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Introductory Course on Model Reduction of Linear Time Invariant Systems

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Outline

Typical Situation

- Fry a steak
- The cook controls the heat at the fireplace
- and observes the process, e.g. via measuring the temperature in the inner

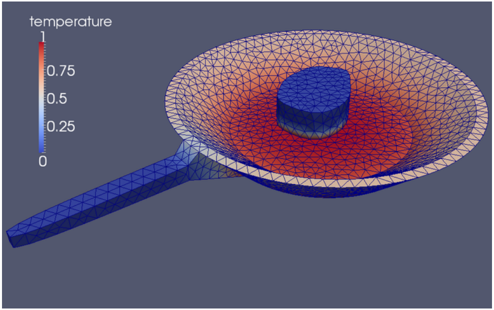
Typical Situation

- The model

$$
\begin{aligned}
\dot{\theta} &= \nabla \cdot (\nu \nabla \theta) && \text{in } (0, \infty) \times \Omega, \\
\theta &= u, && \text{at the plate,} \\
\theta(0) &= 0.
\end{aligned}
$$

- The cook controls the heat at the fireplace, which we denote by u
- and observes the process, e.g. he measures the temperature y in the center: $y = f(\theta)$.

Simulation



- The model:

$$\begin{aligned}\dot{\theta} &= \nabla \cdot (\nu \nabla \theta),\\ \theta &= u,\\ \theta(0) &= 0.\end{aligned}$$

- The cook controls the heat u
- and observes the process via $y = f(\theta)$.

- A *Finite Element* discretization of the problem leads to the finite dimensional model

$$E\dot{\theta}(t) = A\theta(t) + Bu(t), \quad \theta(0) = 0, \tag{1}$$
$$y(t) = C\theta(t), \tag{2}$$

a linear time invariant system.

Linear State Space System

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{3a}$$
$$y(t) = Cx(t) + Du(t), \tag{3b}$$

with

- $E \in \mathbb{R}^{n\times n}$: the identity or the mass matrix
- $A \in \mathbb{R}^{n\times n}$: the system matrix
- $B \in \mathbb{R}^{n\times m}$: the input matrix
- $C \in \mathbb{R}^{q\times n}$: the output matrix
- $D \in \mathbb{R}^{q\times n}$: the throughput

- $x(t) \in \mathbb{R}^{n}$: the system's state
- $u(t) \in \mathbb{R}^{m}$: the input or control
- $y(t) \in \mathbb{R}^{q}$: the output or measurements
- $n, m, q \in \mathbb{N}$: the system dimensions

We will assume that $E = I$ and denote the LTI (3) by (A, B, C, D).

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We will assume that $E = I$ and denote the LTI (3) by (A, B, C, D).

Some Preliminary Thoughts

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

A simple question...

What is x?

- it is a physical state in the model – like the temperature
- in practise, we may not access it – only the measurement $y = Cx$
- it is but a mathematical object as a part of a model
- furthermore, as we will see later, the state x can be severely changed e.g. in the course of model reduction

The state x can be seen...
...as nothing but an artificial object of the model for the input to output behavior

$$\mathbf{G}\colon u \mapsto y$$

of an abstract system $\mathbf{P}$:

$$y \longleftarrow \boxed{\quad \mathbf{P} \quad} \longleftarrow u$$

that maps an input u to the corresponding output y.

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$$\mathbf{G}\colon u \mapsto y$$

of an abstract system $\mathbf{P}$:

$$y \longleftarrow \boxed{\begin{array}{c} y(t) = Cx(t) \\ \dot{x}(t) = Ax(t) + Bu(t) \end{array}} \longleftarrow u$$

that maps an input u to the corresponding output y.

Transfer Function in Time-Domain

If **P** is modelled trough an (A, B, C, D) system, then the function **G** can be defined via

$$\mathbf{G}\colon u \mapsto y\colon y(t) = C\left[e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,\mathrm{d}s\right] + Du(t),$$

known as the formula of *variation of constants*.

This is in **time-domain**: A function u depending on time $t \in [0, \infty)$ is mapped onto a function y depending on time $t \in [0, \infty)$.

Introducing Frequency-Domain

Through the Laplace transform $\mathcal{L}$ and its inverse $\mathcal{L}^{-1}$, we can switch between time-domain and frequency-domain representations of the input and output signals:

$$U(s) := \mathcal{L}\{u\}(s) := \int_0^\infty e^{-st} u(t) \,\mathrm{d}t,$$

where $s \in \mathbb{C}$ is the *frequency* and

$$y(t) := \mathcal{L}^{-1}\{Y\}(t) := \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\gamma - iT}^{\gamma + iT} e^{s} Y(s) \,\mathrm{d}s$$

where $\gamma \in \mathbb{R}$ is chosen such that the contour path of the integration is the domain of convergence of Y.

Laplace Transform of an LTI

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

With the basic properties of the Laplace transform

- $\dot{X}(s) := \mathcal{L}\{\dot{x}\}(s) - x(0) = s\mathcal{L}\{x\}(s) = sX(s) - x(0)$
- and linearity $\mathcal{L}\{Ax\}(s) = AX(s)$

with zero initial value $x(0) = 0$, the (A, B, C, D) system defines the transfer function

$$G(s) := C(sI - A)^{-1}B + D$$

in frequency domain.

Realizations

Fact

An LTI (A, B, C, D) always defines a transfer function

$$G(s) = C(sI - A)^{-1}B + D$$

which is a matrix $G \in \mathbb{R}^{q\times m}$ with coefficients that are rational functions of s.

Question

Given a rational matrix function $s \mapsto G(s) \in \mathbb{R}^{q\times m}$, is there an

$$(A, B, C, D)$$

system, so that $G(s) = C(sI - A)^{-1}B + D$?

Realizations

given G, find (A, B, C, D),
$G(s) = C(sI - A)^{-1}B + D$

If there is one such (A, B, C, D), then there are infinitely many:

- For $T \in \mathbb{R}^{n\times n}$ invertible, also $(TAT^{-1}, TB, CT^{-1}, D)$ is a realization:

$$C(sI - A)^{-1}B + D = CT^{-1}(sI - TAT^{-1})^{-1}TB + D.$$

- Moreover, also

$$(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D)$$

is a realization of G.

Realizations

Facts and Thoughts on Realizations

- If G is *proper*, then there is a realization (A, B, C, D) as a state space system.
- This realization is by no means unique.
- The dimension of the state can be arbitrary large. What is the smallest possible dimension? (cf. *model reduction*)
- What is a good choice for the state?

Remark: A transfer function $G\colon s \mapsto G(s) \in \mathbb{R}^{q\times m}$ with coefficients that are rational functions in s, is *proper*, if in each coefficient the polynomial degree of the numerators does not exceed the degree of denominators.

Controllability and Observability

Based on the previous considerations, we can say that

- The states of an LTI system (A, B, C, D) are just a part of a model that realizes a transfer function G
- The transfer function G describes how controls u lead to outputs y
- As seen above in the example, there can be states that are neither affected (*controlled*) by the inputs nor seen (*observed*) by the outputs
- These states are obviously not needed to realize the input to output behavior of G.

We will give a thorough characterization of the *controllable* and *observable* states of an LTI.

Controllability

Definition

The LTI (A, B, C, D) or the pair (A, B) is said to be *controllable* if, for any initial state $x(0) = x_0$, $t_1 > 0$ and final state x_1, there exists a (piecewise continuous) input u such that the solution of (3) satisfies $x(t_1) = x_1$. Otherwise, the system (A, B, C, D) or the pair (A, B) is said to be *uncontrollable*.

Theorem

The following statements are equivalent:

(i.) *The pair (A, B) is controllable.*

(ii.) *The controllability matrix $\mathcal{C} := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$ has full rank.*

(iii.) *The matrix $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ has full rank for all $\lambda \in \mathbb{C}$.*

Observability

Definition

The LTI (A, B, C, D) or the pair (C, A) is said to be *observable* if, for any $t_1 > 0$, the initial state $x(0) = x_0$ can be determined from the time history of the input u and the output y in the interval of $[0, t_1]$. Otherwise, the system (A, B, C, D), or (C, A), is said to be *unobservable*.

Observability is the dual concept of controllability:

Theorem

The pair (C, A) is observable if and only if the pair $(A^\mathsf{T}, C^\mathsf{T})$ is controllable.

Invariance Under State Space Transformation

Theorem

The LTI (A, B, C, D) is controllable (observable) if, and only if, the transformed LTI $(TAT^{-1}, TB, CT^{-1}, D)$ is controllable (observable), where T is a regular matrix.

- Recall that also a transfer function is invariant with respect to state space transformations on its realization.
- Next, we find the states that are at least necessary for the realization of a transfer function...

Theorem (Kalman Canonical Decomposition)

Given an LTI (A, B, C, D), there is a state space transformation T such that the transformed system $(TAT^{-1}, TB, CT^{-1}, D)$ has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + Du,$$

with the subsystem $(A_{co}, B_{co}, C_{co}, D)$ being controllable and observable, while the remaining states $x_{\bar{c}o}$, $x_{c\bar{o}}$, or $x_{\bar{c}\bar{o}}$ are not controllable, not observable, or neither of them.

For a constructive proof of the Theorem, see Ch. 3.3 of [Zhou, Doyle, Glover '96]

Outcomes of the Kalman Decomposition

For any state space system (A, B, C, D), there is a transformation T so that the transformed states $T^{-1}x$ decompose into

- x_{co} - controllable and observable
- $x_{c\bar{o}}$ - controllable but not observable
- $x_{\bar{c}o}$ - observable but not controllable
- $x_{\bar{c}\bar{o}}$ - not observable and not controllable

Moreover, for the transfer function, it holds that

$$G(s) = C(sI - A)^{-1}B = C_{co}(sI - A_{co})^{-1}B_{co}.$$

Conclusion from the Kalman Decomposition

What does this mean for us and a transfer function $G(s)$?

- The minimal dimension of a realization is the dimension of x_{co} in the *Kalman Canonical Decomposition*
- Such a realization is called minimal realization
- It is the starting point for further model reduction. (Throwing out $x_{\bar{c}o}$ etc. does not effect $G(s)$ and is typically not considered a model reduction)
- There are algorithm to reduce a realization to a minimal one, cf. [Varga '90].
- In practice, the uncontrolled and unobserved states play a role and they may cause troubles. (check the literature for zero dynamics)

Summary

- LTI as model for physical processes (e.g. heat transfer)
- The input/output is often more important than the state
- Moreover, the state need not have a meaning
- State space systems (A, B, C, D) can be seen as realizations of transfer functions
- A transfer function has multiple realizations
- The minimal realizations are of our interest

More on the LTI topics

K. Zhou, J. C. Doyle, and K. Glover.
Robust and Optimal Control. (Chapter 3 for LTI)
Prentice-Hall, Upper Saddle River, NJ, 1996.

A. Varga.
Computation of irreducible generalized state-space realizations.
Kybernetika, 26(2):89–106, 1990.

A. Gaul.
Leckerbraten – a lightweight Python toolbox to solve the heat equation on arbitrary domains
https://github.com/andrenarchy/leckerbraten, 2013.

J. Heiland.
The slides, additional material, and information on this course
https://github.com/highlando/mor-shortcourse-SH, 2015.

Outline

1. Linear Time Invariant Systems
2. Norms of Signals and Systems
   - Norms
   - Norms of Signals
   - Norm of a System
   - Defining a Norm for Systems
   - Relation to Model Reduction
3. Introduction to Model Reduction

Norms of Signals and Systems
Basic Notions of Norms

Ingredients of a normed space $(V, \|\cdot\|)$:

- A linear space V over $\mathbb{C}$ (or $\mathbb{R}$)
- and a functional

$$\|\cdot\|\colon V \to \mathbb{R}$$

  that has the following properties:

  i) $\|\alpha v\| = |\alpha|\,\|v\|$,
  ii) $\|v + w\| \leq \|v\| + \|w\|$, and
  iii) $\|v\| \geq 0$ and $\|v\| = 0$ if, and only if, $v = 0$,

  for any v, $w \in V$ and any $\alpha \in \mathbb{C}$ (or $\mathbb{R}$).

Section
Norms of Linear Operators

If $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$, then for the space of linear maps $\big(V \to W\big)$ a norm is defined via

$$\|G\|_* := \sup_{v \in V, v \neq 0} \frac{\|Gv\|_W}{\|v\|_V}.$$

This is the norm for $G\colon V \to W$ that is induced by $\|\cdot\|_V$ and $\|\cdot\|_W$.
There can be other norms that are not induced.

Norms of Signals and Systems
Norms of Signals

Common norms and spaces for the input or output signals

$$u\colon [0,\infty) \to \mathbb{R}^m \quad \text{or} \quad y\colon [0,\infty) \to \mathbb{R}^q$$

- All definitions work similar for finite time intervals $[0, T]$ or the whole time axis $(-\infty, \infty)$.
- Where it is clear from the context, we will drop the superscripts p and m that denote the dimension of the signals.

Norms of Signals and Systems

Norms of Signals

Definition

The $\mathbf{L}_1^m$ norm

$$\|u\|_{\mathbf{L}_1} := \int_0^\infty \sum_{i=1}^m |u_i(t)| \, \mathrm{d}t$$

defines the $\mathbf{L}_1^m$ space of integrable (summable) functions

$$\mathbf{L}_1^m := \left\{ u\colon [0,\infty) \to \mathbb{R}^m : \|u\|_{\mathbf{L}_1} < \infty \right\}$$

on the positive time axis.

Norms of Signals and Systems

Norms of Signals

Definition

The $\mathbf{L}_\infty^m$ norm

$$\|u\|_{\mathbf{L}_\infty} := \max_{i=\{1,\dots,m\}} \sup_{t>0} |u_i(t)|$$

defines the $\mathbf{L}_\infty^m$ space of bounded functions

$$\mathbf{L}_\infty^m := \{u\colon [0,\infty) \to \mathbb{R}^m : \|u\|_{\mathbf{L}_\infty} < \infty\}.$$

Definition

The $\mathbf{L}_2^q$ norm

$$\|y\|_{\mathbf{L}_2} := \Big(\int_0^\infty \sum_{i=1}^q |y_i(t)|^2 \, \mathrm{d}t\Big)^{\frac{1}{2}}$$

defines the $\mathbf{L}_2^q$ space of square integrable functions

$$\mathbf{L}_2^q := \{y\colon [0,\infty) \to \mathbb{R}^q : \|y\|_{\mathbf{L}_2} < \infty\}$$

Norms of Signals and Systems

Norms of Signals

The $\mathbf{L}_2$ norm can also be evaluated in frequency domain

Theorem

For $u \in \mathbf{L}_2$ it holds that

$$\|u\|_{\mathbf{L}_2} = \Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} U(i\omega)^* U(i\omega)\,\mathrm{d}\omega\Big)^{\frac{1}{2}},$$

where U is the Fourier transform of u.

The Fourier transform $\mathcal{F}$ and the Laplace transform $\mathcal{L}$ coincide for $s = i\omega$, $\omega \in \mathbb{R}$ and $u(t) = 0$ for $t \leq 0$:

$$\mathcal{F}(u)(i\omega) := \int_{-\infty}^{\infty} u(t)e^{-i\omega t}\,\mathrm{d}t = \int_{0}^{\infty} u(t)e^{-st}\,\mathrm{d}t = \mathcal{L}(u)(s)$$

Norms of Signals and Systems

Norm of a System

A system G or (A, B, C, D) transfers inputs to outputs.

Ask yourself...

- What does a norm mean for a system?
- What is a large system, what is a small system?

Norms of Signals and Systems

Norm of a System

From the definition of an operator norm:

$$\|G\| = \sup_{u \neq 0} \frac{\|Gu\|}{\|u\|}$$

we derive that for all u:

$$\|y\| = \|Gu\| \leq \|G\|\|u\|.$$

An Answer

For systems, large refers to what extend an input is amplified.
Therefore, $\|G\|$ is often called the *gain*.

Norms of Signals and Systems

Norm of a System

From the definition of an operator norm:

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With a norm, one can compare two systems G_1 and G_2 via the difference in the output for the same input:

$$\|y_1 - y_2\| = \|G_1 u - G_2 u\| \leq \|G_1 - G_2\|\|u\|.$$

Norms of Signals and Systems

Defining a Norm for Systems

We consider a SISO system $(A, B, C, -)$, i.e $m = q = 1$ and $D = 0$.

Consider $(A, B, C, -)$ a with stable and strictly proper transfer function G is stable. Then the *impulse response* of the system

$$g(t) = C \int_0^t e^{A(t-\tau)} B\delta(\tau) \, \mathrm{d}s = Ce^{At}B$$

decays exponentially and

$$\|g\|_{\mathbf{L}_2} = \Big(\frac{1}{2\pi} \int_{-\infty}^{\infty} G(i\omega)^* G(i\omega) \, \mathrm{d}\omega\Big)^{\frac{1}{2}} =: \|G\|_2 < \infty.$$

A system (A, B, C, D) or A is stable, if there exists a $\lambda > 0$, such that $\|e^A t\| \leq e^{-\lambda t}$, for $t > 0$. This means that all eigenvalues of A must have a negative real part.

Tmpulse response: $\delta(\tau) := \begin{cases} 0, \text{ if } t \neq 0, \\ \text{very large, if } t = 0 \end{cases}$ so that $\int_{-\infty}^{\infty} u(\tau)\delta(\tau) \, \mathrm{d}\tau = u(0)$.

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This defines a norm for systems since (Exercise!)

- $G = C(sI - A)^{-1}B$ is indeed the Laplace transform of g
- the functional $\|\cdot\|_2$ for stable and strictly proper transferfunctions is a norm

Furthermore, $\|y\|_{\mathbf{L}_\infty} \leq \|G\|_2 \|u\|_{\mathbf{L}_\infty}$. (Exercise!)

Norms of Signals and Systems

Defining a Norm for Systems

For MIMO systems $(A, B, C, -)$ with $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$, with a stable and strictly proper transferfunction $\mathcal{G}\colon s \to \mathbb{R}^{q\times m}$, the $\mathcal{H}_2$ norm is defined as

$$\|G\|_2 := \Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\left(G(i\omega)^* G(i\omega)\right)\, \mathrm{d}\omega\Big)^{\frac{1}{2}}.$$

Fact

This is the norm of the *Hardy* space $\mathcal{H}_2$ of matrix functions that are analytic in the open right half of the complex plane. Stable and strictly proper transfer functions are in $\mathcal{H}_2$.

Norms of Signals and Systems

Defining a Norm for Systems

For a stable and proper transfer function one can define the $\mathcal{H}_\infty$ norm:

$$\|G\|_\infty := \sup_{\omega\in\mathbb{R}} \sigma_{\max}\big(G(i\omega)\big),$$

where $\sigma_{\max}\big(G(i\omega)\big)$ is the largest singular value of $G(i\omega)$.

Fact 1

This is the norm of the *Hardy* space $\mathcal{H}_\infty$ of matrix functions that are analytic in the open right half of the complex plane and bounded on the imaginary axis. Stable and strictly proper transfer functions are in $\mathcal{H}_\infty$.

Fact 2

The $\mathcal{H}_\infty$-norm is induced by the $\mathbf{L}_2$ norm:

$$\|G\|_\infty = \sup_{u\in\mathbf{L}_2,u\neq 0} \frac{\|Gu\|_{\mathbf{L}_2}}{\|u\|_{\mathbf{L}_2}}.$$

Relation to Model Reduction

Approximation Problems - Model Reduction

Output errors in time-domain

Comparing the original system G and the reduced system $\hat{G}$:

$$\begin{aligned} \|y-\hat{y}\|_2 &\leq \left\|G-\hat{G}\right\|_\infty \|u\|_2 &\implies \left\|G-\hat{G}\right\|_\infty < \mathrm{tol} \\ \|y-\hat{y}\|_\infty &\leq \left\|G-\hat{G}\right\|_2 \|u\|_2 &\implies \left\|G-\hat{G}\right\|_2 < \mathrm{tol} \end{aligned}$$

Relation to Model Reduction

Approximation Problems - Model Reduction

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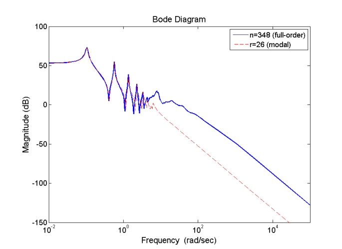
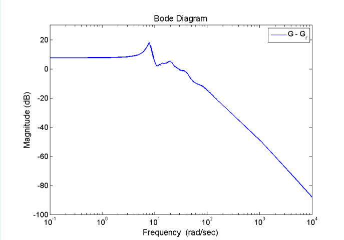
$\mathcal{H}_\infty$-norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable $\mathcal{H}_\infty$-norm bound.
$\mathcal{H}_2$-norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)
Hankel-norm $\|\|G\|\|_H := \sigma_{\max}$	optimal Hankel norm approximation (AAK theory).

Evaluating system norms is computationally very (sometimes too) expensive.

Other measures

- absolute errors $\left|\left|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right|\right|_2$, $\left|\left|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right|\right|_\infty$ $(j = 1, \ldots, N_\omega)$;
- relative errors $\frac{\left|\left|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right|\right|_2}{\left|\left|G(\jmath\omega_j)\right|\right|_2}$, $\frac{\left|\left|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right|\right|_\infty}{\left|\left|G(\jmath\omega_j)\right|\right|_\infty}$;
- "eyeball norm", i.e. look at frequency response/Bode (magnitude) plot: for SISO system, log-log plot frequency vs. $|G(\jmath\omega)|$ (or $|G(\jmath\omega) - \hat{G}(\jmath\omega)|$) in decibels, 1 dB $\simeq 20\log_{10}$(value).





Outline

Introduction to Model Reduction

Model Reduction — Abstract Definition

Problem

Given a model of a physical problem with dynamics described by the states $x(t) \in \mathbb{R}^n$, where n is the dimension of the state space.

The dimension n is large because $x(t)$ typically contains information that

- *is (almost) redundant,*
- *not (really) important,*
- *or not (really) of interest.*

We want to adjust the model such that the new state is of small dimension but still bears all important and interesting information.

This is the task of model reduction (also: dimension reduction, order reduction).

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Introduction to Model Reduction

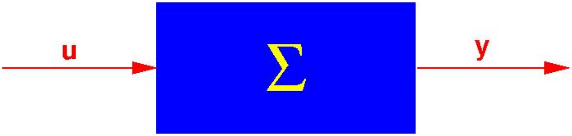
Model Reduction for Dynamical Systems

Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) & = & f(t, x(t), u(t)), \quad x(t_0) = x_0, \\ y(t) & = & g(t, x(t), u(t)) \end{cases}$$

with

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^q$.



Model Reduction for Dynamical Systems

Original System

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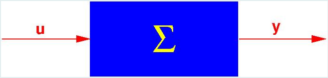

Reduced-Order Model (ROM)

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
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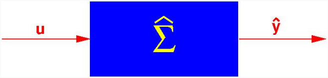

Goal:

$||y - \hat{y}|| <$ tolerance $\cdot \, ||u||$ for all admissible input signals.

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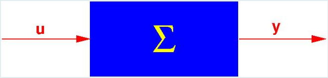

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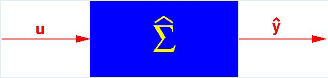

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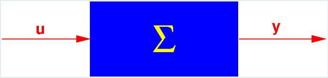

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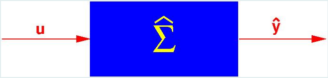

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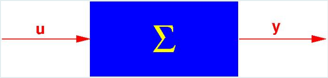

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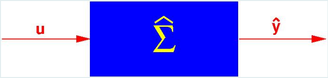

Goal:

$||y - \hat{y}|| <$ tolerance $\cdot\ ||u||$ for all admissible input signals.

Secondary goal: reconstruct approximation of x from $\hat{x}$.

Model Reduction for Dynamical Systems

Linear Systems

Linear, Time-Invariant (LTI) Systems

$$\begin{array}{rclcll} E\dot{x} & = & f(t,x,u) & = & Ax + Bu, \quad E, A \in \mathbb{R}^{n\times n}, & B \in \mathbb{R}^{n\times m}, \\ y & = & g(t,x,u) & = & Cx + Du, \quad C \in \mathbb{R}^{q\times n}, & D \in \mathbb{R}^{q\times m}. \end{array}$$

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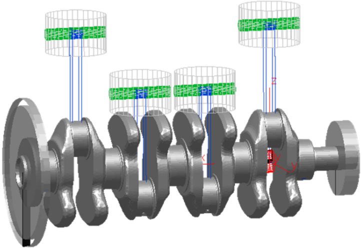
Linear, Time-Invariant Parametric Systems

$$\begin{array}{rcl} E(p)\dot{x}(t;p) & = & A(p)x(t;p)+B(p)u(t),\\ y(t;p) & = & C(p)x(t;p)+D(p)u(t), \end{array}$$

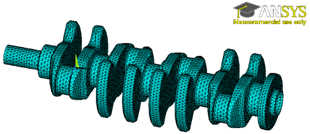
where $A(p), E(p) \in \mathbb{R}^{n\times n}, B(p) \in \mathbb{R}^{n\times m}, C(p) \in \mathbb{R}^{q\times n}, D(p) \in \mathbb{R}^{q\times m}$.

Application Areas

Structural Mechanics / Finite Element Modeling **since $\sim$1960ies**



$\rightsquigarrow$



- Resolving complex 3D geometries $\Rightarrow$ millions of degrees of freedom.
- Analysis of elastic deformations requires many simulation runs for varying external forces.

Standard MOR techniques in structural mechanics: modal truncation, combined with Guyan reduction (static condensation) $\rightsquigarrow$ Craig-Bampton method.

Application Areas

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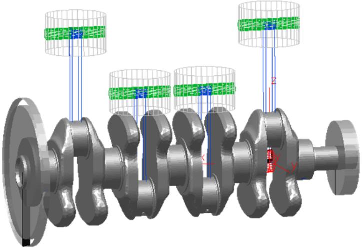

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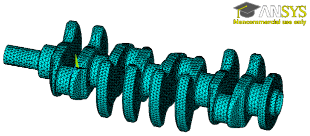

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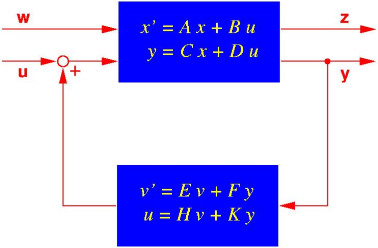
(Optimal) Control **since ∼1980ies**

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input = output of plant,
- output = input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$.



Practical controllers require small N ($N \sim 10$, say) due to

- real-time constraints,
- increasing fragility for larger N.

$\Longrightarrow$ reduce order of plant (n) and/or controller (N).

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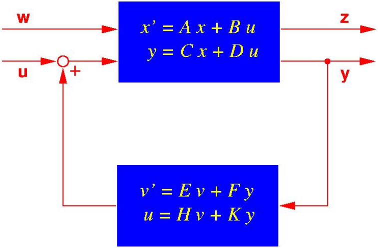

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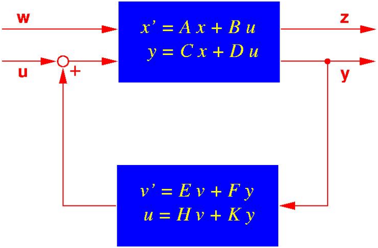

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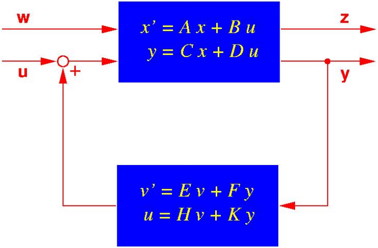

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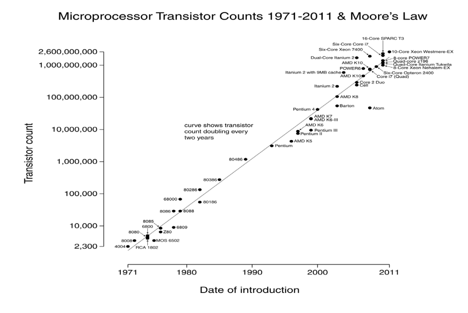
Micro Electronics/Circuit Simulation — since ∼1990ies

Progressive miniaturization

- Verification of VLSI/ULSI chip design needs a large number of simulations.
- **Moore's Law (1965/75)** states that the number of on-chip transistors doubles each 24 months.



Source: http://en.wikipedia.org/wiki/File:Transistor_Count_and_Moore'sLaw_-_2011.svg

Application Areas

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- Increase in packing density and multilayer technology requires modeling of interconnect to ensure that thermic/electro-magnetic effects do not disturb signal transmission.

Intel 4004 (1971)	Intel Core 2 Extreme (quad-core) (2007)
1 layer, 10μ technology	9 layers, $45nm$ technology
2,300 transistors	$> 8,200,000$ transistors
64 kHz clock speed	> 3 GHz clock speed.

Application Areas

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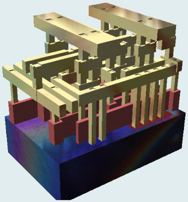

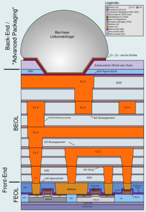

Source: http://en.wikipedia.org/wiki/Image:Silicon_chip_3d.png.

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- Here: mostly MOR for linear systems, they occur in micro electronics through modified nodal analysis (MNA) for RLC networks. e.g., when
  - decoupling large linear subcircuits,
  - modeling transmission lines,
  - modeling pin packages in VLSI chips,
  - modeling circuit elements described by Maxwell's equation using partial element equivalent circuits (PEEC).

Application Areas

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⇝ Clear need for model reduction techniques in order to facilitate or even enable circuit simulation for current and future VLSI design.

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Standard MOR techniques in circuit simulation:
Krylov subspace / Padé approximation / rational interpolation methods.

Application Areas

Many other disciplines in computational sciences and engineering like

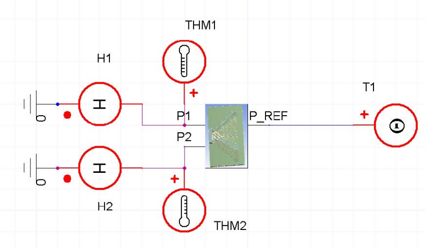
- computational fluid dynamics (CFD),
- computational electromagnetics,
- chemical process engineering,
- design of MEMS/NEMS (micro/nano-electrical-mechanical systems),
- computational acoustics,
- . . .

Motivating Examples

Electro-Thermic Simulation of Integrated Circuit (IC)

[Source: Evgenii Rudnyi, CADFEM GmbH]

- SIMPLORER® test circuit with 2 transistors.



- Conservative thermic sub-system in SIMPLORER: voltage $\rightsquigarrow$ temperature, current $\rightsquigarrow$ heat flow.
- Original model: $n = 270.593$, $m = q = 2 \Rightarrow$ Computing time (on Intel Xeon dualcore 3GHz, 1 Thread):
  - Main computational cost for set-up data $\approx 22min$.
  - Computation of reduced models from set-up data: 44–49sec. ($r = 20$–70).
  - Bode plot (MATLAB on Intel Core i7, 2,67GHz, 12GB): 7.5h for original system, $<$ 1min for reduced system.
  - Speed-up factor: 18 including / $\geq$ 450 excluding reduced model generation!

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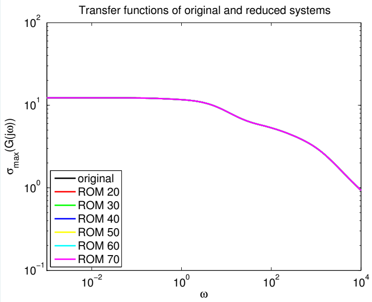
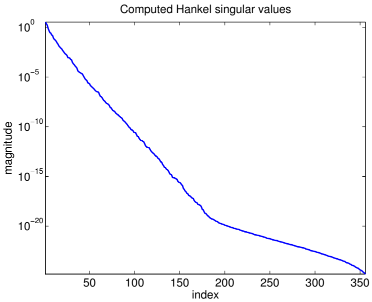
Bode Plot (Amplitude)



Hankel Singular Values



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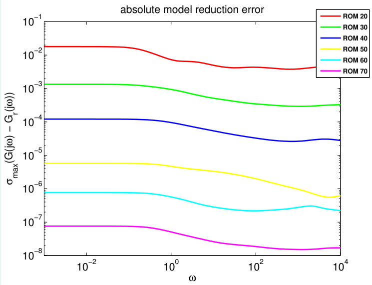
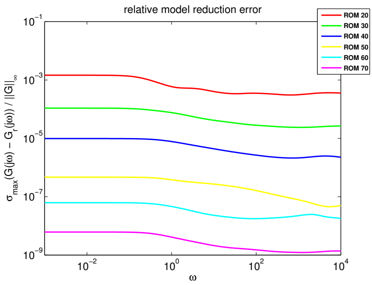
Absolute Error



Relative Error



Motivating Examples

A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

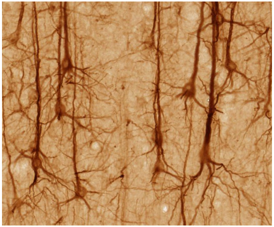
- Simple model for neuron (de-)activation [Chaturantabut/Sorensen 2009]

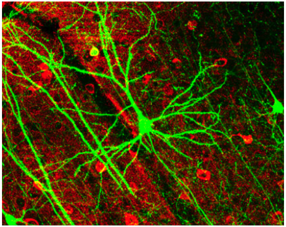
$$\begin{aligned} \epsilon v_t(x,t) &= \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + g, \\ w_t(x,t) &= hv(x,t) - \gamma w(x,t) + g, \end{aligned}$$

with $f(v) = v(v-0.1)(1-v)$ and initial and boundary conditions

$$\begin{aligned} v(x,0) &= 0, & w(x,0) &= 0, & x &\in [0,1] \\ v_x(0,t) &= -i_0(t), & v_x(1,t) &= 0, & t &\geq 0, \end{aligned}$$

where $\epsilon = 0.015, h = 0.5, \gamma = 2, g = 0.05, i_0(t) = 50000t^3 \exp(-15t)$.





Source: http://en.wikipedia.org/wiki/Neuron

Motivating Examples

A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

- Simple model for neuron (de-)activation [CHATURANTABUT/SORENSEN 2009]

$$\begin{aligned} \epsilon v_t(x,t) &= \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + g, \\ w_t(x,t) &= hv(x,t) - \gamma w(x,t) + g, \end{aligned}$$

  with $f(v) = v(v-0.1)(1-v)$ and initial and boundary conditions

$$\begin{aligned} v(x,0) &= 0, & w(x,0) &= 0, & x &\in [0,1] \\ v_x(0,t) &= -i_0(t), & v_x(1,t) &= 0, & t &\geq 0, \end{aligned}$$

  where $\epsilon = 0.015, h = 0.5, \gamma = 2, g = 0.05, i_0(t) = 50000t^3 \exp(-15t)$.
- Parameter g handled as an additional input.
- Original state dimension $n = 2 \cdot 400$, QBDAE dimension $N = 3 \cdot 400$, reduced QBDAE dimension $r = 26$, chosen expansion point $\sigma = 1$.

Motivating Examples

A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

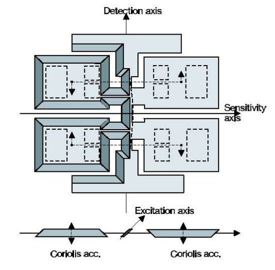
Motivating Examples

Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

- Voltage applied to electrodes induces vibration of wings, resulting rotation due to Coriolis force yields sensor data.
- FE model of second order:
  $N = 17.361 \rightsquigarrow n = 34.722,\ m = 1,\ q = 12.$
- Sensor for position control based on acceleration and rotation.

- Application: inertial navigation.



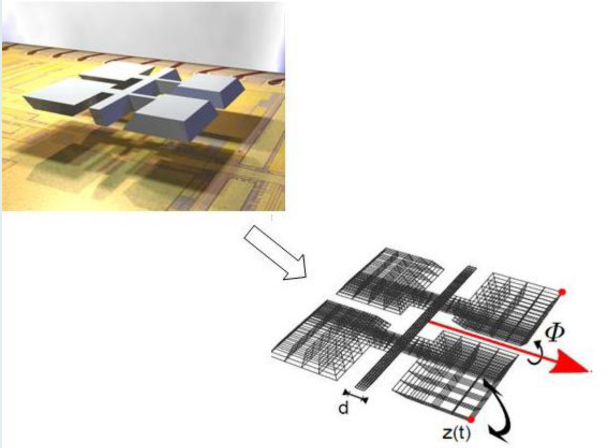
Source: The Oberwolfach Benchmark Collection http://www.imtek.de/simulation/benchmark

Motivating Examples

Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Parametric FE model: $M(d)\ddot{x}(t) + D(\Phi, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t)$.



Motivating Examples

Parametric MOR: Applications in Microsystems/MEMS Design

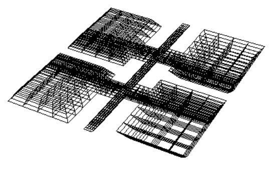
Microgyroscope (butterfly gyro)

Parametric FE model:

$$M(d)\ddot{x}(t) + D(\Phi, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t),$$

wobei

$$\begin{aligned} M(d) &= M_1 + dM_2, \\ D(\Phi, d, \alpha, \beta) &= \Phi(D_1 + dD_2) + \alpha M(d) + \beta T(d), \\ T(d) &= T_1 + \frac{1}{d}T_2 + dT_3, \end{aligned}$$



with

- width of bearing: d,
- angular velocity: Φ,
- Rayleigh damping parameters: α, β.

Motivating Examples

Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Original... and reduced-order model.