Lecture 10 Stoch. Meth. Oct. 27th, 2024

Random vectors

$$X : \Omega \longrightarrow \mathbb{R}^d \qquad d \geq 1$$

Discrete Random Vector

$X : \Omega \to \mathbb{R}^d$ is discrete if $\exists\, N \subseteq \mathbb{R}^d$ finite or at most countable s.t.

$$\mathbb{P}[X \in N] = 1$$

$$\underbrace{p_X(x_1, \dots, x_d)}_{\text{joint (discrete) density}} := \mathbb{P}[X_1 = x_1, X_2 = x_2, \dots, X_d = x_d]$$

$$\mathbb{P}[X \in B] = \sum_{(x_1,\dots,x_d) \in B} p_X(x_1, \dots, x_d)$$

$B \in \mathcal{B}(\mathbb{R}^d)$

$p_X \rightsquigarrow p_{X_1}, \dots, p_{X_d}$ marginal densities? ?

If $X : \Omega \to \mathbb{R}^d$ is discrete, then any component

$X_1, \dots, X_d$ is discrete.

and

$$p_{X_1}(x_1) = \mathbb{P}[\underbrace{X_1 = x_1}] = \qquad \bigcup_{x_2 \in \mathbb{R}} \{X_2 = x_2\}$$

$$= \mathbb{P}[\underbrace{X_1 = x_1, \overbrace{X_2 \in \mathbb{R}}^{\Omega}, \dots, \overbrace{X_d \in \mathbb{R}}^{\Omega}}]$$

$$= \sum_{x_2, \dots, x_d \in \mathbb{R}} \mathbb{P}[X_1 = x_1, X_2 = x_2, \dots, X_d = x_d]$$

$$= \sum_{x_2, x_3, \dots, x_d \in \mathbb{R}} \underbrace{p_X(x_1, x_2, x_3, \dots, x_d)}$$

marginal density

Proposition Let $X : \Omega \to \mathbb{R}^d$ discrete.

Then its components $X_1, X_2, \dots, X_d$ are

independent $\iff$ $\forall\, x_1, \dots, x_d \in \mathbb{R}$

$$\underbrace{p_X(x_1, \dots, x_d)} = \underbrace{\prod_{i=1}^{d} p_{X_i}(x_i)}$$

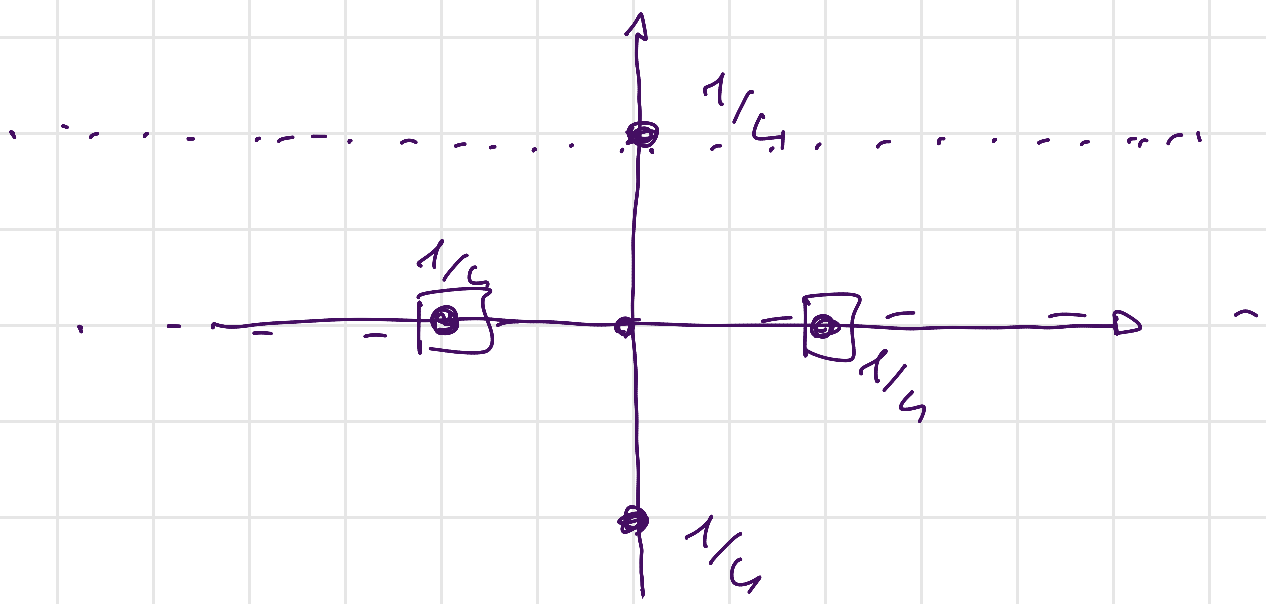
Example: $d=2$ $|N|=4$

$\to (X,Y)$ is discrete $N=\{(1,0),(0,1),(-1,0),(0,-1)\}$

$\cancel{(X_1,X_2)}$ prob. is uniform

$\forall (i,j)\in N$

$\mathbb{P}[(X,Y)=(i,j)]=\frac{1}{4}$



Joint density?

- $P_{X,Y}(x,y)=0 \quad \forall (x,y)\notin N$
- $P_{X,Y}(x,y)=1/4 \quad \forall (x,y)\in N$

$(0,1)$
$(1,0)$
$(0,-1)$
$(-1,0)$

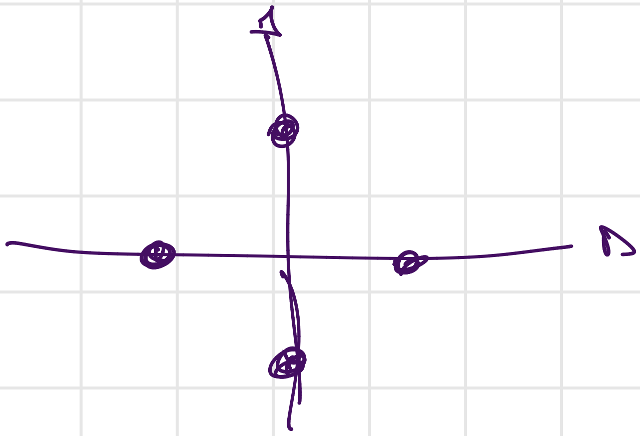
$P_{X,Y} \rightsquigarrow P_X, P_Y$

$$\boxed{P_Y(y)=\sum_{x\in\mathbb{R}} P_{X,Y}(x,y)}$$

$$P_Y(y)=\begin{cases} 0 & y\neq 0,1,-1 \\ P_{X,Y}(0,1)=\frac{1}{4} & y=1,\ y=-1 \\ P_{X,Y}(1,0)+P_{X,Y}(-1,0)=\frac{1}{2} & y=0 \end{cases}$$

$y=-5$ $\quad P_{X,Y}(x,-5) \equiv 0$

$\sum_x P_{X,Y}(x,1)$



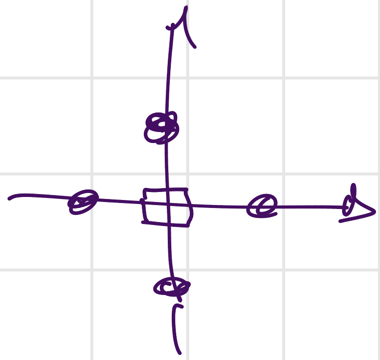
$$P_{X,Y}(x,y) = 1/4 \qquad (x,y) \in N$$

$$P_Y(y) = \begin{cases} 0 & y \neq 0, +1, -1 \\ 1/4 & y = 1,\ y = -1 \\ 1/2 & y = 0 \end{cases}$$

$$P_X(x) = \begin{cases} 0 & x \neq 0, 1, -1 \\ 1/4 & x = 1, -1 \\ 1/2 & x = 0 \end{cases}$$

X and Y are independent?

$$\iff \quad P_{X,Y}(x,y) = P_X(x)\, P_Y(y) \qquad \forall (x,y) \in \mathbb{R}^2$$



$$P_{X,Y}(0,0) = 0 \qquad\qquad \underset{\underset{0}{\shortparallel}}{P_{X,Y}(0,0)} \neq \underbrace{P_X(0) \cdot P_Y(0)}_{1/4}$$

$$P_X(0) = 1/2, \quad P_Y(0) = 1/2$$

X and Y are not independent !!

Absolutely continuous Random Vectors

Def: We say that a random vector $X : \Omega \to \mathbb{R}^d$ is absolutely continuous if there exists $f_X : \mathbb{R}^d \longrightarrow [0,+\infty)$ s.t. $\forall B \in \mathcal{B}(\mathbb{R}^d)$

$$\mathbb{P}[X \in B] = \underbrace{\int \cdots \int}_{B} f_X(t_1, .., t_d)\, dt_1 \dots dt_d$$

For example, $B = [a_1, b_1] \times [a_2, b_2]$,

$$\mathbb{P}[X \in B] = \int_{a_1}^{b_1} dt_1 \int_{a_2}^{b_2} dt_2 \, f_X(t_1, t_2)$$

f_X is called the joint density of $X = (X_1, .., X_d)$

If X is abs. cont. $\Rightarrow X_1, X_2, \dots, X_d$ are abs. cont. random variables

$\forall\, i \in \{1, \dots, d\}$

$$f_{X_i}(x_i) = \int_{\mathbb{R}^{d-1}} \cdots \int f_X(t_1, \dots, t_{i-1}, x_i, t_{i+1}, \dots, t_d)\, dt_1 \dots dt_{i-1}\, dt_{i+1} \dots dt_d$$

density of X_i

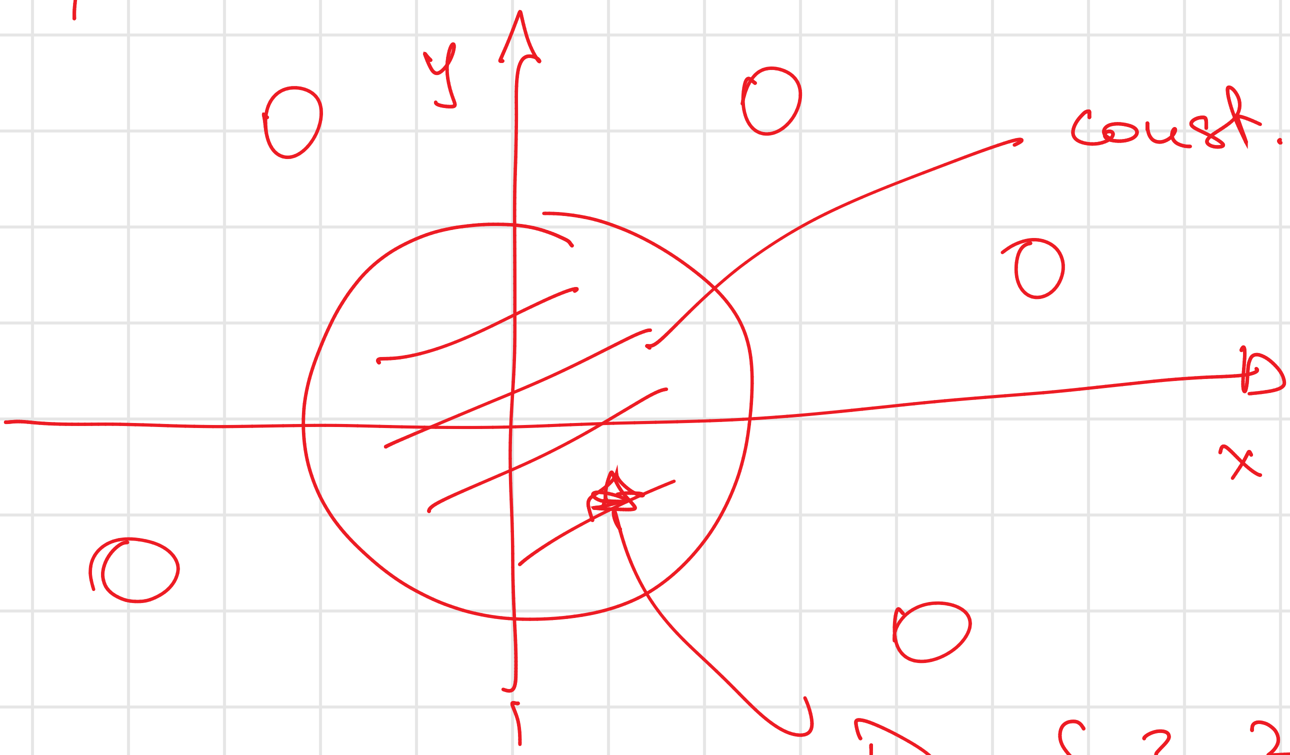
<u>Prop:</u> Let $X_1, \dots, X_d$ be abs. continuous r.v.'s. Then they are independent

$\iff$ $X = (X_1, \dots, X_d)$ is abs. cont. and

$$\underbrace{f_X(x_1, \dots, x_d) = \prod_{i=1}^{d} f_{X_i}(x_i)} \quad \boxed{\text{a.e.}}$$

$f_{XY} > 0 \quad \forall (x,y) \in A$

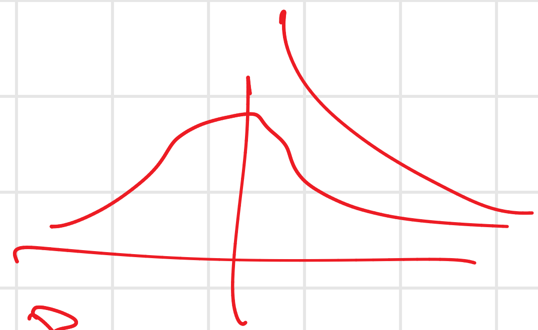
$$f_{X,Y}(x,y) = \begin{cases} \text{const.} > 0 & (x,y) \in \{x^2+y^2 \le 1\} \\ = 0 & \text{otherwise} \end{cases}$$



$D = \{x^2+y^2 \le 1\}$

$(x,y) \longrightarrow f(x,y)$

$(x,y,f(x,y))$



const. = ?

$$\underbrace{\iint_{\mathbb{R}^2} f_{X,Y}\,dx\,dy = 1}$$

$$\iint_{\mathbb{R}^2} f = \iint_D \text{const}\,dx\,dy = \text{const. Area}(D)$$

$$= \text{const} \cdot \pi = 1$$

$$\text{const} = 1/\pi$$

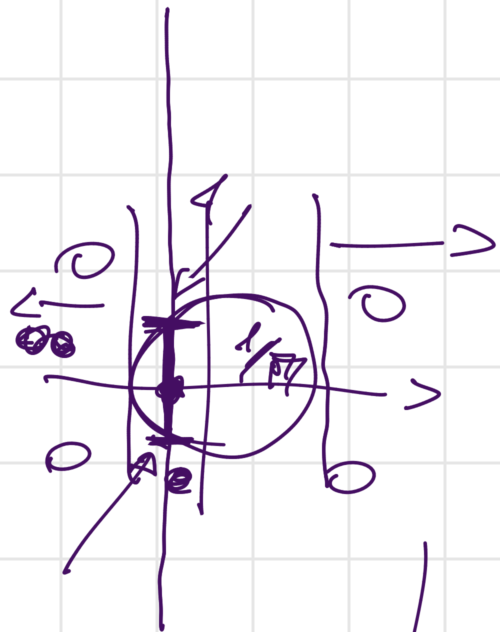
$$\underbrace{f_{X,Y}}(x,y) = \begin{cases} 0 & (x,y) \notin D \\ \underbrace{1/\pi} & (x,y) \in D \end{cases}$$

X, Y are indep.?

X and Y are indep. $\Leftrightarrow f_{X,Y} = f_X \cdot f_Y$

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,t)\,dt$$

$$f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(t,y)\,dt$$



X . $x < -1$, $x > 1$ $\quad f_{X,Y}(x,y) = 0$

$$\boxed{f_X(x) = \int_{\mathbb{R}} \underbrace{f_X(x,t)}_{0}\,dt} = \begin{cases} 0 & x < -1 \text{ or } x > 1 \\ \int_{\mathbb{R}} f_X(x,t)\,dt & -1 < x < 1 \end{cases}$$

$-1 < x < 1$

$$= \int_{-\infty}^{\circ} 0\,dt + \int_{\circ}^{\circ\circ} \frac{1}{\pi}\,dt + \int_{\circ\circ}^{+\infty} 0\,dt$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dt = \boxed{\frac{1}{\pi} 2\sqrt{1-x^2}}$$

$y^2 = 1 - x^2$

$y = \pm\sqrt{1-x^2}$

$\underbrace{\mathbb{E}[X \cdot Y]}\ = \ ?$ $\qquad\qquad$ $\underbrace{X, Y \in L^2 \Rightarrow X \cdot Y \in L^1}$

$\underline{X : \Omega \to \mathbb{R}^d \text{ r.vector}}$, $\quad g : \mathbb{R}^d \to \mathbb{R}$ meas.

$Y := g(X) : \Omega \to \mathbb{R}$ $\quad$ r.v.

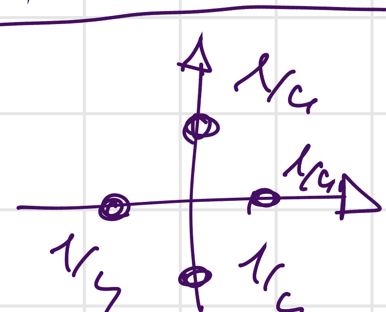
(i) If X is discrete, $\quad \underbrace{\mathbb{E}[g(X)] = \sum_{x \in \mathbb{R}^d} g(x) \cdot P_X(x)}$

(ii) If X is abs. cont.,

$$\mathbb{E}[g(X)] = \int \cdots \int_{\mathbb{R}^d} g(t_1, \dots, t_d) \cdot f_X(t_1, \dots, t_d)\, dt_1 \dots dt_d$$

Ex: (X, Y)

$g(x,y) = x \cdot y$



$\boxed{\begin{matrix} (1,0), (0,1) \\ (-1,0), (0,-1) \end{matrix}}$

$g(x,y) = x \cdot y$

$$0 = \mathbb{E}[X \cdot Y] = \sum_{(x,y) \in \mathbb{R}^2} g(x,y) \cdot P_{X,Y}(x,y) = \underbrace{0}_{} + g(-1,0) P_{X,Y}(-1,0)$$

$$= g(0,1) \cdot P_{X,Y}(0,1) + g(1,0) P_{X,Y}(1,0) + g(0,-1) P_{X,Y}(0,-1)$$

(each of $\mathbb{E}[X \cdot Y]$, $\sum$, $g(0,1)$, $g(1,0)$, $g(0,-1)$, $g(-1,0)$ marked $= 0$)

Prop: Let $\underbrace{X, Y \in L^1(\Omega)}$ and assume that they are independent. Then $X \cdot Y \in L^1$ and

$$\boxed{E[XY] = E[X] \cdot E[Y]}$$

Corollary: If X and Y are indep.

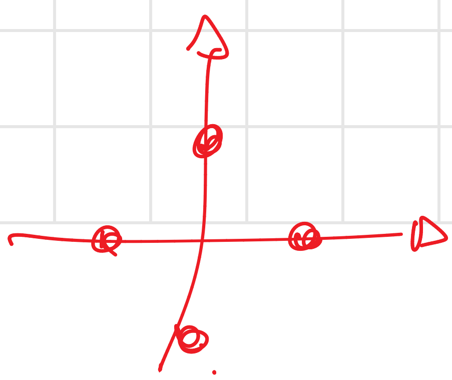
$$\mathrm{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$

$$= E[X \cdot Y] - E[X] \cdot E[Y] = 0$$

$$\Rightarrow \underbrace{\mathrm{Var}[X+Y]} = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}(X,Y)$$

$$= \underbrace{\mathrm{Var}[X] + \mathrm{Var}[Y]}$$

$\underline{E[XY] = E[X] \cdot E[Y]} \not\Rightarrow$ X and Y are indep.?



X and Y are not indep.

X, Y s.t. $\mathrm{Cov}(X,Y)=0$ $\quad \left(\mathbb{E}[XY] = \mathbb{E}[X]\cdot\mathbb{E}[Y] \right)$

are uncorrelated.

$$\underline{\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]}$$

$$\mathbb{E}[\alpha X + \beta Y] = \alpha\,\mathbb{E}[X] + \beta\,\mathbb{E}[Y]$$

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] \qquad (X \perp\!\!\!\perp Y)$$

$$\begin{aligned}\mathrm{Var}[c\cdot X] &= \mathbb{E}\left[\left(c\cdot X - \mathbb{E}[cX]\right)^2\right] \\ &= c^2\cdot\mathbb{E}\left[\left(X-\mathbb{E}[X]\right)^2\right] \\ &= c^2\cdot\mathrm{Var}[X]\end{aligned}$$

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$$\mathrm{Var}[X+Y] = \mathrm{Var}[X-Y]$$

Exercise: $X \sim \mathrm{Exp}(\lambda)$ $\quad X \perp\!\!\!\perp Y$

$Y \sim \mathrm{Bin}(1,p)$ r.v.

$$\underbrace{Z = X\cdot Y}$$

- $\mathbb{E}[Z]$, $\mathrm{Var}[Z]$?
- $F_Z(z) = \mathbb{P}[Z \le z]$