MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Introductory Course on Model Reduction of Linear Time Invariant Systems

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June 21 – July 2, 2021 Shanghai University, Shanghai, China

SHU Remote Course

Overview

Typical Situation

- Fry a steak
- The cook controls the heat at the fireplace
- and observes the process, e.g. via measuring the temperature in the inner

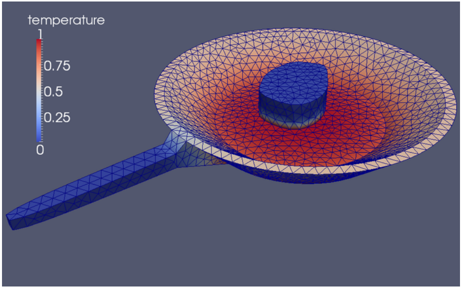
Typical Situation

- The model

$$
\begin{aligned}
\dot{\theta} &= \nabla \cdot (\nu \nabla \theta) \quad \text{in } (0, \infty) \times \Omega, \\
\theta &= u, \quad\quad\quad\quad\quad \text{at the plate}, \\
\theta(0) &= 0.
\end{aligned}
$$

- The cook controls the heat at the fireplace, which we denote by u
- and observes the process, e.g. he measures the temperature y in the center: $y = f(\theta)$.

Simulation



- The model:

$$\dot{\theta} = \nabla \cdot (\nu \nabla \theta),$$
$$\theta = u,$$
$$\theta(0) = 0.$$

- The cook controls the heat u
- and observes the process via $y = f(\theta)$.

- A *Finite Element* discretization of the problem leads to the finite dimensional model

$$E\dot{\theta}(t) = A\theta(t) + Bu(t), \quad \theta(0) = 0, \tag{1}$$
$$y(t) = C\theta(t), \tag{2}$$

a linear time invariant system.

Linear State Space System

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{3a}$$
$$y(t) = Cx(t) + Du(t), \tag{3b}$$

with

- $x(t) \in \mathbb{R}^n$: the system's state
- $u(t) \in \mathbb{R}^m$: the input or control
- $y(t) \in \mathbb{R}^q$: the output or measurements
- $n, m, q \in \mathbb{N}$: the system dimensions

Linear State Space System

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$$y(t) = Cx(t) + Du(t), \tag{3b}$$

with

- $E \in \mathbb{R}^{n\times n}$: the identity or the mass matrix
- $A \in \mathbb{R}^{n\times n}$: the system matrix
- $B \in \mathbb{R}^{n\times m}$: the input matrix
- $C \in \mathbb{R}^{q\times n}$: the output matrix
- $D \in \mathbb{R}^{q\times n}$: the throughput

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- $x(t) \in \mathbb{R}^{n}$: the system's state
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- $y(t) \in \mathbb{R}^{q}$: the output or measurements
- n, m, $q \in \mathbb{N}$: the system dimensions

We will assume that $E = I$ and denote the LTI (3) by (A, B, C, D).

Some Preliminary Thoughts

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

A simple question...

What is x?

- it is a physical state in the model – like the temperature
- in practise, we may not access it – only the measurement $y = Cx$
- it is but a mathematical object as a part of a model
- furthermore, as we will see later, the state x can be severely changed e.g. in the course of model reduction

The state x can be seen...

...as nothing but an artificial object of the model for the input to output behavior

$$\mathrm{G}\colon u \mapsto y$$

of an abstract system P:

$$y \longleftarrow \boxed{\quad \mathrm{P} \quad} \longleftarrow u$$

that maps an input u to the corresponding output y.

The state x can be seen…

…as nothing but an artificial object of the model for the input to output behavior

$$G\colon u \mapsto y$$

of an abstract system P:

$$y \longleftarrow \boxed{\begin{array}{c} y(t) = Cx(t) \\ \dot{x}(t) = Ax(t) + Bu(t) \end{array}} \longleftarrow u$$

that maps an input u to the corresponding output y.

Transfer Function in Time-Domain

If P is modelled trough an (A, B, C, D) system, then the function G can be defined via

$$\mathrm{G}\colon u \mapsto y\colon y(t) = C\Big[e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,\mathrm{d}s\Big] + Du(t),$$

known as the formula of *variation of constants*.

This is in **time-domain**: A function u depending on time $t \in [0, \infty)$ is mapped onto a function y depending on time $t \in [0, \infty)$.

Introducing Frequency-Domain

Through the Laplace transform $\mathcal{L}$ and its inverse $\mathcal{L}^{-1}$, we can switch between time-domain and frequency-domain representations of the input and output signals:

$$U(s) := \mathcal{L}\{u\}(s) := \int_0^\infty e^{-st} u(t)\, \mathrm{d}t,$$

where $s \in \mathbb{C}$ is the *frequency* and

$$y(t) := \mathcal{L}^{-1}\{Y\}(t) := \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\gamma - iT}^{\gamma + iT} e^{s} Y(s)\, \mathrm{d}s$$

where $\gamma \in \mathbb{R}$ is chosen such that the contour path of the integration is the domain of convergence of Y.

Laplace Transform of an LTI

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

With the basic properties of the Laplace transform

- $\dot{X}(s) := \mathcal{L}\{\dot{x}\}(s) - x(0) = s\mathcal{L}\{x\}(s) = sX(s) - x(0)$
- and linearity $\mathcal{L}\{Ax\}(s) = AX(s)$

with zero initial value $x(0) = 0$, the (A, B, C, D) system defines the transfer function

$$G(s) := C(sI - A)^{-1}B + D$$

in frequency domain.

Realizations

Fact

An LTI (A, B, C, D) always defines a transfer function

$$G(s) = C(sI - A)^{-1}B + D$$

which is a matrix $G \in \mathbb{R}^{q\times m}$ with coefficients that are rational functions of s.

Question

Given a rational matrix function $s \mapsto G(s) \in \mathbb{R}^{q\times m}$, is there an

$$(A, B, C, D)$$

system, so that $G(s) = C(sI - A)^{-1}B + D$?

Realizations

given G, find (A, B, C, D),
$G(s) = C(sI - A)^{-1}B + D$

If there is one such (A, B, C, D), then there are infinitely many:

- For $T \in \mathbb{R}^{n \times n}$ invertible, also $(TAT^{-1}, TB, CT^{-1}, D)$ is a realization:

$$C(sI - A)^{-1}B + D = CT^{-1}(sI - TAT^{-1})^{-1}TB + D.$$

- Moreover, also

$$\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D\right)$$

is a realization of G.

Realizations

Facts and Thoughts on Realizations

- If G is *proper*, then there is a realization (A, B, C, D) as a state space system.
- This realization is by no means unique.
- The dimension of the state can be arbitrary large. What is the smallest possible dimension? (cf. *model reduction*)
- What is a good choice for the state?

Remark: A transfer function $G\colon s \mapsto G(s) \in \mathbb{R}^{q\times m}$ with coefficients that are rational functions in s, is *proper*, if in each coefficient the polynomial degree of the numerators does not exceed the degree of denominators.

Controllability and Observability

Based on the previous considerations, we can say that

- The states of an LTI system (A, B, C, D) are just a part of a model that realizes a transfer function G
- The transfer function G describes how controls u lead to outputs y
- As seen above in the example, there can be states that are neither affected (*controlled*) by the inputs nor seen (*observed*) by the outputs
- These states are obviously not needed to realize the input to output behavior of G.

We will give a thorough characterization of the *controllable* and *observable* states of an LTI.

Theorem (Kalman Canonical Decomposition)

Given an LTI (A, B, C, D), there is a state space transformation T such that the transformed system $(TAT^{-1}, TB, CT^{-1}, D)$ has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + Du,$$

with the subsystem $(A_{co}, B_{co}, C_{co}, D)$ being controllable and observable, while the remaining states $x_{\bar{c}o}$, $x_{c\bar{o}}$, or $x_{\bar{c}\bar{o}}$ are not controllable, not observable, or neither of them.

For a constructive proof of the Theorem, see Ch. 3.3 of [Zhou, Doyle, Glover '96]

Outcomes of the Kalman Decomposition

For any state space system (A, B, C, D), there is a transformation T so that the transformed states $T^{-1}x$ decompose into

- x_{co} - controllable and observable
- $x_{c\bar{o}}$ - controllable but not observable
- $x_{\bar{c}o}$ - observable but not controllable
- $x_{\bar{c}\bar{o}}$ - not observable and not controllable

Moreover, for the transfer function, it holds that

$$G(s) = C(sI - A)^{-1}B = C_{co}(sI - A_{co})^{-1}B_{co}.$$

Conclusion from the Kalman Decomposition

What does this mean for us and a transfer function $G(s)$?

- The minimal dimension of a realization is the dimension of x_{co} in the *Kalman Canonical Decomposition*
- Such a realization is called minimal realization
- It is the starting point for further model reduction. (Throwing out $x_{\bar{c}o}$ etc. does not effect $G(s)$ and is typically not considered a model reduction)
- There are algorithm to reduce a realization to a minimal one, cf. [Varga '90].
- In practice, the uncontrolled and unobserved states play a role and they may cause troubles. (check the literature for zero dynamics)

Summary

- LTI as model for physical processes (e.g. heat transfer)
- The input/output behavior is often more important than the state
- Moreover, the state need not have a meaning
- State space systems (A, B, C, D) can be seen as realizations of transfer functions
- A transfer function has multiple realizations
- The minimal realizations are of our interest
- A stable system can have stable realization
- Minimal and stable realization can be balanced

More on the LTI topics

K. Zhou, J. C. Doyle, and K. Glover.
Robust and Optimal Control. (Chapter 3 for LTI)
Prentice-Hall, Upper Saddle River, NJ, 1996.

A. Varga.
Computation of irreducible generalized state-space realizations.
Kybernetika, 26(2):89–106, 1990.

A. Gaul.
Leckerbraten – a lightweight Python toolbox to solve the heat equation on arbitrary domains
`https://github.com/andrenarchy/leckerbraten`, 2013.

J. Heiland.
The slides, additional material, and information on this course
`https://wwww.janheiland.de/20-shu-mor/`, 2020.

Basic Notions of Norms

Ingredients of a normed space $(V, \|\cdot\|)$:

- A linear space V over $\mathbb{C}$ (or $\mathbb{R}$)
- and a functional

  $$\|\cdot\|\colon V \to \mathbb{R}$$

  that has the following properties:

  i) $\|\alpha v\| = |\alpha|\|v\|$,
  ii) $\|v + w\| \leq \|v\| + \|w\|$, and
  iii) $\|v\| \geq 0$ and $\|v\| = 0$ if, and only if, $v = 0$,

  for any v, $w \in V$ and any $\alpha \in \mathbb{C}$ (or $\mathbb{R}$).

Norms of Linear Operators

If $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$, then for the space of linear maps $\bigl(V \to W\bigr)$ a norm is defined via

$$\|G\|_* := \sup_{v\in V, v\neq 0} \frac{\|Gv\|_W}{\|v\|_V}.$$

This is the norm for $G\colon V \to W$ that is induced by $\|\cdot\|_V$ and $\|\cdot\|_W$. There can be other norms that are not induced.

Norms of Signals

Common norms and spaces for the input or output signals

$$u\colon [0,\infty) \to \mathbb{R}^m \quad \text{or} \quad y\colon [0,\infty) \to \mathbb{R}^q$$

- All definitions work similar for finite time intervals $[0, T]$ or the whole time axis $(-\infty, \infty)$.
- Where it is clear from the context, we will drop the superscripts p and m that denote the dimension of the signals.

Norms of Signals

Definition

The $\mathcal{L}_1^m$ norm

$$\|u\|_{\mathcal{L}_1} := \int_0^\infty \sum_{i=1}^m |u_i(t)| \, \mathrm{d}t$$

defines the $\mathcal{L}_1^m$ space of integrable (summable) functions

$$\mathcal{L}_1^m := \left\{ u \colon [0,\infty) \to \mathbb{R}^m : \|u\|_{\mathcal{L}_1} < \infty \right\}$$

on the positive time axis.

Norms of Signals

Definition

The $\mathcal{L}_\infty^m$ norm

$$\|u\|_{\mathcal{L}_\infty} := \max_{i=\{1,\dots,m\}} \sup_{t>0} |u_i(t)|$$

defines the $\mathcal{L}_\infty^m$ space of bounded functions

$$\mathcal{L}_\infty^m := \{u \colon [0,\infty) \to \mathbb{R}^m : \|u\|_{\mathcal{L}_\infty} < \infty\}.$$

Definition

The $\mathcal{L}_2^q$ norm

$$\|y\|_{\mathcal{L}_2} := \Big(\int_0^\infty \sum_{i=1}^{q} |y_i(t)|^2 \, \mathrm{d}t\Big)^{\frac{1}{2}}$$

defines the $\mathcal{L}_2^q$ space of square integrable functions

Norms of Signals

The $\mathcal{L}_2$ norm can also be evaluated in frequency domain

Theorem

For $u \in \mathcal{L}_2$ it holds that

$$\|u\|_{\mathcal{L}_2} = \Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} U(i\omega)^* U(i\omega)\, \mathrm{d}\omega\Big)^{\frac{1}{2}},$$

where U is the Fourier transform of u.

The Fourier transform $\mathcal{F}$ and the Laplace transform $\mathcal{L}$ coincide for $s = i\omega$, $\omega \in \mathbb{R}$ and $u(t) = 0$ for $t \leq 0$:

$$\mathcal{F}(u)(i\omega) := \int_{-\infty}^{\infty} u(t)e^{-i\omega t}\, \mathrm{d}t = \int_{0}^{\infty} u(t)e^{-st}\, \mathrm{d}t = \mathcal{L}(u)(s)$$

Norm of a System

A system G or (A, B, C, D) transfers inputs to outputs.

Ask yourself...

- What does a norm mean for a system?
- What is a large system, what is a small system?

Norm of a System

From the definition of an operator norm:

$$\|G\| = \sup_{u \neq 0} \frac{\|Gu\|}{\|u\|}$$

we derive that for all u:

$$\|y\| = \|Gu\| \leq \|G\|\|u\|.$$

An Answer

For systems, large refers to what extend an input is amplified.
Therefore, $\|G\|$ is often called the *gain*.

Norm of a System

From the definition of an operator norm:

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we derive that for all u:

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With a norm, one can compare two systems G_1 and G_2 via the difference in the output for the same input:

$$\|y_1 - y_2\| = \|G_1 u - G_2 u\| \leq \|G_1 - G_2\|\|u\|.$$

Defining a Norm for Systems

We consider a SISO system $(A, B, C, -)$, i.e $m = q = 1$ and $D = 0$. Consider $(A, B, C, -)$ a with stable and strictly proper transfer function G is stable. Then the *impulse response* of the system

$$g(t) = C \int_0^t e^{A(t-\tau)} B \delta(\tau) \, \mathrm{d}s = Ce^{At}B$$

A system (A, B, C, D) or A is stable, if there exists a $\lambda > 0$, such that $\|e^A t\| \leq e^{-\lambda t}$, for $t > 0$. This means that all eigenvalues of A must have a negative real part.

Tmpulse response: $\delta(\tau) := \begin{cases} 0, \text{ if } t \neq 0, \\ \text{very large, if } t = 0 \end{cases}$ so that $\int_{-\infty}^{\infty} u(\tau)\delta(\tau) \, \mathrm{d}\tau = u(0)$.

Defining a Norm for Systems

We consider a SISO system $(A, B, C, -)$, i.e $m = q = 1$ and $D = 0$.

Consider $(A, B, C, -)$ a with stable and strictly proper transfer function G is stable. Then the *impulse response* of the system

$$g(t) = C \int_0^t e^{A(t-\tau)} B\delta(\tau)\, \mathrm{d}s = Ce^{At}B$$

decays exponentially and

$$\|g\|_{\mathcal{L}_2} = \big(\frac{1}{2\pi} \int_{-\infty}^{\infty} G(i\omega)^* G(i\omega)\, \mathrm{d}\omega\big)^{\frac{1}{2}} =: \|G\|_2 < \infty.$$

Defining a Norm for Systems

We consider a SISO system $(A, B, C, -)$, i.e $m = q = 1$ and $D = 0$. Consider $(A, B, C, -)$ a with stable and strictly proper transfer function G is stable. Then the *impulse response* of the system

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This defines a norm for systems since (Exercise!)

- $G = C(sI - A)^{-1}B$ is indeed the Laplace transform of g
- the functional $\|\cdot\|_2$ for stable and strictly proper transferfunctions is a norm

Furthermore, $\|y\|_{\mathcal{L}_\infty} \leq \|G\|_2 \|u\|_{\mathcal{L}_\infty}$. (Exercise!)

Defining a Norm for Systems

For MIMO systems $(A, B, C, -)$ with $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$, with a stable and strictly proper transferfunction $\mathcal{G}\colon s \to \mathbb{R}^{q\times m}$, the $\mathcal{H}_2$ norm is defined as

$$\|G\|_2 := \Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}\, G(i\omega)^* G(i\omega)\, \mathrm{d}\omega\Big)^{\frac{1}{2}}.$$

Fact

This is the norm of the *Hardy* space $\mathcal{H}_2$ of matrix functions that are analytic in the open right half of the complex plane. Stable and strictly proper transfer functions are in $\mathcal{H}_2$.

Defining a Norm for Systems

For a stable and proper transfer function one can define the $\mathcal{H}_\infty$ norm:

$$\|G\|_\infty := \sup_{\omega\in\mathbb{R}} \sigma_{\max}\big(G(i\omega)\big),$$

where $\sigma_{\max}\big(G(i\omega)\big)$ is the largest singular value of $G(i\omega)$.

Fact 1

This is the norm of the *Hardy* space $\mathcal{H}_\infty$ of matrix functions that are analytic in the open right half of the complex plane and bounded on the imaginary axis. Stable and strictly proper transfer functions are in $\mathcal{H}_\infty$.

Fact 2

The $\mathcal{H}_\infty$-norm is induced by the $\mathcal{L}_2$ norm:

$$\|G\|_\infty = \sup_{u\in\mathcal{L}_2, u\neq 0} \frac{\|Gu\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}}.$$

Approximation Problems - Model Reduction

Output errors in time-domain

Comparing the original system G and the reduced system $\hat{G}$:

$$
\begin{aligned}
\|y-\hat{y}\|_{\mathcal{H}_2} &\leq \|G-\hat{G}\|_{\mathcal{H}_\infty}\|u\|_{\mathcal{H}_2} &&\implies \|G-\hat{G}\|_{\mathcal{H}_\infty} < \mathrm{tol} \\
\|y-\hat{y}\|_{\mathcal{H}_\infty} &\leq \|G-\hat{G}\|_{\mathcal{H}_2}\|u\|_{\mathcal{H}_2} &&\implies \|G-\hat{G}\|_{\mathcal{H}_2} < \mathrm{tol}
\end{aligned}
$$

Approximation Problems - Model Reduction

Output errors in time-domain

Comparing the original system G and the reduced system $\hat{G}$:

$$\begin{aligned} \|y-\hat{y}\|_{\mathcal{H}_2} &\leq \|G-\hat{G}\|_{\mathcal{H}_\infty}\|u\|_{\mathcal{H}_2} & \Longrightarrow \|G-\hat{G}\|_{\mathcal{H}_\infty} < \mathrm{tol} \\ \|y-\hat{y}\|_{\mathcal{H}_\infty} &\leq \|G-\hat{G}\|_{\mathcal{H}_2}\|u\|_{\mathcal{H}_2} & \Longrightarrow \|G-\hat{G}\|_{\mathcal{H}_2} < \mathrm{tol} \end{aligned}$$

$\mathcal{H}_\infty$-norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable $\mathcal{H}_\infty$-norm bound.
$\mathcal{H}_2$-norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)
$\|G\|_H := \sigma_{\max}$	optimal Hankel norm approximation (AAK theory).

Overview

Realizations

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \quad \text{with transfer function} \\ y(t) &= Cx(t) + Du(t), \quad G(s) = C(sI - A)^{-1}B + D, \end{cases}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$ is called a realization of Σ.

Realizations

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the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is called a realization of Σ.

Realizations are not unique!

Transfer function is invariant under state-space transformations,

$$\mathcal{T} : \begin{cases} x &\to Tx, \\ (A, B, C, D) &\to (TAT^{-1}, TB, CT^{-1}, D), \end{cases}$$

Definition

For a linear (time-invariant) system

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the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is called a realization of Σ.

Realizations are not unique!

Transfer function is invariant under addition of uncontrollable/unobservable states:

$$\frac{d}{dt}\begin{bmatrix} x \\ x_1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + \begin{bmatrix} B \\ B_1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + Du(t), \tag{4}$$

$$\frac{d}{dt}\begin{bmatrix} x \\ x_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & C_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + Du(t), \tag{5}$$

for arbitrary $A_j \in \mathbb{R}^{n_j\times n_j}$, $j = 1, 2$, $B_1 \in \mathbb{R}^{n_1\times m}$, $C_2 \in \mathbb{R}^{q\times n_2}$ and any $n_1, n_2 \in \mathbb{N}$.

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \quad \text{with transfer function} \\ y(t) &= Cx(t) + Du(t), \quad G(s) = C(sI - A)^{-1}B + D, \end{cases}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is called a realization of Σ.

Realizations are not unique!

Hence,

$$(A, B, C, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} B \\ B_1 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D \right),$$

$$(TAT^{-1}, TB, CT^{-1}, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & C_2 \end{bmatrix}, D \right),$$

are all realizations of Σ!

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \text{with transfer function} \\ y(t) = Cx(t) + Du(t), & G(s) = C(sI - A)^{-1}B + D, \end{cases}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is called a realization of Σ.

Definition

The McMillan degree of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.
A minimal realization is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

Realizations

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{cases} \quad \begin{array}{l} \text{with transfer function} \\ G(s) = C(sI - A)^{-1}B + D, \end{array}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is called a realization of Σ.

Definition

The McMillan degree of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.
A minimal realization is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

Theorem

A realization (A, B, C, D) of a linear system is minimal $\Longleftrightarrow$
(A, B) is controllable and (A, C) is observable.

Controllability

Definition

The LTI (A, B, C, D) or the pair (A, B) is said to be *controllable* if, for any initial state $x(0) = x_0$, $t_1 > 0$ and final state x_1, there exists a (piecewise continuous) input u such that the solution of (3) satisfies $x(t_1) = x_1$. Otherwise, the system (A, B, C, D) or the pair (A, B) is said to be *uncontrollable*.

Theorem

The following statements are equivalent:

(i.) *The pair (A, B) is controllable.*

(ii.) *The controllability matrix $\mathcal{C} := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$ has full rank.*

(iii.) *The matrix $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ has full rank for all $\lambda \in \mathbb{C}$.*

Observability

Definition

The LTI (A, B, C, D) or the pair (C, A) is said to be *observable* if, for any $t_1 > 0$, the initial state $x(0) = x_0$ can be determined from the time history of the input u and the output y in the interval of $[0, t_1]$. Otherwise, the system (A, B, C, D), or (C, A), is said to be *unobservable*.

Observability is the dual concept of controllability:

Theorem

The pair (C, A) is observable if and only if the pair $(A^\mathsf{T}, C^\mathsf{T})$ is controllable.

Invariance Under State Space Transformation

Theorem

The LTI (A, B, C, D) is controllable (observable) if, and only if, the transformed LTI $(TAT^{-1}, TB, CT^{-1}, D)$ is controllable (observable), where T is a regular matrix.

- Recall that also a transfer function is invariant with respect to state space transformations on its realization.
- Next, we find the states that are at least necessary for the realization of a transfer function. . .

Theorem (Kalman Canonical Decomposition)

Given an LTI (A, B, C, D), there is a state space transformation T such that the transformed system $(TAT^{-1}, TB, CT^{-1}, D)$ has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + Du,$$

with the subsystem $(A_{co}, B_{co}, C_{co}, D)$ being controllable and observable, while the remaining states $x_{\bar{c}o}$, $x_{c\bar{o}}$, or $x_{\bar{c}\bar{o}}$ are not controllable, not observable, or neither of them.

For a constructive proof of the Theorem, see Ch. 3.3 of [Zhou, Doyle, Glover '96]

Overview

Definition

A linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

is stable if its transfer function $G(s)$ has all its poles in the left half plane and it is asymptotically (or Lyapunov or exponentially) stable if all poles are in the open left half plane $\mathbb{C}^- := \{z \in \mathbb{C} \mid \Re(z) < 0\}$.

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Lemma

Sufficient for asymptotic stability is that A is asymptotically stable (or Hurwitz), i.e., the spectrum of A, denoted by $\sigma(A)$, satisfies $\sigma(A) \subset \mathbb{C}^-$.

Note that by abuse of notation, often *stable system* is used for asymptotically stable systems.

Stability

- A system G is stable if all poles of G are located in the left half-plane $\mathbb{C}^-$.

If $m = q = 1$, then $G(s) = \frac{N(s)}{D(s)}$, where $N(s)$ and $D(s)$ are polynomials and the *poles* are the roots of $D(s)$, i.e. those $s \in \mathbb{C}$ for which $D(s) = 0$.

If m, $q > 1$, then one can use the *McMillan* form of G to determine the poles.

Stability

- A system G is stable if all poles of G are located in the left half-plane $\mathbb{C}^-$.
- If (A, B, C, D) is a minimal realization of a stable system G, then the poles of G are the eigenvalues of A.

Mathematical Basics for LTI System 2

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System Norms and System Spaces

Consider a transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathscr{L}_2^m \cong L_2^m(-\infty, \infty)$, with the L_2-norm

$$\|u\|_{\mathcal{H}_2}^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} u(\jmath\omega)^H u(\jmath\omega)\, d\omega.$$

Assume A (asympotically) stable: $\sigma(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} \,:\, \operatorname{re} z < 0\}$.

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$\Longrightarrow y \in \mathscr{L}_2^q \cong L_2^q(-\infty, \infty)$.

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Consequently, the 2-induced operator norm

$$\|G\|_{\mathcal{H}_\infty} := \sup_{\|u\|_{\mathcal{H}_2} \neq 0} \frac{\|Gu\|_{\mathcal{H}_2}}{\|u\|_{\mathcal{H}_2}}$$

is well defined. It can be shown that

$$\|G\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \|G(\jmath\omega)\| = \sup_{\omega \in \mathbb{R}} \sigma_{max}\left(G(\jmath\omega)\right).$$

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System Norms and System Spaces

Consider a transfer function

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Hardy space $\mathcal{H}_\infty$

Function space of matrix-/scalar-valued functions that are analytic and bounded in $\mathbb{C}^+$.

The $\mathcal{H}_\infty$-norm is

$$\|F\|_{\mathcal{H}_\infty} := \sup_{\operatorname{re} s>0} \sigma_{\max}\left(F(s)\right) = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left(F(\jmath\omega)\right).$$

Stable transfer functions are in the Hardy spaces

- $\mathcal{H}_\infty$ in the SISO case (single-input, single-output, $m = q = 1$);
- $\mathcal{H}_\infty^{q\times m}$ in the MIMO case (multi-input, multi-output, $m > 1, q > 1$).

Consider a transfer function

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Paley-Wiener Theorem (Parseval's equation/Plancherel Theorem)

$$L_2(-\infty,\infty) \cong \mathscr{L}_2, \quad L_2(0,\infty) \cong \mathcal{H}_2$$

Consequently, 2-norms in time and frequency domains coincide!

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$\mathcal{H}_\infty$ approximation error

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}$.

$$\|y - \hat{y}\|_{\mathcal{H}_2} = \|Gu - \hat{G}u\|_{\mathcal{H}_2} \leq \|G - \hat{G}\|_{\mathcal{H}_\infty}\|u\|_{\mathcal{H}_2}.$$

$\Longrightarrow$ compute reduced-order model such that $\|G - \hat{G}\|_{\mathcal{H}_\infty} < tol$!

Note: error bound holds in time- and frequency domain due to Paley-Wiener!

Consider stable transfer function

$$G(s) = C\left(sI - A\right)^{-1} B, \quad \text{i.e. } D = 0.$$

Hardy space $\mathcal{H}_2$

Function space of matrix-/scalar-valued functions that are analytic $\mathbb{C}^+$ and bounded w.r.t. the $\mathcal{H}_2$-norm

$$\begin{aligned} \|F\|_{\mathcal{H}_2} :&= \frac{1}{2\pi}\left(\sup_{\operatorname{re}\sigma>0}\int_{-\infty}^{\infty}\|F(\sigma+\jmath\omega)\|_F^2\,d\omega\right)^{\frac{1}{2}} \\ &= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty}\|F(\jmath\omega)\|_F^2\,d\omega\right)^{\frac{1}{2}}. \end{aligned}$$

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$\mathcal{H}_2$ approximation error for impulse response ($u(t) = u_0\delta(t)$)

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$.

$$\|y - \hat{y}\|_{\mathcal{H}_2} = \|Gu_0\delta - \hat{G}u_0\delta\|_{\mathcal{H}_2} \leq \|G - \hat{G}\|_{\mathcal{H}_2}\|u_0\|.$$

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Theorem (Practical Computation of the $\mathcal{H}_2$-norm)

$$\|F\|_{\mathcal{H}_2}^2 = \operatorname{trace} B^T Q B = \operatorname{trace} C P C^T,$$

where P, Q are the controllability and observability Gramians of the corresponding LTI system.

Approximation Problems

Output errors in time-domain

$$
\begin{aligned}
\|y-\hat{y}\|_{\mathcal{H}_2} &\leq \|G-\hat{G}\|_{\mathcal{H}_\infty}\|u\|_{\mathcal{H}_2} && \Longrightarrow \|G-\hat{G}\|_{\mathcal{H}_\infty} < \mathrm{tol} \\
\|y-\hat{y}\|_{\mathcal{H}_\infty} &\leq \|G-\hat{G}\|_{\mathcal{H}_2}\|u\|_{\mathcal{H}_2} && \Longrightarrow \|G-\hat{G}\|_{\mathcal{H}_2} < \mathrm{tol}
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$\mathcal{H}_\infty$-norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable $\mathcal{H}_\infty$-norm bound.
$\mathcal{H}_2$-norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)
Hankel-norm $\|G\|_H := \sigma_{\max}$	optimal Hankel norm approximation (AAK theory).

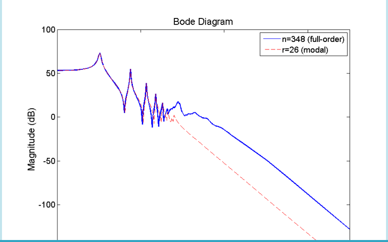
System Norms and System Spaces

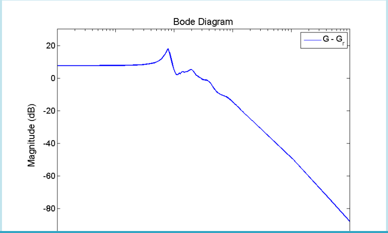
Evaluating system norms is computationally very (sometimes too) expensive.

Other measures

- absolute errors $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_{\mathcal{H}_2}$, $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_{\mathcal{H}_\infty}$ $(j = 1, \ldots, N_\omega)$;
- relative errors $\frac{\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_{\mathcal{H}_2}}{\|G(\jmath\omega_j)\|_{\mathcal{H}_2}}$, $\frac{\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_{\mathcal{H}_\infty}}{\|G(\jmath\omega_j)\|_{\mathcal{H}_\infty}}$;
- "eyeball norm", i.e. look at frequency response/Bode (magnitude) plot: for SISO system, log-log plot frequency vs. $|G(\jmath\omega)|$ (or $|G(\jmath\omega) - \hat{G}(\jmath\omega)|$) in decibels, 1 dB $\simeq 20\log_{10}$(value).

  For MIMO systems, $q \times m$ array of plots G_{ij}.





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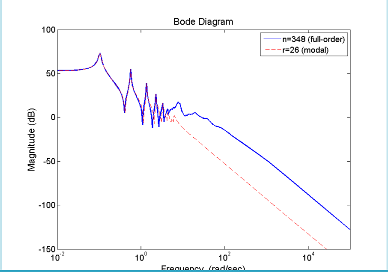

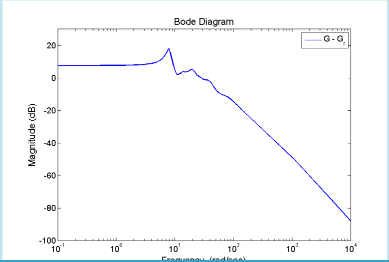

Overview

Outline

Model Reduction — Abstract Definition

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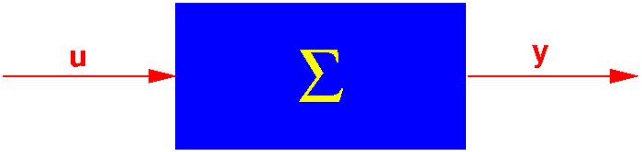
This is the task of model reduction (also: dimension reduction, order reduction).

Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & x(t_0) = x_0, \\ y(t) = g(t, x(t), u(t)) \end{cases}$$

with

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^q$.



Original System

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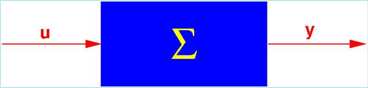

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Model Reduction for Dynamical Systems

Original System

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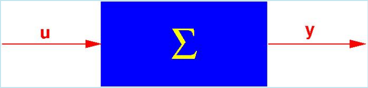
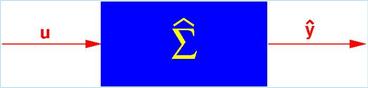


Reduced-Order Model (ROM)

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\hat{y}(t) \in \mathbb{R}^q$.



Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Model Reduction for Dynamical Systems

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$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), \\ y(t) = g(t, x(t), u(t)). \end{cases}$$

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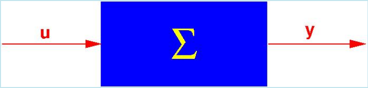

Reduced-Order Model (ROM)

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

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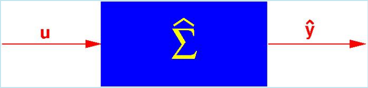

Goal:

$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\|$ for all admissible input signals.

Model Reduction for Dynamical Systems

Original System

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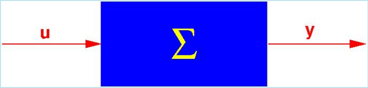

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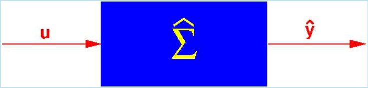

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Secondary goal: reconstruct approximation of x from $\hat{x}$.

Linear, Time-Invariant (LTI) Systems

$$
\begin{array}{rclclll}
E\dot{x} & = & f(t,x,u) & = & Ax + Bu, & E, A \in \mathbb{R}^{n\times n}, & B \in \mathbb{R}^{n\times m}, \\
y & = & g(t,x,u) & = & Cx + Du, & C \in \mathbb{R}^{q\times n}, & D \in \mathbb{R}^{q\times m}.
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$$

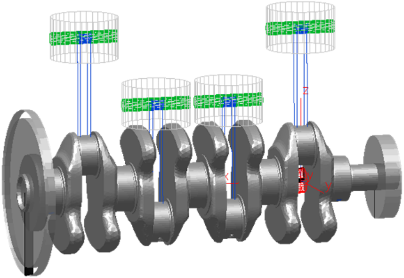
Linear, Time-Invariant Parametric Systems

$$
\begin{aligned}
E(p)\dot{x}(t;p) &= A(p)x(t;p) + B(p)u(t),\\
y(t;p) &= C(p)x(t;p) + D(p)u(t),
\end{aligned}
$$

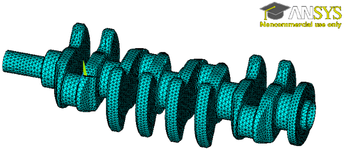
where $A(p), E(p) \in \mathbb{R}^{n\times n}, B(p) \in \mathbb{R}^{n\times m}, C(p) \in \mathbb{R}^{q\times n}, D(p) \in \mathbb{R}^{q\times m}$.

Application Areas

Structural Mechanics / Finite Element Modeling — since ∼1960ies



$\rightsquigarrow$



- Resolving complex 3D geometries $\Rightarrow$ millions of degrees of freedom.
- Analysis of elastic deformations requires many simulation runs for varying external forces.

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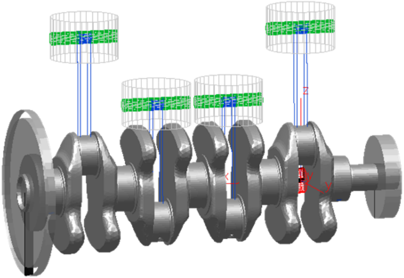

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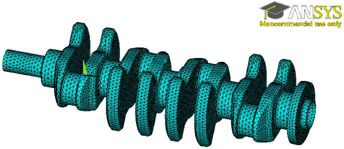

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Standard MOR techniques in structural mechanics: modal truncation, combined with Guyan reduction (static condensation) $\rightsquigarrow$ Craig-Bampton method.

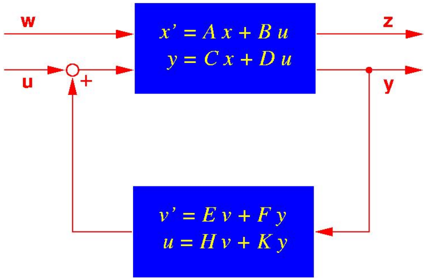
(Optimal) Control since ∼1980ies

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
- output $=$ input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$.



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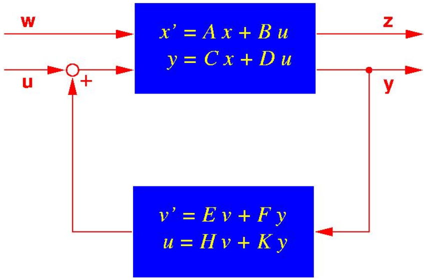

Practical controllers require small N ($N \sim 10$, say) due to

– real-time constraints,
– increasing fragility for larger N.

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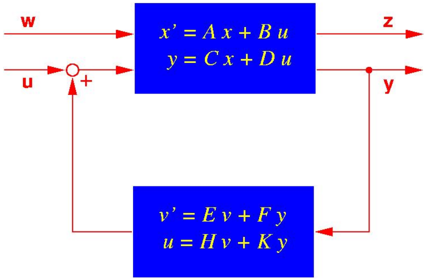

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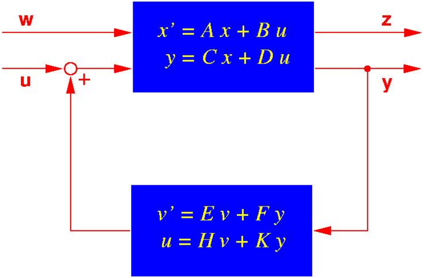

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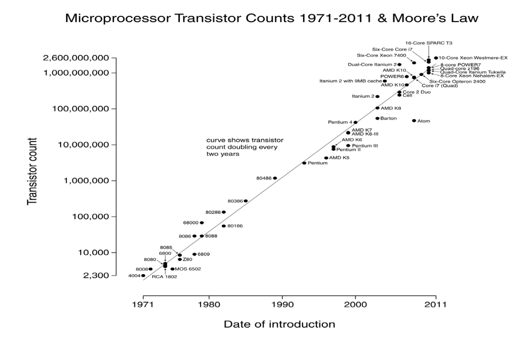
Micro Electronics/Circuit Simulation since ∼1990ies

Progressive miniaturization

- Verification of VLSI/ULSI chip design needs a large number of simulations.
- **Moore's Law (1965/75)** states that the number of on-chip transistors doubles each 24 months.



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- Increase in packing density and multilayer technology requires modeling of interconnect to ensure that thermic/electro-magnetic effects do not disturb signal transmission.

Intel 4004 (1971)	Intel Core 2 Extreme (quad-core) (2007)
1 layer, 10μ technology	9 layers, $45nm$ technology
2,300 transistors	$> 8,200,000$ transistors
64 kHz clock speed	> 3 GHz clock speed.

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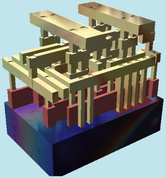

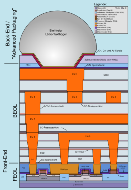

Source: http://en.wikipedia.org/wiki/Image:Silicon_chip_3d.png

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- Here: mostly MOR for linear systems, they occur in micro electronics through modified nodal analysis (MNA) for RLC networks. e.g., when
  - decoupling large linear subcircuits,
  - modeling transmission lines,
  - modeling pin packages in VLSI chips,
  - modeling circuit elements described by Maxwell's equation using partial element equivalent circuits (PEEC).

Application Areas

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$\rightsquigarrow$ Clear need for model reduction techniques in order to facilitate or even enable circuit simulation for current and future VLSI design.

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⇝ Clear need for model reduction techniques in order to facilitate or even enable circuit simulation for current and future VLSI design.

Standard MOR techniques in circuit simulation:
Krylov subspace / Padé approximation / rational interpolation methods.

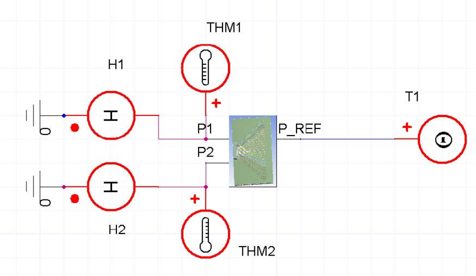
Application Areas

Many other disciplines in computational sciences and engineering like

- computational fluid dynamics (CFD),
- computational electromagnetics,
- chemical process engineering,
- design of MEMS/NEMS (micro/nano-electrical-mechanical systems),
- computational acoustics,
- ...

Electro-Thermic Simulation of Integrated Circuit (IC) [Source: Evgenii Rudnyi, CADFEM GmbH]

- SIMPLORER® test circuit with 2 transistors.



- Conservative thermic sub-system in SIMPLORER:
  voltage $\rightsquigarrow$ temperature, current $\rightsquigarrow$ heat flow.
- Original model: $n = 270.593,\ m = q = 2 \Rightarrow$
  Computing time (on Intel Xeon dualcore 3GHz, 1 Thread):
  - Main computational cost for set-up data $\approx 22min$.
  - Computation of reduced models from set-up data: 44–49sec. ($r = 20$–70).

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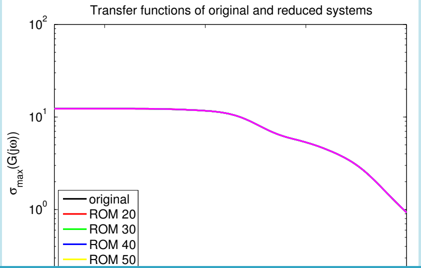
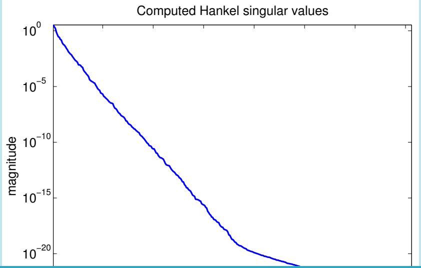
  - Bode plot (MATLAB on Intel Core i7, 2,67GHz, 12GB):
    7.5h for original system, $<$ 1min for reduced system.
  - Speed-up factor: 18 including / $\geq$ 450 excluding reduced model generation!

Bode Plot (Amplitude)



Hankel Singular Values



Motivating Examples

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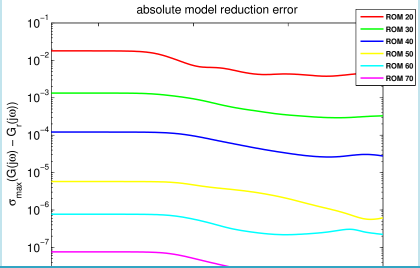
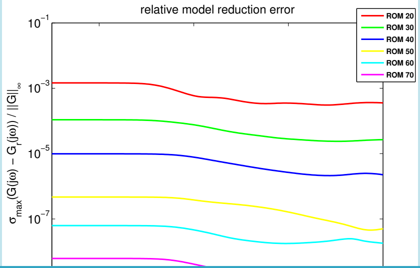
Absolute Error



Relative Error



A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

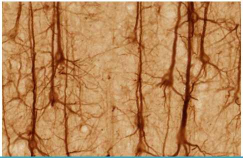
- Simple model for neuron (de-)activation [Chaturantabut/Sorensen 2009]

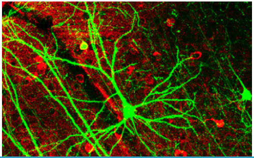
$$\begin{aligned} \epsilon v_t(x,t) &= \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + g, \\ w_t(x,t) &= hv(x,t) - \gamma w(x,t) + g, \end{aligned}$$

with $f(v) = v(v-0.1)(1-v)$ and initial and boundary conditions

$$\begin{aligned} v(x,0) &= 0, & w(x,0) &= 0, & x &\in [0,1] \\ v_x(0,t) &= -i_0(t), & v_x(1,t) &= 0, & t &\geq 0, \end{aligned}$$

where $\epsilon = 0.015, h = 0.5, \gamma = 2, g = 0.05, i_0(t) = 50000t^3 \exp(-15t)$.





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where $\epsilon = 0.015, h = 0.5, \gamma = 2, g = 0.05, i_0(t) = 50000t^3 \exp(-15t)$.

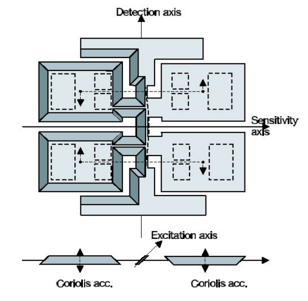
- Parameter g handled as an additional input.
- Original state dimension $n = 2 \cdot 400$, QBDAE dimension $N = 3 \cdot 400$, reduced QBDAE dimension $r = 26$, chosen expansion point $\sigma = 1$.

A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

- Application: inertial navigation.

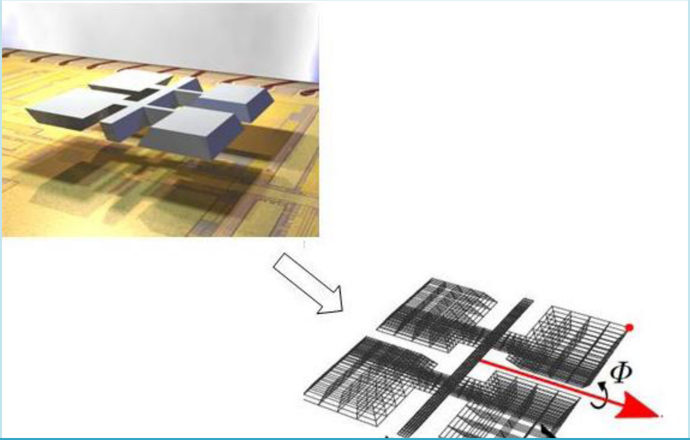


- Voltage applied to electrodes induces vibration of wings, resulting rotation due to Coriolis force yields sensor data.
- FE model of second order: $N = 17.361 \rightsquigarrow n = 34.722,\ m = 1,\ q = 12.$
- Sensor for position control based on acceleration and rotation.

Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Parametric FE model: $M(d)\ddot{x}(t) + D(\Phi, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t).$



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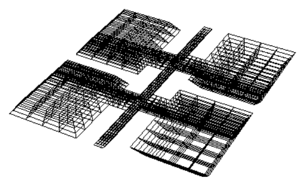
Parametric FE model:

$$M(d)\ddot{x}(t) + D(\Phi, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t),$$

wobei

$$\begin{aligned} M(d) &= M_1 + dM_2, \\ D(\Phi, d, \alpha, \beta) &= \Phi(D_1 + dD_2) + \alpha M(d) + \beta T(d), \\ T(d) &= T_1 + \frac{1}{d}T_2 + dT_3, \end{aligned}$$



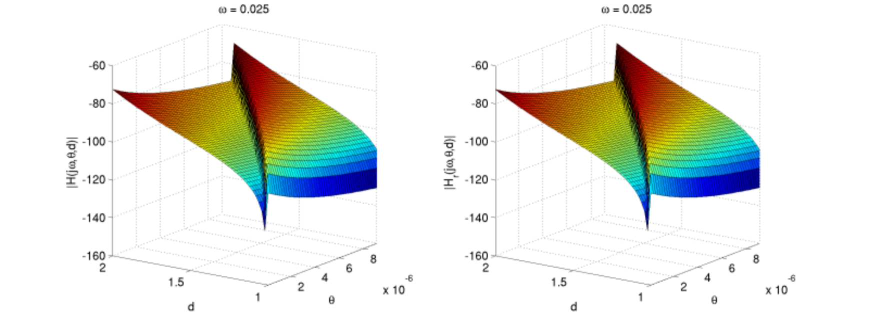
with

- width of bearing: d,
- angular velocity: Φ,
- Rayleigh damping parameters: α, β.

Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Original. . . and reduced-order model.



Overview

Outline

Goals

Requirements: A Model Reduction approach should:

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$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| < \mathrm{tol} \cdot \|u\| \quad \text{for all admissable } u.$$

$\Longrightarrow$ Provide computable error bound/estimate!

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- Preserve physical properties:
  - stability

A G is stable, if all poles of G are in $\mathbb{C}^-$. A system (A, B, C, D) or A is stable, if all eigenvalues of A have a negative real part. Compare: $G(s) = C(sI - A)^{-1}B$

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  $\Longrightarrow$ Provide computable error bound/estimate!
- Preserve physical properties:
  - stability
  - minimum phase

A system G has minimum phase if all zeros of G are in the left half-plane $\mathbb{C}^-$.

Goals

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$\Longrightarrow$ Provide computable error bound/estimate!

- Preserve physical properties:
  - stability
  - minimum phase
  - passivity

A system G is passive if, bluntly speaking, it does not generate energy. Condition for passivity:

$$\int^t u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \text{for all } t \in \mathbb{R}, \quad \text{for all } u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

Definition

A projector $P\colon \mathcal{X} \to \mathcal{X}$ is a linear map, or a matrix, with $P^2 = P$.

Example

- $\mathcal{X} = \mathbb{R}^2$
- $P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is a projector in $\mathcal{X}$

Model Reduction by Projection

Notion and Properties of Projectors

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Model Reduction by Projection

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Model Reduction by Projection

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- For a projector P, the projector $I - P$ onto $\ker P$ is the complementary projector.

Methods:

1. Modal Truncation
2. Rational Interpolation (Padé-Approximation and (rational) Krylov Subspace Methods)
3. Balanced Truncation
4. many more...

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!

The ideal model reduction

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- and given (A, B, C, D), the reduced-order model $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is

$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

Model Reduction by Projection

Model reduction in practise

- Assume that there is a space $\mathcal{V} \subset \mathbb{R}^n$ with $\dim \mathcal{V} = r < n$, such that $x(t) \in \mathcal{V}$ for all time t and input u.
- Take a space $\mathcal{W}$, so that $\mathcal{W}_\perp \oplus \mathcal{V} = \mathbb{R}^n$.
- Galerkin-type projections: $\mathcal{W} = \mathcal{V}$.
- Petrov-Galerkin projections: $\mathcal{W} \neq \mathcal{V}$.
- Find matrices V and W that approximate bases of $\mathcal{V}$ and $\mathcal{W}$, with

$$W^\mathsf{T} V = I_r$$

- Then $V(W^\mathsf{T} V)^{-1} W = VW^\mathsf{T}$ is a projector almost onto $\mathcal{V}$
- Define $\hat{x}(t) := W^\mathsf{T} x(t) \in \mathbb{R}^r$ and define $\tilde{x}(t) := V\hat{x}(t) = VW^\mathsf{T} x(t)$
- If everything is done well, then

$$\|x(t) - \tilde{x}(t)\| = \|x(t) - VW^\mathsf{T} x(t)\| \approx 0$$

- and given (A, B, C, D), the reduced-order model $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is

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...and given an (A, B, C, D) system,

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Why is the ROM defined like this:

It is the (Petrov)-Galerkin condition $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$:

$$W^T \left(\dot{\tilde{x}} - A\tilde{x} - Bu\right) = \qquad W^T \left(VW^T \dot{x} - AVW^T x - Bu\right)$$

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is zero, if, and only if,

$$\dot{\hat{x}} - \hat{A}\hat{x} - \hat{B}u = 0.$$

Projection $\rightsquigarrow$ Rational Interpolation

A Petrov-Galerkin projected ROM interpolates the transfer function:

Theorem 3.3 [Grimme '97, Villemagne/Skelton '87]

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

and $s_* \in \mathbb{C} \setminus (\sigma(A) \cup \sigma(\hat{A}))$, if either

- $(s_* I_n - A)^{-1} B \in \operatorname{range} V$, or
- $(s_* I_n - A)^{-*} C^T \in \operatorname{range} W$,

then the interpolation condition

$$G(s_*) = \hat{G}(s_*).$$

in s_* holds.

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the error transfer function can be written as

$$G(s) - \hat{G}(s) = \quad \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}\right)$$

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&= C\big(I_n - \underbrace{V(sI_r - \hat{A})^{-1}W^T(sI_n - A)}_{=:P(s)}\big)(sI_n - A)^{-1}B.
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If $s_* \in \mathbb{C} \setminus (\sigma(A) \cup \sigma(\hat{A}))$, then $P(s_*)$ is a projector onto $\mathcal{V}$:

range $P(s_*) \subset$ range V, all matrices have full rank $\Rightarrow$ "$=$",

$$P(s_*)^2 = \quad V(s_* I_r - \hat{A})^{-1} W^T (s_* I_n - A) V (s_* I_r - \hat{A})^{-1} W^T (s_* I_n - A)$$

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$$\text{if } (s_* I_n - A)^{-1}B \in \mathcal{V}, \text{ then } (I_n - P(s_*))(s_* I_n - A)^{-1}B = 0,$$

hence

$$G(s_*) - \hat{G}(s_*) = 0 \;\Rightarrow\; G(s_*) = \hat{G}(s_*), \text{ i.e., } \hat{G} \text{ interpolates } G \text{ in } s_*!$$

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$$\text{Analogously,} = \quad C(sI_n - A)^{-1}\Big(I_n - \underbrace{(sI_n - A)V(sI_r - \hat{A})^{-1}W^T}_{=:Q(s)}\Big)B.$$

If $s_* \in \mathbb{C} \setminus (\sigma(A) \cup \sigma(\hat{A}))$, then $Q(s)^H$ is a projector onto $\mathcal{W} \Longrightarrow$

if $(s_* I_n - A)^{-*}C^T \in \mathcal{W}$, *then* $C(s_* I_n - A)^{-1}(I_n - Q(s_*)) = 0,$

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Modal Truncation

Basic method:

Assume A is diagonalizable, $T^{-1}AT = D_A$, project state-space onto A-invariant subspace $\mathcal{V} = \operatorname{span}(t_1, \ldots, t_r)$, $t_k =$ eigenvectors corresp. to "dominant" modes / eigenvalues of A. Then with

$$V = T(:, 1:r) = [\, t_1, \ldots, t_r \,], \quad \tilde{W}^H = T^{-1}(1:r, :), \quad W = \tilde{W}(V^H \tilde{W})^{-1},$$

reduced-order model is

$$\hat{A} := W^H A V = \operatorname{diag}\{\lambda_1, \ldots, \lambda_r\}, \quad \hat{B} := W^H B, \quad \hat{C} = CV$$

Also computable by truncation:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\, \hat{C}, \; C_2 \,], \quad \hat{D} = D.$$

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Properties:

Simple computation for large-scale systems, using, e.g., Krylov subspace methods (Lanczos, Arnoldi), Jacobi-Davidson method.

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Properties:

Error bound:

$$\|G - \hat{G}\|_{\mathcal{H}_\infty} \leq \|C_2\|\|B_2\| \frac{1}{\min_{\lambda \in \sigma(A_2)} |\mathrm{Re}(\lambda)|}.$$

Proof:

$$\begin{aligned}
G(s) = & \quad C(sI - A)^{-1}B + D = CTT^{-1}(sI - A)^{-1}TT^{-1}B + D \\
= & \quad CT(sI - T^{-1}AT)^{-1}T^{-1}B + D \\
= & \quad [\,\hat{C},\ C_2\,] \begin{bmatrix} (sI_r - \hat{A})^{-1} & \\ & (sI_{n-r} - A_2)^{-1} \end{bmatrix} \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix} + D \\
= & \quad \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2,
\end{aligned}$$

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Proof:

$$G(s) = \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2,$$

observing that $\|G - \hat{G}\|_{\mathcal{H}_\infty} = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(C_2(\jmath\omega I_{n-r} - A_2)^{-1}B_2)$, and

$$C_2(\jmath\omega I_{n-r} - A_2)^{-1}B_2 = C_2 \mathrm{diag}\left(\frac{1}{\jmath\omega - \lambda_{r+1}}, \dots, \frac{1}{\jmath\omega - \lambda_n}\right) B_2.$$

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Assume A is diagonalizable, $T^{-1}AT = D_A$, project state-space onto A-invariant subspace $\mathcal{V} = \mathrm{span}(t_1, \ldots, t_r)$, $t_k =$ eigenvectors corresp. to "dominant" modes / eigenvalues of A. Then reduced-order model is

$$\hat{A} := W^H AV = \mathrm{diag}\,\{\lambda_1, \ldots, \lambda_r\}, \quad \hat{B} := W^H B, \quad \hat{C} = CV$$

Also computable by truncation:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\,\hat{C},\ C_2\,], \quad \hat{D} = D.$$

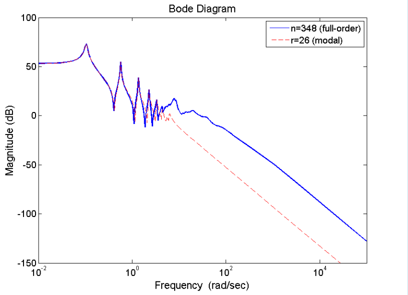
Difficulties:

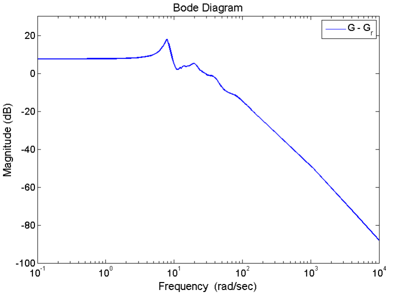
- Eigenvalues contain only limited system information.
- Dominance measures are difficult to compute.
  ([Litz '79] use Jordan canonicial form; otherwise merely heuristic criteria, e.g., [Varga '95]. Recent improvement: dominant pole algorithm.)
- Error bound not computable for really large-scale problems.

Example

BEAM, SISO system from SLICOT Benchmark Collection for Model Reduction, $n = 348$, $m = q = 1$, reduced using 13 dominant complex conjugate eigenpairs, error bound yields $\|G - \hat{G}\|_{\mathcal{H}_\infty} \leq 1.21 \cdot 10^3$

Bode plots of transfer functions and error function





Modal Truncation

Extensions

Base enrichment

Static modes are defined by setting $\dot{x} = 0$ and assuming unit loads, i.e., $u(t) \equiv e_j$, $j = 1, \ldots, m$:

$$0 = Ax(t) + Be_j \quad \Longrightarrow \quad x(t) \equiv -A^{-1}b_j.$$

Projection subspace $\mathcal{V}$ is then augmented by $A^{-1}[\, b_1, \ldots, b_m \,] = A^{-1}B$.

Interpolation-projection framework $\Longrightarrow G(0) = \hat{G}(0)$!

If two sided projection is used, complimentary subspace can be augmented by $A^{-T}C^T \Longrightarrow G'(0) = \hat{G}'(0)$! (If $m \neq q$, add random vectors or delete some of the columns in $A^{-T}C^T$).

Modal Truncation

Extensions

Guyan reduction (static condensation)

Partition states in masters $x_1 \in \mathbb{R}^r$ and slaves $x_2 \in \mathbb{R}^{n-r}$ (FEM terminology)
Assume stationarity, i.e., $\dot{x} = 0$ and solve for x_2 in

$$
\begin{aligned}
0 &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\
\Rightarrow \quad x_2 &= -A_{22}^{-1} A_{21} x_1 - A_{22}^{-1} B_2 u.
\end{aligned}
$$

Inserting this into the first part of the dynamic system

$$
\dot{x}_1 = A_{11} x_1 + A_{12} x_2 + B_1 u, \quad y = C_1 x_1 + C_2 x_2
$$

then yields the reduced-order model

$$
\begin{aligned}
\dot{x}_1 &= (A_{11} - A_{12} A_{22}^{-1} A_{21}) x_1 + (B_1 - A_{12} A_{22}^{-1} B_2) u \\
y &= (C_1 - C_2 A_{22}^{-1} A_{21}) x_1 - C_2 A_{22}^{-1} B_2 u.
\end{aligned}
$$

Pole-Residue Form of Transfer Function

Consider partial fraction expansion of transfer function with $D = 0$:

$$G(s) = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k}$$

with the residues $R_k := (Cx_k)(y_k^H B) \in \mathbb{C}^{q \times m}$.

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Note: this follows using the spectral decomposition $A = XDX^{-1}$, with $X = [\, x_1, \ldots, x_n]$ the right and $X^{-1} =: Y = [y_1, \ldots, y_n]^H$ the left eigenvector matrices:

$$\begin{aligned} G(s) = &\quad C(sI - XDX^{-1})^{-1}B = CX(sI - \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\})^{-1}YB \\ = &\quad [\, Cx_1, \ldots, Cx_n \,] \begin{bmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_n} \end{bmatrix} \begin{bmatrix} y_1^H B \\ \vdots \\ y_n^H B \end{bmatrix} \end{aligned}$$

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Note: $R_k = (Cx_k)(y_k^H B)$ are the residues of G in the sense of the residue theorem of complex analysis:

$$\operatorname{res}(G, \lambda_\ell) = \lim_{s \to \lambda_\ell} (s - \lambda_\ell) G(s) = \sum_{k=1}^{n} \underbrace{\lim_{s \to \lambda_\ell} \frac{s - \lambda_\ell}{s - \lambda_k}}_{= \begin{cases} 0 \text{ for } k \neq \ell \\ 1 \text{ for } k = \ell \end{cases}} R_k = R_\ell.$$

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As projection basis use spaces spanned by right/left eigenvectors corresponding to dominant poles, i.e.. $(\lambda_j,\ x_j,\ y_j)$ with largest

$$\|R_k\|/|\operatorname{re}(\lambda_k)|.$$

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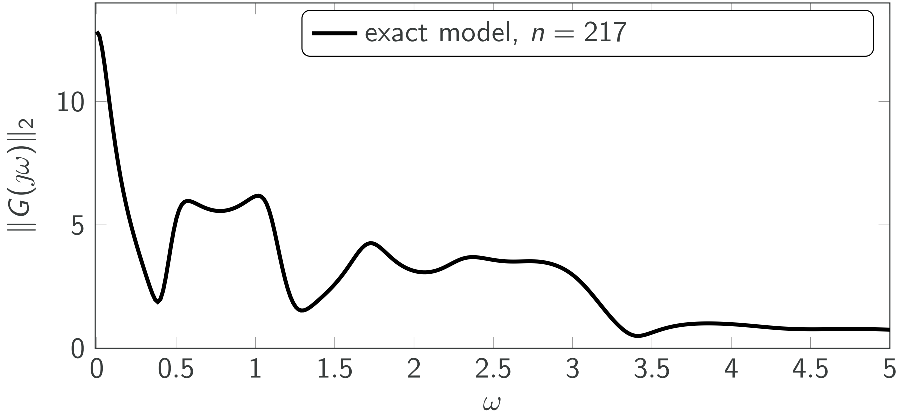
$$\|R_k\|/|\operatorname{re}(\lambda_k)|.$$

Remark

The dominant modes have most important influence on the input-output behavior

Dominant Poles

Random SISO Example ($B,\ C^T \in \mathbb{R}^n$)



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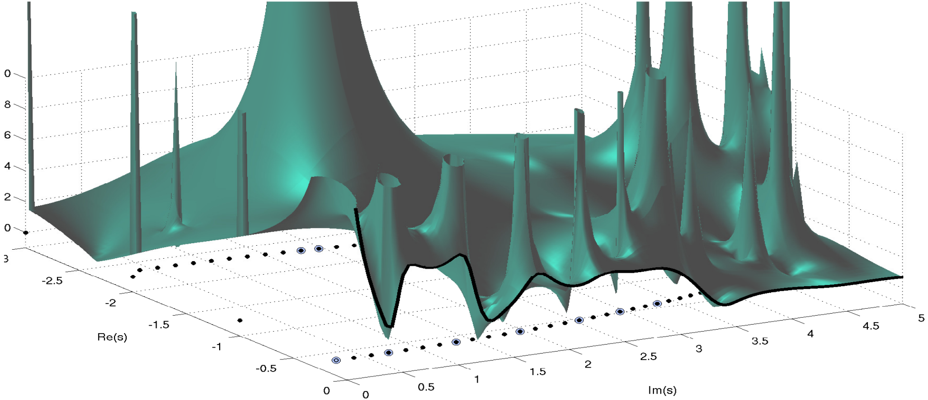

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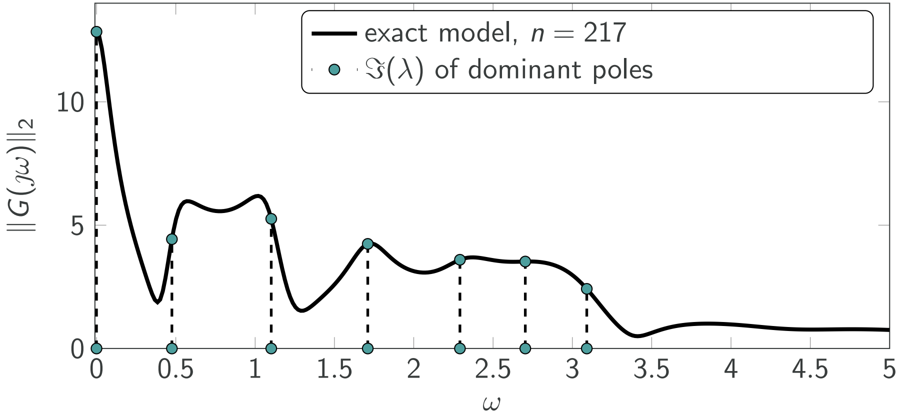

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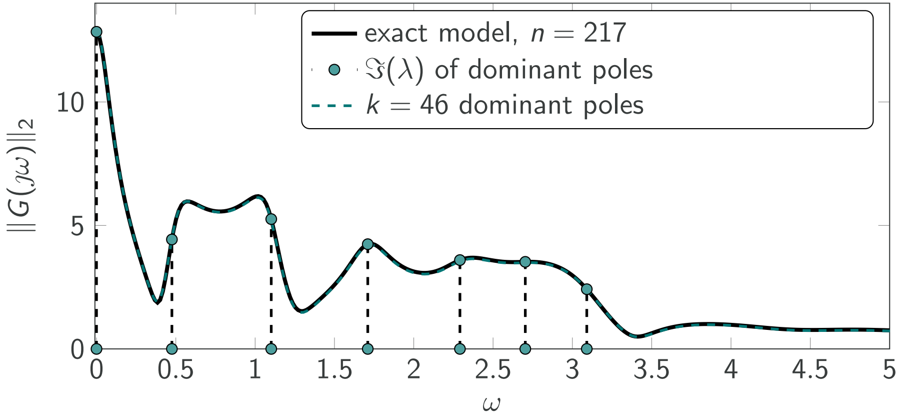

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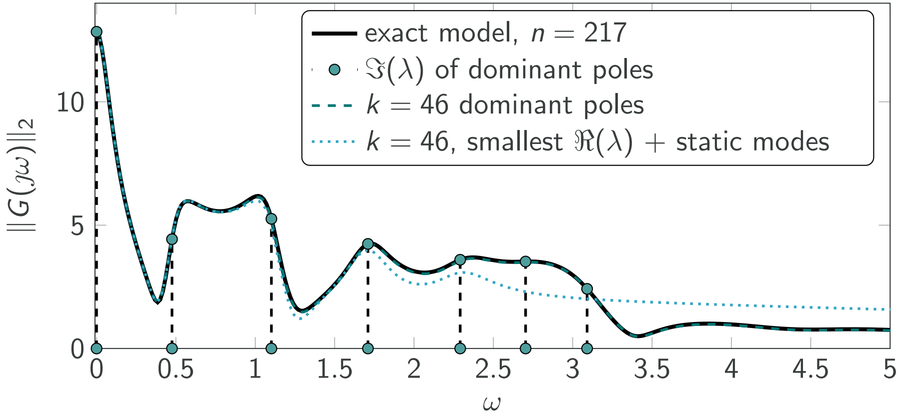
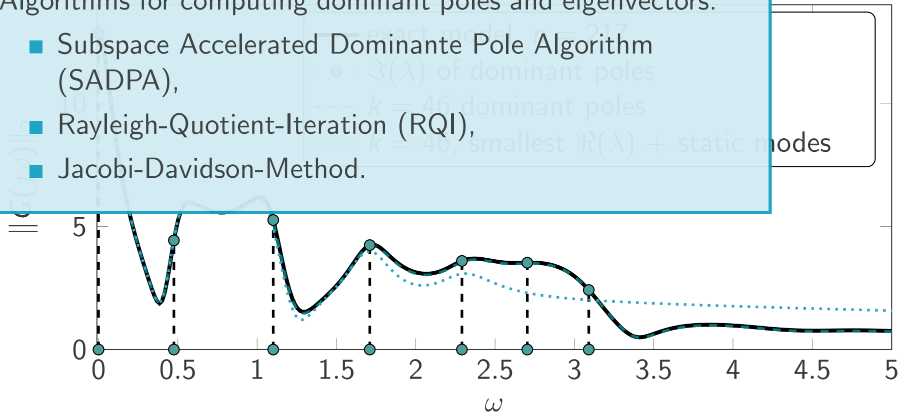


Random SISO Example ($B,\ C^T \in \mathbb{R}^n$)

Algorithms for computing dominant poles and eigenvectors:

- Subspace Accelerated Dominante Pole Algorithm (SADPA),
- Rayleigh-Quotient-Iteration (RQI),
- Jacobi-Davidson-Method.



Overview

1. Introduction to Linear Time Invariant Systems
2. Mathematical Basics for LTI Systems I
3. Mathematical Basics for LTI System 2
4. Introduction to Model Reduction
5. Model Reduction by Projection
6. Gramians and Balanced Realizations
7. Balanced Truncation

Mathematical Basics

If A is stable, then the *Lyapunov* equations

$$A^*P + PA + BB^* = 0$$

and

$$AQ + Q^*A + C^*C = 0$$

have a unique positive definite solutions P and Q, respectively.

- The matrix P is called the the (infinite) controllability Gramian
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- The matrix P is called the the (infinite) controllability Gramian
- and Q is called the (infinite) observability Gramian
- and one can show that P and Q fulfill

$$P = \int_0^\infty e^{A\tau}BB^*e^{A^*\tau}\,\mathrm{d}\tau \quad \text{and} \quad Q = \int_0^\infty e^{A^*\tau}C^*Ce^{A\tau}\,\mathrm{d}\tau.$$

$$A^*P + PA + BB^* = 0$$
$$AQ + Q^*A + C^*C = 0$$

- If P and Q are the Gramians of a stable realization (A, B, C, D),
- then the transformed system $(\hat{A}, \hat{B}, \hat{C}, \hat{D}) = (TAT^{-1}, TB, CT^{-1}, D)$ has the Gramians

$$\hat{P} = TPT^* \quad \text{and} \quad \hat{Q} = (T^{-1})^* QT^{-1}$$

for any regular transformation T.

- For any minimal and stable system (A, B, C, D),
- there are particular transformations T,
- so that the transformed system has Gramians that are equal and diagonal:

$$\hat{P} = \hat{Q} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix},$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$.

These realizations are called Balanced Realizations.

Balanced Realizations

Definition

A realization (A, B, C, D) of a linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \operatorname{diag}\{\sigma_1, \ldots, \sigma_n\} \quad \text{(w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1\text{)}.$$

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When does a balanced realization exist?

Theorem

Given a stable minimal linear system $\Sigma : (A, B, C, D)$, a balanced realization is obtained by the state-space transformation with

$$T_b := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$, $Q = R^T R$ (e.g., Cholesky decompositions) and $SR^T = U \Sigma V^T$ is the SVD of SR^T.

Proof. Exercise!

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$\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

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The Hankel singular values (HSVs) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

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Remark

For non-minimal systems, the Gramians can also be transformed into diagonal matrices with the leading $\hat{n} \times \hat{n}$ submatrices equal to $\operatorname{diag}(\sigma_1, \ldots, \sigma_{\hat{n}})$, and

$$\hat{P}\hat{Q} = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_{\hat{n}}^2, 0, \ldots, 0).$$

see [Laub/Heath/Paige/Ward 1987, Tombs/Postlethwaite 1987].

Overview

Outline

Balanced Truncation

Basic principle:

- Recall: a stable system Σ, realized by (A, B, C, D), is called balanced, if the Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T \;=\; 0, \qquad A^T Q + QA + C^T C \;=\; 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

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- Compute balanced realization of the system via state-space transformation

$$\begin{aligned}\mathcal{T} : (A, B, C, D) \mapsto & \quad (TAT^{-1}, TB, CT^{-1}, D) \\ = & \quad \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right)\end{aligned}$$

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A_{11}, B_1, C_1, D)$.

Balanced Truncation

Motivation:

The HSVs $\sigma(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are system invariants: they are preserved under

$$\mathcal{T} : (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D)$$

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in transformed coordinates, the Gramians satisfy

$$\begin{aligned} (TAT^{-1})(TPT^T) + (TPT^T)(TAT^{-1})^T + (TB)(TB)^T &= 0, \\ (TAT^{-1})^T(T^{-T}QT^{-1}) + (T^{-T}QT^{-1})(TAT^{-1}) + (CT^{-1})^T(CT^{-1}) &= 0 \end{aligned}$$

$$\Rightarrow (TPT^T)(T^{-T}QT^{-1}) = TPQT^{-1},$$

hence $\sigma(PQ) = \sigma((TPT^T)(T^{-T}QT^{-1}))$.

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$\Longrightarrow VW^T$ is a projector, hence BT is a projection method.

Balanced Truncation

Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \dots, \sigma_r$.

Balanced Truncation

Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_{\mathcal{H}_2} \leq \left(2 \sum_{k=r+1}^{n} \sigma_k\right) \|u\|_{\mathcal{H}_2}.$$

Balanced Truncation

Linear, Time-Invariant (LTI) Systems

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu, \qquad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y & = & Cx, \qquad\quad\; C \in \mathbb{R}^{q\times n}, \quad x(-\infty) = 0. \end{array}$$

Alternative to State-Space Operator: Hankel Operator

Instead of

$$\mathcal{S}\colon u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t \in \mathbb{R}.$$

use the Hankel operator: (the future response of the past inputs)

$$\mathcal{H}\colon u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau \quad \text{for } t > 0,$$

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad & A \in \mathbb{R}^{n\times n}, \quad & B \in \mathbb{R}^{n\times m}, \\ y &= Cx, & C \in \mathbb{R}^{q\times n}, \quad & x(-\infty) = 0. \end{aligned}$$

Alternative to State-Space Operator: Hankel Operator

Instead of

$$\mathcal{S}\colon u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\, d\tau \quad \text{for all } t \in \mathbb{R}.$$

use the Hankel operator: (the future response of the past inputs)

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Balanced Truncation

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- The operator $\mathcal{H}$ is compact $\Rightarrow$ $\mathcal{H}$ has discrete SVD
  - $\rightarrow$ The Hankel singular values: $\{\sigma_j\}_{j=1}^{\infty} : \ \sigma_1 \geq \sigma_2 \geq \ldots \geq 0$
  - $\rightarrow$ An SVD-type approximation of the linear map $\mathcal{H}$ is possible!

Balanced Truncation

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$\Downarrow$

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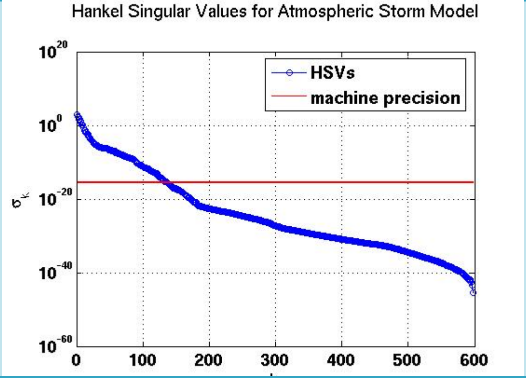

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$\Rightarrow$ solution: Adamjan-Arov-Krein (AAK Theory, 1971/78).

But: computationally unfeasible for large-scale systems.

The *Hankel Singular Values* are Singular Values!

Theorem

Let P, Q be the controllability and observability Gramians of an LTI system Σ. Then the Hankel singular values $\sigma(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the singular values of the Hankel operator associated to Σ.

Assume the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}B_1\\ B_2\end{bmatrix}u, \quad y = [\, C_1,\ C_2\,]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + Du$$

is in balanced coordinates.

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Particularly, if $G(0) = \hat{G}(0)$ ("zero steady-state error") is required, one can apply the same condensation technique as in Guyan reduction: instead of $x_2 = 0$, set $\dot{x}_2 = 0$. This yields the reduced-order model

$$\begin{aligned} \dot{x}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u, \\ y &= (C_1 - C_2A_{22}^{-1}A_{21})x_1 + (D - C_2A_{22}^{-1}B_2)u, \end{aligned}$$

with

- the same properties as the reduced-order model w.r.t. stability, minimality, error bound, but $\hat{D} \neq D$;
- zero steady-state error, $G(0) = \hat{G}(0)$ as desired.

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Note:

- A_{22} invertible as in balanced coordinates, $A_{22}\Sigma_2 + \Sigma_2 A_{22}^T + B_2 B_2^T = 0$ and (A_{22}, B_2) controllable, $\Sigma_2 > 0 \;\Rightarrow\; A_{22}$ stable.
- If the original system is not balanced, first compute a minimal realization by applying balanced truncation with $r = \hat{n}$.

Balancing-Related Methods

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n > 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

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Classical Balanced Truncation (BT) [Mullis/Roberts '76, Moore '81]

- $P =$ controllability Gramian of system given by (A, B, C, D).
- $Q =$ observability Gramian of system given by (A, B, C, D).
- P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0.$$

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LQG Balanced Truncation (LQGBT)

[JONCKHEERE/SILVERMAN '83]

- P/Q = controllability/observability Gramian of closed-loop system based on LQG compensator.
- P, Q solve dual algebraic Riccati equations (AREs)

$$\begin{aligned} 0 &= AP + PA^T - PC^T CP + B^T B, \\ 0 &= A^T Q + QA - QBB^T Q + C^T C. \end{aligned}$$

Balancing-Related Methods

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

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and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Balanced Stochastic Truncation (BST) [Desai/Pal '84, Green '88]

- $P =$ controllability Gramian of system given by (A, B, C, D), i.e., solution of Lyapunov equation $AP + PA^T + BB^T = 0$.
- $Q =$ observability Gramian of right spectral factor of power spectrum of system given by (A, B, C, D), i.e., solution of ARE

$$\hat{A}^T Q + Q\hat{A} + QB_W(DD^T)^{-1}B_W^T Q + C^T(DD^T)^{-1}C = 0,$$

where $\hat{A} := A - B_W(DD^T)^{-1}C$, $B_W := BD^T + PC^T$.

Balancing-Related Methods

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

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and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Positive-Real Balanced Truncation (PRBT) [Green '88]

- Based on positive-real equations, related to positive real (Kalman-Yakubovich-Popov-Anderson) lemma.
- P, Q solve dual AREs

$$\begin{aligned} 0 &= \bar{A}P + P\bar{A}^T + PC^T\bar{R}^{-1}CP + B\bar{R}^{-1}B^T, \\ 0 &= \bar{A}^T Q + Q\bar{A} + QB\bar{R}^{-1}B^T Q + C^T\bar{R}^{-1}C, \end{aligned}$$

where $\bar{R} = D + D^T$, $\bar{A} = A - B\bar{R}^{-1}C$.

Balancing-Related Methods

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n > 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Other Balancing-Based Methods

- Bounded-real balanced truncation (BRBT) – based on bounded real lemma [Opdenacker/Jonckheere '88];
- H_∞ balanced truncation (HinfBT) – closed-loop balancing based on H_∞ compensator [Mustafa/Glover '91].

Both approaches require solution of dual AREs.

- Frequency-weighted versions of the above approaches.

Balancing-Related Methods

- Guaranteed preservation of physical properties like
  - stability (all),
  - passivity (PRBT),
  - minimum phase (BST).
- Computable error bounds, e.g.,

$$\text{BT:} \|G - G_r\|_{\mathcal{H}_\infty} \leq 2 \sum_{j=r+1}^{n} \sigma_j^{BT},$$

$$\text{LQGBT:} \|G - G_r\|_{\mathcal{H}_\infty} \leq 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}}$$

$$\text{BST:} \|G - G_r\|_{\mathcal{H}_\infty} \leq \left(\prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \|G\|_{\mathcal{H}_\infty},$$

- Can be combined with singular perturbation approximation for steady-state performance.

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Overview

Consider

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Cx(t), \end{aligned}$$

where

- $x(t) \in \mathbb{R}^n$: the system's state
- $u(t) \in \mathbb{R}^m$: the input or control
- $y(t) \in \mathbb{R}^q$: the output or measurements

Linear Time-invariant DAEs

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- $E \in \mathbb{R}^{n\times n}$ is *singular*
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System Theoretic Aspects of DAEs

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- We will denote the system by $(E; A, B, C, D)$.
- $(E; A, B, C, D)$ are referred to as descriptor or singular systems.

The transfer function of an $(E; A, B, C, D)$ system in time domain:

$\mathrm{G}\colon u \mapsto y\colon$

$$y(t) = C\big[e^{E^{\mathrm{D}}At}x_0 + \int_0^t e^{E^{\mathrm{D}}A(t-\tau)}E^{\mathrm{D}}Bu(\tau)\,\mathrm{d}\tau - \\ - (I - E^{\mathrm{D}}E)\sum_{i=0}^{\nu-1}(EA^{\mathrm{D}})^i A^{\mathrm{D}}Bu^{(i)}(t)\big] + Du(t),$$

where

- E^{D} is the Drazin inverse of E
- ν is the differentiation index of the DAE $E\dot{x} = Ax$
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Note that if $E = I$, then $E^{\mathrm{D}} = I$ and the transfer function is well-known:

$$\mathrm{G}\colon u \mapsto y\colon y(t) = C\left[e^{At}x_0 + \int^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau\right] + Du(t).$$

- In frequency domain (after a *Laplace* transform) the transfer function is given as

$$G(s) = C(sE - A)^{-1}B + D$$

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- Depending on B and C, the transfer function is likely to be improper.

For an improper it holds that $\|G(s)\| \to \infty \quad \text{as } s \to \infty$.

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- Since standard model reduction approaches return a *proper* reduced system $\hat{G}$, the error will grow for large frequencies.

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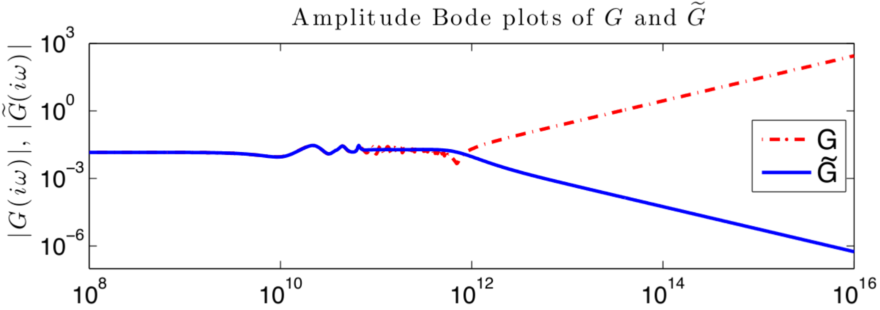

System Theoretic Aspects of DAEs

The general problem is:

- the transfer function can have an improper part (frequency domain)
- the system differentiates the input (time domain)

The general approach is:

1. Project the DAE onto the part that is an ODE, i.e. a standard state space system
2. Keep the remainder, i.e. the algebraic or improper part, as it is

This means: no model reduction on the algebraic part!

We consider linearized Navier-Stokes equations:

$$\begin{aligned} M\dot{v}(t) &= A_1 v(t) + J^T p(t) + B_1 u(t), \\ Jv(t) &= B_2 u(t), \\ y(t) &= C_1 v(t) + C_2 p(t). \end{aligned}$$

- $v(t) \in \mathbb{R}^n$: state (velocity)
- $p(t) \in \mathbb{R}^p$: state (pressure)
- $u(t) \in \mathbb{R}^m$: input or control
- $y(t) \in \mathbb{R}^q$: the output or measurements

- $M \in \mathbb{R}^{n\times n}$: mass matrix (symmetric
- $A_1 \in \mathbb{R}^{n\times n}$: the system matrix
- $J \in \mathbb{R}^{p\times n}$ is another system matrix (full
- $B_1 \in \mathbb{R}^{n\times m}$, $B_2 \in \mathbb{R}^{p\times m}$: input matrices
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Note that this is an $(E; A, B, C, D)$ with

$$
E := \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}, \quad A := \begin{bmatrix} A_1 & -J \\ J^T & 0 \end{bmatrix}, \quad B := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \text{and} \quad C := \begin{bmatrix} C_1 & C_2 \end{bmatrix}.
$$

$$
\begin{aligned}
M\dot{v}(t) &= A_1 v(t) + J^T p(t) + B_1 u(t),\\
Jv(t) &= B_2 u(t),\\
y(t) &= C_1 v(t) + C_2 p(t).
\end{aligned}
$$

Consider the projector

$$
P := I - M^{-1}J^{\mathsf{T}}(JM^{-1}J^{\mathsf{T}})^{-1}J
$$

and see that with $v = Pv + (I-P)v =: v_d + v_a$ the system writes as

$$
\begin{aligned}
M\dot{v}_d(t) &= P^{\mathsf{T}}A_1 v_d(t) + P^{\mathsf{T}}A_1 v_a(t) + P^{\mathsf{T}}B_1 u(t),\\
v_a(t) &= -M^{-1}J^{\mathsf{T}}(JM^{-1}J^{\mathsf{T}})^{-1}JB_2 u(t),\\
p(t) &= -(JM^{-1}J^{\mathsf{T}})^{-1}\big[JM^{-1}[A(v_a(t) + v_d(t)) + B_1 u(t)] - B_2\dot{u}(t)\big],\\
y(t) &= C_1 v_d(t) + C_1 v_a(t) + C_2 p(t).
\end{aligned}
$$

Decoupling Differential and Algebraic Parts

Since v_a and p depend linearly on v_d, u, and $\dot{u}$ is an $(E; A, B, C, D)$ system with the state v_d and

$$
\begin{aligned}
E &:= M, \\
A &:= P^{\mathsf{T}} A, \\
B &:= P^{\mathsf{T}}[B_1 - AM^{-1}J^{\mathsf{T}}(JM^{-1}J^{\mathsf{T}})^{-1}JB_2], \\
C &:= C_1 - C_2(JM^{-1}J^{\mathsf{T}})^{-1}JM^{-1}A, \\
D &:= D_1 + D_2,
\end{aligned}
$$

with

$$
\begin{aligned}
D_1 &:= -C_1 M^{-1}J^{\mathsf{T}}(JM^{-1}J^{\mathsf{T}})^{-1}JB_2 + C_2(JM^{-1}J^{\mathsf{T}})^{-1}JM^{-1}AM^{-1}J^{\mathsf{T}}(JM^{-1}J^{\mathsf{T}})^{-1}JB_1 \\
D_2 &:= -C_2(JM^{-1}J^{\mathsf{T}})^{-1}B_2 \frac{\mathrm{d}}{\mathrm{d}t}.
\end{aligned}
$$

Decoupling Differential and Algebraic Parts

$$D_1 = -C_1 M^{-1} J^{\mathsf{T}} (J M^{-1} J^{\mathsf{T}})^{-1} J B_2 + C_2 (J M^{-1} J^{\mathsf{T}})^{-1} J M^{-1} A M^{-1} J^{\mathsf{T}} (J M^{-1} J^{\mathsf{T}})^{-1} J B_1$$
$$D_2 = -C_2 (J M^{-1} J^{\mathsf{T}})^{-1} B_2 \tfrac{\mathrm{d}}{\mathrm{d}t}.$$

Note that

- The transfer function is given as $G = C(sE - A)^{-1}B + D_1 + sD_2$

Decoupling Differential and Algebraic Parts

$$D_1 = -C_1 M^{-1} J^{\mathsf{T}} (J M^{-1} J^{\mathsf{T}})^{-1} J B_2 + C_2 (J M^{-1} J^{\mathsf{T}})^{-1} J M^{-1} A M^{-1} J^{\mathsf{T}} (J M^{-1} J^{\mathsf{T}})^{-1} J B_1$$
$$D_2 = -C_2 (J M^{-1} J^{\mathsf{T}})^{-1} B_2 \tfrac{\mathrm{d}}{\mathrm{d}t}.$$

Note that

- The transfer function is given as $G = C(sE - A)^{-1}B + D_1 + sD_2$
- if B_2 or C_2 is zero, then D_2 is zero,
  - no $\dot{u}$ in the output
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- if B_2 is zero, then $D_1,\ D_2 = 0$
  - we obtain a standard $(E; A, B, C, D)$ system
  - no improper parts in G

Linear Time-invariant DAEs

If B_2 and C_2 are zero, then we have a standard $(A, B, C, -)$ system:

$$\begin{aligned} M\dot{v}_d &= P^{\mathsf{T}} A v_d + P^{\mathsf{T}} B_1 u, \\ y &= C_1 v. \end{aligned}$$

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- The system is not minimal
  - $\rightarrow$ this is automatically *fixed* by BT, if we can find the right solutions of the nonregular Lyapunov equations like

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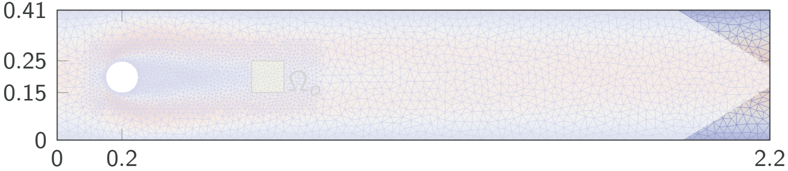
- The system is not stable
  - → Combine BT with *LQG*-stabilization [Benner and Heiland, '15]
- Explicit computation of the projector P is not possible for large scale systems
  - → use algorithms that do not need P explicitly, cf. [Gugercin, Stykel, Wyatt '13], [Heinkenschloss, Sorensen, Sun '08], and [Benner and Heiland, '15]

Numerical Example NSE



- 2D cylinder wake
- Navier-Stokes Equations
- $Re = 100$
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- 30000 velocity nodes

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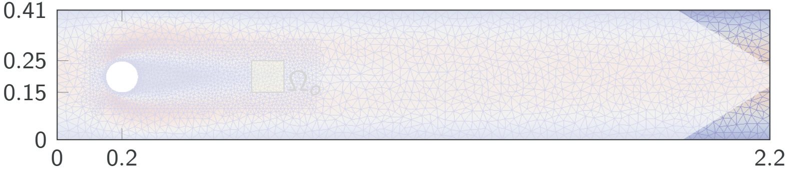

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- Boundary control at 2 outlets
- distributed observation with 6 degrees of freedom
- LQGBT-reduced order observer and controller of state dimension $r = 13$
- Target: stabilization of the steady-state solution

LQGBT Reduction - Bode Plot

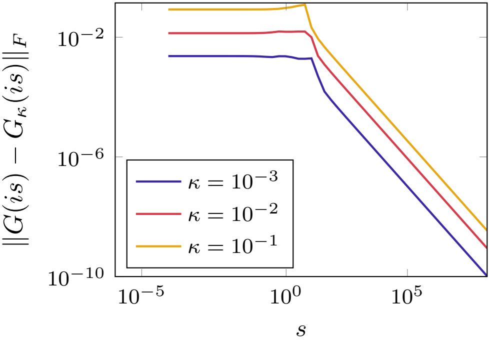


Figure: The error in the frequency response for varying thresholds κ measured in the Frobenius norm with *i* denoting the imaginary unit and the transfer functions in frequency domain as defined, e.g., in [4].

This plot was taken from [Benner and Heiland, '15]. The varying

Linear Time-invariant DAEs

Cylinder Wake Stabilization

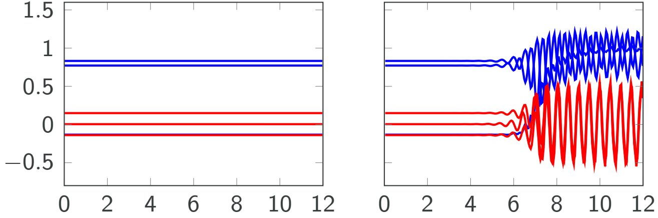
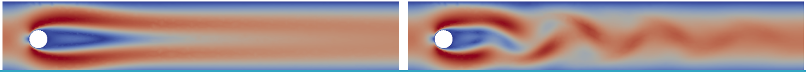


Figure: Measured signal y versus time $t \in [0, 12]$ of the perturbed closed loop system with a reduced controller of dimension $r = 13$ (left), compared to the response of the uncontrolled system (right). Blue corresponds to the x-component of the velocity and red to y-component. Below, a snapshot of the magnitude of the velocity solutions at $t = 12$.



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- One can decouple a DAE to extract the differential/proper part of the system
- The differential part is a standard $(A, B, C, -)$ and can be reduced with standard methods
- The algebraic part must not be reduced
- The efficient implementation requires further effort
- For Navier-Stokes equations there are examples of efficient application of BT related methods

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