

Answers to Caegory Theory by Steve Awodey

Syed Khalid

Nov, 2023

1 Ch1 Categories

1. **Question:** The objects of **Rel** are sets, and an arrow $A \rightarrow B$ is a relation from A to B , that is, a subset $R \subseteq A \times B$. The equality relation $\{\langle a, a \rangle \in A \times A \mid a \in A\}$ is the identity arrow on a set A . Composition in **Rel** is to be given by

$$S \circ R = \{\langle a, c \rangle \in A \times C \mid \exists b (\langle a, b \rangle \in R \ \& \ \langle b, c \rangle \in S)\}$$

for $R \subseteq A \times B$ and $S \subseteq B \times C$.

- (a) Show that **Rel** is a category.
- (b) Show also that there is a functor $G : \mathbf{Sets} \rightarrow \mathbf{Rel}$ taking objects to themselves and each function $f : A \rightarrow B$ to its graph,

$$G(f) = \{\langle a, f(a) \rangle \in A \times B \mid a \in A\}.$$

- (c) Finally, show that there is a functor $C : \mathbf{Rel}^{op} \rightarrow \mathbf{Rel}$ taking each relation $R \subseteq A \times B$ to its converse $R^c \subseteq A \times B$, where,

$$\langle a, b \rangle \in R^c \Leftrightarrow \langle b, a \rangle \in R.$$

Answer:

2. **Question:** Consider the following isomorphisms of categories and determine which hold.
 - (a) $\mathbf{Ref} \cong \mathbf{Ref}^{op}$
 - (b) $\mathbf{Sets} \cong \mathbf{Sets}^{op}$
 - (c) For a fixed set X with powerset $P(X)$, as poset categories $P(X) \cong P(X)^{op}$ (the arrows in $P(X)$ are subset inclusions $A \subseteq B$ for all $A, B \subseteq X$).

Answer:

3. **Question:**

- (a) Show that in **Sets**, the isomorphisms are exactly the bijections.
- (b) Show that in **Monoids**, the isomorphisms are exactly the bijective homomorphisms.
- (c) Show that in **Posets**, the isomorphisms are *not* the same as bijective homomorphisms.

Answer:

4. **Question:** Let X be a topological space and preorder the points by *specialization*: $x \leq y$ iff y is contained in every open set that contains x . Show that this is a preorder, and that it is a poset if X is T_0 (for any two distinct points, there is some open set containing one but not the other). Show that the ordering is trivial if X is T_1 (for any two distinct points, each is contained in an open set not containing the other).

Answer:

5. **Question:**

For any category \mathbf{C} , define a functor $U : \mathbf{C}/C \rightarrow C$ from the slice category over an object C that "forgets about C ." Find a functor $F : \mathbf{C}/C \rightarrow C^{\rightarrow}$ to the arrow category such that $\mathbf{dom} \circ F = U$

Answer:

6. **Question:**

Construct the "coslice category" C/\mathbf{C} of a category \mathbf{C} under an object C from the slice category \mathbf{C}/C and the "dual category" operation $-^{op}$.

Answer:

7. *Question:*

Let $2 = a, b$ be any set with exactly 2 elements a and b . Define a functor $F : \mathbf{Sets}/2 \rightarrow \mathbf{Sets} \times \mathbf{Sets}$ with $F(f : X \rightarrow 2) = (f^{-1}(a), f^{-1}(b))$. Is this an isomorphism of categories? What about the analogous situation with an one-element set $1 = a$ instead of 2 ?

Answer:

8. *Question:*

Any category \mathbf{C} determines a preorder $P(\mathbf{C})$ by defining a binary relation \leq on the objects by

$$A \leq B \text{ if and only if there is an arrow } A \rightarrow B$$

Show that P determines a functor from categories to preorders, by defining its effect on functors between categories and checking the required conditions. Show that P is a (one-sided) inverse to the evident inclusion functor of preorders into categories.

Answer:

9. *Question:*

Describe the free categories, on the following graphs by determining their objects, arrows, and composition operations.

(a) $a \xrightarrow{e} b$

(b) $a \xrightleftharpoons{e} b$

(c)
$$\begin{array}{ccc} a & \xrightarrow{e} & b \\ & \searrow g & \downarrow f \\ & & c \end{array}$$

(d)
$$\begin{array}{ccc} a & \xrightleftharpoons{e} & b \\ & \searrow g & \downarrow f \\ & & c \end{array}$$

Answer:

10. **Question:**

How many free categories on graphs are there which have exactly six arrow? Draw the graphs that generate these categories.

Answer:

11. **Question:**

Show that the free monoid functor

$$M : \mathbf{Sets} \rightarrow \mathbf{Mon}$$

- (a) Assume the particular choice $M(X) = X^*$ and define its effect

$$M(f) : M(A) \rightarrow M(B)$$

on a function $f : A \rightarrow B$ to be

$$M(f)(a_1 \dots a_k) = f(a_1) \dots f(a_k), \quad a_1, \dots, a_k \in A$$

- (b) Assume only the UMP of the free monoid and use it to determine M on functions, showing the result to be a functor.

Reflect on how these two approaches are related

Answer:

12. **Question:**

Verify the UMP for free categories on graphs, defined as above with arrows being sequences of edges. Specifically, let $\mathbf{C}(G)$ be the free category on the graph G , so defined, and $i : G \rightarrow U(\mathbf{C}(G))$ the graph homomorphism taking vertices and edges to themselves, regarded as objects and arrows in $\mathbf{C}(G)$. Show that for any category \mathbf{D} and graph homomorphism $f : G \rightarrow U(\mathbf{D})$, there is a unique functor

$$\bar{h} : \mathbf{C}(G) \rightarrow \mathbf{D}$$

with

$$U(\bar{h}) \circ i = f,$$

where $U : \mathbf{Cat} \rightarrow \mathbf{Graph}$ is the underlying graph functor.

Answer:

13. **Question:**

Use the Cayley representaton to show that every small category is isomorphic to a "concrete" one, that is, one in which the objects are sets and the arrows are functions between them.

Answer:

14. **Question:**

The notion of a category can also be defined with just one sort (arrows) rather than two (arrows and objects); the domains and codomains are taken to be certain *arrows* that act as units under composition, which is partially defined. Read about this definition in section I.1 of Mac Lane's *Categories for the Working Mathematician* , and do the exercise mentioned there, showing that it is equivalent to the usual definition.

Answer:

2 Ch2 Abstract Structures

1. **Question:**

Show that a function between sets is an epimorphism if and only if it is surjective. Conclude that the isos in **Sets** are exactly the epi-monos.

Answer:

2. **Question:**

Show that in a poset category, all arrows are both monic and epic.

Answer:

3. **Question:**

(Inverses are unique.) If an arrow $f : A \rightarrow B$ has inverses $g, g' : B \rightarrow A$ (i.e., $g \circ f = 1_A$ and $f \circ g = 1_B$, and similarly for g'), then $g = g'$.

Answer:

4. **Question:**

With regard to a commutative triangle,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow h & \downarrow g \\ & & C \end{array}$$

in any category \mathbf{C} , show

- (a) if f and g are isos (resp. monos, resp. epis), so is h ;
- (b) if h is monic, so is f ;
- (c) if h is epic, so is g ;
- (d) (by example) if h is monic, g need not be.

Answer:

5. **Question:**

Show that the following are equivalent for an arrow

$$f : A \rightarrow B$$

in any category:

- (a) f is an isomorphism.
- (b) f is both a mono and a split epi.
- (c) f is both a split mono and an epi.
- (d) f is both a split mono and a split epi.

Answer:

6. **Question:**

Show that a homomorphism $h : G \rightarrow H$ of graphs is monic just if it is injective on both edges and vertices.

Answer:

7. *Question:*

Show that in any category, any retract of a projective object is also projective.

Answer:

8. *Question:*

Show that all sets are projective (use the axiom of choice).

Answer:

9. *Question:*

Show that the epis among posets are the surjections (on elements), and that the one-element poset **1** is projective.

Answer:

10. *Question:*

Show that sets, regarded as discrete posets, are projective in the category of posets (use the foregoing exercises). Give an example of a poset that is not projective. Show that every projective poset is discrete, that is, a set. conclude that **Sets** is (isomorphic to) the "full subcategory" of projectives in **Pos**, consisting of all projective posets and all monotone maps between them.

Answer:

11. *Question:*

Let A be a set. Define an A -monoid to be a monoid M equipped with a function $m : A \rightarrow U(M)$ (to the underlying set of M). A morphism $h : (M, m) \rightarrow (N, n)$ of A -monoids is to be a monoid homomorphism $h : M \rightarrow N$ such that $U(h) \circ m = n$ (a commutative triangle). Together with the evident identities and composites, this defines a category $A\text{-}\mathbf{Mon}$ of A -monoids.

Show that an initial object in $A\text{-}\mathbf{Mon}$ is the same thing as a free monoid $M(A)$ on A . (Hint: compare their respective UMPs.)

Answer:

12. **Question:**

Show that for any Boolean algebra B , Boolean homomorphisms $h : B \rightarrow \mathbf{2}$ correspond exactly to ultrafilters in B .

Answer:

13. **Question:**

In any category with binary products, show directly that

$$A \times (B \times C) \cong (A \times B) \times C$$

Answer:

14. **Question:**

- (a) For any index set I , define the product $\prod_{i \in I} X_i$ of an I -indexed family of objects $(X_i)_{i \in I}$ in a category, by giving a UMP generalizing that for binary products (the case $I = 2$).
- (b) Show that in **Sets**, for any set X the set X^I of all functions $f : I \rightarrow X$ has this UMP, with respect to the "constant family" where $X_i = X$ for all $i \in I$, and thus

$$X^I \cong \prod_{i \in I} X$$

Answer:

15. **Question:**

Given a category \mathbf{C} with objects A and B , define the category $\mathbf{C}_{A,B}$ to have objects (X, x_1, x_2) where $x_1 : X \rightarrow A, x_2 : X \rightarrow B$, and with arrows $f : (X, x_1, x_2) \rightarrow (Y, y_1, y_2)$ being arrows $f : X \rightarrow Y$ with $y_1 \circ f = x_1$ and $y_2 \circ f = x_2$.

Show that $\mathbf{C}_{A,B}$ has a terminal object just in case A and B have a product in \mathbf{C} .

Answer:

16. **Question:**

In the category of types $\mathbf{C}(\lambda)$ of the λ -calculus, determine the product functor $A, B \mapsto A \times B$ explicitly. Also show that, for any fixed type A , there is a functor $A \rightarrow (-) : \mathbf{C}(\lambda) \rightarrow \mathbf{C}(\lambda)$, taking any type X to $A \rightarrow X$.

Answer:

17. **Question:**

In any category \mathbf{C} with products, define the *graph* of an arrow $f : A \rightarrow B$ to be the monomorphism

$$\Gamma(f) = \langle 1_A, f \rangle : A \rightarrow A \times B$$

(Why is this monic?). Show that for $\mathbf{C} = \mathbf{Sets}$ this determines a functor $\Gamma : \mathbf{Sets} \rightarrow \mathbf{Rel}$ to the category \mathbf{Rel} of relations, as defined in the exercises to Chapter 1. (To get an actual relation $R(f) \subseteq A \times B$, take the image of $\Gamma(f) : A \rightarrow A \times B$.)

Answer:

18. **Question:**

Show that the forgetful functor $U : \mathbf{Mon} \rightarrow \mathbf{Sets}$ from monoids to sets is representable. Infer that U preserves all (small) products.

Answer: