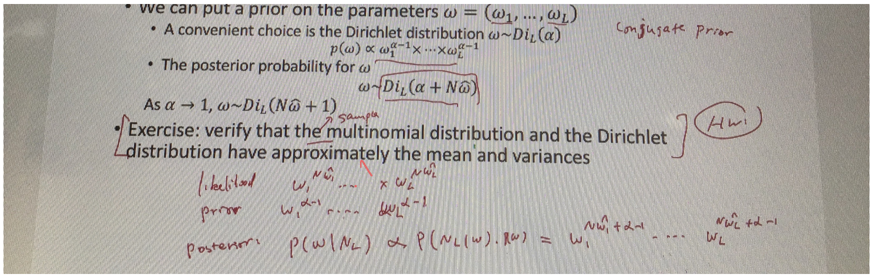
HW 7 Shuang Gao

Question 1



$\hat{\omega}_j$ the observed proportion in category j

ω_j the probability that a sample point falls in category j

① prior $w \sim Di_L(a)$

data $N\hat{w} \sim \text{Multinomial}(N, w)$

$$w \mid N\hat{w} \propto P(N\hat{w} \mid w)\, \pi(w)$$

$$= \frac{N!}{(N\hat{w}_1)! \cdots (N\hat{w}_L)!} w_1^{N\hat{w}_1} \cdots w_L^{N\hat{w}_L} \frac{\Gamma(La)}{\prod_{i=1}^{L} \Gamma(a)} \prod_{i=1}^{L} w_i^{a-1}$$

$$\propto w_1^{N\hat{w}_1 + a - 1} \cdots w_L^{N\hat{w}_L + a - 1}$$

thus $w \mid N\hat{w} \sim \text{Dirichlet}_L(N\hat{w} + a)$

let $a \to 0$ (flat prior), $w \mid N\hat{w} \sim \text{Dirichlet}_L(N\hat{w})$

② bootstrap dist

$$N\hat{w}^* \sim \text{Multi}(N, \hat{w})$$

Comparing ① & ②,

① $w \mid N\hat{w} \sim Dir_L(N\hat{w})$

mean $E(w_i) = \dfrac{N\hat{w}_i}{\sum_{i=1}^{L} N\hat{w}_i} = \hat{w}_i$

$$\text{Cov}(w_i, w_j) = \frac{-N\hat{w}_i N\hat{w}_j}{(\sum_{i=1}^{L} N\hat{w}_i)^2 (\sum_{i=1}^{L} N\hat{w}_i + 1)}$$

$$= \frac{-N^2 \hat{w}_i \hat{w}_j}{N^2 (N+1)}$$

$$= -\frac{\hat{w}_i \hat{w}_j}{N+1}$$

② $N\hat{w}^* \sim \text{Multi}(N, \hat{w})$

mean $E(N\hat{w}_i^*) = N\hat{w}_i$

$\Rightarrow E(\hat{w}_i^*) = \hat{w}_i$

$\text{Cov}(N\hat{w}_i^*, N\hat{w}_j^*)$

$= -N\hat{w}_i \hat{w}_j = N^2 \text{Cov}(\hat{w}_i^*, \hat{w}_j^*)$

$\Rightarrow \text{Cov}(\hat{w}_i^*, \hat{w}_j^*) = -\dfrac{\hat{w}_i \hat{w}_j}{N}$

Conclusion: 2 distribution have same mean and approximately same cov.

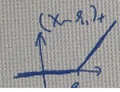
Question 2:

$\beta_1, \beta_4, \beta_5$ and β_6

From simple algebra, it can be shown that

$$(*)\quad f(X) = (\beta_1 + \beta_4 X) + \beta_5 (X - \xi_1)_+ + \beta_6 (X - \xi_2)_+ \quad \text{HW}$$

- So $f(X)$ can be expressed by 4 basis functions:
- $\tilde{h}_1(X) = 1$, $\tilde{h}_2(X) = X$, $\tilde{h}_3(X) = (X - \xi_1)_+$ and $\tilde{h}_4(X) = (X - \xi_2)_+$



$$(X-\xi_1)_+ = \begin{cases} X-\xi_1 & \text{if } X \geq \xi_1 \\ 0 & \text{if } X < \xi_1 \end{cases}$$

original form

$$f(X) = (\beta_1 + \beta_4 X)\, I(X < \xi_1) + (\beta_2 + \beta_5 X)\, I(\xi_1 \leq X < \xi_2) + (\beta_3 + \beta_6 X)\, I(X \geq \xi_2)$$

with 2 boundary condition

$$\begin{cases} \beta_2 = \beta_1 - (\beta_5 - \beta_4)\xi_1 \\ \beta_3 = \beta_1 - (\beta_5 - \beta_4)\xi_1 - (\beta_6 - \beta_5)\xi_2 \end{cases}$$

$$\begin{aligned} f(X) &= (\beta_1 + \beta_4 X)\, I(X < \xi_1) + (\beta_2 + \beta_5 X)\, I(\xi_1 \leq X < \xi_2) + (\beta_3 + \beta_6 X)\, I(X \geq \xi_2) \\ &= (\beta_1 + \beta_4 X)\, I(X < \xi_1) + [\beta_1 - (\beta_5 - \beta_4)\xi_1 + \beta_5 X]\, I(\xi_1 \leq X < \xi_2) \\ &\quad + [\beta_1 - (\beta_5 - \beta_4)\xi_1 - (\beta_6 - \beta_5)\xi_2 + \beta_6 X]\, I(X \geq \xi_2) \end{aligned}$$

① $X < \xi_1$, $f(X) = \beta_1 + \beta_4 X$

② $\xi_1 \leq X < \xi_2$, $f(X) = \beta_1 - (\beta_5 - \beta_4)\xi_1 + \beta_5 X$
$= \beta_1 + \beta_4 \xi_1 + \beta_5 (X - \xi_1)$

③ $X \geq \xi_2$, $f(X) = \beta_1 - (\beta_5 - \beta_4)\xi_1 - (\beta_6 - \beta_5)\xi_2 + \beta_6 X$
$= \beta_1 + \beta_4 \xi_1 + \beta_5(\xi_2 - \xi_1) + \beta_6 (X - \xi_2)$

From above 3 conditions, we can see 4 basis shown in the final form are $h_1(X) = 1$, $h_2(X) = X$,
$h_3(X) = (X - \xi_1)_+$
$h_4(X) = (X - \xi_2)_+$