

## Exercise 3 – Gramians

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We consider the linear time invariant system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}$$

with  $A \in \mathbb{R}^{n,n}$ ,  $B \in \mathbb{R}^{n,p}$ , and  $C \in \mathbb{R}^{q,n}$ .

For  $A$  stable, the infinite *reachability* and *observability* Gramians  $\mathcal{P}$  and  $\mathcal{Q}$  are given as

$$\begin{aligned}\mathcal{P} &= \int_0^\infty e^{A\tau} B B^T e^{A^T \tau} d\tau \quad \text{and} \\ \mathcal{Q} &= \int_0^\infty e^{A^T \tau} C^T C e^{A\tau} d\tau.\end{aligned}$$

Show

1. the Gramians  $\mathcal{P}$  and  $\mathcal{Q}$  satisfy the *Lyapunov* equations

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0$$

and

$$A^T \mathcal{Q} + \mathcal{Q}A + C^T C = 0.$$

2. that they are symmetric positive definite,
3. that the eigenvalues of  $\mathcal{P}\mathcal{Q}$  are invariant under state transformations, and
4. that the state transformation  $x(t) \leftarrow Tx(t)$  with

$$T = \Sigma^{-\frac{1}{2}} V^T R$$

makes the transformed Gramians balanced, e.g.  $\tilde{\mathcal{P}}$  and  $\tilde{\mathcal{Q}}$  are diagonal and equal. Here  $S$  and  $R$  are factors of  $\mathcal{P} = S^T S$  and  $\mathcal{Q} = R^T R$  so that  $V$  and  $\Sigma$  can be defined through the *Singular Value Decomposition* of  $SR^T$ :

$$U\Sigma V^T = SR^T.$$