MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Computational Approaches to $\mathcal{H}_\infty$-robust Controller Design and System Norms for Large-scale Systems

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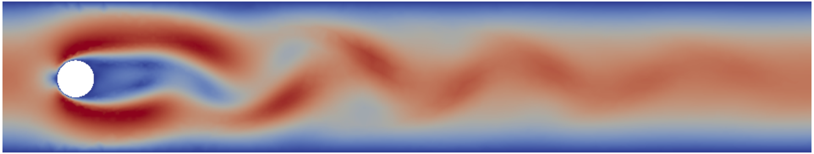
Introduction

Flow Control Task I

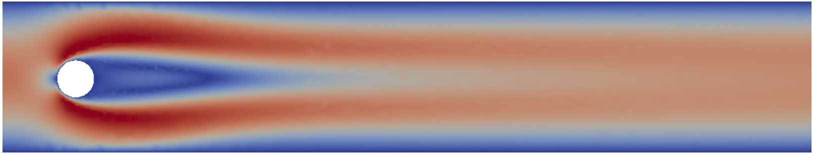


Feedback Control

Problem: The steady state does not persist because of unavoidable system perturbations.

Goal: Stabilizing feedback controller that can handle:

- limited measurements,
- short evaluation time,
- system uncertainties.



Idea: Linearization-based feedback control for stabilization of the steady state.
[Raymond'05,'06&Breiten/Kunisch'14,PB/JH'15]

$$
\begin{aligned}
\dot v + (v \cdot \nabla v) - \frac{1}{Re}\Delta v + \nabla p &= Bu, \\
\nabla \cdot v &= 0, \\
y &= Cv
\end{aligned}
$$

Linearization & Semi-Discretization $\Rightarrow$

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E\dot v - Av - J^{\mathsf{T}} p &= Bu, \\
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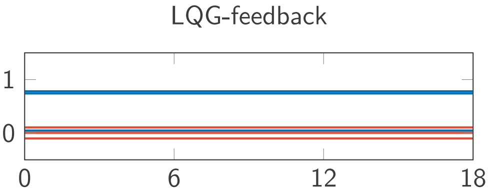
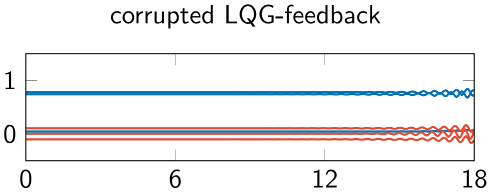
Linearization & Semi-Discretization $\Longrightarrow$

$$E\dot{v} - Av - J^{\mathsf{T}} p = Bu, \qquad Jv = 0, \qquad y = Cv$$

Fragility of Observer-Based Controllers

LQG controllers have no guaranteed robustness margins and will likely fail in the presence of system uncertainties.





Introduction
Flow Control Task II

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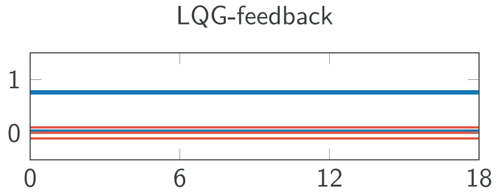

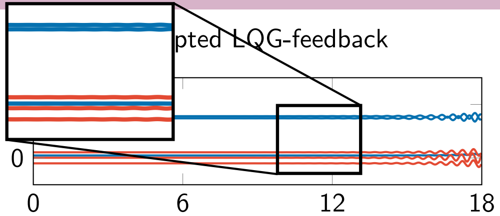

History of $\mathcal{H}_\infty$-robust Controller Design

1978 … **Guaranteed Margins for LQG Regulators**

JOHN C. DOYLE

***Abstract*—There are none.**

1981 … First formulation of the $\mathcal{H}_\infty$-robust control problem.

1987 … State space formulations

2000's … Further developments

- $\mathcal{H}_\infty$-theory for abstract linear systems,
- $\mathcal{H}_\infty$ model reduction,
- solvers for high-dimensional Riccati equations.

today … $\mathcal{H}_\infty$-robust controllers for the stabilization of flows

- in the PDE model of incompressible Navier-Stokes equations
- and in the simulation (this talk).

$\mathcal{H}_\infty$ Riccati Equations [Doyle/Glover/Khargonekar/Francis '89, van Keulen '93]

Under some reasonable assumptions, there exists a $\mathcal{H}_\infty$-robust controller $K(s) \iff$:

1. There exists a stabilizing solution $X_\infty = X_\infty^T \geq 0$ to the regulator Riccati equation

$$A^T X_\infty + X_\infty A + C_1^T C_1 + X_\infty(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty = 0.$$

2. There exists a stabilizing solution $Y_\infty = Y_\infty^T \geq 0$ to the filter Riccati equation

$$A Y_\infty + Y_\infty A^T + B_1 B_1^T + Y_\infty(\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y_\infty = 0.$$

3. It holds $\gamma^2 > \lambda_{\max}(Y_\infty X_\infty)$.

Here,

- $\gamma \in \mathbb{R}$ is a parameter that expresses robustness performance and
- $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times p_1}$, $B_2 \in \mathbb{R}^{n\times p_2}$, $C_1 \in \mathbb{R}^{q_1\times n}$, and $C_2 \in \mathbb{R}^{q_2\times n}$

are matrix coefficients of the considered linear time-invariant system.

This talk

- How to solve the large-scale $\mathcal{H}_\infty$-Riccati equation
  - Riccati iteration
  - using low-rank factors.

- What do we do with the solution?
  - Design a controller.
  - Reduce it.
  - Balance it's robustness with system uncertainties.

Riccati Equations with Indefinite Quadratic Term

$$\mathcal{R}(X) := C^T C + A^T X + XA + X(B_1 B_1^T - B_2 B_2^T)X = 0.$$

Generally

- The solution is $X \in \mathbb{R}^{n \times n}$ – for $n = 50'000$ this means a memory requirement of about 18GB.
- The coefficient $B_1 B_1^T - B_2 B_2^T$ is symmetric but possibly indefinite – for negative definite coefficients, i.e. the “standard” Riccati equation, there exist numerous efficient solution approaches.

Riccati Equations with Indefinite Quadratic Term

$$\mathcal{R}(X) := C^TC + A^TX + XA + X(B_1B_1^T - B_2B_2^T)X = 0.$$

General remarks

- For small sized problems, standard direct methods like the *sign function iteration* or *Schur decomposition approaches* apply.

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- For large-scale problems, resulting, e.g., from H_∞ control, standard methods based on Hamiltonian/even eigenvalue problem can not be used due to $\mathcal{O}(n^3)$ complexity/dense matrix algebra.

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- Krylov subspace methods might be employed, but so far no convergence results, and in case of convergence, no guarantee that stabilizing solution is computed.

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- Krylov subspace methods might be employed, but so far no convergence results, and in case of convergence, no guarantee that stabilizing solution is computed.
- Newton/Newton-ADI method will in general diverge/converge to a non-stabilizing solution.

Riccati Equations with Indefinite Quadratic Term

$$\mathcal{R}(X) := C^T C + A^T X + XA + X(B_1 B_1^T - B_2 B_2^T)X = 0.$$

General remarks

Quick-and-dirty solution: consider $X^{-1}\mathcal{R}(X)X^{-1} = 0$ [DAMM '02]

$\rightsquigarrow$ standard ARE for $\widetilde{X} \equiv X^{-1}$

$$\tilde{\mathcal{R}}(\widetilde{X}) := (B_1 B_1^T - B_2 B_2^T) + \widetilde{X} A^T + A\widetilde{X} + \widetilde{X} C^T C \widetilde{X} = 0.$$

Newton's method will converge to stabilizing solution, Newton-ADI can be employed (with modification for indefinite constant term).

But: low-rank approximation of $\widetilde{X}$ will not yield good approximation of $X \Rightarrow$ not feasible for large-scale problems!

Lyapunov Iterations/Perturbed Hessian Approach

[Cherfi/Abou-Kandil/Bourles '05 (Proc. ACSE 2005)]

Idea

Perturb Hessian to enforce semi-definiteness: write

$$0 = A^T X + XA + Q - XGX = A^T X + XA + Q - XDX + X(D - G)X,$$

where $D = G + \alpha I \geq 0$ with $\alpha \geq \min\{0, -\lambda_{\max}(G)\}$.

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Here: $G = B_2 B_2^T - B_1 B_1^T$

$\Rightarrow$ use $\alpha = ||B_1||^2$ for spectral/Frobenius norm or

$$\alpha = ||B_1||_1 \cdot ||B_1||_\infty.$$

Remark

$W \geq -G$ can be used instead of αI, e.g., $W = \beta B_1 B_1^T$ with $\beta \geq 1$.

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Lyapunov iteration

Based on

$$(A - DX)^T X + X(A - DX) = -Q - XDX - \alpha X^2,$$

iterate

FOR $k = 0, 1, \ldots,$ solve Lyapunov equation

$$(A - DX_k)^T X_{k+1} + X_{k+1}(A - DX_k) = -Q - X_k D X_k - \alpha X_k^2.$$

Theorem [Cherfi/Abou-Kandil/Bourles '05]

If

- $\exists\ \widehat{X}$ such that $\mathcal{R}(\widehat{X}) \geq 0$,
- $\exists\ X_0 = X_0^T \geq \widehat{X}$ such that $\mathcal{R}(X_0) \leq 0$ and $A - DX_0$ is Hurwitz,

then

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then

a) $X_0 \geq \ldots \geq X_k \geq X_{k+1} \geq \ldots \geq \widehat{X}$,

Main problems

- Conditions for initial guess make its computation difficult.
- Observed convergence is linear.

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b) $\mathcal{R}(X_k) \leq 0$ for all $k = 0, 1, \ldots$,

c) $A - DX_k$ is Hurwitz for all $k = 0, 1, \ldots$,

d) $\exists\ \lim_{k\to\infty} X_k =: \underline{X} \geq \widehat{X}$,

Main problems

- Conditions for initial guess make its computation difficult.
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Riccati Iterations

Idea

Consider

$$A^T X + XA + C^T C + X(B_1 B_1^T - B_2 B_2^T)X =: \mathcal{R}(X)$$

and

$$\mathcal{R}(X+Z) = \mathcal{R}(X) + (\underbrace{A + (B_1 B_1^T - B_2 B_2^T)X}_{=:\widehat{A}})^T Z + Z\widehat{A} + Z(B_1 B_1^T - B_2 B_2^T)Z.$$

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Thus, if for some $X = X^T$, the matrix $Z = Z^T$ solves the standard ARE

$$0 = \mathcal{R}(X) + \widehat{A}^T Z + Z\widehat{A} - ZB_2 B_2^T Z,$$

then

$$\mathcal{R}(X+Z) = ZB_1 B_1^T Z$$

which, by the way, is a low-rank factorization of the residual.

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which, by the way, is a low-rank factorization of the residual.

Riccati iteration

1. Set $X_0 = 0$.
2. FOR $k = 1, 2, \ldots,$
   1. Set $A_k := A + B_1(B_1^T X_k) - B_2(B_2^T X_k)$.
   2. Solve the ARE
   $$\mathcal{R}(X_k) + A_k^T Z_k + Z_k A_k - Z_k B_2 B_2^T Z_k = 0.$$
   3. Set $X_{k+1} := X_k + Z_k$.
   4. IF $\left|\left|B_1^T Z_k\right|\right|_2 < \text{tol}$ THEN Stop.

Remark. ARE for $k = 1$ is the standard LQR/H_2 ARE.

Riccati Iterations

Theorem [Lanzon/Feng/B.D.O. Anderson 2007]

If

- (A, B_2) stabilizable,
- (A, C) has no unobservable purely imaginary modes, and
- $\exists$ stabilizing solution X_-,

then

a) $(A + B_1 B_1^T X_k, B_2)$ stabilizable for all $k = 0, 1, \ldots,$
b) $Z_k \geq 0$ for all $k = 0, 1, \ldots,$
c) $A + B_1 B_1^T X_k - B_2 B_2^T X_{k+1}$ is Hurwitz for all $k = 0, 1, \ldots,$
d) $\mathcal{R}(X_{k+1}) = Z_k B_1 B_1^T Z_k$ for all $k = 0, 1, \ldots,$
e) $X_- \geq \ldots \geq X_{k+1} \geq X_k \geq \ldots \geq 0.$
f) If $\exists$ $\lim_{k\to\infty} X_k =: \underline{X}$, then $\underline{X} = X_-$, and
g) convergence is locally quadratic.

Riccati iteration – low-rank version [PB'08&PB/JH/SW'23]

1. Solve the ARE

$$C^TC + A^TZ_0 + Z_0A - Z_0B_2B_2^TZ_0 = 0$$

using low-rank Newton-ADI, yielding Y_0 with $Z_0 \approx Y_0Y_0^T$.

2. Set $V_1 := Y_0$. $\{\%\ V_1V_1^T \approx X_1.\}$

3. FOR $k = 1, 2, \ldots,$
   1. Set $A_k := A + B_1(B_1^TV_k)V_k^T - B_2(B_2^TV_k)V_k^T$.
   2. Solve the ARE

   $$Y_{k-1}(Y_{k-1}^TB_1)(B_1^TY_{k-1})Y_{k-1}^T + A_k^TZ_k + Z_kA_k - Z_kB_2B_2^TZ_k = 0$$

   using low-rank Newton-ADI, yielding Y_k with $Z_k \approx Y_kY_k^T$.
   3. Set $V_{k+1} := \texttt{rank\_revealing\_qr}([\,V_k,\ Y_k\,], \tau)$. $\{\%\ \tau$ truncation tolerance; $V_{k+1}V_{k+1}^T \approx X_{k+1}\}$
   4. IF $\left|\left|(B_1^TY_k)Y_k^T\right|\right|_2 <$ tol THEN Stop.

Multilevel low-rank iteration

- Solution to the $\mathcal{H}_\infty$-Riccati equation

$$X_\infty$$

... use the *Riccati iteration*.

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- Solution at k-th Riccati iteration (a standard ARE)

$$X_\infty \approx X_\infty^{(k)}$$

... use the *Newton-Kleinman iteration*.

Multilevel low-rank iteration

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- Solution at i-th Newton-Kleinman iteration (a Lyapunov equation)

$$^{(i)}X_\infty^{(k)} \approx X_\infty^{(k)}$$

... use the low-rank *ADI iteration*.

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- Solution at j-th low-rank ADI iteration

$${}^{(i)}_{(j)}V_\infty^{(k)}{}^T\,{}^{(i)}_{(j)}V_\infty^{(k)} := {}^{(i)}_{(j)}X_\infty^{(k)} \approx {}^{(i)}X_\infty^{(k)}$$

Riccati Equations with Indefinite Quadratic Term

Generally, feasibility for large-scale comes from a formulation that during the iterations

- expresses approximate solutions
$$X^{(k)} = VV^T$$
and right hand sides in low-rank factorized form,
- preserves the coefficients
$$A^{(k)} = A - BB^T X^{(k)} = A - BK^{(k)}$$
as sparse + low-rank update matrices,
- and resorts to efficient solves of (possibly nonsymmetric or indefinite) Lyapunov equations.

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Possible solvers:

- Standard Krylov subspace solvers in operator form [Hochbruck, Starke, Reichel, Bao, ...].
- Block-Tensor-Krylov subspace methods with truncation [Kressner/Tobler, Bollhöfer/Eppler, B./Breiten, ...].
- Galerkin-type methods based on (extended, rational) Krylov subspace methods [Jaimoukha, Kasenally, Jbilou, Simoncini, Druskin, Knizhermann,...]
- Doubling-type methods [Smith, Chu et al., B./Sadkane/El Khoury, ...].
- ADI methods [Wachspress, Reichel et al., Li, Penzl, B., Saak, Kürschner, ...].

Low-rank Riccati Iteration: Proof of Concept

Table: Results for solving the $\mathcal{H}_\infty$-control Riccati equations for the aircraft ($n = 55$) and cable mass ($n = 76$) benchmarks from [Leibfritz'04].

	aircraft(n=10)			cable mass(=76)		
	LRRI	ICARE	SIGN	LRRI	ICARE	SIGN
Iteration steps	5	—	19	4	—	23
Runtime (s)	0.89996	0.42306	0.07541	31.3999	140.634	5.98172
Rank Z_k	53	55	55	569	758	781
Final res.	5.545e−25	—	—	1.873e−15	—	—
Relative res.	2.599e−07	9.617e−10	1.183e−09	1.910e−09	5.019e−08	2.125e−07
Normalized res.	1.554e−03	5.752e−06	7.074e−06	1.667e−05	4.381e−04	1.855e−03
$\|Z_k^T Z_k\|_2$	1.457e+01	1.457e+01	1.457e+01	1.253e+04	1.253e+04	1.253e+04

Low-rank Riccati Iteration: Efficiency of the method

Table: Results of the LRRI for solving the $\mathcal{H}_\infty$-control Riccati equations for large-scale sparse examples.

	rail	cylinderwake
Dimension n	79 841	47 136
Iteration steps	3	3
Runtime (s)	72.2906	3469.27
Rank Z_k	169	418
Final res.	1.297e−19	2.184e−21
Relative res.	2.125e−21	1.996e−14
Normalized res.	9.766e−11	1.622e−03
$\|Z_k^T Z_k\|_2$	6.866e+11	5.056e+08

OK, what now?

OK, what now?

1. Design the $\mathcal{H}_\infty$-controller.
2. Reduce it.
3. Balance robustness with reduction and linearization errors.

$\mathcal{H}_\infty$ Riccati Equations

[Doyle/Glover/Khargonekar/Francis '89, Van Keulen '93]

Given some simplifying assumptions, there exists an admissible controller $K(s) \iff$:

1. There exists a stabilizing solution $X_\infty = X_\infty^\mathsf{T} \geq 0$ to the regulator Riccati equation
$$A^\mathsf{T} X_\infty + X_\infty A + C_1^\mathsf{T} C_1 + X_\infty(\gamma^{-2} B_1 B_1^\mathsf{T} - B_2 B_2^\mathsf{T}) X_\infty = 0.$$
2. There exists a stabilizing solution $Y_\infty = Y_\infty^\mathsf{T} \geq 0$ to the filter Riccati equation
$$A Y_\infty + Y_\infty A^\mathsf{T} + B_1 B_1^\mathsf{T} + Y_\infty(\gamma^{-2} C_1^\mathsf{T} C_1 - C_2^\mathsf{T} C_2) Y_\infty = 0.$$
3. It holds $\gamma^2 > \lambda_{\max}(Y_\infty X_\infty)$.

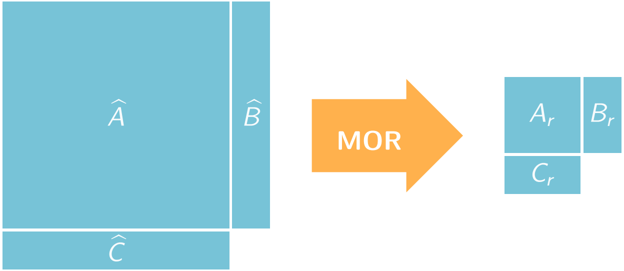
The central (or minimum entropy) controller $\widehat{K}(s) = \widehat{C}(sI_n - \widehat{A})^{-1}\widehat{B}$ has the transfer function

$$\widehat{A} = A + (\gamma^{-2} B_1 B_1^\mathsf{T} - B_2 B_2^\mathsf{T}) X_\infty - Z_\infty Y_\infty C_2^\mathsf{T} C_2, \quad \widehat{B} = Z_\infty Y_\infty C_2^\mathsf{T}, \quad \widehat{C} = -B_2^\mathsf{T} X_\infty,$$

with $Z_\infty = (I_n - \gamma^{-2} X_\infty Y_\infty)^{-1}$.

Challenge

This controller is of the size of the system (i.e. prohibitively large)
$\Rightarrow$ However, can do reduction to size r to enable fast evaluation of the feedback law.



For the normalized $\mathcal{H}_\infty$-robust control problem with $B_1 B_1^T = B_2 B_2^T$:

- Reduction to K_r can be computed from the low-rank factors of X_∞, Y_∞ [Mustafa&Glover'91]
- Proven: The controller will be still stabilizing but with a slightly worse robustness margin.

- Notation: normalized left coprime factorizations transferfunctions of the full and the reduced system:

$$G = M^{-1}N \quad \text{and} \quad G_r = M_r^{-1}N_r$$

(for computation see below or [PB/JH/W. '19]),

- The approximation error of the $\mathcal{H}_\infty$ balanced truncation is given by

$$\left\| \begin{bmatrix} \beta(N - N_r) & M - M_r \end{bmatrix} \right\|_{\mathcal{H}_\infty} =: \beta\hat{\epsilon} \leq \beta\epsilon$$

  - for $\beta = \sqrt{1 - \gamma^{-2}}$.
  - for a theoretical threshhold $\hat{\epsilon}$,
  - and a computable estimate ϵ.
  - (see below).

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Theorem [Mustafa/Glover '91]

The reduced-order $\mathcal{H}_\infty$ controller is guaranteed to stabilize the full-order system if

$$\hat{\epsilon}(\beta + \gamma) < 1 \quad \text{or} \quad \epsilon(\beta + \gamma) < 1.$$

- for $\beta = \sqrt{1 - \gamma^{-2}}$.
- for a theoretical threshhold $\hat{\epsilon}$,
- and a computable estimate ϵ.
- (see below).

- A perturbation A_Δ in the linear system

$$\dot{x} = (A + A_\Delta)x + B_2 u$$

- smoothly [H.22] transfers into a coprime factor perturbation

$$G \approx G_\Delta = M_\Delta^{-1} N_\Delta$$

.

Theorem [Mustafa/Glover '91, PB/JH/W. '19]

Any stabilizing controller K with $\mathcal{H}_\infty$-performance that satisfies γ is guaranteed to stabilize the disturbed system if

$$\left\| \begin{bmatrix} N - N_\Delta & M - M_\Delta \end{bmatrix} \right\|_{\mathcal{H}_\infty} < \gamma^{-1}.$$

Application to Incompressible Nonlinear Flows

For consistent initial values, i.e., $Jv_0 = 0$, the semi-discretized Navier-Stokes equation can be realized by an ODE system:

$$\begin{aligned} E\dot{v} &= Av + J^{\mathsf{T}} p + Bu, \\ 0 &= Jv, \\ y &= Cv, \end{aligned} \quad \Rightarrow \quad \begin{aligned} E\dot{v} &= \mathbf{\Pi}^{\mathsf{T}} A \mathbf{\Pi} v + \mathbf{\Pi}^{\mathsf{T}} B, \\ y &= C\mathbf{\Pi} v, \end{aligned}$$

where $\mathbf{\Pi} = I_{n_v} - E^{-1}J^{\mathsf{T}}(JE^{-1}J^{\mathsf{T}})^{-1}J$ is the discrete Leray projection.

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Implicit Realization [Heinkenschloss/Sorensen/Sun '08, Bänsch/PB/Saak/Weichelt '15, and many more...]

The explicit projection $\boldsymbol{\Pi}$ can be avoided in the numerical methods by solving saddle point problems of the type

$$\begin{bmatrix} A + s_i E & J^{\mathsf{T}} \\ J & 0 \end{bmatrix} \begin{bmatrix} X \\ * \end{bmatrix} = \begin{bmatrix} Y \\ 0 \end{bmatrix}.$$

Given the normalized left coprime factorizations: $G = M^{-1}N$, $G_\Delta = M_\Delta^{-1}N_\Delta$, with

$$\begin{bmatrix} N_{\boldsymbol{\Delta}}(s) & M_{\boldsymbol{\Delta}}(s) \end{bmatrix} = \mathcal{C}(s\mathcal{E} - \widetilde{\mathcal{A}} - \boldsymbol{\mathcal{A}}_{\boldsymbol{\Delta}})^{-1} \begin{bmatrix} \mathcal{B} & -\widetilde{\mathcal{L}} \end{bmatrix} + \begin{bmatrix} 0 & I_p \end{bmatrix},$$

where

$$\mathcal{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad \widetilde{\mathcal{A}} - \boldsymbol{\mathcal{A}}_{\boldsymbol{\Delta}} = \begin{bmatrix} A + \boldsymbol{A}_{\boldsymbol{\Delta}} - (1-\gamma^{-2})EY_{\mathcal{H}_\infty}C^\mathsf{T}C & J^\mathsf{T} \\ J & 0 \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \widetilde{\mathcal{L}} = \begin{bmatrix} (1-\gamma^{-2})EY_{\mathcal{H}_\infty}C^\mathsf{T} \\ 0 \end{bmatrix}$$

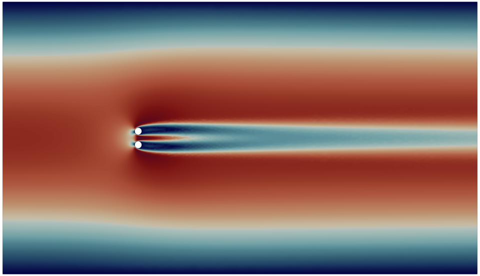
is a realization that can be used to compute, e.g.,

$$\| \begin{bmatrix} N - N_{\boldsymbol{\Delta}} & M - M_{\boldsymbol{\Delta}} \end{bmatrix} \|_{\mathcal{H}_\infty} < \gamma^{-1}?$$

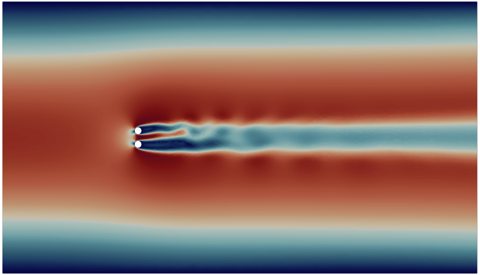
by Navier-Stokes simulation tools.

Numerical Example

Numerical Example
Double Cylinder: Setup



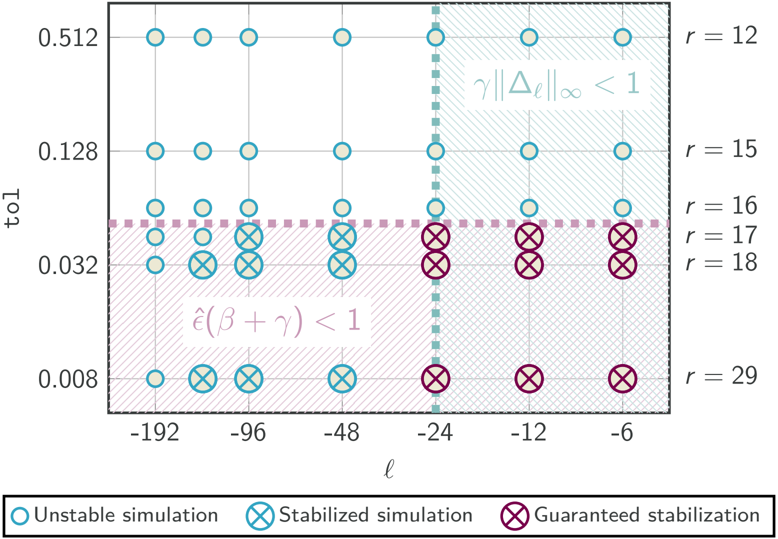
(a) Steady state.



(b) Disturbed flow.

- Navier-Stokes equations discretized by Taylor-Hood finite elements
- system order $n = 51\,337$
- boundary control: individual rotation of both cylinders
- observations: 3 velocity sensors in the wake behind the cylinders

- Reynolds number 60
- K with robustness margin: $\gamma = 12.5418$
- linearization error: disturbed Reynolds number


0.512
0.128
0.032
0.008
tol
-192
-96
-48
-24
-12
-6
ℓ
$r = 12$
$r = 15$
$r = 16$
$r = 17$
$r = 18$
$r = 29$
$\gamma\|\Delta_\ell\|_\infty < 1$
$\hat{\epsilon}(\beta + \gamma) < 1$
Unstable simulation
Stabilized simulation
Guaranteed stabilization

Conclusions

Conclusions

Summary

- The low-rank formulation of the Riccati iteration (LRRI) enables the computation of solution to, e.g., the $\mathcal{H}_\infty$-Riccati equation for large-scale systems.
- Proof of concept: LRRI competes well with dense routines for small system sizes and shows fast convergence for large system sizes.
- Once the $\mathcal{H}_\infty$-Riccati solutions are at hand, low-order $\mathcal{H}_\infty$-controller design comes at little extra cost.
- Application to incompressible flows using implicit realizations of projections.
- Outlook: Theory on balancing errors in the multilevel iteration.

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P. Benner, J. Heiland, S.W.R. Werner: Robust output-feedback stabilization for incompressible flows using low-dimensional $\mathcal{H}_\infty$-controllers. arXiv:2103.01608. (comes with codes)

- Low-rank solvers for (in-)definite Riccati equations are available in M-M.E.S.S.
- HINFBT and LQGBT implementations can be found in the MORLAB toolbox.

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