Exercise 3 – Gramians

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July 16, 2020

We consider the linear time invariant system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned}$$

with $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,p}$, and $C \in \mathbb{R}^{q,n}$.

For A stable, the infinite *reachability* and *observability* Gramians $\mathcal{P}$ and $\mathcal{Q}$ are given as

$$\begin{aligned}\mathcal{P} &= \int_0^\infty e^{A\tau}BB^*e^{A^*\tau}d\tau \quad \text{and}\\ \mathcal{Q} &= \int_0^\infty e^{A^*\tau}C^*Ce^{A\tau}d\tau.\end{aligned}$$

Show

1. the Gramians $\mathcal{P}$ and $\mathcal{Q}$ satisfy the *Lyapunov* equations

   $$A\mathcal{P} + \mathcal{P}A^* + BB^* = 0$$

   and

   $$A^*\mathcal{Q} + \mathcal{Q}\mathcal{A} + C^*C = 0.$$

2. that they are symmetric positive definite,
3. that the eigenvalues of $\mathcal{P}\mathcal{Q}$ are invariant under state transformations, and
4. that the state transformation $x(t) \leftarrow Tx(t)$ with

   $$T = \Sigma^{-\frac{1}{2}}V^*R$$

   makes the transformed Gramians balanced, e.g. $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{Q}}$ are diagonal and equal. Here S and R are factors of $\mathcal{P} = S^*S$ and $\mathcal{Q} = R^*R$ so that V and Σ can be defined through the *Singular Value Decomposition* of SR^*:

   $$U\Sigma V^* = SR^*.$$