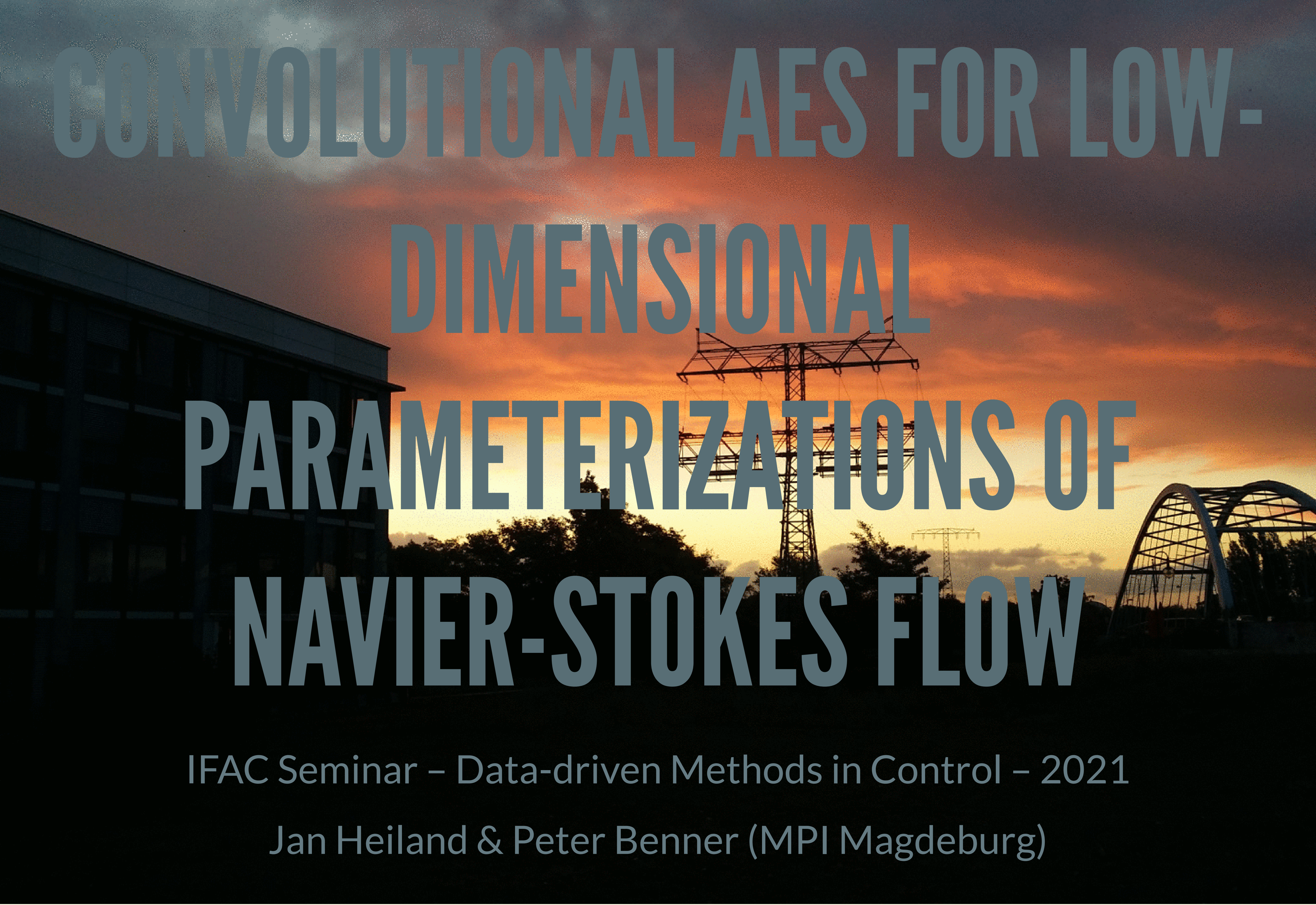


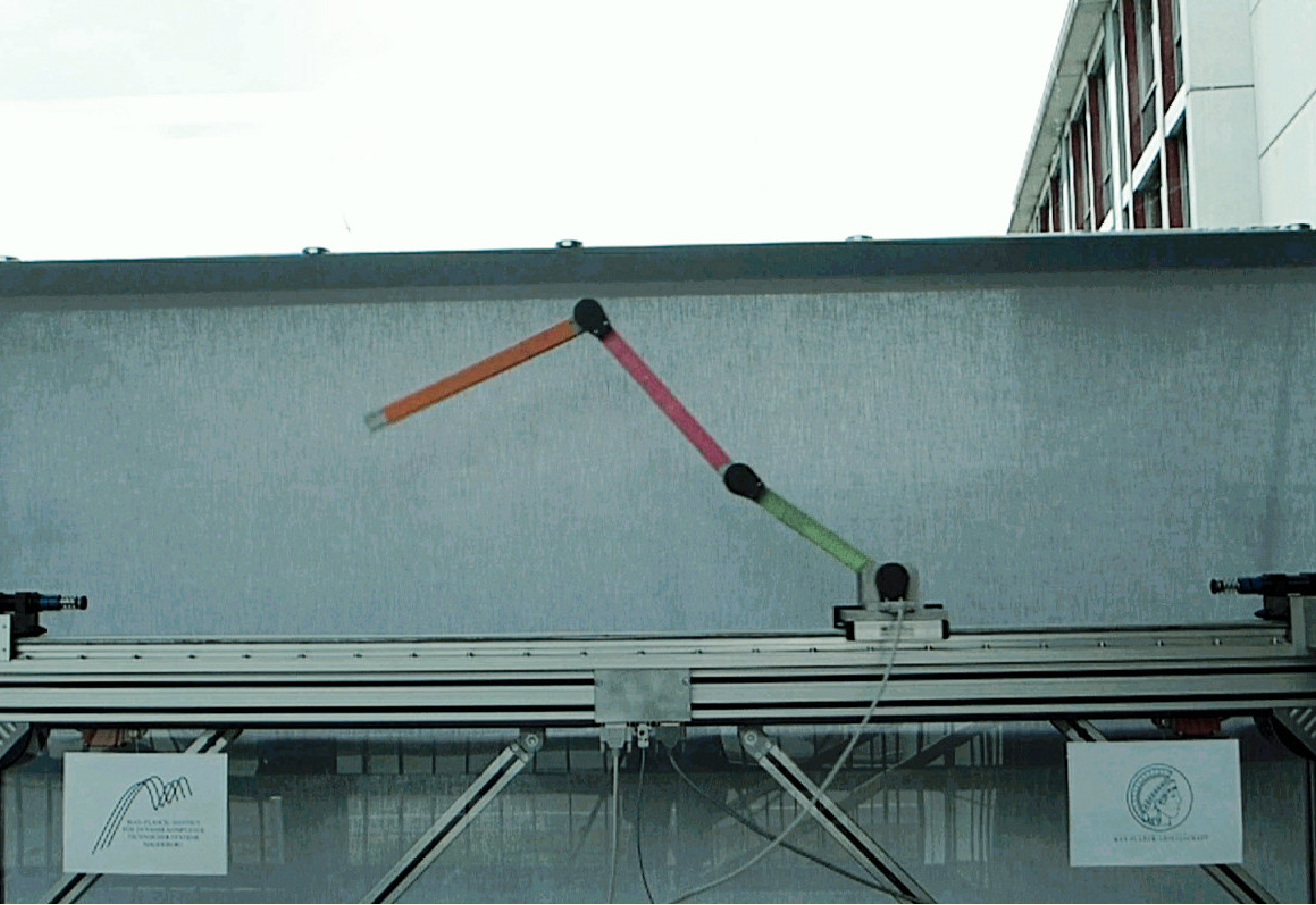
CONVOLUTIONAL AES FOR LOW-DIMENSIONAL PARAMETERIZATIONS OF NAVIER-STOKES FLOW

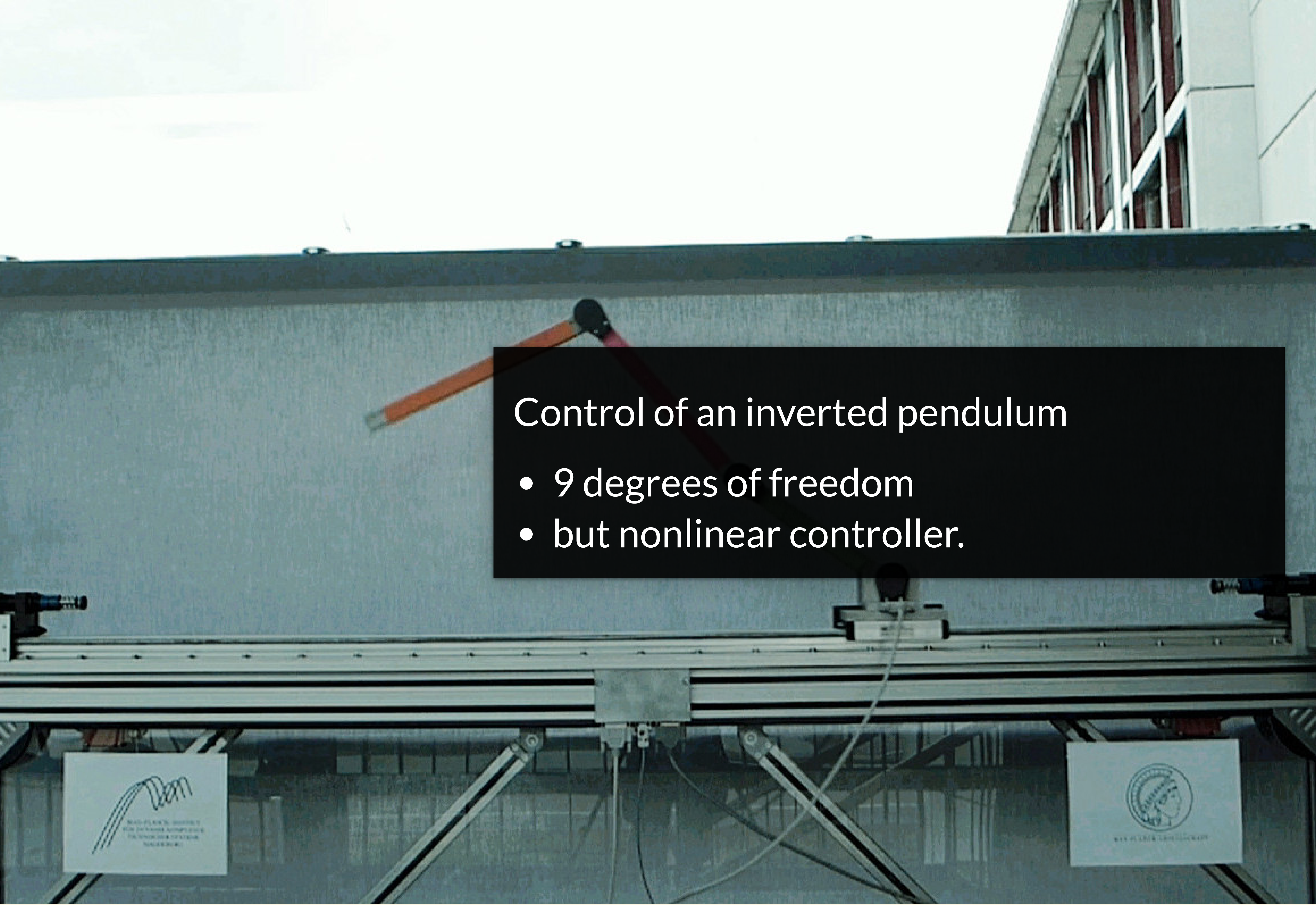
IFAC Seminar – Data-driven Methods in Control – 2021

Jan Heiland & Peter Benner (MPI Magdeburg)

INTRODUCTION

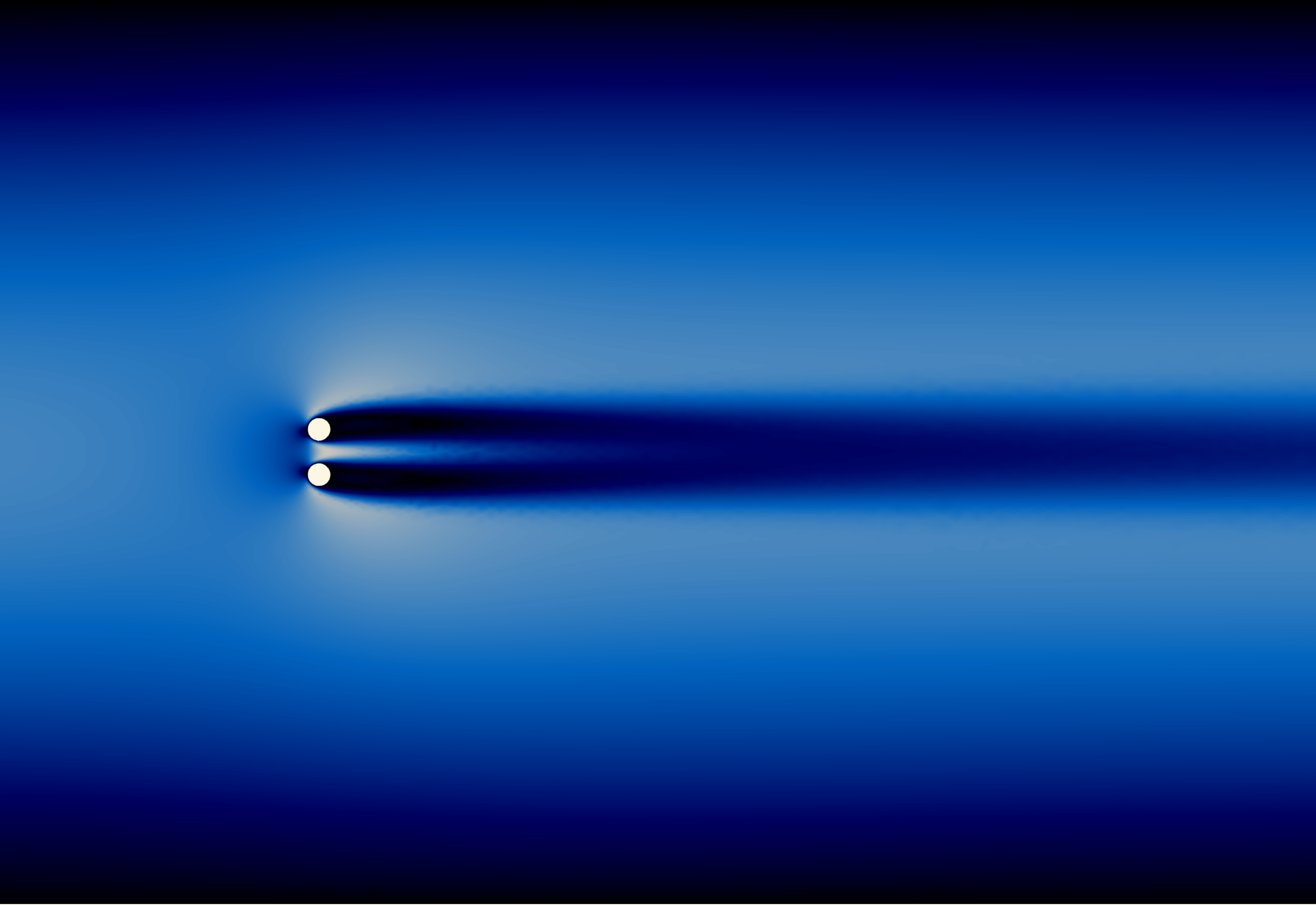
$$\dot{x} = f(x) + Bu$$

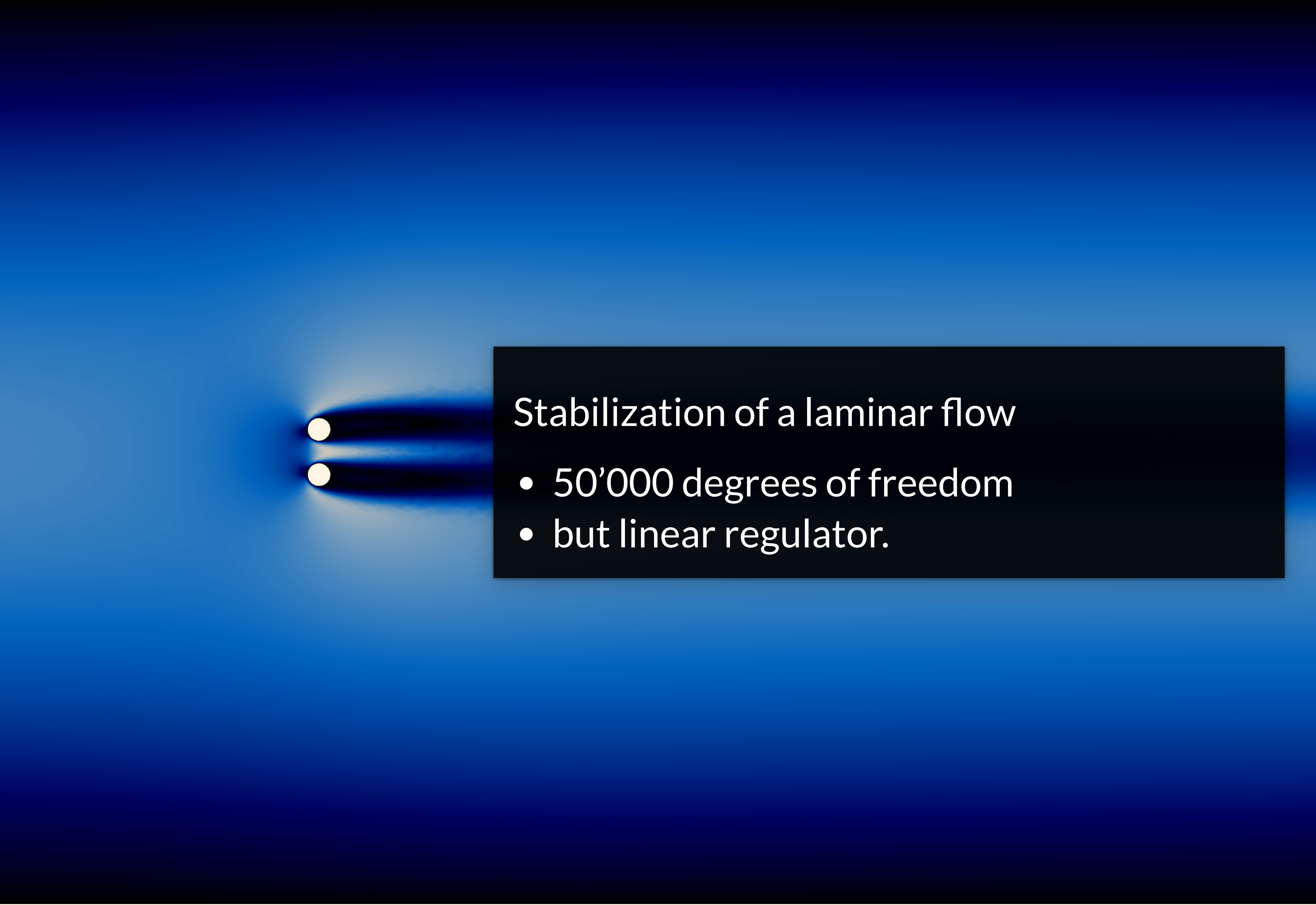




Control of an inverted pendulum

- 9 degrees of freedom
- but nonlinear controller.





Stabilization of a laminar flow

- 50’000 degrees of freedom
- but linear regulator.

CONTROL OF NONLINEAR & LARGE-SCALE SYSTEMS

A general approach would include

- powerful backends (linear algebra / optimization)
- exploitation of general structures
- data-driven surrogate models
- all of it?!

SDC REPRESENTATION

$$\dot{x} = [A(x)]\, x + Bu$$

- Under mild conditions, the flow $f(x)$ can be factorized

$$\dot{x} = [A(x)]\, x + Bu$$

– a *state dependent coefficient* system – with some

$$A\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}.$$

- Control through a *state-dependent state-feedback law*

$$u = -[B^* P(x)]\, x.$$

NONLINEAR SDRE FEEDBACK

- Set

$$u = -[B^T P(x)]\, x.$$

- with $P(x)$ as the solution to the state-dependent Riccati equation

$$A(x)^T P + PA(x) - PBB^T P + C^T C = 0$$

- the system

$$\dot{x} = f(x) + Bu \ = [A(x) - BB^T P(x)]\, x$$

can be controlled towards an equilibrium; see, e.g., Banks, Lewis, and Tran (2007).

LINEAR UPDATES AS AN ALTERNATIVE

Theorem Benner and Heiland (2018)

- ...
- If P_0 is the Riccati solution for $x = x_0$
- and if E solves the **linear** equation

$$A(x)E + E(A(x_0) - BB^T P_0) = A(x_0) - A(x)$$

- with $\|E\| \leq \epsilon < 1$,
- then $u = -B^T P_0 (I + E)^{-1}$ stabilizes the system.

LPV REPRESENTATION

$$\dot{x} \approx \left[A_0 + \Sigma\, \rho_k(x) A_k\right] x + Bu$$

The *linear parameter varying* (LPV) representation/approximation

$$\dot{x} = f(x) + Bu = [\tilde{A}(\rho(x))]\, x + Bu \approx [A_0 + \Sigma\, \rho_k(x) A_k]\, x + Bu$$

with **affine parameter dependency** can be exploited for designing nonlinear controller through scheduling.

SCHEDULING OF H_∞ CONTROLLERS

- If $\rho(x) \in \mathbb{R}^k$ can be confined to a bounded polygon,
- there is globally stabilizing H_∞ controller
- that can be computed
- through solving k **coupled LMI** in the size of the state dimension;

see Apkarian, Gahinet, and Becker (1995) .

SERIES EXPANSION OF SDRE SOLUTION

For $A(x) = \sum_{k=1}^{r} \rho_k(x) A_k$, the solution P to the SDRE

$$A(x)^T P + PA(x) - PBB^T P + C^T C = 0$$

can be expanded in a series

$$P(x) = P_0 + \sum_{|\alpha|>0} \rho(x)^{(\alpha)} P_\alpha$$

where P_0 solves a Riccati equation and P_α solve Lyapunov (linear!) equations;

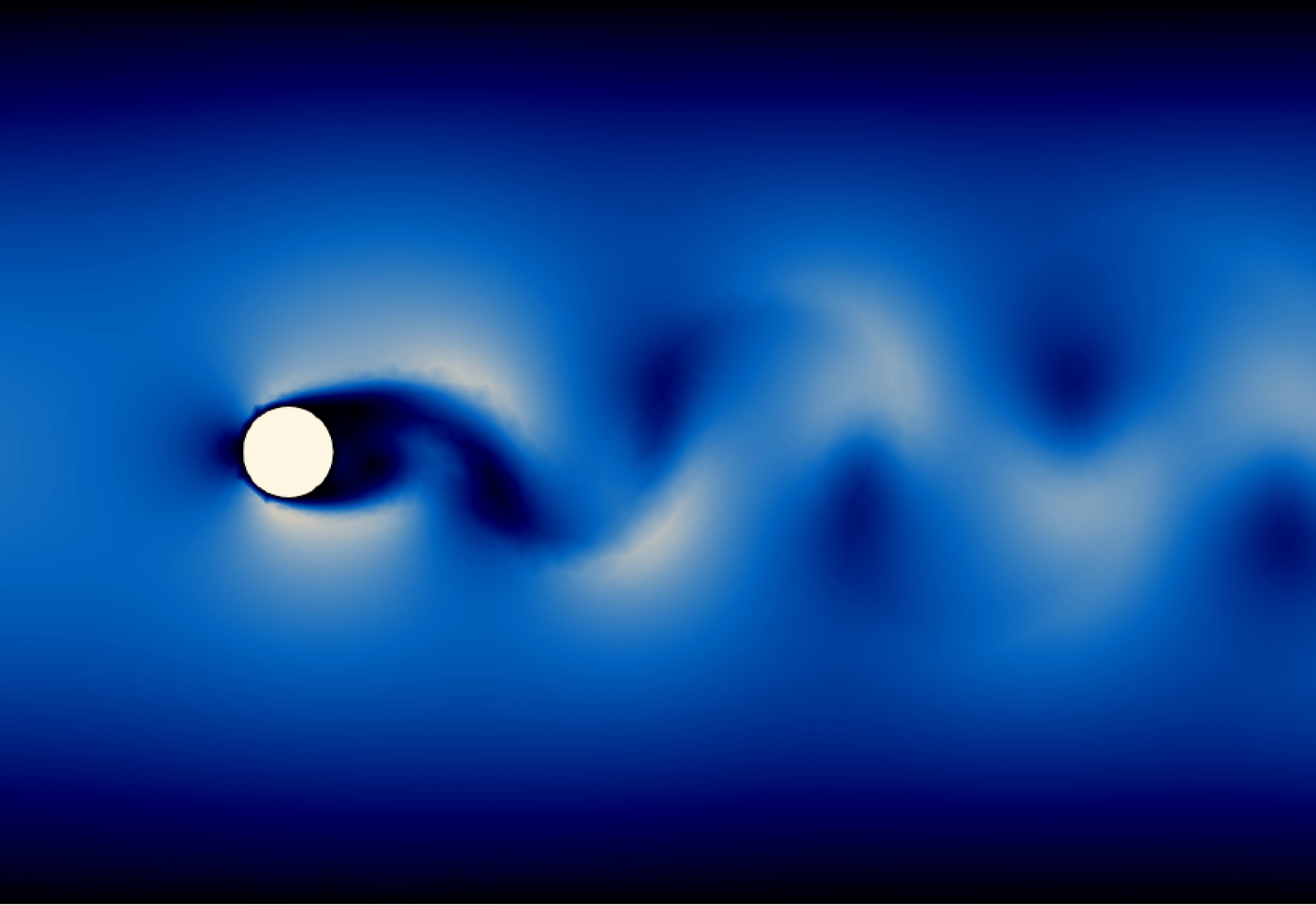
see Beeler, Tran, and Banks (2000).

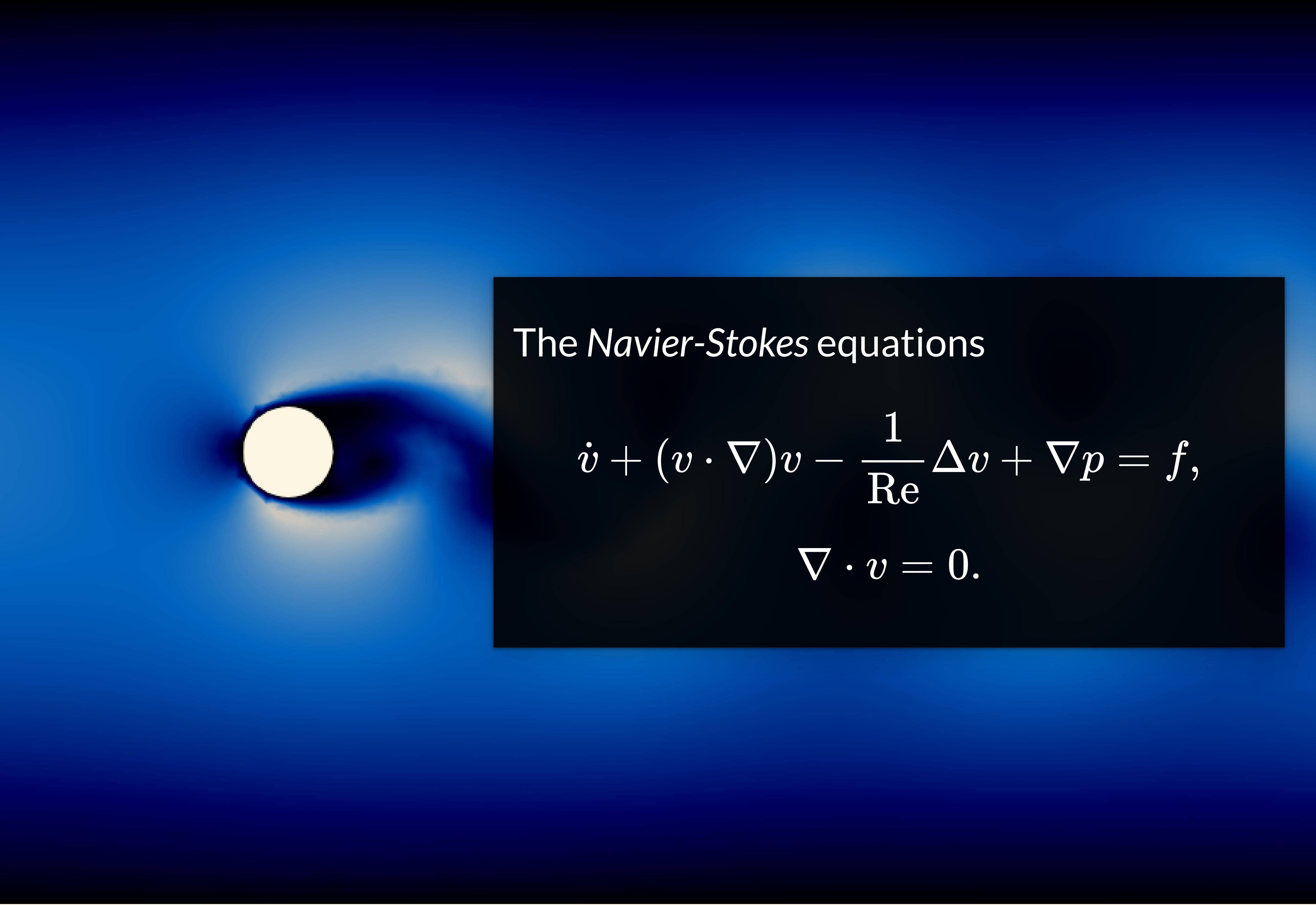
WE SEE

Manifold opportunities if only k was small.

LOW-DIMENSIONAL LPV

Approximation of *Navier-Stokes Equations* by *Convolutional Neural Networks*





The *Navier-Stokes* equations

$$\dot{v} + (v \cdot \nabla) v - \frac{1}{\mathrm{Re}} \Delta v + \nabla p = f,$$

$$\nabla \cdot v = 0.$$

- Let v be the velocity solution and let

$$V = [V_1 \quad V_2 \quad \cdots \quad V_r]$$

  be a, say, *POD* basis with

$$v(t) \approx VV^T v(t) =: \tilde{v}(t),$$

- then

$$\rho(v(t)) = V^T v(t)$$

  is a parametrization.

- And with

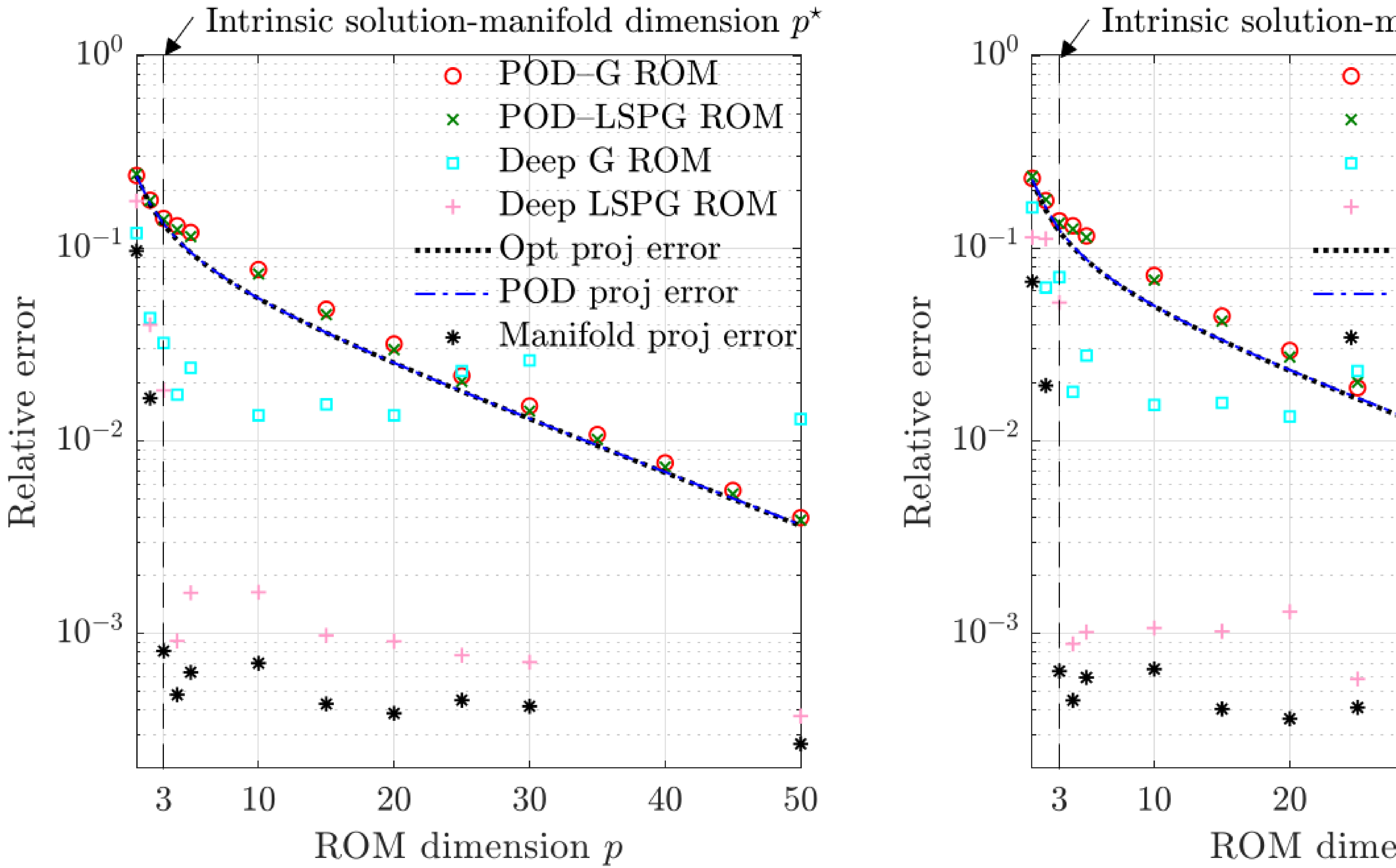
$$\tilde{v} = VV^T v = V\rho = \sum_{k=1}^{r} V_k \rho_k,$$

- the NSE has the low-dimensional LPV representation via

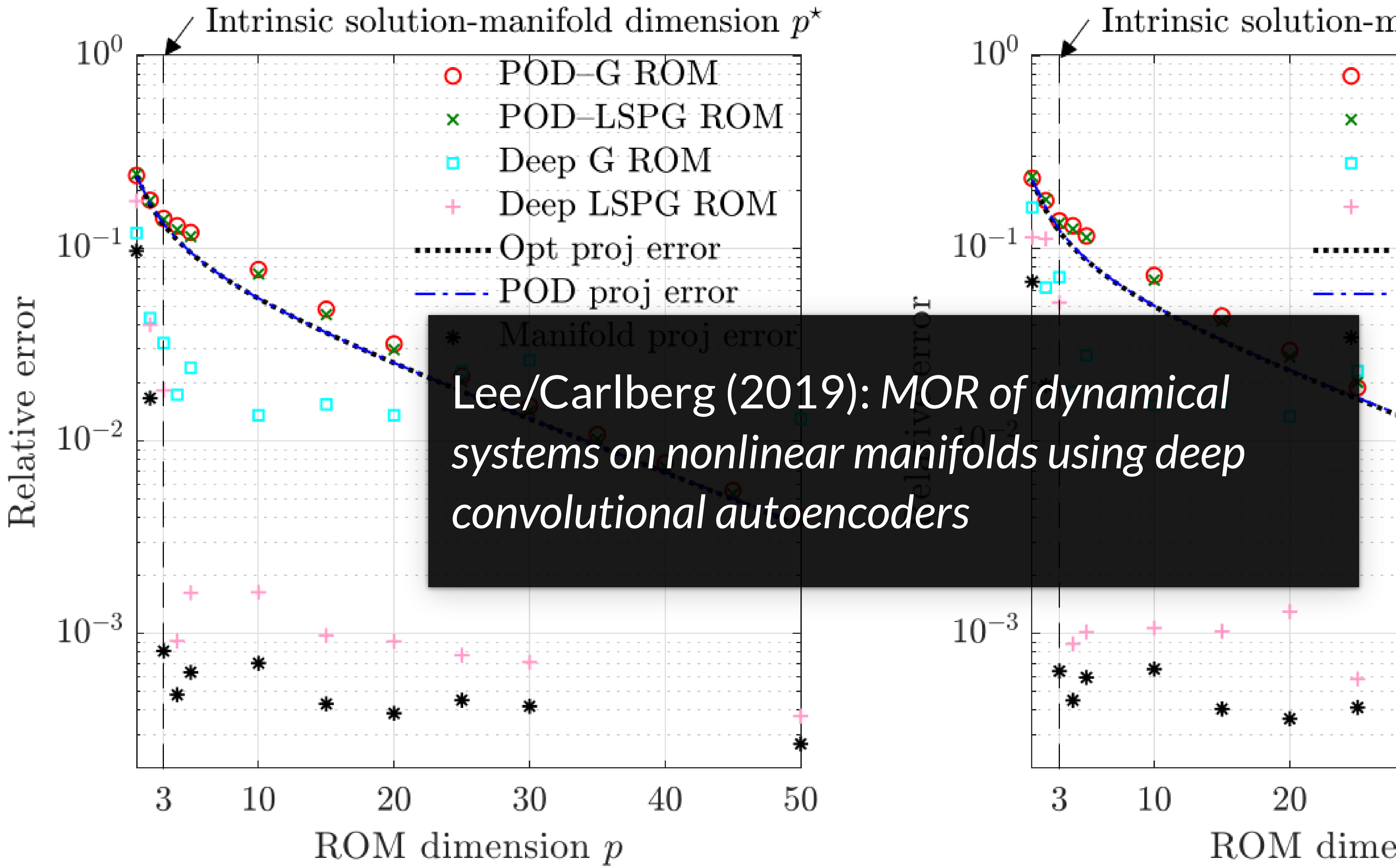
$$(v \cdot \nabla)v \approx (\tilde{v} \cdot \nabla)v = [\sum_{k=1}^{r} \rho_k (V_k \cdot \nabla)] \, v.$$

QUESTION

Can we do better than POD?

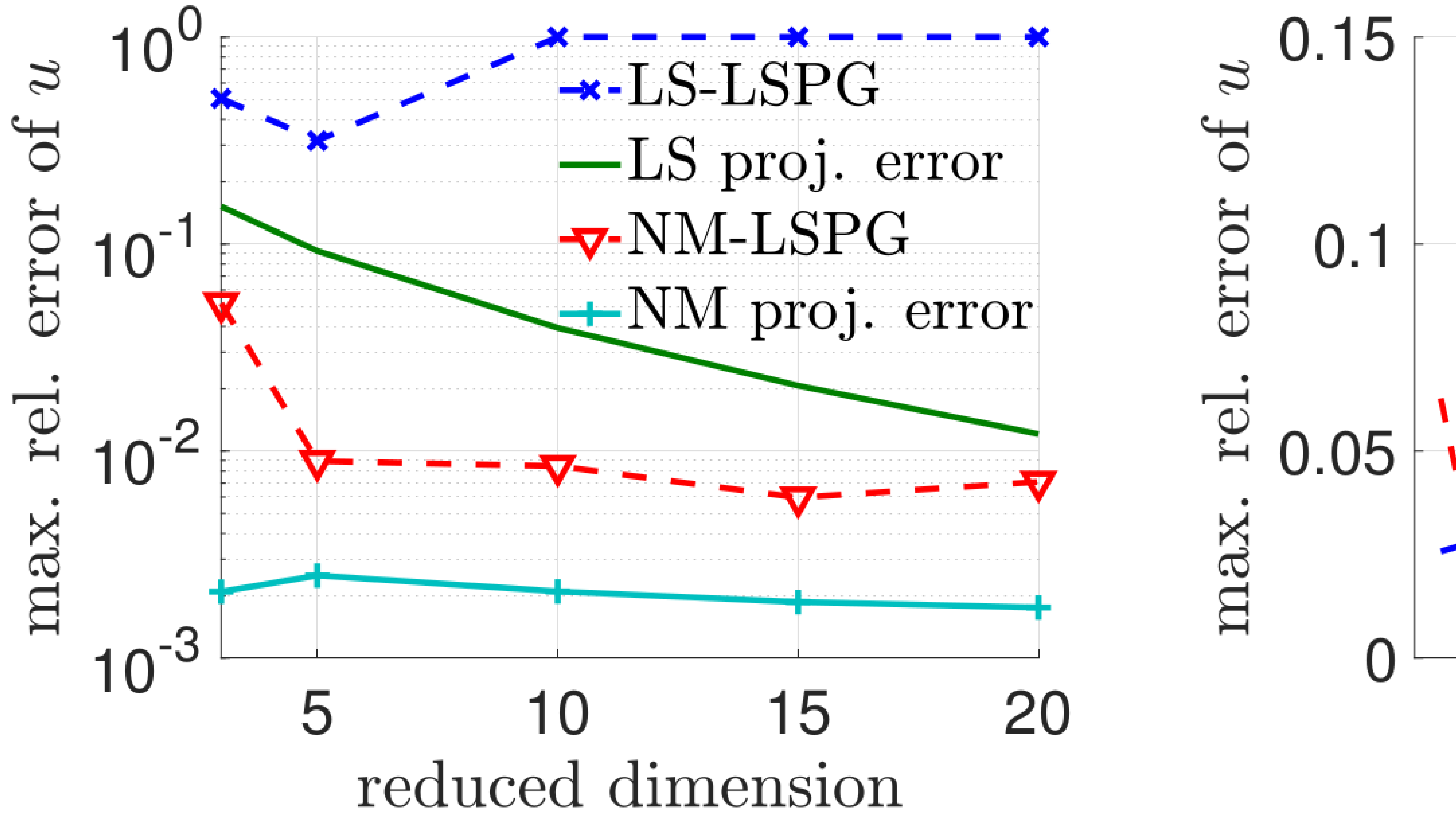


(a) Online-parameter instance 1, $\boldsymbol{\mu}^1_{\text{test}} = (4.3, 0.021)$ (b) Online-parameter instance

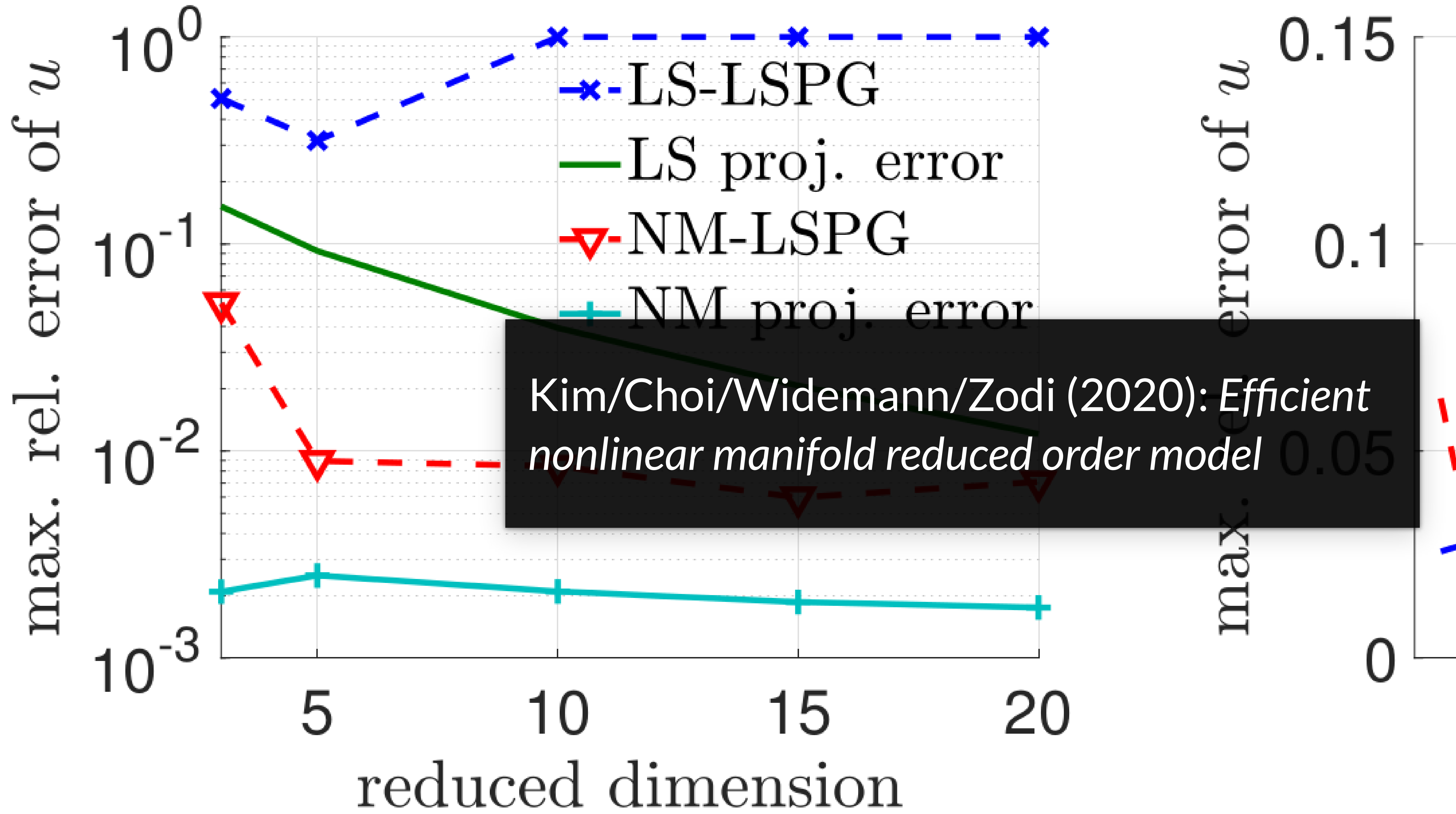


Lee/Carlberg (2019): *MOR of dynamical systems on nonlinear manifolds using deep convolutional autoencoders*

(a) Online-parameter instance 1, $\boldsymbol{\mu}^1_{\text{test}} = (4.3, 0.021)$ (b) Online-parameter instance



(a) Relative errors vs reduced dimensions



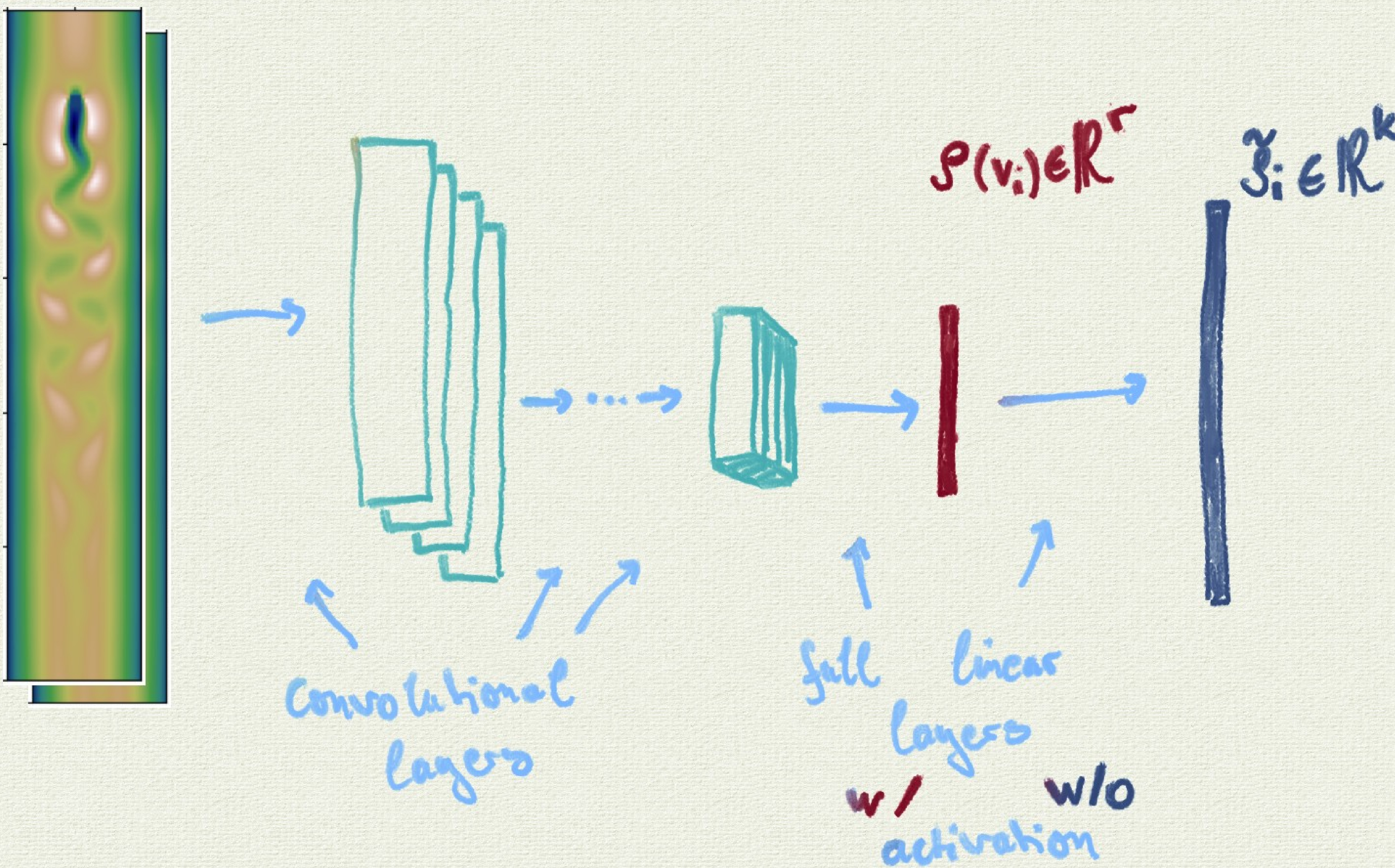
Kim/Choi/Widemann/Zodi (2020): *Efficient nonlinear manifold reduced order model*

(a) Relative errors vs reduced dimensions

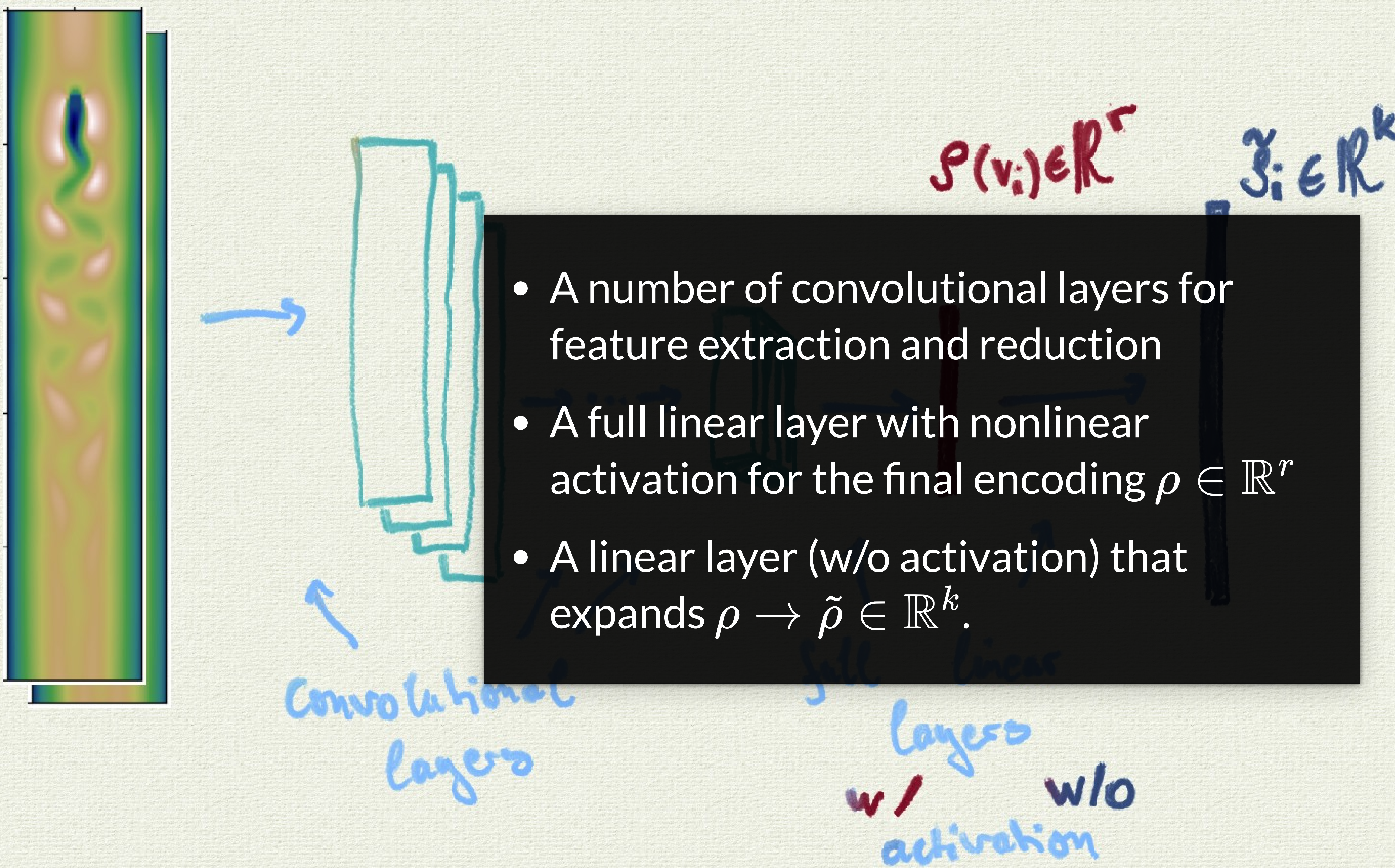
CONVOLUTION AUTOENCODERS FOR NSE

1. Consider solution snapshots $v(t_k)$ as pictures.
2. Learn convolutional kernels to extract relevant features.
3. While extracting the features, we reduce the dimensions.
4. Encode $v(t_k)$ in a low-dimensional ρ_k.

OUR EXAMPLE ARCHITECTURE IMPLEMENTATION



$\rho(v_i) \in \mathbb{R}^r$
$z_i \in \mathbb{R}^k$
convolutional layers
full linear layers
w/ w/o
activation



- A number of convolutional layers for feature extraction and reduction
- A full linear layer with nonlinear activation for the final encoding $\rho \in \mathbb{R}^r$
- A linear layer (w/o activation) that expands $\rho \to \tilde{\rho} \in \mathbb{R}^k$.

INPUT:

- Velocity snapshots v_i of an FEM simulation with

$$n = 50'000$$

  degrees of freedom
- interpolated to two pictures with 63x95 pixels each
- makes a 2x63x69 tensor.

TRAINING FOR MINIMIZING:

$$\|v_i - VW\rho(v_i)\|_M^2$$

which includes

1. the POD modes $V \in \mathbb{R}^{n\times k}$,
2. a learned weight matrix $W \in \mathbb{R}^{k\times r}$: $\rho \mapsto \tilde{\rho}$,
3. the mass matrix M of the FEM discretization.

GOING PINN

Outlook: the induced low-dimensional affine-linear LPV representation of the convection

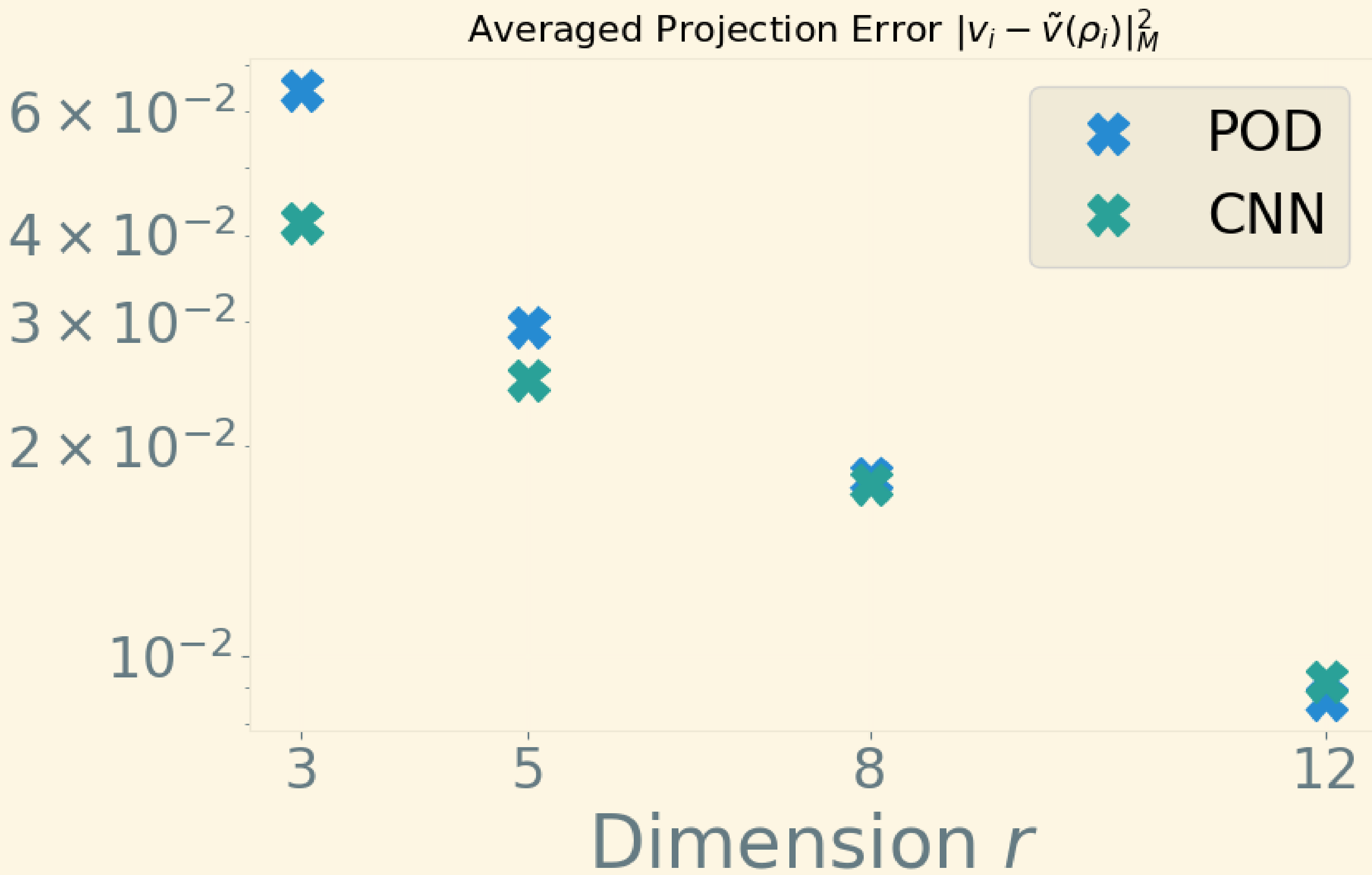
$$\|(v_i \cdot \nabla)v_i - (VW\rho_i \cdot \nabla)v_i\|^2_{M^{-1}}$$

as the target of the optimization.

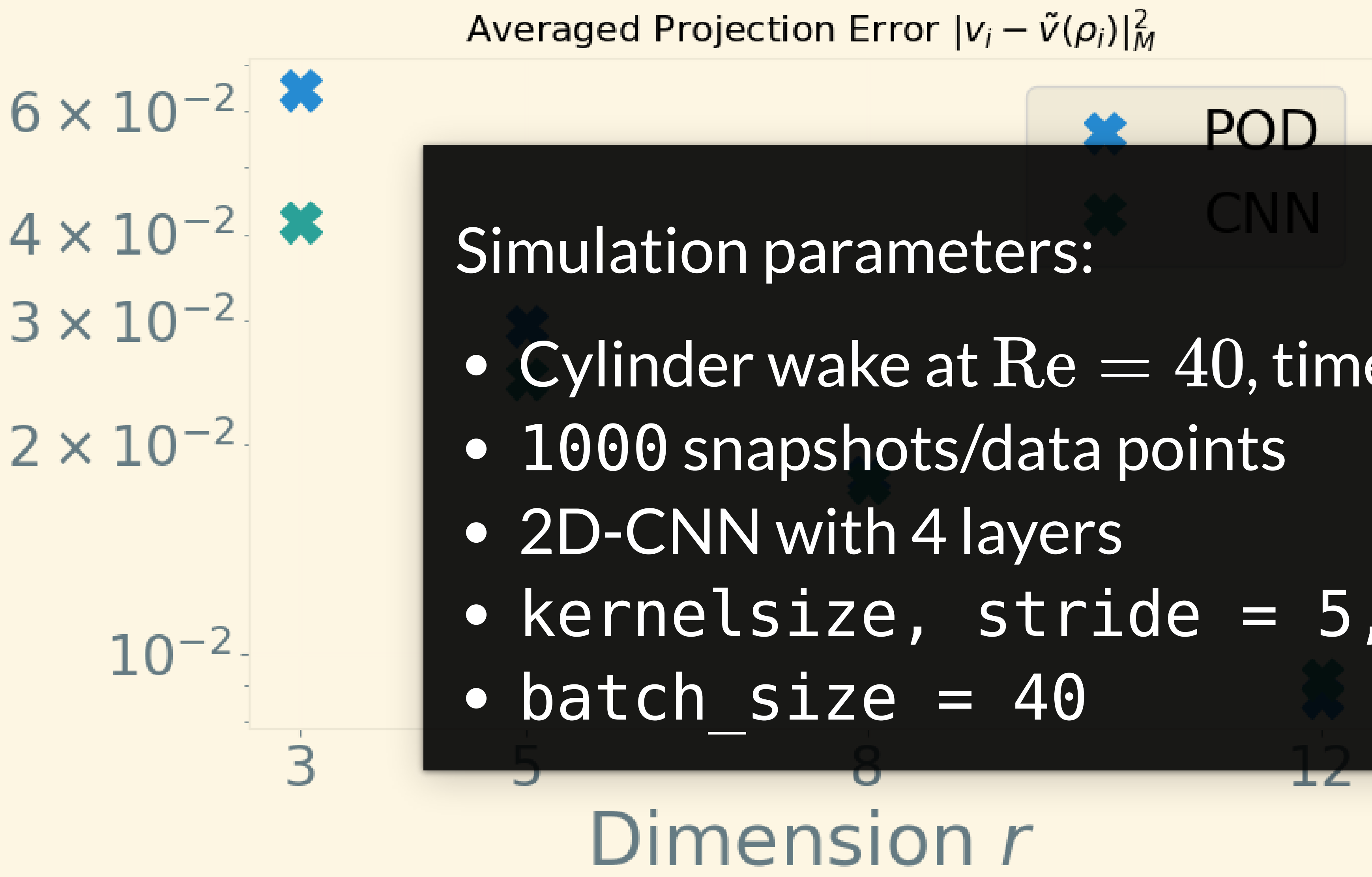
Implementation issues:

- Include FEM operators while
- maintaining the *backward* mode of the training.

RESULTS



Averaged Projection Error $|v_i - \tilde{v}(\rho_i)|_M^2$
6×10^{-2}
4×10^{-2}
3×10^{-2}
2×10^{-2}
10^{-2}
POD
CNN
3
5
8
12
Dimension r



Simulation parameters:

- Cylinder wake at $\mathrm{Re} = 40$, time in $[0, 8]$
- 1000 snapshots/data points
- 2D-CNN with 4 layers
- `kernelsize, stride = 5, 2.`
- `batch_size = 40`

CONCLUSION

... AND OUTLOOK

- LPV with affine-linear dependencies are attractive if only k is small.
- Proof of concept that CNN can *improve* POD at very low dimensions.
- Next: Include the parametrized convection in the training.
- Outlook: Use for nonlinear controller design.

… AND OUTLOOK

- LPV with affine-linear dependencies are attractive if only k is small.
- Proof of concept that CNN can *improve* POD at very low dimensions.
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- Outlook: Use for nonlinear controller design.

Thank You!

Apkarian, Pierre, Pascal Gahinet, and Greg Becker. 1995. "Self-Scheduled H_∞ Control of Linear Parameter-Varying Systems: A Design Example." *Autom.* 31 (9): 1251–61. https://doi.org/10.1016/0005-1098(95)00038-X.

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