Fiber circuits - E.Coli

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1 Circuits $|n = 0, \ell\rangle$

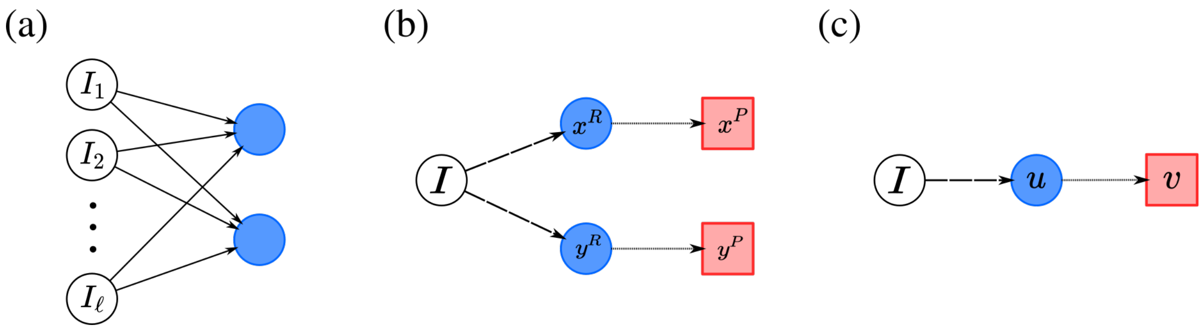

Figure 1: (a) Biological representation, (b) Mathematical representation, and (c) quotient network.

1.1 General admissible equations

$$\begin{aligned}\dot{x}^R &= f(x^R, I)\\ \dot{x}^P &= g(x^P, x^R)\\ \dot{y}^R &= f(y^R, I)\\ \dot{y}^P &= g(y^P, y^R),\end{aligned} \tag{1}$$

with vector of coordinates $\vec{x} = (x^R, x^P, y^R, y^P)$.

1.2 Jacobian and eigenvalues

Considering the partial derivatives obtained from Eq. 1, we have: ${\partial \dot{x}^R}/{\partial x^R} = f_1$; ${\partial \dot{x}^P}/{\partial x^R} = g_2$; ${\partial \dot{x}^P}/{\partial x^P} = g_1$; ${\partial \dot{y}^R}/{\partial x^R} = f_1$; ${\partial \dot{y}^P}/{\partial x^P} = g_1$; ${\partial \dot{y}^P}/{\partial y^R} = g_2$. From these derivatives at the equilibrium point, the Jacobian is given as

$$J = \begin{pmatrix} f_1 & 0 & 0 & 0\\ g_2 & g_1 & 0 & 0\\ 0 & 0 & f_1 & 0\\ 0 & 0 & g_2 & g_1 \end{pmatrix}\Bigg|_{eq.} = \begin{pmatrix} a & 0 & 0 & 0\\ c & b & 0 & 0\\ 0 & 0 & a & 0\\ 0 & 0 & c & b \end{pmatrix}. \tag{2}$$

Eigenvalues of J:
$\lambda_1 = a,\ \lambda_2 = a,\ \lambda_3 = b,\ \lambda_4 = b$

Eigenvectors of J:
$v_1 = \left(0, 0, \frac{a-b}{c}, 1\right)$
$v_2 = \left(\frac{a-b}{c}, 1, 0, 0\right)$
$v_3 = \left(0, 0, 0, 1\right)$
$v_4 = \left(0, 1, 0, 0\right)$

1.3 Realizations in E. coli

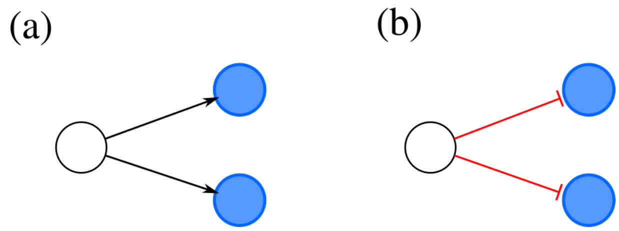

Figure 2: $|n = 0, \ell = 1\rangle$

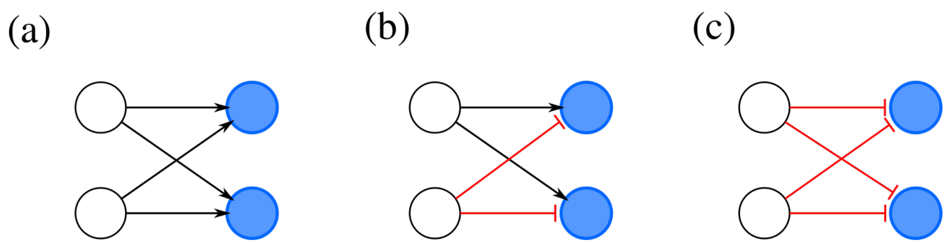

Figure 3: $|n = 0, \ell = 2\rangle$

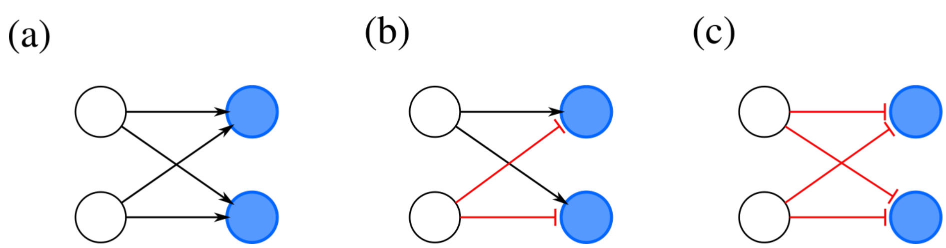

Figure 4: $|n = 0, \ell = 3\rangle$

1.4 Special model

Since all inputs $I_1, I_2, \ldots, I_\ell$ are symmetric, then we can represent all them by $I = \sum_j^\ell I_j$, which preserves the symmetry. Therefore, the special model for all circuits above is given by

$$f(u, I) = -\alpha u + I$$
$$g(u, v) = -\delta u + \beta v' \tag{3}$$

where we can observe that I cannot act as a bifurcation parameter.

1.5 Bifurcation conditions

Performing the partial derivatives on the special model we obtain that

$$\frac{\partial f}{\partial u} = -\alpha; \quad \frac{\partial g}{\partial u} = -\delta; \quad \frac{\partial g}{\partial v} = \beta, \tag{4}$$

for which we have

$$\lambda_{1,2} = -\alpha < 0; \quad \lambda_{3,4} = -\delta < 0. \tag{5}$$

In this case, we expect synchrony-preserving ($\lambda_{1,2} < 0$) bifurcations, but no oscillations ($\lambda_{3,4} < 0$).

2 Circuits $|n = 1, \ell\rangle$

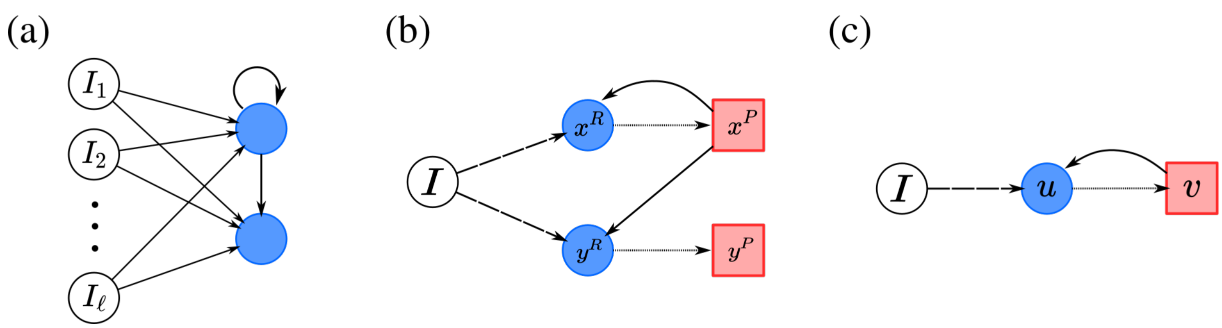

Figure 5: (a) Biological representation, (b) Mathematical representation, and (c) quotient network.

2.1 General admissible equations

$$\begin{aligned}
\dot{x}^R &= f(x^R, x^P, I) \\
\dot{x}^P &= g(x^P, x^R) \\
\dot{y}^R &= f(y^R, x^P, I) \\
\dot{y}^P &= g(y^P, y^R),
\end{aligned} \tag{6}$$

with vector of coordinates $\vec{x} = (x^R, x^P, y^R, y^P)$.

2.2 Jacobian and eigenvalues

Considering the partial derivatives obtained from Eq. 6, we have: ${\partial \dot{x}^R}/{\partial x^R} = f_1$, ${\partial \dot{x}^R}/{\partial x^P} = f_2$, ${\partial \dot{x}^P}/{\partial x^R} = g_2$, ${\partial \dot{x}^P}/{\partial x^P} = g_1$, ${\partial \dot{y}^R}/{\partial x^R} = f_1$, ${\partial \dot{y}^R}/{\partial x^P} = f_2$, ${\partial \dot{y}^P}/{\partial x^P} = g_1$, ${\partial \dot{y}^P}/{\partial y^R} = g_2$, From these derivatives at the equilibrium point, the Jacobian is given as

$$J = \begin{pmatrix} f_1 & f_2 & 0 & 0 \\ g_2 & g_1 & 0 & 0 \\ 0 & f_2 & f_1 & 0 \\ 0 & 0 & g_2 & g_1 \end{pmatrix}\Bigg|_{eq.} = \begin{pmatrix} a & b & 0 & 0 \\ d & c & 0 & 0 \\ 0 & b & a & 0 \\ 0 & 0 & d & c \end{pmatrix} \tag{7}$$

Eigenvalues of J:
$\lambda_1 = a$, $\lambda_2 = c$, $\lambda_{3,4} = \frac{1}{2}(a + c \mp \sqrt{(a-c)^2 + 4bd})$

Eigenvectors of J:
$v_1 = \Big(0, 0, \frac{a-c}{d}, 1\Big)$
$v_2 = \Big(0, 0, 0, 1\Big)$
$v_3 = \Big(-\frac{(-a+c+\sqrt{(a-c)^2+4bd})}{2d}, 1, -\frac{(-a+c+\sqrt{(a-c)^2+4bd})}{2d}, 1\Big)$
$v_4 = \Big(-\frac{(-a+c-\sqrt{(a-c)^2+4bd})}{2d}, 1, -\frac{(-a+c-\sqrt{(a-c)^2+4bd})}{2d}, 1\Big)$

2.3 Realizations in E. coli

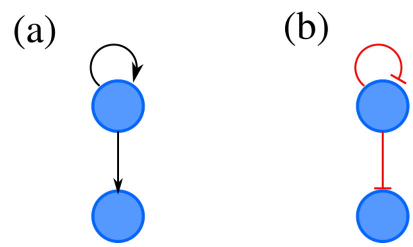

Figure 6: $|n = 1, \ell = 0\rangle$

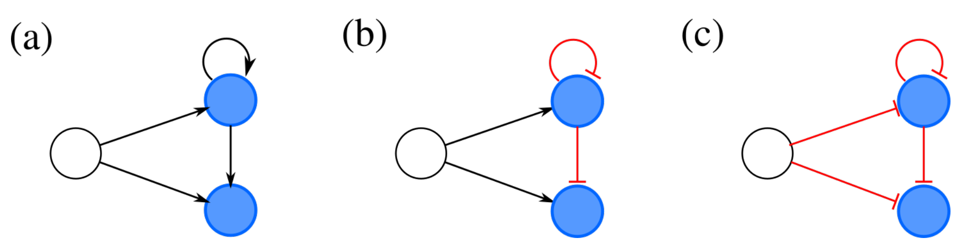

Figure 7: $|n = 1, \ell = 1\rangle$

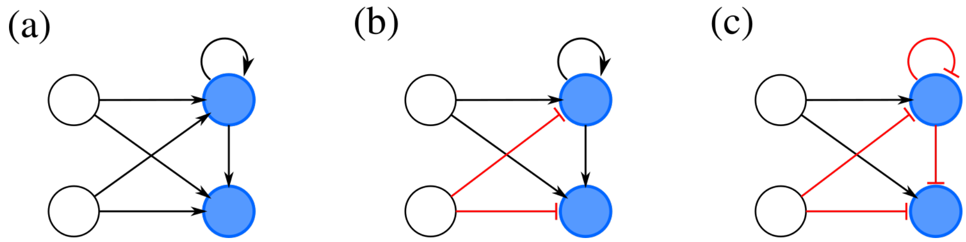

Figure 8: $|n = 1, \ell = 2\rangle$

2.4 Special model

Regarding the inputs $I_1, I_2, \ldots, I_\ell$ we know that they must be symmetric in order to keep the fibration symmetry. Therefore, we represent all the inputs by $I = \sum_j^\ell I_j$, which should preserve the symmetry. Now, considering the special model for the circuits above, we have two cases:

1. UNSAT-FFF Circuits: Fig. 6(b), Fig. 7(b) and (c), and Fig. 8(c).

$$\begin{aligned} f(u, v, I) &= -\alpha u + S(v) + I \\ g(u, v) &= -\delta u + \beta v \end{aligned} \tag{8}$$

2. SAT-FFF Circuits: Fig. 6(a), Fig. 7(a), and Fig. 8(a) and (b).

$$\begin{aligned} f(u, v, I) &= -\alpha u + (1 - S(v)) + I \\ g(u, v) &= -\delta u + \beta v \end{aligned}, \tag{9}$$

where in both cases I cannot act as bifurcation parameter.

2.5 Bifurcation conditions

Performing the partial derivatives on the special models above we obtain that for case 1

$$\frac{\partial f}{\partial u} = -\alpha; \quad \frac{\partial f}{\partial v} = S'(v) < 0; \quad \frac{\partial g}{\partial u} = -\delta; \quad \frac{\partial g}{\partial v} = \beta, \tag{10}$$

for which we have

$$a = -\alpha = \lambda_1; \quad b = S'(v); \quad c = -\delta = \lambda_2; \quad d = \beta. \tag{11}$$

In this case, we expect synchrony-preserving ($\lambda_{1,2} < 0$) bifurcations, and decaying oscillations since $\lambda_{3,4}$ are complex with negative real parts. If $a + c \approx 0$, we have approximately stable oscillations.

For case 2, we have

$$\frac{\partial f}{\partial u} = -\alpha; \quad \frac{\partial f}{\partial v} = -S'(v) > 0; \quad \frac{\partial g}{\partial u} = -\delta; \quad \frac{\partial g}{\partial v} = \beta, \tag{12}$$

for which we have

$$a = -\alpha = \lambda_1; \quad b = -S'(v); \quad c = -\delta = \lambda_2; \quad d = \beta. \tag{13}$$

In this case, we expect synchrony-preserving ($\lambda_{1,2} < 0$) bifurcations, but no oscillations since $(a-c)^2 + 4bd$ is positive.

3 Circuits $|n = 2, \ell\rangle$

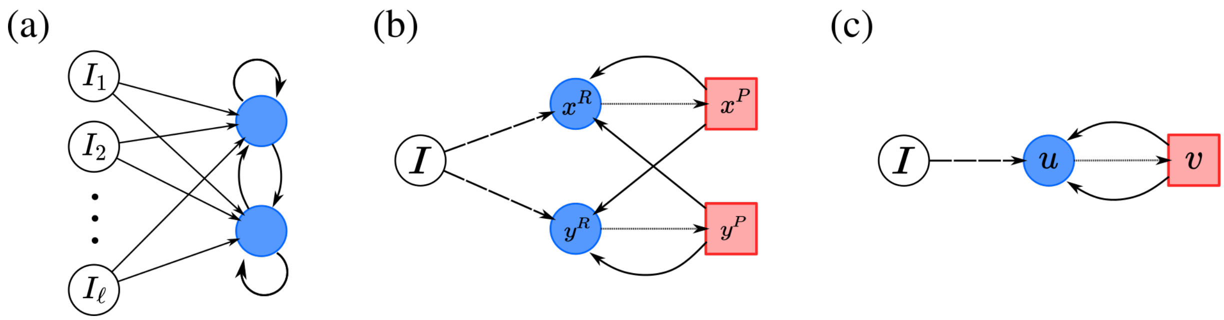

Figure 9: (a) Biological representation, (b) Mathematical representation, and (c) quotient network.

3.1 General admissible equations

$$\begin{aligned} \dot{x}^R &= f(x^R, \overline{x^P, y^P}, I) \\ \dot{x}^P &= g(x^P, x^R) \\ \dot{y}^R &= f(y^R, \overline{x^P, y^P}, I) \\ \dot{y}^P &= g(y^P, y^R), \end{aligned} \tag{14}$$

Obs: The bar $\overline{x^P, y^P}$ representing the vertex symmetry is only valid when all regulations are of the same type.

3.2 Jacobian and eigenvalues

Considering the partial derivatives obtained from Eq. 14, we have: $\partial \dot{x}^R / \partial x^R = f_1$, $\partial \dot{x}^R / \partial x^P = f_2$, $\partial \dot{x}^R / \partial y^P = f_3$, $\partial \dot{x}^P / \partial x^R = g_2$, $\partial \dot{x}^P / \partial x^P = g_1$, $\partial \dot{y}^R / \partial x^R = f_1$, $\partial \dot{y}^R / \partial x^P = f_2$, $\partial \dot{y}^R / \partial y^P = f_3$, $\partial \dot{y}^P / \partial x^P = g_1$, $\partial \dot{y}^P / \partial y^R = g_2$. From these derivatives at the equilibrium point, the Jacobian is given as

$$J = \begin{pmatrix} f_1 & f_2 & 0 & f_3 \\ g_2 & g_1 & 0 & 0 \\ 0 & f_2 & f_1 & f_3 \\ 0 & 0 & g_2 & g_1 \end{pmatrix}\Bigg|_{eq.} = \begin{pmatrix} a & b & 0 & c \\ e & d & 0 & 0 \\ 0 & b & a & c \\ 0 & 0 & e & d \end{pmatrix} \tag{15}$$

Eigenvalues of J:
$\lambda_1 = a,\ \lambda_2 = d,\ \lambda_{3,4} = \frac{1}{2}(a + d \mp \sqrt{(a-d)^2 + 4e(b+c)})$

Eigenvectors of J:
$v_1 = \left(\frac{c(d-a)}{be}, -\frac{c}{b}, \frac{a-d}{e}, 1\right)$
$v_2 = \left(0, -\frac{c}{b}, 0, 1\right)$
$v_3 = \left(-\frac{(d-a+\sqrt{(a-d)^2+4e(b+c)})}{2e}, 1, -\frac{(d-a+\sqrt{(a-d)^2+4e(b+c)})}{2e}, 1\right)$

$v_4 = \left(-\frac{(d-a-\sqrt{(a-d)^2+4e(b+c)})}{2e}, 1, -\frac{(d-a-\sqrt{(a-d)^2+4e(b+c)})}{2e}, 1\right)$

Obs: When all regulations are the same, then $f_2 = f_3$ at the equilibrium due to the vertex symmetry.

3.3 Realizations in E. coli

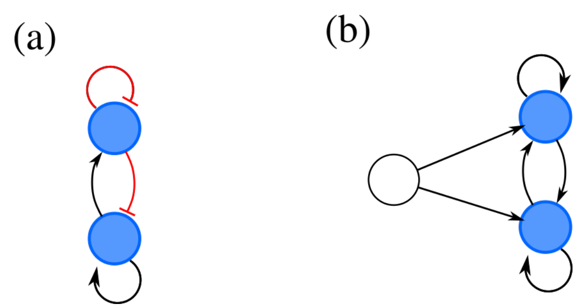

Figure 10: (a) $|n=2, \ell=0\rangle$. (b) $|n=2, \ell=1\rangle$

3.4 Special model

Following Prof. Ian's notes, we have for Fig. 10(a)

$$\begin{aligned} f(u,v,w,I) &= -\alpha u + 1 - S(v) + T(w) \\ g(u,v) &= -\delta u + \beta v \end{aligned}, \tag{16}$$

where $S(v)$ and $T(w)$ are both Hill functions.

For Fig. 10(b) we have

$$\begin{aligned} f(u,v,w,I) &= -\alpha u + (1 - S(v)) + (1 - S(w)) \\ g(u,v) &= -\delta u + \beta v \end{aligned} \tag{17}$$

3.5 Bifurcation conditions

Performing the partial derivatives on the special models above we obtain for Fig. 10(a) that

$$\frac{\partial f}{\partial u} = -\alpha; \quad \frac{\partial f}{\partial v} = -S'(v) > 0; \quad \frac{\partial f}{\partial w} = T'(w) < 0; \quad \frac{\partial g}{\partial u} = -\delta; \quad \frac{\partial g}{\partial v} = \beta, \tag{18}$$

for which we have

$$a = -\alpha = \lambda_1; \quad b = -S'(v); \quad c = T'(w); \quad d = -\delta = \lambda_2; \quad e = \beta. \tag{19}$$

In this case, we expect synchrony-preserving ($\lambda_{1,2} < 0$) bifurcations, and decaying oscillations depending on the magnitude of c. For these oscillations, they are approximately stable if $a + d \approx 0$.

For Fig. 10(b), we have

$$\frac{\partial f}{\partial u} = -\alpha; \quad \frac{\partial f}{\partial v} = -S'(v) > 0; \quad \frac{\partial f}{\partial w} = -S'(w) > 0; \quad \frac{\partial g}{\partial u} = -\delta; \quad \frac{\partial g}{\partial v} = \beta, \tag{20}$$

for which we have

$$a = -\alpha = \lambda_1; \quad b = c = -S'(v); \quad d = -\delta = \lambda_2; \quad e = \beta. \tag{21}$$

In this case, we expect synchrony-preserving ($\lambda_{1,2} < 0$) bifurcations, but no oscillations since $(a-d)^2 + 4e(b+c)$ is positive.

4 2-Fibo Fiber on $|n = 1, \ell = 2\rangle$ - Simple input

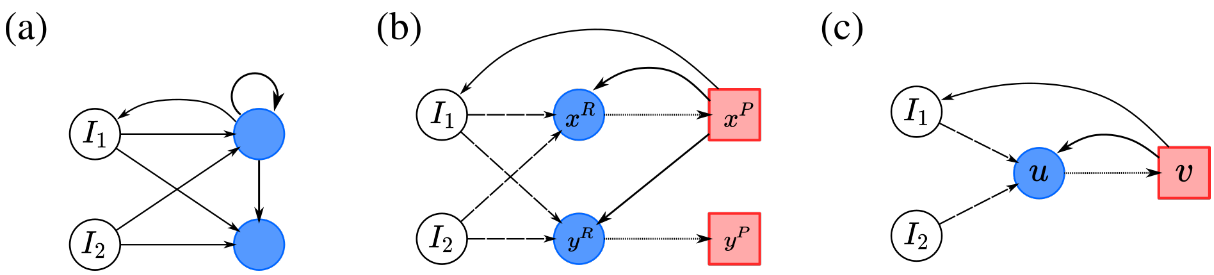

Figure 11: (a) Biological representation, (b) Mathematical representation, and (c) quotient network.

4.1 General admissible equations

$$\begin{aligned}
\dot{x}^R &= f(x^R, x^P, I_1(x^P), I_2) \\
\dot{x}^P &= g(x^P, x^R) \\
\dot{y}^R &= f(y^R, x^P, I_1(x^P), I_2) \\
\dot{y}^P &= g(y^P, y^R),
\end{aligned} \tag{22}$$

with vector of coordinates $\vec{x} = (x^R, x^P, y^R, y^P)$.

4.2 Jacobian and eigenvalues

Considering the partial derivatives obtained from Eq. 22, we have: ${}^{\partial \dot{x}^R}/_{\partial x^R} = f_1$, ${}^{\partial \dot{x}^R}/_{\partial x^P} = f_2$, ${}^{\partial \dot{x}^P}/_{\partial x^R} = g_2$, ${}^{\partial \dot{x}^P}/_{\partial x^P} = g_1$, ${}^{\partial \dot{y}^R}/_{\partial x^R} = f_1$, ${}^{\partial \dot{y}^R}/_{\partial x^P} = f_2$, ${}^{\partial \dot{y}^P}/_{\partial x^P} = g_1$, ${}^{\partial \dot{y}^P}/_{\partial y^R} = g_2$, From these derivatives at the equilibrium point, the Jacobian is given as

$$J = \begin{pmatrix} f_1 & f_2 & 0 & 0 \\ g_2 & g_1 & 0 & 0 \\ 0 & f_2 & f_1 & 0 \\ 0 & 0 & g_2 & g_1 \end{pmatrix}\Bigg|_{eq.} = \begin{pmatrix} a & b & 0 & 0 \\ d & c & 0 & 0 \\ 0 & b & a & 0 \\ 0 & 0 & d & c \end{pmatrix} \tag{23}$$

Eigenvalues of J:
$\lambda_1 = a$, $\lambda_2 = c$, $\lambda_{3,4} = \frac{1}{2}(a + c \mp \sqrt{(a-c)^2 + 4bd})$

Eigenvectors of J:
$v_1 = \Big(0, 0, \frac{a-c}{d}, 1\Big)$
$v_2 = \Big(0, 0, 0, 1\Big)$
$v_3 = \Big(-\frac{(-a+c+\sqrt{(a-c)^2+4bd})}{2d}, 1, -\frac{(-a+c+\sqrt{(a-c)^2+4bd})}{2d}, 1\Big)$
$v_4 = \Big(-\frac{(-a+c-\sqrt{(a-c)^2+4bd})}{2d}, 1, -\frac{(-a+c-\sqrt{(a-c)^2+4bd})}{2d}, 1\Big)$

4.3 Realizations in E. coli

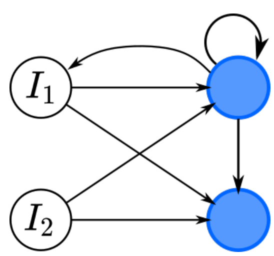

Figure 12: 2-Fibonnaci Fiber $|\phi_2 = 1.6180..., \ell = 2\rangle$

4.4 Special model

$$\begin{aligned} f(u, v, I_1, I_2) &= -\alpha u + (1 - S(v)) + I_1(v) + I_2 \\ g(u, v) &= -\delta u + \beta v \end{aligned} \tag{24}$$

Here we need to decide the expression of $I_1(v)$. For instance, $I_1(v) = Iv$ with I constant. In this case, we have ${}^{\partial f}/_{\partial v} = -S'(v) + I = b$ at the equilibrium point.

In this case, $b > 0$ if $I < S'(v)$; $b < 0$ if $I > S'(v)$, and $b = 0$ if $I = S'(v)$. Therefore, in this simple case I is a bifurcation parameter.

5 2-Fibonacci Fiber on $|n = 1, \ell = 2\rangle$

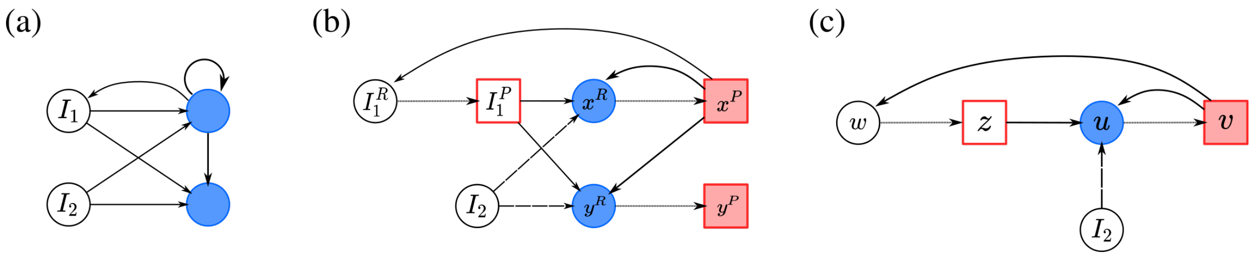

Figure 13: (a) Biological representation, (b) Mathematical representation, and (c) quotient network.

5.1 General admissible equations

$$\begin{aligned}
\dot{x}^R &= f(x^R, x^P, I_1, I_2) \\
\dot{x}^P &= g(x^P, x^R) \\
\dot{y}^R &= f(y^R, x^P, I_1, I_2) \\
\dot{y}^P &= g(y^P, y^R) \\
\dot{I}_1^R &= f(I_1^R, x^P) \\
\dot{I}_1^P &= g(I_1^P, I_1^R)
\end{aligned} \tag{25}$$

with vector of coordinates $\vec{x} = (x^R, x^P, y^R, y^P, I_1^R, I_1^P)$.

5.2 Jacobian and eigenvalues

Considering the partial derivatives obtained from Eq. 25, we have: $\partial\dot{x}^R/\partial x^R = f_1$, $\partial\dot{x}^R/\partial x^P = f_2$, $\partial\dot{x}^R/\partial I_1{}^P = f_3$, $\partial\dot{x}^P/\partial x^R = g_2$, $\partial\dot{x}^P/\partial x^P = g_1$, $\partial\dot{y}^R/\partial x^R = f_1$, $\partial\dot{y}^R/\partial x^P = f_2$, $\partial\dot{y}^R/\partial I_1{}^P = f_3$, $\partial\dot{y}^P/\partial x^P = g_1$, $\partial\dot{y}^P/\partial y^R = g_2$, $\partial\dot{I_1}^R/\partial I_1{}^R = i_1$, $\partial\dot{I_1}^R/\partial x^P = i_2$, $\partial\dot{I_1}^P/\partial I_1{}^R = j_2$, $\partial\dot{I_1}^P/\partial I_1{}^P = j_1$. From these derivatives at the equilibrium point, the Jacobian is given as

$$J = \begin{pmatrix} f_1 & f_2 & 0 & 0 & f_3 & 0 \\ g_2 & g_1 & 0 & 0 & 0 & 0 \\ 0 & f_2 & f_1 & 0 & f_3 & 0 \\ 0 & 0 & g_2 & g_1 & 0 & 0 \\ 0 & i_2 & 0 & 0 & i_1 & 0 \\ 0 & 0 & 0 & 0 & j_2 & j_1 \end{pmatrix}\Bigg|_{eq.} = \begin{pmatrix} a & b & 0 & 0 & c & 0 \\ e & d & 0 & 0 & 0 & 0 \\ 0 & b & a & 0 & c & 0 \\ 0 & 0 & e & d & 0 & 0 \\ 0 & q & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & l & k \end{pmatrix} \tag{26}$$