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Introductory Course on Model Reduction of Linear Time Invariant Systems

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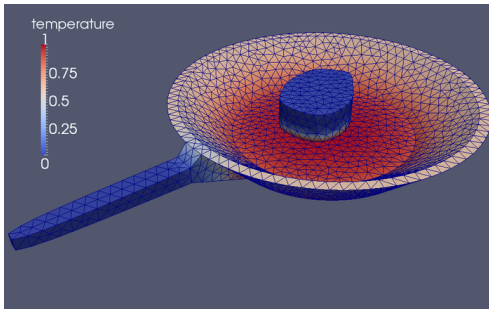
Outline

- 1 Linear Time Invariant Systems
- 2 Norms of Signals and Systems
- 3 Introduction to Model Reduction
- 4 Model Reduction by Projection

- Fry a steak
- The cook controls the heat at the fireplace
- and observes the process, e.g. via measuring the temperature in the inner

- The cook controls the heat at the fireplace, which we denote by u
- and observes the process, e.g. he measures the temperature y in the center: $y = f(\theta)$.

Simulation



- The model:

$$\dot{\theta} = \nabla \cdot (\nu \nabla \theta),$$

$$\theta = u,$$

$$\theta(0) = 0.$$

- The cook controls the heat u
- and observes the process via $y = f(\theta)$.

- A *Finite Element* discretization of the problem leads to the finite dimensional model

$$E\dot{\theta}(t) = A\theta(t) + Bu(t), \quad \theta(0) = 0, \quad (1)$$

$$y(t) = C\theta(t), \quad (2)$$

a linear time invariant system.

Linear State Space System

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad (3a)$$

$$y(t) = Cx(t) + Du(t), \quad (3b)$$

with

- $E \in \mathbb{R}^{n \times n}$: the identity or the mass matrix
- $A \in \mathbb{R}^{n \times n}$: the system matrix
- $B \in \mathbb{R}^{n \times m}$: the input matrix
- $C \in \mathbb{R}^{q \times n}$: the output matrix
- $D \in \mathbb{R}^{q \times n}$: the throughput
- $x(t) \in \mathbb{R}^n$: the system's state
- $u(t) \in \mathbb{R}^m$: the input or control
- $y(t) \in \mathbb{R}^q$: the output or measurements
- $n, m, q \in \mathbb{N}$: the system dimensions

We will assume that $E = I$ and denote the LTI (3) by (A, B, C, D) .

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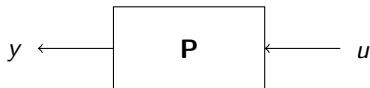
What is x ?

- it is a physical state in the model – like the temperature
- in practise, we may not access it – only the measurement $y = Cx$
- it is but a mathematical object as a part of a model
- furthermore, as we will see later, the state x can be severely changed e.g. in the course of model reduction

...as nothing but an artificial object of the model for the input to output behavior

$$\mathbf{G}: u \mapsto y$$

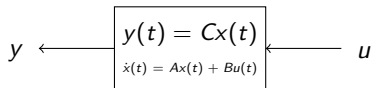
of an abstract system **P**:



that maps an input u to the corresponding output y .

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Realizations

Fact

An LTI (A, B, C, D) always defines a transfer function

$$G(s) = C(sI - A)^{-1}B + D$$

which is a matrix $G \in \mathbb{R}^{q \times m}$ with coefficients that are rational functions of s .

Question

Given a rational matrix function $s \mapsto G(s) \in \mathbb{R}^{q \times m}$, is there an

$$(A, B, C, D)$$

system, so that $G(s) = C(sI - A)^{-1}B + D$?

Realizations

Facts and Thoughts on Realizations

- If G is *proper*, then there is a realization (A, B, C, D) as a state space system.
- This realization is by no means unique.
- The dimension of the state can be arbitrary large. What is the smallest possible dimension? (cf. *model reduction*)
- What is a good choice for the state?

Remark: A transfer function $G: s \mapsto G(s) \in \mathbb{R}^{q \times m}$ with coefficients that are rational functions in s , is *proper*, if in each coefficient the polynomial degree of the numerators does not exceed the degree of denominators.

Controllability and Observability

Based on the previous considerations, we can say that

- The states of an LTI system (A, B, C, D) are just a part of a model that realizes a transfer function G
- The transfer function G describes how controls u lead to outputs y
- As seen above in the example, there can be states that are neither affected (*controlled*) by the inputs nor seen (*observed*) by the outputs
- These states are obviously not needed to realize the input to output behavior of G .

We will give a thorough characterization of the *controllable* and *observable* states of an LTI.

Controllability

Definition

The LTI (A, B, C, D) or the pair (A, B) is said to be *controllable* if, for any initial state $x(0) = x_0$, $t_1 > 0$ and final state x_1 , there exists a (piecewise continuous) input u such that the solution of (3) satisfies $x(t_1) = x_1$. Otherwise, the system (A, B, C, D) or the pair (A, B) is said to be *uncontrollable*.

Theorem

The following statements are equivalent:

- (i.) *The pair (A, B) is controllable.*
- (ii.) *The controllability matrix $\mathcal{C} := [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$ has full rank.*
- (iii.) *The matrix $[A - \lambda I \quad B]$ has full rank for all $\lambda \in \mathbb{C}$.*

Observability

Definition

The LTI (A, B, C, D) or the pair (C, A) is said to be *observable* if, for any $t_1 > 0$, the initial state $x(0) = x_0$ can be determined from the time history of the input u and the output y in the interval of $[0, t_1]$. Otherwise, the system (A, B, C, D) , or (C, A) , is said to be *unobservable*.

Observability is the dual concept of controllability:

Theorem

The pair (C, A) is observable if and only if the pair (A^T, C^T) is controllable.

Invariance Under State Space Transformation

Theorem

The LTI (A, B, C, D) is controllable (observable) if, and only if, the transformed LTI $(TAT^{-1}, TB, CT^{-1}, D)$ is controllable (observable), where T is a regular matrix.

- Recall that also a transfer function is invariant with respect to state space transformations on its realization.
- Next, we find the states that are at least necessary for the realization of a transfer function...

Theorem (Kalman Canonical Decomposition)

Given an LTI (A, B, C, D) , there is a state space transformation T such that the transformed system $(TAT^{-1}, TB, CT^{-1}, D)$ has the form

$$\frac{d}{dt} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + Du,$$

with the subsystem $(A_{co}, B_{co}, C_{co}, D)$ being controllable and observable, while the remaining states $x_{\bar{c}o}$, $x_{c\bar{o}}$, or $x_{\bar{c}\bar{o}}$ are not controllable, not observable, or neither of them.

For a constructive proof of the Theorem, see Ch. 3.3 of [ZHOU, DOYLE, GLOVER '96]

Outcomes of the Kalman Decomposition

For any state space system (A, B, C, D) , there is a transformation T so that the transformed states $T^{-1}x$ decompose into

- x_{co} - controllable and observable
- $x_{c\bar{o}}$ - controllable but not observable
- $x_{\bar{c}o}$ - observable but not controllable
- $x_{\bar{c}\bar{o}}$ - not observable and not controllable

Moreover, for the transfer function, it holds that

$$G(s) = C(sI - A)^{-1}B = C_{co}(sI - A_{co})^{-1}B_{co}.$$

- LTI as model for physical processes (e.g. heat transfer)
- The input/output is often more important than the state
- Moreover, the state need not have a meaning
- State space systems (A, B, C, D) can be seen as realizations of transfer functions
- A transfer function has multiple realizations
- The minimal realizations are of our interest

More on the LTI topics



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- 1 Linear Time Invariant Systems
- 2 Norms of Signals and Systems
 - Norms
 - Norms of Signals
 - Norm of a System
 - Defining a Norm for Systems
 - Relation to Model Reduction
- 3 Introduction to Model Reduction
- 4 Model Reduction by Projection

Basic Notions of Norms

- A linear space V over \mathbb{C} (or \mathbb{R})
- and a functional

$$\|\cdot\|: V \rightarrow \mathbb{R}$$

that has the following properties:

- i) $\|\alpha v\| = |\alpha| \|v\|$,
- ii) $\|v + w\| \leq \|v\| + \|w\|$, and
- iii) $\|v\| \geq 0$ and $\|v\| = 0$ if, and only if, $v = 0$,

for any $v, w \in V$ and any $\alpha \in \mathbb{C}$ (or \mathbb{R}).

Section

Norms of Linear Operators

If $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$, then for the space of linear maps $(V \rightarrow W)$ a norm is defined via

$$\|G\|_* := \sup_{v \in V, v \neq 0} \frac{\|Gv\|_W}{\|v\|_V}.$$

This is the norm for $G: V \rightarrow W$ that is induced by $\|\cdot\|_V$ and $\|\cdot\|_W$.
There can be other norms that are not induced.

Norms of Signals and Systems

Norms of Signals

Common norms and spaces for the input or output signals

$$u: [0, \infty) \rightarrow \mathbb{R}^m \quad \text{or} \quad y: [0, \infty) \rightarrow \mathbb{R}^q$$

- All definitions work similar for finite time intervals $[0, T]$ or the whole time axis $(-\infty, \infty)$.
- Where it is clear from the context, we will drop the superscripts p and m that denote the dimension of the signals.

Norms of Signals and Systems

Norms of Signals

Definition

The \mathbf{L}_1^m norm

$$\|u\|_{\mathbf{L}_1} := \int_0^\infty \sum_{i=1}^m |u_i(t)| \, dt$$

defines the \mathbf{L}_1^m space of **integrable (summable) functions**

$$\mathbf{L}_1^m := \{u: [0, \infty) \rightarrow \mathbb{R}^m : \|u\|_{\mathbf{L}_1} < \infty\}$$

on the positive time axis.

Norms of Signals and Systems

Norms of Signals

Definition

The \mathbf{L}_∞^m norm

$$\|u\|_{\mathbf{L}_\infty} := \max_{i=\{1,\dots,m\}} \sup_{t>0} |u_i(t)|$$

defines the \mathbf{L}_∞^m space of **bounded functions**

$$\mathbf{L}_\infty^m := \{u: [0, \infty) \rightarrow \mathbb{R}^m : \|u\|_{\mathbf{L}_\infty} < \infty\}.$$

Definition

The \mathbf{L}_2^q norm

$$\|y\|_{\mathbf{L}_2} := \left(\int_0^\infty \sum_{i=1}^q |y_i(t)|^2 dt \right)^{\frac{1}{2}}$$

defines the \mathbf{L}_2^q space of **square integrable functions**

$$\mathbf{L}_2^q := \{y: [0, \infty) \rightarrow \mathbb{R}^q : \|y\|_{\mathbf{L}_2} < \infty\}$$

Norms of Signals and Systems

Norms of Signals

The \mathbf{L}_2 norm can also be evaluated in frequency domain

Theorem

For $u \in \mathbf{L}_2$ it holds that

$$\|u\|_{\mathbf{L}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} U(i\omega)^* U(i\omega) d\omega \right)^{\frac{1}{2}},$$

where U is the Fourier transform of u .

The Fourier transform \mathcal{F} and the Laplace transform \mathcal{L} coincide for $s = i\omega$, $\omega \in \mathbb{R}$ and $u(t) = 0$ for $t \leq 0$:

$$\mathcal{F}(u)(i\omega) := \int_{-\infty}^{\infty} u(t)e^{-i\omega t} dt = \int_0^{\infty} u(t)e^{-st} dt = \mathcal{L}(u)(s)$$

Norms of Signals and Systems

Norm of a System

A system G or (A, B, C, D) transfers inputs to outputs.

Ask yourself. . .

- What does a norm mean for a system?
- What is a large system, what is a small system?

Norms of Signals and Systems

Norm of a System

From the definition of an operator norm:

$$\|G\| = \sup_{u \neq 0} \frac{\|Gu\|}{\|u\|}$$

we derive that for all u :

$$\|y\| = \|Gu\| \leq \|G\| \|u\|.$$

An Answer

For systems, large refers to what extent an input is amplified. Therefore, $\|G\|$ is often called the *gain*.

Norm of a System

$$\|G\| = \sup_{u \neq 0} \frac{\|Gu\|}{\|u\|}$$
$$\|y\| = \|Gu\| \leq \|G\| \|u\|.$$
$$\|y_1 - y_2\| = \|G_1 u - G_2 u\| \leq \|G_1 - G_2\| \|u\|.$$

Norms of Signals and Systems

Defining a Norm for Systems

We consider a SISO system $(A, B, C, -)$, i.e. $m = q = 1$ and $D = 0$.

Consider $(A, B, C, -)$ a with stable and strictly proper transfer function G is stable. Then the *impulse response* of the system

$$g(t) = C \int_0^t e^{A(t-\tau)} B \delta(\tau) \, ds = C e^{At} B$$

decays exponentially and

$$\|g\|_{\mathbf{L}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} G(i\omega)^* G(i\omega) \, d\omega \right)^{\frac{1}{2}} =: \|G\|_2 < \infty.$$

A system (A, B, C, D) or A is stable, if there exists a $\lambda > 0$, such that $\|e^{At}\| \leq e^{-\lambda t}$, for $t > 0$. This means that all eigenvalues of A must have a negative real part.

Impulse response: $\delta(\tau) := \begin{cases} 0, & \text{if } t \neq 0, \\ \text{very large,} & \text{if } t = 0 \end{cases}$ so that $\int_{-\infty}^{\infty} u(\tau) \delta(\tau) \, d\tau = u(0)$.

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This defines a norm for systems since (Exercise!)

- $G = C(sI - A)^{-1}B$ is indeed the Laplace transform of g
- the functional $\|\cdot\|_2$ for stable and strictly proper transferfunctions is a norm

Furthermore, $\|y\|_{\mathbf{L}_\infty} \leq \|G\|_2 \|u\|_{\mathbf{L}_\infty}$. (Exercise!)

Norms of Signals and Systems

Defining a Norm for Systems

For MIMO systems $(A, B, C, -)$ with $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$, with a stable and strictly proper transferfunction $\mathcal{G}: s \rightarrow \mathbb{R}^{q \times m}$, the \mathcal{H}_2 norm is defined as

$$\|G\|_2 := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}(G(i\omega)^* G(i\omega)) \, d\omega \right)^{\frac{1}{2}}.$$

Fact

This is the norm of the *Hardy* space \mathcal{H}_2 of matrix functions that are analytic in the open right half of the complex plane. Stable and strictly proper transfer functions are in \mathcal{H}_2 .

Norms of Signals and Systems

Defining a Norm for Systems

For a stable and proper transfer function one can define the \mathcal{H}_∞ norm:

$$\|G\|_\infty := \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(i\omega)),$$

where $\sigma_{\max}(G(i\omega))$ is the largest singular value of $G(i\omega)$.

Fact 1

This is the norm of the *Hardy* space \mathcal{H}_∞ of matrix functions that are analytic in the open right half of the complex plane and bounded on the imaginary axis. Stable and strictly proper transfer functions are in \mathcal{H}_∞ .

Fact 2

The \mathcal{H}_∞ -norm is induced by the \mathbf{L}_2 norm:

$$\|G\|_\infty = \sup_{u \in \mathbf{L}_2, u \neq 0} \frac{\|Gu\|_{\mathbf{L}_2}}{\|u\|_{\mathbf{L}_2}}.$$

Approximation Problems - Model Reduction

Comparing the original system G and the reduced system \hat{G} :

$$\|y - \hat{y}\|_\infty \leq \|G - \hat{G}\|_2 \|u\|_2 \implies \|G - \hat{G}\|_2 < \text{tol}$$

Relation to Model Reduction

Approximation Problems - Model Reduction

Output errors in time-domain

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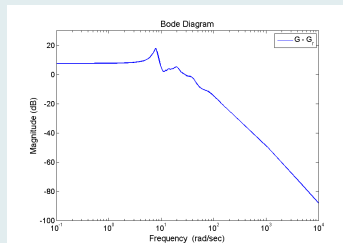
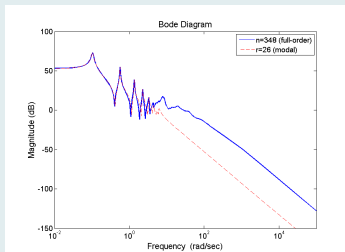
$$\|y - \hat{y}\|_\infty \leq \|G - \hat{G}\|_2 \|u\|_2 \implies \|G - \hat{G}\|_2 < \text{tol}$$

\mathcal{H}_∞ -norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable \mathcal{H}_∞ -norm bound.
\mathcal{H}_2 -norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)
Hankel-norm $\ G\ _H := \sigma_{\max}$	optimal Hankel norm approximation (AAK theory).

Evaluating system norms is computationally very (sometimes too) expensive.

Other measures

- absolute errors $\left\| G(j\omega_j) - \hat{G}(j\omega_j) \right\|_2, \left\| G(j\omega_j) - \hat{G}(j\omega_j) \right\|_\infty$
($j = 1, \dots, N_\omega$);
- relative errors $\frac{\left\| G(j\omega_j) - \hat{G}(j\omega_j) \right\|_2}{\left\| G(j\omega_j) \right\|_2}, \frac{\left\| G(j\omega_j) - \hat{G}(j\omega_j) \right\|_\infty}{\left\| G(j\omega_j) \right\|_\infty}$;
- "eyeball norm", i.e. look at **frequency response/Bode (magnitude) plot**:
for SISO system, log-log plot frequency vs. $|G(j\omega)|$ (or $|G(j\omega) - \hat{G}(j\omega)|$)
in decibels, $1 \text{ dB} \simeq 20 \log_{10}(\text{value})$.



Introduction to Model Reduction

Model Reduction — Abstract Definition

Problem

Given a model of a physical problem with dynamics described by the states $x(t) \in \mathbb{R}^n$, where n is the dimension of the state space.

The dimension n is large because $x(t)$ typically contains information that

- *is (almost) redundant,*
- *not (really) important,*
- *or not (really) of interest.*

We want to adjust the model such that the new state is of small dimension but still bears all important and interesting information.

This is the task of model reduction (also: dimension reduction, order reduction).

Introduction to Model Reduction

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Model Reduction for Dynamical Systems

$$\Sigma: \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)), & x(t_0) = x_0, \\ y(t) &= g(t, x(t), u(t)) \end{cases}$$

- **states** $x(t) \in \mathbb{R}^n$,
- **inputs** $u(t) \in \mathbb{R}^m$,
- **outputs** $y(t) \in \mathbb{R}^q$.



Model Reduction for Dynamical Systems

Original System

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Reduced-Order Model (ROM)

$$\hat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \hat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \hat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\hat{y}(t) \in \mathbb{R}^q$.



Goal:

$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\|$ for all admissible input signals.

Model Reduction for Dynamical Systems

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- outputs $y(t) \in \mathbb{R}^q$.



Reduced-Order Model (ROM)

$$\hat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \hat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \hat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\hat{y}(t) \in \mathbb{R}^q$.



Goal:

$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\|$ for all admissible input signals.

Linear Systems

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} E\dot{x} &= f(t, x, u) = Ax + Bu, & E, A &\in \mathbb{R}^{n \times n}, & B &\in \mathbb{R}^{n \times m}, \\ y &= g(t, x, u) = Cx + Du, & C &\in \mathbb{R}^{q \times n}, & D &\in \mathbb{R}^{q \times m}. \end{aligned}$$

Model Reduction for Dynamical Systems

Linear Systems

Linear, Time-Invariant (LTI) Systems

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Linear, Time-Invariant Parametric Systems

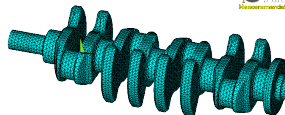
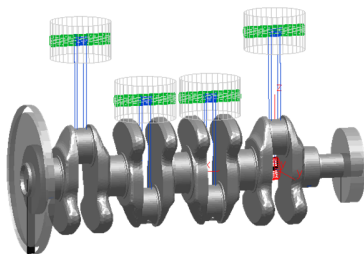
$$\begin{aligned} E(p)\dot{x}(t; p) &= A(p)x(t; p) + B(p)u(t), \\ y(t; p) &= C(p)x(t; p) + D(p)u(t), \end{aligned}$$

where $A(p), E(p) \in \mathbb{R}^{n \times n}$, $B(p) \in \mathbb{R}^{n \times m}$, $C(p) \in \mathbb{R}^{q \times n}$, $D(p) \in \mathbb{R}^{q \times m}$.

Application Areas

Structural Mechanics / Finite Element Modeling

since ~1960ies



 ANSYS
Noncommercial use only

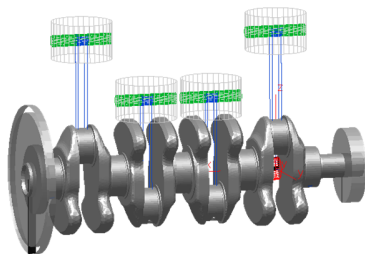
- Resolving complex 3D geometries \Rightarrow millions of degrees of freedom.
- Analysis of elastic deformations requires many simulation runs for varying external forces.

Standard MOR techniques in structural mechanics: modal truncation, combined with Guyan reduction (static condensation) \rightsquigarrow Craig-Bampton method.

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Application Areas

(Optimal) Control

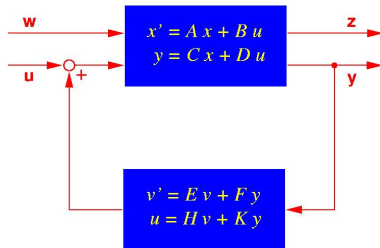
since ~1980ies

Feedback Controllers

A feedback controller (**dynamic compensator**) is a linear system of order N , where

- input = output of plant,
- output = input of plant.

Modern (LQG-/ \mathcal{H}_2 -/ \mathcal{H}_∞ -) control design: $N \geq n$.



Practical controllers require small N ($N \sim 10$, say) due to

- real-time constraints,
- increasing fragility for larger N .

\implies reduce order of plant (n) and/or controller (N).

Standard MOR techniques in systems and control: **balanced truncation** and related methods.

Application Areas

(Optimal) Control

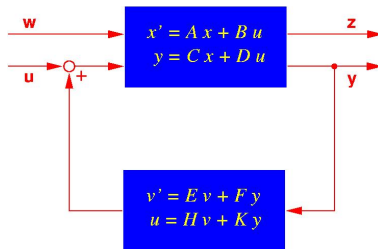
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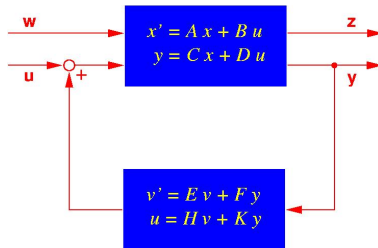
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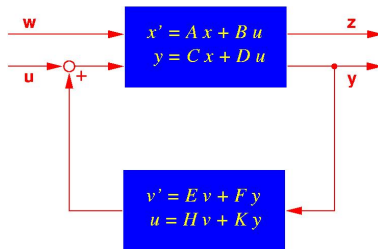
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Micro Electronics/Circuit Simulation

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Progressive miniaturization

- Verification of VLSI/ULSI chip design needs a large number of simulations.
- **Moore's Law (1965/75)** \rightsquigarrow steady increase of describing equations, i.e., network topology (Kirchhoff's laws) and characteristic element/semi-conductor equations.

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- Increase in **packing density** and multilayer technology requires modeling of **interconnect** to ensure that thermic/electro-magnetic effects do not disturb signal transmission.

Intel 4004 (1971)	Intel Core 2 Extreme (quad-core) (2007)
1 layer, 10μ technology	9 layers, $45nm$ technology
2,300 transistors	$> 8,200,000$ transistors
64 kHz clock speed	> 3 GHz clock speed.

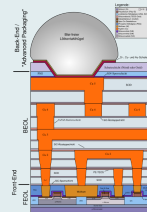
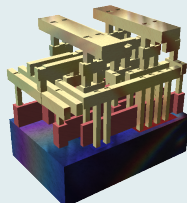
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Source: http://en.wikipedia.org/wiki/Image:Silicon_chip_3d.png.

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- Here: mostly MOR for linear systems, they occur in micro electronics through modified nodal analysis (MNA) for RLC networks. e.g., when
 - decoupling large **linear subcircuits**,
 - modeling **transmission lines**,
 - modeling **pin packages** in VLSI chips,
 - modeling circuit elements described by Maxwell's equation using partial element equivalent circuits (**PEEC**).

Micro Electronics/Circuit Simulation

since ~ 1990 ies

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Standard MOR techniques in circuit simulation:

Krylov subspace / Padé approximation / rational interpolation methods.

Application Areas

Many other disciplines in **computational sciences and engineering** like

- computational fluid dynamics (CFD),
- computational electromagnetics,
- chemical process engineering,
- design of MEMS/NEMS (micro/nano-electrical-mechanical systems),
- computational acoustics,
- ...

Motivating Examples

Electro-Thermic Simulation of Integrated Circuit (IC)

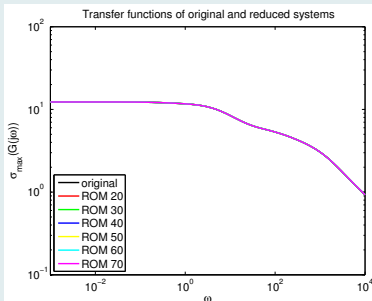
[Source: Evgenii Rudnyi, CADFEM GmbH]

- Original model: $n = 270.593$, $m = q = 2 \Rightarrow$

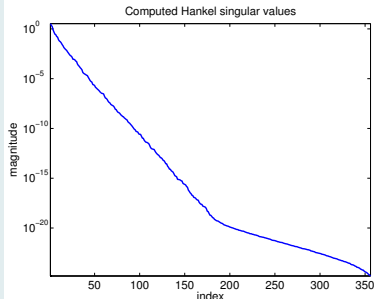
Computing time (on Intel Xeon dualcore 3GHz, 1 Thread):

- Main computational cost for set-up data $\approx 22min$.
- Computation of reduced models from set-up data: 44–49sec. ($r = 20$ –70).
- Bode plot (MATLAB on Intel Core i7, 2,67GHz, 12GB):
7.5h for original system, < 1min for reduced system.
- Speed-up factor: 18 including / ≥ 450 excluding reduced model generation!

Bode Plot (Amplitude)



Hankel Singular Values



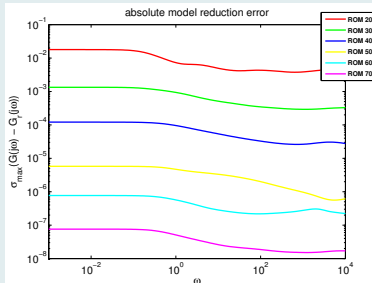
Motivating Examples

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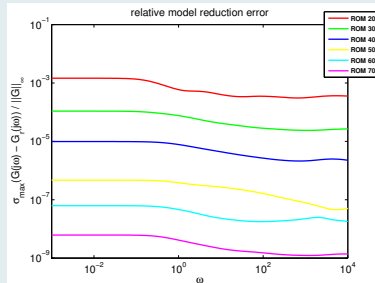
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Absolute Error



Relative Error



Motivating Examples

A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

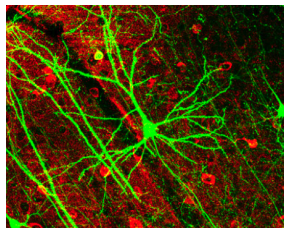
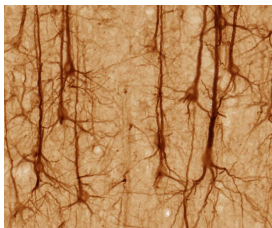
- Simple model for neuron (de-)activation [CHATURANTABUT/SORENSEN 2009]

$$\begin{aligned}\epsilon v_t(x, t) &= \epsilon^2 v_{xx}(x, t) + f(v(x, t)) - w(x, t) + g, \\ w_t(x, t) &= hv(x, t) - \gamma w(x, t) + g,\end{aligned}$$

with $f(v) = v(v - 0.1)(1 - v)$ and initial and boundary conditions

$$\begin{aligned}v(x, 0) &= 0, & w(x, 0) &= 0, & x &\in [0, 1] \\ v_x(0, t) &= -i_0(t), & v_x(1, t) &= 0, & t &\geq 0,\end{aligned}$$

where $\epsilon = 0.015$, $h = 0.5$, $\gamma = 2$, $g = 0.05$, $i_0(t) = 50000t^3 \exp(-15t)$.



Source: <http://en.wikipedia.org/wiki/Neuron>

A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

- Parameter g handled as an additional input.
- Original state dimension $n = 2 \cdot 400$, QBDAE dimension $N = 3 \cdot 400$, reduced QBDAE dimension $r = 26$, chosen expansion point $\sigma = 1$.

Motivating Examples

A Nonlinear Model from Computational Neurosciences: the FitzHugh-Nagumo System

Motivating Examples

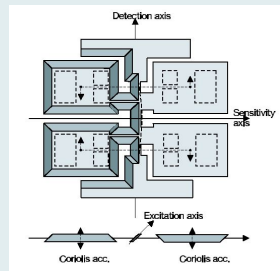
Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)



- Voltage applied to electrodes induces vibration of wings, resulting rotation due to Coriolis force yields sensor data.
- FE model of second order:
 $N = 17.361 \rightsquigarrow n = 34.722, m = 1, q = 12.$
- Sensor for position control based on acceleration and rotation.

- Application: inertial navigation.



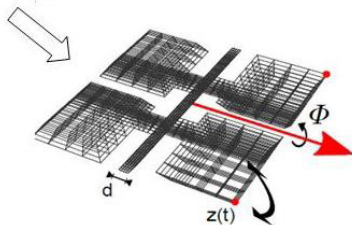
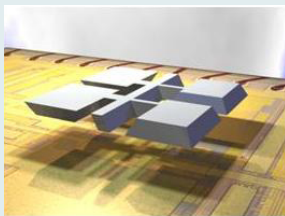
Source: The Oberwolfach Benchmark Collection <http://www.imtek.de/simulation/benchmark>

Motivating Examples

Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Parametric FE model: $M(d)\ddot{x}(t) + D(\Phi, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t)$.



Motivating Examples

Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

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$$M(d)\ddot{x}(t) + D(\Phi, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t),$$

wobei

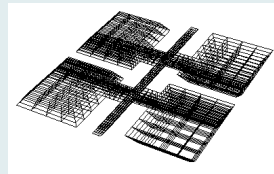
$$M(d) = M_1 + dM_2,$$

$$D(\Phi, d, \alpha, \beta) = \Phi(D_1 + dD_2) + \alpha M(d) + \beta T(d),$$

$$T(d) = T_1 + \frac{1}{d}T_2 + dT_3,$$

with

- width of bearing: d ,
- angular velocity: Φ ,
- Rayleigh damping parameters: α, β .

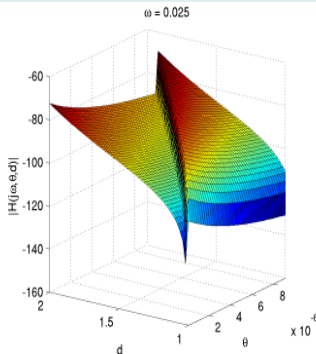


Motivating Examples

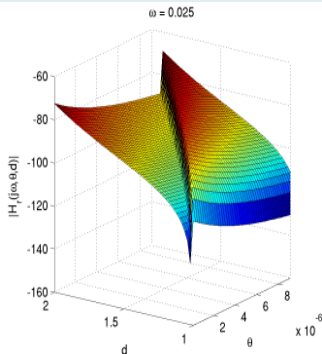
Parametric MOR: Applications in Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Original. . .



and reduced-order model.



Outline

- 1 Linear Time Invariant Systems
- 2 Norms of Signals and Systems
- 3 Introduction to Model Reduction
- 4 Model Reduction by Projection
 - Projection and Interpolation
 - Modal Truncation
 - Rational Interpolation
 - \mathcal{H}_2 -Optimal Model Reduction

Model Reduction by Projection

Goals

Requirements: A Model Reduction approach should:

- Automatically generate compact models \hat{G} from a given model G
- Satisfy desired error tolerance tol for all admissible input signals u

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| < \text{tol} \cdot \|u\| \quad \text{for all admissible } u.$$

⇒ Provide computable error bound/estimate!

- Preserve physical properties:
 - stability
 - minimum phase
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A G is **stable**, if all poles of G are in \mathbb{C}^- . A system (A, B, C, D) or A is **stable**, if all eigenvalues of A have a negative real part. Compare: $G(s) = C(sI - A)^{-1}B$

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- Preserve physical properties:
 - stability
 - **minimum phase**
 - passivity

A system G has **minimum phase** if all zeros of G are in the left half-plane \mathbb{C}^- .

Model Reduction by Projection

Projection Basics

Definition

A projector $P: \mathcal{X} \rightarrow \mathcal{X}$ is a linear map, or a matrix, with $P^2 = P$.

Example

- $\mathcal{X} = \mathbb{R}^2$
- $P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is a projector in \mathcal{X}

Model Reduction by Projection

Notion and Properties of Projectors

- A projector is a linear map $P: \mathcal{X} \rightarrow \mathcal{X}$ with $P^2 = P$.
- If $\mathcal{X} = \mathbb{R}^n$, a projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$.
- Let $\mathcal{V} = \text{range}(P)$, then P is called a projector **onto** \mathcal{V} .
- If $\{v_1, \dots, v_r\}$ is a basis of some $\mathcal{V} \in \mathcal{X}$ and $V = [v_1, \dots, v_r]$, then

$$P := V(V^T V)^{-1} V^T$$

defines the **orthogonal** projector onto \mathcal{V} .

- If $\mathcal{W} \subset \mathcal{X}$ is another r -dimensional subspace with a basis matrix $W = [w_1, \dots, w_r]$ so that $W^T V$ is not singular, then

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defines the **oblique** projector onto \mathcal{V} along the orthogonal complement \mathcal{W}_\perp of \mathcal{W} .

- For a projector P , the projector $I - P$ onto $\ker P$ is the **complementary** projector.

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Model Reduction by Projection

Notion and Properties of Projectors

- A projector is a linear map $P: \mathcal{X} \rightarrow \mathcal{X}$ with $P^2 = P$.
- If $\mathcal{X} = \mathbb{R}^n$, a projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$.
- Let $\mathcal{V} = \text{range}(P)$, then P is called a projector **onto** \mathcal{V} .
- If $\{v_1, \dots, v_r\}$ is a basis of some $\mathcal{V} \in \mathcal{X}$ and $V = [v_1, \dots, v_r]$, then

$$P := V(V^T V)^{-1} V^T$$

defines the **orthogonal** projector onto \mathcal{V} .

- If $\mathcal{W} \subset \mathcal{X}$ is another r -dimensional subspace with a basis matrix $W = [w_1, \dots, w_r]$ so that $W^T V$ is not singular, then

$$P = V(W^T V)^{-1} W^T$$

defines the **oblique** projector onto \mathcal{V} along the orthogonal complement \mathcal{W}_\perp of \mathcal{W} .

- For a projector P , the projector $I - P$ onto $\ker P$ is the **complementary** projector.

Projection and Interpolation

- 1 Modal Truncation
- 2 Rational Interpolation (Padé-Approximation and (rational) Krylov Subspace Methods)
- 3 Balanced Truncation
- 4 many more...

computation of reduced-order model (ROM) by projection!

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- There is a space $\mathcal{V} \subset \mathbb{R}^n$ with $\dim \mathcal{V} = r < n$, such that $x \in \mathcal{V}$ for all time t and input u .
- Take a space \mathcal{W} , so that $\mathcal{W}_\perp \oplus \mathcal{V} = \mathbb{R}^n$.
- Galerkin-type projections: $\mathcal{W} = \mathcal{V}$.
- Petrov-Galerkin projections: $\mathcal{W} \neq \mathcal{V}$.
- Take matrices V and W that form bases of \mathcal{V} and \mathcal{W} , with

$$W^T V = I_r$$

- Then $V(W^\top V)^{-1}W = VW^\top$ is a projector onto \mathcal{V}
- Define $\hat{x} := W^\top x \in \mathbb{R}^r$ and define $\tilde{x} := V\hat{x} = VW^\top x$
- If everything is exact, then

$$\|x - \tilde{x}\| = \|x - VW^T x\| = 0$$

- and given (A, B, C, D) , the **reduced-order model** $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is

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Model Reduction by Projection

Definition of the reduced model

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the **reduced-order model** $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is

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Why is the ROM defined like this:

It is the (Petrov)-Galerkin condition $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$:

$$W^T (\dot{\tilde{x}} - A\tilde{x} - Bu) = W^T (VW^T \dot{x} - AVW^T x - Bu)$$

is zero, if, and only if,

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Projection \rightsquigarrow Rational Interpolation

Theorem 3.3 [GRIMME '97, VILLEMAGNE/SKELTON '87]

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = C V, \quad (\hat{D} = D),$$

- $(s_* I_n - A)^{-1} B \in \text{range}(V)$, or
- $(s_* I_n - A)^{-*} C^T \in \text{range}(W)$,

$$G(s_*) = \hat{G}(s_*).$$

Note: extension to Hermite interpolation conditions later!

Projection \rightsquigarrow Rational Interpolation

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = C V, \quad (\hat{D} = D),$$
$$G(s) - \hat{G}(s) = (C(sl_n - A)^{-1}B + D) - (\hat{C}(sl_r - \hat{A})^{-1}\hat{B} + \hat{D})$$

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Model Reduction by Projection

Projection \rightsquigarrow Rational Interpolation

Given the ROM

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = C V, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$\begin{aligned} G(s) - \hat{G}(s) &= (C(sI_n - A)^{-1}B + D) - (\hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}) \\ &= C \left((sI_n - A)^{-1} - V(sI_r - \hat{A})^{-1}W^T \right) B \\ &= C \left(I_n - \underbrace{V(sI_r - \hat{A})^{-1}W^T}_{=: P(s)} (sI_n - A) \right) (sI_n - A)^{-1} B. \end{aligned}$$

Model Reduction by Projection

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If $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, then $P(s_*)$ is a projector onto \mathcal{V} :

$\text{range}(P(s_*)) \subset \text{range}(V)$, all matrices have full rank $\Rightarrow "="$,

$$\begin{aligned} P(s_*)^2 &= V(s_*I_r - \hat{A})^{-1}W^T(s_*I_n - A)V(s_*I_r - \hat{A})^{-1}W^T(s_*I_n - A) \\ &= V(s_*I_r - \hat{A})^{-1} \underbrace{(s_*I_r - \hat{A})(s_*I_r - \hat{A})^{-1}}_{=I_r} W^T(s_*I_n - A) = P(s_*). \end{aligned}$$

Projection \rightsquigarrow Rational Interpolation

Modal Truncation

Basic method:

Assume A is diagonalizable, $T^{-1}AT = D_A$, project state-space onto A -invariant subspace $\mathcal{V} = \text{span}(t_1, \dots, t_r)$, $t_k =$ eigenvectors corresp. to “dominant” modes / eigenvalues of A . Then with

$$V = T(:, 1:r) = [t_1, \dots, t_r], \quad \tilde{W}^H = T^{-1}(1:r, :), \quad W = \tilde{W}(V^H \tilde{W})^{-1},$$

reduced-order model is

$$\hat{A} := W^H A V = \text{diag}\{\lambda_1, \dots, \lambda_r\}, \quad \hat{B} := W^H B, \quad \hat{C} = C V$$

Also computable by truncation:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\hat{C}, C_2], \quad \hat{D} = D.$$

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Properties:

Simple computation for large-scale systems, using, e.g., Krylov subspace methods (Lanczos, Arnoldi), Jacobi-Davidson method.

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Properties:

Error bound:

$$\|G - \hat{G}\|_{\infty} \leq \|C_2\| \|B_2\| \frac{1}{\min_{\lambda \in \Lambda(A_2)} |\operatorname{Re}(\lambda)|}.$$

Proof:

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D = CTT^{-1}(sI - A)^{-1}TT^{-1}B + D \\ &= \hat{C}(sI - \hat{A})^{-1}\hat{B} + C_2(sI_{n-r} - A_2)^{-1}B_2 + D \\ &= [\hat{C}, C_2] \begin{bmatrix} (sI_r - \hat{A})^{-1} & \\ & (sI_{n-r} - A_2)^{-1} \end{bmatrix} \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix} + D \\ &= \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2, \end{aligned}$$

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Proof:

$$G(s) = \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2,$$

observing that $\left\| G - \hat{G} \right\|_{\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(C_2(j\omega I_{n-r} - A_2)^{-1}B_2)$, and

$$C_2(j\omega I_{n-r} - A_2)^{-1}B_2 = C_2 \operatorname{diag} \left(\frac{1}{j\omega - \lambda_{r+1}}, \dots, \frac{1}{j\omega - \lambda_n} \right) B_2.$$

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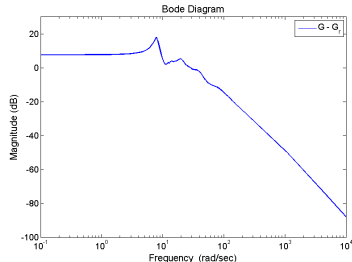
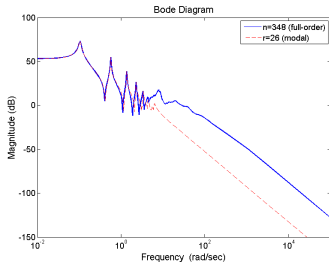
Difficulties:

- Eigenvalues contain only limited system information.
- Dominance measures are difficult to compute.
([LITZ '79] use Jordan canonical form; otherwise merely heuristic criteria, e.g., [VARGA '95]. Recent improvement: **dominant pole algorithm**.)
- Error bound not computable for really large-scale problems.

Example

conjugate eigenpairs, error bound yields $\|G - \hat{G}\|_{\infty} \leq 1.21 \cdot 10^3$

Bode plots of transfer functions and error function



Extensions

Extensions

Modal Truncation

Dominant Poles

Pole-Residue Form of Transfer Function

Consider partial fraction expansion of transfer function with $D = 0$:

$$G(s) = \sum_{k=1}^n \frac{R_k}{s - \lambda_k}$$

with the **residues** $R_k := (Cx_k)(y_k^H B) \in \mathbb{C}^{q \times m}$.

Dominant Poles

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Note: this follows using the spectral decomposition $A = XDX^{-1}$, with $X = [x_1, \dots, x_n]$ the right and $X^{-1} =: Y = [y_1, \dots, y_n]^H$ the left eigenvector matrices:

$$\begin{aligned} G(s) &= C(sI - XDX^{-1})^{-1}B = CX(sI - \text{diag}\{\lambda_1, \dots, \lambda_n\})^{-1}YB \\ &= [C x_1, \dots, C x_n] \begin{bmatrix} \frac{1}{s - \lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s - \lambda_n} \end{bmatrix} \begin{bmatrix} y_1^H B \\ \vdots \\ y_n^H B \end{bmatrix} \\ &= \sum_{k=1}^n \frac{(C x_k)(y_k^H B)}{s - \lambda_k}. \end{aligned}$$

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Note: $R_k = (C x_k)(y_k^H B)$ are the residues of G in the sense of the residue theorem of complex analysis:

$$\begin{aligned} \text{res}(G, \lambda_\ell) &= \lim_{s \rightarrow \lambda_\ell} (s - \lambda_\ell) G(s) = \sum_{k=1}^n \underbrace{\lim_{s \rightarrow \lambda_\ell} \frac{s - \lambda_\ell}{s - \lambda_k}}_{R_k} R_k = R_\ell. \end{aligned}$$

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As projection basis use spaces spanned by right/left eigenvectors corresponding to **dominant poles**, i.e.. (λ_j, x_j, y_j) with largest

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Modal Truncation

Dominant Poles

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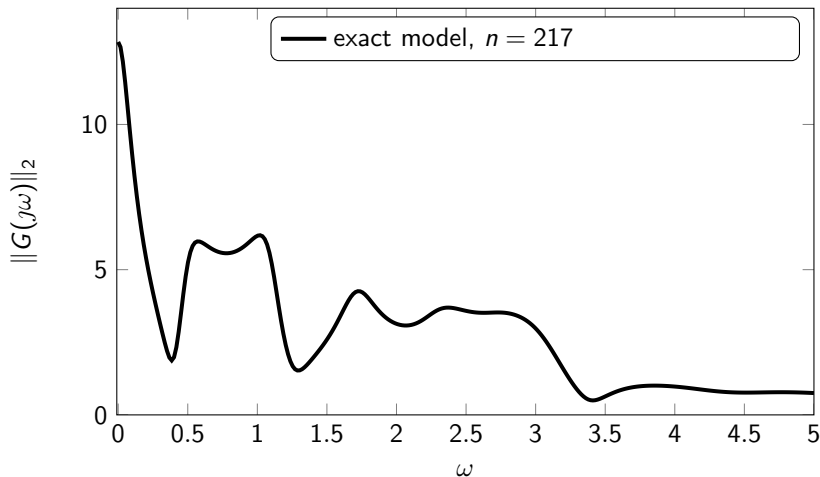
$$\|R_k\| / |\operatorname{re}(\lambda_k)|.$$

Remark

The dominant modes have most important influence on the input-output behavior of the system and are responsible for the "peaks" in the frequency response.

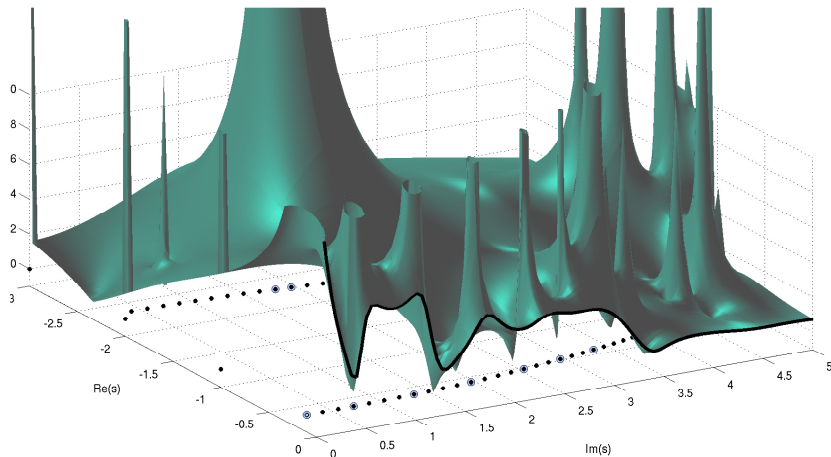
Dominant Poles

Random SISO Example ($B, C^T \in \mathbb{R}^n$)



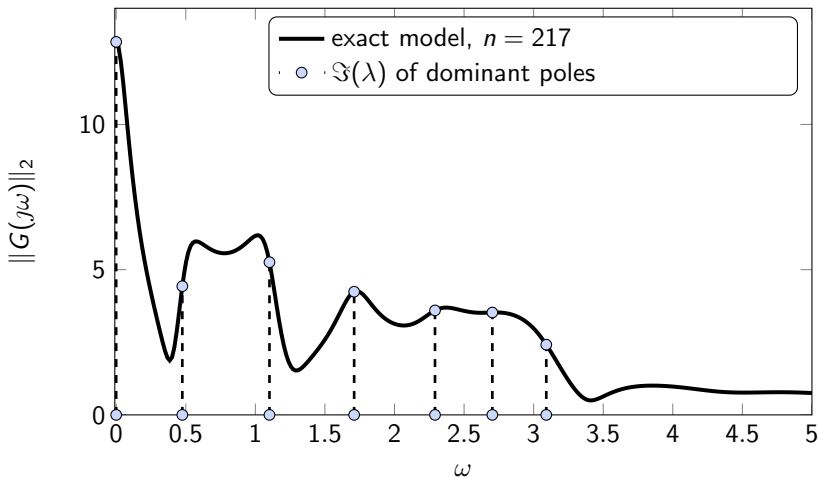
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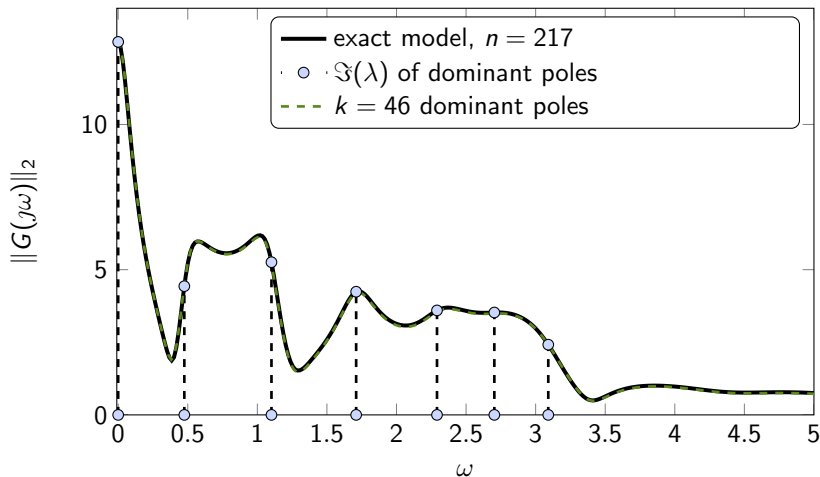
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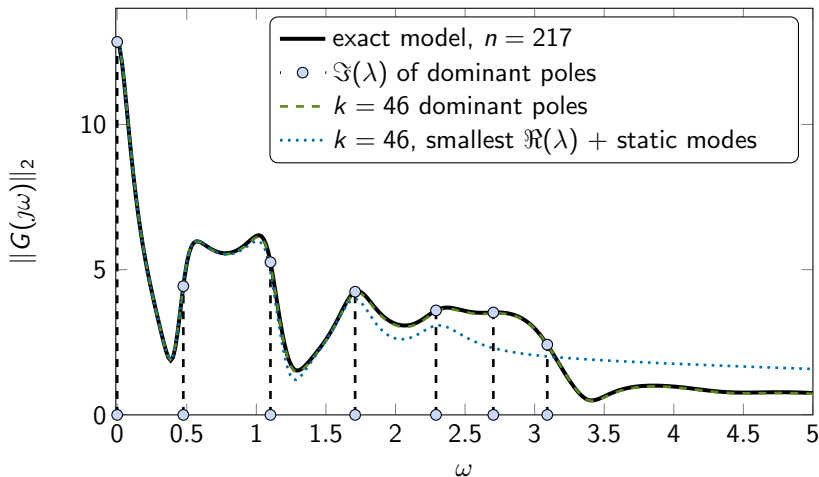
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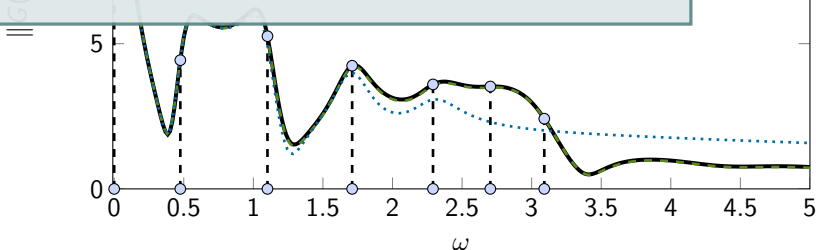


Dominant Poles

Random SISO Example ($B, C^T \in \mathbb{R}^n$)

Algorithms for computing dominant poles and eigenvectors:

- Subspace Accelerated Dominant Pole Algorithm (SADPA),
- Rayleigh-Quotient-Iteration (RQI),
- Jacobi-Davidson-Method.



Outline

- 1 Linear Time Invariant Systems
- 2 Norms of Signals and Systems
- 3 Introduction to Model Reduction
- 4 Model Reduction by Projection
 - Projection and Interpolation
 - Modal Truncation
 - Rational Interpolation
 - \mathcal{H}_2 -Optimal Model Reduction

Rational Interpolation

Given an LTI system $\dot{x} = Ax + Bu, y = Cx$ with transfer function $G(s) = C(sI_n - A)^{-1}B$, a reduced-order model is obtained using projection approach with $V, W \in \mathbb{R}^{n \times r}$ and $W^T V = I_r$ by computing

Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

Galerkin-type (one-sided) projection: $W = V$.

Model Reduction by Projection

Rational Interpolation

Computation of reduced-order model by projection

Given an LTI system $\dot{x} = Ax + Bu, y = Cx$ with transfer function $G(s) = C(sI_n - A)^{-1}B$, a reduced-order model is obtained using projection approach with $V, W \in \mathbb{R}^{n \times r}$ and $W^T V = I_r$ by computing

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = C V.$$

Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

Galerkin-type (one-sided) projection: $W = V$.

Rational Interpolation/Moment-Matching

Choose V, W such that

$$G(s_j) = \hat{G}(s_j), \quad j = 1, \dots, k,$$

and

$$\frac{d^i}{ds^i} G(s_j) = \frac{d^i}{ds^i} \hat{G}(s_j), \quad i = 1, \dots, K_j, \quad j = 1, \dots, k.$$

Rational Interpolation

If

$$\begin{aligned} \text{span} \{ (s_1 I_n - A)^{-1} B, \dots, (s_k I_n - A)^{-1} B \} &\subset \text{Ran}(V), \\ \text{span} \{ (s_1 I_n - A)^{-T} C^T, \dots, (s_k I_n - A)^{-T} C^T \} &\subset \text{Ran}(W), \end{aligned}$$

then

$$G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds} G(s_j) = \frac{d}{ds} \hat{G}(s_j), \quad \text{for } j = 1, \dots, k.$$

Model Reduction by Projection

Rational Interpolation

Theorem (simplified) [GRIMME '97, VILLEMAGNE/SKELTON '87]

If

$$\begin{aligned} \text{span} \{ (s_1 I_n - A)^{-1} B, \dots, (s_k I_n - A)^{-1} B \} &\subset \text{Ran}(V), \\ \text{span} \{ (s_1 I_n - A)^{-T} C^T, \dots, (s_k I_n - A)^{-T} C^T \} &\subset \text{Ran}(W), \end{aligned}$$

then

$$G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds} G(s_j) = \frac{d}{ds} \hat{G}(s_j), \quad \text{for } j = 1, \dots, k.$$

Remarks:

using Galerkin/one-sided projection yields $G(s_j) = \hat{G}(s_j)$, but in general

$$\frac{d}{ds} G(s_j) \neq \frac{d}{ds} \hat{G}(s_j).$$

Model Reduction by Projection

Rational Interpolation

Theorem (simplified) [GRIMME '97, VILLEMAGNE/SKELTON '87]

If

$$\begin{aligned}\text{span} \{ (s_1 I_n - A)^{-1} B, \dots, (s_k I_n - A)^{-1} B \} &\subset \text{Ran}(V), \\ \text{span} \{ (s_1 I_n - A)^{-T} C^T, \dots, (s_k I_n - A)^{-T} C^T \} &\subset \text{Ran}(W),\end{aligned}$$

then

$$G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds} G(s_j) = \frac{d}{ds} \hat{G}(s_j), \quad \text{for } j = 1, \dots, k.$$

Remarks:

$k = 1$, standard Krylov subspace(s) of dimension $K \rightsquigarrow$ moment-matching methods/Padé approximation,

$$\frac{d^i}{ds^i} G(s_1) = \frac{d^i}{ds^i} \hat{G}(s_1), \quad i = 0, \dots, K - 1(+K).$$

Model Reduction by Projection

Rational Interpolation

Theorem (simplified) [GRIMME '97, VILLEMAGNE/SKELTON '87]

If

$$\begin{aligned} \text{span} \{ (s_1 I_n - A)^{-1} B, \dots, (s_k I_n - A)^{-1} B \} &\subset \text{Ran}(V), \\ \text{span} \{ (s_1 I_n - A)^{-T} C^T, \dots, (s_k I_n - A)^{-T} C^T \} &\subset \text{Ran}(W), \end{aligned}$$

then

$$G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds} G(s_j) = \frac{d}{ds} \hat{G}(s_j), \quad \text{for } j = 1, \dots, k.$$

Remarks:

computation of V, W from **rational Krylov subspaces**, e.g.,

- dual rational Arnoldi/Lanczos [GRIMME '97],
- **Iterative Rational Krylov- Algo.** [ANTOULAS/BEATTIE/GUGERCIN '07].

Best \mathcal{H}_2 -norm approximation problem

Find $\arg \min_{\hat{G} \in \mathcal{H}_2 \text{ of order } \leq r} \|G - \hat{G}\|_2$.

\mathcal{H}_2 -Optimal Model Reduction

The Basic IRKA Algorithm

Algorithm 1 IRKA (MIMO version/MIRIAM)

Input: A stable, B , C , \hat{A} stable, \hat{B} , \hat{C} , $\delta > 0$.

Output: A^{opt} , B^{opt} , C^{opt}

- 1: **while** $(\max_{j=1,\dots,r} \left\{ \frac{|\mu_j - \mu_j^{\text{old}}|}{|\mu_j|} \right\} > \delta)$ **do**
 - 2: $\text{diag}\{\mu_1, \dots, \mu_r\} := T^{-1}\hat{A}T = \text{spectral decomposition,}$
 $\tilde{B} = \hat{B}^H T^{-T}$, $\tilde{C} = \hat{C}T$.
 - 3: $V = \left[(-\mu_1 I - A)^{-1} B \tilde{B}_1, \dots, (-\mu_r I - A)^{-1} B \tilde{B}_r \right]$
 - 4: $W = \left[(-\mu_1 I - A^T)^{-1} C^T \tilde{C}_1, \dots, (-\mu_r I - A^T)^{-1} C^T \tilde{C}_r \right]$
 - 5: $V = \text{orth}(V)$, $W = \text{orth}(W)$, $W = W(V^H W)^{-1}$
 - 6: $\hat{A} = W^H A V$, $\hat{B} = W^H B$, $\hat{C} = C V$
 - 7: **end while**
 - 8: $A^{opt} = \hat{A}$, $B^{opt} = \hat{B}$, $C^{opt} = \hat{C}$
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