lecture 2 Stoch. Meth. 4 oct. 2024

Prob. space $(\Omega, \mathcal{A}, \mathbb{P})$ $\boxed{\mathbb{P} : \mathcal{A} \to [0,1]}$

$\mathcal{A}$ - σ-field $\mathcal{A} \subseteq 2^{\Omega}$

$A \in \mathcal{A}$ A event $\mathbb{P}[A] \in [0,1]$

- $\mathbb{P}[\Omega] = 1$
- $(A_n)_{n \in \mathbb{N}}$, pairwise disjoint $(A_n \cap A_m = \emptyset \ \forall n \neq m)$

$$\mathbb{P}\left[\bigcup_n A_n\right] = \sum_n \mathbb{P}[A_n]$$

Ex: $|\Omega| = n$ $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$

$\mathbb{P}[\{\omega_1\}] \in [0,1]$ $\mathcal{A} = 2^{\Omega}$

$\mathbb{P}[\{\omega_1\}] = p_1$

$\mathbb{P}[\{\omega_2\}] = p_2$, ..., $\mathbb{P}[\{\omega_n\}] = p_n$

$p_1, p_2, \ldots, p_n \in [0,1]$ • $\mathbb{P}[\Omega] = 1$

$$\boxed{1} = \mathbb{P}[\Omega] = \mathbb{P}\Big[\bigcup_{k=1}^{n} \underbrace{\{\omega_k\}}\Big] \underset{(2)}{=} \sum_{k=1}^{n} \mathbb{P}[\{\omega_k\}] \boxed{= \sum_{k=1}^{n} p_k}$$

$$\begin{cases} p_1, \dots, p_n \in [0,1] \\ \sum_{k=1}^{n} p_k = 1 \end{cases}$$

$(\Omega, 2^{\Omega})$

$n-1$ parameters

$$p_n = 1 - \sum_{k=1}^{n-1} p_k$$

gos

$$p_1 = p_2 = \dots = p_n = p \in [0,1]$$

$$\sum_{k=1}^{n} p_k = n \cdot p = 1 \Rightarrow p = 1/n$$

$\mathbb{P}[A]$

$A \in 2^{\Omega}$

$A = \{\omega_2, \omega_4, \dots, \omega_8\}$

$$\mathbb{P}[A] = \sum_{\cdot} \mathbb{P}[\omega_\cdot] = \sum_{\cdot} 1/n = \frac{|A|}{n} = \frac{|A|}{|\Omega|}$$

Uniform probability

In general

$\omega \in \Omega$

$$\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\{\omega\}]$$

Properties

$A \in \mathcal{A}$, $\boxed{\mathbb{P}[A] + \mathbb{P}[A^c] = 1}$

Proof

$$1 = \mathbb{P}[\Omega] = \mathbb{P}[A \cup A^c] = \mathbb{P}[A] + \mathbb{P}[A^c]$$

disjoint

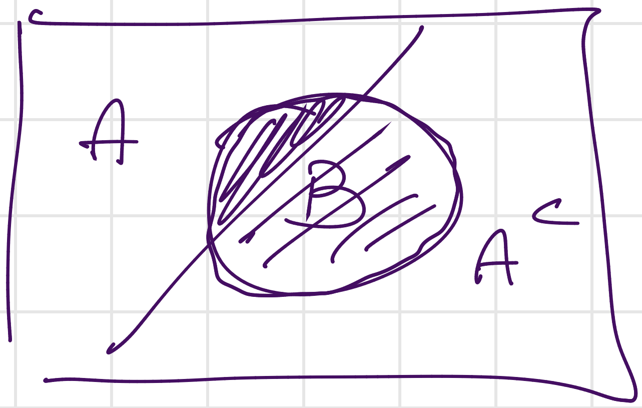
$A \in \mathcal{A}$ $\qquad \mathbb{P}[A] = 1 - \mathbb{P}[A^c]$

$A, B \in \mathcal{A}$ $\qquad \mathbb{P}[A \cup B] = 1 - \mathbb{P}[(A \cup B)^c]$

$$= 1 - \mathbb{P}[A^c \cap B^c]$$

- $\mathbb{P}[\emptyset] = 1 - \mathbb{P}[\Omega] = 1 - 1 = 0$

- $A, B \in \mathcal{A}$ $\qquad \mathbb{P}[B] = \mathbb{P}[A \cap B] + \mathbb{P}[A^c \cap B]$



$B = (A \cap B) \cup (A^c \cap B)$

disjoint

$\rightarrow 2^{\Omega}$

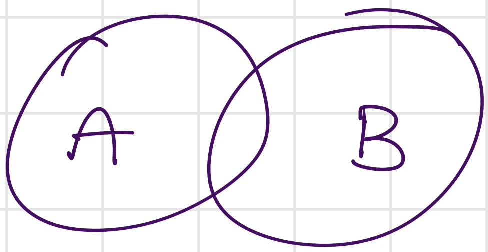
- if $A \subseteq B$, then $\mathbb{P}[A] \leq \mathbb{P}[B]$

monotonicity of the probability

$(2^\Omega, \subseteq)$ $\quad \subseteq$ is partial ordering

$A \subseteq B,\ B \subseteq A \Rightarrow A = B$

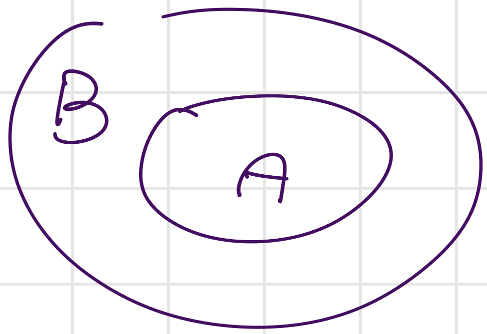
$A \subseteq B,\ B \subseteq C \Rightarrow A \subseteq C$



$P[B] = P[A \cap B] + P[A^c \cap B]$

$A \subseteq B \Rightarrow \boxed{P[A] \leq P[B]}$

Proof:



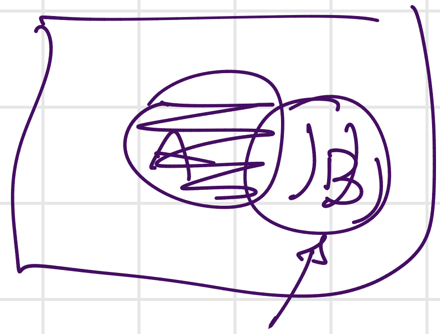
$\underbrace{A \cap B = A}$

$P[B] = P[A \cap B] + P[A^c \cap B] = \underbrace{P[A] + P[A^c \cap B]}_{\geq P[A]}$

• if $A, B \in \mathcal{A}$

$$\boxed{\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]}$$

$$\mathbb{P}[A \cup B] \le \mathbb{P}[A] + \mathbb{P}[B]$$



Proof: $A \cup B = A \cup (A^c \cap B)$

disjoint

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[A^c \cap B] =$$

$$= \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$$

(2) $\mathbb{P}[B] = \mathbb{P}[A \cap B] + \mathbb{P}[A^c \cap B]$

$$\Downarrow$$

$$\mathbb{P}[A^c \cap B] = \mathbb{P}[B] - \mathbb{P}[A \cap B]$$

• $A, B, C \in \mathcal{A}$

$$\mathbb{P}[A \cup B \cup C]$$

$$\mathbb{P}[A \cup B \cup C] = \mathbb{P}[(A \cup B) \cup C] = \underbrace{\mathbb{P}[A \cup B]} + \underbrace{\mathbb{P}[C]}$$

$$- \mathbb{P}[\underbrace{\underline{(A \cup B)} \cap \underline{C}}]$$

$$= \mathbb{P}[A] + \mathbb{P}[B] - \underline{\mathbb{P}[A \cap B]} + \mathbb{P}[C]$$

$$- \underbrace{\mathbb{P}[(A \cap C) \cup (B \cap C)]}$$

$$= \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C]$$

$$- \mathbb{P}[A \cap B] - \mathbb{P}[A \cap C] - \mathbb{P}[B \cap C]$$

$$- (- \mathbb{P}[A \cap C \cap B \cap C)])$$

$$\boxed{\begin{aligned} &= \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C] \\ &- \mathbb{P}[A \cap B] - \mathbb{P}[A \cap C] - \mathbb{P}[B \cap C] \\ &+ \mathbb{P}[A \cap B \cap C] \quad = \mathbb{P}[A \cup B \cup C] \end{aligned}}$$

$A_1, \dots, A_n$

$$\underbrace{\mathbb{P}\Big[\bigcup_{k=1}^{n} A_k\Big]} = \sum_{k=1}^{n} \mathbb{P}[A_k] - \sum_{k \neq j} \mathbb{P}[A_k \cap A_j]$$

$$+ \sum_{k \neq j \neq i} \mathbb{P}[A_k \cap A_j \cap A_i] - \dots$$

$$\dots + (-1)^{n+1} \mathbb{P}[A_1 \cap A_2 \cap \dots \cap A_n]$$

<u>Problem</u>: We throw n balls randomly into 3 boxes, initially empty.

Compute

$$\mathbb{P}[\{\text{at least one box remains empty}\}]$$

<u>Solution</u>:

$$A_i = \{\text{the } i\text{-th box is empty}\}$$

$$\mathbb{P}[A_1 \cup A_2 \cup A_3]$$

$$= \underline{\mathbb{P}[A_1]} + \underline{\mathbb{P}[A_2]} + \underline{\mathbb{P}[A_3]}$$

$$- \mathbb{P}[\underline{A_1 \cap A_2}] - \mathbb{P}[\underline{A_1 \cap A_3}] - \mathbb{P}[\underline{A_2 \cap A_3}]$$

$$+ \mathbb{P}[\underline{A_1 \cap A_2 \cap A_3}] = 0 \qquad A_1 \cap A_2 \cap A_3 = \emptyset$$

$$= 3\ \mathbb{P}[A_1] - 3\ \mathbb{P}[A_1 \cap A_2]$$

$\mathbb{P}[A_1]$

$\mathbb{P}[A_1 \cap A_2]$

$$\Omega = \{ (\overset{\{1,2,3\}}{\overset{\in}{\omega_1}}, \omega_2, \dots, \omega_n) : \omega_i \in \{1,2,3\} \}$$

uniform prob.

$$\mathbb{P}[A] = \frac{|A|}{|\Omega|}$$

$|\Omega| = 3^n$ $\quad \omega = (\omega_1, \omega_2, \dots, \omega_n)$

$\omega_i \in \{1,2,3\}$

$$\Omega = \{1,2,3\}^n$$

$$A_1 = \{\ \omega \in \Omega : \quad \omega_i \in \{2,3\}\ \} \qquad |A_1| = 2^n$$

$$\mathbb{P}[A_1] = \frac{|A_1|}{|\Omega|} = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$$

$$A_1 \cap A_2 = \{\underbrace{(3,3,3,\dots,3)}\} \qquad \mathbb{P}[A_1 \cap A_2] = \frac{1}{3^n}$$

$$\mathbb{P}[A_1 \cup A_2 \cup A_3] = 3\left(\frac{2^n}{3^n} - \frac{1}{3^n}\right) = $$

$$= \frac{2^n - 1}{3^{n-1}}$$

To calculate the numerical solution, let's proceed by breaking it down step by step.

Step 1: Total number of possible outcomes

The total number of ways to distribute n balls into 3 boxes is:

$$\text{Total outcomes} = 3^n$$

Step 2: Number of favorable outcomes (i.e., at least one empty box)

We will calculate the complement: the number of ways in which no box is empty. This can be done by using inclusion-exclusion.

Inclusion-Exclusion Formula:

Let $N(n)$ represent the number of ways to place n balls into 3 boxes such that all boxes are non-empty.

- There are 3^n ways to place all n balls into 3 boxes (the total number of outcomes).
- Let A_1, A_2, and A_3 represent the events that a particular box is empty.

By inclusion-exclusion, the number of favorable outcomes where no box is empty is:

$$N(n) = 3^n - 3 \cdot 2^n + 3 \cdot 1^n$$

Where:

- 3^n counts all possible outcomes.
- $3 \cdot 2^n$ subtracts the cases where one box is empty (leaving 2 boxes for the n balls).
- $3 \cdot 1^n$ adds back the cases where two boxes are empty (since we subtracted them twice in the previous step).

Step 3: Probability calculation

The number of ways to have at least one box empty is:

$$\text{Favorable outcomes} = 3^n - N(n) = 3^n - (3^n - 3 \cdot 2^n + 3 \cdot 1^n) = 3 \cdot 2^n - 3$$

Thus, the probability of having at least one empty box is:

$$P(\text{at least one empty box}) = \boxed{\frac{3 \cdot 2^n - 3}{3^n}}$$

Example Calculation:

Let's compute the probability for a few values of n.

- **For** $n = 1$:

$$P(\text{at least one empty box}) = \frac{3 \cdot 2^1 - 3}{3^1} = \frac{3 \cdot 2 - 3}{3} = \frac{6 - 3}{3} = 1$$

Example Calculation:

Let's compute the probability for a few values of n.

- **For** $n = 1$:

$$P(\text{at least one empty box}) = \frac{3 \cdot 2^1 - 3}{3^1} = \frac{3 \cdot 2 - 3}{3} = \frac{6 - 3}{3} = 1$$

- **For** $n = 2$:

$$P(\text{at least one empty box}) = \frac{3 \cdot 2^2 - 3}{3^2} = \frac{3 \cdot 4 - 3}{9} = \frac{12 - 3}{9} = \frac{9}{9} = 1$$

- **For** $n = 3$:

$$P(\text{at least one empty box}) = \frac{3 \cdot 2^3 - 3}{3^3} = \frac{3 \cdot 8 - 3}{27} = \frac{24 - 3}{27} = \frac{21}{27} = \frac{7}{9} \approx 0.7778$$

- **For** $n = 4$:

$$P(\text{at least one empty box}) = \frac{3 \cdot 2^4 - 3}{3^4} = \frac{3 \cdot 16 - 3}{81} = \frac{48 - 3}{81} = \frac{45}{81} = \frac{5}{9} \approx 0.5556$$

So, you can compute the probability for any number n using this method!

$\boxed{\mathbb{P}: \mathcal{A} \longrightarrow [0,1]}$

σ-field

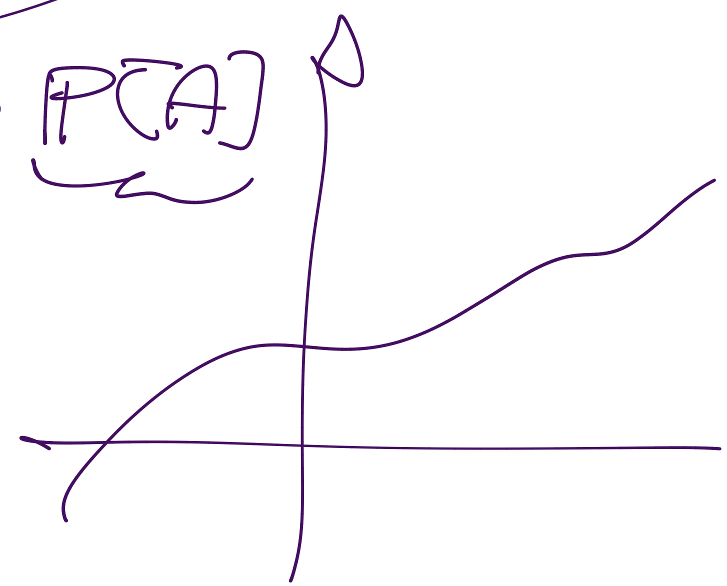
"continuity of the Probability"

$f: \mathbb{R} \to \mathbb{R}$ cont. in $x \in \mathbb{R}$

$x_n \longrightarrow x \implies f(x_n) \longrightarrow f(x)$ easier

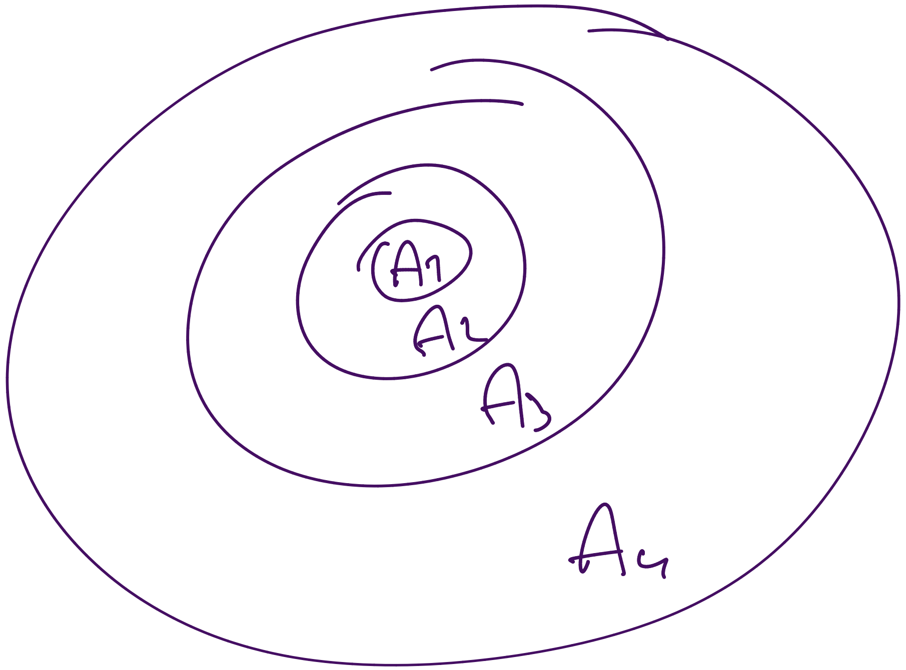
$\boxed{A_n \longrightarrow A}$

$\mathbb{P}[A_n] \longrightarrow \mathbb{P}[A]$



$(A_n)_{n\in\mathbb{N}}$ increasing sequences of events

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots \subseteq A_n \subseteq A_{n+1} \subseteq \dots \qquad \forall n$$



$$\lim_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} A_n$$

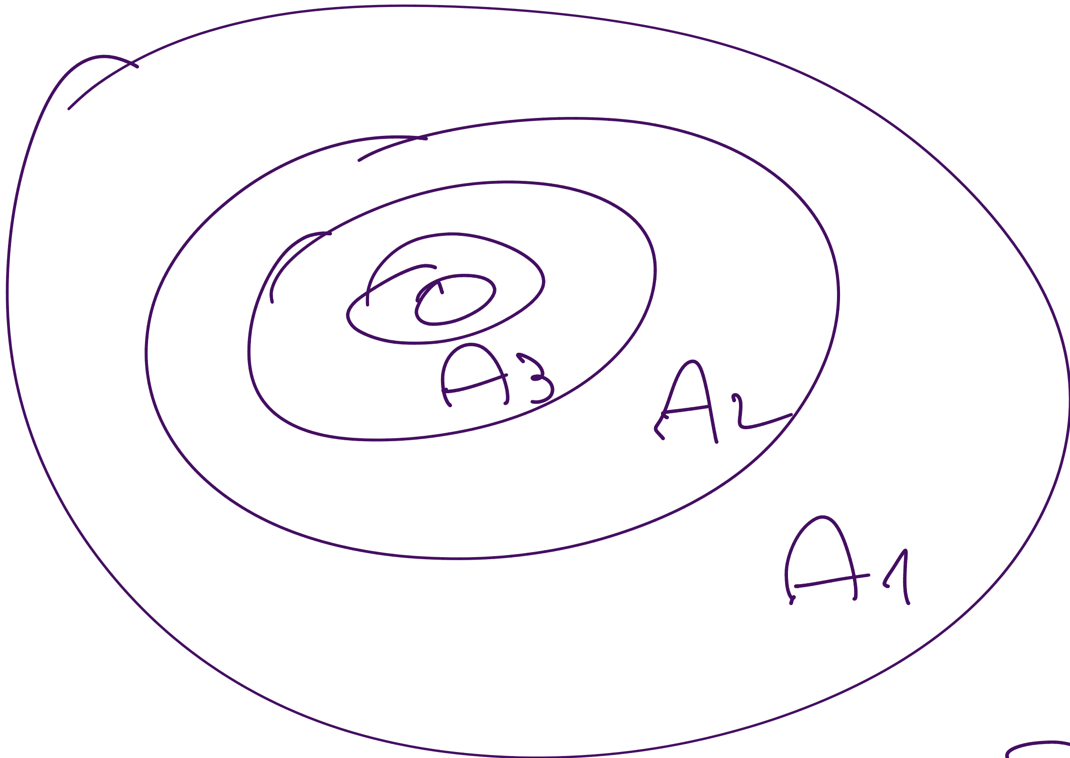
It is possible to prove that in this case

$$\boxed{\mathbb{P}\left[\lim_{n\to\infty} A_n\right] = \lim_{n\to\infty} \mathbb{P}[A_n]}$$

continuity of the probability

(A_n) decreasing sequences of events

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots \supseteq \boxed{A_n \supseteq A_{n+1} \quad \forall n \in \mathbb{N}} \supseteq \dots$$



$$\boxed{\lim_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} A_n}$$

$$\mathbb{P}[\lim_{n\to\infty} A_n] = \lim_{n\to\infty} \mathbb{P}[A_n]$$

... to be continued.