MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Optimal Control Approach to Multi-Body Systems with Servo Constraints

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Outline

Introduction

Multibody systems with servo constraints

- Example: End effector of a robot should follow prescribed trajectory

→ Cranes

→ Cable suspension manipulator

- Find input for a given output (inverse dynamics, feed-forward control)

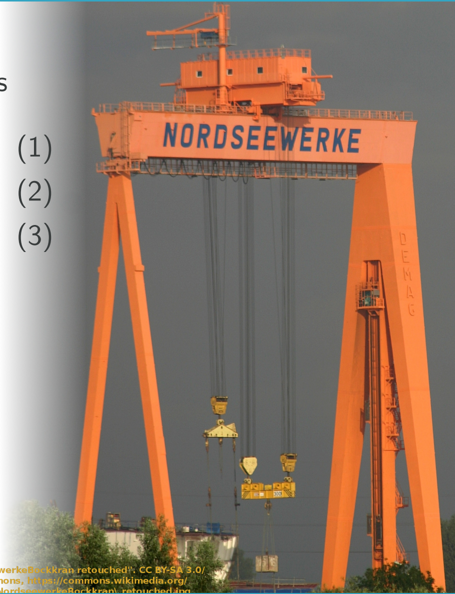
Introduction

Multibody systems with servo constraints

$$M\ddot{x} = Ax + G^T(x)p + Bu, \qquad (1)$$
$$g(x) = 0, \qquad (2)$$
$$Cx = y. \qquad (3)$$

Typical DAE structure:

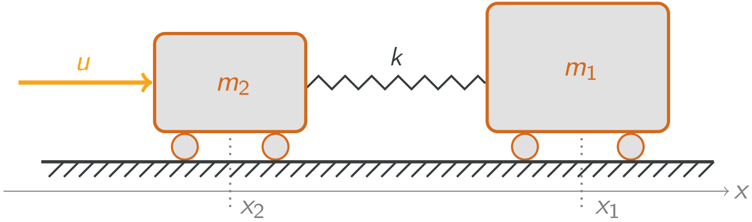
- Holonomic constraints (2)
  - $\rightarrow$ Index 3
- Servo constraints (3)
  - $\rightarrow$ Index ≥ 5



Linear Example

A linear example for illustration:

- 2 cars connected by a spring
- Variables: x_1, x_2, and input force u
- Trajectory of x_1 prescribed by a y (e.g. a rest-to-rest maneuver)



Linear Example

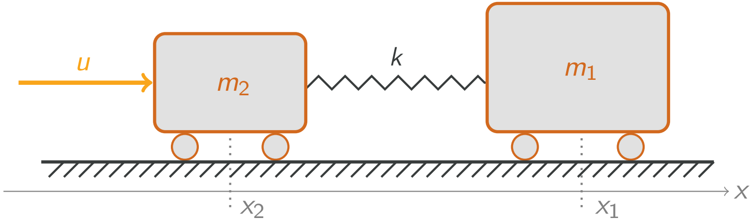
Formulation as DAE

- Dynamics of system

$$
\begin{aligned}
m_1\ddot{x}_1 &= -k(x_1 - x_2 - d), \\
m_2\ddot{x}_2 &= k(x_1 - x_2 - d) + u
\end{aligned}
$$

- Constraint to enforce x_1 to stay on trajectory y

$$
x_1 = y
$$



Linear Example

Formulation as DAE

- Dynamics of system

$$\begin{aligned} m_1\ddot{x}_1 &= -k(x_1 - x_2 - d), \\ m_2\ddot{x}_2 &= \quad k(x_1 - x_2 - d) + u, \end{aligned}$$

constrained by

$$x_1 = y.$$

- No holonomic constraints but DAE of index 5
  $\rightarrow$ $y \in \mathcal{C}^4([0, T])$ needed for continuous input u
- Numerical solution requires index reduction,
  cf. [Altmann, Betsch & Yang '14] and [Blajer & Kołodziejczyk '04]

Linear Example

Formulation as Optimal Control Problem

- Replace servo constraint $x_1 = y$ by cost functional

$$\mathcal{J}(x,u) := \frac{1}{2}\|x_1(T) - y(T)\|^2 + \frac{1}{2}\int_0^T \|x_1 - y\|^2 + \beta_0\|u\|^2 \,\mathrm{d}t$$

- Task: minimize $\mathcal{J}(x,u)$ under the constraint

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Linear Example

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- $y \in \mathcal{C}([0,T])$ sufficient for existence of continuous input u
  $\rightarrow$ not the same solution!
- May also penalize derivatives of u

Linear Example

Formulation as Optimal Control Problem

- More general case: $M\ddot{x} = Ax + Bu + f, \quad Cx = y$
  Replace servo constraint $Cx = y$ by cost functional

$$\mathcal{J}(x,u) := \mathcal{S}(x(T)) + \int_0^T \mathcal{Q}(x) + \mathcal{R}(u)\,\mathrm{d}t$$

with the quadratic performance criteria

$$\mathcal{Q}(x) := \frac{1}{2}\|Cx - y\|^2, \quad \mathcal{R}(u) := \frac{1}{2}\sum_{i=0}^{\kappa} \beta_i \|u^{(i)}\|^2,$$

$$\text{and} \quad \mathcal{S}(x(T)) := \gamma\frac{1}{2}\|Cx(T) - y(T)\|^2$$

- Task: minimize $\mathcal{J}(x,u)$ under the constraint

$$M\ddot{x} = Ax + Bu + f$$

Linear Example

Optimality Systems

- Formal optimality system

$$
\begin{aligned}
M\ddot{x} &= Ax + Bu + f, \\
M^{\mathsf{T}}\ddot{\lambda} &= A^{\mathsf{T}}\lambda - C^{\mathsf{T}}Cx + C^{\mathsf{T}}y, \\
0 &= \sum_{i=0}^{\kappa}(-1)^i\beta_i u^{(2i)} - B^{\mathsf{T}}\lambda
\end{aligned}
$$

$+$ boundary conditions for x, $\dot{x}$, λ, $\dot{\lambda}$, and u and its derivatives

- Case $\kappa = 0$: $\quad 0 = \beta_0 u - B^{\mathsf{T}}\lambda \qquad \rightarrow$ DAE of index 1
- Case $\kappa = 1$: $\quad 0 = \beta_0 u - \beta_1 \ddot{u} - B^{\mathsf{T}}\lambda \quad \rightarrow$ ODE

Linear Example

What is happening

Consider the general linear case with $\kappa = 0$.

Lemma (LQ case, w/o holonomic constraints [ALTMANN&JH '15]**)**

If the system defined by (A, B, C) is controllable and observable, then, for $\beta_0 = 0$, the exact solution and the optimal solution coincide, provided y is sufficiently smooth.

What if y is not smooth enough and $\beta_0 \to 0$ ...

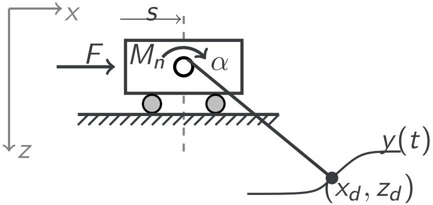
- $\to$ no convergence $\|x_1 - y\|$ possible (in sup-norm)
- $\to$ strong peaks in the derivatives of u

Indeed, for the 2-car example we compute for the optimal solution (x_1, u)

$$x_1 - y = \beta_0(2\ddot{u} - \frac{1}{k}u^{(4)})$$

Nonlinear Case

e.g., The Overhead Crane



$$\begin{bmatrix} m_t & & & \\ & J & & \\ & & m & \\ & & & m \end{bmatrix} \begin{bmatrix} \ddot{s} \\ \ddot{\alpha} \\ \ddot{x}_d \\ \ddot{z}_d \end{bmatrix} - G^{\mathsf{T}}(\bar{x})p - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u = \begin{bmatrix} 0 \\ -rmg \\ 0 \\ mg \end{bmatrix},$$

$$(x_d - s)^2 + z_d^2 - (r\alpha)^2 = 0, \qquad (g(\bar{x}))$$

$$\begin{bmatrix} x_d \\ z_d \end{bmatrix} - y = 0.$$

Nonlinear Case

Formulation as Optimal Control Problem

With a costfunctional of the same type, the DAE formulation can be relaxed and a solution be sought through the first order optimality conditions

$$
\begin{aligned}
M\ddot{x} &= Ax + G^{\mathsf{T}}(x)p + Bu + f, \\
0 &= g(x), \\
M^{\mathsf{T}}\ddot{\lambda} &= A^{\mathsf{T}}\lambda + \frac{\partial}{\partial x}\big(G(x)^{\mathsf{T}}p\big)\lambda - G^{\mathsf{T}}(x)\mu - C^{\mathsf{T}}QCx + C^{\mathsf{T}}Qy, \\
0 &= G(x)\lambda, \\
0 &= \sum_{i=0}^{\kappa}(-1)^i R_i u^{(2i)} - B^{\mathsf{T}}\lambda,
\end{aligned}
$$

plus initial and terminal conditions.

Nonlinear Case

- In the case of holonomic constraints, the first order optimality conditions are DAEs of index ≥ 3
- For the existence of solutions, consistency of the initial and terminal values is necessary
- The terminal values of λ are defined through the optimal state x^* and the cost functional

Theorem (Ensuring consistency [ALTMANN&JH '15]**)**

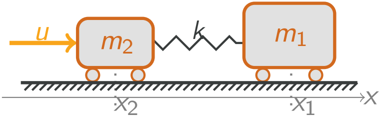
Let $P_{x^*(T)}$ be a projector onto the kernel of $G(x^*(T))$, that satisfies $M^{-\mathsf{T}}P_{x^*(T)}^{\mathsf{T}} = P_{x^*(T)}M^{-\mathsf{T}}$. Then, the projected formal terminal conditions

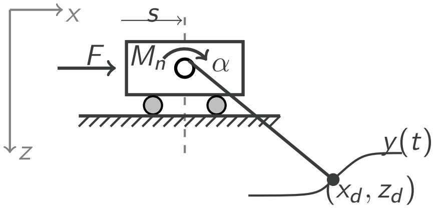
$$M^{\mathsf{T}}\lambda(T) = 0, \quad M^{\mathsf{T}}\dot{\lambda}(T) = P_{x^*(T)}C^{\mathsf{T}}S(Cx^*(T) - y(T)),$$

ensure consistency of the terminal conditions for λ. Moreover, if $(x^*, p^*, u^*, \lambda, \mu)$ solve the optimality system with the projected terminal conditions, then u^* is a stationary point of the associated Lagrange functional.

Numerical Examples

- $m_1 = 2\text{kg},\ m_2 = 1kg,$
- $(m_{12} = 1kg$, third cart)
- $k = 10\text{N/m}$



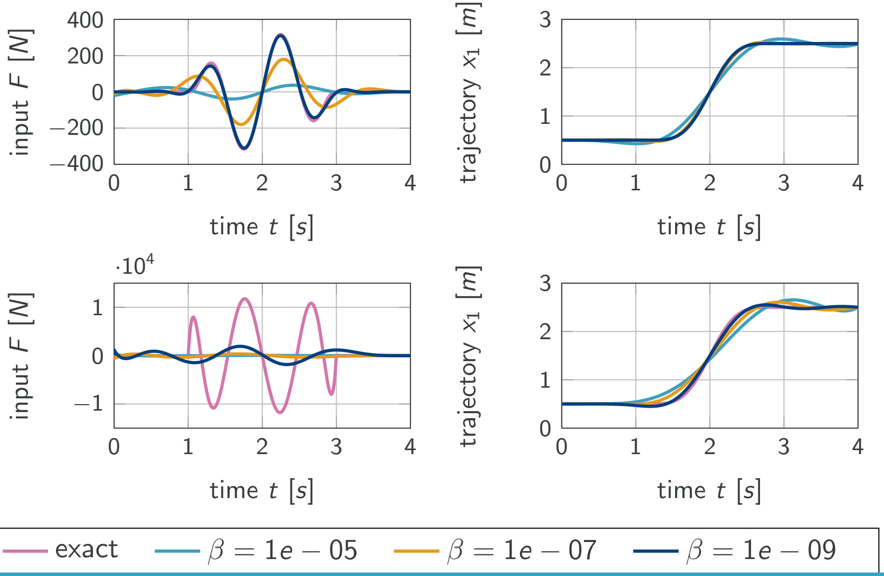


- $m_t = 10kg,\ m = 1kg,$
- $J = 0.1Nm$, (moment of inertia)
- $r = 0.1m$

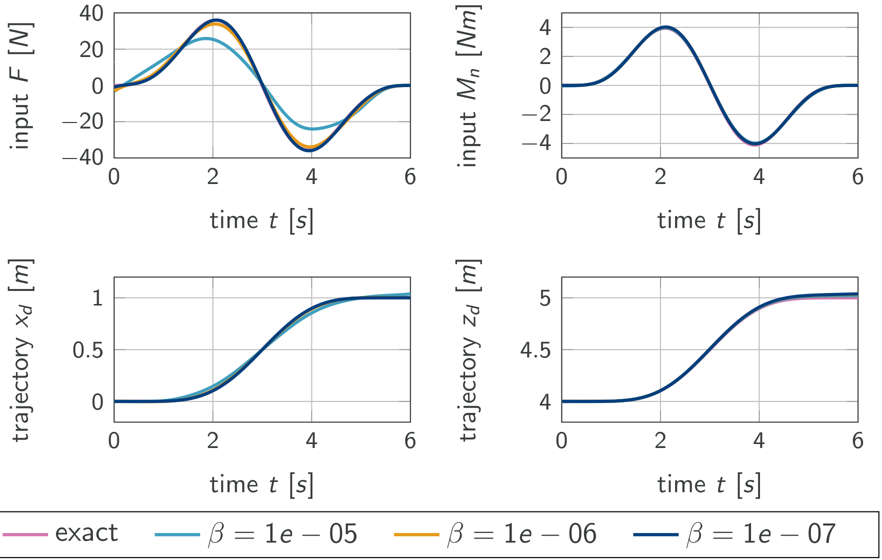
Optimization parameters

- $\kappa = 0$, and different values of β_0
- $y \in \mathcal{C}^7$ – rest-to-rest maneuver (piecewise polynomial)

Numerical Examples – The 2/3-car Chain



Numerical Examples – The Overhead Crane



Conclusions and Outlook

Optimal control approach to servo control problems

- Relaxes smoothness conditions on the target trajectory
- Includes cost of inputs
- Allows for feedback representations
- Requires the solution of (nonstandard) boundary value problems

Future work:

- Prove existence of solutions of the optimality system
- Apply the resulting controller in experiments

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Multibody Syst. Dyn., 11(4):343–364, 2004.

Thank you!

Thank you for your attention!

I am always open for discussion

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