Lecture 19 Stoch. Methods Nov. 29th, 2024

Chernoff's bound

$Z_1, \dots, Z_n \sim N(0,1)$ independent

$X_1, \dots, X_n \qquad X_i = Z_i^2$ Chi-square dist.

$$m(t) = \mathbb{E}[e^{tX_1}] = \begin{cases} +\infty & t \geq 1/2 \\ (1-2t)^{-1/2} & t < 1/2 \end{cases}$$

$\mu = 1$

$$\mathbb{P}[\bar{X}_n \geq \mu + \varepsilon] \leq e^{-n g(t)}$$

$$g(t) = t(\mu+\varepsilon) - \log m(t)$$

t^* the max. $\qquad t^* = \dfrac{\varepsilon}{2(1+\varepsilon)}$

$$g(t^*) = \frac{1}{2}\left(\varepsilon - \underbrace{\log(1+\varepsilon)}\right)$$

$\boxed{\varepsilon > 0}$ $\qquad \underbrace{\log(1+\varepsilon) \leq \varepsilon - \varepsilon^2 / 2(1+\varepsilon)}$

$a(\varepsilon) = \log(1+\varepsilon)$

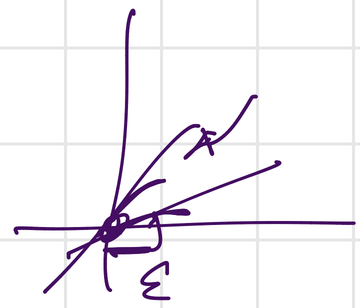
$b(\varepsilon) = \varepsilon - \dfrac{\varepsilon^2}{2(1+\varepsilon)}$ $\qquad a(\varepsilon) \leq b(\varepsilon) \quad \varepsilon \geq 0$

$$a(\varepsilon) = \underbrace{\lg(1+\varepsilon)}_{} \qquad b(\varepsilon) = \varepsilon - \frac{\varepsilon^2}{2(1+\varepsilon)}$$

$\varepsilon = 0$ $\qquad a(0) = \lg(1) = 0$

$$\frac{d}{dt}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

$b(0) = 0$



$\rightsquigarrow a', b'$

$$a'(\varepsilon) = \boxed{\frac{1}{1+\varepsilon}}, \quad b'(\varepsilon) = 1 - \frac{2\varepsilon\,(\overbrace{2(1+\varepsilon)}^{2+2\varepsilon}) - \varepsilon^2\,2}{4(1+\varepsilon)^2}$$

$$= 1 - \frac{4\varepsilon + 4\varepsilon^2 - 2\varepsilon^2}{4(1+\varepsilon)^2} = 1 - \frac{\cancel{4}^{2}\varepsilon + \cancel{2}\varepsilon^2}{\cancel{4}^{2}(1+\varepsilon)^2}$$

$$= \frac{2(1+\varepsilon)^2 - 2\varepsilon - \varepsilon^2}{2(1+\varepsilon)^2} = \frac{2 + 2\varepsilon^2 + 4\varepsilon - 2\varepsilon - \varepsilon^2}{2(1+\varepsilon)^2}$$

$$= \boxed{\frac{\varepsilon^2 + 2\varepsilon + 2}{2(1+\varepsilon)^2}} = b'(\varepsilon)$$

If for small $\varepsilon > 0$ $\quad \boxed{a'(\varepsilon) < b'(\varepsilon)}$

$$\Rightarrow \boxed{a(\varepsilon) \le b(\varepsilon)} \qquad a(\varepsilon) = \int_0^\varepsilon \underbrace{a'(t)}_{}\,dt$$

$$b(\varepsilon) = \int_0^\varepsilon \underbrace{b'(t)}_{}\,dt$$

$$\boxed{\underbrace{\frac{1}{1+\varepsilon}} \overset{?}{<} \frac{\varepsilon^2+2\varepsilon+2}{2(1+\varepsilon)^2}}$$

$$\boxed{\varepsilon > 0}$$

$$\Leftrightarrow \quad 2(1+\varepsilon) < \varepsilon^2 + 2\varepsilon + 2$$

$$\Leftrightarrow \quad 0 < \varepsilon^2 + \cancel{2\varepsilon} + \cancel{2} - \cancel{2} - \cancel{2\varepsilon}$$

$$\Leftrightarrow \quad \boxed{0 < \varepsilon^2}$$

YES

for small ε $\quad \underbrace{\lg(1+\varepsilon) \leq \varepsilon - \frac{\varepsilon^2}{2(1+\varepsilon)}}$

$$g(t^*) = \frac{1}{2}\left(\varepsilon - \underbrace{\lg(1+\varepsilon)}\right)$$

$$\geq \frac{1}{2}\left(\cancel{\varepsilon} - \cancel{\varepsilon} + \frac{\varepsilon^2}{2(1+\varepsilon)}\right)$$

$$= \frac{\varepsilon^2}{4(1+\varepsilon)}$$

<u>Upper tail estimate</u>

$$\mathbb{P}\left[\bar{X}_n \geq 1+\varepsilon\right] \leq e^{-n \cdot \frac{\varepsilon^2}{4(1+\varepsilon)}}$$

Lower tail estimate

$$\mathbb{P}[\overline{X}_n \le 1-\varepsilon] \le e^{-n\,h(t)}$$

$$m(t) = (1-2t)^{-1/2}$$

$$\underbrace{m(-t)} = (1+2t)^{-1/2}$$

$$h(t) = -t(1-\varepsilon) - \underbrace{\log(m(-t))}$$

$$= \underbrace{-t(1-\varepsilon)} + \frac{1}{2}\log(1+2t)$$

t^* $\qquad h(0) = 0 + 1/2 \log(1) = 0$

$$\underbrace{h'(t) = 0} \iff -(1-\varepsilon) + \frac{1}{\cancel{2}} \cdot \frac{1}{1+2t} \cdot \cancel{2} = 0$$

$$\iff \frac{1}{1+2t} = 1-\varepsilon \iff 1+2t = \frac{1}{1-\varepsilon}$$

$$\iff 2t = \frac{1}{1-\varepsilon} - 1 = \frac{1-1+\varepsilon}{1-\varepsilon} = \frac{\varepsilon}{1-\varepsilon}$$

$$\iff t^* = \boxed{\frac{\varepsilon}{2(1-\varepsilon)}}$$

$$\boxed{h(t) = -t(1-\varepsilon) + \frac{1}{2}\log(1+2t)}$$

$$h(t^*) \ge h(t)$$

$$1 + \frac{\varepsilon}{1-\varepsilon} = \frac{1-\varepsilon+\varepsilon}{1-\varepsilon} = \frac{1}{1-\varepsilon}$$

$$\boxed{h(t^*)} = -\frac{\varepsilon}{2\cancel{(1-\varepsilon)}}\cancel{(1-\varepsilon)} + \frac{1}{2}\log\left(1 + \frac{\varepsilon}{1-\varepsilon}\right)$$

$$= -\varepsilon/2 - 1/2 \log(1+\varepsilon) = -1/2\,(\varepsilon + \log(1+\varepsilon))$$

$$h(t^*) = -\frac{1}{2}\left(\varepsilon + \lg(1-\varepsilon)\right)$$

$\Rightarrow \lg(1-x) \leq -x - x^2/2$ (prove as an exercise)!

$$h(t^*) = -1/2\left(\varepsilon + \lg(1-\varepsilon)\right) \geq$$

$$\geq -1/2\left(\cancel{\varepsilon} - \cancel{\varepsilon} - \varepsilon^2/2\right) = \varepsilon^2/4$$

Lower tail estimate

$$\mathbb{P}[\overline{X}_n \leq 1-\varepsilon] \leq \underbrace{e^{-n\,\varepsilon^2/4}} \leq \underbrace{e^{-n\,\varepsilon^2/8}}$$

Upper tail estimate

$$\mathbb{P}[\overline{X}_n \geq 1+\varepsilon] \leq \underbrace{e^{-n\frac{\varepsilon^2}{4(1+\varepsilon)}}} \leq \underbrace{e^{-n\,\varepsilon^2/8}}$$

$$\frac{\varepsilon^2}{4(1+\varepsilon)}$$

$$\mathbb{P}[|\overline{X}_n - 1| \geq \varepsilon] = \text{Upper tail} + \text{Lower tail}$$

$$1+\varepsilon < 2 \Rightarrow \frac{1}{1+\varepsilon} > \frac{1}{2} \Rightarrow \boxed{-\frac{1}{1+\varepsilon} < -\frac{1}{2}} \qquad 1+\varepsilon < \boxed{2}$$

if $\boxed{0 < \varepsilon < 1}$

$$\leq \underbrace{e^{-n\frac{\varepsilon^2}{4\cdot 2}}} = e^{-n\,\varepsilon^2/8}$$

$$\underbrace{\mathbb{P}[\,|\bar{X}_n - 1| \geq \varepsilon\,]}_{\text{Chernoff's bound}} \leq 2 \cdot e^{-n\varepsilon^2/8} \qquad \underbrace{0 < \varepsilon < 1}$$

Chernoff's bound

Normal distributions $X_i \sim \mathcal{N}(0,1)$

$$\mathbb{P}[\bar{X}_n \geq \varepsilon] \leq e^{-n\,\varepsilon^2/2}$$

$f(x)$

$$\mathbb{P}[X \in B] = \int_B f(t)\,dt$$

$$\underbrace{\phi(x)} = \frac{1}{\sqrt{2\pi}} \cdot \underbrace{e^{-x^2/2}}$$

$$\phi'(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \cdot \left(-\frac{x^2}{2}\right)'$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}} \cdot (-x)$$

$$\boxed{\phi'(x) = (-x)\,\phi(x)}$$

differential equation

$$\mathbb{P}[X > x] = \mathbb{P}[X \in (x,+\infty)] = \int_{(x,+\infty)} \phi(t)\,dt$$

$$\mathbb{P}[X > x] = \int_x^{+\infty} \phi(t)\,dt = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-t^2/2}\,dt$$

$$= \underbrace{\int_{\boxed{x}}^{+\infty} t \cdot \boxed{\frac{1}{t}} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt}$$

$$x < t \iff \boxed{\frac{1}{t} < \frac{1}{x}}$$

$$< \underbrace{\frac{1}{x}} \cdot \int_x^{+\infty} t \cdot \underbrace{\frac{1}{\sqrt{2\pi}} e^{-t^2/2}}_{\phi(t)}\,dt$$

$$\boxed{t \cdot \phi(t) = -\phi'(t)}$$

$$\mathbb{P}[X > x] < \frac{1}{x} \cdot \int_x^{+\infty} \left(-\phi'(t)\right) dt$$

$$= \frac{1}{x} \left[-\phi(t)\right]_x^{+\infty} = \frac{1}{x} \cdot \underbrace{\phi(x)}$$

$$\boxed{\mathbb{P}[X > x] < \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}} \qquad X \sim N(0,1)$$

$$x > 0$$

$$\overline{X}_n = \frac{1}{n}\left(X_1 + \dots + X_n\right)$$

$X_1, \dots, X_n \sim N(0,1)$ independent

$$X_1 + \dots + X_n \sim N(0, n)$$

$$\overline{X}_n \sim N\left(0, \frac{n}{n^2}\right) = N(0, 1/n)$$

$$\frac{X_1 + \dots + X_n}{n}$$

$$N(0,1) \sim \sqrt{n} \cdot \overline{X}_n$$

$$P[\overline{X}_n \geq \varepsilon] = P[\underbrace{\overline{X}_n \cdot \sqrt{n}}_{Z \sim N(0,1)} \geq \sqrt{n} \cdot \varepsilon]$$

$$\leq \frac{1}{\varepsilon \cdot \sqrt{n}} \cdot \frac{1}{\sqrt{2\pi}} \, e^{-n \varepsilon^2/2}$$

Chernoff

$$\frac{1}{\sqrt{n}}$$

$$\phi' = -t\,\phi$$

<u>Exercise</u>: prove $X \sim N(0,1)$

$$P[X \geq x] \geq \left(\frac{1}{x} - \frac{1}{x^3}\right) \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$

$$\int_x^{+\infty} \phi(t)\,dt = \int_x^{+\infty} \left(-\frac{1}{t}\phi'(t)\right) dt \quad \dots$$

Chernoff's bounds discrete r.v's

$X_1, \dots, X_n \sim Bin(1, p)$ indep.

$p = \mathbb{P}[X_1 = 1] \Rightarrow p = \mathbb{E}[X_1]$

$$\boxed{m(t) = p e^{t} + (1-p)}$$

$\forall t \in \mathbb{R}$

$\mu = p$

1. Upper tail estimate

$g(t^*)$

$$\mathbb{P}[\overline{X}_n \geq \mu + \varepsilon] \leq e^{-n \boxed{g(t)}}$$

$$g(t) = t(p+\varepsilon) - \underbrace{\lg(pe^{t} + (1-p))}$$

$$g'(t) = p + \varepsilon - \frac{1}{pe^{t} + (1-p)} \cdot pe^{t}$$

$$g'(t) = 0 \Leftrightarrow p + \varepsilon - \frac{1}{pe^{t} + (1-p)} = 0$$

....

$$g'(t)=0 \iff t=t^* = \lg\left(\frac{(1-p)(1+\delta)}{1-p-\delta p}\right)$$

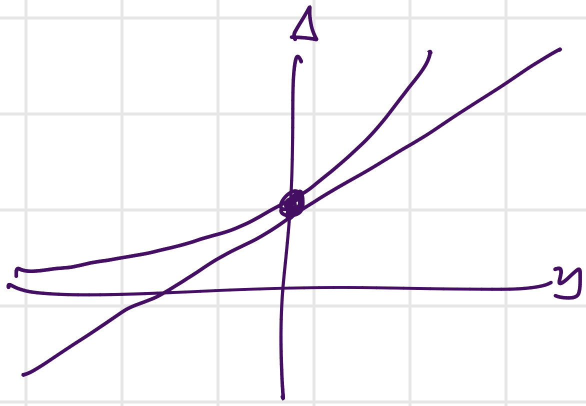
$$\boxed{\varepsilon = \delta p}$$

$$g(t^*) \; \ldots$$

$$g(t) = t\cdot p(1+\delta) - \underbrace{\lg(1-p+pe^{t})}_{}$$

$$\underbrace{1+x \le e^{x}}_{}$$

$\forall x$



$$e^{-u g(t)}$$

$$(\underbrace{1-p}_{} + pe^{t}) = 1 + p(e^{t}-1)$$
$$\le e^{p(e^{t}-1)}$$

$$\underbrace{\lg(1-p+pe^{t})}_{} \le \lg\left(e^{p(e^{t}-1)}\right) = p\cdot(e^{t}-1)$$

$$g(t) \ge tp(1+\delta) - p(e^{t}-1) = \underbrace{\bar{g}(t)}_{}$$

$$\bar{g}'(t) = \ldots \qquad \bar{t}^* = \lg(1+\delta)$$

$X_1, \dots, X_n \sim \mathrm{Bin}(1,p)$ indep.

Upper tail estimate $\quad \varepsilon = \delta p$

$$\underbrace{\mathbb{P}[\overline{X}_n \geq (1+\delta)p] \leq e^{-np\frac{\delta^2}{2+\delta}}}$$

Central Limit Theorem

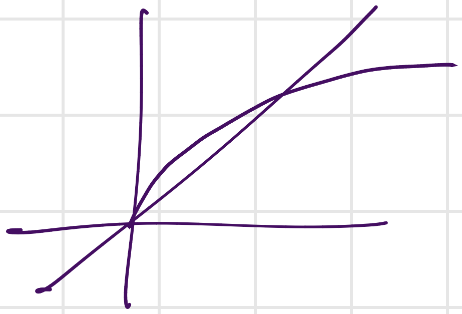
$\overline{X}_n = \frac{1}{n}(X_1 + \dots + X_n)$

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. square integrable r.v.'s, with expect. μ and variance $\sigma^2 > 0$. Let

$$\underbrace{Z_n} = \underbrace{\sqrt{n}\cdot\frac{\overline{X}_n - \mu}{\sigma}} = \underbrace{\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\,\sigma}}$$

Then $\underbrace{Z_n} \xrightarrow[n\to\infty]{D} \underbrace{Z} \sim N(0,1)$

Remark: $\mu = 0$, $\sigma^2 = 1$



LLN $\qquad \dfrac{\sum_{i=1}^{n} X_i}{n} \xrightarrow[\text{a.s.}]{P} 0$

$$\longrightarrow \quad \frac{X_1(\omega) + \dots + X_n(\omega)}{n} \longrightarrow 0$$

CLT $\qquad \dfrac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \xrightarrow{D} N(0,1)$

$$\longrightarrow$$

$$\frac{\sum_{i=1}^{n} X_i}{\lg(\lg n)} \longrightarrow 1$$

Proof: The cf characterizes the distribution.

$$\boxed{\text{Remark: } X_n \xrightarrow{D} X \iff \varphi_{X_n}(u) \longrightarrow \varphi_X(u)}$$

$\lim F_{X_n}(x) = F_X(x)$ $\forall x$ cont. of F_X

$\forall u \in \mathbb{R}$

$$\boxed{Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i}$$

$$Y_i = \frac{X_i - \mu}{\sigma}$$

$$\boxed{\begin{array}{l} \mathbb{E}[Y_1] = 0 \\ Var[Y_1] = 1 \end{array}}$$

$\boxed{Y_i \text{ iid}}$

$$\varphi_{Z_n}(u) = \mathbb{E}\left[e^{iu \cdot Z_n}\right] =$$

$$= \mathbb{E}\left[e^{iu \cdot \frac{1}{\sqrt{n}} \cdot \sum_{i=1}^{n} Y_i}\right] \overset{IND}{=}$$

$$= \prod_{i=1}^{n} \mathbb{E}\left[e^{iu \frac{1}{\sqrt{n}} Y_i}\right]$$

$$= \prod_{i=1}^{n} \varphi_{Y_i}(u/\sqrt{n}) = \left(\varphi_{Y_1}(u/\sqrt{n})\right)^n$$

$$\varphi_{Y_1}(x) = 1 + \cancel{\underbrace{\varphi'_{Y_1}(0)}} \cdot x + \underbrace{\varphi''_{Y_1}(0)} \frac{x^2}{2!} + o(x^2)$$

Taylor expand.

$$\varphi'_{Y_1}(0) = i\, \mathbb{E}[Y_1] = 0$$

$$\varphi''_{Y_1}(0) = (i)^2 \cdot \mathbb{E}[Y_i^2] = -1$$

$$\underbrace{\left(1+\frac{1}{x}\right)^x \to e}$$

$$\varphi_{Y_1}(x) = 1 - x^2/2! + o(x^2)$$

$$\varphi_{Y_1}\left(\frac{u}{\sqrt{n}}\right) = 1 - \frac{1}{2}\frac{u^2}{n} + o\left(\frac{1}{n}\right)$$

$$\left(\varphi_{Y_1}\left(\frac{u}{\sqrt{n}}\right)\right)^n = \left(1 - \frac{1}{2} u^2/n + o\left(\frac{1}{n}\right)\right)^n$$

$$\xrightarrow[n\to\infty]{} e^{-u^2/2} = \varphi_Z(u)$$

$$Z \sim N(0,1)$$