

Exercise 3 – Gramians

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We consider the linear time invariant system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}$$

with $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,p}$, and $C \in \mathbb{R}^{q,n}$.

For A stable, the infinite *reachability* and *observability* Gramians \mathcal{P} and \mathcal{Q} are given as

$$\begin{aligned}\mathcal{P} &= \int_0^\infty e^{A\tau} B B^T e^{A^T \tau} d\tau \quad \text{and} \\ \mathcal{Q} &= \int_0^\infty e^{A^T \tau} C^T C e^{A\tau} d\tau.\end{aligned}$$

Show

1. the Gramians \mathcal{P} and \mathcal{Q} satisfy the *Lyapunov* equations

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0$$

and

$$A^T \mathcal{Q} + \mathcal{Q}A + C^T C = 0.$$

2. that they are symmetric positive definite,
3. that the eigenvalues of $\mathcal{P}\mathcal{Q}$ are invariant under state transformations, and
4. that the state transformation $x(t) \leftarrow Tx(t)$ with

$$T = \Sigma^{-\frac{1}{2}} V^T R$$

makes the transformed Gramians balanced, e.g. $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{Q}}$ are diagonal and equal. Here S and R are factors of $\mathcal{P} = S^T S$ and $\mathcal{Q} = R^T R$ so that V and Σ can be defined through the *Singular Value Decomposition* of SR^T :

$$U\Sigma V^T = SR^T.$$