A61 $\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$

gesucht $S_n = \sum_{k=1}^{n} \frac{1}{(k+1)(k+2)}$

und $\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)} = \lim_{n\to\infty} S_n$.

Partialsummen: $S_1 = \sum_{k=1}^{1} \frac{1}{(k+1)(k+2)} = \frac{1}{(1+1)(1+2)} = \frac{1}{6}$

$S_2 = \sum_{k=1}^{2} \frac{1}{(k+1)(k+2)} = \frac{1}{6} + \frac{1}{(2+1)(2+2)} = \frac{1}{6} + \frac{1}{12} = \frac{2}{12} + \frac{1}{12} = \frac{1}{4}$

$S_3 = \sum_{k=1}^{3} \frac{1}{(k+1)(k+2)} = \frac{1}{6} + \frac{1}{12} + \frac{1}{(3+1)(3+2)} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{1}{4} + \frac{1}{20} = \frac{3}{10}$

"Ich erkenne keine Struktur... :("

Ansatz :)

Partialbruchzerlegung:

beide Seiten haben die gleichen "Pole"

$$\frac{1}{(k+1)(k+2)} \stackrel{?}{=} \frac{a_1}{(k+1)} + \frac{a_2}{(k+2)}$$

$$\Updownarrow$$

$$1 = a_1(k+2) + a_2(k+1)$$
$$= k(a_1+a_2) + 2a_1 + a_2$$

diese Gleichung muss für alle $k \in \mathbb{N}$ gelten

$\Rightarrow a_1 + a_2 = 0$ bzw. $a_1 = -a_2$

$\Rightarrow 1 = 2a_1 + a_2 = 2a_1 - a_1 = a_1$

$\Rightarrow a_1 = 1$

$a_2 = -1$

und somit $\frac{1}{(k+1)(k+1)} = \frac{1}{k+1} - \frac{1}{k+2}$

mit $\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2}$ gilt

$$S_1 = \sum_{k=1}^{1} \frac{1}{(k+1)(k+2)} = \sum_{k=1}^{1} \left(\frac{1}{k+1} - \frac{1}{k+2} \right) = \frac{1}{2} - \frac{1}{3}$$

$$S_2 = \sum_{k=1}^{2} \dots = \sum_{k=1}^{2} \text{—} \;''\; \text{—} = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = \frac{1}{2} - \frac{1}{4}$$

$$S_3 = \sum_{k=1}^{3} (\dots) = \sum_{k=1}^{3} (\text{—} + \text{—}) = \frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} = \frac{1}{2} - \frac{1}{5}$$

$$\vdots$$

$$S_n = \sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \sum_{k=1}^{n} \left(\frac{1}{k+1} - \frac{1}{k+2} \right) = \underbrace{\frac{1}{2} - \frac{1}{3}}_{k=1} + \underbrace{\frac{1}{3} - \frac{1}{4}}_{k=2} + \underbrace{\frac{1}{4} - \frac{1}{5}}_{k=3} + \dots + \frac{1}{n+1} - \frac{1}{n+2}$$

(with $=0$ over each cancelling pair $-\frac{1}{3}+\frac{1}{3}$, $-\frac{1}{4}+\frac{1}{4}$, $-\frac{1}{5}+\dots$, $\dots+\frac{1}{n+1}$)

$$= \frac{1}{2} - \frac{1}{n+2}$$

also $S_n = \frac{1}{2} - \frac{1}{n+2}$

und damit

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2} - \frac{1}{n+2} = \frac{1}{2}$$

$0, \; n \to \infty$

∎