Lecture 4 Stoch. Math. Oct. 7, 2024

Independence of n-events ✓

Independence of σ-fields ✓

Bayes formula ✓

$\boxed{\limsup A_n, \ \liminf A_n}$

Borel Cantelli lemma

$A \perp\!\!\!\perp B$ are indep. $\Leftrightarrow$ $\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]$

$\underbrace{A_1, \dots, A_n}$ are independent $\Leftrightarrow$ $\forall \ \underbrace{J} \subseteq \{1, 2, \dots, n\}$

then $\mathbb{P}\left[\bigcap_{j \in J} A_j\right] = \prod_{j \in J} \mathbb{P}[A_j]$

$(n = 2, 3)$

Indep. of any family of events $\{A_t, t \in T\}$

These events are independent $\Leftrightarrow$ the events of every finite subset of these events, are independent

Thm: If A, B are independent, then A^c and B are independent.

Proof:

$$\mathbb{P}[A^c \cap B] \overset{?}{=} \mathbb{P}[A^c] \cdot \mathbb{P}[B]$$

$$\parallel$$

$$\mathbb{P}[B] - \mathbb{P}[A \cap B]$$

$\mathbb{P}[B] = \mathbb{P}[A \cap B] + \mathbb{P}[A^c \cap B]$

$$\parallel$$

$$\mathbb{P}[B] - \mathbb{P}[A] \cdot \mathbb{P}[B]$$

$$\parallel$$

$$\mathbb{P}[B]\,(1 - \mathbb{P}[A]) = \mathbb{P}[A^c] \cdot \mathbb{P}[B] \quad \blacksquare$$

σ_A = minimum σ-field that contains A

$= \{\emptyset, \Omega, A, A^c\}$

$\sigma_B = \{\emptyset, \Omega, B, B^c\}$

If A and B are indep., we have that F_A and σ_B are "independent", i.e.

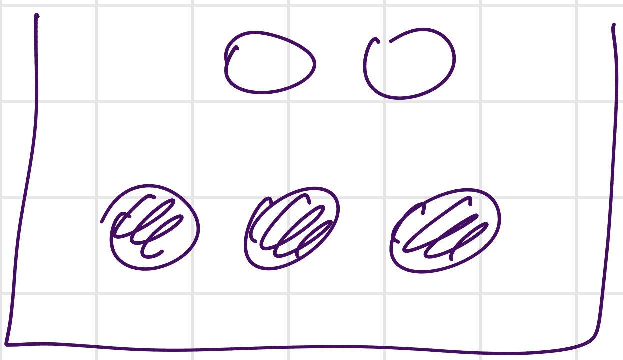
$\forall\, C \in \mathcal{F}_A$ and $D \in \mathcal{F}_B$, $C \perp\!\!\!\perp D$

<u>Definition</u> : Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be σ-fields on Ω , we say that they are independent

$\Leftrightarrow \forall\, A_1 \in \mathcal{A}_1 , A_2 \in \mathcal{A}_2, \ldots, A_n \in \mathcal{A}_n$

we have $\mathbb{P}\left[\bigcap_{k=1}^{n} A_k\right] = \prod_{k=1}^{n} \mathbb{P}[A_k]$.

$$\boxed{\mathbb{P}[A \mid \underbrace{B}_{\mathcal{A}}]} \rightsquigarrow \mathbb{P}[A \mid \mathcal{A}]$$



$B_1 = \{$ black 1^{st} extrac. $\}$

$B_2 = \{$ black 2^{nd} extrac. $\}$

$$\mathbb{P}[B_2] = \mathbb{P}[B_2 \mid B_1] \cdot \mathbb{P}[B_1] + \mathbb{P}[B_2 \mid B_1^c] \cdot \mathbb{P}[B_1^c]$$

$\mathbb{P}[B_1]$

$$\mathbb{P}[B_1 \mid B_2] = \,?$$

$$\boxed{\begin{array}{l}\mathbb{P}[B_1] > 0 \\ \mathbb{P}[B_2] > 0\end{array}}$$

$$\mathbb{P}[B_1 \cap B_2] = \underbrace{\mathbb{P}[B_1 \mid B_2]}\cdot \mathbb{P}[B_2]$$

$$\|$$

$$\mathbb{P}[B_2 \mid B_1]\cdot \mathbb{P}[B_1]$$

$$\boxed{\mathbb{P}[B_1 \mid B_2]} = \frac{\mathbb{P}[B_2 \mid B_1]\cdot \mathbb{P}[B_1]}{\mathbb{P}[B_2]}$$

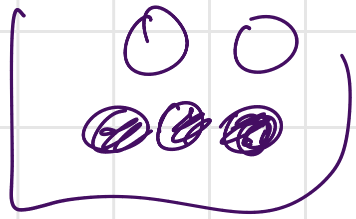
$$= \boxed{\frac{\mathbb{P}[B_2 \mid B_1]\cdot \mathbb{P}[B_1]}{\mathbb{P}[B_2 \mid B_1]\cdot \mathbb{P}[B_1] + \mathbb{P}[B_2 \mid B_1^c]\cdot \mathbb{P}[B_1^c]}}$$

Bayes' formula

$$\mathbb{P}[B_1] = \frac{3}{5} \quad \substack{< \\ = \\ >} \quad \mathbb{P}[B_1 \mid B_2]$$

$$\mathbb{P}[B_1 \mid B_2] = \frac{\mathbb{P}[B_2 \mid B_1] \cdot \mathbb{P}[B_1]}{\mathbb{P}[B_2 \mid B_1] \cdot \mathbb{P}[B_1] + \mathbb{P}[B_2 \mid B_1^c] \cdot \mathbb{P}[B_1^c]}$$

$$\mathbb{P}[B_1] = 3/5 = \mathbb{P}[B_2]$$



$$= \frac{1/2 \cdot 3/5}{3/5} = 1/2$$

$$\mathbb{P}[B_1] = 3/5 > 1/2 = \mathbb{P}[B_1 \mid B_2]$$

$$\mathbb{P}[B_1 \mid B_2 \cap B_3 \cap B_4] = 0$$

$\boxed{\text{Bayes' formula}}$

$A \in \mathcal{A}$ and $B_1, \dots, B_n \in \mathcal{A}$ s.t.

$$B_i \cap B_j = \emptyset \quad \forall i \neq j \quad \text{and} \quad \bigcup_i B_i = \Omega$$

$\{B_1, \dots, B_n\}$ a disjoint partition of Ω

$$\mathbb{P}[B_i \mid A] = \frac{\mathbb{P}[A \mid B_i] \cdot \mathbb{P}[B_i]}{\sum_{j=1}^{n} \mathbb{P}[A \mid B_j] \cdot \mathbb{P}[B_j]}$$

$\mathbb{P}[B_i]$ - prior probabilities

$\mathbb{P}[A \mid B_i]$ - likelihoods

$\mathbb{P}[B_i \mid A]$ - posterior probabilities

Medical tests H.I.V. E.L.I.S.A.

Healthy

(M, S) (Pos, Neg)

Ill

$\mathbb{P}[M \mid Pos) \stackrel{?}{=}$

$\mathbb{P}[S \mid Neg) \stackrel{?}{=}$

$$\mathbb{P}[M \mid Pos] = \frac{\mathbb{P}[Pos \mid M] \cdot \mathbb{P}[M]}{\mathbb{P}[Pos \mid M] \cdot \mathbb{P}[M] + \mathbb{P}[Pos \mid S] \mathbb{P}[S]}$$

$\mathbb{P}[M] = 1 - \mathbb{P}[S] = 0.000025$

$\mathbb{P}[S] = 0.999975$

$$\underbrace{\mathbb{P}[Pos|M] = 0.993}$$ sensibility of the test

$$\mathbb{P}[Neg|M] = 0.007$$

$$\boxed{\mathbb{P}[Neg|S] = 0.9999}$$ specificity of the test

$$\mathbb{P}[Pos|S] = 0.0001$$

$$\underbrace{\mathbb{P}[M] = 0{,}000025}$$

$$\mathbb{P}[M|Pos] = \frac{\mathbb{P}[Pos|M]\cdot\mathbb{P}[M]}{\mathbb{P}[Pos|M]\,\mathbb{P}[M] + \mathbb{P}[Pos|S]\cdot\mathbb{P}[S]}$$

$$= \frac{0.993 \cdot 0.000025}{0.993\cdot 0.000025 + 0.0001 \cdot 0.999975}$$

$$\simeq \underbrace{0.1988}$$

20 %

$$\underbrace{\mathbb{P}[S|Neg]} = \frac{\mathbb{P}[Neg|S]\cdot\mathbb{P}[S]}{\mathbb{P}[Neg|S]\cdot\mathbb{P}[S] + \mathbb{P}[Neg|M]\cdot\mathbb{P}[M]}$$

$$= \frac{0.9999 \cdot 0.999975}{0.9999\cdot 0.999975 + 0.007 \cdot 0.000025} \simeq \underbrace{0.99999}$$

Limit of a sequence of events

① Increasing sequences, $A_n \subseteq A_{n+1}$ $\forall n$

$$\lim_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} A_n$$

② Decreasing sequences, $B_n \supseteq B_{n+1}$ $\forall n$

$$\lim_{n\to\infty} B_n := \bigcap_{n=1}^{\infty} B_n$$

$\limsup A_n$, $\liminf A_n$

If $(A_n)_{n\in\mathbb{N}}$ is any sequence of events

decreasing sequence

$$A \ni \underbrace{\limsup_{n\to\infty} A_n}_{} := \lim_{n\to\infty}\left(\bigcup_{m=n}^{+\infty} A_m\right)$$

$$= \bigcap_{n=1}^{\infty}\left(\bigcup_{m=n}^{+\infty} A_m\right)$$

incres.

$$A \ni \liminf_{n\to\infty} A_n := \lim_{n\to+\infty}\left(\bigcap_{m=n}^{\infty} A_m\right)$$

$$= \bigcup_{n=1}^{\infty}\left(\bigcap_{m=n}^{\infty} A_m\right)$$

Properties:

$C \subseteq D$

$\forall x \in C \Rightarrow x \in D$

- $\liminf\limits_{n\to\infty} A_n \in \mathcal{A}$
- $\limsup\limits_{n\to\infty} A_n \in \mathcal{A}$

$A_n \subseteq \Omega$

- $\liminf A_n \subseteq \limsup A_n$

Proof: $\omega \in \liminf A_n \iff \omega \in \bigcup\limits_{n=1}^{\infty} \underbrace{\left(\bigcap\limits_{m=n}^{\infty} A_m \right)}_{B_n}$

$\iff \exists\, \bar{n} : \omega \in B_{\bar{n}} = \bigcap\limits_{m=n}^{\infty} A_m$

$\iff \exists\, \bar{n} : \omega \in A_m \quad \forall m \geq \bar{n}$

$\liminf = \emptyset$

$$\underbrace{\omega \in \limsup A_n}\quad \Longleftrightarrow \quad \omega \in \bigcap_{n=1}^{\infty} \Big(\underbrace{\bigcup_{m=n}^{\infty} A_m}_{C_n} \Big)$$

$$\Longleftrightarrow \forall n \in \mathbb{N}, \quad \omega \in C_n = \bigcup_{m=n}^{\infty} A_m$$

$$\Longleftrightarrow \underbrace{\forall n \in \mathbb{N}}, \ \exists\, m_n \geq n, \ \text{s.t.}\ \underbrace{\omega \in A_{m_n}}$$

ω that belongs to the sequence of events infinitely often

$$\limsup_n A_n = \{ \quad A_n \text{ i.o.} \}$$

$$\liminf A_n \subseteq \limsup A_n$$

$$\begin{matrix} \cup \\ \omega \end{matrix} \longrightarrow \in$$

$$\Rightarrow \exists \bar{n} : \omega \in A_m \quad \forall m \geq n$$

$$\Rightarrow \omega \in \{A_n\} \text{ i.o.}$$

Definition: Given $(A_n)_{n\in\mathbb{N}}$ of events

We say that $A = \lim_{n\to\infty} A_n$, where $A \in \mathcal{A}$,

$$\Longleftrightarrow \quad A = \liminf_n A_n = \limsup_n A_n$$

Exercise: If $(A_n)_{n\in\mathbb{N}}$ is a family of events show that

$$\underbrace{\limsup_{n\to\infty} \mathbb{1}_{A_n}} - \underbrace{\liminf_{n\to\infty} \mathbb{1}_{A}} = \_\_\_\_$$

$$= \mathbb{1}_{\left(\underbrace{\limsup_{n\to\infty} A_n} \setminus \underbrace{\liminf_{n\to\infty} A_n}\right)}$$

$$\boxed{\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}} \qquad \chi_A$$

Indicator function

Excercise 2

Given a family $(A_n)_{n\in\mathbb{N}}$

$$A \in \lim_{n\to\infty} A_n \iff \lim_{n\to\infty} \mathbb{1}_{A_n} = \mathbb{1}_A$$

$$\limsup_{n\to\infty} x_n = \inf_{n\geq 0} \left(\sup_{m\geq n} x_m \right)$$

$$\liminf_{n\to\infty} x_n = \sup_{n\geq 0} \left(\inf_{m\geq n} x_m \right)$$