Calculus Module 1-3 Formulae Sheet

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Single Variable Calculus

1. If $f(x)$ is continous when $x \in [a,b]$ and is differentiable when $x \in (a,b)$
   Rolle's Theorem: and if $f(a) = f(b)$ then $\exists c \in (a,b)$ such that $f'(c) = 0$
   Langrange's Mean Value Theorem: then $\exists c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$
2. First Derivative Test : Find critical points c from $f'(x) = 0$. $\forall \epsilon > 0$ we have,
   If $f(x-\epsilon) < 0$ and $f(x+\epsilon) > 0$ then c is a local minimum.
   If $f(x-\epsilon) > 0$ and $f(x+\epsilon) < 0$ then c is a local maximum.
3. Second Derivative Test: If $f'(c) = 0$
   and if $f''(c) > 0$ then c is a local minimum.
   and if $f''(c) < 0$ then c is a local maximum.
   and if $f''(c) = 0$ and if $\forall \epsilon > 0$, $\text{sign}(f''(c-\epsilon)) = -\text{sign}(f+\epsilon)$ then c is an inflection point.
4. Average Value of Function $f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$
   MVT for Integrals: if $f(x)$ is continous on $[a,b]$ then $\exists c \in [a,b]$ such that $f(c) = f_{avg}$
5. Area Between Curves $A = \int_a^b |f(x) - g(x)|dx$ where $|f(x)-g(x)| = \begin{cases} f(x) - g(x), & \text{when} f(x) \geq g(x) \\ g(x) - f(x) & \text{when} g(x) \geq f(x) \end{cases}$ (split the integrals)
6. Volumes of Solids of Revolutions: If $f(x) \geq g(x)$, $x \in [a,b]$ and $g^{-1}(y) \geq f^{-1}(y)$, $y \in [c,d]$:
   - About $y = q$ using disks- $I = \int_a^b \pi[(f(x)-q)^2 - (g(x)-q)^2]dx$
   - About $x = p$ using disks - $I = \int_c^d \pi[(g^{-1}(y)-p)^2 - (f^{-1}(y)-p)^2]dy$
   - About $x = p$ using shells - $I = \int_a^b 2\pi|x-p|[f(x) - g(x)]dx$
   - About $y = q$ using shells- $I = \int_c^d 2\pi|y-q|[g^{-1}(y) - f^{-1}(y)]dy$
7. Taylor Expansion for $f(x)$ around a point a:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a^2)\cdots \implies \frac{f(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 \cdots = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

   For Maclaurin's series put $a = 0$.

Multivariable Calculus

1. Limits of $F(x,y)$
   To show that limit *doesnt exist* we find the limit along two different paths and find differing limit values.
   To *prove the existence* of a limit we use Epsilon-Delta Method: $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that if assumed $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \epsilon$ (manipulate $|f(x,y) - L|$ to get inequality with δ then express δ as a function of ϵ)
2. Continuity of $F(x,y)$: f is continous at point (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$
3. Partial Derivatives for a function $f(x,y)$
   (a) $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = f_{xx}$

(b) $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$

(c) for $f(x(t), y(t))$: $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$

4. Total Differential Value for $z = f(x, y)$ - $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

5. Jacobian $J = \frac{\partial(x,y)}{\partial(u,v)} \implies \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$

   Also if $J^{-1} = \frac{\partial(u,v)}{\partial(x,y)}$ we have, $J^{-1}J = 1$

   Two functions $u(x, y), v(x, y)$ are functionally dependent iff $\frac{\partial(u,v)}{\partial(x,y)} = 0$

Application of Multivariable Calculus

1. Taylor Expansion for two variables for approximating f around the point (a, b):

$$f(x, y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} \frac{1}{i!j!}\frac{\partial^{(i+j)} f(a, b)}{\partial x^i \partial y^j}(x - a)^i (y - b)^j$$

   First Order Approximation $(n = 1)$:$f(x, y) \approx L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$
   Second Order Approximation $(n = 2)$:$f \approx L(x, y) + \frac{1}{2}f_{xx}(a, b)(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{1}{2}f_{yy}(a, b)(y - b)^2$

2. For Maxima and Minima of $f(x, y)$, find stationary points from the equations $f_x = 0$ and $f_y = 0$. Then evaluate f_{xx}, f_{yy}, f_{xy} at stationary point $P(a, b)$

   - If $f_{xx}f_{yy} > (f_{xy})^2$ and $f_{xx} < 0$ or $f_{yy} < 0$ then P is a local Maxima
   - If $f_{xx}f_{yy} > (f_{xy})^2$ and $f_{xx} > 0$ or $f_{yy} > 0$ then P is a local Minima
   - If $f_{xx}f_{yy} < (f_{xy})^2$ then P is a saddle point(neither a minima nor a maxima).
   - If $f_{xx}f_{yy} = (f_{xy})^2$ then anything is possible (*RIP!*).

3. Constrained Maxima and Minima of $F(x, y)$

4. Langrange's Multiplier Method