Lecture 6 Stoc. Meth. Oct. 14, 2024

Random variable (function)

$$\Omega \xrightarrow{X} \mathbb{R} \qquad \forall B \in \mathcal{B}(\mathbb{R}),\ X^{-1}(B) \in \mathcal{A}$$

$$(\Omega, \mathcal{A}, \mathbb{P}) \qquad (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_X)$$

$$\forall B,\quad \mathbb{P}[X^{-1}(B)] = \mu_X(B)$$

μ_X distribution of X, or law of X.

You can prove that $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_X)$ is a prob. space.

Discrete r.v. $\exists N$ finite or countable $N \subseteq \mathbb{R}$ s.t. $\mathbb{P}[X \in N] = 1$

$$p(x) = \mathbb{P}[X = x] \quad \forall x \in \mathbb{R}$$

density of X (discrete density)

- $p(x) = 0 \quad \forall x \in \mathbb{R} \setminus N$

- $p(x) \geq 0 \quad \forall x \in N$

- $\sum_{x \in \mathbb{R}} p(x) = \sum_{x \in N} p(x) = 1$

if p is a density, the previous properties are satisfied

On the converse, any function p that satisfy •, •, •, is a density.

μ_X discrete $\Longleftrightarrow$ $p(x)$

Example

$N = \{0, 1, \dots, n\}$ $|N| = n+1$ finite

is a density?

$$\begin{cases} p(k) = 0 & \text{if } k \neq 0, 1, \dots, n \\ p(k) = \binom{n}{k} p^k (1-p)^{n-k} \end{cases}$$

$p \in [0,1]$

$k \in N$

$$p(k) = \binom{n}{k} \underbrace{p^k}_{\geq 0} \underbrace{(1-p)^{n-k}}_{\geq 0} \qquad p \in [0,1]$$

$$k \in \{0, 1, \dots, n\}$$

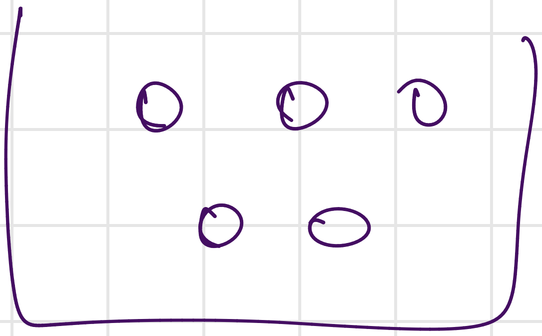
$$k \geq 0$$

$$k \leq n$$

$$\Rightarrow \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\begin{cases} n! = n(n-1)(n-2)\cdots 2 \cdot 1 \\ \quad n \in \mathbb{N} \\ 0! = 1 \end{cases}$$

__Remark__

$D =$ 

$|D| = n$

$0 \leq k \leq n$

$$\underbrace{\binom{n}{k}}_{\text{number of subsets of cardinality equal to } k}$$

X is a binomial (n, p) r.v. $(n \in \mathbb{N}$, $p \in [0,1])$

iff
$$\begin{cases} p(k) = 0 & k \notin \{0, \dots, n\} \\ p(k) = \binom{n}{k} p^k (1-p)^{n-k} & k \in \{0, 1, \dots, n\} \end{cases}$$

$$\boxed{\sum_{k=0}^{n} p(k) = 1}$$

$$1 \overset{??}{=} \sum_{k=0}^{n} \binom{n}{k} \overset{a^k}{\underbrace{p}}{}^{k} \, \overset{b^{n-k}}{\underbrace{(1-p)}}{}^{n-k}$$

$$= (\underset{a}{p} \underset{+}{+} \underset{b}{(1-p)})^n = 1^n = 1$$

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}}$$

$n = 2$, $n = 3$, $\dots$ $\boxed{n} \Rightarrow \boxed{n+1}$

Remarke

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k \, b^{n-k}$$

$$\boxed{a = b = 1}$$

$$\Rightarrow \quad 2^n = \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 1^{n-k}$$

$$\boxed{2^n = \sum_{k=0}^{n} \binom{n}{k}}$$

$$|\Omega| = n \qquad |2^\Omega| = 2^{|\Omega|} = 2^n$$

$$\binom{n}{0} = 1 \qquad \emptyset$$

100

$$\binom{n}{1} = n \qquad \binom{100}{2} = \frac{100 \cdot 99}{2}$$

$$\binom{n}{2} = \frac{n(n-1)}{2} \quad \ldots \quad \binom{n}{n} = 1$$

Ω

$X : \Omega \to \mathbb{R}$ Law

$\mu_X : \mathcal{B}(\mathbb{R}) \to [0,1]$

Borel sets

is a probability

$A \subseteq B \Rightarrow \mathbb{P}[A] \le \mathbb{P}[B]$

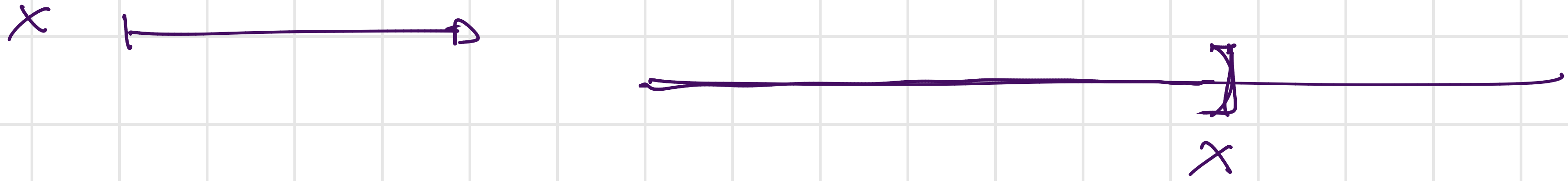
continuity

$\lim_{n \to \infty} A_n = A \Rightarrow \lim_{n \to \infty} \mathbb{P}[A_n] = \mathbb{P}[A]$

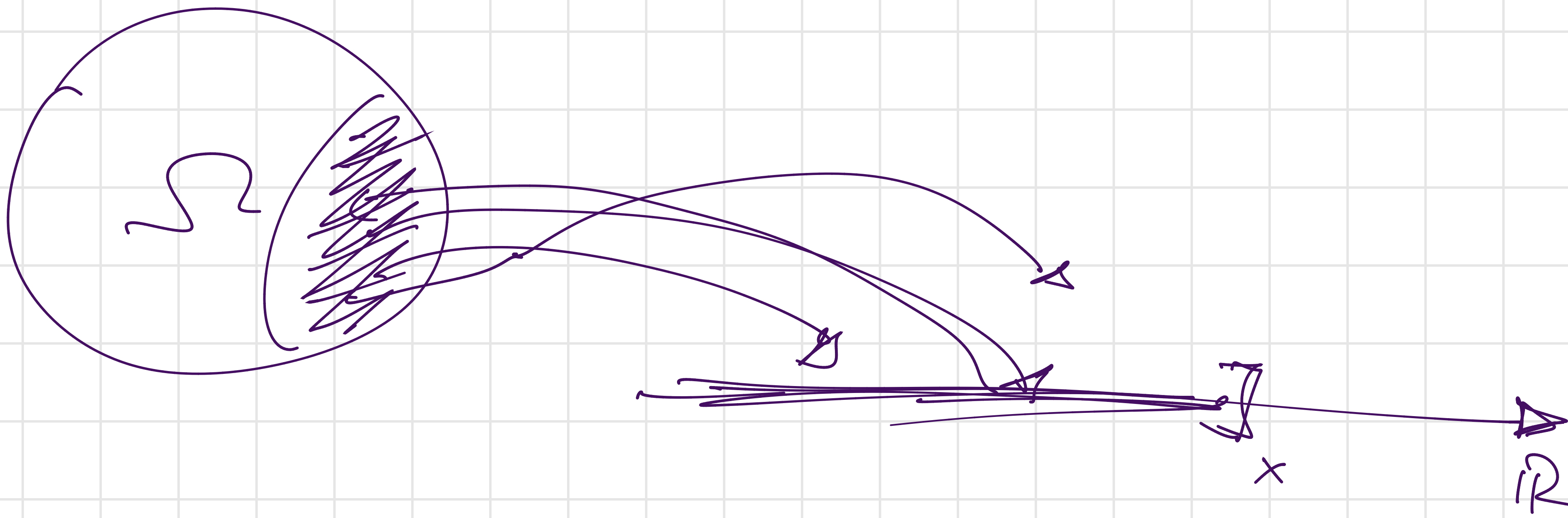
$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_X)$

$F_X : \mathbb{R} \to [0,1]$

$$x \longmapsto F_X(x) = \mathbb{P}[X \le x]$$
$$= \mathbb{P}[X \in (-\infty, x]]$$
$$= \mu_X((-\infty, x])$$



$$F_X(x) = \mathbb{P}[X \in (-\infty, x]]$$



F_X is the <u>distribution function</u> of X

μ_X Law of X

$$F_X(x) = \mathbb{P}[X \le x]$$

$$F(0) = \mathbb{P}[X \le 0]$$

$$F(1) = \mathbb{P}[X \le 1]$$

It is possible to prove that: F_X

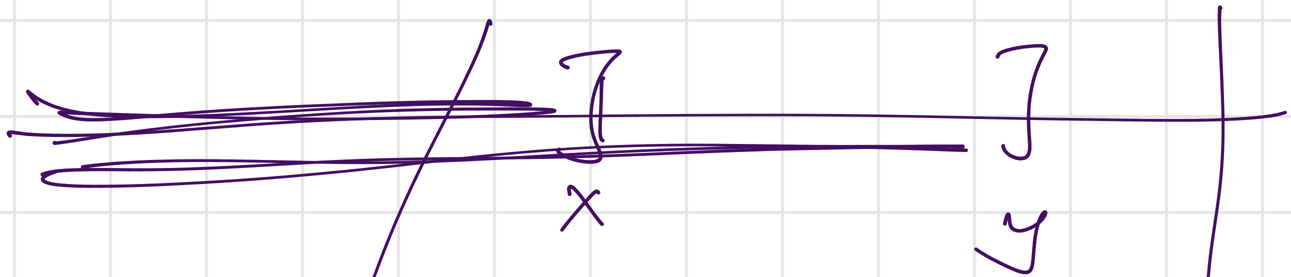
1. $F_{X}(x)$ is non decreasing, i.e.

$$\boxed{x < y} \Rightarrow F_X(x) \le F_Y(y)$$

$x, y \in \mathbb{R}$

$$\boxed{\begin{aligned} F_X(x) &= \mu_X((-\infty, x]) \\ F_X(y) &= \mu_X((-\infty, y]) \end{aligned}}$$

$$x < y \Rightarrow (-\infty, x] \subseteq (-\infty, y]$$



Since μ_X is a probability

$$\Rightarrow \mu_X((-\infty, x]) \le \mu_X((-\infty, y])$$

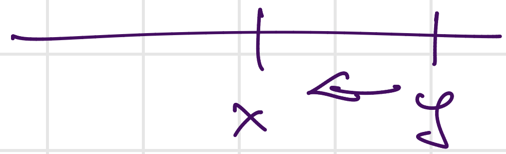
monotonicity

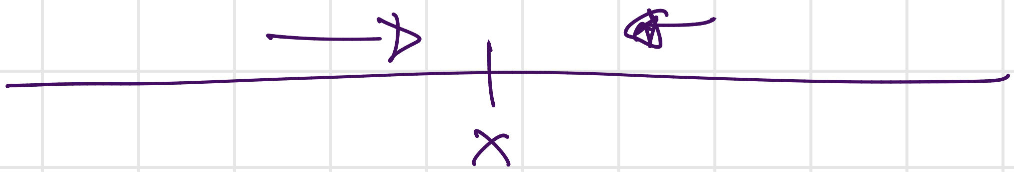
$$\Rightarrow \underbrace{F_X}\overset{\in \mathbb{R}}{(x)} \le \underbrace{F_X}\overset{\in \mathbb{R}}{(y)}$$

∎

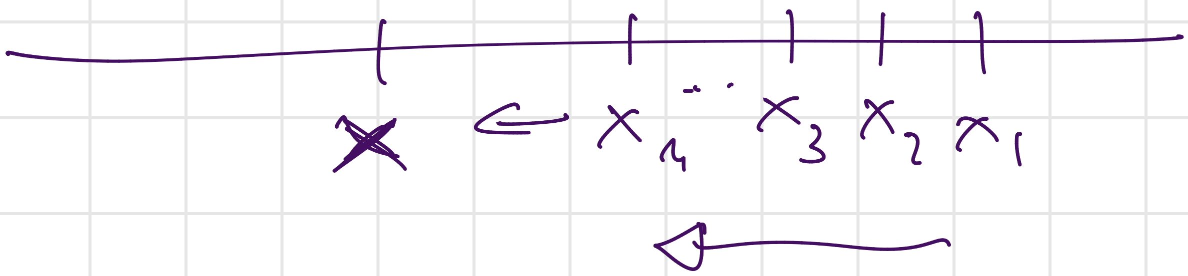
2. $\boxed{F_X \text{ is right continuous}}$, i.e.

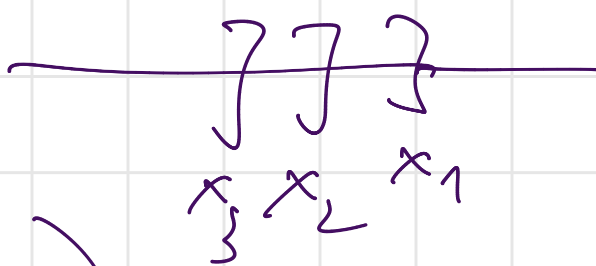
$$\lim_{y \downarrow x} \underbrace{F_X(y)} = \underbrace{F_X(x)}$$





$$\lim_{y \downarrow x} F_X(x) = \underbrace{\lim_{\substack{x_n \to x \\ x \le x_n}} F_X(x_n)} \qquad \forall \text{ sequence } x_n$$





$$F_X(x_n) = \mu_X\left((-\infty, x_n]\right)$$

$A_n = (-\infty, x_n]$ $\quad (A_n)_{n \in \mathbb{N}}$ is a

<u>decreasing</u> or increasing family of sets?

$$\underbrace{\lim_{n\to\infty} \mu_X(A_n)} = \underbrace{\mu_X\left[\lim_{n\to\infty} A_n\right]}$$

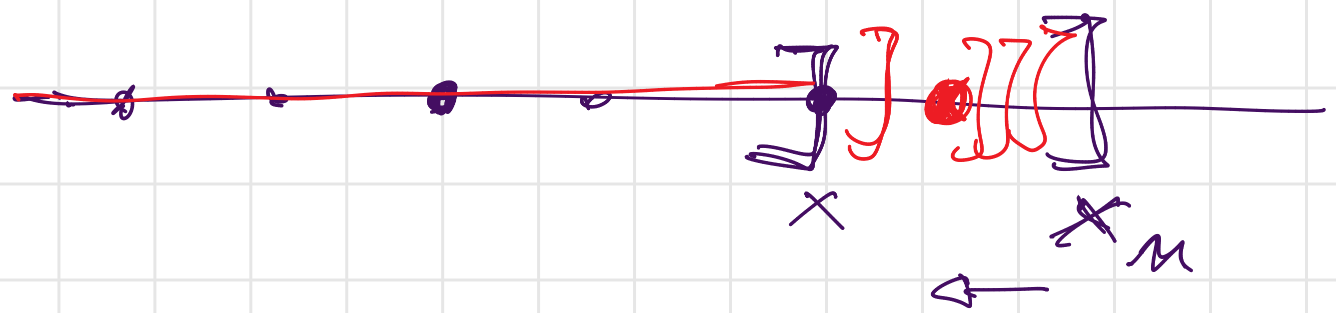
where $\underbrace{\lim_{n\to\infty} A_n} = \bigcap_{n=1}^{\infty} A_n$

decreasing sequence

if $\boxed{\lim_{n\to\infty} A_n = (-\infty, x]}$

$$\boxed{\lim_{n\to\infty} F_{X_n}(x_n) = F_{X_n}(x)}$$

$$\underbrace{\bigcap_{n=1}^{\infty} (-\infty, x_n]}_{A} \;\overset{\subseteq}{\underset{\supseteq}{=}}\; (-\infty, x]$$



$\omega \in (-\infty, x] \Rightarrow \omega \in (-\infty, x_n] \quad \forall n$

$\omega \in \bigcap_{n=1}^{\infty} (-\infty, x_n]$

3. $\lim_{x \to -\infty} F_X(x) = 0$, $\lim_{x \to +\infty} F_X(x) = 1$

F_X distribution function satisfies 1., 2., 3.

Thm 1: The distribution function F_X characterizes the law μ_X, i.e. if X and Y are two real r.v.'s with $\boxed{F_X = F_Y}$, then $\mu_X = \mu_Y$.

$\mu_X : \mathcal{B}(\mathbb{R}) \longrightarrow [0,1]$

$F_X : \underbrace{\mathbb{R}} \longrightarrow [0,1]$

Thm 2. A function F is the distribution function of a (unique) r.v. if and only if it satisfies the properties 1., 2., 3.

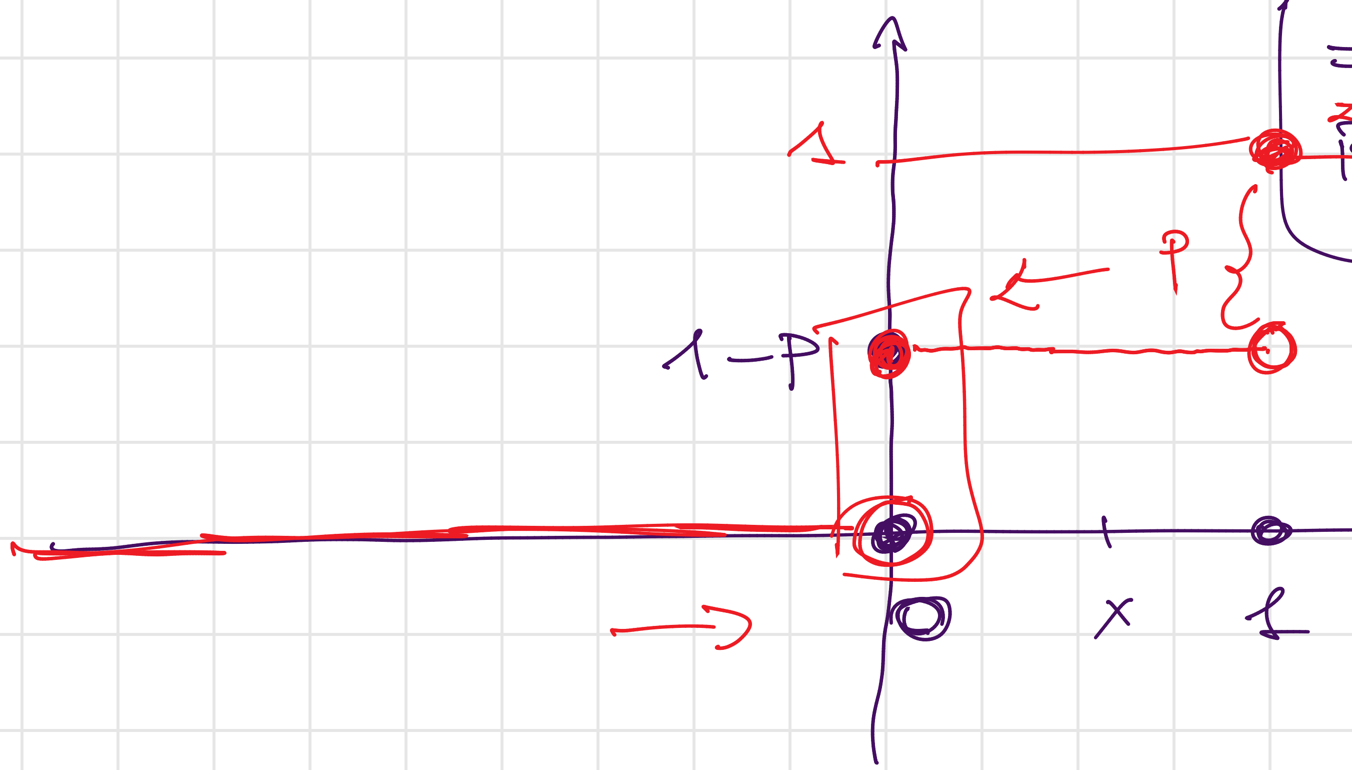
Ex. 1

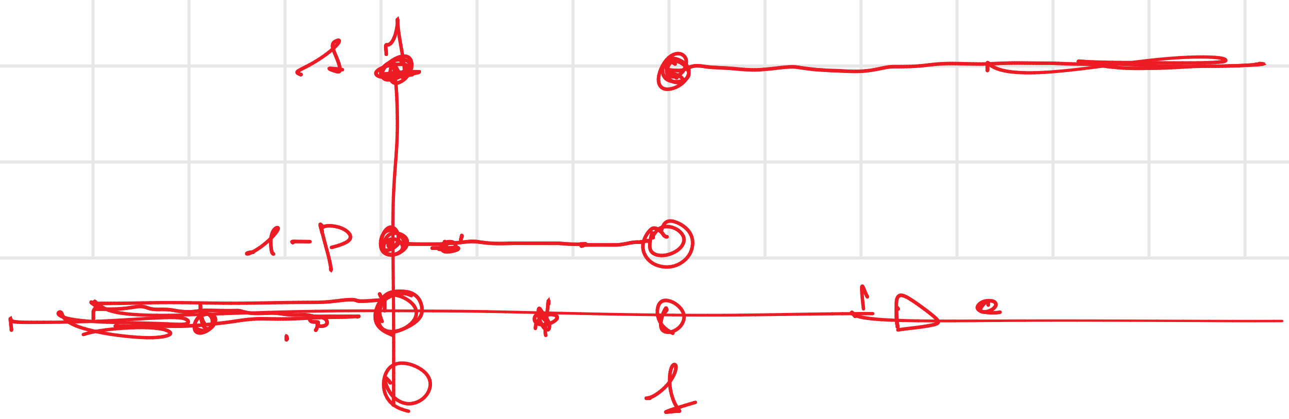
X is Binomial $(1, p)$ random variable

$$N = \{0, 1\}$$

$$\begin{cases} \mathbb{P}[X=0] = 1-p \\ \mathbb{P}[X=1] = p \\ \mathbb{P}[X=x] = 0 & \forall x \neq 0, 1 \end{cases}$$

$$F_X(x) = \mathbb{P}[X \leq x] = \begin{cases} 0 & x<0 \\ \mathbb{P}[X \leq 0] = \mathbb{P}[X<0] + \mathbb{P}[X=0] = 0 + 1 - p & x=0 \\ \mathbb{P}[X \leq x] = \mathbb{P}[X=0] & 0<x<1 \\ = 1 & x \geq 1 \end{cases}$$





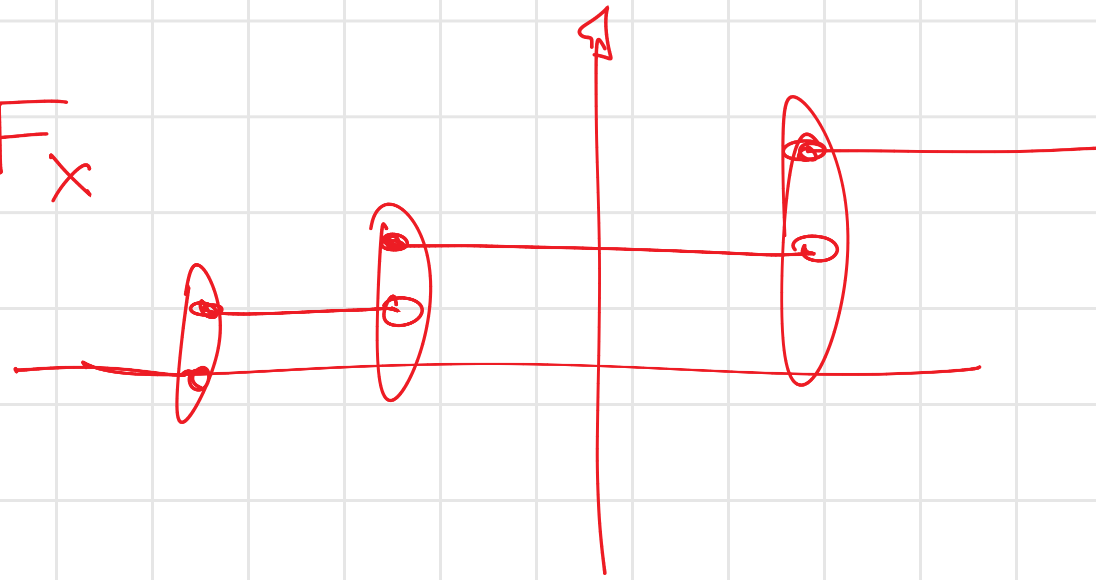
1. YES

2. YES

3. YES

X discrete $\Rightarrow$

F_X



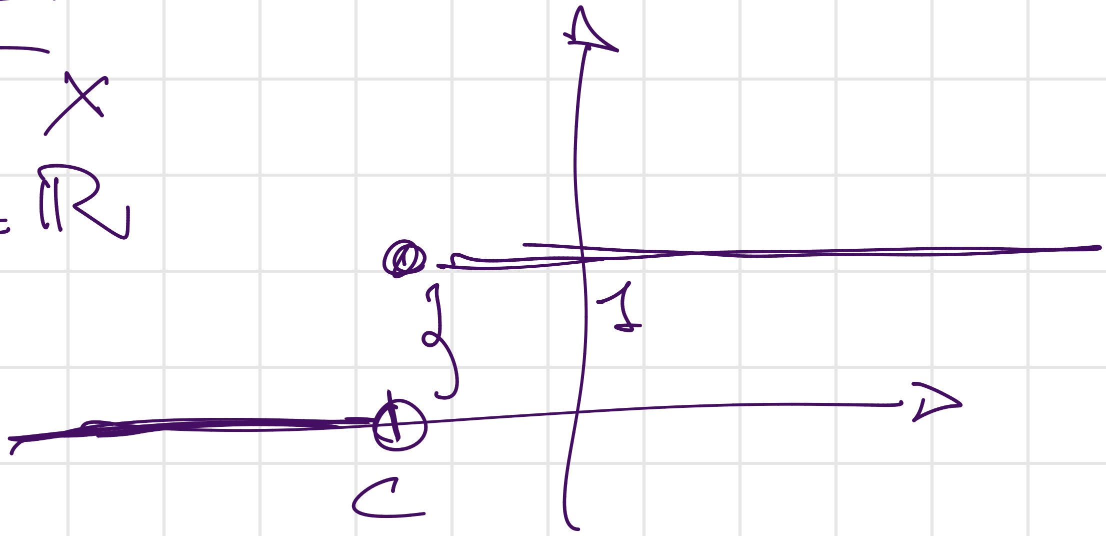
r.v.

$X = \boxed{c}$

F_X

$c \in \mathbb{R}$

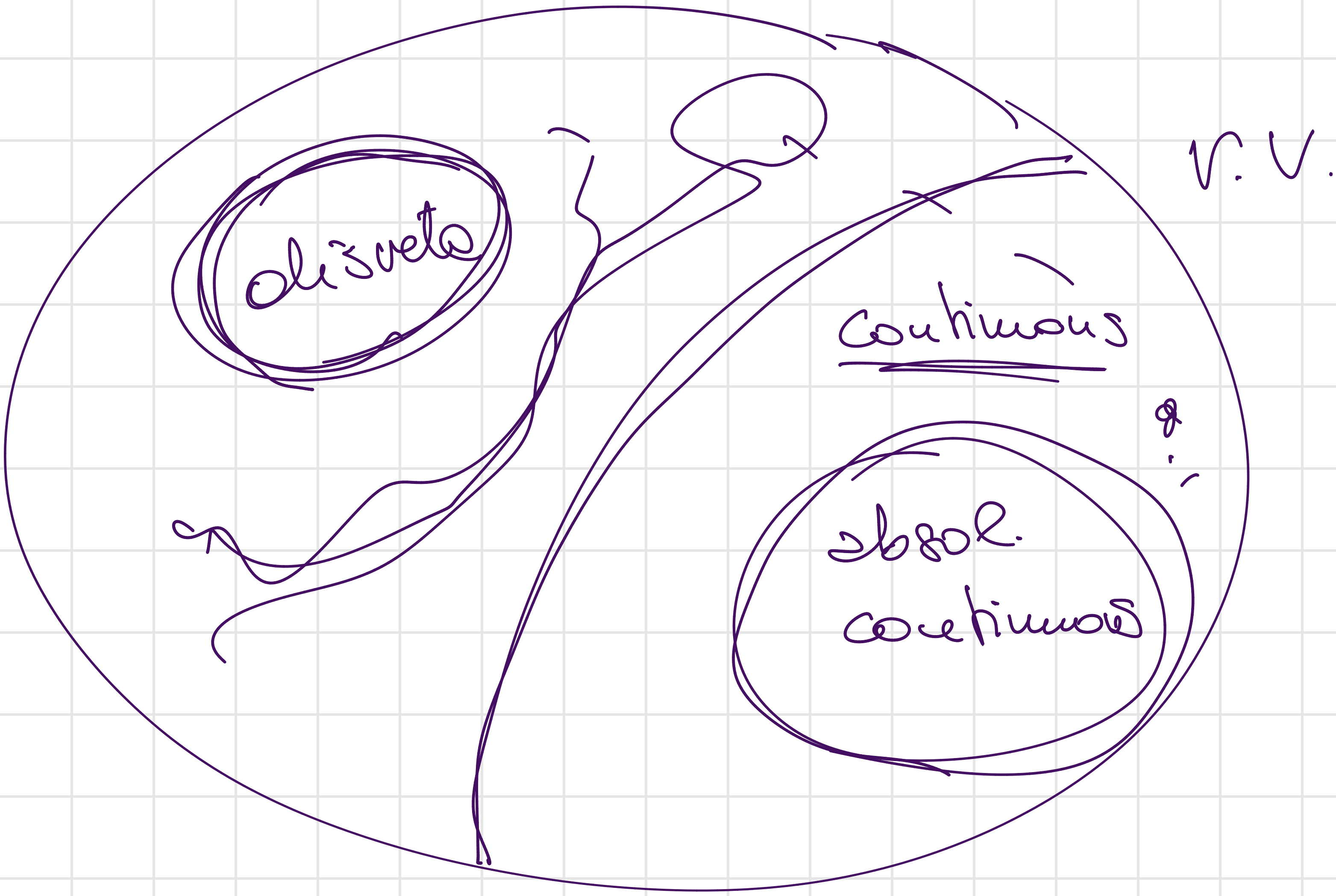
$P[X = \boxed{c}] = 1$



$N = \{c\}$

$$\boxed{c} : \Omega \longrightarrow \mathbb{R}$$
$$\omega \longmapsto c$$

Absolutely continuous random variables



X is a continuous r.v. $\iff$ F_X is continuous

Absolutely continuous r.v.

Def: Let X be a real valued r.v.
We say that X is absolutely continuous if there exists a function $f_X : \mathbb{R} \to [0, +\infty)$

s.t. $\forall x \in \mathbb{R}$

$$F_X(x) = \mathbb{P}[X \leq x] = \int_{-\infty}^{x} f_X(t)\,dt$$

f_X is called the density of the r.v. X.