Physics Presentation

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Postulates of QM

1. System represented by vector $|\Psi(t)\rangle \in \mathbf{H}$, Hilbert Space - complete vector space with an inner product.
2. Dynamical variables $f(x,p)$ correspond to Hermitian Operators $\hat{F}(\hat{x},\hat{p})$, where $\langle x'|\,\hat{x}\,|x\rangle = x\delta(x'-x)$ and $[\hat{x},\hat{p}] = i\hbar$
3. Measurement of $\hat{A}$ will yield a with probability $\propto |\langle\Psi_a|\Psi\rangle|^2$ and $|\Psi\rangle \to |\Psi_a\rangle$ where $\hat{A}\,|\psi_a\rangle = a\,|\psi_a\rangle$
4. System obeys Schrodinger's Equation. $i\hbar\frac{d}{dt}\,|\Psi(t)\rangle = \hat{H}\,|\Psi(t)\rangle$

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar\hat{I}$$

$$\left\langle x' \right| [\hat{x}, \hat{p}] \left| x \right\rangle = i\hbar \left\langle x' \middle| x \right\rangle = i\hbar\delta(x - x') =$$

$$\left\langle x' \right| \hat{x}\hat{p} \left| x \right\rangle - \left\langle x' \right| \hat{p}\hat{x} \left| x \right\rangle = \left\langle x' \right| \hat{p} \left| x \right\rangle (x' - x)$$

$$\left\langle x' \right| \hat{p} \left| x \right\rangle = i\hbar \frac{\delta(x' - x)}{x' - x} = -i\hbar \frac{\partial\delta(x' - x)}{\partial x} = \left\langle x' \right| (-i\hbar \frac{\partial}{\partial x}) \left| x \right\rangle$$

$\therefore \hat{p} = -i\hbar \frac{\partial}{\partial x}$ in the position basis.

Similarly in the momentum basis, $\hat{x} = i\hbar \frac{\partial}{\partial p}$

Density Matrix

For a pure state $|\psi\rangle = \sum_i c_i |\phi\rangle$, the density matrix representation is $\hat{\rho} = |\psi\rangle\langle\psi|$ with matrix elements $\rho_{ij} = \langle\phi_i|\psi\rangle|\langle\psi|\phi_j\rangle = c_i c_j^*$.
In a mixed ensemble of systems, let p_k be the probability of the state of a system chosen at random being $|\psi_k\rangle$.Then the density matrix of the ensemble is-

$$\hat{\rho} = \sum_k p_k |\psi_k\rangle\langle\psi_k| = \sum_k p_k \hat{\rho}_k$$

$$\rho_{ij} = \sum_k p_k c_{ik} c_{jk}^* = \left\langle c_i c_j^* \right\rangle .$$

Say, $\hat{A}|a\rangle = a|a\rangle$ & projection operator $\hat{P}_a = |a\rangle\langle a|$. Then probability of getting measurement a is $P(a) = \mathrm{Tr}\left(\hat{\rho}\hat{P}_a\right)$

Normalization: $\mathrm{Tr}(\hat{\rho}) = 1$

Expectation Value: $\left\langle \hat{A} \right\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{A}\right)$

Von Neumann Eqn: $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}]$

For a pure state $\hat{\rho}^2 = \hat{\rho}$, but for a mixed state $Tr(\hat{\rho}^2) < 1$

Quantum Optics

For a two level system of atoms, let $|\psi_1\rangle, |\psi_2\rangle$ represent the ground state and excited state having energies E_1, E_2 respectively. Let the incident light have $\omega = \omega_0 + \delta\omega \approx \frac{E_2 - E_1}{\hbar}$. Then $|\Psi(\vec{r}, t)\rangle = c_1\psi_1(\vec{r})e^{-\frac{iE_1 t}{\hbar}} + c_1\psi_1(\vec{r})e^{-\frac{iE_1 t}{\hbar}}$.
Let the hamiltonian be split into two terms, one for non-interacting atom and the other pertubation term for light atom interaction $\hat{H} = \hat{H}_0(\vec{r}) + \hat{V}(t)$. Semi-classically, $\hat{V}(t) = e\vec{r} \cdot \vec{E}(t) = exE_0 \cos\omega t$. We have the following coupled equations controlling the system:

$$\frac{\mathrm{d}c_1(t)}{\mathrm{d}t} = \frac{i}{2}\Omega_R c_2(t)(e^{i(\omega-\omega_0)t} + e^{-i(\omega+\omega_0)t})$$
$$\frac{\mathrm{d}c_2(t)}{\mathrm{d}t} = \frac{i}{2}\Omega_R c_1(t)(e^{i(\omega+\omega_0)t} + e^{-i(\omega-\omega_0)t}).$$

Rabi Oscillations Defined as $\Omega_R = \frac{1}{\hbar}|\mu_{12}E_0|$ where $\mu_{ij} = -e\langle i|\, x\, |j\rangle$ is the dipole matrix element.