



MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG



COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Introductory Course on Model Reduction of Linear Time Invariant Systems

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SHU Remote Course





Overview

1. Introduction to Linear Time Invariant Systems
2. Mathematical Basics for LTI Systems I
3. Mathematical Basics for LTI System 2
4. Introduction to Model Reduction
5. Model Reduction by Projection
6. Gramians and Balanced Realizations
7. Balanced Truncation



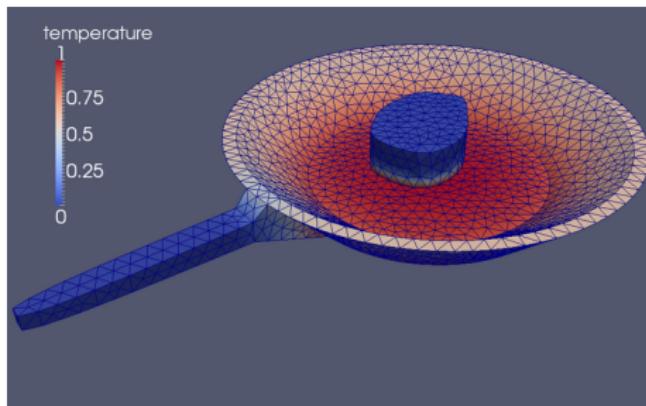
- Fry a steak
- The cook controls the heat at the fireplace
- and observes the process, e.g. via measuring the temperature in the inner



- The model

$$\begin{aligned}\dot{\theta} &= \nabla \cdot (\nu \nabla \theta) && \text{in } (0, \infty) \times \Omega, \\ \theta &= u, && \text{at the plate,} \\ \theta(0) &= 0.\end{aligned}$$

- The cook controls the heat at the fireplace, which we denote by u
- and observes the process, e.g. he measures the temperature y in the center: $y = f(\theta)$.



- The model:

$$\dot{\theta} = \nabla \cdot (\nu \nabla \theta),$$

$$\theta = u,$$

$$\theta(0) = 0.$$

- The cook controls the heat u
- and observes the process via $y = f(\theta)$.

- A *Finite Element* discretization of the problem leads to the finite dimensional model

$$E\dot{\theta}(t) = A\theta(t) + Bu(t), \quad \theta(0) = 0, \tag{1}$$

$$y(t) = C\theta(t), \tag{2}$$

a linear time invariant system.

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad (3a)$$

$$y(t) = Cx(t) + Du(t), \quad (3b)$$

with

- $x(t) \in \mathbb{R}^n$: the system's state
- $u(t) \in \mathbb{R}^m$: the input or control
- $y(t) \in \mathbb{R}^q$: the output or measurements
- $n, m, q \in \mathbb{N}$: the system dimensions

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$$y(t) = Cx(t) + Du(t), \quad (3b)$$

with

- $E \in \mathbb{R}^{n \times n}$: the identity or the mass matrix
- $A \in \mathbb{R}^{n \times n}$: the system matrix
- $B \in \mathbb{R}^{n \times m}$: the input matrix
- $C \in \mathbb{R}^{q \times n}$: the output matrix
- $D \in \mathbb{R}^{q \times n}$: the throughput

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We will assume that $E = I$ and denote the LTI (3) by (A, B, C, D) .



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Some Preliminary Thoughts

$$\begin{aligned}E\dot{x}(t) &= Ax(t) + Bu(t), \\y(t) &= Cx(t) + Du(t)\end{aligned}$$

A simple question...

What is x ?

- it is a physical state in the model – like the temperature
- in practise, we may not access it – only the measurement $y = Cx$
- it is but a mathematical object as a part of a model
- furthermore, as we will see later, the state x can be severely changed
e.g. in the course of model reduction

The state x can be seen...

...as nothing but an artificial object of the model for the input to output behavior

$$G: u \mapsto y$$

of an abstract system P :



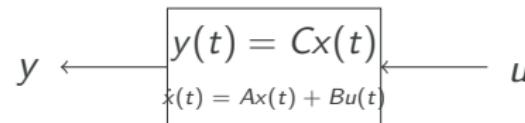
that maps an input u to the corresponding output y .

The state x can be seen...

...as nothing but an artificial object of the model for the input to output behavior

$$G: u \mapsto y$$

of an abstract system P :



that maps an input u to the corresponding output y .

If P is modelled through an (A, B, C, D) system, then the function G can be defined via

$$G: u \mapsto y: y(t) = C \left[e^{At} x_0 + \int_0^t e^{A(t-s)} B u(s) \, ds \right] + D u(t),$$

known as the formula of *variation of constants*.

This is in **time-domain**: A function u depending on time $t \in [0, \infty)$ is mapped onto a function y depending on time $t \in [0, \infty)$.

Through the **Laplace transform** \mathcal{L} and its inverse \mathcal{L}^{-1} , we can switch between time-domain and frequency-domain representations of the input and output signals:

$$U(s) := \mathcal{L}\{u\}(s) := \int_0^\infty e^{-st} u(t) \, dt,$$

where $s \in \mathbb{C}$ is the *frequency* and

$$y(t) := \mathcal{L}^{-1}\{Y\}(t) := \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} e^{st} Y(s) \, ds$$

where $\gamma \in \mathbb{R}$ is chosen such that the contour path of the integration is the domain of convergence of Y .



$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

With the basic properties of the Laplace transform

- $\dot{X}(s) := \mathcal{L}\{\dot{x}\}(s) - x(0) = s\mathcal{L}\{x\}(s) = sX(s) - x(0)$
- and linearity $\mathcal{L}\{Ax\}(s) = AX(s)$

with zero initial value $x(0) = 0$, the (A, B, C, D) system defines the transfer function

$$G(s) := C(sl - A)^{-1}B + D$$

in frequency domain.

Fact

An LTI (A, B, C, D) always defines a transfer function

$$G(s) = C(sl - A)^{-1}B + D$$

which is a matrix $G \in \mathbb{R}^{q \times m}$ with coefficients that are rational functions of s .

Question

Given a rational matrix function $s \mapsto G(s) \in \mathbb{R}^{q \times m}$, is there an

$$(A, B, C, D)$$

system, so that $G(s) = C(sl - A)^{-1}B + D$?

given G , find (A, B, C, D) ,
 $G(s) = C(sl - A)^{-1}B + D$

If there is **one** such (A, B, C, D) , then there are **infinitely** many:

- For $T \in \mathbb{R}^{n \times n}$ invertible, also $(TAT^{-1}, TB, CT^{-1}, D)$ is a realization:

$$C(sl - A)^{-1}B + D = CT^{-1}(sl - TAT^{-1})^{-1}TB + D.$$

- Moreover, also

$$\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D \right)$$

is a realization of G .

Facts and Thoughts on Realizations

- If G is *proper*, then there is a realization (A, B, C, D) as a state space system.
- This realization is by no means unique.
- The dimension of the state can be arbitrary large. What is the smallest possible dimension? (cf. *model reduction*)
- What is a good choice for the state?

Remark: A transfer function $G: s \mapsto G(s) \in \mathbb{R}^{q \times m}$ with coefficients that are rational functions in s , is *proper*, if in each coefficient the polynomial degree of the numerators does not exceed the degree of denominators.

Based on the previous considerations, we can say that

- The states of an LTI system (A, B, C, D) are just a part of a model that realizes a transfer function G
- The transfer function G describes how controls u lead to outputs y
- As seen above in the example, there can be states that are neither affected (*controlled*) by the inputs nor seen (*observed*) by the outputs
- These states are obviously not needed to realize the input to output behavior of G .

We will give a thorough characterization of the *controllable* and *observable* states of an LTI.

Theorem (Kalman Canonical Decomposition)

Given an LTI (A, B, C, D) , there is a state space transformation T such that the transformed system $(TAT^{-1}, TB, CT^{-1}, D)$ has the form

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} &= \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u \\ y &= [C_{co} \quad 0 \quad C_{\bar{c}o} \quad 0] \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + Du, \end{aligned}$$

with the subsystem $(A_{co}, B_{co}, C_{co}, D)$ being controllable and observable, while the remaining states $x_{\bar{c}o}$, $x_{c\bar{o}}$, or $x_{\bar{c}\bar{o}}$ are not controllable, not observable, or neither of them.

For a constructive proof of the Theorem, see Ch. 3.3 of [ZHOU, DOYLE, GLOVER '96]

For any state space system (A, B, C, D) , there is a transformation T so that the transformed states $T^{-1}x$ decompose into

- x_{co} - controllable and observable
- $x_{c\bar{o}}$ - controllable but not observable
- $x_{\bar{c}o}$ - observable but not controllable
- $x_{\bar{c}\bar{o}}$ - not observable and not controllable

Moreover, for the transfer function, it holds that

$$G(s) = C(sl - A)^{-1}B = C_{co}(sl - A_{co})^{-1}B_{co}.$$

What does this mean for us and a transfer function $G(s)$?

- The minimal dimension of a realization is the dimension of x_{co} in the *Kalman Canonical Decomposition*
- Such a realization is called **minimal realization**
- It is the starting point for further model reduction. (Throwing out $x_{\bar{c}o}$ etc. does not effect $G(s)$ and is typically not considered a model reduction)
- There are algorithms to reduce a realization to a minimal one, cf. [VARGA '90].
- In practice, the uncontrolled and unobserved states play a role and they may cause troubles. (check the literature for **zero dynamics**)



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Summary

- LTI as model for physical processes (e.g. heat transfer)
- The **input/output** behavior is often more important than the state
- Moreover, the state need not have a meaning
- State space systems (A, B, C, D) can be seen as **realizations** of transfer functions
- A transfer function has **multiple** realizations
- The **minimal realizations** are of our interest
- A **stable** system can have stable realization
- Minimal and stable realization can be balanced



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More on the LTI topics

- Book icon: K. Zhou, J. C. Doyle, and K. Glover.

Robust and Optimal Control. (Chapter 3 for LTI)
Prentice-Hall, Upper Saddle River, NJ, 1996.



- A. Varga.

Computation of irreducible generalized state-space realizations.
Kybernetika, 26(2):89–106, 1990.



- A. Gaul.

Leckerbraten – a lightweight Python toolbox to solve the heat equation on arbitrary domains

<https://github.com/andrenarchy/leckerbraten>, 2013.



- J. Heiland.

The slides, additional material, and information on this course

<https://www.janheiland.de/20-shu-mor/>, 2020.

Basic Notions of Norms

Ingredients of a normed space $(V, \|\cdot\|)$:

- A linear space V over \mathbb{C} (or \mathbb{R})
- and a functional

$$\|\cdot\|: V \rightarrow \mathbb{R}$$

that has the following properties:

- i) $\|\alpha v\| = |\alpha| \|v\|,$
- ii) $\|v + w\| \leq \|v\| + \|w\|,$ and
- iii) $\|v\| \geq 0$ and $\|v\| = 0$ if, and only if, $v = 0,$

for any $v, w \in V$ and any $\alpha \in \mathbb{C}$ (or \mathbb{R}).

Norms of Linear Operators

If $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$, then for the space of linear maps $(V \rightarrow W)$ a norm is defined via

$$\|G\|_* := \sup_{v \in V, v \neq 0} \frac{\|Gv\|_W}{\|v\|_V}.$$

This is the norm for $G: V \rightarrow W$ that is induced by $\|\cdot\|_V$ and $\|\cdot\|_W$. There can be other norms that are not induced.

Norms of Signals

Common norms and spaces for the input or output signals

$$u: [0, \infty) \rightarrow \mathbb{R}^m \quad \text{or} \quad y: [0, \infty) \rightarrow \mathbb{R}^q$$

- All definitions work similar for finite time intervals $[0, T]$ or the whole time axis $(-\infty, \infty)$.
- Where it is clear from the context, we will drop the superscripts p and m that denote the dimension of the signals.

Norms of Signals

Definition

The \mathcal{L}_1^m norm

$$\|u\|_{\mathcal{L}_1} := \int_0^\infty \sum_{i=1}^m |u_i(t)| dt$$

defines the \mathcal{L}_1^m space of integrable (summable) functions

$$\mathcal{L}_1^m := \{ u: [0, \infty) \rightarrow \mathbb{R}^m : \|u\|_{\mathcal{L}_1} < \infty \}$$

on the positive time axis.

Norms of Signals

Definition

The \mathcal{L}_∞^m norm

$$\|u\|_{\mathcal{L}_\infty} := \max_{i=\{1,\dots,m\}} \sup_{t>0} |u_i(t)|$$

defines the \mathcal{L}_∞^m space of **bounded functions**

$$\mathcal{L}_\infty^m := \{u: [0, \infty) \rightarrow \mathbb{R}^m : \|u\|_{\mathcal{L}_\infty} < \infty\}.$$

Definition

The \mathcal{L}_2^q norm

$$\|y\|_{\mathcal{L}_2} := \left(\int_0^\infty \sum_{i=1}^q |y_i(t)|^2 dt \right)^{\frac{1}{2}}$$

defines the \mathcal{L}_2^q space of **square integrable functions**

Norms of Signals

The \mathcal{L}_2 norm can also be evaluated in frequency domain

Theorem

For $u \in \mathcal{L}_2$ it holds that

$$\|u\|_{\mathcal{L}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} U(i\omega)^* U(i\omega) d\omega \right)^{\frac{1}{2}},$$

where U is the Fourier transform of u .

The Fourier transform \mathcal{F} and the Laplace transform \mathcal{L} coincide for $s = i\omega$, $\omega \in \mathbb{R}$ and $u(t) = 0$ for $t \leq 0$:

$$\mathcal{F}(u)(i\omega) := \int_{-\infty}^{\infty} u(t) e^{-i\omega t} dt = \int_0^{\infty} u(t) e^{-st} dt = \mathcal{L}(u)(s)$$

Norm of a System

A system G or (A, B, C, D) transfers inputs to outputs.

Ask yourself...

- What does a norm mean for a system?
- What is a large system, what is a small system?

Norm of a System

From the definition of an operator norm:

$$\|G\| = \sup_{u \neq 0} \frac{\|Gu\|}{\|u\|}$$

we derive that for all u :

$$\|y\| = \|Gu\| \leq \|G\|\|u\|.$$

An Answer

For systems, large refers to what extend an input is amplified.
Therefore, $\|G\|$ is often called the *gain*.

Norm of a System

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With a norm, one can compare two systems G_1 and G_2 via the difference in the output for the same input:

$$\|y_1 - y_2\| = \|G_1 u - G_2 u\| \leq \|G_1 - G_2\| \|u\|.$$

Defining a Norm for Systems

We consider a SISO system $(A, B, C, -)$, i.e $m = q = 1$ and $D = 0$.

Consider $(A, B, C, -)$ with stable and strictly proper transfer function G is stable. Then the *impulse response* of the system

$$g(t) = C \int_0^t e^{A(t-\tau)} B \delta(\tau) \, d\tau = Ce^{At}B$$

A system (A, B, C, D) or A is stable, if there exists a $\lambda > 0$, such that $\|e^{At}\| \leq e^{-\lambda t}$, for $t > 0$. This means that all eigenvalues of A must have a negative real part.

Impulse response: $\delta(\tau) := \begin{cases} 0, & \text{if } t \neq 0, \\ \text{very large,} & \text{if } t = 0 \end{cases}$ so that $\int_{-\infty}^{\infty} u(\tau)\delta(\tau) \, d\tau = u(0)$.

Defining a Norm for Systems

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$$g(t) = C \int_0^t e^{A(t-\tau)} B \delta(\tau) \, ds = Ce^{At}B$$

decays exponentially and

$$\|g\|_{\mathcal{L}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} G(i\omega)^* G(i\omega) \, d\omega \right)^{\frac{1}{2}} =: \|G\|_2 < \infty.$$

Defining a Norm for Systems

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This defines a norm for systems since (Exercise!)

- $G = C(sl - A)^{-1}B$ is indeed the Laplace transform of g
- the functional $\|\cdot\|_2$ for stable and strictly proper transferfunctions is a norm

Furthermore, $\|y\|_{\mathcal{L}_{\infty}} \leq \|G\|_2 \|u\|_{\mathcal{L}_{\infty}}$. (Exercise!)

Defining a Norm for Systems

For MIMO systems $(A, B, C, -)$ with $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$, with a stable and strictly proper transferfunction $\mathcal{G}: s \rightarrow \mathbb{R}^{q \times m}$, the \mathcal{H}_2 norm is defined as

$$\|G\|_2 := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace } G(i\omega)^* G(i\omega) \, d\omega \right)^{\frac{1}{2}}.$$

Fact

This is the norm of the *Hardy* space \mathcal{H}_2 of matrix functions that are analytic in the open right half of the complex plane. Stable and strictly proper transfer functions are in \mathcal{H}_2 .

Defining a Norm for Systems

For a stable and proper transfer function one can define the \mathcal{H}_∞ norm:

$$\|G\|_\infty := \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(i\omega)),$$

where $\sigma_{\max}(G(i\omega))$ is the largest singular value of $G(i\omega)$.

Fact 1

This is the norm of the *Hardy* space \mathcal{H}_∞ of matrix functions that are analytic in the open right half of the complex plane and bounded on the imaginary axis. Stable and strictly proper transfer functions are in \mathcal{H}_∞ .

Fact 2

The \mathcal{H}_∞ -norm is induced by the \mathcal{L}_2 norm:

$$\|G\|_\infty = \sup_{u \in \mathcal{L}_2, u \neq 0} \frac{\|Gu\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}}.$$

Approximation Problems - Model Reduction

Output errors in time-domain

Comparing the original system G and the reduced system \hat{G} :

$$\begin{aligned}\|y - \hat{y}\|_{\mathcal{H}_2} &\leq \|G - \hat{G}\|_{\mathcal{H}_\infty} \|u\|_{\mathcal{H}_2} \quad \Rightarrow \|G - \hat{G}\|_{\mathcal{H}_\infty} < \text{tol} \\ \|y - \hat{y}\|_{\mathcal{H}_\infty} &\leq \|G - \hat{G}\|_{\mathcal{H}_2} \|u\|_{\mathcal{H}_2} \quad \Rightarrow \|G - \hat{G}\|_{\mathcal{H}_2} < \text{tol}\end{aligned}$$

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\mathcal{H}_∞ -norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable \mathcal{H}_∞ -norm bound.
\mathcal{H}_2 -norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)
$\ G\ _H := \sigma_{\max}$	optimal Hankel norm approximation (AAK theory).

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Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \text{ with transfer function } G(s) = C(sI - A)^{-1}B + D,$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$ is called a realization of Σ .

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Realizations are not unique!

Transfer function is invariant under state-space transformations,

$$\mathcal{T} : \begin{cases} \begin{matrix} x \\ (A, B, C, D) \end{matrix} \rightarrow \begin{matrix} Tx \\ (TAT^{-1}, TB, CT^{-1}, D) \end{matrix} \end{cases}$$

Definition

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the quadruple $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$ is called a **realization** of Σ .

Realizations are not unique!

Transfer function is invariant under addition of uncontrollable/unobservable states:

$$\frac{d}{dt} \begin{bmatrix} x \\ x_1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} x \\ x_1 \end{bmatrix} + \begin{bmatrix} B \\ B_1 \end{bmatrix} u(t), \quad y(t) = [C \quad 0] \begin{bmatrix} x \\ x_1 \end{bmatrix} + Du(t), \quad (4)$$

$$\frac{d}{dt} \begin{bmatrix} x \\ x_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \quad y(t) = [C \quad C_2] \begin{bmatrix} x \\ x_2 \end{bmatrix} + Du(t), \quad (5)$$

for arbitrary $A_j \in \mathbb{R}^{n_j \times n_j}$, $j = 1, 2$, $B_1 \in \mathbb{R}^{n_1 \times m}$, $C_2 \in \mathbb{R}^{q \times n_2}$ and any $n_1, n_2 \in \mathbb{N}$.

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \text{ with transfer function } G(s) = C(sl - A)^{-1}B + D,$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$ is called a realization of Σ .

Realizations are not unique!

Hence,

$$(A, B, C, D), \quad \left(\begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} B \\ B_1 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D \right),$$

$$(TAT^{-1}, TB, CT^{-1}, D), \quad \left(\begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & C_2 \end{bmatrix}, D \right),$$

are all realizations of Σ !

Definition

For a linear (time-invariant) system

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the quadruple $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$ is called a realization of Σ .

Definition

The **McMillan degree** of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.

A **minimal realization** is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order \hat{n} .

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \text{ with transfer function } G(s) = C(sI - A)^{-1}B + D,$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$ is called a realization of Σ .

Definition

The **McMillan degree** of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.

A **minimal realization** is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order \hat{n} .

Theorem

A realization (A, B, C, D) of a linear system is minimal \iff (A, B) is controllable and (A, C) is observable.

Definition

The LTI (A, B, C, D) or the pair (A, B) is said to be *controllable* if, for any initial state $x(0) = x_0$, $t_1 > 0$ and final state x_1 , there exists a (piecewise continuous) input u such that the solution of (3) satisfies $x(t_1) = x_1$. Otherwise, the system (A, B, C, D) or the pair (A, B) is said to be *uncontrollable*.

Theorem

The following statements are equivalent:

- (i.) The pair (A, B) is controllable.
- (ii.) The controllability matrix $\mathcal{C} := [B \ AB \ A^2B \ \dots \ A^{n-1}B]$ has full rank.
- (iii.) The matrix $[A - \lambda I \ B]$ has full rank for all $\lambda \in \mathbb{C}$.

Definition

The LTI (A, B, C, D) or the pair (C, A) is said to be *observable* if, for any $t_1 > 0$, the initial state $x(0) = x_0$ can be determined from the time history of the input u and the output y in the interval of $[0, t_1]$. Otherwise, the system (A, B, C, D) , or (C, A) , is said to be *unobservable*.

Observability is the dual concept of controllability:

Theorem

The pair (C, A) is observable if and only if the pair (A^T, C^T) is controllable.

Theorem

The LTI (A, B, C, D) is controllable (observable) if, and only if, the transformed LTI $(TAT^{-1}, TB, CT^{-1}, D)$ is controllable (observable), where T is a regular matrix.

- Recall that also a transfer function is invariant with respect to state space transformations on its realization.
- Next, we find the states that are at least necessary for the realization of a transfer function...

Theorem (Kalman Canonical Decomposition)

Given an LTI (A, B, C, D) , there is a state space transformation T such that the transformed system $(TAT^{-1}, TB, CT^{-1}, D)$ has the form

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} &= \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u \\ y &= [C_{co} \quad 0 \quad C_{\bar{c}o} \quad 0] \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + Du, \end{aligned}$$

with the subsystem $(A_{co}, B_{co}, C_{co}, D)$ being controllable and observable, while the remaining states $x_{\bar{c}o}$, $x_{c\bar{o}}$, or $x_{\bar{c}\bar{o}}$ are not controllable, not observable, or neither of them.

For a constructive proof of the Theorem, see Ch. 3.3 of [ZHOU, DOYLE, GLOVER '96]

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Definition

A linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

is **stable** if its transfer function $G(s)$ has all its poles in the left half plane and it is **asymptotically (or Lyapunov or exponentially) stable** if all poles are in the open left half plane $\mathbb{C}^- := \{z \in \mathbb{C} \mid \Re(z) < 0\}$.

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Lemma

Sufficient for asymptotic stability is that A is **asymptotically stable** (or **Hurwitz**), i.e., the spectrum of A , denoted by $\sigma(A)$, satisfies $\sigma(A) \subset \mathbb{C}^-$.

Note that by abuse of notation, often *stable system* is used for asymptotically stable systems.

Stability

- A system G is **stable** if all **poles** of G are located in the left half-plane \mathbb{C}^- .

If $m = q = 1$, then $G(s) = \frac{N(s)}{D(s)}$, where $N(s)$ and $D(s)$ are polynomials and the *poles* are the roots of $D(s)$, i.e. those $s \in \mathbb{C}$ for which $D(s) = 0$.

If $m, q > 1$, then one can use the *McMillan form* of G to determine the poles.

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- A system G is **stable** if all **poles** of G are located in the left half-plane \mathbb{C}^- .
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Consider a transfer function

$$G(s) = C(sI - A)^{-1}B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty, \infty)$, with the L_2 -norm

$$\|u\|_{\mathcal{H}_2}^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} u(j\omega)^H u(j\omega) d\omega.$$

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 $\implies y \in \mathcal{L}_2^q \cong L_2^q(-\infty, \infty)$.

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$$\|G\|_{\mathcal{H}_\infty} := \sup_{\|u\|_{\mathcal{H}_2} \neq 0} \frac{\|Gu\|_{\mathcal{H}_2}}{\|u\|_{\mathcal{H}_2}}$$

is well defined. It can be shown that

$$\|G\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \|G(j\omega)\| = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(j\omega)).$$

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Hardy space \mathcal{H}_∞

Function space of matrix-/scalar-valued functions that are analytic and bounded in \mathbb{C}^+ .

The \mathcal{H}_∞ -norm is

$$\|F\|_{\mathcal{H}_\infty} := \sup_{\operatorname{Re} s > 0} \sigma_{\max}(F(s)) = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(F(j\omega)).$$

Stable transfer functions are in the Hardy spaces

- \mathcal{H}_∞ in the SISO case (single-input, single-output, $m = q = 1$);
- $\mathcal{H}_\infty^{q \times m}$ in the MIMO case (multi-input, multi-output, $m > 1, q > 1$).

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Paley-Wiener Theorem (Parseval's equation/Plancherel Theorem)

$$L_2(-\infty, \infty) \cong \mathcal{L}_2, \quad L_2(0, \infty) \cong \mathcal{H}_2$$

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\mathcal{H}_∞ approximation error

Reduced-order model \Rightarrow transfer function $\hat{G}(s) = \hat{C}(sl_r - \hat{A})^{-1}\hat{B} + \hat{D}$.

$$\|y - \hat{y}\|_{\mathcal{H}_2} = \|Gu - \hat{G}u\|_{\mathcal{H}_2} \leq \|G - \hat{G}\|_{\mathcal{H}_\infty} \|u\|_{\mathcal{H}_2}.$$

\Rightarrow compute reduced-order model such that $\|G - \hat{G}\|_{\mathcal{H}_\infty} < tol$!

Note: error bound holds in time- and frequency domain due to Palev-Wiener!

Consider stable transfer function

$$G(s) = C(sI - A)^{-1}B, \quad \text{i.e. } D = 0.$$

Hardy space \mathcal{H}_2

Function space of matrix-/scalar-valued functions that are analytic \mathbb{C}^+ and bounded w.r.t. the \mathcal{H}_2 -norm

$$\begin{aligned} \|F\|_{\mathcal{H}_2} &:= \frac{1}{2\pi} \left(\sup_{\operatorname{re} \sigma > 0} \int_{-\infty}^{\infty} \|F(\sigma + j\omega)\|_F^2 d\omega \right)^{\frac{1}{2}} \\ &= \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \|F(j\omega)\|_F^2 d\omega \right)^{\frac{1}{2}}. \end{aligned}$$

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\mathcal{H}_2 approximation error for impulse response ($u(t) = u_0\delta(t)$)

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$$\|y - \hat{y}\|_{\mathcal{H}_2} = \|Gu_0\delta - \hat{G}u_0\delta\|_{\mathcal{H}_2} \leq \|G - \hat{G}\|_{\mathcal{H}_2} \|u_0\|.$$

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Theorem (Practical Computation of the \mathcal{H}_2 -norm)

$$\|F\|_{\mathcal{H}_2}^2 = \text{trace } B^T Q B = \text{trace } C P C^T,$$

where P, Q are the controllability and observability Gramians of the corresponding LTI system.

Output errors in time-domain

$$\begin{aligned}\|y - \hat{y}\|_{\mathcal{H}_2} &\leq \|G - \hat{G}\|_{\mathcal{H}_\infty} \|u\|_{\mathcal{H}_2} \quad \Rightarrow \|G - \hat{G}\|_{\mathcal{H}_\infty} < \text{tol} \\ \|y - \hat{y}\|_{\mathcal{H}_\infty} &\leq \|G - \hat{G}\|_{\mathcal{H}_2} \|u\|_{\mathcal{H}_2} \quad \Rightarrow \|G - \hat{G}\|_{\mathcal{H}_2} < \text{tol}\end{aligned}$$

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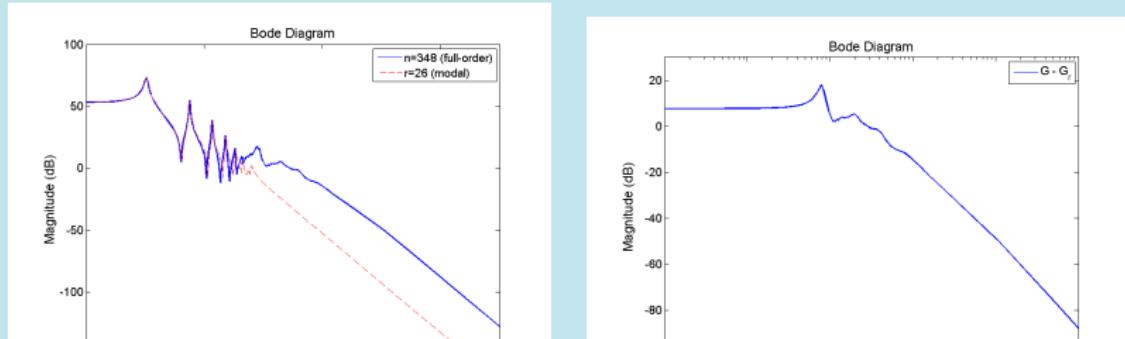
\mathcal{H}_∞ -norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable \mathcal{H}_∞ -norm bound.
\mathcal{H}_2 -norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)
Hankel-norm $\ G\ _H := \sigma_{\max}$	optimal Hankel norm approximation (AAK theory).

Evaluating system norms is computationally very (sometimes too) expensive.

Other measures

- absolute errors $\|G(j\omega_j) - \hat{G}(j\omega_j)\|_{\mathcal{H}_2}, \|G(j\omega_j) - \hat{G}(j\omega_j)\|_{\mathcal{H}_\infty}$ ($j = 1, \dots, N_\omega$);
- relative errors $\frac{\|G(j\omega_j) - \hat{G}(j\omega_j)\|_{\mathcal{H}_2}}{\|G(j\omega_j)\|_{\mathcal{H}_2}}, \frac{\|G(j\omega_j) - \hat{G}(j\omega_j)\|_{\mathcal{H}_\infty}}{\|G(j\omega_j)\|_{\mathcal{H}_\infty}}$;
- "eyeball norm", i.e. look at frequency response/Bode (magnitude) plot: for SISO system, log-log plot frequency vs. $|G(j\omega)|$ (or $|G(j\omega) - \hat{G}(j\omega)|$) in decibels, $1 \text{ dB} \simeq 20 \log_{10}(\text{value})$.

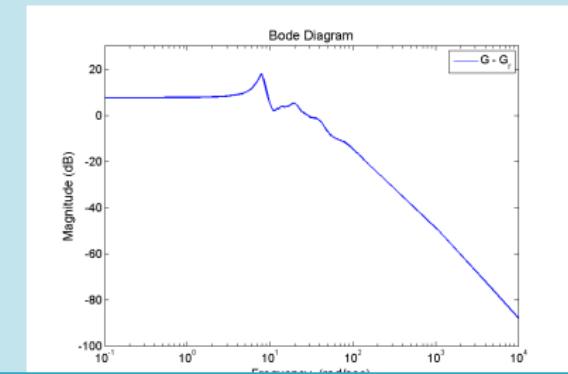
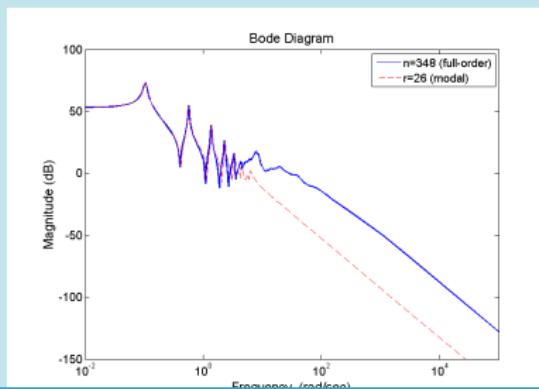
For MIMO systems, $q \times m$ array of plots G_{ij} .



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Overview

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Introduction to Model Reduction

Model Reduction — Abstract Definition

Problem

Given a model of a physical problem with dynamics described by the states $x(t) \in \mathbb{R}^n$, where n is the dimension of the state space.



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The dimension n is large because $x(t)$ typically contains information that

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This is the task of **model reduction** (also: **dimension reduction**, **order reduction**).

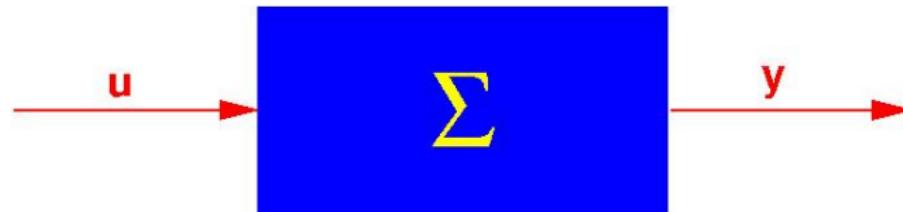
Model Reduction for Dynamical Systems

Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & x(t_0) = x_0, \\ y(t) = g(t, x(t), u(t)) \end{cases}$$

with

- **states** $x(t) \in \mathbb{R}^n$,
- **inputs** $u(t) \in \mathbb{R}^m$,
- **outputs** $y(t) \in \mathbb{R}^q$.



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$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \text{ for all admissible input signals.}$$

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Reduced-Order Model (ROM)

$$\hat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \hat{f}(t, \hat{x}(t), \textcolor{violet}{u}(t)), \\ \hat{y}(t) = \hat{g}(t, \hat{x}(t), \textcolor{violet}{u}(t)). \end{cases}$$

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- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\hat{y}(t) \in \mathbb{R}^q$.



Goal:

$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\|$ for all admissible input signals.

Secondary goal: reconstruct approximation of x from \hat{x} .

Linear Systems

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} E\dot{x} &= f(t, x, u) = Ax + Bu, \quad E, A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \\ y &= g(t, x, u) = Cx + Du, \quad C \in \mathbb{R}^{q \times n}, \quad D \in \mathbb{R}^{q \times m}. \end{aligned}$$

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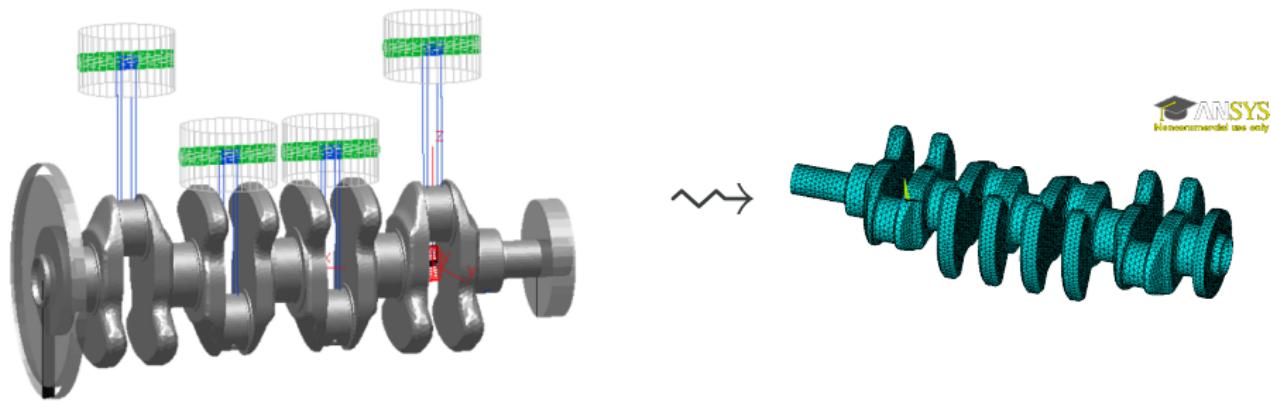
Linear, Time-Invariant Parametric Systems

$$\begin{aligned} E(p)\dot{x}(t; p) &= A(p)x(t; p) + B(p)u(t), \\ y(t; p) &= C(p)x(t; p) + D(p)u(t), \end{aligned}$$

where $A(p), E(p) \in \mathbb{R}^{n \times n}, B(p) \in \mathbb{R}^{n \times m}, C(p) \in \mathbb{R}^{q \times n}, D(p) \in \mathbb{R}^{q \times m}$.

Structural Mechanics / Finite Element Modeling

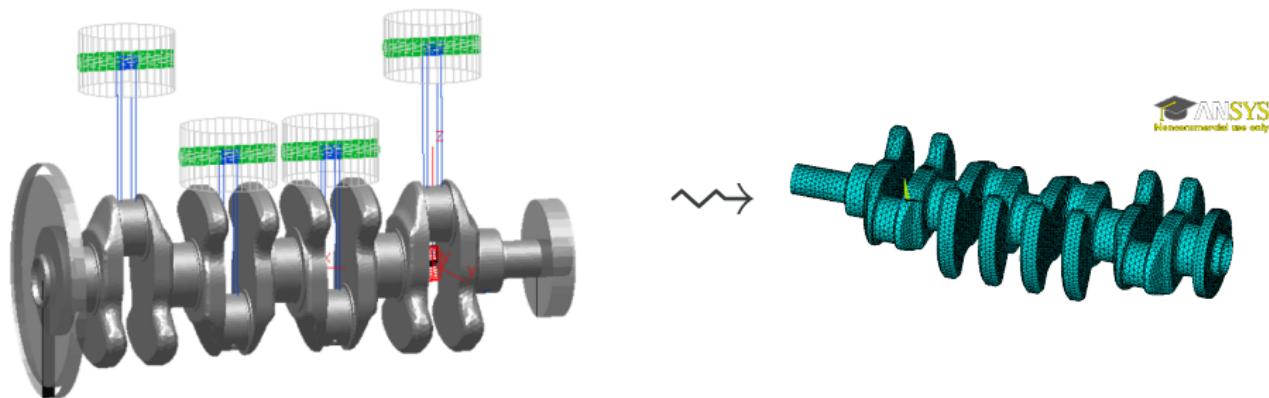
since ~1960ies



- Resolving complex 3D geometries \Rightarrow millions of degrees of freedom.
- Analysis of elastic deformations requires many simulation runs for varying external forces.

Structural Mechanics / Finite Element Modeling

since ~1960ies



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- Analysis of elastic deformations requires many simulation runs for varying external forces.

Standard MOR techniques in structural mechanics: **modal truncation**, combined with **Guyan reduction (static condensation)** \rightsquigarrow **Craig-Bampton method**.

(Optimal) Control

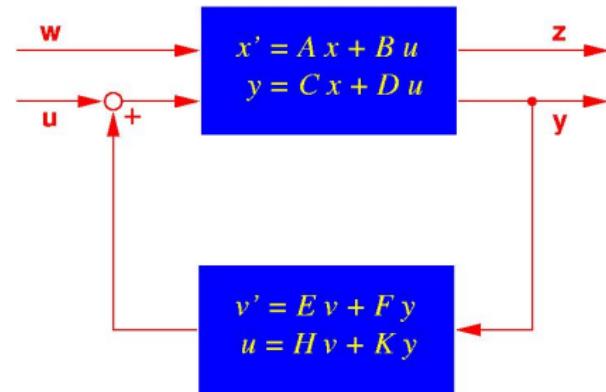
since ~1980ies

Feedback Controllers

A feedback controller (**dynamic compensator**) is a linear system of order N , where

- input = output of plant,
- output = input of plant.

Modern (LQG-/ \mathcal{H}_2 -/ \mathcal{H}_∞ -) control design: $N \geq n$.





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Application Areas

(Optimal) Control

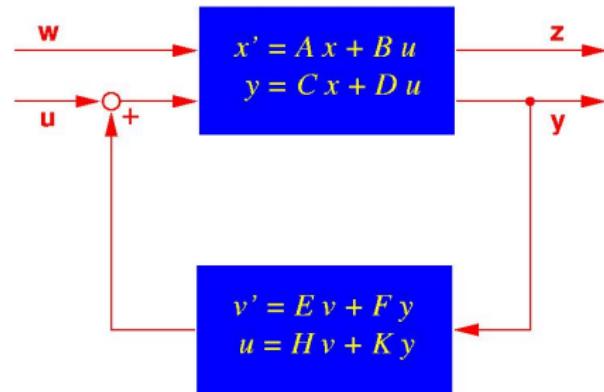
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Practical controllers require small N ($N \sim 10$, say) due to

- real-time constraints,
- increasing fragility for larger N .



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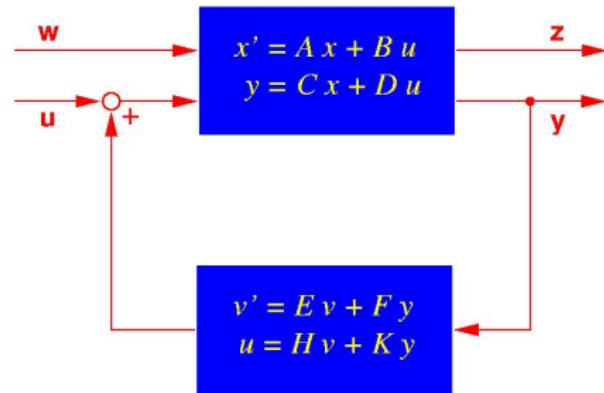
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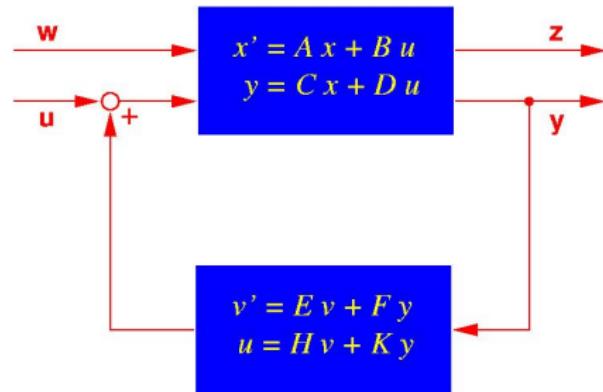
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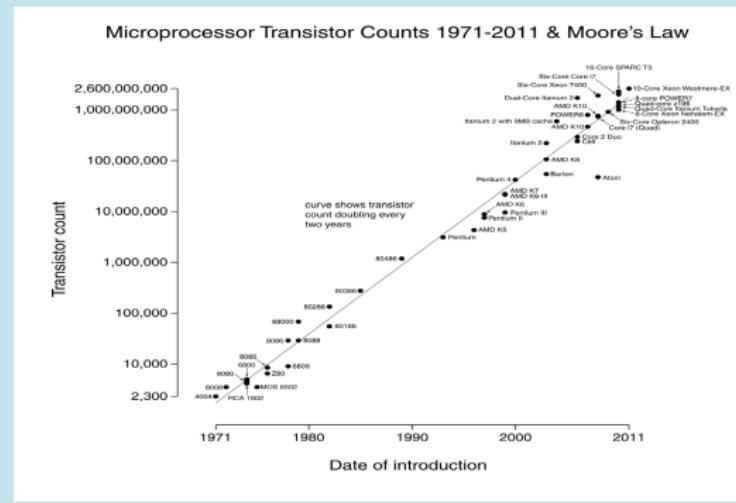
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Micro Electronics/Circuit Simulation

since ~1990ies

Progressive miniaturization

- Verification of VLSI/ULSI chip design needs a large number of simulations.
- **Moore's Law (1965/75)** states that the number of on-chip transistors doubles each 24 months.





Application Areas

Micro Electronics/Circuit Simulation

since ~1990ies

Progressive miniaturization

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- **Moore's Law (1965/75)** \rightsquigarrow steady increase of describing equations, i.e., network topology (Kirchhoff's laws) and characteristic element/semiconductor equations.



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- Increase in **packing density** and multilayer technology requires modeling of **interconnect** to ensure that thermic/electro-magnetic effects do not disturb signal transmission.

Intel 4004 (1971)

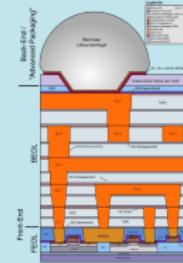
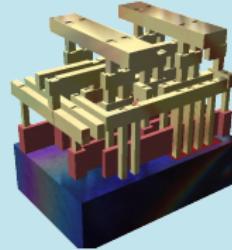
1 layer, 10μ technology
2,300 transistors
64 kHz clock speed

Intel Core 2 Extreme (quad-core) (2007)

9 layers, $45nm$ technology
 $> 8,200,000$ transistors
 > 3 GHz clock speed.

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Source: http://en.wikipedia.org/wiki/Image:Silicon_chip_3d.png

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- Here: mostly MOR for linear systems, they occur in micro electronics through modified nodal analysis (MNA) for RLC networks. e.g., when
 - decoupling large **linear subcircuits**,
 - modeling **transmission lines**,
 - modeling **pin packages** in VLSI chips,
 - modeling circuit elements described by Maxwell's equation using partial element equivalent circuits (**PEEC**).

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since ~1990ies

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Standard MOR techniques in circuit simulation:

Krylov subspace / Padé approximation / rational interpolation methods.



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Application Areas

Many other disciplines in **computational sciences and engineering** like

- computational fluid dynamics (CFD),
- computational electromagnetics,
- chemical process engineering,
- design of MEMS/NEMS (micro/nano-electrical-mechanical systems),
- computational acoustics,
- ...

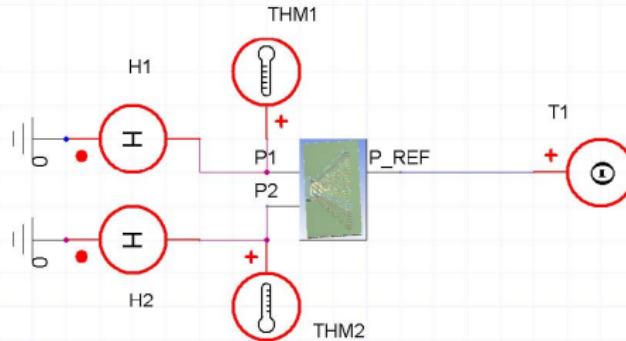


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Motivating Examples

Electro-Thermic Simulation of Integrated Circuit (IC) [Source: Evgenii Rudnyi, CADFEM GmbH]

- SIMPLORER® test circuit with 2 transistors.

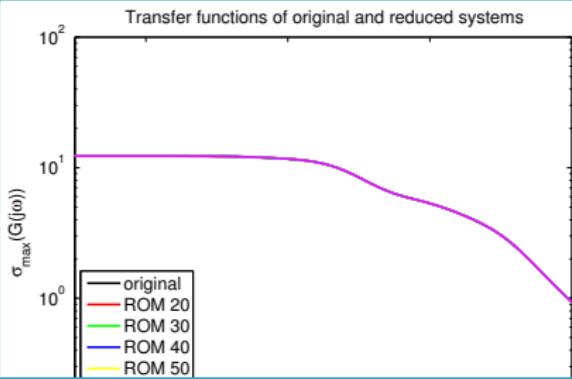


- Conservative thermic sub-system in SIMPLORER:
voltage \rightsquigarrow temperature, current \rightsquigarrow heat flow.
- Original model: $n = 270.593$, $m = q = 2 \Rightarrow$
Computing time (on Intel Xeon dualcore 3GHz, 1 Thread):
 - Main computational cost for set-up data $\approx 22\text{min.}$
 - Computation of reduced models from set-up data: 44–49sec. ($r = 20\text{--}70$).

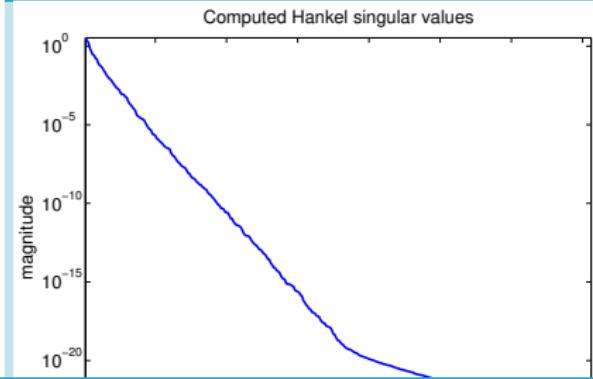
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 - Bode plot (MATLAB on Intel Core i7, 2.67GHz, 12GB):
7.5h for original system, < 1min for reduced system.
 - Speed-up factor: **18 including / ≥ 450 excluding** reduced model generation!

Bode Plot (Amplitude)



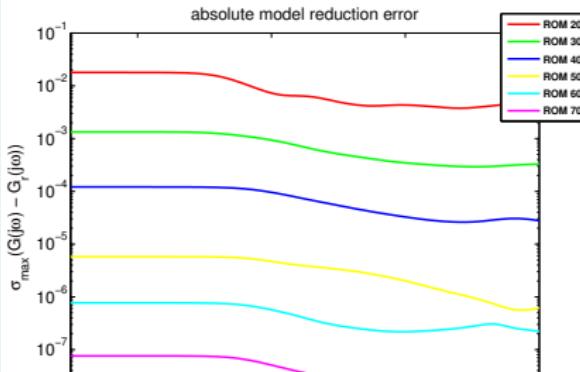
Hankel Singular Values



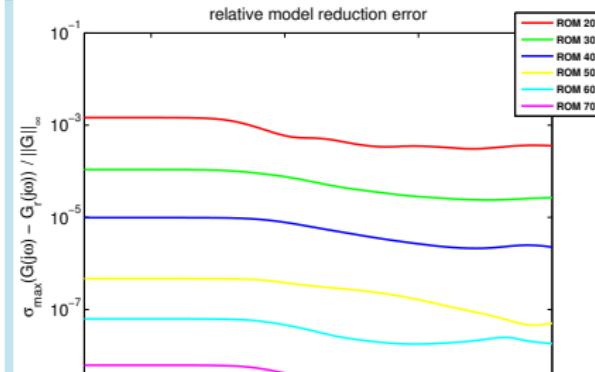
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Absolute Error



Relative Error



A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

- Simple model for neuron (de-)activation

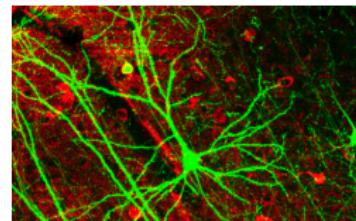
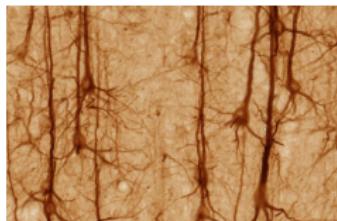
[CHATURANTABUT/SORENSEN 2009]

$$\begin{aligned}\epsilon v_t(x, t) &= \epsilon^2 v_{xx}(x, t) + f(v(x, t)) - w(x, t) + g, \\ w_t(x, t) &= hv(x, t) - \gamma w(x, t) + g,\end{aligned}$$

with $f(v) = v(v - 0.1)(1 - v)$ and initial and boundary conditions

$$\begin{aligned}v(x, 0) &= 0, & w(x, 0) &= 0, & x \in [0, 1] \\ v_x(0, t) &= -i_0(t), & v_x(1, t) &= 0, & t \geq 0,\end{aligned}$$

where $\epsilon = 0.015$, $h = 0.5$, $\gamma = 2$, $g = 0.05$, $i_0(t) = 50000t^3 \exp(-15t)$.



A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

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[CHATURANTABUT/SORENSEN 2009]

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where $\epsilon = 0.015$, $h = 0.5$, $\gamma = 2$, $g = 0.05$, $i_0(t) = 50000t^3 \exp(-15t)$.

- Parameter g handled as an additional input.
- Original state dimension $n = 2 \cdot 400$, QBDAE dimension $N = 3 \cdot 400$, reduced QBDAE dimension $r = 26$, chosen expansion point $\sigma = 1$.



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Motivating Examples

A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System



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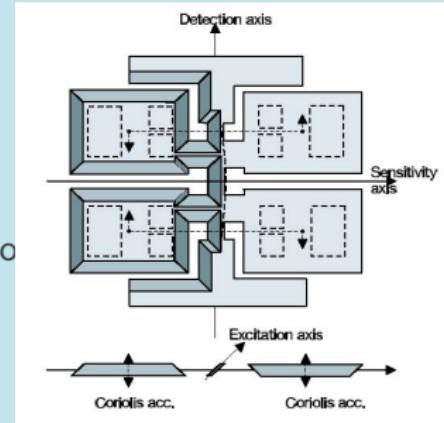
Motivating Examples

Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)



- Application: inertial navigation.



- Voltage applied to electrodes induces vibration of wings, resulting rotation due to Coriolis force yields sensor data.
- FE model of second order:
 $N = 17.361 \rightsquigarrow n = 34.722, m = 1, q = 12.$
- Sensor for position control based on acceleration and rotation.



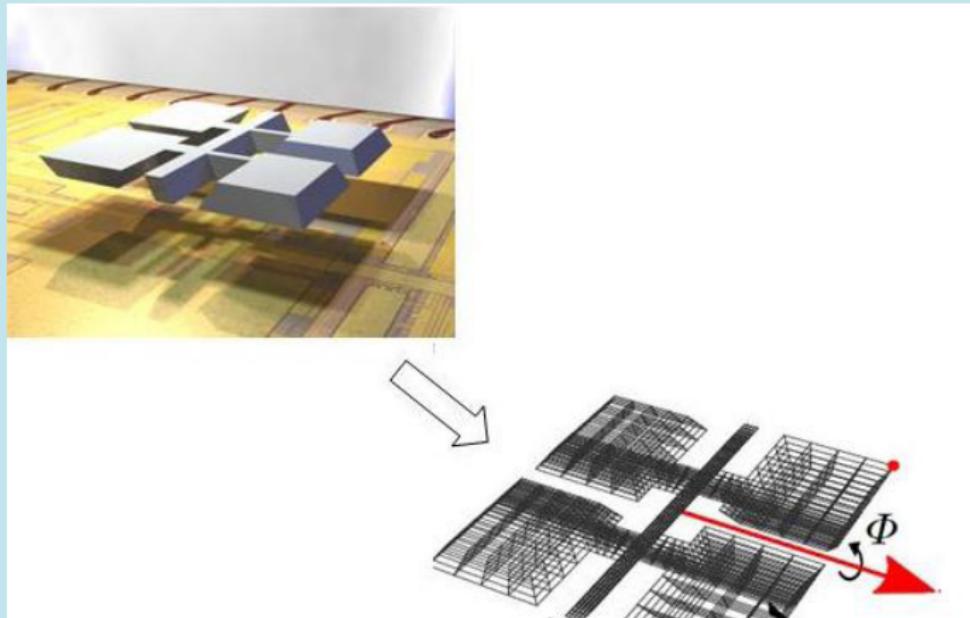
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Motivating Examples

Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Parametric FE model: $M(d)\ddot{x}(t) + D(\Phi, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t).$



Parametric MOR: Applications in Microsystems/MEMS Design

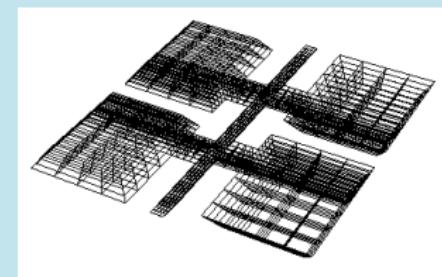
Microgyroscope (butterfly gyro)

Parametric FE model:

$$M(d)\ddot{x}(t) + D(\Phi, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t),$$

wobei

$$\begin{aligned} M(d) &= M_1 + dM_2, \\ D(\Phi, d, \alpha, \beta) &= \Phi(D_1 + dD_2) + \alpha M(d) + \beta T(d), \\ T(d) &= T_1 + \frac{1}{d}T_2 + dT_3, \end{aligned}$$



with

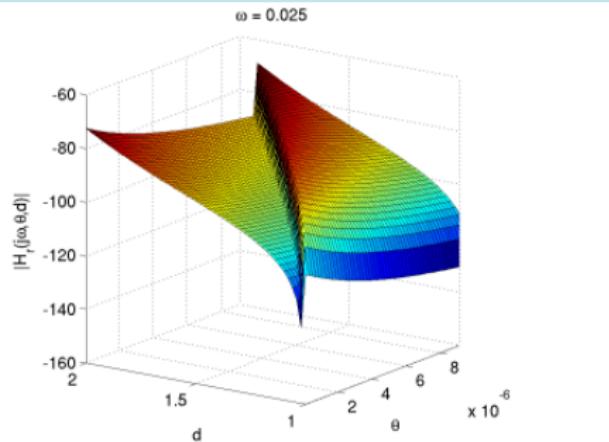
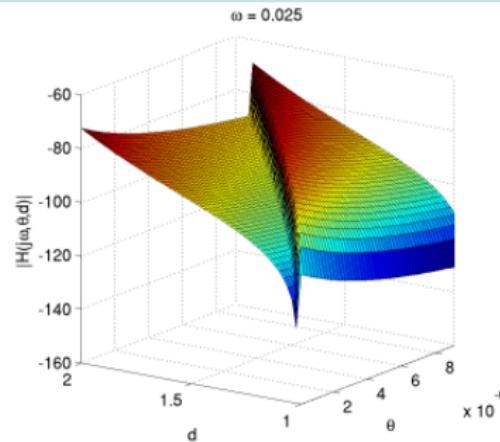
- width of bearing: d ,
- angular velocity: Φ ,
- Rayleigh damping parameters: α, β .

Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Original...

and reduced-order model.





Overview

1. Introduction to Linear Time Invariant Systems
2. Mathematical Basics for LTI Systems I
3. Mathematical Basics for LTI System 2
4. Introduction to Model Reduction
5. Model Reduction by Projection
6. Gramians and Balanced Realizations
7. Balanced Truncation

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 - Projection and Interpolation
 - Modal Truncation
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Model Reduction by Projection

Goals

Requirements: A Model Reduction approach should:

- Automatically generate compact models \hat{G} from a given model G

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$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| < \text{tol} \cdot \|u\| \quad \text{for all admissible } u.$$

⇒ Provide computable error bound/estimate!

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- Preserve physical properties:
 - stability

A G is **stable**, if all poles of G are in \mathbb{C}^- . A system (A, B, C, D) or A is **stable**, if all eigenvalues of A have a negative real part. Compare: $G(s) = C(sI - A)^{-1}B$

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- Preserve physical properties:
 - stability
 - minimum phase

A system G has **minimum phase** if all zeros of G are in the left half-plane \mathbb{C}^- .



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- Preserve physical properties:
 - stability
 - minimum phase
 - **passivity**

A system G is **passive** if, bluntly speaking, it does not generate energy. Condition for passivity:

$$\int_0^t u(\tau)^T y(\tau) d\tau \geq 0 \quad \text{for all } t \in \mathbb{R}, \quad \text{for all } u \in L_2(\mathbb{R}, \mathbb{R}^m).$$



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Model Reduction by Projection

Projection Basics

Definition

A projector $P: \mathcal{X} \rightarrow \mathcal{X}$ is a linear map, or a matrix, with $P^2 = P$.

Example

- $\mathcal{X} = \mathbb{R}^2$
- $P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is a projector in \mathcal{X}



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Model Reduction by Projection

Notion and Properties of Projectors

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- If $\{v_1, \dots, v_r\}$ is a basis of some $\mathcal{V} \subset \mathcal{X}$ and $V = [v_1, \dots, v_r]$, then

$$P := V(V^T V)^{-1} V^T$$

defines the **orthogonal** projector onto \mathcal{V} .



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- If $\mathcal{W} \subset \mathcal{X}$ is another r -dimensional subspace with a basis matrix $W = [w_1, \dots, w_r]$ so that $W^T V$ is not singular, then

$$P = V(W^T V)^{-1} W^T$$

defines the **oblique** projector onto \mathcal{V} along the orthogonal complement \mathcal{W}_\perp of \mathcal{W} .



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- For a projector P , the projector $I - P$ onto $\ker P$ is the **complementary** projector.

Projection and Interpolation

Methods:

1. Modal Truncation
2. Rational Interpolation (Padé-Approximation and (rational) Krylov Subspace Methods)
3. Balanced Truncation
4. many more...

Joint feature of these methods:

computation of reduced-order model (ROM) by projection!



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Model Reduction by Projection

The ideal model reduction

- There is a space $\mathcal{V} \subset \mathbb{R}^n$ with $\dim \mathcal{V} = r < n$, such that $x(t) \in \mathcal{V}$ for all time t and input u .



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Model Reduction by Projection

The ideal model reduction

- There is a space $\mathcal{V} \subset \mathbb{R}^n$ with $\dim \mathcal{V} = r < n$, such that $x(t) \in \mathcal{V}$ for all time t and input u .
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$$W^T V = I_r$$



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- If everything is exact, then

$$\|x(t) - \tilde{x}(t)\| = \|x(t) - VW^T x(t)\| = 0$$



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- Then $V(W^T V)^{-1} W = VW^T$ is a projector onto \mathcal{V}
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$$\|x(t) - \tilde{x}(t)\| = \|x(t) - VW^T x(t)\| = 0$$

- and given (A, B, C, D) , the reduced-order model $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is

$$\hat{A} := W^T A V, \quad \hat{B} := W^T B, \quad \hat{C} := C V, \quad (\hat{D} := D).$$

Model reduction in practise

- Assume that there is a space $\mathcal{V} \subset \mathbb{R}^n$ with $\dim \mathcal{V} = r < n$, such that $x(t) \in \mathcal{V}$ for all time t and input u .
- Take a space \mathcal{W} , so that $\mathcal{W}_\perp \oplus \mathcal{V} = \mathbb{R}^n$.
- Galerkin-type projections: $\mathcal{W} = \mathcal{V}$.
- Petrov-Galerkin projections: $\mathcal{W} \neq \mathcal{V}$.
- Find matrices V and W that approximate bases of \mathcal{V} and \mathcal{W} , with

$$W^T V = I_r$$

- Then $V(W^T V)^{-1} W = VW^T$ is a projector almost onto \mathcal{V}
- Define $\hat{x}(t) := W^T x(t) \in \mathbb{R}^r$ and define $\tilde{x}(t) := V\hat{x}(t) = VW^T x(t)$
- If everything is done well, then

$$\|x(t) - \tilde{x}(t)\| = \|x(t) - VW^T x(t)\| \approx 0$$

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Definition of the reduced model

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the **reduced-order model** $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is

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Why is the ROM defined like this:

It is the (Petrov)-Galerkin condition $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$:

$$W^T (\dot{\tilde{x}} - A\tilde{x} - Bu) = \quad W^T (VW^T \dot{x} - AVW^T x - Bu)$$



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is zero, if, and only if,

$$\dot{\tilde{x}} - \hat{A}\hat{x} - \hat{B}u = 0.$$



CSC

Model Reduction by Projection

Projection \rightsquigarrow Rational Interpolation

A Petrov-Galerkin projected ROM interpolates the transfer function:

Theorem 3.3

[GRIMME '97, VILLEMAGNE/SKELTON '87]

Given the ROM

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = C V, \quad (\hat{D} = D),$$

and $s_* \in \mathbb{C} \setminus (\sigma(A) \cup \sigma(\hat{A}))$, if either

- $(s_* I_n - A)^{-1} B \in \text{range } V$, or
- $(s_* I_n - A)^{-*} C^T \in \text{range } W$,

then the interpolation condition

$$G(s_*) = \hat{G}(s_*).$$

in s_* holds.

Given the ROM

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = C V, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$G(s) - \hat{G}(s) = (C(sI_n - A)^{-1}B + D) - \left(\hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D} \right)$$

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If $s_* \in \mathbb{C} \setminus (\sigma(A) \cup \sigma(\hat{A}))$, then $P(s_*)$ is a projector onto \mathcal{V} :

$\text{range } P(s_*) \subset \text{range } V$, all matrices have full rank $\Rightarrow " = "$,

$$P(s_*)^2 = V(s_* I_r - \hat{A})^{-1} W^T (s_* I_n - A) V(s_* I_r - \hat{A})^{-1} W^T (s_* I_n - A)$$

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If $s_* \in \mathbb{C} \setminus (\sigma(A) \cup \sigma(\hat{A}))$, then $P(s_*)$ is a projector onto $\mathcal{V} \implies$

if $(s_* I_n - A)^{-1}B \in \mathcal{V}$, then $(I_n - P(s_*))(s_* I_n - A)^{-1}B = 0$,

hence

$G(s_*) - \hat{G}(s_*) = 0 \Rightarrow G(s_*) = \hat{G}(s_*)$, i.e., \hat{G} interpolates G in s_* !

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$$\text{Analogously, } = C(sl_n - A)^{-1} \left(I_n - \underbrace{(sl_n - A)V(sl_r - \hat{A})^{-1} W^T}_{=:Q(s)} \right) B.$$

If $s_* \in \mathbb{C} \setminus (\sigma(A) \cup \sigma(\hat{A}))$, then $Q(s)^H$ is a projector onto $\mathcal{W} \implies$

if $(s_* I_n - A)^{-*} C^T \in \mathcal{W}$, then $C(s_* I_n - A)^{-1} (I_n - Q(s_*)) = 0$,

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CSC

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then the interpolation condition

$$G(s_*) = \hat{G}(s_*).$$

in s_* holds.

Basic method:

Assume A is diagonalizable, $T^{-1}AT = D_A$, project state-space onto A -invariant subspace $\mathcal{V} = \text{span}(t_1, \dots, t_r)$, t_k = eigenvectors corresp. to “dominant” modes / eigenvalues of A . Then with

$$V = T(:, 1:r) = [t_1, \dots, t_r], \quad \tilde{W}^H = T^{-1}(1:r, :), \quad W = \tilde{W}(V^H \tilde{W})^{-1},$$

reduced-order model is

$$\hat{A} := W^H A V = \text{diag}\{\lambda_1, \dots, \lambda_r\}, \quad \hat{B} := W^H B, \quad \hat{C} = C V$$

Also computable by truncation:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\hat{C}, C_2], \quad \hat{D} = D.$$

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Properties:

Simple computation for large-scale systems, using, e.g., Krylov subspace methods (Lanczos, Arnoldi), Jacobi-Davidson method.

Basic method:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\hat{C}, C_2], \quad \hat{D} = D.$$

Properties:

Error bound:

$$\|G - \hat{G}\|_{\mathcal{H}_\infty} \leq \|C_2\| \|B_2\| \frac{1}{\min_{\lambda \in \sigma(A_2)} |\operatorname{Re}(\lambda)|}.$$

Proof:

$$\begin{aligned} G(s) &= C(sl - A)^{-1}B + D = CTT^{-1}(sl - A)^{-1}TT^{-1}B + D \\ &= CT(sl - T^{-1}AT)^{-1}T^{-1}B + D \\ &= [\hat{C}, C_2] \begin{bmatrix} (sl_r - \hat{A})^{-1} & \\ & (sl_{n-r} - A_2)^{-1} \end{bmatrix} \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix} + D \\ &= \hat{G}(s) + C_2(sl_{n-r} - A_2)^{-1}B_2, \end{aligned}$$

Basic method:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\hat{C}, C_2], \quad \hat{D} = D.$$

Properties:

Error bound:

$$\|G - \hat{G}\|_{\mathcal{H}_\infty} \leq \|C_2\| \|B_2\| \frac{1}{\min_{\lambda \in \sigma(A_2)} |\operatorname{Re}(\lambda)|}.$$

Proof:

$$G(s) = \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2,$$

observing that $\|G - \hat{G}\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(C_2(j\omega I_{n-r} - A_2)^{-1}B_2)$, and

$$C_2(j\omega I_{n-r} - A_2)^{-1}B_2 = C_2 \operatorname{diag} \left(\frac{1}{j\omega - \lambda_{r+1}}, \dots, \frac{1}{j\omega - \lambda_n} \right) B_2.$$

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Assume A is diagonalizable, $T^{-1}AT = D_A$, project state-space onto A -invariant subspace $\mathcal{V} = \text{span}(t_1, \dots, t_r)$, t_k = eigenvectors corresp. to “dominant” modes / eigenvalues of A . Then reduced-order model is

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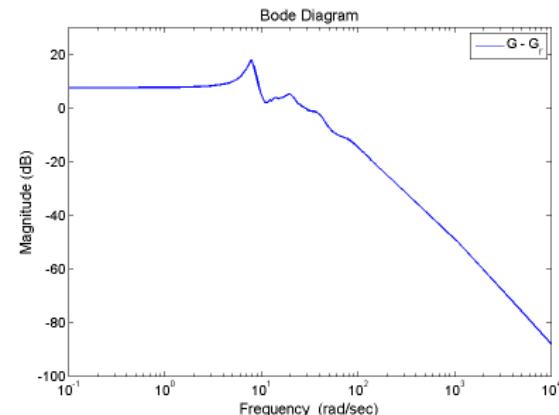
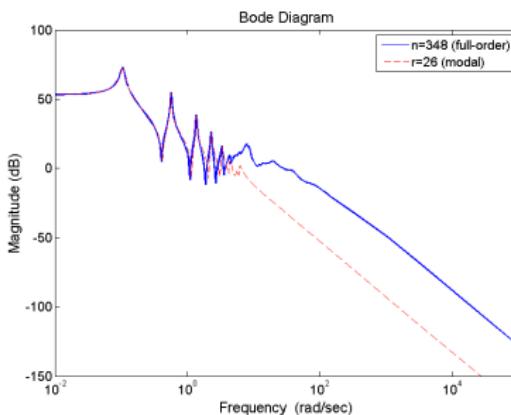
Difficulties:

- Eigenvalues contain only limited system information.
- Dominance measures are difficult to compute.
([LITZ '79] use Jordan canonical form; otherwise merely heuristic criteria, e.g., [VARGA '95]. Recent improvement: **dominant pole algorithm**.)
- Error bound not computable for really large-scale problems.

Example

BEAM, SISO system from **SLICOT Benchmark Collection for Model Reduction**, $n = 348$, $m = q = 1$, reduced using 13 dominant complex conjugate eigenpairs, error bound yields $\|G - \hat{G}\|_{\mathcal{H}_\infty} \leq 1.21 \cdot 10^3$

Bode plots of transfer functions and error function



Extensions

Base enrichment

Static modes are defined by setting $\dot{x} = 0$ and assuming unit loads, i.e., $u(t) \equiv e_j, j = 1, \dots, m$:

$$0 = Ax(t) + Be_j \implies x(t) \equiv -A^{-1}b_j.$$

Projection subspace \mathcal{V} is then augmented by $A^{-1}[b_1, \dots, b_m] = A^{-1}B$.

Interpolation-projection framework $\implies G(0) = \hat{G}(0)$!

If two sided projection is used, complimentary subspace can be augmented by $A^{-T}C^T \implies G'(0) = \hat{G}'(0)$! (If $m \neq q$, add random vectors or delete some of the columns in $A^{-T}C^T$).

Extensions

Guyan reduction (static condensation)

Partition states in **masters** $x_1 \in \mathbb{R}^r$ and **slaves** $x_2 \in \mathbb{R}^{n-r}$ (FEM terminology)

Assume stationarity, i.e., $\dot{x} = 0$ and solve for x_2 in

$$\begin{aligned} 0 &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ \Rightarrow x_2 &= -A_{22}^{-1} A_{21} x_1 - A_{22}^{-1} B_2 u. \end{aligned}$$

Inserting this into the first part of the dynamic system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u, \quad y = C_1x_1 + C_2x_2$$

then yields the reduced-order model

$$\begin{aligned} \dot{x}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y &= (C_1 - C_2A_{22}^{-1}A_{21})x_1 - C_2A_{22}^{-1}B_2u. \end{aligned}$$

Dominant Poles

Pole-Residue Form of Transfer Function

Consider partial fraction expansion of transfer function with $D = 0$:

$$G(s) = \sum_{k=1}^n \frac{R_k}{s - \lambda_k}$$

with the residues $R_k := (Cx_k)(y_k^H B) \in \mathbb{C}^{q \times m}$.

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with the residues $R_k := (Cx_k)(y_k^H B) \in \mathbb{C}^{q \times m}$.

Note: this follows using the spectral decomposition $A = XDX^{-1}$, with $X = [x_1, \dots, x_n]$ the right and $X^{-1} =: Y = [y_1, \dots, y_n]^H$ the left eigenvector matrices:

$$\begin{aligned} G(s) &= C(sl - XDX^{-1})^{-1}B = CX(sl - \text{diag}\{\lambda_1, \dots, \lambda_n\})^{-1}YB \\ &= [Cx_1, \dots, Cx_n] \begin{bmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_n} \end{bmatrix} \begin{bmatrix} y_1^H B \\ \vdots \\ y_n^H B \end{bmatrix} \end{aligned}$$

Dominant Poles

Pole-Residue Form of Transfer Function

Consider partial fraction expansion of transfer function with $D = 0$:

$$G(s) = \sum_{k=1}^n \frac{R_k}{s - \lambda_k}$$

with the residues $R_k := (Cx_k)(y_k^H B) \in \mathbb{C}^{q \times m}$.

Note: $R_k = (Cx_k)(y_k^H B)$ are the residues of G in the sense of the residue theorem of complex analysis:

$$\begin{aligned} \text{res}(G, \lambda_\ell) &= \lim_{s \rightarrow \lambda_\ell} (s - \lambda_\ell) G(s) = \sum_{k=1}^n \underbrace{\lim_{s \rightarrow \lambda_\ell} \frac{s - \lambda_\ell}{s - \lambda_k}}_{R_k = R_\ell} \\ &= \begin{cases} 0 & \text{for } k \neq \ell \\ 1 & \text{for } k = \ell \end{cases} \end{aligned}$$

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As projection basis use spaces spanned by right/left eigenvectors corresponding to dominant poles, i.e.. (λ_j, x_j, y_j) with largest

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Remark

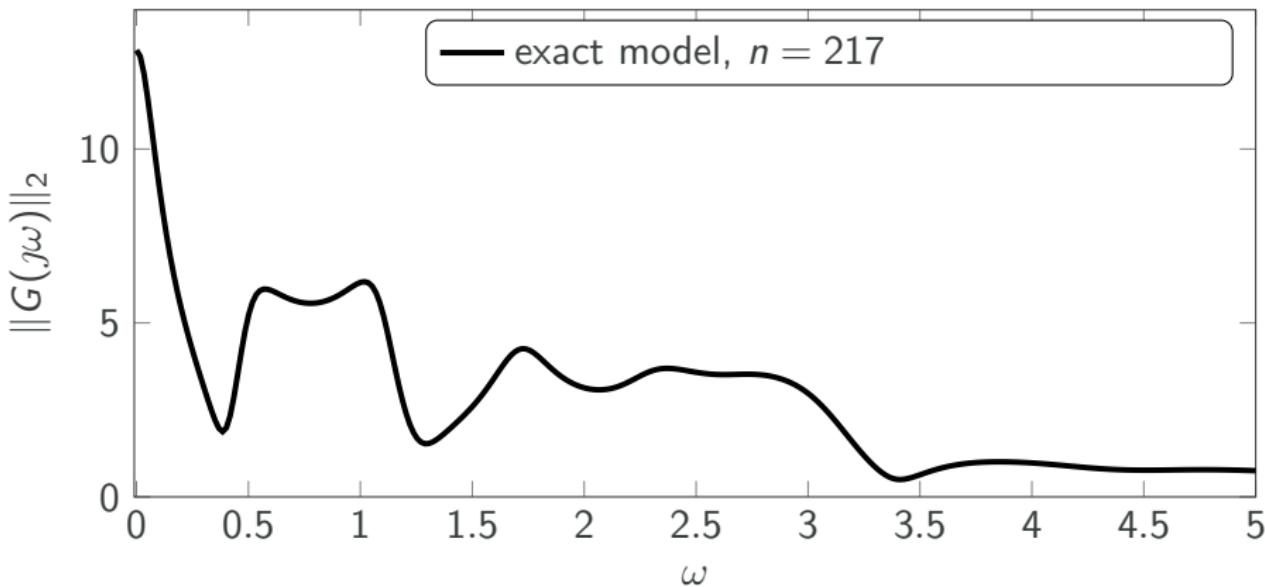
The dominant modes have most important influence on the input-output behavior



CSC

Dominant Poles

Random SISO Example ($B, C^T \in \mathbb{R}^n$)

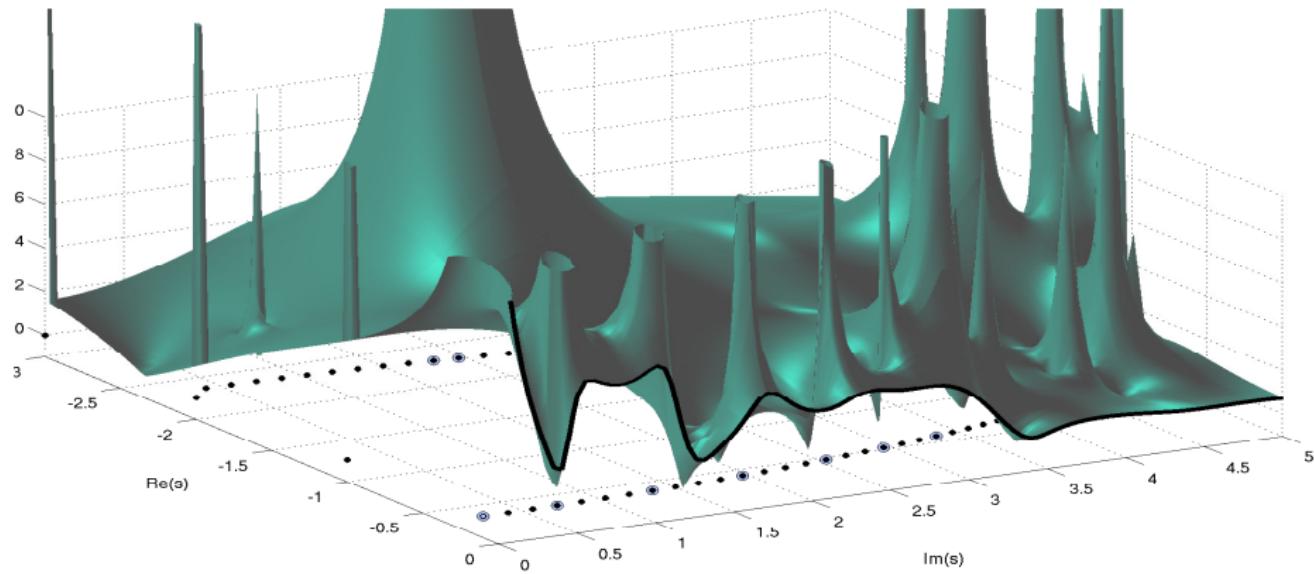




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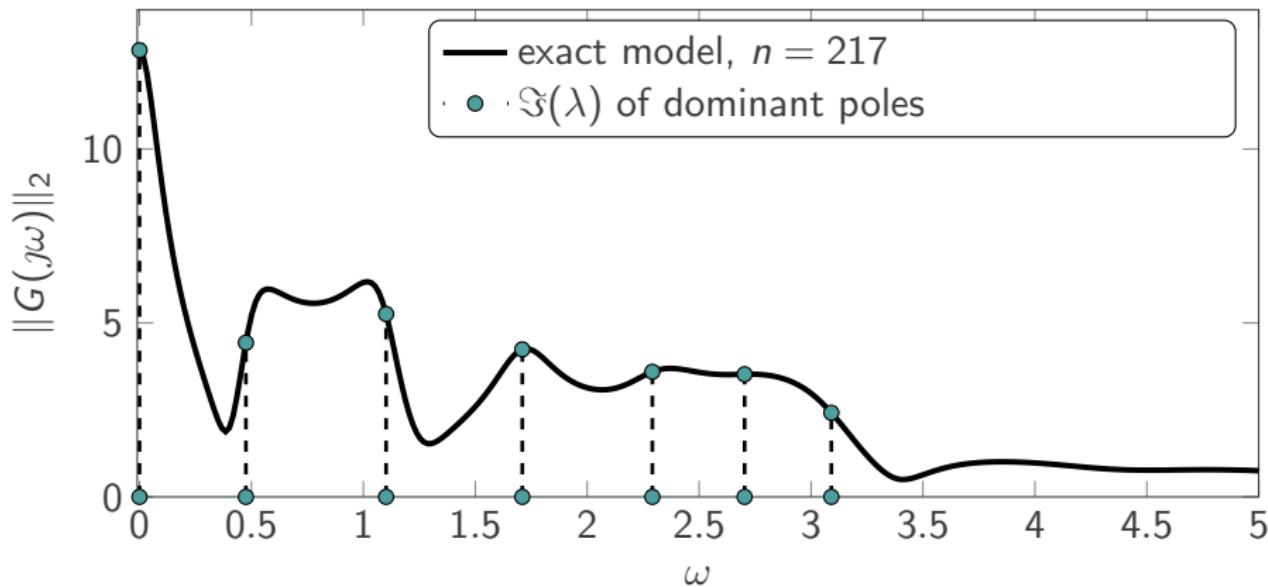




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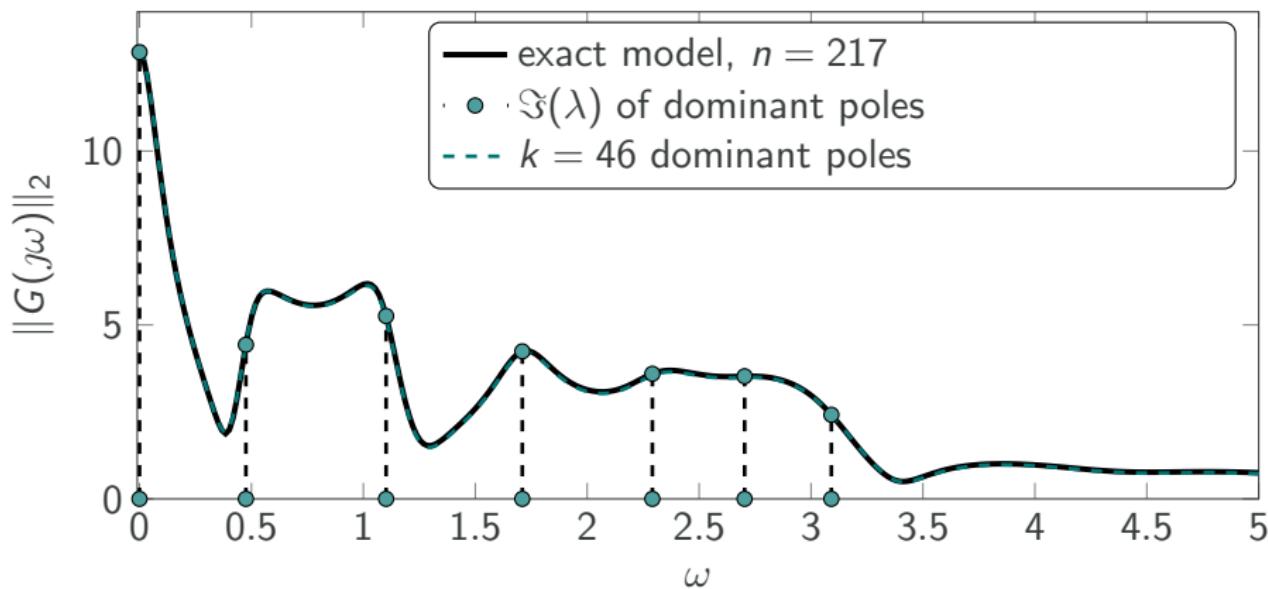




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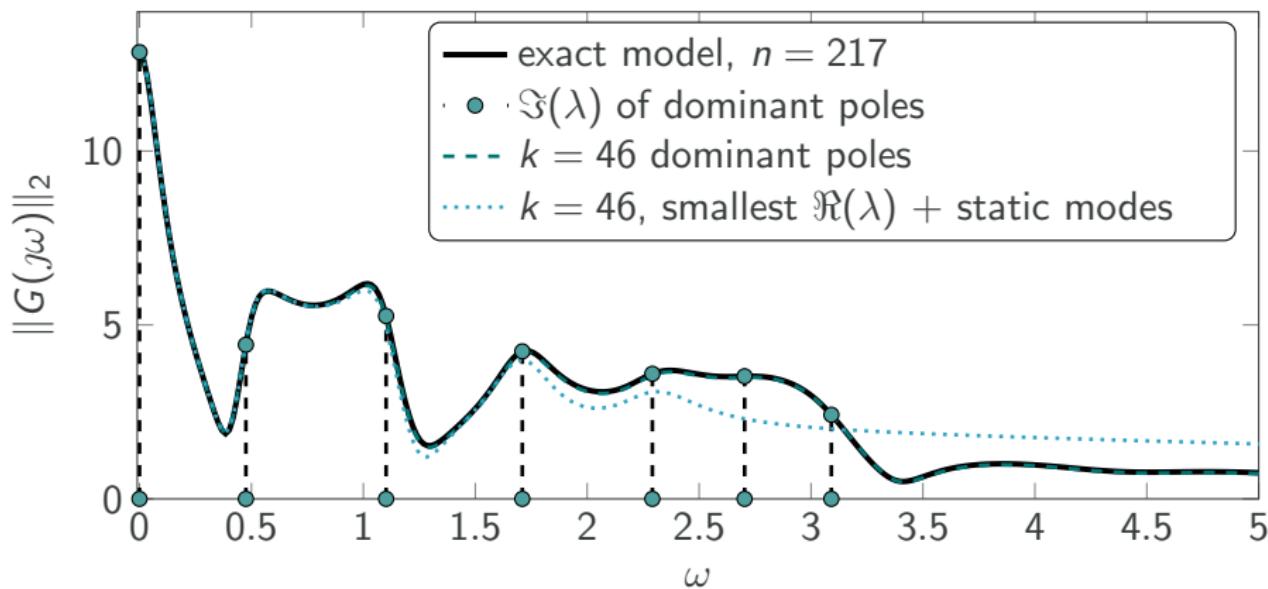




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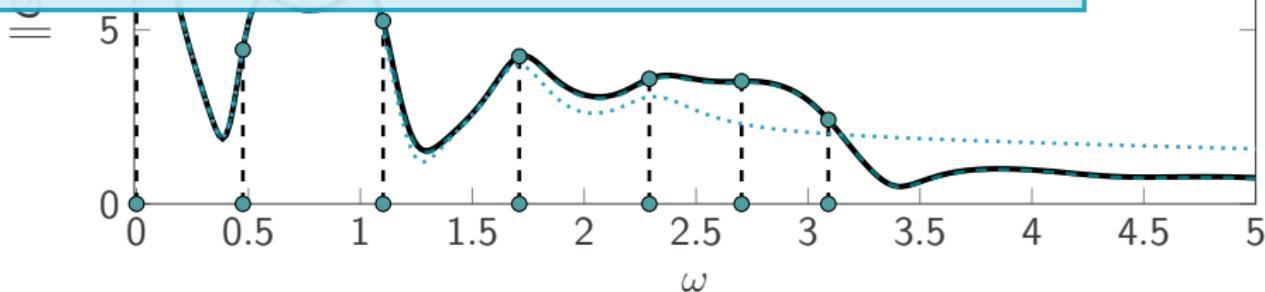
CSC

Dominant Poles

Random SISO Example ($B, C^T \in \mathbb{R}^n$)

Algorithms for computing dominant poles and eigenvectors:

- Subspace Accelerated Dominante Pole Algorithm (SADPA),
- Rayleigh-Quotient-Iteration (RQI),
- Jacobi-Davidson-Method.





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2. Mathematical Basics for LTI Systems I
3. Mathematical Basics for LTI System 2
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7. Balanced Truncation

Mathematical Basics

If A is stable, then the *Lyapunov* equations

$$AP + PA^* + BB^* = 0$$

and

$$A^*Q + QA + C^*C = 0$$

have a unique positive definite solutions P and Q , respectively.

- The matrix P is called the the (infinite) **controllability Gramian**
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- The matrix P is called the the (infinite) **controllability Gramian**
- and Q is called the (infinite) **observability Gramian**
- and one can show that P and Q fulfill

$$P = \int_0^\infty e^{A\tau} BB^* e^{A^*\tau} d\tau \quad \text{and} \quad Q = \int_0^\infty e^{A^*\tau} C^* C e^{A\tau} d\tau.$$

Mathematical Basics

$$\begin{aligned} A^*P + PA + BB^* &= 0 \\ AQ + Q^*A + C^*C &= 0 \end{aligned}$$

- If P and Q are the Gramians of a stable realization (A, B, C, D) ,
- then the transformed system $(\hat{A}, \hat{B}, \hat{C}, \hat{D}) = (TAT^{-1}, TB, CT^{-1}, D)$ has the Gramians

$$\hat{P} = TPT^* \quad \text{and} \quad \hat{Q} = (T^{-1})^*QT^{-1}$$

for **any** regular transformation T .

Mathematical Basics

- For any **minimal and stable** system (A, B, C, D) ,
- there are particular transformations T ,
- so that the transformed system has Gramians that are **equal** and **diagonal**:

$$\hat{P} = \hat{Q} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix},$$

with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$.

These realizations are called **Balanced Realizations**.

Definition

A realization (A, B, C, D) of a linear system Σ is **balanced** if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \text{diag} \{ \sigma_1, \dots, \sigma_n \} \quad (\text{w.l.o.g. } \sigma_j \geq \sigma_{j+1}, j = 1, \dots, n-1).$$

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When does a balanced realization exist?

Theorem

Given a **stable** minimal linear system $\Sigma : (A, B, C, D)$, a balanced realization is obtained by the state-space transformation with

$$T_b := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$, $Q = R^T R$ (e.g., Cholesky decompositions) and $S R^T = U \Sigma V^T$ is the SVD of $S R^T$.

Proof. Exercise!

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$\sigma_1, \dots, \sigma_n$ are the **Hankel singular values** of Σ .

Note: $\sigma_1, \dots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \dots, \sigma_n > 0$ in case of minimality!

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Remark

For non-minimal systems, the Gramians can also be transformed into diagonal matrices with the leading $\hat{n} \times \hat{n}$ submatrices equal to $\text{diag}(\sigma_1, \dots, \sigma_{\hat{n}})$, and

$$\hat{P}\hat{Q} = \text{diag}(\sigma_1^2, \dots, \sigma_{\hat{n}}^2, 0, \dots, 0).$$

see [LAUB/HEATH/PAIGE/WARD 1987, TOMBS/POSTLETHWAITE 1987].



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 - The Basic Method
 - Theoretical Background
 - Singular Perturbation Approximation
 - Balanced Pole-Matching

Basic principle:

- Recall: a stable system Σ , realized by (A, B, C, D) , is called **balanced**, if the **Gramians**, i.e., solutions P, Q of the **Lyapunov equations**

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- Compute balanced realization of the system via **state-space transformation**

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A_{11}, B_1, C_1, D)$.

Motivation:

The HSVs $\sigma(PQ)^{\frac{1}{2}} = \{\sigma_1, \dots, \sigma_n\}$ are **system invariants**: they are preserved under

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in transformed coordinates, the Gramians satisfy

$$\begin{aligned}(TAT^{-1})(TPT^T) + (TPT^T)(TAT^{-1})^T + (TB)(TB)^T &= 0, \\ (TAT^{-1})^T(T^{-T}QT^{-1}) + (T^{-T}QT^{-1})(TAT^{-1}) + (CT^{-1})^T(CT^{-1}) &= 0 \\ \Rightarrow (TPT^T)(T^{-T}QT^{-1}) &= TPQT^{-1},\end{aligned}$$

hence $\sigma(PQ) = \sigma((TPT^T)(T^{-T}QT^{-1}))$.



CSC

Balanced Truncation

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$\implies VW^T$ is a projector, hence BT is a **projection method**.

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- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_{\mathcal{H}_2} \leq \left(2 \sum_{k=r+1}^n \sigma_k \right) \|u\|_{\mathcal{H}_2}.$$

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned}\dot{x} &= Ax + Bu, & A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \\ y &= Cx, & C \in \mathbb{R}^{q \times n}, \quad x(-\infty) = 0.\end{aligned}$$

Alternative to State-Space Operator: Hankel Operator

Instead of

$$\mathcal{S}: u \mapsto y, \quad y(t) = \int_{-\infty}^t Ce^{A(t-\tau)} Bu(\tau) d\tau \quad \text{for all } t \in \mathbb{R}.$$

use the **Hankel operator**: (the future response of the past inputs)

$$\mathcal{H}: u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^0 Ce^{A(t-\tau)} Bu(\tau) d\tau \quad \text{for } t > 0,$$

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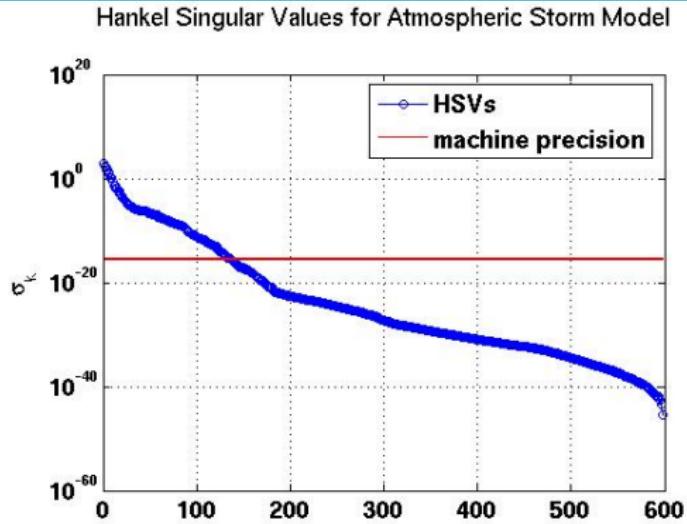
- The operator \mathcal{H} is compact $\Rightarrow \mathcal{H}$ has discrete SVD
 - The **Hankel singular values**: $\{\sigma_j\}_{j=1}^{\infty} : \sigma_1 \geq \sigma_2 \geq \dots \geq 0$
 - An **SVD-type** approximation of the linear map \mathcal{H} is possible!

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 \Rightarrow solution: Adamjan-Arov-Krein (AAK Theory, 1971/78).

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned}\dot{x} &= Ax + Bu, & A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \\ y &= Cx, & C \in \mathbb{R}^{q \times n}, \quad x(-\infty) = 0.\end{aligned}$$

Alternative to State-Space Operator: Hankel Operator

$$\mathcal{H}: u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^0 Ce^{A(t-\tau)} Bu(\tau) d\tau \quad \text{for all } t > 0.$$

\mathcal{H} compact $\Rightarrow \mathcal{H}$ has discrete SVD

\Rightarrow Best approximation problem w.r.t. 2-induced operator norm well-posed

\Rightarrow solution: Adamjan-Arov-Krein (AAK Theory, 1971/78).

But: computationally unfeasible for large-scale systems.

The *Hankel Singular Values* are Singular Values!

Theorem

Let P, Q be the controllability and observability Gramians of an LTI system Σ . Then the Hankel singular values $\sigma(PQ)^{\frac{1}{2}} = \{\sigma_1, \dots, \sigma_n\}$ are the singular values of the Hankel operator associated to Σ .



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Singular Perturbation Approximation(aka Balanced Residualization)

Assume the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \quad y = [C_1, C_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D u$$

is in balanced coordinates.



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Balanced truncation would set $x_2 = 0$ and use (A_{11}, B_1, C_1, D) as reduced-order model, thereby the information present in the remaining model is ignored!



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Particularly, if $G(0) = \hat{G}(0)$ ("zero steady-state error") is required, one can apply the same condensation technique as in Guyan reduction: instead of $x_2 = 0$, set $\dot{x}_2 = 0$. This yields the reduced-order model

$$\begin{aligned} \dot{x}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u, \\ y &= (C_1 - C_2A_{22}^{-1}A_{21})x_1 + (D - C_2A_{22}^{-1}B_2)u, \end{aligned}$$

with

- the same properties as the reduced-order model w.r.t. stability, minimality, error bound, but $\hat{D} \neq D$;
- zero steady-state error, $G(0) = \hat{G}(0)$ as desired.



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Note:

- A_{22} invertible as in balanced coordinates, $A_{22}\Sigma_2 + \Sigma_2A_{22}^T + B_2B_2^T = 0$ and (A_{22}, B_2) controllable, $\Sigma_2 > 0 \Rightarrow A_{22}$ stable.
- If the original system is not balanced, first compute a minimal realization by applying balanced truncation with $r = \hat{n}$.



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Balancing-Related Methods

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \text{diag}(\sigma_1, \dots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \dots \geq \sigma_n > 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.



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Classical Balanced Truncation (BT)

[MULLIS/ROBERTS '76, MOORE '81]

- P = controllability Gramian of system given by (A, B, C, D) .
- Q = observability Gramian of system given by (A, B, C, D) .
- P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0.$$

Basic Principle

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LQG Balanced Truncation (LQGBT)

[JONCKHEERE/SILVERMAN '83]

- P/Q = controllability/observability Gramian of closed-loop system based on LQG compensator.
- P, Q solve dual algebraic Riccati equations (AREs)

$$0 = AP + PA^T - PC^T CP + B^T B,$$

$$0 = A^T Q + QA - QBB^T Q + C^T C.$$

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

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and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Balanced Stochastic Truncation (BST)

[DESAI/PAL '84, GREEN '88]

- P = controllability Gramian of system given by (A, B, C, D) , i.e., solution of **Lyapunov equation** $AP + PA^T + BB^T = 0$.
- Q = observability Gramian of right spectral factor of power spectrum of system given by (A, B, C, D) , i.e., solution of **ARE**

$$\hat{A}^T Q + Q\hat{A} + QB_W(DD^T)^{-1}B_W^T Q + C^T(DD^T)^{-1}C = 0,$$

$$\text{where } \hat{A} := A - B_W(DD^T)^{-1}C, \quad B_W := BD^T + PC^T.$$



Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

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and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Positive-Real Balanced Truncation (PRBT)

[GREEN '88]

- Based on positive-real equations, related to positive real (Kalman-Yakubovich-Popov-Anderson) lemma.
- P, Q solve dual AREs

$$\begin{aligned} 0 &= \bar{A}P + P\bar{A}^T + PC^T\bar{R}^{-1}CP + B\bar{R}^{-1}B^T, \\ 0 &= \bar{A}^TQ + Q\bar{A} + QB\bar{R}^{-1}B^TQ + C^T\bar{R}^{-1}C, \end{aligned}$$

where $\bar{R} = D + D^T$, $\bar{A} = A - B\bar{R}^{-1}C$.

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \text{diag}(\sigma_1, \dots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \dots \geq \sigma_n > 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Other Balancing-Based Methods

- Bounded-real balanced truncation (BRBT) – based on bounded real lemma [OPDENACKER/JONCKHEERE '88];
- H_∞ balanced truncation (HinfBT) – closed-loop balancing based on H_∞ compensator [MUSTAFA/GLOVER '91].

Both approaches require solution of dual AREs.

- Frequency-weighted versions of the above approaches.



- Guaranteed preservation of physical properties like
 - stability (all),
 - passivity (PRBT),
 - minimum phase (BST).
- Computable error bounds, e.g.,

$$\text{BT: } \|G - G_r\|_{\mathcal{H}_\infty} \leq 2 \sum_{j=r+1}^n \sigma_j^{BT},$$

$$\text{LQGBT: } \|G - G_r\|_{\mathcal{H}_\infty} \leq 2 \sum_{j=r+1}^n \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}}$$

$$\text{BST: } \|G - G_r\|_{\mathcal{H}_\infty} \leq \left(\prod_{j=r+1}^n \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \|G\|_{\mathcal{H}_\infty},$$

- Can be combined with singular perturbation approximation for steady-state performance.



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Overview

1. Introduction to Linear Time Invariant Systems
2. Mathematical Basics for LTI Systems I
3. Mathematical Basics for LTI System 2
4. Introduction to Model Reduction
5. Model Reduction by Projection
6. Gramians and Balanced Realizations
7. Balanced Truncation



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Linear Time-invariant DAEs

System Theoretic Aspects of DAEs

Consider

$$\begin{aligned}Ex(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\y(t) &= Cx(t),\end{aligned}$$

where

- $x(t) \in \mathbb{R}^n$: the system's state
- $u(t) \in \mathbb{R}^m$: the input or control
- $y(t) \in \mathbb{R}^q$: the output or measurements

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- $E \in \mathbb{R}^{n \times n}$ is *singular*
- $A \in \mathbb{R}^{n \times n}$: the system matrix
- $B \in \mathbb{R}^{n \times m}$: the input matrix
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- $x(t) \in \mathbb{R}^n$: the system's state
- $u(t) \in \mathbb{R}^m$: the input or control
- $y(t) \in \mathbb{R}^q$: the output or measurements
- We will denote the system by $(E; A, B, C, D)$.
- $(E; A, B, C, D)$ are referred to as **descriptor** or **singular** systems.
- $E \in \mathbb{R}^{n \times n}$ is *singular*
- $A \in \mathbb{R}^{n \times n}$: the system matrix
- $B \in \mathbb{R}^{n \times m}$: the input matrix
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System Theoretic Aspects of DAEs

The transfer function of an $(E; A, B, C, D)$ system in time domain:

$G: u \mapsto y$:

$$\begin{aligned} y(t) = & C \left[e^{E^D A t} x_0 + \int_0^t e^{E^D A(t-\tau)} E^D B u(\tau) \, d\tau - \right. \\ & \left. - (I - E^D E) \sum_{i=0}^{\nu-1} (EA^D)^i A^D B u^{(i)}(t) \right] + Du(t), \end{aligned}$$

where

- E^D is the **Drazin inverse** of E
- ν is the **differentiation index** of the DAE $E\dot{x} = Ax$
- $u^{(i)}$ denotes the i -th derivative of u

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- $u^{(i)}$ denotes the i -th derivative of u

Note that if $E = I$, then $E^D = I$ and the transfer function is well-known:

$$G: u \mapsto y: y(t) = C \left[e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) \, d\tau \right] + Du(t).$$



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Linear Time-invariant DAEs

System Theoretic Aspects of DAEs

- In frequency domain (after a *Laplace* transform) the transfer function is given as

$$G(s) = C(sE - A)^{-1}B + D$$

.



System Theoretic Aspects of DAEs

- In frequency domain (after a *Laplace* transform) the transfer function is given as

$$G(s) = C(sE - A)^{-1}B + D$$

- Depending on B and C , the transfer function is likely to be **improper**.

For an **improper** it holds that $\|G(s)\| \rightarrow \infty$ as $s \rightarrow \infty$.



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Linear Time-invariant DAEs

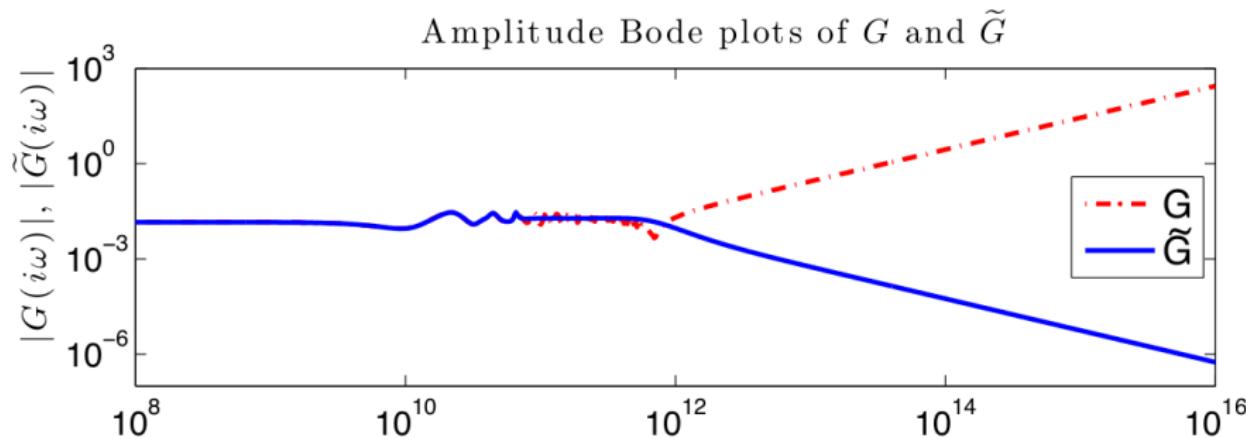
System Theoretic Aspects of DAEs

- In frequency domain (after a *Laplace* transform) the transfer function is given as $G(s) = C(sE - A)^{-1}B + D$.
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- Since standard model reduction approaches return a *proper* reduced system \hat{G} , the error will grow for large frequencies.



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Linear Time-invariant DAEs

System Theoretic Aspects of DAEs

The general problem is:

- the transfer function can have an improper part (frequency domain)
- the system differentiates the input (time domain)

The general approach is:

1. Project the DAE onto the part that is an ODE, i.e. a standard state space system
2. Keep the remainder, i.e. the algebraic or improper part, as it is

This means: no model reduction on the algebraic part!



We consider linearized Navier-Stokes equations:

$$\begin{aligned}M\dot{v}(t) &= A_1 v(t) + J^T p(t) + B_1 u(t), \\Jv(t) &= B_2 u(t), \\y(t) &= C_1 v(t) + C_2 p(t).\end{aligned}$$

- $v(t) \in \mathbb{R}^n$: state (velocity)
- $p(t) \in \mathbb{R}^p$: state (pressure)
- $u(t) \in \mathbb{R}^m$: input or control
- $y(t) \in \mathbb{R}^q$: the output or measurements
- $M \in \mathbb{R}^{n \times n}$: mass matrix (symmetric)
- $A_1 \in \mathbb{R}^{n \times n}$: the system matrix
- $J \in \mathbb{R}^{p \times n}$ is another system matrix (full)
- $B_1 \in \mathbb{R}^{n \times m}, B_2 \in \mathbb{R}^{p \times m}$: input matrices
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Note that this is an $(E; A, B, C, D)$ with

$$E := \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}, \quad A := \begin{bmatrix} A_1 & -J \\ J^T & 0 \end{bmatrix}, \quad B := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \text{and} \quad C := [C_1 \quad C_2].$$

$$\begin{aligned} M\dot{v}(t) &= A_1 v(t) + J^T p(t) + B_1 u(t), \\ Jv(t) &= B_2 u(t), \\ y(t) &= C_1 v(t) + C_2 p(t). \end{aligned}$$

Consider the projector

$$P := I - M^{-1} J^T (J M^{-1} J^T)^{-1} J$$

and see that with $v = Pv + (I - P)v =: v_d + v_a$ the system writes as

$$\begin{aligned} M\dot{v}_d(t) &= P^T A_1 v_d(t) + P^T A_1 v_a(t) + P^T B_1 u(t), \\ v_a(t) &= -M^{-1} J^T (J M^{-1} J^T)^{-1} J B_2 u(t), \\ p(t) &= -(J M^{-1} J^T)^{-1} [J M^{-1} [A(v_a(t) + v_d(t)) + B_1 u(t)] - B_2 \dot{u}(t)], \\ y(t) &= C_1 v_d(t) + C_1 v_a(t) + C_2 p(t). \end{aligned}$$

Since v_a and p depend linearly on v_d , u , and \dot{u} is an $(E; A, B, C, D)$ system with the state v_d and

$$E := M,$$

$$A := P^T A,$$

$$B := P^T [B_1 - AM^{-1}J^T(JM^{-1}J^T)^{-1}JB_2],$$

$$C := C_1 - C_2(JM^{-1}J^T)^{-1}JM^{-1}A,$$

$$D := D_1 + D_2,$$

with

$$D_1 := -C_1M^{-1}J^T(JM^{-1}J^T)^{-1}JB_2 + C_2(JM^{-1}J^T)^{-1}JM^{-1}AM^{-1}J^T(JM^{-1}J^T)^{-1}JB_1$$

$$D_2 := -C_2(JM^{-1}J^T)^{-1}B_2 \frac{d}{dt}.$$

Decoupling Differential and Algebraic Parts

$$\begin{aligned}D_1 &= -C_1 M^{-1} J^T (JM^{-1} J^T)^{-1} J B_2 + C_2 (JM^{-1} J^T)^{-1} J M^{-1} A M^{-1} J^T (JM^{-1} J^T)^{-1} J B_1 \\D_2 &= -C_2 (JM^{-1} J^T)^{-1} B_2 \frac{d}{dt}.\end{aligned}$$

Note that

- The transfer function is given as $G = C(sE - A)^{-1}B + D_1 + sD_2$

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- The transfer function is given as $G = C(sE - A)^{-1}B + D_1 + sD_2$
- if B_2 or C_2 is zero, then D_2 is zero,
 - no \dot{u} in the output
 - no obviously improper part sD_2 in G
- if B_2 is zero, then $D_1, D_2 = 0$
 - we obtain a standard $(E; A, B, C, D)$ system
 - no improper parts in G

If B_2 and C_2 are zero, then we have a standard $(A, B, C, -)$ system:

$$M\dot{v}_d = P^T A v_d + P^T B_1 u,$$
$$y = C_1 v.$$

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If we want to apply Balanced Truncation, we need to cope with the following difficulties:

- The system is not minimal
 - this is automatically *fixed* by BT, if we can find the right solutions of the nonregular Lyapunov equations like

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 - Combine BT with *LQG*-stabilization [BENNER AND HEILAND, '15]

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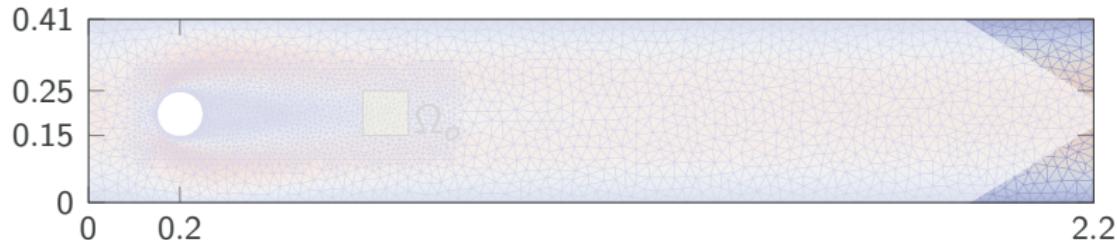
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- The system is not stable
 - Combine BT with *LQG*-stabilization [BENNER AND HEILAND, '15]
- Explicit computation of the projector P is not possible for large scale systems
 - use algorithms that do not need P explicitly, cf. [GUGERCIN, STYKEL, WYATT '13], [HEINKENSCHLOSS, SORENSEN, SUN '08], and [BENNER AND HEILAND, '15]



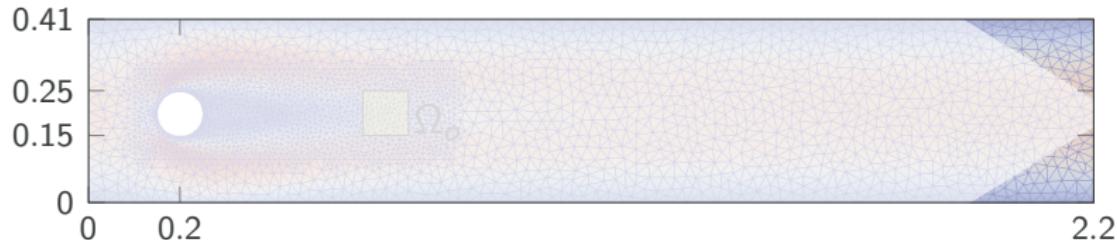
Numerical Example NSE



- 2D cylinder wake
- Navier-Stokes Equations
- $Re = 100$
- *Taylor-Hood* finite elements
- 30000 velocity nodes



Numerical Example NSE



- 2D cylinder wake
- Navier-Stokes Equations
- $Re = 100$
- *Taylor-Hood* finite elements
- 30000 velocity nodes
- Boundary control at 2 outlets
- distributed observation with 6 degrees of freedom
- LQGBT-reduced order observer and controller of state dimension $r = 13$
- Target: stabilization of the steady-state solution



LQGBT Reduction - Bode Plot

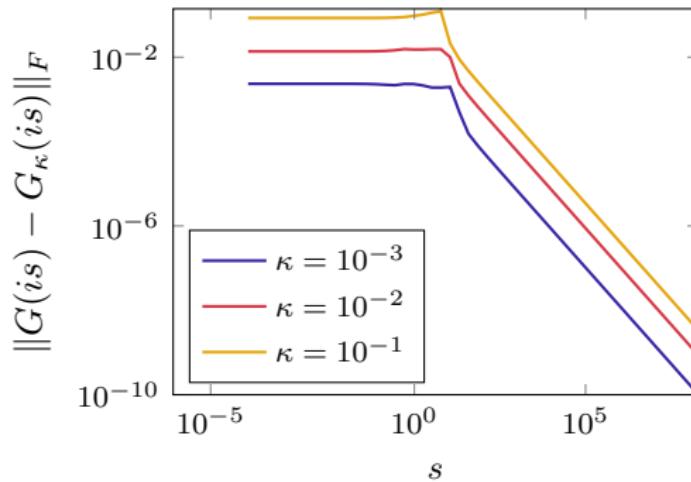


Figure: The error in the frequency response for varying thresholds κ measured in the Frobenius norm with i denoting the imaginary unit and the transfer functions in frequency domain as defined, e.g., in [4].

This plot was taken from [BENNER AND HEILAND '15]. The varying

Cylinder Wake Stabilization

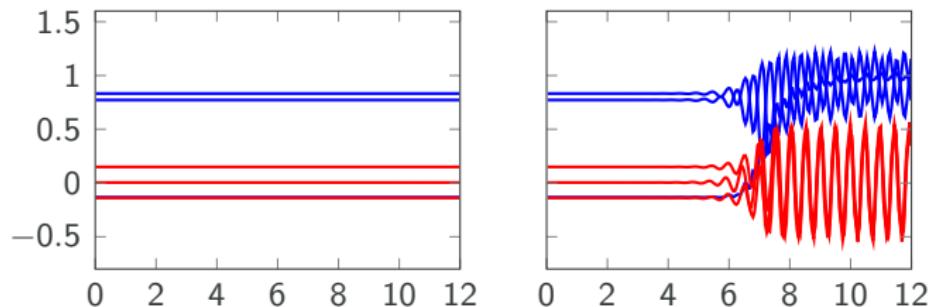
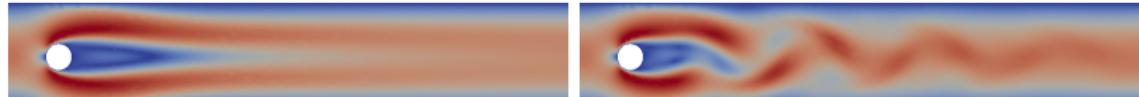


Figure: Measured signal y versus time $t \in [0, 12]$ of the perturbed closed loop system with a reduced controller of dimension $r = 13$ (left), compared to the response of the uncontrolled system (right). Blue corresponds to the x -component of the velocity and red to y -component. Below, a snapshot of the magnitude of the velocity solutions at $t = 12$.



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- The efficient implementation requires further effort
- For Navier-Stokes equations there are examples of efficient application of BT related methods

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CSC

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