Lecture 3 Stoch. Meth. Oct. 4, 2024

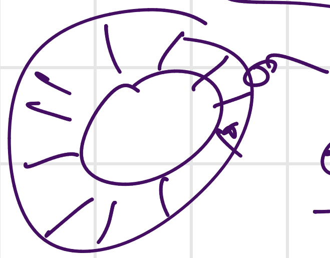
$(\Omega, \mathcal{A}, \mathbb{P})$, properties of $\mathbb{P}$, continuity of a prob. $\mathbb{P}$

$\mathcal{A} \subsetneq 2^{\Omega}$

Roulette

green

$\boxed{0}, \underbrace{1, \dots, 36}$



even, odd

$\{0\}$, $\{\text{even}\}$, $\{\text{odd}\}$

$$\boxed{\Omega = \{0, 1, \dots, 36\}}$$

$$\mathbb{P}[\{1\}] = p_1$$

$$\boxed{\begin{array}{l} p, q \in [0,1] \\ p + q \leq 1 \end{array}}$$

$$\boxed{\begin{array}{l} \mathbb{P}[\{\text{even}\}] = p, \quad \mathbb{P}[\{\text{odd}\}] = q \\ \mathbb{P}[\{0\}] = 1 - p - q \end{array}}$$

$\boxed{\mathcal{A}} = \{\emptyset, \Omega, \{0\}, \{\text{even}\}, \{\text{odd}\},$

$\{\text{even}\} \cup \{\text{odd}\}, \{0\} \cup \{\text{odd}\}, \{0\} \cup \{\text{even}\}\}$

$\{0\}, \{\text{even}\}, \{\text{odd}\}$ set of generators of $\mathcal{A}$

$|\mathcal{A}| = 2^3$

$(\Omega, \mathcal{A}, \mathbb{P})$

- Conditional Probability
- Independence

Def: $(\Omega, \mathcal{A}, \mathbb{P})$ Let $A, B \in \mathcal{A}$ with $\underline{\mathbb{P}[B] > 0}$

We define the <u>conditional probability</u> of A given B as

$$\mathbb{P}[A \mid B] := \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

$$A \longmapsto \mathbb{P}[A \mid B] \quad \text{fixed } B$$

<u>this is a probability</u> on $\mathcal{A}$.

$$\mathcal{A} \longrightarrow [0,1]$$

- $\mathbb{P}[\emptyset \mid B] = 0$

Ex 1 Toss two dice: what is the prob. that the first die is less or equal to 2, given that the sum of the two dice is equal to 4?

$B = \{ \text{the sum is } 4 \}$

$A = \{ \text{the first die is less or equal to } 2 \}$

$$\Omega = \{1,2,3,4,5,6\}^2 = \{ (i,j) : i,j \in \{1,\dots,6\} \}$$

$$\mathcal{A} = 2^{\Omega}, \quad \mathbb{P}[\{(i,j)\}] = \frac{1}{|\Omega|} = \frac{1}{36}$$

$$B = \{ \underbrace{(1,3), (2,2), (3,1)} \}, \quad \mathbb{P}[B] = \frac{3}{36}$$

$$A = \left\{ \begin{matrix} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6) \end{matrix} \right\}$$

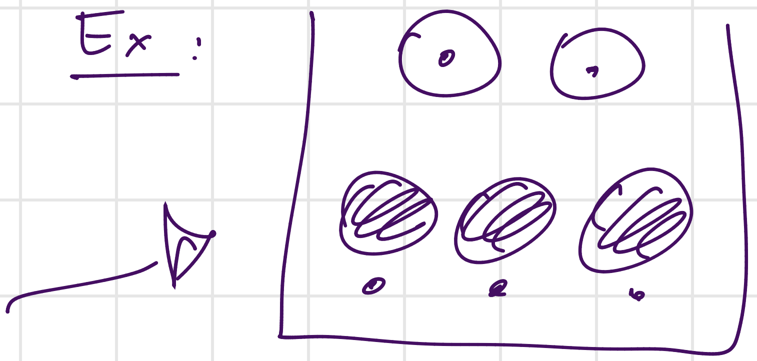
$$\boxed{\mathbb{P}[A] = \frac{12}{36} = \frac{1}{3}} \qquad A \cap B = \{(1,3), (2,2)\}$$

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{2/36}{3/36} = \boxed{2/3 = \mathbb{P}[A|B]}$$

$$\mathbb{P}[A \mid C] = \frac{\mathbb{P}[A \cap C]}{\mathbb{P}[C]} = 0 \qquad A \cap C = \emptyset$$

$C = \{$ the sum is equal to 12 $\} = \{(6,6)\}$

<u>Ex</u>:



2 white

3 black

withdraw two balls <u>without</u> replacement

$B_i = \{$ the i-th ball is black $\}$

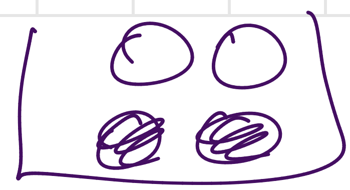
<u>Remark</u>

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

$$\boxed{\mathbb{P}[A \cap B] = \mathbb{P}[B] \cdot \mathbb{P}[A \mid B]}$$

$$\mathbb{P}[B_1 \cap B_2] = \mathbb{P}[B_1] \cdot \mathbb{P}[B_2 \mid B_1]$$

$$= \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}$$



$A_1, A_2, A_3 \in \mathcal{A}$

$$\mathbb{P}[\underbrace{A_1 \cap A_2} \cap \underbrace{A_3}] = \mathbb{P}[A_3 \mid A_1 \cap A_2] \cdot \mathbb{P}[\underbrace{A_1 \cap A_2}]$$

$$= \mathbb{P}[A_3 \mid A_1 \cap A_2] \cdot \mathbb{P}[A_2 \mid A_1] \cdot \mathbb{P}[A_1]$$

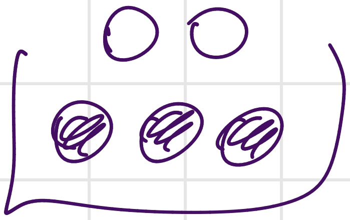
$$1 \longrightarrow 2 \longrightarrow 3$$

$A_1, \dots, A_n \in \mathcal{A}$ assuming that $\mathbb{P}[A_1 \cap A_2 \cap \dots \cap A_{n-1}] > 0$

$$\mathbb{P}[A_1, \dots, A_n] = \mathbb{P}[A_1] \cdot \mathbb{P}[A_2 \mid A_1] \cdot \mathbb{P}[A_3 \mid A_1 \cap A_2] \cdot \dots \cdot \mathbb{P}[A_n \mid A_1 \cap A_2 \cap \dots \cap A_{n-1}]$$

$$\mathbb{P}[B] = \mathbb{P}[A \cap B] + \mathbb{P}[A^c \cap B]$$

$$= \mathbb{P}[A] \cdot \mathbb{P}[B \mid A] + \mathbb{P}[A^c] \cdot \mathbb{P}[B \mid A^c]$$



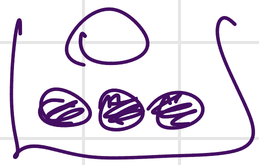
B_1, B_2

$$\boxed{P[B_1] = 3/5}$$

$$P[B_2] = P[B_1 \cap B_2] + P[B_1^c \cap B_2]$$

$$= P[B_1] \cdot P[B_2 | B_1] + P[B_1^c] \cdot P[B_2 | B_1^c]$$

$$= \frac{3}{5} \cdot \frac{1}{2} + \frac{2}{5} \cdot \frac{3}{4}$$



$$= \frac{3}{10} + \frac{3}{10} = \frac{6}{10} = \boxed{\frac{3}{5} = P[B_2]}$$

$$P[B_3] = \frac{3}{5}$$

B_4, B_5

Independence

$A \perp\!\!\!\perp B$, $A_1, \dots, A_n$ are independent ,

$\underbrace{\mathcal{A}_1 \perp\!\!\!\perp \mathcal{A}_2}$ two σ-fields are indep.

<u>Def</u>: $A, B \in \mathcal{A}$, $\underbrace{\mathbb{P}[B] > 0}$

A and B are independent if

$$\boxed{\mathbb{P}[A \mid B] = \mathbb{P}[A]} \Leftrightarrow$$

$$\frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \mathbb{P}[A]$$

$$\Rightarrow \boxed{\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]}$$

Remarks 1: If $\underbrace{\mathbb{P}[B]=0}$, B is independent of any other event.

$$0=\underbrace{\mathbb{P}[A\cap B]=\mathbb{P}[A]\cdot \mathbb{P}[B]}_{\text{YES}}=0 \qquad \forall A \in \mathcal{A}$$

$$\mathbb{P}[B] = 0$$

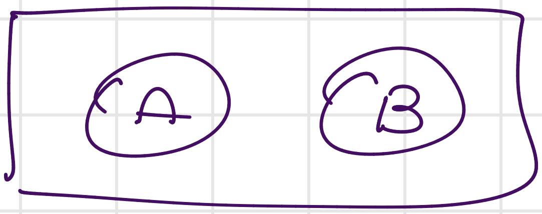
$$A\cap B \subseteq B$$

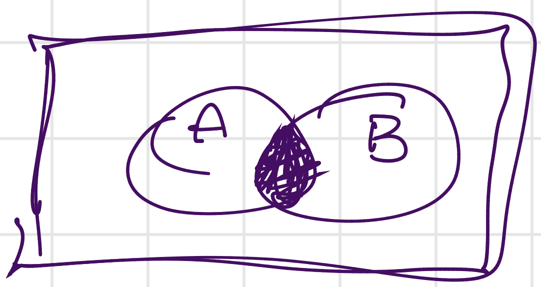
$$0 \leq \underbrace{\mathbb{P}[A\cap B]} \leq \mathbb{P}[B]=0 \quad (\text{monotonicity})$$

$$\Rightarrow \mathbb{P}[A\cap B]=0$$

Remark 2

disjoint $\xleftrightarrow{\text{for}}$ independent





$$A\cap B=\emptyset$$

$$0=\mathbb{P}[A\cap B]=\mathbb{P}[A]\cdot\mathbb{P}[B]$$

$\Leftrightarrow$ at least one of the two events has prob. 0

$$\mathbb{P}[A\cap B] = \mathbb{P}[A]\cdot\mathbb{P}[B]$$

Ex: toss two dice

$A = \{\text{the first die gives } 1\}$

$B = \{\text{the second die gives } 6\}$

$\boxed{A \perp\!\!\!\perp B \ ?}$

$A = \{(1, j) : j \in \{1, \dots, 6\}\}$ $\quad |A| = 6, \ \mathbb{P}[A] = \frac{6}{36}$

$B = \{(i, 6), \ i \in \{1, \dots, 6\}\}$ $\quad |B| = 6, \ \mathbb{P}[B] = 1/6$

$$\boxed{\underset{\underset{1/36}{\shortparallel}}{\mathbb{P}[A \cap B]} \underset{\text{YES}}{=} \mathbb{P}[A] \cdot \mathbb{P}[B] = 1/36}$$

$A \cap B = \{(1, 6)\}$ $\quad \mathbb{P}[A \cap B] = \frac{1}{36}$

$C = \{\text{the sum of the two dice is } \boxed{7}\}$ $\quad \mathbb{P}[C] = \frac{1}{6}$

$C = \{\underline{(1, 6)}, (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

$\boxed{A \perp\!\!\!\perp C \ ? \ \text{YES}}$ $\quad A \cap C = \{(1, 6)\}$

$B \perp\!\!\!\perp C$ YES $\quad B \cap C = \{(1, 6)\}$

$D = \{ \text{the sum is equal to } 6 \}$

$A \perp\!\!\!\perp D$?

$D = \{ (1,5), (2,4), (3,3), (4,2), (5,1) \}$

$\mathbb{P}[D] = 5/36$

$$\underset{1/36}{\underset{\shortparallel}{\mathbb{P}[A \cap D]}} \neq \underset{1/6}{\underset{\shortparallel}{\mathbb{P}[A]}} \cdot \underset{5/36}{\underset{\shortparallel}{\mathbb{P}[D]}} = \frac{5}{216}$$

$A \cap D = \{ (1,5) \}$

A and D are not $\underline{\underline{\text{indep.}}}$

A, B, C are independent $\Longleftrightarrow$

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]$$

$$\mathbb{P}[A \cap C] = \mathbb{P}[A] \cdot \mathbb{P}[C]$$

$$\mathbb{P}[B \cap C] = \mathbb{P}[B] \cdot \mathbb{P}[A]$$

$$+ \quad \boxed{\mathbb{P}[A \cap B \cap C] = \mathbb{P}[A] \cdot \mathbb{P}[B] \cdot \mathbb{P}[C]}$$

Previous example

A, B, C are "pairwise" independent

$$\mathbb{P}[A \cap B \cap C] \neq \underbrace{\mathbb{P}[A] \cdot \mathbb{P}[B] \cdot \mathbb{P}[C]}_{= \frac{1}{216}} \quad ?? \ldots$$

$$\mathbb{P}[A \cap B \cap C] = \mathbb{P}[\{(1,6)\}] = 1/36, \qquad \mathbb{P}[A] = 1/6, \quad \mathbb{P}[B] = 1/6, \quad \mathbb{P}[C] = 1/6$$

$$B \cap C = \{(1,6)\}$$

A, B and C are <u>not</u> independent

$$\mathbb{P}[A \mid B] = \mathbb{P}[A] \qquad \mathbb{P}[A] = 1/6$$

$$\mathbb{P}[A \mid B \cap C] = 1$$