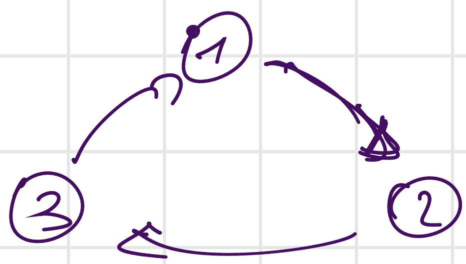
Lecture 21 — Stoch. Math. — Dec. 5^{th}, 2024

Markov Chain

$$X_0, X_1, X_2, \dots, X_n, \dots$$

discrete-time stochastic process

$X_n \in S$ finite, countable space



transitions prob. $\mathbb{P}[X_{n+1} = x \mid X_n = y] = P_{yx}$

$P = (P_{yx})_{y,x \in S}$ is a matrix

Stoch. matrix
$$\begin{cases} P_{yx} \geq 0 & \forall x, y \in S \\ \sum_{x \in S} P_{yx} = 1 \end{cases}$$

Chapman - Kolm. equation

$$\underbrace{P^{m+n}_{yx} := \mathbb{P}[X_{m+n} = x \mid X_0 = y]}$$ m+n step transition prob.

$$P^{m+n} = P^m \cdot P^n$$

$$\underbrace{P^{m+n}_{yx}} = \sum_{z\in S} \underbrace{P^m_{yz}}\,\underbrace{P^n_{zx}}$$

n-times
$\overbrace{P\cdot P \cdots P}$
$=$

$$\underbrace{P^2_{yx}} = (\underbrace{P^2})_{\underbrace{yx}} \quad \ldots \quad \underbrace{P^n_{yx}} = (\underbrace{P^n})_{yx}$$

Exercise: prove that $\forall n \in \mathbb{N}$ P^n is a stochastic matrix $\boxed{(\nu_1, .., \nu_k)\, P^n}$ ν

$$\nu_y := \mathbb{P}[X_0 = y]$$

$$\underbrace{\mathbb{P}[X_n = x]} = \sum_{y\in S} \underbrace{\mathbb{P}[X_n = x \mid X_0 = y]} \cdot \underbrace{\mathbb{P}[X_0 = y]}$$

$$= \sum_y (P^n)_{yx} \cdot \nu_y$$

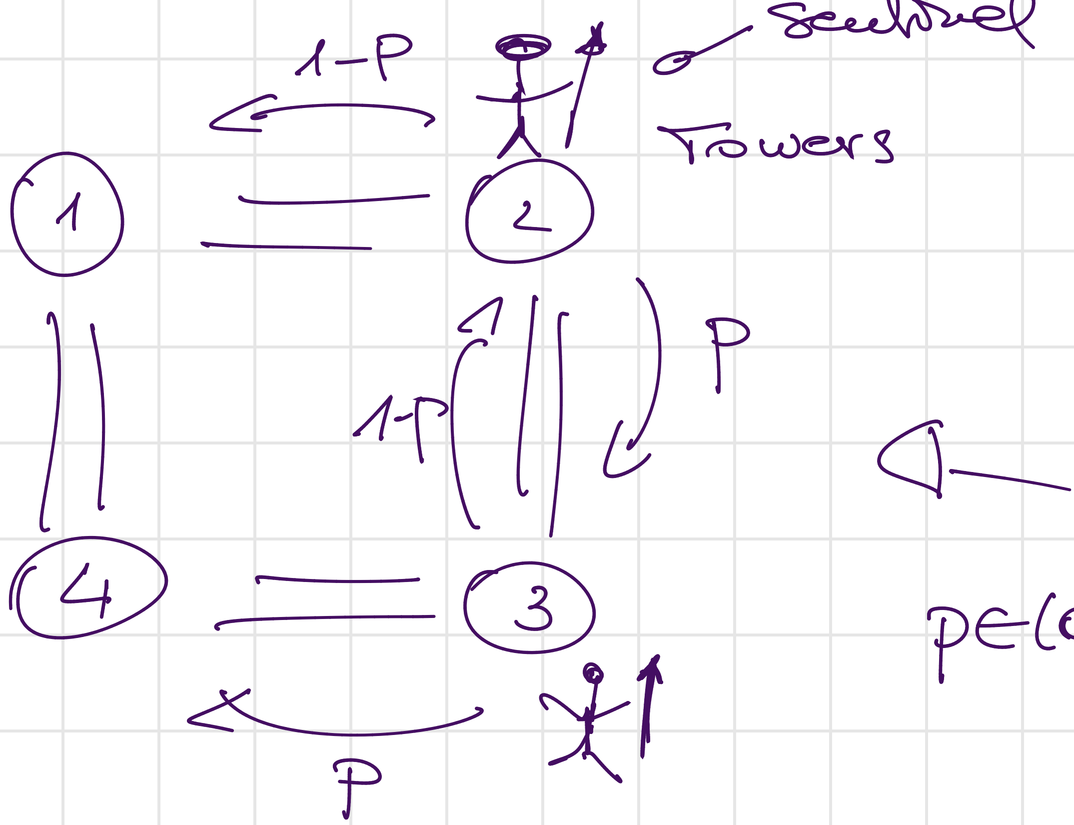
$$= \sum_y \nu_y \cdot (P^n)_{yx} = ((\nu P^n))_x$$

$$\mathbb{P}[X_n = x] = \Big(\underbrace{\mu}\cdot\underbrace{P^n}_{\uparrow}\Big)_x$$

product of matrices

$$(P^2)_{yx} \neq (P_{yx})^2$$

Example 3



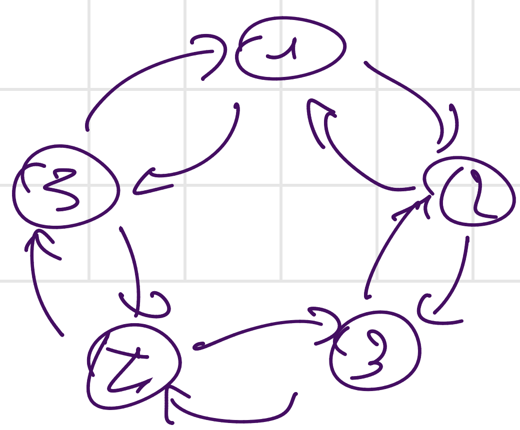
$p \in (0,1)$

$$S = \{1,2,3,4\}$$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \end{bmatrix} \end{array}$$

Exercise

$$S = \{1,2,3,4,5\}$$



Make class. of the states S

1. Communication

(i) We say that $j \in S$ is accessible from $i \in S$ $\quad i \longrightarrow j$

if $\exists\, n \in \mathbb{N}$ s.t. $P^{n}_{ij} > 0$

$\forall i \in S$, $\underline{P^0_{ii} = 1}$

(ii) We say that i and j communicate

$$\boxed{i \longleftrightarrow j \iff i \longrightarrow j \text{ and } j \longrightarrow i}$$

Communication is an equivalent relation of elements of S: $i \sim j$

$$\boxed{i \longleftrightarrow j}$$

✓ (1) $i \sim i \quad \forall i \in S \qquad i \leftrightarrow i$

$$P^0_{ii} = I$$

✓ (2) $i \sim j \Rightarrow j \sim i$

$$i \leftrightarrow j \iff \underbrace{j \leftrightarrow i}$$

(3) $i \sim j$ and $j \sim k \Rightarrow i \sim k$

$$\underbrace{i \leftrightarrow j} \text{ and } j \leftrightarrow k \Rightarrow \boxed{i \leftrightarrow k} \;??$$

$$\boxed{i \longrightarrow k \;?}$$

$$i \longrightarrow j \Rightarrow \exists\, n : \underbrace{P^n_{ij} > 0}$$

$$j \longrightarrow k \Rightarrow \exists\, m : \underbrace{P^m_{jk} > 0}$$

$$\Rightarrow \underbrace{P^{n+m}_{ik}} \geq \underbrace{\underbrace{P^n_{ij}} \cdot P^m_{jk}} > 0$$

$$\Rightarrow i \longrightarrow k$$

$$i \rightsquigarrow j \rightsquigarrow k$$

YES

(In a similar way you can prove that $k \to i$)

Ex:

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1/2 & \boxed{1/2} & 0 \\ \boxed{0} & 1/3 & 2/3 \\ 1/3 & 1/2 & 1/6 \end{pmatrix}$$

$S = \{1, 2, 3\}$

$\boxed{1 \longleftrightarrow 2} \leftarrow$ YES

$1 \longleftarrow 2$ Yes

$2 \longrightarrow 3 \longrightarrow 1$

$1 \longleftrightarrow 3$

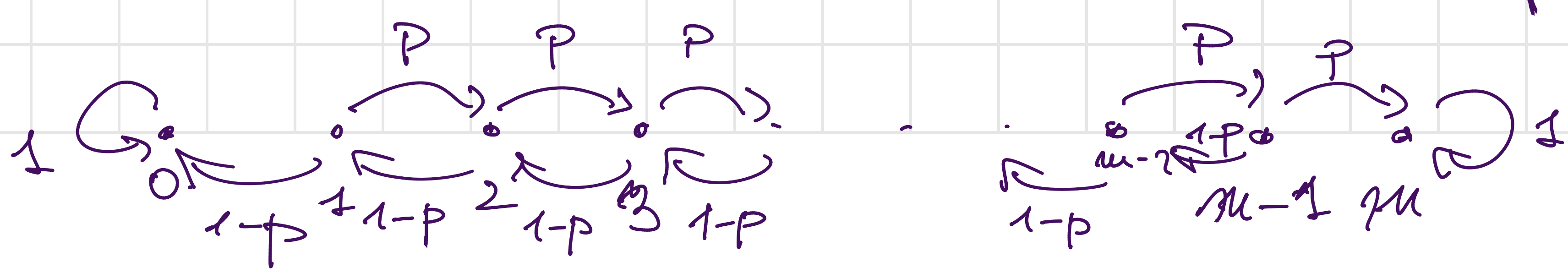
$C_0 = \{1, 2, 3\}$

In these cases, when $S = C_0$, we say that the Markov Chain is <u>irreducible</u>.

<u>Exercise</u>: prove that the MC of the seaweed is also irreducible.

$S = \{0, 1, 2, \dots, m\}$

<u>Example</u>: Gambler's Ruin $\quad [X_n = k]$



$$0 \xrightarrow{1} 0 \qquad C_0 = \{0\}$$

$$C_1 = \{m\}$$

$$m \xrightarrow{1} m$$

$$C_2 = \{\underbrace{1, \dots, m-1}\}$$

$i \in \{1, \dots, m-1\}$

$$C_2 = \{\underbrace{i}, \underbrace{i+1}, \underbrace{i+2}, \dots, \underbrace{m-1}, i-1, i-2, \dots, 1\}$$

In these cases, 0, m, we call these state absorbing states.

$\boxed{2}$ Periodicity of the states

- If $\{n > 0 : P_{ii}^{n} > 0\}$ has no common divisor other than 1 then we say that i is aperiodic

• If $\underbrace{\{n > 0 : P_{ii}^{n} > 0\}}$ has common divisor greater than 1

Then the greatest common divisor $d(i)$ is called the period of the state i.

Remark 1: If $P_{ii} > 0$, then i is aperiodic.

Remark 2: If $i, j \in C_0$ $(i \leftrightarrow j)$ and i is aperiodic (or has period $= d$) then the same holds true for j.

Ex: $S = \{1, 2, 3\}$ all the states are aperiodic since $P_{11} = 1/2 > 0$

$$\begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & \cdot & \cdot\cdot \\ \cdot & - & \cdot \end{pmatrix}$$

$S = \{1, 2, 3, 4\}$ $\quad p = 1/2$

$p = 1/2 = 1 - p$

$$P = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$

$1/2$

$$P^2 = P \cdot P = \begin{pmatrix} \boxed{0 + \frac{1}{4} + 0 + 1/4} & 0 & \boxed{1/2} & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & \frac{1}{2} & 0 & 1/2 \end{pmatrix}$$

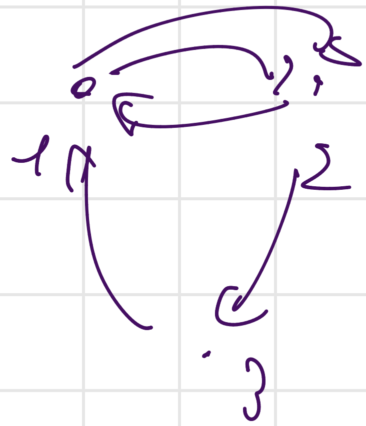
$P^3 = P$, $P^4 = P^2$

$P^0 = Id$, $P^{2n} = P^2 \quad \forall n$

$P^{2n+1} = P \quad \forall n$

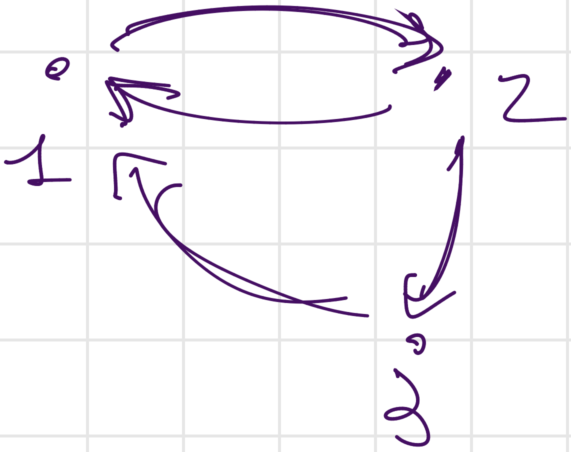
$\Rightarrow$ 1 is periodic of period $\boxed{2}$

$\{n : P_{11}^n > 0\} = \{2, 4, 6, 8, 10, \dots\}$



$P_{11}^2 > 0$

$P_{11}^3 > 0$



$P_{11}^2 > 0$

$P_{11}^3 > 0$

$2, 3 \in \{n : P_{11}^n > 0\}$

$\boxed{1 \text{ is aperiodic}}$

$$S = C_0 \cup C_1 \cup \dots \cup C_n$$

Def: An equivalence class is <u>closed</u>

if $\forall\, i \to j,\ i \in C \Rightarrow j \in C$

Gamblers' ruin
- $\to C_0 = \{0\}$ closed
- $\to C_1 = \{n\}$ closed
- $\to C_2 = \{1, 2, \dots, n-1\}$ not closed

3) Recurrence and Transience

Let $\underbrace{f_i}_{} = \mathbb{P}[X_n = i \text{ for some } n \geq 1 \mid X_0 = i]$

- $i \in S$ is recurrent $\iff$ $\underbrace{f_i = 1}_{}$
- $i \in S$ is transient $\iff$ $f_i < 1$

Let $N_i := \left| \{ n \geq 0 : X_n = i \} \right|$

(i) if i is recurrent, then $\mathbb{P}[N_i = +\infty \mid X_0 = i] = 1$

(ii) if i is transient, then $\mathbb{P}[N_i = +\infty \mid X_0 = i] = 0$

Moreover, $N_i \mid X_0 = i \sim \text{Geo}(1 - f_i)$

Prop: If i is recurrent (transient) and $i \leftrightarrow j$, then j is recurrent (transient).

Example Gambler's ruin: $\{0\}$ is recurrent, $\{m\}$ is rec. $\{1, \dots, m-1\}$ transient states.

Prop. 1: Every recurrent class is closed

Prop. 2. Every finite closed class is recurrent.

$|S| < +\infty$