq \varepsilon] \leq \frac{\sigma^2}{n\varepsilon^2} \xrightarrow{n\to\infty} 0}$$

Markov inequality

If $\underbrace{Y \geq 0}$ and $\delta > 0$,

$$\boxed{\mathbb{P}[Y \geq \delta] \leq \frac{\mathbb{E}[Y]}{\delta}}$$

$A \in \mathcal{A}$

$$\mathbb{1}_A = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

$$\mathbb{E}[\mathbb{1}_A] = \mathbb{P}[A]$$

Proof:

$$\boxed{Y \geq \delta \cdot \underbrace{\mathbb{1}_{\{Y \geq \delta\}}}}$$

$\mathbb{E}[\cdot]$ monotone

$$\mathbb{E}[Y] \geq \mathbb{E}[\delta \cdot \mathbb{1}_{\{Y \geq \delta\}}]$$

$$= \delta \cdot \mathbb{E}[\mathbb{1}_{\{Y \geq \delta\}}] =$$

$$= \delta \cdot \mathbb{P}[Y \geq \delta]$$

Chebischev inequality

$X \in L^2(\Omega)$ square integ. r.v.

$\mathbb{E}[X^2] < +\infty$

We apply the Markov inequality to

$$Y = (X-\mu)^2 \geq 0$$

$$\mathbb{P}[\,|X-\mu| \geq \varepsilon\,] = \mathbb{P}[\,\underbrace{(X-\mu)^2}_{Y \geq 0} \geq \underbrace{\varepsilon^2}_{\delta > 0}\,]$$

$$\overset{\text{Markov ineq.}}{\leq} \frac{\mathbb{E}[Y]}{\delta} = \frac{\mathbb{E}[(X-\mu)^2]}{\varepsilon^2} = \frac{\operatorname{Var} X}{\varepsilon^2}$$

Chebischev ineq. Let $X \in L^2$; $\forall \varepsilon > 0$

$$\boxed{\mathbb{P}[\,|X-\mu| \geq \varepsilon\,] \leq \frac{\operatorname{Var}[X]}{\varepsilon^2}}$$

$$\boxed{\mathbb{P}\left[\,|\overline{X}_n - \mu| \geq \varepsilon\right]} \leq \frac{Var[X]}{\varepsilon^2}$$

$$\mu = \mathbb{E}[X_i]$$

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad \boxed{\mathbb{E}[\overline{X}_n]} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[X_i]$$

$$= \frac{1}{n} \cdot n \cdot \mu = \boxed{\mu}$$

$$\boxed{Var[\overline{X}_n]} = Var\left[\frac{1}{n} \sum_{i=1}^{n} X_i\right] =$$

$$= \frac{1}{n^2} Var\left[\sum_{i=1}^{n} X_i\right] \overset{IND.}{=}$$

$$= \frac{1}{n^2} \cdot \sum_{i=1}^{n} \underbrace{Var[X_i]}_{\sigma^2} = \frac{n \cdot \sigma^2}{n^2} = \boxed{\frac{\sigma^2}{n}}$$

Cheb. ineq.

$$\mathbb{P}\left[\,|\overline{X}_n - \mu| \geq \varepsilon\right] \leq \frac{Var[\overline{X}_n]}{\varepsilon^2}$$

$$= \boxed{\frac{\sigma^2}{n \cdot \varepsilon^2}}$$

WLLN