Answers to Caegory Theory by Steve Awodey

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1 Ch1 Categories

1. ***Question:*** The objects of **Rel** are sets, and an arrow $A \to B$ is a relation from A to B, that is, a subset $R \subseteq A \times B$. The equality relation $\{\langle a, a\rangle \in A \times A | a \in A\}$ is the identity arrow on a set A. Compositoin in **Rel** is to be given by

$$S \circ R = \{\langle a, c\rangle \in A \times C \mid \exists b(\langle a, b\rangle \in R \;\&\; \langle b, c\rangle \in S)\}$$

for $R \subseteq A \times B$ and $S \subseteq B \times C$.

(a) Show that **Rel** is a category.

(b) Show also that there is a functor $G : \mathbf{Sets} \to \mathbf{Rel}$ taking objects to themselves and each function $f : A \to B$ to its graph,

$$G(f) = \{\langle a, f(a)\rangle \in A \times B \mid a \in A\}.$$

(c) Finally, show that there is a functor $C : \mathbf{Rel}^{op} \to \mathbf{Rel}$ taking each relation $R \subseteq A \times B$ to its converse $R^c \subseteq A \times B$, where,

$$\langle a, b\rangle \in R^c \Leftrightarrow \langle b, a\rangle \in R.$$

Answer:

(a) To show that definition of identity when composed with an arbtirary morphism gives back the same morphism.

$$\begin{aligned} R &= \{\langle b, a\rangle \in B \times A\} \\ I &= \{\langle a, a\rangle \in A \times A | a \in A\} \\ R \circ I &= \{\langle b, a\rangle \in B \times A \mid \exists a(\langle b, a\rangle \in B \times A \;\&\; \langle a, a\rangle \in A \times A)\} \\ &= I \end{aligned}$$

(b) Identity and composition are already defined. To show that composition is associative.

$$\begin{aligned} \text{LHS} &= S \circ (R \circ Q) \\ \text{RHS} &= (S \circ R) \circ Q \end{aligned}$$

2. ***Question:*** Consider the following isomorphisms of categories and determine which hold.

   (a) $\mathbf{Ref} \cong \mathbf{Ref}^{op}$

   (b) $\mathbf{Sets} \cong \mathbf{Sets}^{op}$

   (c) For a fixed set X with powerset $P(X)$, as poset categories $P(X) \cong P(X)^{op}$ (the arrows in $P(X)$ are subset inclusions $A \subseteq B$ for all $A, B \subseteq X$).

   Answer:

3. ***Question:***

   (a) Show that in **Sets**, the isomorphisms are exactly the bijections.

   (b) Show that in **Monoids**, the isomorphisms are exactly the bijective homomorphisms.

   (c) Show that in **Posets**, the isomorphisms are *not* the same as bijective homomorphisms.

   Answer:

4. ***Question:*** Let X be a topological space and preorder the points by *specializatoin:* $x \leq y$ iff y is contained in every open set that contains x. Show that this is a preorder, and that it is a poset if X is T_0 (for any two distinct points, there is some open set containing one but not the other). Show that the ordering is trivial is X is T_1 (for any two distinct points, each is contained in an open set not containing the other).

   Answer:

5. ***Question:***

   For any category $\mathbf{C}$, define a functor $U : \mathbf{C}/C \to C$ from the slice category over an object C that "forgets about C." Find a functor $F : \mathbf{C}/C \to C^{\to}$ to the arrow category such that $\mathbf{dom} \circ F = U$

   Answer:

6. ***Question:***

   Construct the ”coslice category” $C/\mathbf{C}$ of a category $\mathbf{C}$ under an object C from the slice category $\mathbf{C}/C$ and the ”dual category” operation $-^{op}$.

   Answer:

7. ***Question:***

   Let $2 = a, b$ be any set with exactly 2 elements a and b. Define a functor $F : \mathbf{Sets}/2 \to \mathbf{Sets} \times \mathbf{Sets}$ with $F(f : X \to 2) = (f^{-1}(a), f^{-1}(b))$. Is this an isomorphism of categories? What about the analogous situation with an one-element set $1 = a$ instead of 2?

   Answer:

8. ***Question:***

   Any category $\mathbf{C}$ determines a preorder $P(\mathbf{C})$ by defining a binary relation $\leq$ on the objects by

   $$A \leq B \text{ if and only if there is an arrow } A \to B$$

   Show that P determines a functor from categories to preorders, by defining its effect on functors between categories and checking the required conditions. Show that P is a (one-sided) inverse to the evident inclusion functor of preorders into categories.

   Answer:

9. ***Question:***

   Describe the free categories, on the following graphs by determining their objects, arrows, and composition operations.

   (a) $a \xrightarrow{e} b$

   (b) $\begin{tikzcd} a \arrow[r, shift left, "e"] & b \arrow[l, shift left] \end{tikzcd}$

   (c) $\begin{tikzcd} a \arrow[r, "e"] \arrow[dr, "g"'] & b \arrow[d, "f"] \\ & c \end{tikzcd}$

(d) $$\begin{tikzcd} a \arrow[r, shift left, "e"] \arrow[dr, "g"'] & b \arrow[l, shift left, "h"] \arrow[d, "f"] \\ & c \end{tikzcd}$$

Answer:

10. ***Question:***

    How many free categories on graphs are there which have exactly six arrow? Draw the graphs that generate these categories.

    Answer:

11. ***Question:***

    Show that the free monoid functor

    $$M : \mathbf{Sets} \to \mathbf{Mon}$$

    (a) Assume the particular choice $M(X) = X^*$ and define its effect

    $$M(f) : M(A) \to M(B)$$

    on a function $f : A \to B$ to be

    $$M(f)(a_1...a_k) = f(a_1)...f(a_k), \quad a_1, ...a_k \in A$$

    (b) Assume only the UMP of the free monoid and use it to determine M on functions, showing the result to be a functor.

    Reflect on how these two approaches are related

    Answer:

12. ***Question:***

    Verify the UMP for free categories on graphs, defined as above with arrows being sequences of edges. Specifically, let $\mathbf{C}(G)$ be the free category on the graph G , so defined, and $i : G \to U(\mathbf{C}(G))$ the graph homomorphism taking vertices and edges to themselves, regarded as objects and arrows in $\mathbf{C}(G)$. Show that for any category $\mathbf{D}$ and graph homomorphism $f : G \to U(\mathbf{D})$, there is a unique functor

$$\overline{h} : \mathbf{C}(G) \to D$$

with

$$U((\bar{h})) \circ i = h,$$

where $U : \mathbf{Cat} \to \mathbf{Graph}$ is the underlying graph functor.

Answer:

13. ***Question:***

    Use the Cayley representaton to show that every small category is isomorphic to a "concrete" one, that is, one in which the objects are sets and the arrows are functions between them.

    Answer:

14. ***Question:***

    The notion of a category can also be defined with just one sort (arrows) rather than two (arrows and objects); the domains and codomains are taken to be certain *arrows* that act as units under composition, which is partially defined. Read about this definition in section I.1 of Mac Lane's *Categories for the Working Mathematician* , and do the exercise mentioned there, showing that it is equivalent to the usual definition.

    Answer:

2 Ch2 Abstract Structures

1. ***Question:***

   Show that a function between sets is an epimorphism if and only if it is surjective. Conclude that the isos in **Sets** are exactly the epi-monos.

   Answer:

2. ***Question:***

   Show that in a poset category, all arrows are both monic and epic.

   Answer:

3. ***Question:***

   (Inverses are unique.) If an arrow $f : A \to B$ has inverses $g, g' : B \to A$ (i.e., $g \circ f = 1_A$ and $f \circ g = 1_B$, and similarly for g'), then $g = g'$.

   Answer:

4. ***Question:***

   With regard to a commutative triangle,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

   in any category $\mathbf{C}$, show

   (a) if f and g are isos (resp. monos, resp. epis), so is h ;

   (b) if h is monic, so is f ;

   (c) if h is epice, so is g ;

   (d) (by example) if h is monic, g need not be.

   Answer:

5. ***Question:***

   Show that the following are equivalent for an arrow

$$f : A \to B$$

   in any category:

   (a) f is an isomorphism.

   (b) f is both a mono and a split epi.

   (c) f is both a split mono and an epi.

(d) f is both a split mono and a split epi.

Answer:

6. ***Question:***

   Show that a homomorphism $h : G \to H$ of graphs is monic just if it is injective on both edges and vertices.

   Answer:

7. ***Question:***

   Show that in any category, any retract of a projective object is also projective.

   Answer:

8. ***Question:***

   Show that all sets are projective (use the axiom of choice).

   Answer:

9. ***Question:***

   Show that the epis among posets are the surjections (on elements), and that the one-element poset $\mathbf{1}$ is projective.

   Answer:

10. ***Question:***

    Show that sets, regarded as discrete posets, are projective in the category of posets (use the foregoing exercises). Give an example of a poset that is not projective. Show that every projective poset is discrete, that is, a set.

conclude that **Sets** is (isomorphic to) the ”full subcategory” of projectives in **Pos**, conisting of all projective posets and all monotone maps between them.

Answer:

11. ***Question:***

    Let A be a set. Define an A-*monoid* to be a monoid M equipped with a function $m : A \to U(M)$ (to the underlying set of M). A morphism $h : (M, m) \to (N, n)$ of A-monoids is to be a monoid homomorphism $h : M \to N$ such that $U(h) \circ m = n$ (a commutative triangle). Together with the evident identities and composites, this defines a category A-**Mon** of A-monoids.

    Show that an initial object in A-**Mon** is the same thing as a free monoid $M(A)$ on A. (Hint: compare their respective UMPs.)

    Answer:

12. ***Question:***

    Show that for any Boolean algebra B, Boolean homomorphisms $h : B \to \mathbf{2}$ correspond exactly to ultrafilters in B .

    Answer:

13. ***Question:***

    In any category with binary products, show directly that

    $$A \times (B \times C) \cong (A \times B) \times C$$

    Answer:

14. ***Question:***

(a) For any index set I , define the product $\prod_{i \in I} X_i$ of an I-indexed family of objects $(X_i)_{i \in I}$ in a category, by giving a UMP generalizing that for binary products (the case $I = 2$).

(b) Show that in **Sets** , for any set X the set X^I of all functions $f : I \to X$ has this UMP, with respect to the "constant family" where $X_i = X$ for all $i \in I$, and thus

$$X^I \cong \prod_{i \in I} X$$

Answer:

15. ***Question:***

Given a category $\mathbf{C}$ with objects A and B, define the category $\mathbf{C}_{A,B}$ to have objects (X, x_1, x_2) where $x_1 : X \to A, x_2 : X \to B$, and with arrows $f : (X, x_1, x_2) \to (Y, y_1, y_2)$ being arrows $f : X \to Y$ with $y_1 \circ f = x_1$ and $y_2 \circ f = x_2$.

Show that $\mathbf{C}_{A,B}$ has a terminal object just in case A and B have a product in $\mathbf{C}$.

Answer:

16. ***Question:***

In the category of types $\mathbf{C}(\lambda)$ of the λ-calculus, determine the product functor $A, B \mapsto A \times B$ explicitly. Also show that, for any fixed type A, there is a functor $A \to (-) : \mathbf{C}(\lambda) \to \mathbf{C}(\lambda)$, taking any type X to $A \to X$.

Answer:

17. ***Question:***

In any category $\mathbf{C}$ with products, define the *graph* of an arrow $f : A \to B$ to be the monomorphism

$$\Gamma(f) = \langle 1_A, f \rangle : A \rightarrowtail A \times B$$

(Why is this monic?). Show that for $\mathbf{C} = \mathbf{Sets}$ this determines a functor $\Gamma : \mathbf{Sets} \to \mathbf{Rel}$ to the category $\mathbf{Rel}$ of relations, as defined in the exercises to Chapter 1. (To get an actual relation $R(f) \subseteq A \times B$, take the image of $\Gamma(f) : A \rightarrowtail A \times B$.)

Answer:

18. ***Question:***

    Show that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Sets}$ from monoids to sets is representable. Infer that U preserves all (small) products.

    Answer: