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Introductory Course on Model Reduction of Linear Time Invariant Systems

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Outline

1 Linear Time Invariant Systems

Typical Situation

- Fry a steak
- The cook controls the heat at the fireplace
- and observes the process, e.g. via measuring the temperature in the inner

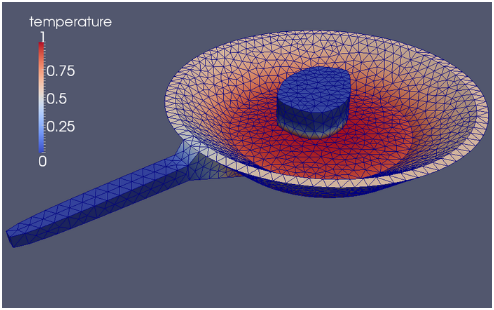
Typical Situation

- The model

$$
\begin{aligned}
\dot{\theta} &= \nabla \cdot (\nu \nabla \theta) \quad &&\text{in } (0, \infty) \times \Omega, \\
\theta &= u, &&\text{at the plate,} \\
\theta(0) &= 0.
\end{aligned}
$$

- The cook controls the heat at the fireplace, which we denote by u
- and observes the process, e.g. he measures the temperature y in the center: $y = f(\theta)$.

Simulation



- The model:

$$\begin{aligned} \dot{\theta} &= \nabla \cdot (\nu \nabla \theta), \\ \theta &= u, \\ \theta(0) &= 0. \end{aligned}$$

- The cook controls the heat u
- and observes the process via $y = f(\theta)$.

- A *Finite Element* discretization of the problem leads to the finite dimensional model

$$E\dot{\theta}(t) = A\theta(t) + Bu(t), \quad \theta(0) = 0, \tag{1}$$

$$y(t) = C\theta(t), \tag{2}$$

a linear time invariant system.

Linear State Space System

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{3a}$$
$$y(t) = Cx(t) + Du(t), \tag{3b}$$

with

- $E \in \mathbb{R}^{n \times n}$: the identity or the mass matrix
- $A \in \mathbb{R}^{n \times n}$: the system matrix
- $B \in \mathbb{R}^{n \times m}$: the input matrix
- $C \in \mathbb{R}^{q \times n}$: the output matrix
- $D \in \mathbb{R}^{q \times n}$: the throughput
- $x(t) \in \mathbb{R}^n$: the system's state
- $u(t) \in \mathbb{R}^m$: the input or control
- $y(t) \in \mathbb{R}^q$: the output or measurements
- n, m, $q \in \mathbb{N}$: the system dimensions

We will assume that $E = I$ and denote the LTI (3) by (A, B, C, D).

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Some Preliminary Thoughts

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

A simple question...

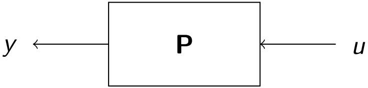
What is x?

- it is a physical state in the model – like the temperature
- in practise, we may not access it – only the measurement $y = Cx$
- it is but a mathematical object as a part of a model
- furthermore, as we will see later, the state x can be severely changed e.g. in the course of model reduction

The state x can be seen...
...as nothing but an artificial object of the model for the input to output behavior

$$\mathbf{G}\colon u \mapsto y$$

of an abstract system $\mathbf{P}$:



that maps an input u to the corresponding output y.

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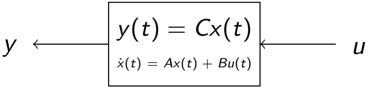

that maps an input u to the corresponding output y.

Transfer Function in Time-Domain

If **P** is modelled trough an (A, B, C, D) system, then the function **G** can be defined via

$$\mathbf{G}\colon u \mapsto y\colon y(t) = C\left[e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,\mathrm{d}s\right] + Du(t),$$

known as the formula of *variation of constants*.

This is in **time-domain**: A function u depending on time $t \in [0, \infty)$ is mapped onto a function y depending on time $t \in [0, \infty)$.

Introducing Frequency-Domain

Through the Laplace transform $\mathcal{L}$ and its inverse $\mathcal{L}^{-1}$, we can switch between time-domain and frequency-domain representations of the input and output signals:

$$U(s) := \mathcal{L}\{u\}(s) := \int_0^\infty e^{-st} u(t)\, \mathrm{d}t,$$

where $s \in \mathbb{C}$ is the *frequency* and

$$y(t) := \mathcal{L}^{-1}\{Y\}(t) := \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\gamma - iT}^{\gamma + iT} e^{s} Y(s)\, \mathrm{d}s$$

where $\gamma \in \mathbb{R}$ is chosen such that the contour path of the integration is the domain of convergence of Y.

Laplace Transform of an LTI

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

With the basic properties of the Laplace transform

- $\dot{X}(s) := \mathcal{L}\{\dot{x}\}(s) - x(0) = s\mathcal{L}\{x\}(s) = sX(s) - x(0)$
- and linearity $\mathcal{L}\{Ax\}(s) = AX(s)$

with zero initial value $x(0) = 0$, the (A, B, C, D) system defines the transfer function

$$G(s) := C(sI - A)^{-1}B + D$$

in frequency domain.

Realizations

Fact

An LTI (A, B, C, D) always defines a transfer function

$$G(s) = C(sI - A)^{-1}B + D$$

which is a matrix $G \in \mathbb{R}^{q\times m}$ with coefficients that are rational functions of s.

Question

Given a rational matrix function $s \mapsto G(s) \in \mathbb{R}^{q\times m}$, is there an

$$(A, B, C, D)$$

system, so that $G(s) = C(sI - A)^{-1}B + D$?

Realizations

given G, find (A, B, C, D),
$G(s) = C(sI - A)^{-1}B + D$

If there is one such (A, B, C, D), then there are infinitely many:

- For $T \in \mathbb{R}^{n\times n}$ invertible, also $(TAT^{-1}, TB, CT^{-1}, D)$ is a realization:

$$C(sI - A)^{-1}B + D = CT^{-1}(sI - TAT^{-1})^{-1}TB + D.$$

- Moreover, also

$$\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D\right)$$

is a realization of G.

Realizations

Facts and Thoughts on Realizations

- If G is *proper*, then there is a realization (A, B, C, D) as a state space system.
- This realization is by no means unique.
- The dimension of the state can be arbitrary large. What is the smallest possible dimension? (cf. *model reduction*)
- What is a good choice for the state?

Remark: A transfer function $G\colon s \mapsto G(s) \in \mathbb{R}^{q\times m}$ with coefficients that are rational functions in s, is *proper*, if in each coefficient the polynomial degree of the numerators does not exceed the degree of denominators.

Controllability and Observability

Based on the previous considerations, we can say that

- The states of an LTI system (A, B, C, D) are just a part of a model that realizes a transfer function G
- The transfer function G describes how controls u lead to outputs y
- As seen above in the example, there can be states that are neither affected (*controlled*) by the inputs nor seen (*observed*) by the outputs
- These states are obviously not needed to realize the input to output behavior of G.

We will give a thorough characterization of the *controllable* and *observable* states of an LTI.

Controllability

Definition

The LTI (A, B, C, D) or the pair (A, B) is said to be *controllable* if, for any initial state $x(0) = x_0$, $t_1 > 0$ and final state x_1, there exists a (piecewise continuous) input u such that the solution of (3) satisfies $x(t_1) = x_1$. Otherwise, the system (A, B, C, D) or the pair (A, B) is said to be *uncontrollable*.

Theorem

The following statements are equivalent:

(i.) *The pair (A, B) is controllable.*

(ii.) *The controllability matrix $\mathcal{C} := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$ has full rank.*

(iii.) *The matrix $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ has full rank for all $\lambda \in \mathbb{C}$.*

Observability

Definition

The LTI (A, B, C, D) or the pair (C, A) is said to be *observable* if, for any $t_1 > 0$, the initial state $x(0) = x_0$ can be determined from the time history of the input u and the output y in the interval of $[0, t_1]$. Otherwise, the system (A, B, C, D), or (C, A), is said to be *unobservable*.

Observability is the dual concept of controllability:

Theorem

The pair (C, A) is observable if and only if the pair $(A^{\mathsf{T}}, C^{\mathsf{T}})$ is controllable.

Invariance Under State Space Transformation

Theorem

The LTI (A, B, C, D) is controllable (observable) if, and only if, the transformed LTI $(TAT^{-1}, TB, CT^{-1}, D)$ is controllable (observable), where T is a regular matrix.

- Recall that also a transfer function is invariant with respect to state space transformations on its realization.
- Next, we find the states that are at least necessary for the realization of a transfer function...

Theorem (Kalman Canonical Decomposition)

Given an LTI (A, B, C, D), there is a state space transformation T such that the transformed system $(TAT^{-1}, TB, CT^{-1}, D)$ has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + Du,$$

with the subsystem $(A_{co}, B_{co}, C_{co}, D)$ being controllable and observable, while the remaining states $x_{\bar{c}o}$, $x_{c\bar{o}}$, or $x_{\bar{c}\bar{o}}$ are not controllable, not observable, or neither of them.

For a constructive proof of the Theorem, see Ch. 3.3 of [Zhou, Doyle, Glover '96]

Outcomes of the Kalman Decomposition

For any state space system (A, B, C, D), there is a transformation T so that the transformed states $T^{-1}x$ decompose into

- x_{co} - controllable and observable
- $x_{c\bar{o}}$ - controllable but not observable
- $x_{\bar{c}o}$ - observable but not controllable
- $x_{\bar{c}\bar{o}}$ - not observable and not controllable

Moreover, for the transfer function, it holds that

$$G(s) = C(sI - A)^{-1}B = C_{co}(sI - A_{co})^{-1}B_{co}.$$

Conclusion from the Kalman Decomposition

What does this mean for us and a transfer function $G(s)$?

- The minimal dimension of a realization is the dimension of x_{co} in the *Kalman Canonical Decomposition*
- Such a realization is called minimal realization
- It is the starting point for further model reduction. (Throwing out $x_{\bar{c}o}$ etc. does not effect $G(s)$ and is typically not considered a model reduction)
- There are algorithm to reduce a realization to a minimal one, cf. [Varga '90].
- In practice, the uncontrolled and unobserved states play a role and they may cause troubles. (check the literature for zero dynamics)

Summary

- LTI as model for physical processes (e.g. heat transfer)
- The input/output is often more important than the state
- Moreover, the state need not have a meaning
- State space systems (A, B, C, D) can be seen as realizations of transfer functions
- A transfer function has multiple realizations
- The minimal realizations are of our interest

More on the LTI topics

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