Dynamic optimization: the Ramsey-Cass-Koopman model

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Introduction

The Ramsey-Cass-Koopmans (RCK) model

- Ramsey (1928) introduced a model of optimal savings. The model was not well-received at that time because it was too mathematically demanding.
- Almost thirty years later, the model was getting tractions when Koopmans (1963) and Cass (1965) formalized and made extensions to the original model.

This model is commonly refered to as the Ramsey-Cass-Koopmans model of optimal growth.

Why this model

- The RCK model is iconic in neoclassical growth theories, and provides a natural benchmark case.
- The RCK model examine dynamics optimizations of over an infinite horizon, which makes it much easier to approach from the optimal control PoV
- The RCK model endogenize consumption, which makes it one of the first micro-founded models.

Building blocks of the model

Production function

There is a large number of identical firms, producing a common production . Each has access to the production function:

$$Y = \Phi(K, L)$$

Which is homogeneous with both K and L.

We use the lowercase form y and k to denote output per labour and capital per labour:

$$y = \phi(k)$$

where $\phi'(\cdot) > 0$ and $\phi''(\cdot) < 0$, $\lim_{k \to 0} \phi^{'}(k) = \infty$ and $\lim_{k \to \infty} \phi^{'}(k) = 0$

Firms

Firms face the constraints in capital:

$$\dot{K} = I - \delta K = Y - C - \delta K$$

Expanding $\dot{K}$ into $\frac{d(kL)}{dt}$, then use the product rule and divide L from both sides in the above equation give us:

$$\dot{k} = \phi(k) - c - (n + \delta)k$$

Also consumptions should be less than income from the same time period: $0 \leq c(t) \leq \phi[k(t)]$

Utility function i

The household's utility function takes the form

$$U = \int_{t=0}^{\infty} e^{-\rho t} U(c(t)) L(t) dt$$

where

- ρ is the discount rate, the greater is ρ, the less the household values futures consumption relative to current consumption.
- $U(c(t))$ is the instantatneous utility function,

$$\int_0^{\infty} U(c) L(t) e^{-\rho t} dt = \int_0^{\infty} U(c) L_0 e^{nt} e^{-} \rho t dt = L_0 \int_0^{\infty} U(c) e^{-(\rho - n)} dt$$

Utility function ii

$U(c(t))$ usually take the constant-relative-risk-aversion form (Arrow 1965; Pratt 1964):

$$U(c(t)) = \frac{c(t)^{1-\theta}}{1-\theta}, \quad \theta 0, \quad \rho - n - (1-\theta) > 0$$

where:

- θ represents the household's willingness to shift consumption between different periods: elasticity of subtitution between consumption at any two point in time is $\frac{1}{\theta^2}$
- The relative risk aversion $RRA = -\frac{cU''(c)}{U'(c)} = \theta$ is constant
- $\rho - n - (1-\theta) > 0$ ensures that lifetime utilities do not diverge.

In special case of $\theta \to 1$, the instantaneous utility function simplifies to $ln(c)$.

Model Analysis

The optimization problem

$$
\begin{aligned}
\text{Maximize} \quad & \int_0^\infty U(c)e^{-rt}dt \\
\text{s.t.} \quad & \dot{k} = \phi(k) - c - (n+\delta)k \\
& k(0) = k_0 \\
\text{and} \quad & 0 \leq c(t) \leq \phi[k(t)]
\end{aligned}
$$

The optimal control problem

For the Hamiltonian:

$$H = U(c)e^{rt} + \lambda\left[\phi(k) - c - (n+\delta)k\right]$$

The equations-of-motion conditions are written as:

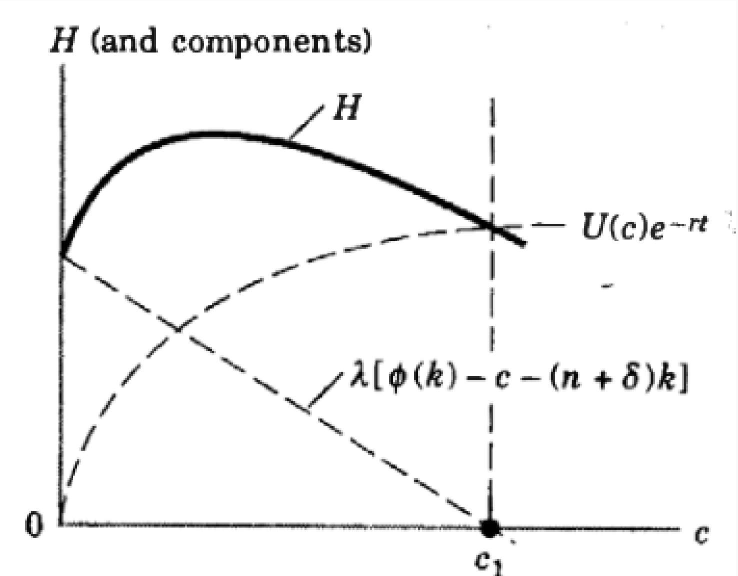
$$\dot{\lambda} = -\frac{\partial H}{\partial k} \Leftrightarrow \dot{\lambda} = -\lambda[\theta^{'}(k) - (n+\delta)] \tag{1}$$

$$\dot{k} = \theta(k) - c - (n+\delta)k \tag{2}$$

Since H is convex and c is unrestricted, we can accordingly find the maximum of H by setting:

$$\frac{\partial H}{\partial c} = U'(c)e^{-rt} - \lambda = 0 \Leftrightarrow U'(c) = \lambda e^{rt} \tag{3}$$

Examination on convexity of H (Fig. 9.2 Chiang 2000):



The Current-Value Hamiltonian i

In economics, the integrand function F often contains a discount factor $e^{\rho t}$:

$$F(t, y, u) = G(t, y, u)e^{\rho t}$$

We define a new multiplier m such that:

$$m = \lambda e^{\rho t}$$

then $H_c \equiv He^{\rho t} = G(t, y, u) + mf(t, y, u)$

G is called the Instanteous Utility Function.

The Current-Value Hamiltonian ii

The new conditions can be rearranged as

1. $\max_u H_c$ for all $t \in [0, T]$
2. $\frac{\partial H_c}{\partial y} = -\dot{m} + \rho m$
3. $\frac{\partial H}{\partial u} = 0$
4. And a transversality condition

The optimal control problem (rephrased)

For the current-value Hamiltonian:

$$H_c = U(c) + m[\phi(k) - c - (n+\delta)k]$$

The derivative conditions are as follow:

$$\frac{\partial H_c}{\partial c} = U'(c) - m = 0 \tag{4}$$

$$\dot{k} = \frac{\partial H_c}{\partial m} = \phi(k) - c - (n+\delta)k \tag{5}$$

$$\dot{m} = -\frac{\partial H_c}{\partial k} + rm = -m[\phi'(k) - (n+\delta)] + rm$$

$$\Leftrightarrow \dot{m} = -m[\phi'(k) - (n+\delta+r)] \tag{6}$$

Constructing the Phase Diagram

From (4) and (6) we have:

$$\dot{c} = -\frac{U'(c)}{U''(c)} \left[\phi'(k) - (n + \delta + r)\right] \tag{7}$$

Along with (5):

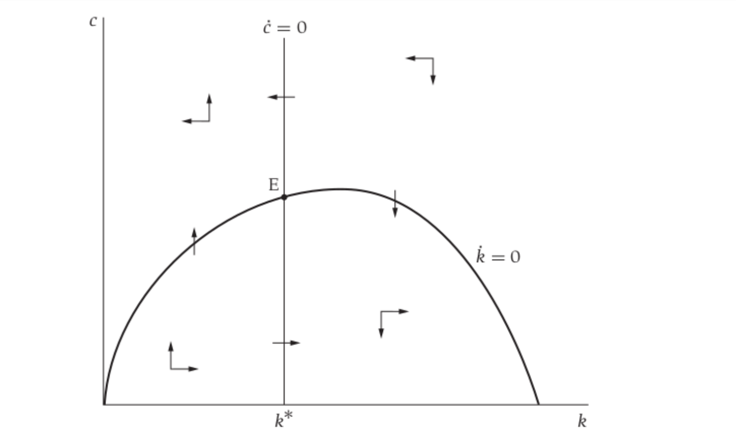
$$\dot{k} = \frac{\partial H_c}{\partial m} = \phi(k) - c - (n + \delta)k$$

We calculate the value where $\dot{c} = 0$ and $\dot{k} = 0$:

$$\begin{cases} (5) & \Rightarrow \dot{k} = 0 \Leftrightarrow c = \phi(k) - (n + \delta)k \\ (7) & \Rightarrow \dot{c} = 0 \Leftrightarrow \phi'(k) = n + \delta + r \end{cases} \tag{8}$$

Phase diagram analysis

The $c-k$ phase diagram (Fig. 2.3 Romer 2019):



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