Lecture 16 Stoch. Math. Nov. 21st, 2020

Problem: $X_1, X_2, \dots, X_n$ iid. $\mathcal{U}([0,1])$ r.v.'s

$$Z := X_1 + X_2 + \dots + X_n$$

Prove that $\mathbb{P}[Z \leq x] = \dfrac{x^n}{n!}$ $\quad 0 \leq x \leq 1$ $\quad (*)$

$n=2$ $\quad \mathbb{P}[X_1 + X_2 \leq x] = \dfrac{x^2}{2!}$ $\quad 0 \leq x \leq 1$

By induction on n, let us prove that $(*)$ is true.

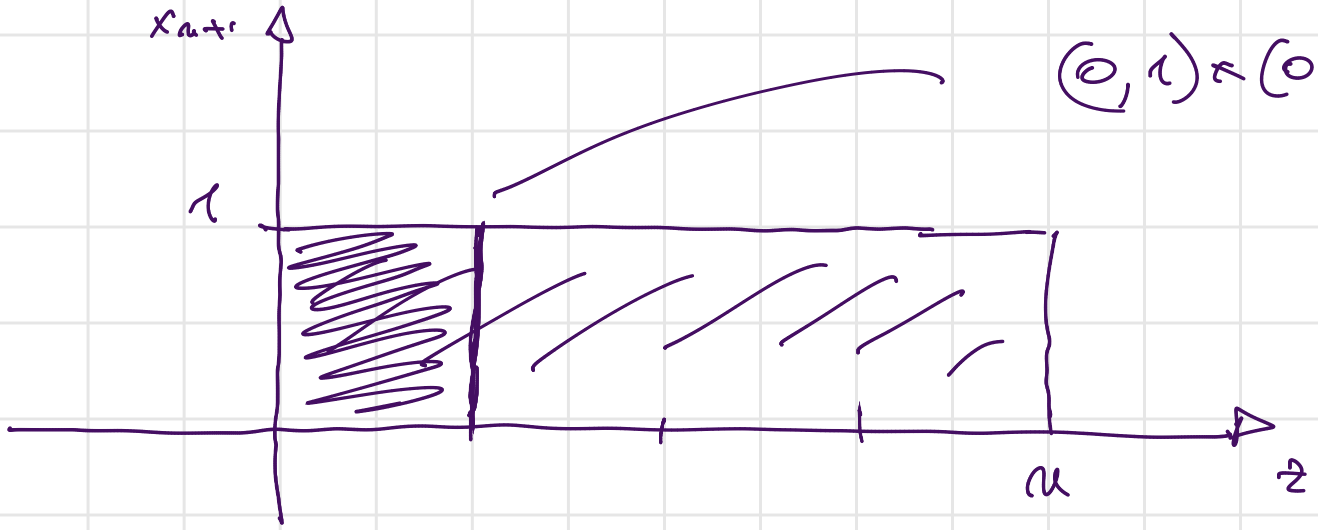
We assume $\boxed{\mathbb{P}[Z \leq x] = \dfrac{x^n}{n!}}$ $\forall x \in [0,1]$, and we want to prove $W := X_1 + X_2 + \dots + X_{n+1}$

$$\mathbb{P}[W \leq x] = \frac{x^{n+1}}{(n+1)!} \qquad 0 \leq x \leq 1$$

Proof: $W = Z + X_{n+1}$ $\qquad Z \perp\!\!\!\perp X_{n+1} \sim \mathcal{U}([0,1])$

$X_{n+1} \in (0,1)$, $Z \in (0,n)$ $\qquad (Z, X_{n+1}) \in (0,n) \times (0,1)$

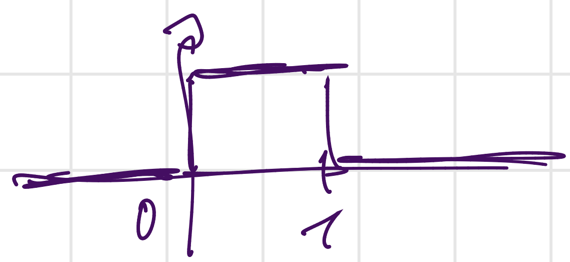
(Z, X_{n+1})



$(0,1)\times(0,1)$

$$f_{Z,X_{n+1}}(z,x) = \begin{cases} \underbrace{f_Z(z)\cdot f_{X_{n+1}}(x)} & (0,n)\times(0,1) \\ 0 & \text{otherwise} \end{cases}$$

$x \in (0,1)$ $\quad f_{X_{n+1}}(x) = 1$



$z \in (0,1)$ $\quad f_Z(z) = F_Z'(z) =$

$=$

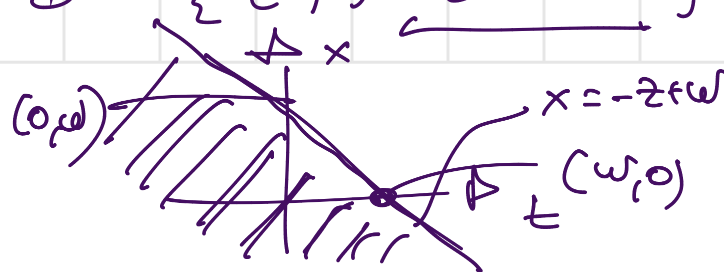
$$F_Z(z) = \frac{z^n}{n!} \rightsquigarrow f_Z(z) = F_Z'(z) = \frac{n\cdot z^{n-1}}{n!} = \frac{z^{n-1}}{(n-1)!}$$

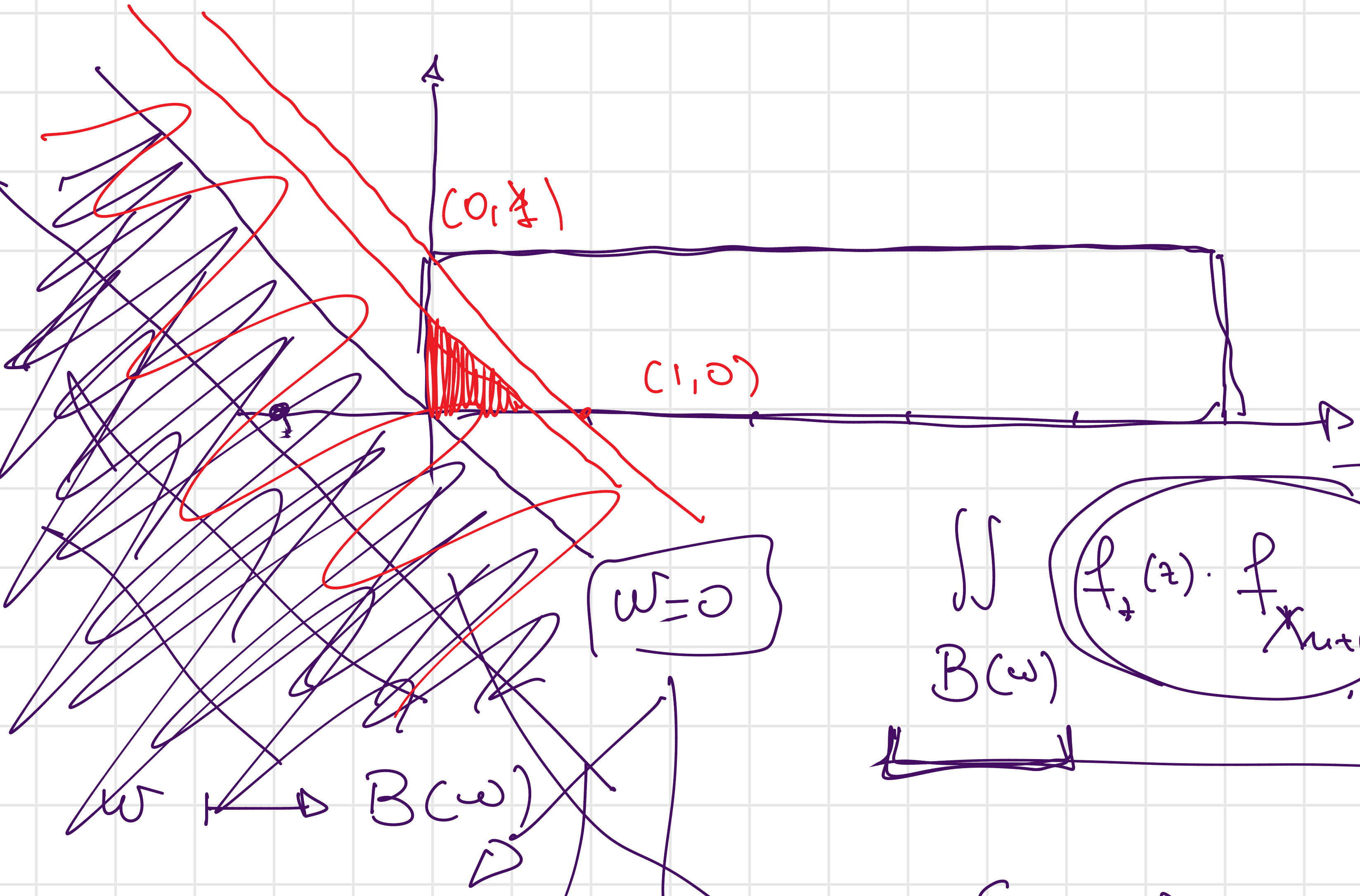
$\underbrace{w \in (0,1)}$

$$W \rightsquigarrow \boxed{F_W(w) = P[W \le w] = P[Z + X_{n+1} \le w]}$$

$$= P[\underbrace{(Z, X_{n+1})} \in B(w)] \qquad B(w) = \{(z,x) : \underline{z + x \le w}\}$$

$$= \iint_{B(w)} f_{Z,X_{n+1}}(z,x)\,dz\,dx \qquad \overset{\cap}{\mathbb{R}^2}$$





$$\iint_{B(w)} f_Z(z)\cdot f_{X_{n+1}}(x)\, dz\, dx$$

$\boxed{w=0}$

$$B(0) := \{(z,x) : z+x \le 0\}$$

$w<0$

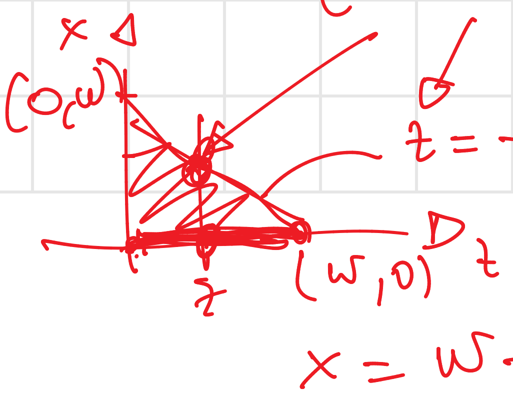
$w=-1$

$w=0 \qquad \mathbb{P}[Z + X_{n+1} \le 0] = 0$

$w<0 \qquad \mathbb{P}[Z + X_{n+1} \le w] = 0$

$$0 \le w \le 1 \qquad \mathbb{P}[Z + X_{n+1} \le w] = \iint 1 \cdot \frac{z^{n-1}}{(n-1)!}\, dx\, dz$$

$w=1 \quad (w-z, z) \quad z + x = 1$



$z = -x + w$

$$= \int_0^w dz \left(\int_0^{w-z} dx \right) \frac{z^{n-1}}{(n-1)!} =$$

$\boxed{w-z}$

$x = w - t$

$$= \int_0^{w} \frac{z^{n-1}}{(n-1)!} \cdot (w-z)\, dz = \frac{z^n}{(n-1)!}$$

$$= \int_0^{w} \frac{z^{n-1}}{(n-1)!} (w)\, dz + \int_0^{w} \frac{z^{n-1} \cdot (-z)}{(n-1)!}\, dz$$

$$= w \left[\frac{z^n}{n!} \right]_0^{w} - \left[\frac{z^{n+1}}{(n+1)(n-1)!} \right]_0^{w}$$

$$\frac{d}{dz}\left(\frac{z^n}{n!}\right) = \frac{z^{n-1}}{(n-1)!}$$

$$= \boxed{\frac{w \cdot w^n}{n!}} - \frac{w^{n+1}}{(n+1)(n-1)!} =$$

$$= \frac{w^{n+1}}{(n-1)!}\left[\frac{1}{n} - \frac{1}{n+1}\right] = \frac{w^{n+1}}{(n-1)!} \cdot \frac{\cancel{n}+1 \cancel{-n}}{n(n+1)}$$

$0 \leq w \leq 1$ $$= \frac{w^{n+1}}{(n+1)!} = \mathbb{P}[W \leq w]$$

Exercise: Let (X,Y) be an abs. cont. random vector whose density is

$$f_{(X,Y)}(x,y) = \begin{cases} \alpha(x+y) & (x,y)\in[0,1]^2 \\ 0 & \text{otherwise} \end{cases}$$

(i) det. the value of α $\qquad \alpha \in \mathbb{R}$

(ii) X and Y are independent?

NO. It is suff. to prove $f_{(X,Y)} \neq f_X f_Y$

(iii) Define $Z = \max(X,Y)$

compute $\mathbb{P}[Z \le z] \quad \forall z \in \mathbb{R}$

$\mathbb{P}[(X,Y) \in C(z)] = \iint_{C(z)} f \, dx\, dy$

subsets of $\mathbb{R}^2$

② Determine the expected number of independent $\mathcal{U}(0,1)$ r.v.'s that need to be summed to exceed 1.

$X_1 \le 1$ $\qquad$ $\mathbb{P}[X_1 > 1] = 0$

X_1, X_2 $\qquad$ $X_1 + X_2 < 1$, $\quad X_1 + X_2 > 1$

$\{\omega : X_1(\omega) + X_2(\omega) > 1\}$ $\qquad$ $N(\omega) = 2$

$\mathbb{E}[N]$ $\qquad$ $N := \underbrace{\min\{n \in \mathbb{N} : X_1 + X_2 + \dots + X_n > 1\}}$

$\mathbb{P}[N = 0] = 0$ $\qquad$ $\boxed{\mathbb{E}[N] = e}$

$\mathbb{P}[N = 1] = 0$

$\mathbb{P}[N = 2] = \mathbb{P}[X_1 + X_2 > 1]$

$\mathbb{P}[\underbrace{N = 3}] = \mathbb{P}[X_1 + X_2 \le 1, \underbrace{X_1 + X_2 + X_3 > 1}]$

$\overbrace{\mathbb{P}[N = n]} = \mathbb{P}[\underbrace{X_1 + \dots + X_{n-1} \le 1}, \underbrace{X_1 + \dots + X_n > 1}]$

$$\mathbb{E}[N] = \sum_{n=0}^{\infty} n \cdot \underbrace{\mathbb{P}[N = n]}$$

$\mathbb{P}[X_1 + \dots + X_n \le 1]$ ← Part ①

$$= \sum_{n=0}^{\infty} \underbrace{\mathbb{P}[N > n]} \underset{\text{prove}}{=} \sum_{n=0}^{+\infty} \frac{1}{n!} = e^1 = e$$

mgf, ch

X – d-dim. r.v., $A \in M(n \times d)$, $b \in \mathbb{R}^n$

$$\boxed{Y := AX + b}$$

$t \in \mathbb{R}^n$

$$\boxed{m_Y(t) = e^{\langle t, b\rangle} \cdot m_X(A^T \cdot t)}$$

$u \in \mathbb{R}^n$

$$\varphi_Y(u) = e^{i\langle u, b\rangle} \cdot \varphi_X(A^T u)$$

Multivariate Normal Distribution

Let $Z_1, \dots, Z_d \sim N(0,1)$ <u>independent</u>

$$Z = (Z_1, \dots, Z_d)^T \qquad \sum_{i=1}^{d} u_i^2 = \|u\|^2 \qquad e^{-u_i^2/2}$$

$$\boxed{\varphi_{Z_d}(u)} = \varphi_Z(u_1, u_2, \dots, u_d) = \prod_{i=1}^{d} \varphi_{Z_i}(u_i)$$

$$= \prod_{i=1}^{d} e^{-u_i^2/2} = e^{-\frac{1}{2}\sum_{i=1}^{d} u_i^2} = \boxed{e^{-1/2 \|u\|^2}}$$

$A \in M(n \times d), \quad \mu \in \mathbb{R}^n$

$$X := \underbrace{A \cdot Z + \mu} \in \mathbb{R}^n$$

$v = (v_1, \dots, v_n) \in \mathbb{R}^n$

$$\varphi_X(v) = e^{i\langle v, \mu\rangle} \cdot \underbrace{\varphi_Z(A^T v)}$$

$$= e^{i\langle v, \mu\rangle} \cdot e^{-\frac{\|A^T v\|^2}{2}}$$

$$= e^{i\langle v, \mu\rangle} \cdot e^{-\frac{1}{2}\langle \Sigma v, v\rangle}$$

Σ is a sym. matrix

$$\|A^T v\|^2 = \langle A^T v, \underbrace{A^T} v\rangle = \langle (A^T)^T A^T v, v\rangle$$

$$= \langle A \cdot A^T v, v\rangle = \langle \Sigma v, v\rangle$$

$(A^T)^T = A$

$$\underbrace{\Sigma} := A A^T \in M(n \times n)$$

$$(n \times d) \cdot (d \times n)$$

$$\underbrace{\Sigma \text{ is symmetric}} \iff \Sigma = \Sigma^T = (A A^T)^T = (A^T)^T \cdot A^T$$

$$= A \cdot A^T$$

$$\underbrace{X = A \cdot Z + \mu}_{d \rightsquigarrow n} \qquad \mathbb{E}[Z] = (\mathbb{E}[Z_1], \dots, \mathbb{E}[Z_d]) = (0, \dots, 0)$$

$$\mathbb{E}[X] = (\mathbb{E}[X_1], \mathbb{E}[X_2], \dots, \mathbb{E}[X_n])$$

$$\mathbb{E}[X_i] = \mathbb{E}\Big[\sum_{j=1}^{d} A_{ij} Z_j + \mu_i\Big] =$$

$$X_i = \sum_{j=1}^{d} A_{ij} Z_j + \mu_i$$

$$= \mathbb{E}\Big[\sum_{j=1}^{d} A_{ij} Z_j\Big] + \mu_i = \sum_{j=1}^{d} \mathbb{E}[A_{ij} Z_j] + \mu_i$$

$$= \sum_{j=1}^{d} A_{ij} \cdot \underbrace{\mathbb{E}[Z_j]}_{= 0} + \mu_i = \mu_i$$

$$X = A \cdot Z + \mu$$

$$\mathbb{E}[X] = \mu$$

$$\mathrm{Cov}(X_i, X_j) = \mathbb{E}\big[(X_i - \overbrace{\mathbb{E}[X_i]}^{\mu_i})(X_j - \overbrace{\mathbb{E}[X_j]}^{\mu_j})\big]$$

$$= \mathrm{Cov}\Big(\underbrace{\sum_{k=1}^{d} A_{ik} Z_k}, \underbrace{\sum_{h=1}^{d} A_{jh} Z_h}\Big)$$

$$= \sum_{k=1}^{d} \sum_{h=1}^{d} A_{ik} A_{jh} \underbrace{Cov(z_k, z_h)}$$

$$Cov(z_k, z_h) = \begin{cases} 0 & h \neq k \\ Var[z_k] = 1 & h = k \end{cases}$$

$h = k$ $\neq 0$

$$= \underbrace{\sum_{k=1}^{d}} \underset{=}{A_{ik}} \underset{=}{A_{jk}}$$

$$\boxed{= \Sigma_{ij}}$$

$$\boxed{= Cov(X_i, X_j)}$$

$$\Sigma = \underbrace{A \cdot A^T}$$

Σ_{ij}

So, Σ is called the covariance matrix of X.

$$\Sigma = A \cdot A^T$$

Σ is symmetric, but it is also a

$\underbrace{\text{non-negative definite}}$ matrix

$$\Rightarrow \underbrace{\langle \Sigma x, x \rangle} \geq 0 \quad \forall x \in \mathbb{R}^n$$

$$\langle \Sigma x, x \rangle = \langle A \cdot A^T x, x \rangle = $$
$$= \langle A^T x, A^T x \rangle = \| A^T x \|^2 \geq 0$$

__Definition:__ A random vector X is called Normal with mean μ and covariance matrix Σ (Σ symm. and non-neg. definite) if its characteristic function is equal to

$$\varphi_X(v) = e^{i\langle v, \mu \rangle} \cdot e^{-\frac{1}{2}\langle \Sigma v, v \rangle}$$

and we write $X \sim \mathcal{N}(\mu, \Sigma)$

(Z standard $Z \sim \mathcal{N}(\underline{0}, Id)$)

Prop. 1: Let $X \sim N(\mu, \Sigma)$ and set

$$Y := B \cdot X + b \quad \text{Then}$$

$$Y \sim N\left(B\mu + b,\ B \Sigma B^T\right)$$

Prop 2: If (X, Y) is a two dim. Normal distr.

then $X \perp\!\!\!\perp Y \iff \boxed{Cov(X, Y) = 0}$

Proof: $\Rightarrow$ always true.

$\Leftarrow$ $\Sigma = \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}$, Moreover, $\mathbb{E}[X] = \mu_x$, $\mathbb{E}[Y] = \mu_y$

$$\varphi_{(X,Y)}(u,v) = e^{i\langle (u,v), (\mu_x, \mu_y) \rangle} \cdot e^{-\frac{1}{2}\langle \Sigma \binom{u}{v}, \binom{u}{v} \rangle}$$

$$= e^{i(u \cdot \mu_x + v \cdot \mu_y)} \cdot e^{-\frac{1}{2}(u^2 \sigma_x^2 + v^2 \sigma_y^2)}$$

$$= e^{iu\mu_x} \cdot e^{iv\mu_y} \cdot e^{-\frac{1}{2} u^2 \sigma_x^2} \cdot e^{-\frac{1}{2} v^2 \sigma_y^2}$$

$$= e^{iu\mu_x} \cdot e^{-\frac{1}{2} u^2 \sigma_x^2} \cdot e^{iv\mu_y} \cdot e^{-\frac{1}{2} v^2 \sigma_y^2}$$

$$= \varphi_X(u) \cdot \varphi_Y(v) \iff \boxed{X \perp\!\!\!\perp Y}$$