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COMPUTATIONAL METHODS IN
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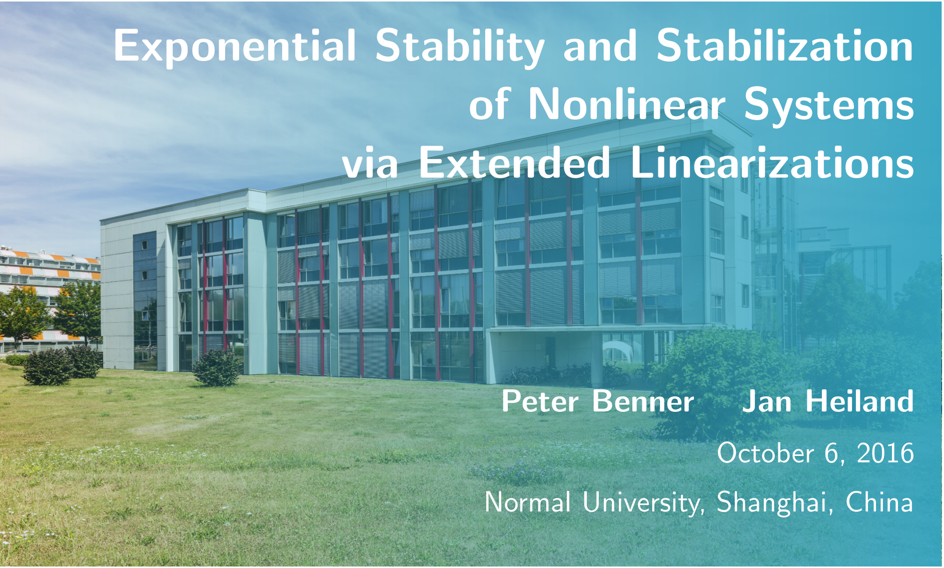

Exponential Stability and Stabilization of Nonlinear Systems via Extended Linearizations

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Outline

Introduction

Set-point control of an autonomous system

$$\left\{\dot\zeta = f(\zeta) + Bu,\ \zeta(0) = z_0\right\} \quad \to \quad \zeta(t) \to z^*,\ \text{with } f(z^*) = 0$$

is the same as

$$\left\{\dot\xi = \tilde f(\xi) + Bu,\ \xi(0) = x_0\right\} \quad \to \quad \xi(t) \to 0,\ \text{with } \tilde f(0) = 0,$$

where $\xi := \zeta - z^*$, and, accordingly, $\tilde f(\xi) = f(\xi + z^*)$.

If f is Lipshitz-continuous, then (since $\tilde f(0) = 0$) there is a matrix valued function A, such that

$$\tilde f(\xi) = A(\xi)\xi.$$

Introduction

Subject:

- *Extended linearizations* or *state-dependent coefficient* systems

$$\dot{\xi}(t) = A(\xi(t))\xi(t), \quad \xi(0) = x_0 \in X_0 \subset \mathbb{R}^n,$$

where $A\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$.

Application:

- Set point control of nonlinear autonomous systems.

Related Work:

- Extends the approach of linearization based stabilization of *set points* as in [B./H. '15, Breiten/Kunisch '14]
- [Banks et al. '07] – nonlinear feedback laws via state-dependent Riccati equations
- [Hill/Ilchmann '11] – stability of linear time-varying systems

Stability of SDC Systems

To describe exponential stability for the considered type of SDC systems

$$\dot{\xi}(t) = A(\xi(t))\xi(t), \tag{SDC}$$

we adapt a definition for time varying systems:

Definition

System (SDC) is called *uniformly exponentially stable* if there exist positive constants K and ω such that for any $x_0 \in \mathbb{R}$, a solution ξ of (SDC) with $\xi(0) = x_0$ satisfies

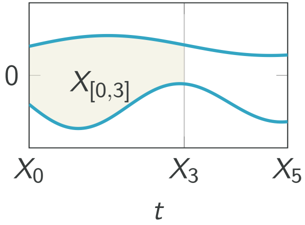
$$\|\xi(t)\| \leq Ke^{-\omega t}\|x_0\|, \quad \text{for } t \geq 0.$$

It is called *uniformly exponentially stable on* X_0, if, for some $X \subset \mathbb{R}^n$, stability is given for any $x_0 \in X_0$.

Stability of SDC Systems

Key observations:

- The coefficient A is stabilized together with the state.
- Linear time-varying system are much more arbitrary.



Ideas:

- Consider the set $X_{[0,T]}$ in which the system evolves until a time T,
- Assume that $\|e^{A(x)t}\| \leq Ke^{-\omega t}$ for any $x \in X_{[0,T]}$ and $t \leq 0$,
- If there is a t^*, such that $X_{t^*} \subsetneq X_0$, then all is good.

Stability of SDC Systems

Lemma ([B./H. '16])

For a given $T > 0$, let A be differentiable and uniformly stable with constants K and ω on $X_{[0,T]}$ and for $0 \leq t \leq T$, let

$$m_t := \inf_{\rho \in \mathbb{R}_{\geq 0}} \sup_{\xi \in \Xi_{[0,t]}} \frac{\int_0^t e^{-\omega(t-s)} \|A(\xi(s)) - A(\xi(\rho))\| \|\xi(s)\| \, ds}{\int_0^t e^{-\omega(t-s)} |s-\rho| \|\xi(s)\| \, ds}.$$

If for a t^*, with $0 < t^* \leq T$,

$$-\omega_{t^*} := \sqrt{K m_{t^*} \log 2} - \omega \quad \textit{and} \quad -\omega^* := \frac{\log K}{t^*} - \omega_{t^*}$$

are negative, then the snapshots $\xi(t)$ of any solution ξ to (SDC) with $\xi(0) = x_0 \in X_0$ on the discrete grid $\mathcal{T}^* := \{t \colon t = N \cdot t^*,\ N = 0, 1, \dots\}$ decay exponentially in the sense that

$$\|\xi(t)\| \leq \|x_0\| e^{-\omega^* t}, \quad \textit{for all } t \in \mathcal{T}^*.$$

Riccati-based Stabilization

New subject:

- Feedback stabilization of (SDC)

$$\dot{\xi}(t) = A(\xi(t))\xi(t) + Bu(t) :$$

  find a (state-dependent) feedback gain F, such, that the closed loop system

$$\dot{\xi}(t) = [A(\xi(t)) - BF(\xi(t))]\xi(t)$$

  is stable.
- By the preceding theory, we need bounds on decay and transient behavior uniformly in $x \in X$:

Definition

We say that $A \in \mathbb{R}^{n,n}$ is in class $\mathcal{S}_{K,\omega}$ for given constants K and ω, if

$$\|e^{At}\| \leq Ke^{-\omega t},$$

for $t > 0$.

Riccati-based Stabilization

Regression:

- If $A(x) - BF(x) \in \mathcal{S}_{K,\omega}$, where $F(x) = R^{-1}B^\mathsf{T}P$ and where $P = P(x)$ solves the Riccati equation

$$PA(x) + A(x)^\mathsf{T}P - PBR^{-1}B^\mathsf{T}P + Q = 0,$$

for given $R \succ 0$ and $Q \succcurlyeq 0$,

- then

$$\begin{bmatrix} A(x) & -BR^{-1}B^\mathsf{T} \\ -Q & A(x)^\mathsf{T} \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} Z,$$

and $Z = A(x) - BR^{-1}B^\mathsf{T}P \in \mathcal{S}_{K,\omega}$.

Riccati-based Stabilization

Lemma ([B./H. '16])

Consider the Riccati based feedback so that $A(x) - BR^{-1}B^{\mathsf{T}}P = Z \in \mathcal{S}_{K,\omega}$. If for a $\delta \in \mathbb{R}^{n,n}$, there exist $Q_\delta \in \mathbb{R}^{n,n}$, R_δ^{-1}, and $E \in \mathbb{R}^{n,n}$ such that

$$\begin{bmatrix} A(x) + \delta & -B[R^{-1} + R_\delta^{-1}]B^{\mathsf{T}} \\ -Q - Q_\delta & A(x)^{\mathsf{T}} + \delta^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} I + E \\ P \end{bmatrix} = \begin{bmatrix} I + E \\ P \end{bmatrix} Z,$$

and if $\|E\| < 1$, then $(I + E)$ is invertible and with $P_\delta := P(I+E)^{-1}$ it holds that

$$A(x) + \delta - B[R^{-1} + R_\delta^{-1}]B^{\mathsf{T}}P_\delta \in \mathcal{S}_{\tilde{K},\omega},$$

with $\tilde{K} = \frac{1+\|E\|}{1-\|E\|}K$.

Riccati-based Stabilization

$$\begin{bmatrix} A(x)+\delta & -B[R^{-1}+R_\delta^{-1}]B^\mathsf{T} \\ -Q-Q_\delta & A(x)^\mathsf{T}+\delta^\mathsf{T} \end{bmatrix} \begin{bmatrix} I+E \\ P \end{bmatrix} = \begin{bmatrix} I+E \\ P \end{bmatrix} Z$$

- If there is a solution E with $\|E\| < 1$, then Q_δ is not a design parameter.
- Thus, it is sufficient to find R_δ^{-1} such that

$$(A(x)+\delta)E - EZ = -\delta + BR_\delta^{-1}B^\mathsf{T}P$$

  has a small solution E.
- Known estimates indicate that E is small if δ and R_δ^{-1} are small.
- Existence of E is a harder problem.

Numerical Example

A *5D example* from [Banks et al. '07] with the SDC system $\dot{\xi} = A\xi + Bu$:

$$\begin{bmatrix} \dot{\xi}_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \xi_4^2 & 0 \\ -\xi_1 & 0 & 0 & \xi_4^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u, \quad \xi(0) = x_0 \in \mathbb{R}^5.$$

We add the observation $\eta = C\xi$, defined as

$$\eta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{bmatrix}.$$

And integrate on $(0, 3]$ starting from

$$x_0 = \begin{bmatrix} -1.3 & -1.4 & -1.1 & -2.0 & 0.3 \end{bmatrix}^T.$$

Numerical Example

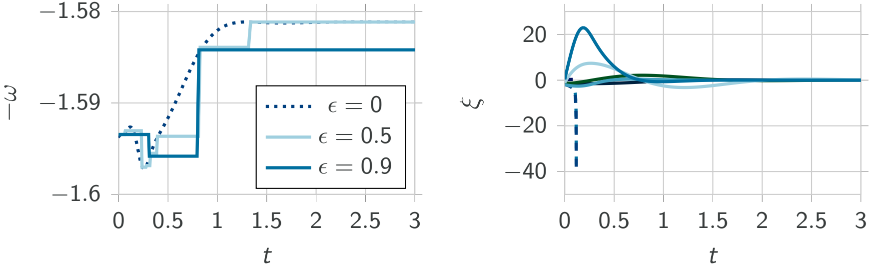
- The p-update-Algorithm:
  (1) Start with a Riccati feedback P_0.
  (2) At every time instance, solve a Sylvester equation for an update E of the feedback.
  (2a) If $\|E\|$ exceeds a threshhold ϵ, reset the current feedback with a new Riccati feedback $P_0 \leftarrow P_{\text{new}}$.

- The parameters for the definition of the feedback and the updates:

$$R = 10^{-3} I_{2\times 2}, \quad Q = CC^{\mathsf{T}}, \quad \text{and} \quad R_\delta^{-1} = 0.$$

- The code for the example is available from gitlab.mpi-magdeburg.mpg.de/heiland/code-ext-lin-stab.

Numerical Example



Scheme	ϵ	#fb-switches	#f-eva	comp-time
sdre	0	—	339	$0.079s$
p-update	0.1	33	1577	$0.330s$
p-update	0.5	7	619	$0.130s$
p-update	0.9	2	425	$0.090s$

Numerical Example

Chaffee Infante equation

For the spatial coordinate $z \in (0,2)$ and time $t \in (0,3]$, consider the autonomous PDE

$$\begin{aligned} \dot{\xi} &= \partial_{zz}\xi + 5(1-\xi^2)\xi, \quad x_0 = 0.2\sin(0.5\pi z), \\ \partial_z \xi(t)\big|_{z=2} &= u(t) \\ \xi(t)\big|_{z=0} &= 0 \end{aligned}$$

which we turn into a SDRC system of order N via a spatial finite element discretization with N degrees of freedom.

Numerical Example

Set point control of *Chaffee Infante* equation

The *Chaffee Infante* equation has an unstable set point at $\xi(t, z) = 0$ and to stable set points.

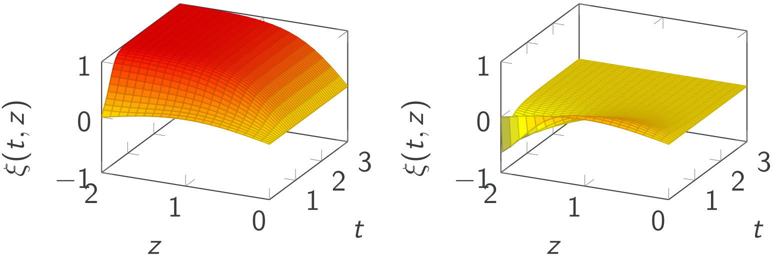


Figure : The uncontrolled (left) and the stabilized (right) evolution of the solution to the *Chaffee Infante* equation.

Task: force the system into the unstable set point.

Numerical Example

	Scheme	ϵ	#fb-switches	#f-eva	comp-time
$N = 20$	sdre	0	—	442	$1.921s$
	p-update	0.5	2	838	$3.266s$
	p-update	0.9	0	451	$1.756s$
$N = 60$	sdre	0	—	1194	$15.426s$
	p-update	0.5	4	2240	$18.379s$
	p-update	0.9	2	1770	$14.140s$
$N = 100$	sdre	0	—	2106	$90.148s$
	p-update	0.5	7	3778	$68.080s$
	p-update	0.9	4	2423	$43.816s$

Table : Influence of ϵ on the number of switches #fb-switches in the feedback definition, on the number of function evaluations #f-eva in the time integrator, and on the overall computation time comp-time for the simulation of the stabilized *Chaffee Infante* equation with finite element discretizations on varying mesh sizes N.

Conclusion

on the numerical results

- The p-update scheme
  - comes with theoretical backing and
  - bases on (linear) Sylvester rather than on (nonlinear) Riccati equations,
  - wich pays of for larger systems.

Conclusion

and Outlook

- Sufficient conditions for stability of SDC systems:
  - Adaption of known results for linear time-varying systems with
  - localized estimates
  - based on integral mean values.
- The needed uniform bounds on the decay and the transient behavior
  - can be achieved by updating Riccati based feedback
  - by solutions to Sylvester equations.
- Proof of concept for a numerical example.
- Next steps:
  - Solvability of the Sylvester equation for the updates,
  - implementation in large scale settings.

Thank you!

Thank you for your attention!

I am always open for discussion

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