# Drop Formation and Beads-on-String of a Fluid Thread

## A105 Project Report

[redacted]

2023-04-19

# 1 Theory

In this project, we look at the equation (1) which describe the drop formation under a one-dimensional approximation for a thin axisymmetric fluid thread with a free surface. It was derived by Eggers and Dupont[1].

$$\begin{cases} \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{3}{h^2}\nu\frac{\partial}{\partial z}\left(h^2\frac{\partial v}{\partial z}\right) - g \\ \frac{\partial h}{\partial t} + v\frac{\partial h}{\partial z} = -\frac{1}{2}h\frac{\partial v}{\partial z} \\ p = \gamma\left(\frac{1}{h\left(1+\left(\frac{\partial h}{\partial z}\right)^2\right)^{1/2}} - \frac{\frac{\partial^2 h}{\partial z^2}}{\left(1+\left(\frac{\partial h}{\partial z}\right)^2\right)^{3/2}}\right) \end{cases} \tag{1}$$

In equation (1) we have the surface tension $\gamma$, the kinematic viscosity $\nu$, the fluid density $\rho$ and gravity $g$.

As a first side remark, we note that the viscosity $\nu$ consists of a viscous term $3\frac{\partial^2 v}{\partial z^2}$ and an additional term $\frac{6}{h}\frac{\partial v}{\partial z}\frac{\partial h}{\partial z}$. This second term couples height to velocity even in the absence of surface tension.

As a second side remark, we note that we have three parameters: $\frac{\gamma}{\rho}$ with units $[L^3 \cdot T^{-2}]$, $\nu$ with units $[L^2 \cdot T^{-1}]$, and $g$ with units $[L \cdot T^{-2}]$. Additionally, we can and will define an initial height $h_0$ with units $[L]$. By dimensional analysis we have two non-dimensional variables. However, non-dimensionalizing our equation does not yield a significant advantage and we proceed with the above.

## 1.1 Inviscid and surface tension-free limit

Following the lecture, we start by neglecting viscosity $\nu = 0$ and additionally assume the surface tension is negligible. Equivalently, we can assume that the fraction $\frac{\gamma}{\rho} \ll 1$ and neglect the pressure term. Naturally, this loses much of the physics of the original problem but provides more manageable equations.

### 1.1.1 Quasi-linear form

Ignoring viscosity and surface tension brings equation (1) into a quasi-linear form:

$$\frac{\partial}{\partial t}\begin{bmatrix} v \\ h \end{bmatrix} + \begin{bmatrix} v & 0 \\ \frac{1}{2}h & v \end{bmatrix} \frac{\partial}{\partial z}\begin{bmatrix} v \\ h \end{bmatrix} = \begin{bmatrix} -g \\ 0 \end{bmatrix} \tag{2}$$

which is in the shape of

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{q}}{\partial z} = \mathbf{s}.$$

In accordance with the lecture, we must diagonalize the operator $\mathbf{A}$ which will provide the speeds for the characteristics. Unfortunately, this operator is not diagonalizable and only has the eigenvalue $\lambda_1 = v$, with geometric multiplicity of one. Our system (2) is therefore not hyperbolic by definition.

Taking a closer look at equation (2), we recognize that the equation for $v$ is decoupled and simply the inviscid Burgers' equation with a constant driving force. This is consistent with the eigenvalue $\lambda_1 = v$ that we have found. We may use the results from the lecture where we have derived the homogeneous solution. However, finding the general particular solution is by no means easy and will note be done here (though, as suggested after the presentation, rescaling the time may provide a solution).

Nevertheless, we can guess one solution that allows us to somewhat validate the code. If we start from a uniform state $v(0, z) = v_0$, we can find a position independent solution:

$$v(t, z) = v(t) = v_0 - gt \quad \Rightarrow \quad \frac{\partial h}{\partial t} + \big(v_0 - gt\big)\frac{\partial h}{\partial z} + 0 = 0$$

where we recognize the characteristic with speed $v_0 - gt$ for $h$. The generic solution for the initial condition $h(0, z) = h_0(z)$ is:

$$h(t, z) = h_0\Big(z - v_0 t + \frac{1}{2}gt^2\Big) \tag{3}$$

This just corresponds to a falling fluid thread.

### 1.1.2 Conservation form: an auxiliary problem

To identify shock speeds, we need the conservation form:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial z} = \mathbf{s}.$$

For the Burgers' equation this is $f_1(v, h) = \frac{1}{2}v^2$. Unfortunately, this is not possible the rest of equation (2) as there does not exists a $f_2(v, h)$ such that $\mathrm{d}f_2 = \frac{1}{2}h\mathrm{d}v + v\mathrm{d}h$. If we instead had the equation

$$\frac{\partial}{\partial t}\begin{bmatrix} v \\ h \end{bmatrix} + \begin{bmatrix} v & 0 \\ \frac{1}{2}h & \frac{1}{2}v \end{bmatrix} \frac{\partial}{\partial z}\begin{bmatrix} v \\ h \end{bmatrix} = \begin{bmatrix} -g \\ 0 \end{bmatrix}, \tag{4}$$

this would be straight forward and $f_2(v, h) = \frac{1}{2}vh$. Since it is often useful to solve adjacent problems, let us have a closer look at this auxiliary problem.

In this case, the second eigenvalue is $\lambda_2 = \frac{v}{2}$. The system is thus hyperbolic and more over totally hyperbolic (different eigenvalues). We may also identify the associated eigenvectors such that $\mathbf{A}^\intercal \mathbf{l}_i = \lambda_i \mathbf{l}_i$. These provide the ordinary differential equations $\mathbf{l}_i \cdot \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathbf{l}_i \cdot \mathbf{s}$ along the characteristic curves $\mathrm{d}x = \lambda_i \mathrm{d}t$.

$$\mathbf{l}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{l}_2 = \begin{bmatrix} -\frac{h}{v} \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t} = -g & \text{along} \quad \mathrm{d}x = v\mathrm{d}t \\ \frac{\mathrm{d}h}{\mathrm{d}t} - \frac{h}{v}\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{h}{v}g & \text{along} \quad \mathrm{d}x = \frac{v}{2}\mathrm{d}t \end{cases}$$

Furthermore, we can write for the boundary conditions,

$$\frac{\partial}{\partial t}\begin{bmatrix} v \\ h \end{bmatrix} + \begin{bmatrix} \frac{h}{v}(\mathcal{L}_1 - 2\mathcal{L}_2) \\ \frac{h^2}{2v^2}(\mathcal{L}_1 - 2\mathcal{L}_2) + \mathcal{L}_2 \end{bmatrix} = \begin{bmatrix} -g \\ 0 \end{bmatrix}, \quad \mathcal{L}_1 = \frac{v^2}{h}\frac{\partial v}{\partial z} + v\frac{\partial h}{\partial z}, \quad \mathcal{L}_2 = \frac{v}{2}\frac{\partial h}{\partial z}$$

though since both characteristics are in the direction of the velocity, *i.e.*, the same direction, it is not needed.

Finally, we find from the conservation form the shock speeds:

$$v_{s1} = \frac{1}{2}\frac{v_L^2 - v_R^2}{v_L - v_R} \qquad \text{and} \qquad v_{s2} = \frac{1}{2}\frac{h_L v_L - h_R v_R}{h_L - h_R}$$

Note that if there is no jump in velocity, $v_{s2}$ follows the characteristic curve for $\lambda_2$.

Finally, we should also note that any factor $\alpha \neq 1$ works to make $\mathbf{A}$ diagonalizable and it might be useful to look at the limits $\alpha \to 1^+$ and $\alpha \to 1^-$. However, these do not need to have a conservation form either in general and thus exhibit similar problems to equation (2).

## 1.2 Dispersion

We follow the original paper[1] and now re-derive a dispersion relation using a first-order approximation and no gravity, *i.e.*, $g = 0$. In particular, we look at the decay or growth in time using an exponential and a complex wave in space with a phase shift $\delta$. This differs slightly from the paper, but the same result is obtained. The reason we consider decay or growth in time will be apparent from the final result. Our Ansatz is thus

$$h(t,z) = h_0 \cdot (1 + \varepsilon e^{\mathrm{i}\,kz + \sigma t})$$
$$v(t,z) = v_0 \varepsilon \delta e^{\mathrm{i}\,kz + \sigma t}$$

with the unknown phase shift $\delta$ ($|\delta| = 1$) and an unspecified amplitude $v_0$. The initial height $h_0$ is set by us and the perturbation $\varepsilon \ll 1$ is small.

$$\frac{\partial h}{\partial t} = -v\frac{\partial h}{\partial z} - \frac{1}{2}h\frac{\partial v}{\partial z}$$

$$\Rightarrow \quad \sigma h_0 \varepsilon e^{\mathrm{i}\,kz + \sigma t} = -\frac{1}{2}\,\mathrm{i}\,k v_0 h_0 \varepsilon \delta e^{\mathrm{i}\,kz + \sigma t} + \mathcal{O}(\varepsilon^2)$$
$$\Rightarrow \quad \sigma = -\frac{1}{2}\,\mathrm{i}\,v_0 k \delta + \mathcal{O}(\varepsilon) \quad \Rightarrow \quad \frac{1}{v_0 \delta} = -\,\mathrm{i}\,\frac{k}{2\sigma} + \mathcal{O}(\varepsilon)$$

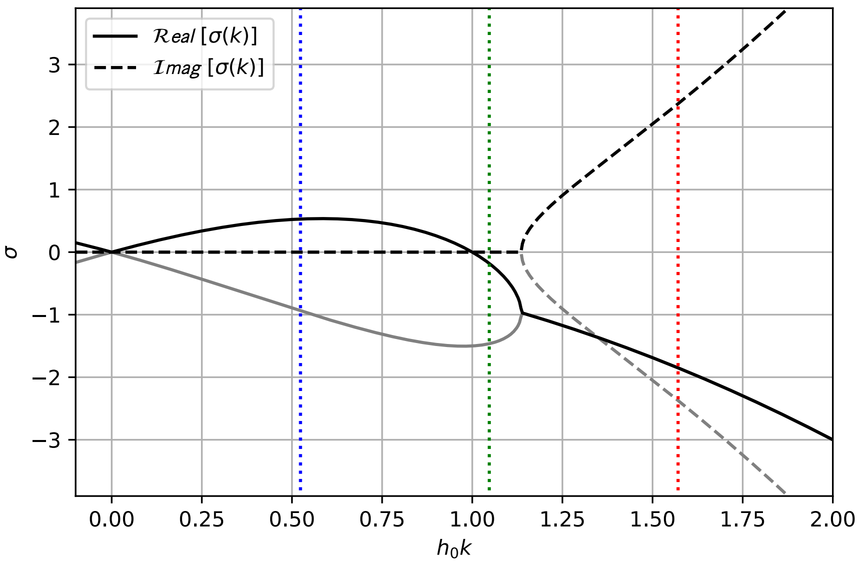


Figure 1: Dispersion relation using both solutions given $\nu = \frac{1}{2}$, $\frac{\gamma}{\rho} = 5$ and $h_0 = 1$. Note the positive value for small wave numbers $k$. At larger wave numbers, oscillations appear but they still decay.

This first results gives us a relation for $v_0\delta$ in terms of $\sigma$, and thus also resolves the unknown phase shift $\delta$ and amplitude $v_0$. The second equation is more involved and thus Mathematica was used to obtain:

$$\begin{aligned}\sigma &= -3\nu k^2 + \mathrm{i}\,\frac{\gamma}{\rho}\frac{k}{h_0 v_0 \delta}\left(1 - h_0^2 k^2\right) + \mathcal{O}(\varepsilon)\\ &\approx -3\nu k^2 - \frac{1}{2}\frac{\gamma}{\rho}\frac{k^2}{\sigma h_0}\left(1 - h_0^2 k^2\right)\end{aligned}$$

Finally, we now solve for $\sigma$ to find the dispersion relation (under the assumption of dimensionless parameters):

$$\sigma \approx -\frac{3}{2}k^2\nu \pm \frac{1}{2}\sqrt{\frac{\gamma}{\rho}\frac{k^2}{h_0}2(1 - h_0^2k^2) + 9k^4\nu^2} \tag{5}$$

It should now be clear why we considered decay or growth in time: if we had considered oscillations in time, the right-hand side of this result would include an imaginary unit. We also note that this is identical to the paper[1], except that the paper ignores one set of the solution. This is because we really only care in this analysis about growth, that is, any $k$ such that $\sigma > 0$.

The dispersion relation (5) describes a stability boundary at $h_0k = 1$. An example is shown in figure 1. The viscosity acts as a stabilizer, but cannot completely counteract when surface tension is present, *i.e.*, $\gamma > 0$. These instabilities occur at small wave numbers $k$ ($\sigma > 0$ or, equivalently, $h_0k < 1$) and thus large wave lengths. The largest growing wave number is given in the paper[1], and was again re-derived using Mathematica (with much struggle and re-definitions

$\nu \to \frac{\nu}{h_0^2}$, $\frac{\gamma}{\rho} \to \frac{\gamma}{\rho}\frac{1}{h_0^3}$, and $k \to kh_0$, thus eliminating $h_0$ from the equations):

$$k_{\max} = \frac{1}{h_0}\sqrt{\frac{1}{2+\sqrt{18\frac{\nu}{h_0}\frac{\rho}{\gamma}}}} \tag{6}$$

It should also be noted that this dispersion relation holds for the auxiliary problem (4), since the term differing is quadratic in $\varepsilon$.

For completeness, we state the stability boundary for the inviscid case and the case of large viscosity (in the limit $\nu \to \infty$), again identical to the paper[1]:

$$\sigma_{\nu=0} = \frac{1}{2}\sqrt{\frac{\gamma}{\rho}\frac{k^2}{h_0}2(1-h_0^2k^2)}, \qquad \sigma_{\nu\to\infty} = \frac{\gamma}{\rho}\frac{1}{6}\frac{1}{\nu h_0}\left(1-h_0^2k^2\right)$$

### 1.3 Reformulation

The height $h$ must always be non-negative. Let us therefore define a new function $f(t,z) = \log h(t,z)$ and note that $f : T \times \mathbb{R} \to \mathbb{R}$. We find:

$$\begin{cases} \frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} + 3e^{-2f}\nu\frac{\partial}{\partial z}\left(e^{2f}\frac{\partial v}{\partial z}\right) - g \\ \frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial z} - \frac{1}{2}\frac{\partial v}{\partial z} \\ p = \gamma e^{-2f}\left(\frac{1}{\left(e^{-2f}+\left(\frac{\partial f}{\partial z}\right)^2\right)^{1/2}} - \frac{\left(\frac{\partial f}{\partial z}\right)^2+\frac{\partial^2 f}{\partial z^2}}{\left(e^{-2f}+\left(\frac{\partial f}{\partial z}\right)^2\right)^{3/2}}\right) \end{cases} \tag{7}$$

This 'logarithmic' formulation can help tremendously with numerical stability in many, but not all, situations. For all results shown we have used this formulation. Note that adding artificial viscosity to the height $h$ will lead to a factor $\mu\left(\frac{\partial^2 f}{\partial z^2} + \left(\frac{\partial f}{\partial z}\right)^2\right)$.

## 2 Results

### 2.1 Falling fluid thread

We start with the idea of a falling fluid thread, that is no surface tension $\gamma = 0$, inviscid $\nu = 0$, and with the initial condition $v_0 = 0$. Furthermore, we set $g = -10$ with periodic boundaries and some random initial height $h_0(z)$. The result is shown in figure 2 and corresponds to our analytic result.

### 2.2 Validation of dispersion

Next, we validate the dispersion relation. In particular, note that we have the exact solution for small perturbations $\varepsilon$ at small times. In general, the solution is written as a superposition of the two $\sigma$ values, but only the potentially-unstable solution $\sigma^+$ is of interest here. We have the following analytic solutions:

$$h(t,z) = h_0 \cdot \left(1 + \varepsilon e^{\mathrm{i}\,kz+\sigma^+t}\right), \qquad v(t,z) = \frac{2\sigma^+}{\mathrm{i}\,k}\varepsilon e^{\mathrm{i}\,kz+\sigma^+t}$$

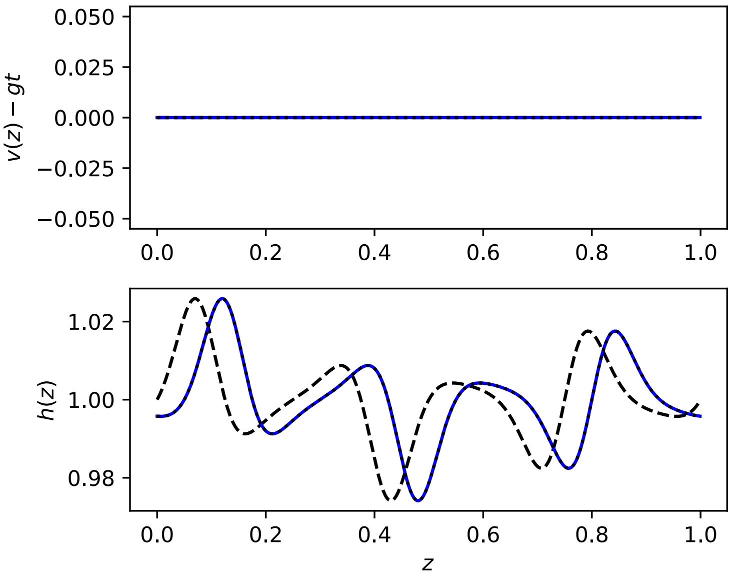


Figure 2: Numerical (blue line) and analytic (dotted) solution at $t = 0.1$ for $v_0 = 0$ without surface tension and viscosity. Periodic boundaries are applied and $g = -10$. The initial height is shown as a dashed line.

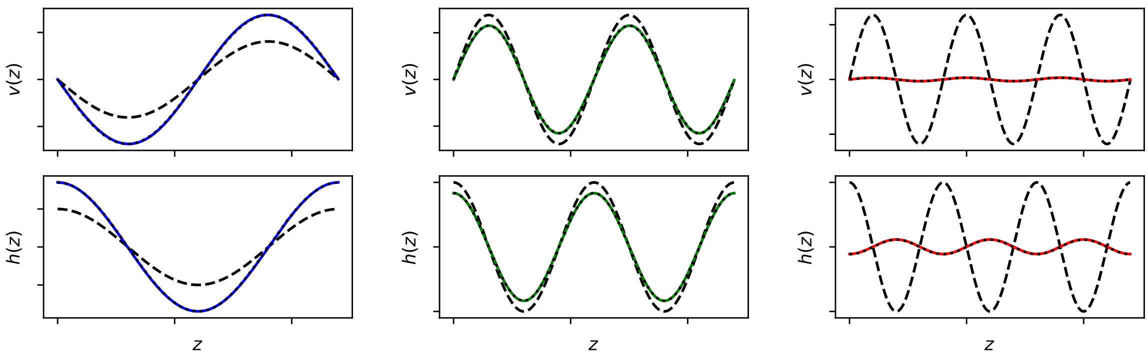


Figure 3: Dispersion for different wave numbers at $t = 0$ (dashed) and at $t = 1$ (colored lines for numerical, black dotted for analytical) for a perturbation of $\varepsilon = 0.01$. Colors and parameters match figure 1.

For convenience in the simulation we use trigonometric functions by looking, without loss of generality, only at the real part. The results are shown in figure 3 with wave numbers corresponding to the colors indicated in figure 1. As expected, the numerical results match the analytic solution. In particular, we see the growth at a low wave number (blue), the decay at a moderate wave number (green), and the decay plus phase shift at a high wave number (red).

## 2.3 Riemann-like problems

In this section we look at Riemann-like problems with the initial condition

$$v_0(z) = \begin{cases} 1 & z \in (0.2, 0.4) \\ \frac{1}{2} & \text{else} \end{cases} \qquad \text{and} \qquad h_0 = 1.$$

We consider the surface tension-free limit, *i.e.*, no pressure term. We start with the auxiliary problem, where we add the regular artificial viscosity, and finish with the original problem, where we add the proper viscous term.

### 2.3.1 The auxiliary problem

From our analysis on the auxiliary problem (4), we have derived the shock velocities and thus can draw the shock lines in the $x$-$t$ diagram. For the Burgers' equation they are given by:

$$t_1(z) = \frac{v_0^L + v_0^R - \sqrt{\left(v_0^L + v_0^R\right)^2 - 8gz}}{2g}, \quad t_1|_{g=0} = \frac{2}{v_0^L + v_0^R} z$$

The shocks arising from the height are:

$$t_2(z) = \frac{v_0^L h_0^L - v_0^R h_0^R - \sqrt{\left(v_0^L h_0^L - v_0^R h_0^R\right)^2 - 4\left(h_0^L - h_0^R\right)^2 z}}{\left(h_0^L - h_0^R\right) g}, \quad t_2|_{g=0} = 2\frac{\left(h_0^L - h_0^R\right)}{v_0^L h_0^L - v_0^R h_0^R} z$$

Finally, as we have noted, without a jump in velocity $v$, the shock from the height is equal to the characteristic for $\lambda_2$:

$$t_c(z) = \frac{v - \sqrt{v^2 - 4gz}}{g}, \quad t_c|_{g=0} = \frac{2}{v} z$$

The numerical results match these theoretical predictions as shown in figure 4. The deviations at large times are due to not considering the rarefaction and the associated velocity decrease properly.

### 2.3.2 The original problem

Let us now apply the same to the original problem (2), using the proper viscous term from (1). The result is shown in figure 5 for the case without gravity. The case with gravity follows the same principles as was seen for the Burgers' equation in the auxiliary problem.

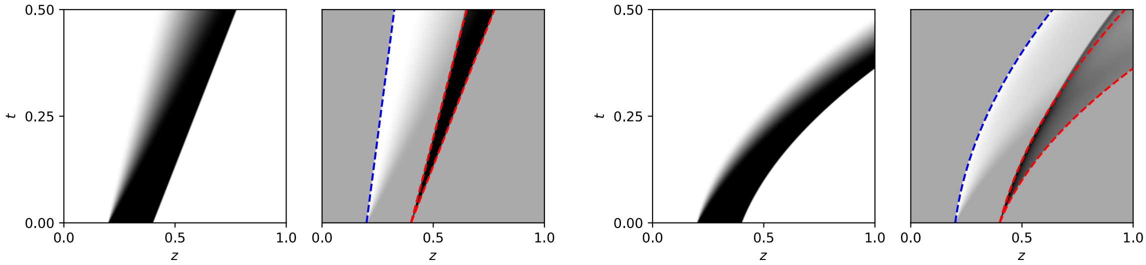


Figure 4: Step in velocity evolving over time using the auxiliary problem (4) without and with gravity. The blue line shows the characteristic and shock on the left side, which are identical due to the lack of a jump in velocity from the rarefaction solution. The red lines show the shock lines on the right.

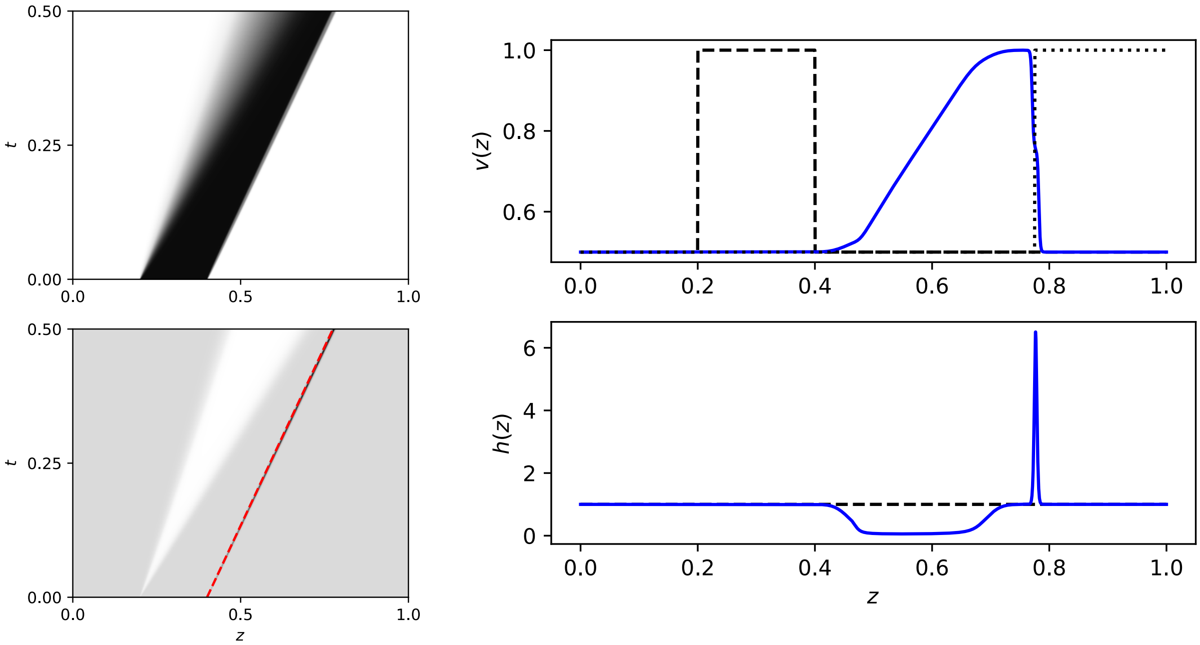


Figure 5: Step in velocity evolving over time using the original problem (1) without gravity. Only one shock line is shown in the $x$-$t$ diagram. Also shown is the result after some time.

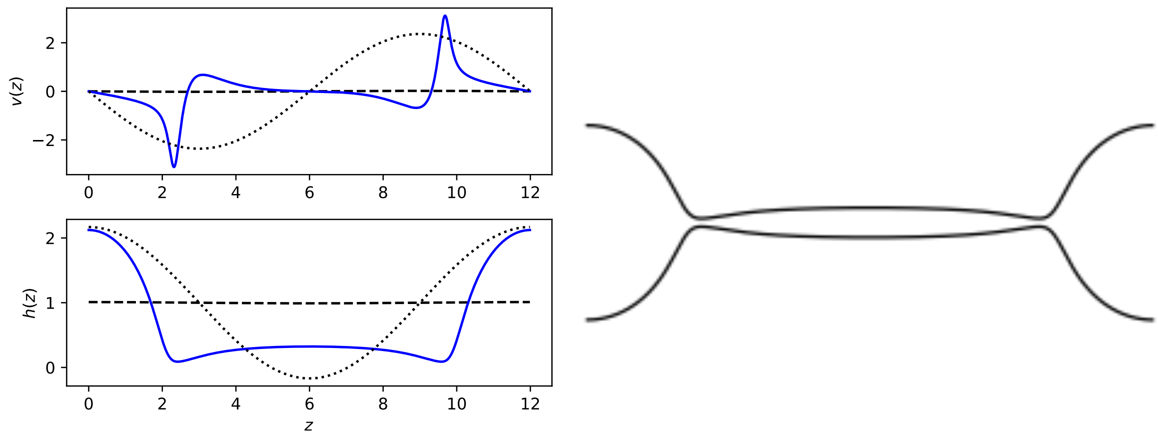


Figure 6: Dispersion for the unstable wave number in figure 1 after longer times, before numerically unstable. Also shown separately is the height profile mirrored around 0.

We recognize at least one shock, though the peak in the height and second step in the Burgers' equation suggest that these are likely two shocks. I am not sure if these are numerical artifacts or if indeed we have two shocks in this system. If it is indeed the latter case, then these shock likely move at the same speed and at the same location. They separated here due to the artificial viscosity and numerical errors.

We also note that the height decreases to zero over the rarefaction solution of the Burgers' equation, indicating a separation of the fluid. Indeed, the size of this decrease matches the rarefaction solution and moves with the velocity of the flow.

## 2.4 Drop formation

Let us now turn back to the drop formation. For drops to form, we perturb the flow velocity upstream while the upstream height is fixed and can be though of as a nozzle. We now look at the evolution downstream. This proved to be a very difficult task as the simulation tends to become unstable with the formation of drops. In the original paper by Eggers and Dupont[1], they solved this issue by using an adaptive mesh. This is beyond the scope of this project but we actually already observed drops forming.

In the validation of the dispersion relation, we looked at the growing mode. By running this until shortly before it blows up, we observe what is shown in figure 6. We recognize the clear drop structure found in the paper. Therefore, we conclude that the paper observes the growth of the mode as the fluid travels downstream and it is a matter of fine-tuning the parameters to obtain drop formation.

In figure 7 we show our result after tuning the parameters. Note that as the drops become more pronounced, the simulation becomes unstable and an increase in resolution and time-stepping would be necessary. The parameters must be tuned such that the numerical instability are just outside the domain while still showing the drop formation. We also note that from the dispersion relation (5) we have a dependence on the height $h_0$. Gravity stretches the fluid, decreasing the height and allowing stable modes to turn into unstable.


Figure 7: Drops forming using our code with $\nu = 1.32 \times 10^5$, $\frac{\gamma}{\rho} = 3.35 \times 10^{-6}$, $g = -0.01$ and $\mu = 2 \times 10^{-5}$. These parameters where chosen for numerical stability and do not represent a specific fluid.

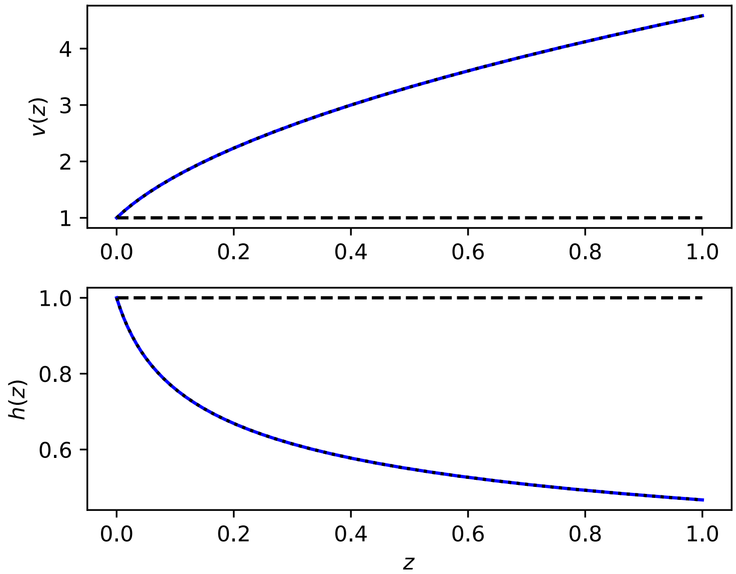


Figure 8: Effect of gravity for a steady flow from the nozzle with $v_0 = 1$ and a radius $h_0 = 1$. Solid blue lines are the numerical result, the dotted the analytic solutions and the dashed lines are the initial conditions.

### 2.4.1 Adding gravity

We conclude this project by looking at the effect of gravity for a nozzle. As mentioned for the drop, gravity stretches the fluid and here we show how. We consider the flow out of a fixed nozzle with initial velocity $v_0 \neq 0$ and height $h_0$, which is the nozzle radius. We further ignore viscosity and surface tension. Finally, we search for the steady state.

Under these assumptions, we can solve the equations and find

$$v\frac{\mathrm{d}v}{\mathrm{d}z} = -g \quad \Rightarrow \quad v(z) = \sqrt{v_0^2 - 2gz}$$

and

$$v\frac{\mathrm{d}h}{\mathrm{d}z} + \frac{1}{2}h\frac{\mathrm{d}v}{\mathrm{d}z} = 0 \quad \Rightarrow \quad \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}z} = \frac{1}{2}\frac{g}{v_0^2 - 2gz}$$

$$\Rightarrow \quad h(z) = h_0\sqrt[4]{\frac{v_0^2}{v_0^2 - 2gz}}. \tag{8}$$

The result is shown in figure 8, with both the numerical and analytic results overlapping.

The condition that $v_0 \neq 0$ is essential as the boundary condition cannot be enforced on $h_0$ otherwise. Numerically, this shows by a dependence on the step size $\Delta z$. As $\Delta z \to 0$ we obtain a step function with $h(z > 0) = 0$ and no smooth solution exists.

# References

[1] Jens Eggers and Todd F. Dupont. Drop formation in a one-dimensional approximation of the Navier–Stokes equation. *Journal of Fluid Mechanics*, 262:205–221, 1994.

# Addendum

The linearized differential operator for $\mathcal{L}$ defined by $\frac{\partial q}{\partial t} = \mathcal{L}(q)$ is (without some potentially missed terms):

$$
\begin{aligned}
\partial_v \mathcal{L}_v(v,h) &= -\frac{\partial v}{\partial z} - v\frac{\partial}{\partial z} + 3\nu\frac{\partial^2}{\partial z^2} \\
\partial_h \mathcal{L}_v(v,h) &= 6\nu\frac{1}{h}\frac{\partial v}{\partial z}\frac{\partial}{\partial z} - 6\nu\frac{1}{h^2}\frac{\partial h}{\partial z}\frac{\partial v}{\partial z} \\
&\quad - \frac{\gamma}{\rho}\left[\frac{\partial}{\partial z}\left(\frac{1}{h\left(1+\frac{\partial h}{\partial z}^2\right)^{3/2}}\frac{\partial h}{\partial z}\right)\frac{\partial}{\partial z} + \frac{1}{h\left(1+\frac{\partial h}{\partial z}^2\right)^{3/2}}\frac{\partial h}{\partial z}\frac{\partial^2}{\partial z^2}\right. \\
&\qquad + \frac{\partial}{\partial z}\left(\frac{1}{h^2\sqrt{1+\frac{\partial h}{\partial z}^2}}\right) + \frac{1}{h^2\sqrt{1+\frac{\partial h}{\partial z}^2}}\frac{\partial}{\partial z} \\
&\qquad + \frac{\partial}{\partial z}\left(\frac{1}{\left(1+\frac{\partial h}{\partial z}^2\right)^{3/2}}\right)\frac{\partial^2}{\partial z^2} + \frac{1}{\left(1+\frac{\partial h}{\partial z}^2\right)^{3/2}}\frac{\partial^3}{\partial z^3} \\
&\qquad \left. - \frac{\partial}{\partial z}\left(\frac{3}{\left(1+\frac{\partial h}{\partial z}^2\right)^{5/2}}\frac{\partial^2 h}{\partial z^2}\frac{\partial h}{\partial z}\right)\frac{\partial}{\partial z} - \frac{3}{\left(1+\frac{\partial h}{\partial z}^2\right)^{5/2}}\frac{\partial^2 h}{\partial z^2}\frac{\partial h}{\partial z}\frac{\partial^2}{\partial z^2}\right] \\
\partial_v \mathcal{L}_h(v,h) &= -\frac{\partial h}{\partial z} - \frac{1}{2}h\frac{\partial}{\partial z} \\
\partial_h \mathcal{L}_h(v,h) &= -v\frac{\partial}{\partial z} - \frac{1}{2}\frac{\partial v}{\partial z}
\end{aligned}
$$

The eigenvalues of this operator, given some stationary solution $[v, h]$, provide the linear stability of the stationary solution.

The pressure gradient is explicitly:

$$
\begin{aligned}
\frac{1}{\gamma}\frac{\partial p}{\partial z} &= \frac{1}{h^2\left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]^{1/2}}\frac{\partial h}{\partial z} + \frac{1}{h\left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]^{3/2}}\frac{\partial h}{\partial z}\frac{\partial^2 h}{\partial z^2} \\
&\quad + \frac{1}{\left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]^{3/2}}\frac{\partial^3 h}{\partial z^3} - \frac{3}{\left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]^{5/2}}\left(\frac{\partial^2 h}{\partial z^2}\right)^2\frac{\partial h}{\partial z}
\end{aligned}
$$