A105

Burgers' Equation

[redacted]
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Consider the inhomogeneous Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g, \tag{1}$$

with a constant forcing g.

Lemma. *Let $v(x', t)$ be a solution to the homogeneous Burgers' equation. Then*

$$u(x,t) := v\Big(x + \frac{1}{2}gt^2, t\Big) - gt$$

solves equation (1), given $v(x,0) = u(x,0)$.

Proof. Note that $\frac{\partial x'}{\partial x} = 1$ and $\frac{\partial (gt)}{\partial x} = 0$. It then follows that,

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \\
&= \frac{\partial}{\partial t}\left[v\Big(x + \frac{1}{2}gt^2, t\Big) - gt\right] + \left[v\Big(x + \frac{1}{2}gt^2, t\Big) - gt\right]\frac{\partial}{\partial x}\left[v\Big(x + \frac{1}{2}gt^2, t\Big) - gt\right] \\
&= \frac{\partial v}{\partial t} + gt\frac{\partial v}{\partial x'} - g + v\frac{\partial v}{\partial x'} - gt\frac{\partial v}{\partial x'} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x'} - g = 0 - g = -g
\end{aligned}$$

which was to show.

We note that this also holds for the viscous Burgers' equation since,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left[v\Big(x + \frac{1}{2}gt^2, t\Big) - gt\right] = \frac{\partial^2}{\partial x^2}\left[v\Big(x + \frac{1}{2}gt^2, t\Big)\right] = \frac{\partial^2 v}{\partial x'^2}\left(\frac{\partial x'}{\partial x}\right)^2 + \frac{\partial v}{\partial x'}\frac{\partial^2 x'}{\partial x^2} = \frac{\partial^2 v}{\partial x'^2}$$

due to $\frac{\partial x'}{\partial x} = 1$. Additionally, we can extend this to any time-dependent forcing f where the second anti-derivative exists.

Theorem. *Let $v(x', t)$ be a solution to the homogeneous Burgers' equation. Then*

$$u(x,t) := v\Big(x - F^{(2)}(t), t\Big) + F^{(1)}(t) \tag{2}$$

solves the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = f(t), \tag{3}$$

with $\frac{\mathrm{d}F^{(2)}}{\mathrm{d}t} = F^{(1)}$ and $\frac{\mathrm{d}F^{(1)}}{\mathrm{d}t} = f(t)$, such that

$$u(x,0) = v\left(x - F^{(2)}(0), 0\right) + F^{(1)}(0).$$

Proof. Note that $\frac{\partial x'}{\partial x} = 1$ and $\frac{\partial F^{(1)}(t)}{\partial x} = 0$. It then follows that,

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \\
&= \frac{\partial}{\partial t}\left[v\Big(x - F^{(2)}(t), t\Big) + F^{(1)}(t)\right] + \left[v\Big(x - F^{(2)}(t), t\Big) + F^{(1)}(t)\right]\frac{\partial}{\partial x}\left[v\Big(x - F^{(2)}(t), t\Big) + F^{(1)}(t)\right] \\
&= \frac{\partial v}{\partial t} - F^{(1)}(t)\frac{\partial v}{\partial x'} + f(t) + v\frac{\partial v}{\partial x'} + F^{(1)}(t)\frac{\partial v}{\partial x'} \\
&= \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x'} + f(t) = f(t)
\end{aligned}$$

hence, together with the matching initial condition, (2) solves (3).

Note that the two constants in $F^{(2)}(t)$ and $F^{(1)}(t)$ arising from the anti-derivative are determined by the initial condition and depend on the initial condition taken for $v(x', t)$. In general, they correspond to a Galilean transformation and thus different reference frames.