Differential Algebraic Equations
Exercise Sheet I – Introductory Considerations and Basic Notions

A Multi-body systems
Many multibody system can be modelled

$$M\dot{x} = Mv \tag{1a}$$
$$M\dot{v} = Ax + Kv - G(x)^T\lambda + f, \tag{1b}$$
$$0 = g(x), \tag{1c}$$

with $x(t) \in \mathbb{R}^n$, $\lambda(t) \in \mathbb{R}^m$, $M \in \mathbb{R}^{n,n}$ symmetric strictly positive definite, $A \in \mathbb{R}^{n,n}$, $g\colon \mathbb{R}^n \to \mathbb{R}^m$, and

$$G(x) := \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(x) & \cdots & \frac{\partial g_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1}(x) & \cdots & \frac{\partial g_m}{\partial x_n}(x) \end{bmatrix},$$

is the Jacobian of g at x, with $m < n$ and $G(x)$ having full rank for any x with $g(x) = 0$.

1. Show that the pendulum (l: Exa. 1.?) can be brought into the form (1).

2. Find, write down, and explain another multibody system that can be modelled in the form of (1).

B Separation of differential and algebraic parts
Under certain regularity assumptions, a general DAE

$$\mathcal{F}(t, x, \dot{x}) = 0$$

can locally be brought into the form

$$\dot{x}_1 = \mathcal{L}(t, x_1, x_2), \tag{2a}$$
$$x_2 = \mathcal{R}(t, x_1), \tag{2b}$$

by means of differentiation, elimination, and the splitting $x = [x_1, x_2]$.

1. Bring the *circuit* example (lecture: Exa. 1.?) into the form (2) with $\mathcal{L}$ and $\mathcal{R}$ defined explicitly. How many differentiations did you need? What were the necessary regularity conditions?

2. Bring the *pendulum* example (lecture: Exa. 1.?) into the form (2). Here, $\mathcal{L}$ and $\mathcal{R}$ may be defined implicitly. How many differentiations did you need? What were the necessary regularity conditions?

3. How can one express consistency conditions for an intial value x_0 by means of formulation (2).

C Modelling the pendulum anew
The pendulum can also be modelled as a pure ODE, e.g., by means of certain *generalized coordinates*. Present such a model and discuss advantages of the different formulations in view of a general multibody system or *Automatic Modelling* (L1).

D Spatially discretized linearized Navier-Stokes equations
In simulations of flows, equations of the form

$$M\dot{v} = Av - B^T p, \quad v(0) = v_0 \in \mathbb{R}^n \tag{3a}$$
$$0 = Bv \tag{3b}$$

with $v(t) \in \mathbb{R}^n$, $p(t) \in \mathbb{R}^m$, A, $M \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{m,n}$, such that M is invertible as is $BM^{-1}B^T$. Equation (2) typically represents a spatially discretized and linearized Navier-Stokes equation.

1. Reformulate (3) in the form of (2).How many differentiations did you need? What were the necessary regularity conditions?

2. Write down an ODE initial value problem for v and express its solution explicitly. (Hint: *Variation of Constants*)

3. Prove that any v that solves the initial value problem (3) also solves the ODE from **D**.2. What about the converse direction?