Dynamic Optimization in Economics: Optimal Control Theory

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Introduction

Why should we care about dynamics optimization?

Dynamic economic models (according to Sargent 1987):

- People with purposes, beliefs, constraints
- Governments with powers to tax, spend, borrow, redistribute
- Technologies for producing goods, services, physical and human capital
- Stochastic processes describing information flows and economics shocks
- An equilibrium concept describing how marrkets and rules and regualtions reconcile people's diverse purposes and possibilities

Why do we care about dynamic optimizations

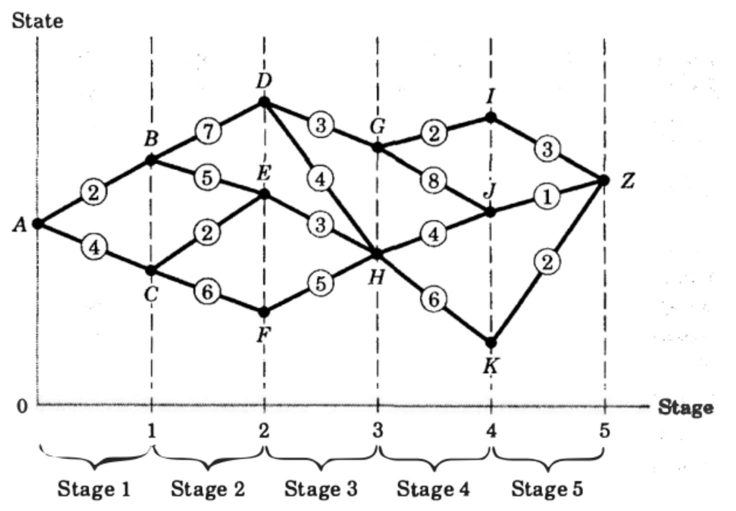
The solution sought in *classical calculus* methods of finding free and constrained extrema and *mathematical programming* usually consists of a single optimal magnitude for every choice variable, such as the optimal level of output per week and the optimal price to charge for a product. It does not call for a schedule of optimal sequential action (Chiang 2000).

The dynamic optimization problem poses the question of what is the optimal magnitude of a choice variable:

- in each period of time within the planning-period (discrete-time)
- each point of time in an interval (continous-time)
- or even an infinite planning horizon

Notation the optimal time path of a (continous-time) variable y will be denoted by $y^*(t)$.

Calculus of Variations: the Shortest Path Problem



The optmimal solution is the path $ACEHJZ$. An one-stage-at-a-time optimization procedure will *not* yield the optimal path.

Calculus of Variations: Some terminology

Notation the general notation of for mapping from path to points (the functional): $V\left[y(t)\right]$.

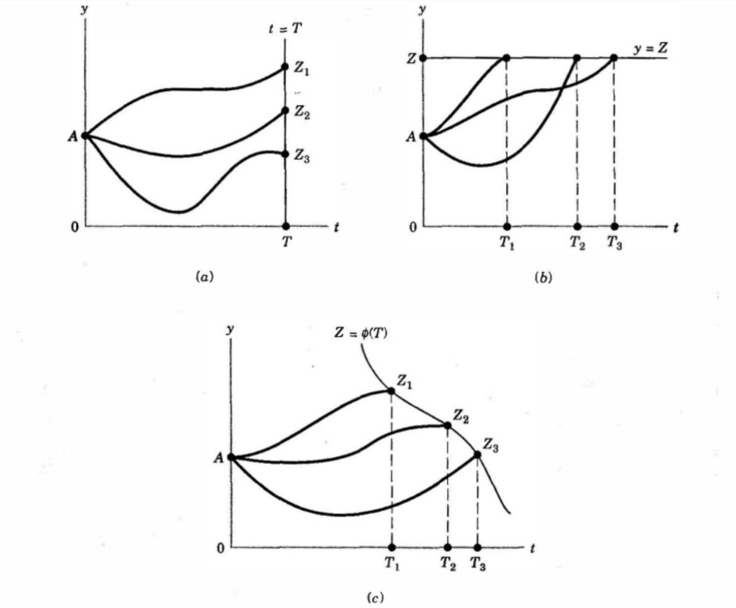
Notation to denote time intervals or segment of a path, we use $y\left[0, T\right]$ or $y\left[0, \tau\right]$.

Notation Optimal path: $y^*(t)$ or y^* path

Notation initial point (time and state): tuple $(0, A)$

Notation terminal point (time and state): tuple (T, Z)

Calculus of Variations: Some terminology



Calculus of Variations: Some terminology

Type of variable terminal points problems:

1. The fixed-time-horizon problem, or fixed-time problem, or vertical-terminal-line problem means that the terminal time T is fixed, terminal state Z may vary.
2. Fixed-endpoint problem or horizontal-terminal-line problem: terminal state Z is fixed, terminal time T may vary. Problems of this type in which T is minimized is called a time-optimal problem
3. terminal-curve or terminal surface problems: T and Z are tied together by a constrant $Z = \Phi(T)$. Such relation plots a terminal curve (or a terminal surface, in higher dimensions).
4. infinite planning horizon problems

In *variable-terminal-points* problems, the planner has one more degree of freedom than *fixed-terminal-point* problems, so an extra condition is needed: the transversality condition, because it normally appears as a description of how the optimal path 'transverse' the terminal line/curve.

The Objective functional

An optimal path is, by definition, one that maximizes or minimizes the path value $V[y]$, which is equivalent to the sum of state values (discrete-time), or the intergrals of state values (continuous-time).

In continuous time, since each arc is infinitesimal in length, 3 pieces of information is needed to for arc identification:

1. t: starting stage (time)
2. $y(t)$: starting state
3. $y'(t) \equiv dy/dt$: The direction in which the arc proceeds

For that, the general expression for ar values is $F[t, y(t), y'(t)]$, and the path-value functional can be written as:

$$V[y] = \int_0^T F[t, y(t), y\prime(t)]dt$$

Optimal control vs. Dynamic Programing

Optimal control theory and Dynamic Programming are two modern extensions of the Caculus of Variations:

- The single most significant development in optimal control theory is known as the Maximum Principle. This principle is commonly associated with the Russian mathematician Lev Pontryagin (Pontryagin 2018), although an American mathematician, Magnus R. Hestenes, independently produced comparable work in a Rand Corp. report in 1949.
- The dynamic programming approach, developed by Richard E. Bellman (Bellman 1954), breaks the dynamic optimization into a sequence of easier problem, as the Bellman's Principle of Optimality prescribes.

The Optimal Control Approach to Dynamic Optimization

The simplest optimization problem in OC

$$
\begin{aligned}
\text{Maximize} \quad & V = \int_0^T (t, y, u) \\
\text{s.t.} \quad & \dot{y} = f(t, y, u) \\
& y(0) = A y(T) \quad \text{free} \qquad \text{A, T given} \\
\text{and} \quad & u(t) \in \mathscr{U} \qquad \text{for all } t \in [0, T]
\end{aligned}
$$

Notation the dotted $\dot{y}$ denotes the first order derivation in t

Notation lowercase f denotes the function symbol in the equation of motion

Notation capital F denotes the intergrand function in the objective function

If $\dot{y} = u$, the problem is precisely the vertical terminal line problem in calculus of variations.

Pontryagin's Maximum Principle

For a Hamiltonian:

$$H = F(t, y_t, u_t) + \lambda_t f(t, y_t, u_t)$$

Notation λ denotes the costate variable or auxiliary variable.

The Pontryagin Maximum Principle introduced the necessary conditions for the optimization problem:

1. $\max_u H(t, y_t, u_t, \lambda_t)$ for all $t \in [0, T]$
2. $\frac{\partial H}{\partial y_t} = -\dot{\lambda}_t = -\frac{\partial \lambda_t}{\partial t}$
3. $\frac{\partial H}{\partial \lambda_t} = \dot{y}_t$
4. And a transversality condition (such as: $\dot{\lambda}_T = 0$)

The transversality condition

This condition specifies what would happen if we *transverse* outside of the planning horizon.

Different variations of the terminal conditions:

- Horizontal terminal line: $[H]_{t=T} = 0$
- Terminal Curve $y_T = \phi(T)$: $[H - \lambda\phi']_{t=T} = 0$

Case study: bang-bang problem

$$
\begin{aligned}
\text{Maximize} \quad & V = \int_0^2 (2y - 3u)dt \\
\text{s.t.} \quad & \dot{y} = y + u \\
& y(0) = 4; \quad y(2) \text{ free} \\
\text{and} \quad & u(t) \in \mathscr{U} = [0, 2]
\end{aligned}
$$

Sufficient conditions

There are two of such sufficient theorems: the Mangasarian theorem and the Arrow theorem.

Mangasarian sufficience theorem

1. both the F and f functions are differentiable and concave in the variable (y, u) jointly
2. in the optimal solution it is true that:

$$\lambda(t) \geq 0 \text{ for all } t \in [0, T] \quad \text{if } f \text{ is nonlinear in } y \text{ or in } u$$

Sufficient conditions

At any t, given y and λ, the H maximized by a particular u, u^*, which depends on t, y, and λ:

$$u^* = u^*(t, y, \lambda)$$

When we subtitute this into the Halmonian, we obtain:

$$H^0(t, y, \lambda) = F(t, y, u^*) + \lambda f(t, y, u^*)$$

Note that H^0 is evaluated along u^* only, not t and y.

Arrow sufficience theorem
H^0 is concave in y for all t in time interval $[0, T]$, for a given λ.

Case study: Capital theory

The following problem is from Dorfman (1969):

$$\begin{aligned} \text{Maximize} \quad & \Pi = \int_0^T \pi(t, K, u)dt \\ \text{s.t.} \quad & \dot{K} = f(t, K, u) \\ \text{and} \quad & K(0) = K_0 \quad K(T) \text{ free} \quad (K_0, T \text{ given}) \end{aligned}$$

Particularly we examine the case that:

$$\pi = K_t(au_t - \frac{b}{2}u_t^2)dt$$

$$\dot{K} = cK_t$$

Case study: Capital theory

FOC	Equations	Interpretion
Choice	$dH/du = 0$	Find the optimal balance between current welfares and future consequences
State	$dH/dk = -\dot{\lambda}$	The marginal value of the state variable is decreasing at the same rate at which it is generating benefit. OR Along the optimal path, the loss that would be suffered if we delayed acquisition of a marginal unit of capital for an instant must equal the instantaneous marginal value of that unit of capital.
Co-state	$dH/d\lambda = \dot{k}$	The state equation must hold.

The Current-Value Hamiltonian

In economics, the integrand function F often contains a discount factor $e^{\rho t}$:

$$F(t, y, u) = G(t, y, u)e^{\rho t}$$

We define a new multiplier m such that:

$$m = \lambda e^{\rho t}$$

then $H_c \equiv He^{\rho t} = G(t, y, u) + mf(t, y, u)$

G is called the Instanteous Utility Function.

The Current-Value Hamiltonian

The new conditions can be rearranged as

1. $\max_u H_c$ for all $t \in [0, T]$
2. $\frac{\partial H_c}{\partial y} = -\dot{m} + \rho m$
3. $\frac{\partial H}{\partial u} = 0$
4. And a transversality condition

Case study: The Ramsey-Cass-Koopmans model

$$\begin{aligned}
\text{Maximize} \quad & \int_0^\infty U(c)e^{-rt}dt \\
\text{s.t.} \quad & \dot{k} = \phi(k) - c - (n+\delta)k \\
& k(0) = k_0 \\
\text{and} \quad & 0 \leq c(t) \leq \phi[k(t)]
\end{aligned}$$

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