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CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Time-dependent Dirichlet Boundary Conditions in Finite Element Discretizations

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Introduction

Motivation

- A controlled physical processes are controlled via interfaces is modelled via PDEs for the state υ with boundary control u

$$\mathcal{F}(\dot{\upsilon}, \upsilon; u)$$

- and approximated through, e.g., finite elements

$$F(\dot{v}, v; u).$$

- For further treatment like optimal control or model reduction, we want to have a standard state space form of F:

$$\dot{v} = f(v; u) \quad \text{or} \quad \dot{v} = Av + Bu.$$

Introduction

Or a bit more concrete

- The PDE

$$\dot{v} = \mathcal{A}v, \quad \text{in the domain,}$$
$$v = u \quad \text{at the boundary}$$

- is spatially discretized to give an ODE with a constraint

$$M\dot{v} = Av, \quad \text{at the inner nodes,}$$
$$v = u \quad \text{at the boundary nodes}$$

- however, elimination like $v = v_i + v_u$, where $v_u = G^{-1}u$ fulfills the boundary values, does lead to

$$M\dot{v}_i = Av_i + AG^{-1}u - MG^{-1}\dot{u},$$

which is not of the standard form.

Functional Analytical Framework

- $\Omega \in \mathbb{R}^d$, $d \in \{2, 3\}$ – a bounded and smooth domain with boundary Γ.
- $\mathcal{V} := W^{1,2}(\Omega)$ and $\mathcal{H} := L^2(\Omega)$ – the state spaces
- $\mathcal{V}'$ – the dual space of $\mathcal{V}$ with respect to the embedding $\mathcal{V} \hookrightarrow \mathcal{H}$,
- the trace spaces

$$\mathcal{Q}' = W^{\frac{1}{2},2}(\Gamma) \quad \text{and} \quad \mathcal{Q} = \mathcal{Q}'' := \mathcal{L}(\mathcal{Q}', \mathbb{R}),$$

- and the trace operator

$$\gamma \colon \mathcal{V} \to \mathcal{Q}'.$$

Introduction

A generic PDE

Problem

Let $T > 0$ and consider $\mathcal{A} \colon \mathcal{V} \to \mathcal{V}'$.
For $\mathcal{F} \in L^2(0, T; \mathcal{V}')$, for $v_0 \in \mathcal{H}$, and $\mathcal{U} \in L^2(0, T; \mathcal{Q}')$, find v with $v(t) \in \mathcal{V}$ and $\dot{v}(t) \in \mathcal{V}'$, a.e. on $(0, T)$, so that

$$\begin{aligned} \dot{v}(t) - \mathcal{A}v(t) &= \mathcal{F}(t), \\ \gamma v(t) &= \mathcal{U}(t), \end{aligned}$$

holds for almost all $t \in (0, T)$, and so that $v(0) = v_0$ in $\mathcal{H}$.

- We will not discuss time regularity here.
- For illustration purposes, we consider only the linear case.

Introduction

Convection-diffusion Example

Problem

Given a diffusion parameter ν, a convection wind β, and a function g prescribing the boundary conditions, find a function ρ that satisfies

$$\begin{aligned}\dot\rho(t) + \beta \cdot \nabla \rho(t) - \nu \Delta \rho(t) &= 0, \\ \rho|_\Gamma(t) &= g(t).\end{aligned}$$

In standard weak formulations, the convection diffusion problem resembles the generic PDE with $\upsilon(t) \in W^{1,2}(\Omega)$ and, e.g., $\mathcal{A}$ defined via

$$\langle \mathcal{A}\upsilon(t), \phi\rangle_{\mathcal{V}',\mathcal{V}} = \int_\Omega (\beta\cdot\nabla\upsilon(t), \phi) + \nu(\nabla\upsilon(t), \nabla\phi)\, \mathrm{d}\omega - \int_\Gamma \nu(\frac{\partial\upsilon}{\partial n}(t), \phi)\, \mathrm{d}\gamma,$$

for all $\phi \in \mathcal{V} := W^{1,2}(\Omega)$ and with $\frac{\partial}{\partial n}$ denoting the normal derivative.

Introduction

Finite Element Discretization

- Finite element space $V = \operatorname{span}\{\psi_i\}_{i=1}^{n_v} \subset \mathcal{V}$
- Assume that the basis $\{\psi_i\}_{i=1}^{n_v}$ is a nodal basis, i.e. the basis functions are associated with nodes of a mesh and they have local support.

We consider the decomposition

$$V = V_I \oplus V_\Gamma, \quad \text{with dimensions } n_v = n_I + n_d$$

- where $V_I = \operatorname{span}\{\psi_i\}_{i=1}^{n_I}$ is the space spanned by the basis functions of the inner
- and $V_\Gamma \subset V$ is the space of the basis functions $\{\psi_i\}_{i=n_I+1}^{n_v}$ that live on the boundary

Introduction

Finite Element Discretization

The splitting of the FEM space

$$V = V_I \oplus V_\Gamma,$$

is applied for ansatz and test spaces. Accordingly the mass matrix

$$M := \left[\left(\psi_i, \psi_j \right)_{\mathcal{H}} \right]_{i,j=1\cdots,n_v}$$

- is split with respect to the test space: $M = \begin{bmatrix} M_I \\ M_\Gamma \end{bmatrix}$,
- and, once more, with respect to the trial space, $M_I = \begin{bmatrix} M_{II} & M_{I\Gamma} \end{bmatrix}$,
- i.e. M_{II} and $M_{I\Gamma}$ are the part associated with the inner dofs and the part relates to the boundary dofs tested against the inner nodes, respectively.

Subsection

- similarly for the system matrix A approximating $\mathcal{A}$ is decomposed

$$A = \begin{bmatrix} A_I \\ A_\Gamma \end{bmatrix} \quad \text{and} \quad A_I = \begin{bmatrix} A_{II} & A_{I\Gamma} \end{bmatrix}$$

- as is a right hand side.

- Also, we write the FEM solution as

$$v = v_I + v_\Gamma = \sum_{i=1}^{n_I} v_i \psi_i + \sum_{i=n_I+1}^{n_v} v_i \psi_i,$$

- and freely assign it with the coefficient vector $v = \begin{bmatrix} v_I \\ v_\Gamma \end{bmatrix}$.

Introduction

Finite Element Discretization

- To assign the boundary values, we simply assign the dofs associated with the corresponding boundaries via

$$Gv = u, \quad \text{where } G = \begin{bmatrix} 0 & I \end{bmatrix} \in \mathbb{R}^{n_v - n_I, n_v}$$

Eventually, the discrete equations read:

Problem

Find $v \in \big([0, T] \to \mathbb{R}^{n_v}\big)$ *so that*

$$\begin{aligned} M_I \dot{v}(t) - A_I v(t) &= f(t), \quad v(0) = \alpha, \\ Gv(t) &= u(t), \end{aligned}$$

where α *is a given initial value and* u *is the control variable.*

Possible Approaches

The direct approach

Note that we cannot simply eliminate the boundary nodes in

$$M_I \dot{v} - A_I v = 0,$$
$$Gv = v_\Gamma = u.$$

- with the partitioning of M_I and A_I with respect to the inner and boundary nodes, we obtain

$$\begin{bmatrix} M_{II} & M_{I\Gamma} \end{bmatrix} \begin{bmatrix} \dot{v}_I \\ \dot{v}_\Gamma \end{bmatrix} = A_{II} v_I + A_{I\Gamma} v_\Gamma$$

- and, having inserted $v_\Gamma = u$,

$$M_{II} \dot{v}_I = A_{II} v_I + A_{I\Gamma} u - M_{I\Gamma} \dot{u}.$$

Possible Approaches

Lifting

Define a function that fulfills the boundary values for all time:

$$\tilde{v}(t) = \begin{bmatrix} \tilde{v}_I(t) \\ u(t) \end{bmatrix}.$$

Then the difference to the actual solution $\hat{v}_I = v_I - \tilde{v}_I$ solves

$$M_{II}\dot{\hat{v}}_I = A_{II}\hat{v}_I + Bu, \quad \hat{v}_I(0) = \alpha_I + M_{II}^{-1}M_{I\Gamma}u(0),$$

with

$$B = [A_{II}M_I^{-1}M_{I\Gamma} - A_{I\Gamma}].$$

- the obtained system is independent choice of the lifting $\tilde{v}_I$
- some choices of $\tilde{v}_I$ are readily extended to nonlinear equations

Possible Approaches

Multipliers

If the constraint is incorporated via a multiplier:

$$\begin{aligned} M\dot{v}(t) - Av(t) - G^T\lambda(t) &= 0, \\ Gv(t) &= u(t), \end{aligned}$$

one can apply a projection scheme:

- Define $P := I - M^{-1}G^TS^{-1}G$, where $S := GM^{-1}G^T$,
- consider $v = Pv + (I - P)v =: v_i + v_g$,
- and find that $v_g = M^{-1}G^TS^{-1}u$ and

  $$M\dot{v}_i - P^TAv_i = P^TBu,$$

  with $B := AM^{-1}G^TS^{-1}$.

Possible Approaches

Ultra Weak Formulations

- Let $\Phi = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and consider the pure diffusion equation.
- υ a is called *very weak* solution if

$$\int_\Omega (\dot{\upsilon}, \phi)\ \mathsf{d}\omega - \int_\Omega \nu(\upsilon, \Delta\phi)\ \mathsf{d}\omega = -\int_\Gamma \nu\big(u, \frac{\partial \phi}{\partial n}\big)\ \mathsf{d}\gamma$$

  for all $\phi \in \Phi$.
- The abstract equations indicate that a spatial discretization may lead to a standard system
- The difficulty however, lies in the definition of matching test functions of high regularity with zero boundary values and suitable ansatz functions.

Possible Approaches

Ultra Weak Formulations

- With the nonconforming ansatz spaces $V \subset W_0^{1,2}(\Omega)$, the *ultra weak* solution is approximated

$$\int_\Omega \big(\dot{v}, \phi\big)\ \mathrm{d}\omega + \int_\Omega \nu\big(\nabla v, \nabla\phi\big)\ \mathrm{d}\omega = \int_\Omega \big(f, \phi\big)\ \mathrm{d}\omega - \nu\int_\Gamma \big(u, \frac{\partial\phi}{\partial n}\big)\ \mathrm{d}\gamma$$

  for all $\phi \in V$.
- which is doable by in standard FEM packages.
- Note that the solution is approximated by a functions with a zero trace, i.e. $v\big|_\Gamma = 0$.

Possible Approaches

Robin Relaxation

- The Dirichlet conditions can be approximated by a Robin type condition

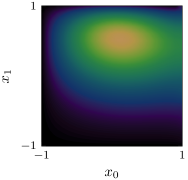
$$v \approx \alpha \frac{\partial v}{\partial n} + v = g \quad \text{or} \quad \frac{\partial v}{\partial n} \approx \frac{1}{\alpha}(g - v) \qquad \text{on } \Gamma,$$

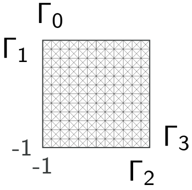
- with a parameter α that is intended to go to zero
- which are incorporated *naturally* in the weak formulation

Numerical Studies

Test Setup

For $\beta_1(x) = \frac{1}{10}\begin{bmatrix} x_0 + 1 \\ -(x_1 + 1) \end{bmatrix}$ and $\nu_1 = 0.1$,





find approximations to the scalar function ρ satisfying

$$\begin{aligned} \dot{\rho}(t) + \beta_1 \cdot \nabla\rho(t) - \nu_1 \Delta\rho(t) &= 0, \\ \rho\big|_{\Gamma_0}(t) &= u(t), \\ \rho\big|_{\Gamma_1 \cup \Gamma_2}(t) &= 0, \\ \frac{\partial\rho}{\partial\nu}\Big|_{\Gamma_3}(t) &= 0, \end{aligned}$$

on given discretizations N_h and N_τ of the spatial domain $\Omega = [-1, 1]^2$ and of the time interval $[0, 0.2]$.

Numerical Studies

Test Setup

We consider the following schemes

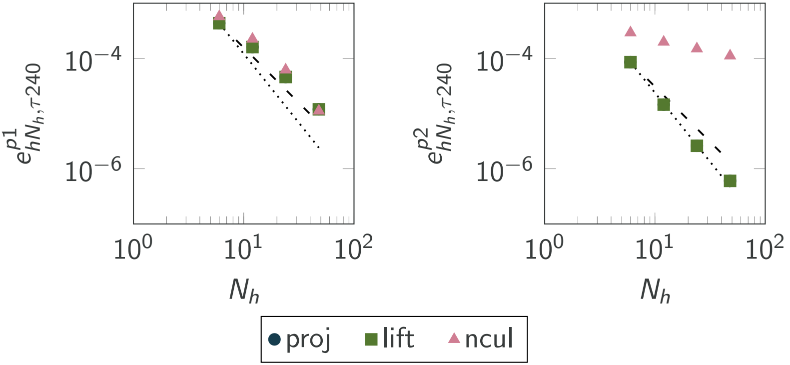
- lift – lifting of the boundary conditions via split mass matrix
- proj – incorporation of the constraint via Lagrange multiplier and projections
- ncul – nonconforming approximation of *ultra week* solutions
- pero – relaxation via Robin approximation

and for a varying space discretization N_h, time discretization N_τ, and polynomial degrees cg, and a numerically computed reference solution, we investigate the errors

$$e^{pcg}_{hN_h,\tau N_\tau} := \rho^{pcg}_{hN_h,\tau N_\tau} - \rho_{\mathrm{ref}}$$

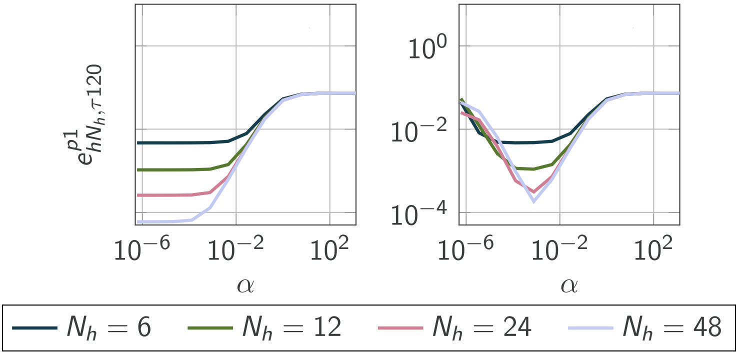
measured in a numerical approximation of the $L^2(0,1;L^2([-1,1]^2))$ norm.

Estimated Order of Convergence



- Convergence tests for the consistent implementations
- the dashed lines indicate the slope of a quadratic convergence
- the dotted lines indicate a convergence of order 2.5.

Penalty versus Accuracy



- Errors for pero for different values of the penalization parameter α
- for exact system solves (right)
- and approximate system solves (left)

Numerical Studies

Peformance in GMRES

	av.#its	$e_{h48,\tau120}^{p1,\text{tol}1e-7}$
proj	10.6	$1.2 \cdot 10^{-5}$
lift	10.6	$1.2 \cdot 10^{-5}$
pero	14.2	$7.8 \cdot 10^{-6}$
ncul	10.6	$1.1 \cdot 10^{-5}$

- Performance of *GMRES* within the various formulations for $N_h = 48$, $N_\tau = 120$, and linear elements (cg $= 1$)
- The averaged number of iterations per time-step av.#its and the approximation error in the case that the resulting linear equations are solved using *GMRES* up to a relative residual of tol $= 10^{-7}$.
- The colored cells contain the lowest measured values.

Conclusion

and an outlook

- There are numerous approaches that approximate a Dirichlet boundary control problem into a standard state space system
- Penalization schemes are easy to implement but require a wise choice of the parameter
- Consistent schemes need but a little extra implementation efforts

Future work:

- Investigate the performance of the schemes in optimal control or model reduction setups
- Analyse the consistency of the reformulations with the infinite dimensional PDE

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and an outlook

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- Investigate the performance of the schemes in optimal control or model reduction setups
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Thank you for your attention.

Literature

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