Communication $i \longleftrightarrow j$

$|S| < +\infty$

$$S = \underbrace{C_0 \cup \dots \cup C_m}_{\text{closed classes}} \cup T$$

transient

recurrent

$\underline{i \text{ is recurrent}} \iff f_i = \mathbb{P}[X_n = i \text{ for some } n \mid X_0 = i] = 1$

$\underline{i \text{ is transient}} \iff f_i < 1$

$\boxed{|S| = +\infty}$

$N_i \in \mathbb{N} \cup \{\infty\}$

$N_i := |\{ n \geq 0 : X_n = i \}|$ r.v.

(i) if $\underline{i \text{ is recurrent}}$ then $\mathbb{P}[N_i = +\infty \mid X_0 = i] = 1$

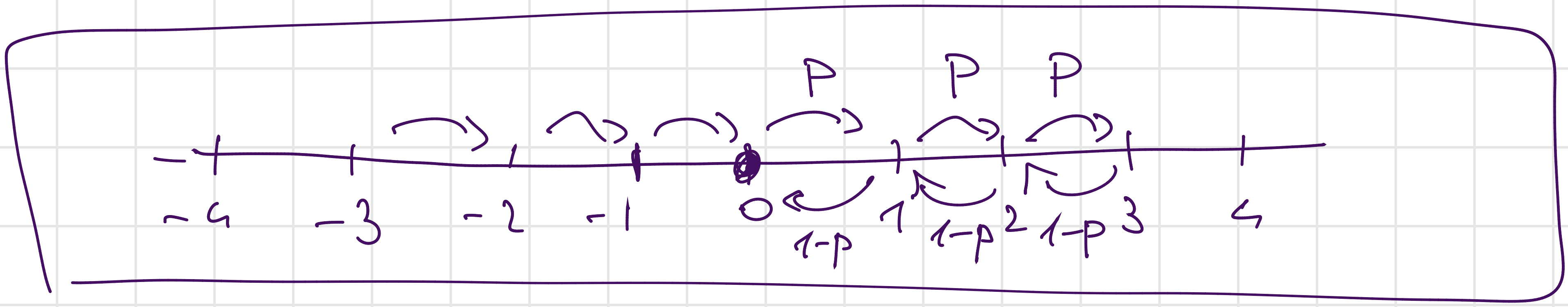
(ii) if i is transient then $\mathbb{P}[N_i = +\infty \mid X_0 = i] = 0$

$N_i \mid X_0 = i \sim Geo(1 - f_i)$

$|S| = +\infty$

$p \in [0,1]$

Random Walk on $\mathbb{Z}$



$X_1, X_2, X_3, \dots$ iid $\qquad X_n = \begin{cases} 1 & p \\ -1 & 1-p \end{cases}$

$$\begin{cases} \boxed{S_0 = 0} \\ S_1 = S_0 + X_1 \\ S_2 = S_1 + X_2 \\ \vdots \\ S_{n+1} = S_n + X_{n+1} = X_1 + X_2 + \dots + X_{n+1} \end{cases}$$

$p \in [0,1]$

MC $(S_n)_{n \geq 0}$

(S_n) is irreducible !! $\quad p \in (0,1)$

$\mathbb{Z}$ state space

$X_1, X_2, \dots$ iid $\quad \mathbb{E}[X_1] =$

$$X_1 = \begin{cases} 1 & p \\ -1 & 1-p \end{cases}$$

$$= 1 \cdot p + (-1)(1-p)$$

$$= p - 1 + p = 2p - 1$$

(ω)

SLLN

$$\boxed{\frac{S_n}{n}} = \frac{X_1 + \dots + X_n}{n} \xrightarrow[n\to\infty]{a.s.} 2p-1$$

$$\frac{S_n(\omega)}{n} \longrightarrow 2p-1 \quad \text{for almost every } \omega \in \Omega.$$

If $p > 1/2$ $\quad 2p-1 > 1-1 = 0$ $\qquad \frac{S_n}{n} \to 2p-1$

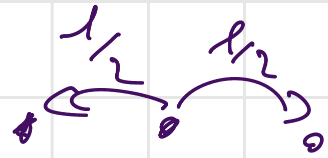
$2p > 1$ $\qquad \Rightarrow$ a.s. $(S_n) \to +\infty$

0 is transient

If $p < 1/2$ $\qquad S_n \to -\infty$ a.e. $\Rightarrow$ 0 is transient

$\boxed{p = 1/2}$ $\qquad 2p - 1 = 0 \qquad \frac{S_n}{n} \to 0 \qquad \cancel{\Rightarrow}$

$p = 1/2$ symmetric R.W.



$$N_0 = |\{n \geq 0 : S_n = 0\}|$$

0 is recurrent $\iff$ $\boxed{\mathbb{P}[\underbrace{N_0 = +\infty} \mid S_0 = 0] = 1}$

0 is transient $\iff$ $\mathbb{P}[N_0 = +\infty \mid S_0 = 0] = 0$

$N_0 \in \mathbb{N} \cup \{\infty\}$

$$\mathbb{E}[X] = \sum k \cdot \underbrace{\mathbb{P}[X = k]}$$

$$\mathbb{E}[X \mid S_0 = 0] = \sum k \cdot \underbrace{\mathbb{P}[X = k \mid S_0 = 0]}$$

$$N_0 \sim Geo(1 - f_0)$$

$$\underbrace{\mathbb{E}[N_0 \mid S_0 = 0]} = \begin{cases} +\infty & 0 \ \underline{\text{recurrent}} \\ \dfrac{1}{1 - f_0} & 0 \ \underline{\text{transient}} \end{cases}$$

$$\underbrace{N_0 = \sum_{n=0}^{+\infty} I_n} \to I_n = \begin{cases} 1 & \text{if } S_n = 0 \\ 0 & \text{if } S_n \neq 0 \end{cases}$$

$$\mathbb{E}[N_0 \mid S_0 = 0] = \mathbb{E}\left[\sum_{n=0}^{+\infty} I_n \mid S_0 = 0\right] = \sum_{n=0}^{+\infty} \mathbb{E}[I_n \mid S_0 = 0]$$

$$\mathbb{E}[\mathbb{1}_A] = \mathbb{P}[A]$$

$S_n = 0$

$$= \sum_{n=0}^{+\infty} \mathbb{E}[I_n \mid S_0 = 0] = \sum_{n=0}^{+\infty} \mathbb{P}[S_n = 0 \mid S_0 = 0]$$

$$= \sum_{n=0}^{+\infty} P^n_{00} = \mathbb{E}[N_0 \mid S_0 = 0] \qquad (P^n)_{00}$$

i is recurrent $\iff \displaystyle\sum_{n=0}^{+\infty} P^n_{ii} = +\infty$

i is transient $\iff \displaystyle\sum_{n=0}^{+\infty} P^n_{ii} < +\infty$

$$\mathbb{P}[S_n = 0 \mid S_0 = 0] = \begin{cases} 0 & n \text{ odd} \\ \times & n \text{ even} \end{cases}$$

$n = 2j \qquad j \in \mathbb{N}$

$$\mathbb{P}[S_n = 0 \mid S_0 = 0] = \mathbb{P}[S_{2j} = 0 \mid S_0 = 0]$$

$$P[S_{2J}=0 \mid S_0=0] = \binom{2J}{J} p^J \underbrace{(1-p)^J}_{2}$$

$$P_{00}^{2J} = \binom{2J}{J} p^J (1-p)^J$$

$$= \frac{\boxed{(2J)!}}{J!\,J!} p^J (1-p)^J$$

$$\boxed{\sum_{J=0}^{\infty} P_{00}^{2J} \begin{cases} <+\infty \\ =+\infty \end{cases}}$$

$$\boxed{p = 1/2}$$

$$4p(1-p) = 1$$

$$4 \cdot 1/2 \cdot 1/2 = 1$$

Stirling's formula

$$\underbrace{n!} \sim n^n e^n \sqrt{2\pi n}$$

$$P_{00}^{2J} \sim \frac{(2\cancel{J})^{2J} \cdot \cancel{e^{2J}} \cdot \sqrt{\cancel{2\pi}\, 2J}}{\cancel{(J)^J} \cdot \cancel{e^J} \cdot \cancel{\sqrt{2\pi J}} \cdot \cancel{(J^J)} \cdot \cancel{e^J} \sqrt{\cancel{2}\pi J}} p^J (1-p)^J$$

2^{2J} $\sqrt{2}$

$$\int \frac{1}{\sqrt{\pi} \cdot \sqrt{J}}$$

$$J^J \cdot J^J = J^{2J}$$

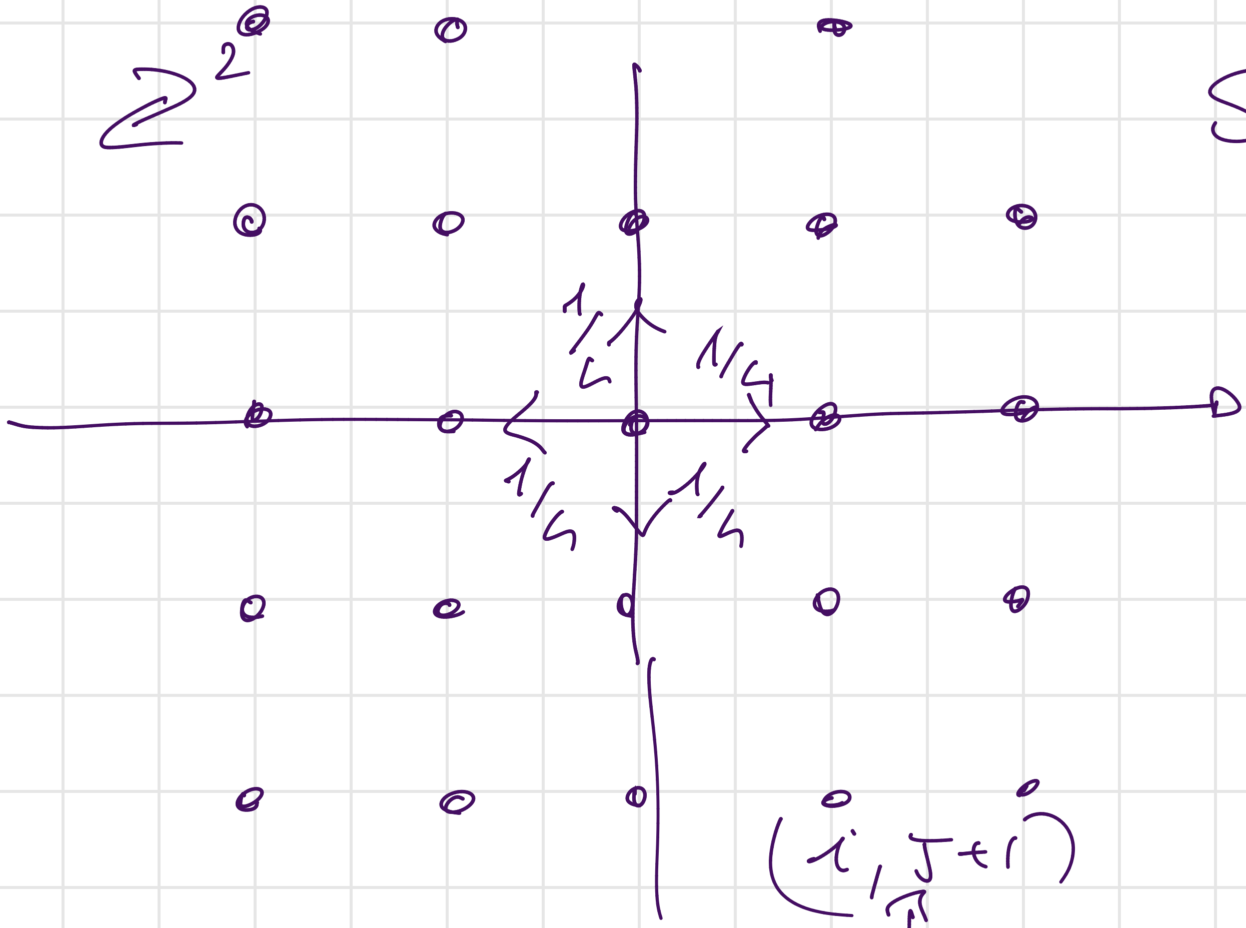
$$= \frac{2^{2J}}{\sqrt{\pi} \cdot \sqrt{J}} p^J (1-p)^J = \frac{(4p(1-p))^J}{\sqrt{\pi} \cdot \sqrt{J}}$$

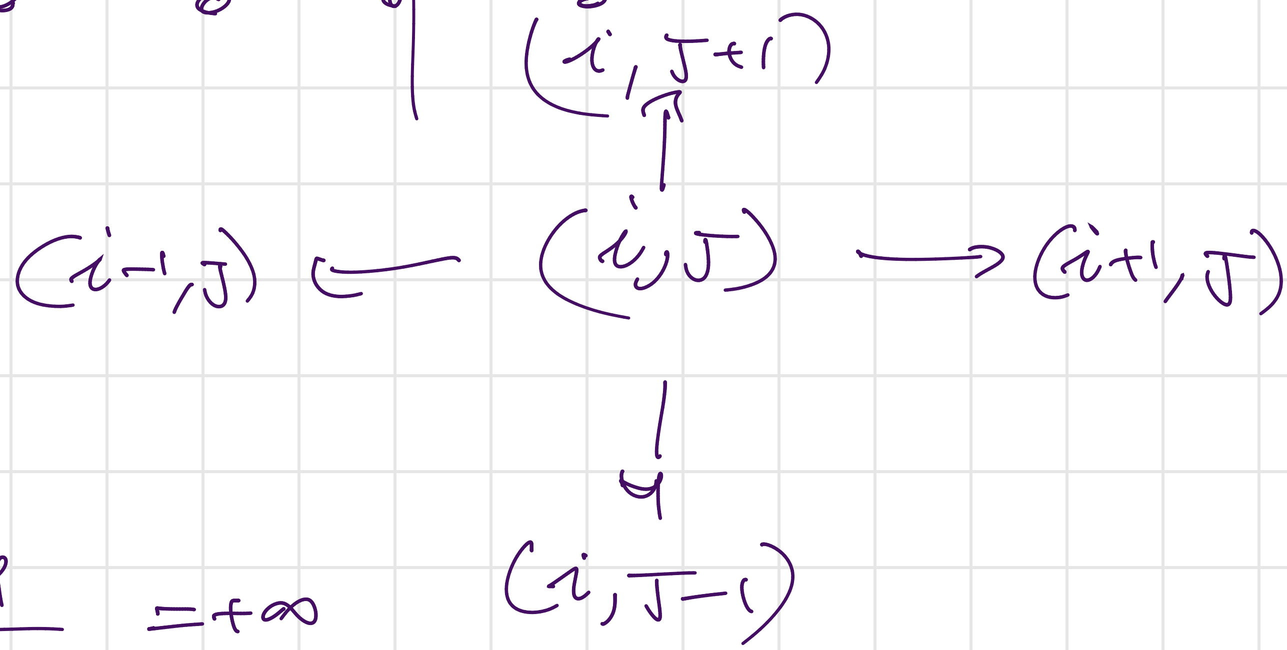
4^J

$$\sum_J \frac{1}{\sqrt{J}} = +\infty$$

$\Rightarrow$ 0 is recurrent

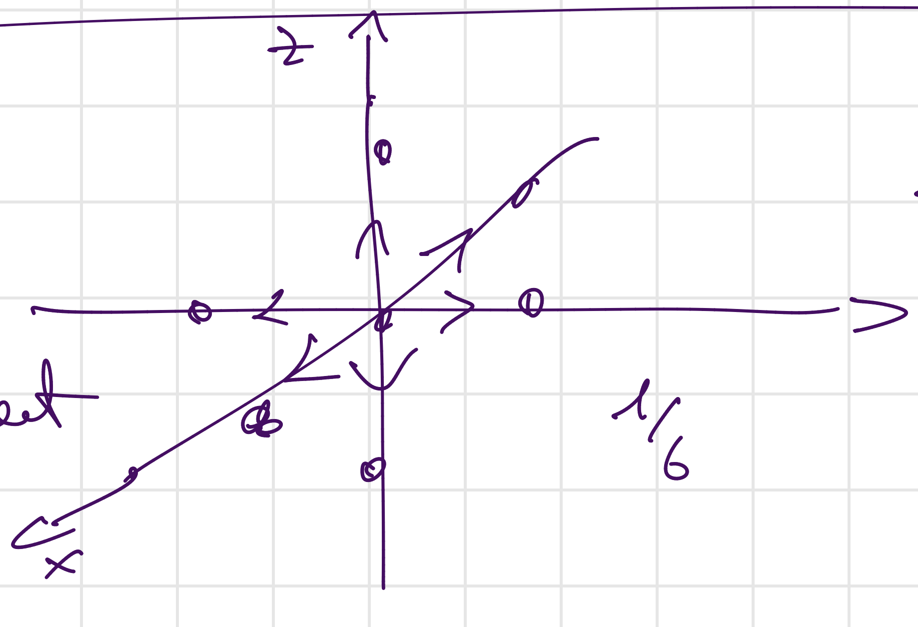
$n=2$ $\quad \mathbb{Z}^2$ $\quad$ Symmetric





$$\sum_j P_{00}^{2j} \sim \sum \frac{1}{j} = +\infty$$

$\Longrightarrow$ $\mathbb{Z}^3$



$$\sum_j P_{00}^{2j} \sim \sum \frac{1}{j^{3/2}} < +\infty$$

0 is transient

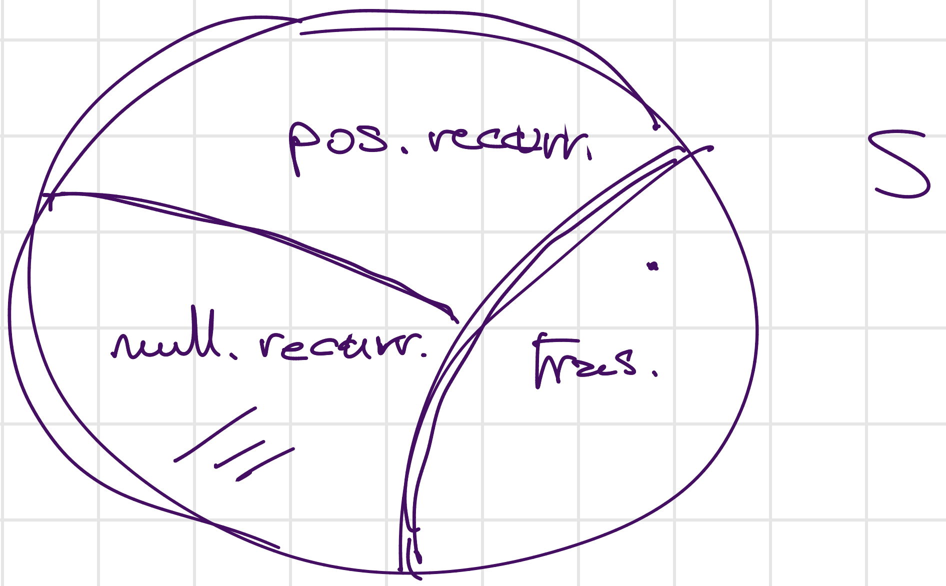
Ergodic MC

<u>Def</u>: Let $T_i := \inf \{ \underline{\underline{n \geq 1}} : X_n = i \}$

and $\underbrace{m_i}_{i \text{ recurrent}} := \mathbb{E}[T_i \mid X_0 = i]$

A state i is <u>positive recurrent</u> if $m_i < +\infty$

A state i is <u>null recurrent</u> if $m_i = +\infty$



<u>Def.</u> (1) An aperiodic, positive recurrent state is called ERGODIC.

(2) An ERGODIC M.C. is a M.C whose states are all ergodic

Theorem For a irreducible, ergodic

M.C. $(X_n)_{n\geq 0}$ exists

$$\boxed{\underline{\pi_j} := \lim_{n\to\infty} P^n_{ij}} \quad \forall j \in S$$

and is independent of i

In addition:

(a) $\pi = (\pi_0, \pi_1, \dots, \pi_n, \dots)$ is the unique

solution of

$$\rightarrow (A) \begin{cases} \boxed{\underline{\pi} = \underline{\pi}\, \underline{P}} \\ \sum_{i\in S} \pi_i = 1 \end{cases} \leftarrow \pi \text{ is a distribution}$$

(b) $\pi_j = \frac{1}{m_j} \Rightarrow \pi_j > 0 \quad \forall j$

(c) $\pi_j = \lim_{n\to\infty} \frac{\#\text{ visits to state } j \text{ by time } n}{n}$

= long run proportion of time the chain spends in state j

π invariant distribution (stationary distribution) $\quad \pi = \pi P$

$\boxed{|S| < +\infty}$

$X_0 \sim \pi \qquad \mathbb{P}[X_0 = i] = \pi_i \quad \forall i \in S$

$$\mathbb{P}[X_1 = i] = \sum_{j \in S} \underbrace{\mathbb{P}[X_0 = j]}_{\pi} \cdot \underbrace{\mathbb{P}[X_1 = i \mid X_0 = j]}_{P}$$

$$= \pi_i$$

$X_0 \sim \pi, \; X_1 \sim \pi, \; X_2 \sim \pi, \; \ldots, \; X_n \sim \pi$

Es:

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/3 & 2/3 \\ 1/3 & 1/2 & 1/6 \end{pmatrix}$$

irred. $\quad \underbrace{|S| = 3}$

recurr. $\Leftrightarrow$ positive recurr.

aperiodic

ERGODIC

$$\begin{cases} \pi = \pi P \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$$(\pi_1, \pi_2, \pi_3)^T = (\pi_1, \pi_2, \pi_3) \cdot \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/3 & 2/3 \\ 1/3 & 1/2 & 1/6 \end{pmatrix}$$

$$\begin{cases} \frac{1}{2}\pi_1 + \frac{1}{3}\pi_3 = \pi_1 \\ \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3 = \pi_2 \\ \frac{2}{3}\pi_2 + \frac{1}{6}\pi_3 = \pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$$\ldots\ldots \longmapsto \qquad \pi = \left(\frac{8}{35}, \frac{15}{35}, \frac{12}{35}\right)$$