MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
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COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

LQG-BT Low-Order Controller for Stabilization of Laminar Flows

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Outline

Constrained Riccati Equations

$$
\begin{aligned}
M\dot v - Av + J^T p &= Bu \\
Jv &= 0 \\
y &= Cv
\end{aligned}
$$

?

$$
\begin{aligned}
M\dot v - \Pi A v &= \Pi B u \\
y &= Cv
\end{aligned}
$$

$\mathcal{R}(X)$

Model Reduction

Optimal Control

with A, M, $\Pi \in \mathbb{R}^{n_x,n_x}$, $J \in \mathbb{R}^{n_v,n_p}$, $B \in \mathbb{R}^{n_x,n_u}$, and $C \in \mathbb{R}^{n_y,n_x}$.

For Flow Equations

$$\begin{aligned} M\dot{v} - Av + J^T p &= Bu \\ Jv \qquad &= 0 \\ y &= Cv \end{aligned}$$

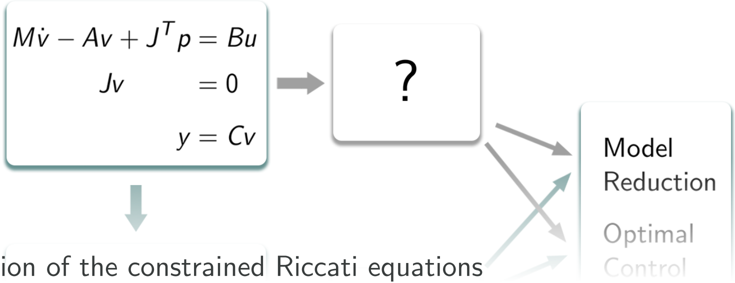
$$\Downarrow$$

$$\begin{aligned} M\dot{v} - \Pi A v &= \Pi B u \\ y &= Cv \end{aligned}$$

- Projection onto the manifold of the constraints
- gives an ODE
- equivalent in theory
- but eventually avoided in practice
  - numerically infeasible
  - systematic errors may be introduced
  - structure is not preserved

Constrained Riccati Equations

for constrained dynamics



Derivation of the constrained Riccati equations

- directly via optimality conditions,
  $\rightarrow$ [Kunkel, Mehrmann '08], [Kurina, März '07], [JH '14]
- reformulation of the ODE related system,
  $\rightarrow$ see below, [PB, JH '14]
- or reformulation of the numerical schemes
  $\rightarrow$ [Heinkenschloss, Sorensen, Sun '08], [Gugercin, Stykel, Wyatt '13], [Bänsch, PB, Weichelt, Saak '15].

Projected Riccati Equation

To define, e.g., the *Linear-Quadratic Regulator*, one needs a solution to the associated *control* Riccati equation of the form

$$\Pi A^T \Pi^T X M + M^T X \Pi A \Pi^T - M^T X \Pi B B^T \Pi^T X M + \Pi C^T C \Pi^T = 0$$

for $X \in \mathbb{R}^{n_v, n_v}$.

Equivalence to Projected Riccati Equations

Lemma

Let M be invertible, J have full rank, and $\Pi := I - J^T(JM^{-1}J^T)^{-1}JM^{-1}$. The matrix $X \in \mathbb{R}^{n_v,n_v}$ solves,

$$\Pi A^T \Pi^T XM + M^T X \Pi A \Pi^T - M^T X \Pi BB^T \Pi^T XM + \Pi C^T C \Pi^T = 0$$

if it solves

$$\begin{aligned} A^T XM + M^T XA - M^T XBB^T XM + \\ MYJ^T + JY^T M^T + CC^T &= 0, \\ JXM^T &= 0, \\ MXJ^T &= 0, \end{aligned}$$

for a suitable $Y \in \mathbb{R}^{n_v,n_p}$.

Low-Rank Approximations

How to obtain approximations to a symmetric solution of

$$
\begin{aligned}
A^T XM + M^T XA - M^T XBB^T XM + \\
MYJ^T + JY^T M^T + CC^T &= 0, \\
JXM^T &= 0.
\end{aligned}
$$

1. Factorize the solution $X = ZZ^H$,
2. apply a *low-rank Newton-ADI iteration* [PB, Li, Penzl '08] to the constrained Riccati equation [JH '14], and
3. obtain skinny factors Z_{n_k}, that approximate $X \approx Z_{n_k} Z_{n_k}^H$.

Applications

Same idea and result for

- Lyapunov equations,
  - e.g. for Balanced Truncation,
- *Filter* Riccati equations,
  - e.g. for observer design or LQG-Balanced Truncation,
- and Differential Riccati equations,
  - e.g. for finite time-horizon control.

Applications

We consider spatially discretized *Navier-Stokes* equations with control u and observation $y = Cv$

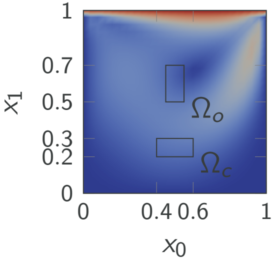
$$
\begin{aligned}
M\dot{v} &= -N(v)v - \frac{1}{Re}Lv + J^T p - Bu + f, \\
0 &= Jv - g, \\
v(0) &= \alpha, \\
y &= Cv,
\end{aligned}
$$

where

- α is the steady-state solution and
- and input operator B that models a distributed control or Dirichlet boundary control via approximating Robin conditions

Linearized Model for a Driven Cavity

- Driven Cavity
- Navier-Stokes Equations
- $Re = 200$
- Linearized about the steady state
- *Taylor-Hood* finite elements
- 6000 velocity nodes



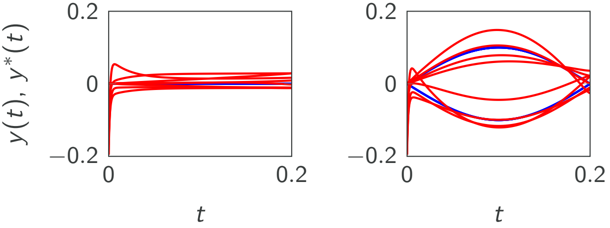
- Distributed control with 8 degrees of freedom
- distributed observation with 8 degrees of freedom

Velocity Output Tracking

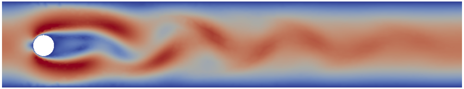
- Target: tracking of a velocity output trajectory

$$\frac{1}{2}[Cv - y^*]^\mathsf{T} V [Cv - y^*]\Big|_{t=T} + \frac{1}{2}\int_0^T [Cv - y^*]^\mathsf{T} W [Cv - y^*] + u^\mathsf{T} R u \,\mathrm{d}t,$$

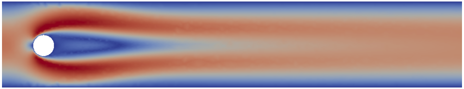
- subject to linearized Navier-Stokes equation
- The optimal control is given as a feedback defined through a low-rank solution trajectory of a constrained DRE



Stabilization of the Cylinder Wake



- Cylinder wake at moderate *Reynolds* numbers
- The steady state is a solution, but unstable
- Goal: Stabilizing feedback controller that works in experiments
- Thus, the simulation needs to cope with:
  - ➙ limited measurements
  - ➙ short evaluation times
  - ➙ system uncertainties
  - ➙ actuation at the boundary



LQGBT Reduced Controller

The LQG-Balanced Truncation controller is a simultaneous application of

- a linearization about the steady state
  - ➙ to directly attack the deviations
- a *Kalman filter*
  - ➙ estimate the state using a few measurements
- an *LQG regulator*
  - ➙ stabilize the linearized system
- and *Balanced Truncation*
  - ➙ reduce the linearized and stable system

In Short – for Navier-Stokes Equations

1. Consider the linearization about α

$$\begin{aligned} M\dot{v} &= A_\alpha v + J^T p - Bu + f, \quad v(0) = \alpha, \\ 0 &= Jv, \\ y &= Cv. \end{aligned}$$

2. Compute X_c and X_o which solve the associated *control* and *filter Riccati equations* to define the state estimate $\hat{x}$ and the regulator u as

$$\begin{aligned} M\dot{\hat{x}} &= \hat{A}_\alpha \hat{x} + X_o M C^T (y - C\alpha), \\ u &= -B^T M X_c \hat{x}, \end{aligned}$$

   with $\hat{x}(0) = 0$ and $\hat{A}_\alpha$ denoting the observer dynamics.

3. Balance and truncate X_o and X_c to define a reduced observer

Expectations and Limitations

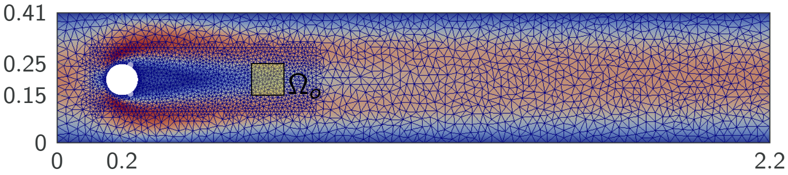
The proposed controller is based on a linearized model

- ➙ we expect a good performance for small deviations

and is designed to work for

- ✓ limited state information
- ✓ fast and unstable dynamics
- ✓ high dimensionality
- ✓ boundary control
- ✗ but still awaits robustification

Simulation Setup



- 2D cylinder wake
- Navier-Stokes Equations
- $Re = 100$
- *Taylor-Hood* finite elements
- 30000 velocity nodes

- Boundary control at 2 outlets
- distributed observation with 6 degrees of freedom
- LQGBT-reduced order observer and controller of state dimension $n_k = 13$
- Target: stabilization of the steady-state solution

Definition of the Input Operator

- Control through injection and suction at outlets Γ_{c_1}, Γ_{c_2} located at the cylinder periphery at $\pm\pi/3$.
- Prescribe Dirichlet conditions for the velocity

$$v = g_1(x)u_1(t), \quad v = g_2(x)u_2(t)$$

at Γ_{c_1} and Γ_{c_2}, where $g_{1/2}$ are the shape functions and $u_{1/2}$ are the magnitudes of the controls.

- Use a small γ to relax the Dirichlet conditions to Robin conditions at $\Gamma_{1/2}$:

$$v \approx g_{1/2}u_{1/2} + \gamma(\frac{1}{Re}\frac{\partial v}{\partial n} - pn)$$

- that are *naturally* included in Finite Element discretizations.
- For other approaches see [PB, JH '15].

Simulation Results

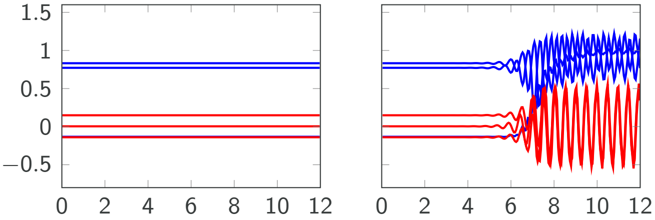
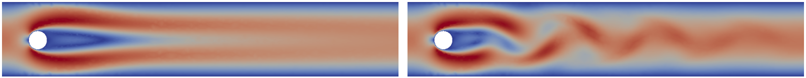


Figure : Measured signal y versus time $t \in [0, 12]$ of the perturbed closed loop system with a reduced controller of dimension $n_k = 13$ (left), compared to the response of the uncontrolled system (right). Blue corresponds to the x-component of the velocity and red to y-component. Below, a snapshot of the magnitude of the velocity solutions at $t = 12$.



Summary and Conclusion

- The DAE structure of flow equations can be handled by constraint Riccati equations
- There are numerical approximation methods to solve large-scale constrained Riccati equations
- Thus, the general LQGBT approach can be applied to controller design for Navier-Stokes equations
- Also velocity tracking in a large scale setup is doable
- Canonical application – feedback stabilization of an open loop control
- Next task: robustification

Thank you for your attention!

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