# Exercise 3 – Gramians

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We consider the linear time invariant system

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{aligned}
$$

with $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,p}$, and $C \in \mathbb{R}^{q,n}$.

For $A$ stable, the infinite *reachability* and *observability* Gramians $\mathcal{P}$ and $\mathcal{Q}$ are given as

$$
\begin{aligned}
\mathcal{P} &= \int_0^\infty e^{A\tau} B B^T e^{A^T \tau} d\tau \quad \text{and} \\
\mathcal{Q} &= \int_0^\infty e^{A^T\tau} C^T C e^{A\tau} d\tau.
\end{aligned}
$$

Show

1. the Gramians $\mathcal{P}$ and $\mathcal{Q}$ satisfy the *Lyapunov* equations
$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0$$
and
$$A^T\mathcal{Q} + \mathcal{Q}\mathcal{A} + C^TC = 0.$$
2. that they are symmetric positive definite,
3. that the eigenvalues of $\mathcal{P}\mathcal{Q}$ are invariant under state transformations, and
4. that the state transformation $x(t) \leftarrow Tx(t)$ with
$$T = \Sigma^{-\frac{1}{2}} V^T R$$
makes the transformed Gramians balanced, e.g. $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{Q}}$ are diagonal and equal. Here $S$ and $R$ are factors of $\mathcal{P} = S^TS$ and $\mathcal{Q} = R^TR$ so that $V$ and $\Sigma$ can be defined through the *Singular Value Decomposition* of $SR^T$:
$$U\Sigma V^T = SR^T.$$