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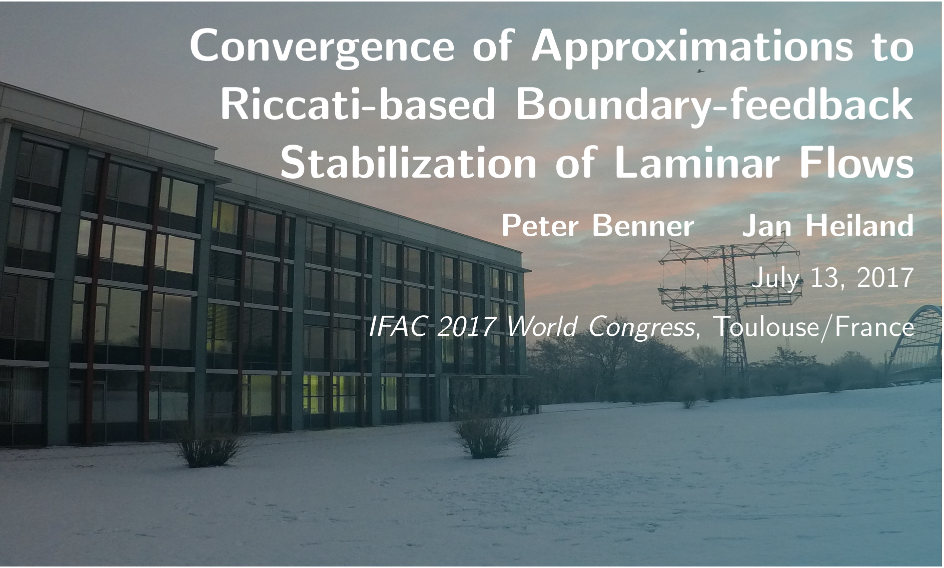
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY



Convergence of Approximations to Riccati-based Boundary-feedback Stabilization of Laminar Flows

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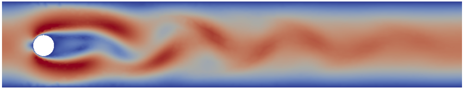
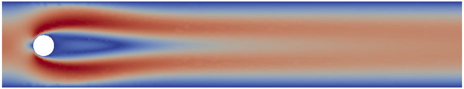
Introduction

Introduction

Stabilization of the Cylinder Wake



- Cylinder wake at moderate *Reynolds* numbers
- The steady state is a solution, but unstable
- Goal: Stabilizing feedback controller that works in experiments
- Thus, the simulation needs to cope with:
  - → limited measurements
  - → short evaluation times
  - → system uncertainties
  - → actuation at the boundary



Introduction

Linearization based feedback

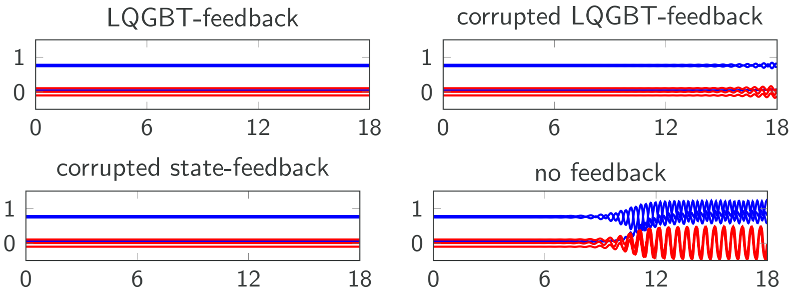
- see [Raymond'05, '06] for theory concerning flows
- see [Bänsch,PB&al'15, PB&JH'15] for numerical results
- see also [Breiten&Kunisch'14]

has two major disadvantages when it comes to applications

1. small domain of attraction
   - ➙ the target state is unstable
   - ➙ the system is unlikely in a state in which the controller would work
2. fragility against system uncertainties
   - ➙ observer-based controller will fail if the system parameters change

Introduction

Cylinderwake, $Re = 80$, velocity measurements in the wake



- corrupted linearization – about the not quite converged steady state
  - visually undistinguishable from the *right* linearization point
  - relative difference in norm: 5%

Introduction

Consider the Oseen linearization

$$\begin{aligned} \dot{v} + A_\alpha v &= Bu \quad \text{in } \mathcal{H}_{\mathrm{df}}, \\ y &= Cv \end{aligned}$$

where

- $\mathcal{H}_{\mathrm{df}} \subset L^2(\Omega)$ is the space of divergence-free functions (+ boundary conditions)
- $A_\alpha \colon D(A_\alpha) \subset \mathcal{H}_{\mathrm{df}} \to \mathcal{H}_{\mathrm{df}}$ is the *Oseen* operator
  - ➙ linearized incompressible Navier–Stokes Equations
  - ➙ about a steady state v_α, i.e. $A_\alpha = A(v_\alpha)$
- $B \colon \mathbb{R}^p \to \mathcal{H}_{\mathrm{df}}$ is the input operator
- $C \colon \mathcal{H}_{\mathrm{df}} \to \mathbb{R}^q$ is the output operator

Introduction

Consider the Oseen linearization

$$\begin{aligned} \dot{v} + A_\alpha v &= Bu \quad \text{in } \mathcal{H}_{\mathrm{df}}, \\ y &= Cv \end{aligned}$$

as an input u to output map y with the transfer function:

$$G(v_\alpha)(s) = C(sI - A_\alpha)^{-1}B \in \mathbb{C}^{q,r}$$

Introduction

1. The Oseen System has the transfer function

$$G(v_\alpha)(s) = C(sI - A_\alpha)^{-1}B \in \mathbb{C}^{q,r}$$

2. An finite-dimensional approximation to it has the transfer function

$$G(v_\alpha)_N(s) = C(sI - A(v_\alpha)_N)^{-1}B_N \in \mathbb{C}^{q,r}$$

3. Uncertainty in the linearization point gives another transfer function

$$G(v_\alpha + \delta)(s) = C(sI - A(v_\alpha + \delta))^{-1}B_N \in \mathbb{C}^{q,r}$$

4. Typically, one has both – an approximation to the linearization point and to the system:

$$G(v_{\alpha;N})_N(s) = C(sI - A(v_{\alpha;N})_N)^{-1}B_N \in \mathbb{C}^{q,r}$$

Introduction

Bad news [IEEE Transaction on Automatic Control ('78)]:

> **Guaranteed Margins for LQG Regulators**
>
> JOHN C. DOYLE
>
> ***Abstract*—There are none.**

Good news: The transfer functions of

- [Curtain'03]: Galerkin approximations of evolution systems
- [PB&JH'16]: Oseen systems with changes in the linearization point
- [this talk]: stable mixed-FEM approximation of the Oseen system

differ from the exact transfer function through a coprime factor perturbation.

➙ Cf. the observer/controller robustification as proposed by [Curtain&Zwart'95, Curtain'03].

Sufficient conditions for robustified LQG controllers

We generalize the conditions given in [Curtain'03]

- for Galerkin approximations of a generic linear system

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

  with

  - X – Hilbert space
  - $A\colon D(A) \subset X \to X$ – generator of a C^0-semigroup
  - $B\colon \mathbb{R}^p \to X$ – bounded

→ towards mixed-FEM approximations of Oseen systems.

Mixed FEM for incompressible flow equations

State-space of the Oseen equation:

$$\mathcal{H}_{\mathrm{df}} := \{z \in L^2(\Omega) : \operatorname{div} z = 0 \text{ and } z \cdot n\big|_{\Gamma_d} = 0\} \subset L^2(\Omega),$$

In mixed FEM, the divergence condition is relaxed through

$$\operatorname{div}^N v^N = 0, \quad \text{if} \quad \int_\Omega \operatorname{div} v^N \cdot q^N \,\mathrm{d}x = 0,$$

for $v^N \in \mathcal{V}^N$ for all basis functions $q \in \mathcal{Q}^N$, where

- $\mathcal{V}^N$ and $\mathcal{Q}^N$ – FEM spaces for the velocity and the pressure.

Note that, in general, for the state-space

$$\mathcal{H}_{\mathrm{df}}^N := \ker(\operatorname{div}^N) \subset L^2(\Omega)$$

of the approximation it holds that $\mathcal{H}_{\mathrm{df}}^N \not\subset \mathcal{H}_{\mathrm{df}}$.

Stable Approximation of the Leray Projector

Let

- P^N – the orthogonal projector onto $\mathcal{V}^N$
- $\Pi^N \colon \mathcal{V}^N \to \mathcal{V}^N$ – orthogonal projection onto $\mathcal{H}^N_{\mathrm{df}}$
  - ➝ the discrete *Leray*-projector

Assumption (A0)

The restriction

$$R^N := \Pi^N P^N \colon \mathcal{V}^N \to \mathcal{H}^N_{\mathrm{df}}$$

is bounded independent of N and

$$R^N z \to z, \text{ for any } z \in \mathcal{H}_{\mathrm{df}} \text{ as } N \to \infty.$$

Stable Approximation of the Leray Projector

Assumption (A0)

The restriction $R^N := \Pi^N P^N \colon \mathcal{H}_{\mathrm{df}} \to \mathcal{H}^N_{\mathrm{df}}$ is bounded independent of N and $R^N z \to z$, for any $z \in \mathcal{H}_{\mathrm{df}}$ as $N \to \infty$.

For Galerkin schemes: $R^N = P^N$ ✓. For mixed FEM, we have:

Lemma ([PB&JH'17])

*(**A0**) holds, if (and only if [Girault&Raviart'86(Lem.II.1.1)]) the FEM spaces $\mathcal{V}^N$ and $\mathcal{Q}^N$ with the refinement parameter N fulfill the condition that*

$$\inf_{0 \neq q^N \in \mathcal{Q}^N} \sup_{0 \neq h^N \in \mathcal{H}^N} \frac{\int_\Omega q^N \cdot \mathrm{div}^N h^N \, \mathrm{d}x}{\|q^N\|_{H^1(\Omega)} \|h^N\|_{L^2(\Omega)}} \geq \beta > 0,$$

with a constant β independent of N.

The following assumptions are then standard; see, e.g., [ITO'87].

(A1) Convergence of the semigroups:

$$S^N(t)R^N z \to S(t)z \quad \text{and} \quad (S^N)^*(t)R^N z \to S^*(t)z,$$

for each $z \in \mathcal{H}_{\mathrm{df}}$ uniformly for t in bounded subsets of $[0, \infty)$.

(A2) ✗ Convergence of the input operator:
for each $u \in \mathbb{R}^2$, $\Pi^N B^N u \to \Pi B u$.

(A3) The family of pairs $(\Pi^N A^N, \Pi^N B^N)$ is uniformly stabilizable, and the family of pairs $(\Pi^N A^N, C)$ is uniformly detectable.

Theorem ([CURTAIN'03, PB&JH'17])

*Let assumptions (**A0**)–(**A3**) be satisfied. Then, there exists a finite-dimensional LQG-based controller that stabilizes the ∞-dimensional Oseen system with a robustness margin with respect to coprime factor perturbations.*

On the convergence of the boundary control operator

- We relax the Dirichlet control $v\big|_{\Gamma_c} = g_c u - \varepsilon\left(\frac{1}{Re}\frac{\partial v}{\partial n} - pn\right)$
- The input operator: we formally define

$$\langle Bu, w\rangle = -\frac{1}{\varepsilon}\int_{\Gamma_c} n g_c w \, \mathrm{ds} \cdot u, \quad w \in \mathcal{H}_{\mathrm{df}}$$

- $B\colon \mathbb{R}^p \to L^2(\Omega)$ is unbounded but ΠB is bounded
  - ➙ Π is the continuous *Leray-projector*
- We have that $\mathcal{H}_{\mathrm{df}}^N \to \mathcal{H}_{\mathrm{df}}$, but "in" the L^2 space.
  - ➙ the convergence of

$$\Pi^N B^N u = Bu\big|_{\mathcal{H}_{\mathrm{df}}^N} \to Bu\big|_{\mathcal{H}_{\mathrm{df}}} = \Pi Bu$$

is not immediate.

Conclusion and Outlook

- Observer-based controllers are fragile
- For Oseen equations the fragility comes from uncertainties through the linearization point or approximation
  - ➙ which are coprime factor perturbations
  - ➙ need particular robustification
  - ✗ Curtain/Zwart robustification did not work out of the box
- Sufficient conditions for the existence of robustified controllers for the Oseen system
- Future work:
  - ➙ close gaps in theory:
    - ➙ adapt results from [BADRA'06]
  - ➙ design and test robust controllers

Thank you!

Thank you for your attention!

Questions? Now or anytime later...

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Sufficient Conditions

Ideas of the proof:

1. Let $v \in \mathcal{H}_{\mathrm{df}}$, then $P^N v = v_{\mathrm{df}}^N + v_{\mathrm{c}}^N$,
   where $v_{\mathrm{df}}^N = R^N v \in \mathcal{H}_{\mathrm{df}}^N$ and $v_{\mathrm{c}}^N \in (\mathcal{H}_{\mathrm{df}}^N)_\perp$.
2. We have $\operatorname{div}^N v = 0$ and $\operatorname{div}^N(R^N v) = 0$
   (since $v \in \mathcal{H}_{\mathrm{df}}$ is, in particular, discretely div-free)
3. so that $\operatorname{div}^N(v - P^N v) = \operatorname{div}^N(v - R^N v - v_{\mathrm{c}}^N) = -\operatorname{div}^N v_{\mathrm{c}}^N$.
4. by LBB-stability, div^N has a uniformly bounded right inverse[1]:

$$\|v_{\mathrm{c}}^N\| \leq \eta \|I - P^N\| \|v\|$$

5. $\|R^N v\| = \|P^N v - v_{\mathrm{c}}^N\| \leq \|P^N v\| + \|v_{\mathrm{c}}^N\| \leq (1 + \eta)\|v\|$.

[1] [Girault&Raviart '86]