Chapter 9
Default Correlations and Credit Portfolio Risk

INTRODUCTION

Banks hold large portfolios of loans and, similarly to asset management, correlations or more generally dependencies are the main drivers of the risk of the portfolio. A bank commonly uses credit portfolio models to make assessments about the risks of a portfolio in terms of probability distributions of potential credit losses. Popular approaches used in the industry include actuarial and mathematical, as well as models that use computer simulations for generating the loss distribution. Usually, the outcome of such a model is a highly skewed probability distribution for the potential losses of the portfolio, as shown in Exhibit 9.1. The distribution is often characterized by some important parameters, namely the expected loss (EL), the value at risk (VaR, which is a quantile), and the conditional value at risk (CVaR) or expected shortfall(ES), which is the expectation of the losses that are greater than the value at risk. The expected loss is typically covered by provisions. The required economic capital for the bank to stay solvent is then the difference between the VaR and the expected loss (or CVaR and expected loss).

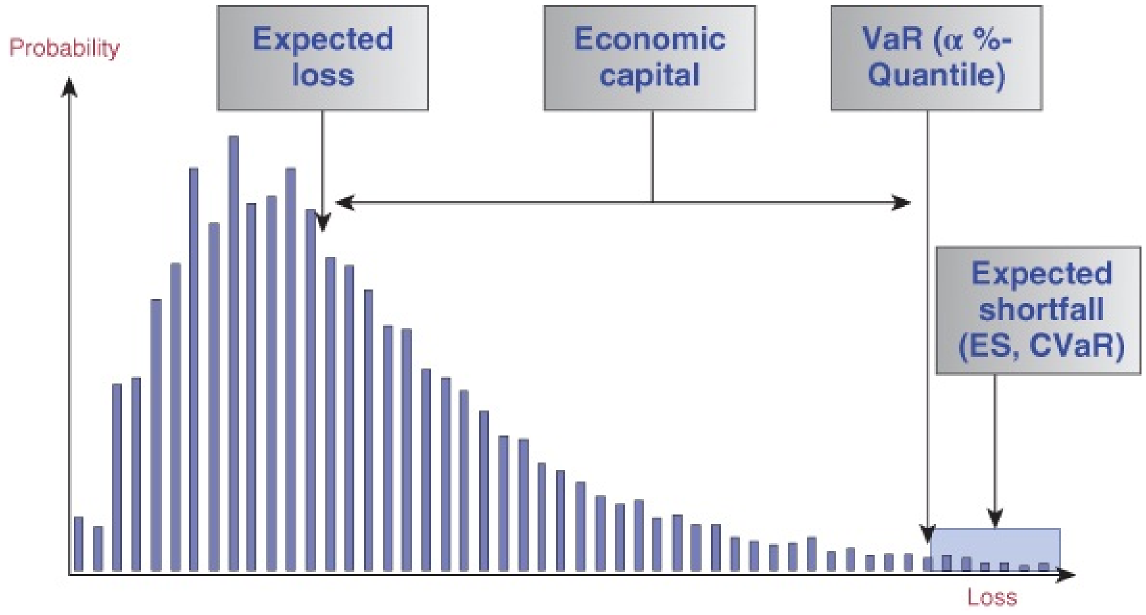

Exhibit 9.1 Stylized Loss Distribution

Let L_i be the random loss for a credit risky instrument i where $i = 1, \ldots, n$ in a certain time period. The portfolio loss is then the sum over the losses of all instruments (i.e., $L = \sum_{i=1}^{n} L_i$). Let $F(L)$ be the cumulative distribution function (CDF) of L. The risk measures then become:

- Expected loss:

$$E(L) = E\left(\sum_{i=1}^{n} L_i\right)$$

- Value at risk (α -quantile):

  $$VaR_\alpha(L) = \inf\{F^{-1}(\alpha)\}$$

  where $F^{-1}(\cdot)$ is the generalized inverse of the CDF.
- Economic capital:

  $$EC_\alpha = VaR_\alpha(L) - E(L)$$
- Expected shortfall:

  $$ES_\alpha(L) = E(L|L > VaR_\alpha(L))$$

Although there are different models used in the industry, a seminal paper by Gordy (2000) showed that the most widely applied approaches can be unified and, if adequately parameterized, yield very similar results. Therefore, in this chapter we focus on one particular specification, which is widely used in practice and also implemented in the Basel Accord for regulatory capital. At the end of the chapter, we briefly show, similarly to Gordy (2000), that the risk of choosing the wrong model is not a big issue on a portfolio level. Hence, we focus in this chapter on ways to estimate correlations from empirical data, which is often a key shortcoming of industry models.

MODELING LOSS DISTRIBUTIONS WITH CORRELATED DEFAULTS

In order to show the issue with correlation, let the loss of instrument i , $i = 1, \ldots, n$, be:

$$L_i = EAD_i \cdot LGD_i \cdot D_i$$

where EAD_i and LGD_i are the exposure at default and the loss given default, and D_i is a default indicator variable. Let us for now assume that EAD and LGD are deterministic. The randomness of the loss is then driven by the default indicator, which is given as in the earlier chapter by:

$$D_i = \begin{cases} 1 & \text{borrower } i \text{ defaults} \\ 0 & \text{otherwise} \end{cases}$$

Now consider a simple portfolio of two obligors with portfolio loss

$$L = L_1 + L_2 = EAD_1 \cdot LGD_1 \cdot D_1 + EAD_2 \cdot LGD_2 \cdot D_2$$

Let $P(D_i = 1)$ be the probability of borrower i defaulting ($i \in \{1, 2\}$) and let $P(D_i = 1 \cap D_j = 1)$ be the joint probability of both borrowers defaulting ($i \neq j$). There are four potential scenarios for the default behavior of both borrowers (and therefore for the portfolio loss), as depicted in the following table.

Borrower 1	Borrower 2 No Default	Default	
No Default	$P(D_1 = 0 \cap D_2 = 0)$	$P(D_1 = 0 \cap D_2 = 1)$	$P(D_1 = 0)$
Default	$P(D_1 = 1 \cap D_2 = 0)$	$P(D_1 = 1 \cap D_2 = 1)$	$P(D_1 = 1)$
	$P(D_2 = 0)$	$P(D_2 = 1)$	1

The outcome combinations are: (1) neither borrower defaults, (2) Borrower 1 defaults, (3) Borrower 2 defaults, or (4) both borrowers default. The respective probabilities for these events are given in the table. The probabilities at the margins of the table are the marginal probabilities of default and nondefault for each borrower. In the special case of independence of the two borrowers, the probability that both borrowers default (i.e., their joint default probability $P(D_1 = 1 \cap D_2 = 1)$) equals the product of their marginal probabilities (which is per definition of independence). However, if the defaults are positively dependent (e.g., due to macroeconomic factors, which are the same for all borrowers in a given period), the joint default probability is higher than in the case of independence. As the joint default probability gives the probability for the highest possible loss of the portfolio (namely a total loss), we can see that, ceteris paribus, the extreme risk of the bank increases with default dependence. The default correlation can be computed as:

$$\rho^D = Corr(D_1, D_2) = \frac{E(D_1 \cdot D_2) - E(D_1) \cdot E(D_2)}{\sqrt{\text{Var}(D_1)\text{Var}(D_2)}}$$
$$= \frac{P(D_1 = 1 \cap D_2 = 1) - P(D_1 = 1) \cdot P(D_2 = 1)}{\sqrt{P(D_1 = 1)(1 - P(D_1 = 1))P(D_2 = 1)(1 - P(D_2 = 1))}}$$

For two borrowers, it is hard to empirically quantify a value for the default correlation or the joint default probability in practice. For a higher number of borrowers this becomes burdensome, if not impossible. For two borrowers, there are four probabilities in the table. For n borrowers, this would require 2^n probabilities, which is challenging for most credit portfolios. Therefore, we introduce a model that makes some simplifying assumptions.

Basic Model Framework

The Bernoulli default indicator variable can be characterized using some underlying metric variable that renders $D_i = 1$ if it crosses some threshold, and $D_i = 0$ otherwise. Let R_i be a metric variable following a continuous distribution. We then have:

$$D_i = \begin{cases} 1 & \text{if } R_i < c_i \\ 0 & \text{if } R_i \geq c_i \end{cases}$$

where c_i is the threshold value. The probability of default can then be expressed as $PD_i = P(D_i = 1) = P(R_i < c_i)$. If one assumes a structural default model and interprets R_i as a normalized asset return (i.e., normally distributed and standardized), then the PD is simply given as $PD_i = P(R_i < c_i) = \Phi(c_i)$, which is obviously the approach taken in the probit model as discussed in the PD chapter.

To introduce correlation of defaults, we let the underlying metric variables be correlated by a common stochastic variable. The R_i variable is split into an idiosyncratic component ϵ_i and a (systematic) component X (common to all i) via

$$R_i = \sqrt{\rho}X + \sqrt{1-\rho}\epsilon_i$$

where $X \sim N(0,1)$, $\epsilon_i \sim N(0,1)$ i.i.d. The correlation between R_i and $R_j, j \neq i$, is given by the parameter ρ. Because R_i can be thought of as the asset return in an underlying structural framework, ρ is often called the “asset (return) correlation.” The PD can still be given by $PD_i = P(R_i < c_i)$, as R_i is standard normally distributed.

Under this extension, a stochastic, conditional PD (CPD), can be given that is conditioned on the systematic random factor X as

$$CPD_i(X) = \Phi\left(\frac{c_i - \sqrt{\rho}X}{\sqrt{1-\rho}}\right)$$

Analytical Solution

The CPD has expectation

$$\begin{aligned} E(CPD_i(X)) &= \int_{-\infty}^{\infty} \Phi\left(\frac{c_i - \sqrt{\rho}x}{\sqrt{1-\rho}}\right)\phi(x)dx \\ &= PD_i \end{aligned}$$

where $\phi(\cdot)$ is the standard normal probability density function (PDF), and variance

$$\begin{aligned} \text{Var}(CPD_i(X)) &= E(CPD_i(X)^2) - [E(CPD_i(X))]^2 \\ &= \Phi(c_i, c_i, \rho) - PD_i^2 \end{aligned}$$

where $\Phi(c_i, c_i, \rho)$ is the bivariate normal CDF with correlation ρ and standardized margins; see Gordy (2000). The density $g(p_i)$ of $P_i = CPD_i(X)$ is given by:

$$\begin{aligned} g(p_i) &= \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \cdot \frac{\phi\left(\frac{c_i - \Phi^{-1}(p_i)\sqrt{1-\rho}}{\sqrt{\rho}}\right)}{\phi(\Phi^{-1}(p_i))} \\ &= \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \cdot \exp\left(\frac{1}{2}(\Phi^{-1}(p_i))^2 - \frac{1}{2\rho}(c_i - \sqrt{1-\rho} \cdot \Phi^{-1}(p_i))^2\right) \end{aligned}$$

(see Vasicek 1987, 1991; Koyluoglu and Hickman 1998). The CDF $G(p_i)$ and α-quantile are:

$$G(p_i) = \Phi\left(\frac{\sqrt{1-\rho}\cdot\Phi^{-1}(p_i) - c_i}{\sqrt{\rho}}\right)$$

$$q_\alpha = \Phi\left(\frac{c_i + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)$$

The joint probability of default for two obligors now becomes:

$$\begin{aligned} P(D_1 = 1 \cap D_2 = 1) &= \int_{-\infty}^{\infty} \Phi\left(\frac{c_1 - \sqrt{\rho}x}{\sqrt{1-\rho}}\right)\Phi\left(\frac{c_2 - \sqrt{\rho}x}{\sqrt{1-\rho}}\right)\phi(x)dx \\ &= \Phi(c_1, c_2, \rho) \end{aligned}$$

where $\phi(x)$ is the standard normal PDF. The default correlation is:

$$\rho^D = Corr(D_1, D_2) = \frac{\Phi(c_1, c_2, \rho) - PD_1 PD_2}{\sqrt{PD_1(1-PD_1)}\sqrt{PD_2(1-PD_2)}}$$

It can be shown that if a portfolio has homogeneous parameters (i.e., PD and correlation) and infinitely many borrowers, the density $g(p_i)$ represents its default rate distribution (see Vasicek 1987, 1991). This model is sometimes called the asymptotic single risk factor (ASRF) model or Vasicek model. It is implemented in the internal ratings based approach (IRBA) of the Basel Accord and has some nice properties. The risk contribution of any loan to the risk of a portfolio can be computed by considering the properties of the loan only and does not require considering the portfolio in which the loan is held; for technical details see Gordy (2003). Thus, risk contributions are portfolio invariant. Hence, the risk contribution of a loan in terms of value at risk is directly given by q_α. So, if the correlation and the confidence level α are given, the risk contribution and hence capital can be computed as a function of the probability of default (PD). This is exactly the approach followed in the Basel IRBA formula, which uses q_α as the key constituent (see Basel Committee on Banking Supervision 2006, 2011). There the capital requirements are computed via

$$C = LGD \cdot \left[\Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}\Phi^{-1}(0.999)}{\sqrt{1-\rho}}\right) - PD\right] \cdot \frac{1 + (M - 2.5)b}{1 - 1.5b}$$

where the last factor is a maturity adjustment with $b = (0.11852 - 0.05478 \cdot \ln\ (PD))$ to account for the fact that long-term credits are more risky than short-term credits and maturity effects are more pronounced for obligors with low PDs. In the capital requirement formula for corporates, sovereigns, and banks, the correlation is a function of the PD as follows:

$$\rho \equiv \rho(PD) = 0.12\frac{1 - \exp^{-50\cdot PD}}{1 - \exp^{-50}} + 0.24\left(1 - \frac{1 - \exp^{-50\cdot PD}}{1 - \exp^{-50}}\right)$$

The first expression in square brackets of the capital formula is the 99.9th percentile of the CPD from which the expected loss is subtracted. The result is then multiplied by the loss given default (LGD) and the maturity adjustment. The correlation is assumed to be a decreasing function of the PD with lower bound of 12 percent for high PDs and upper bound of 24 percent for low PDs. The decreasing form of the correlation leads to a flattening of the capital curve for higher PDs. Risk-weighted assets (RWA) are then computed as:

$$RWA = C \cdot 12.5 \cdot EAD$$

The maturity M represents the nominal or effective maturity and is between 1 and 5 years. For exposures to large financial sector entities, ρ is multiplied by 1.25. For retail exposures, there is no maturity adjustment. For residential mortgages, the asset correlation has been set to 15 percent, for qualifying revolving exposures to 4 percent. For other retail exposures the asset correlation is a decreasing function of the PD with lower bound of 3 percent for high PDs and upper bound of 16 percent for low PDs.

$$\rho \equiv \rho(PD) = 0.03 \frac{1 - \exp^{-35 \cdot PD}}{1 - \exp^{-35}} + 0.16 \left(1 - \frac{1 - \exp^{-35 \cdot PD}}{1 - \exp^{-35}}\right)$$

In the United States, the latter has been set to: $\rho = 0.03 + 0.13\exp^{-35 \cdot PD}$. We analyze these asset correlations further in our stress testing chapter.

It is important to emphasize again that the risk weight functions cover unexpected loss, since expected loss should be covered by provisions. Also note that the risk-weighted assets are not explicitly calculated. They can, however, be backed out using the relationship that regulatory capital equals 8 percent of the risk-weighted assets. This implies that the risk-weighted assets are equal to 12.50 times the regulatory capital. Also, as a result of various quantitative impact studiesrun by the Basel group, an additional scaling factor of 1.06 is applied to the risk-weighted assets. See Articles 153 and 154 of the EU directive for more details (European Union 2013).

Remember the crucial role of the asset correlation term in the capital requirements formula. The question is: How are these values determined? For corporates, the assets can be unambiguously quantified by inspecting balance sheets, and various financial models have been introduced to quantify asset correlations in this context. For retail exposures, it becomes considerably more difficult, as the assets are less tangible, with the exception of real estate values, which are generally reasonably well described. The asset correlations have been determined using some empirical but not published procedure. They reflect a combination of supervisory judgment and empirical evidence. It is assumed that they are set based on reverse engineering of economic capital models from a selection of large banks worldwide. Note that, as you can see in the Vasicek model, the asset correlations also measure how the asset class is dependent upon the state of the economy.

The loss density of the model can be computed for two homogeneous portfolios in SAS by the following IML code. The PDs are 1 percent and 5 percent and the asset correlations are 10 percent and 20 percent for portfolio 1 and portfolio 2.

```
PROC IML;
/*Set parameters portfolio 1*/
```

```
PD1 = 0.01;
rho1 = 0.1;
/*Set parameters portfolio 2*/
PD2 = 0.05;
rho2 = 0.2;
/*Generate skeleton for loss distribution*/
DO p_i = 0.0001 TO 0.2 BY 0.0001;
/*Density portfolio 1*/
g_p_i1 = SQRT(1-rho1)/ SQRT(rho1)
* EXP(0.5*(PROBIT(p_i)**2) -0.5/rho1
* (PROBIT(PD1)-SQRT(1-rho1) * PROBIT(p_i))**2 );
/*Density portfolio 2*/
g_p_i2 = SQRT(1-rho2)/ SQRT(rho2)
* EXP(0.5*(PROBIT(p_i)**2) -0.5/rho2
* (PROBIT(PD2)-SQRT(1-rho2) * PROBIT(p_i))**2 );
/*Generate output data*/
out = out//(p_i||g_p_i1||g_p_i2);
END;
/*Name output variables*/
x={'loss','PD_1', 'PD_5'};
/*Export output data set*/
CREATE  analytic FROM out [colname=x];
APPEND FROM out;
QUIT;
```

The following code generates the PDF for the portfolio loss. The vertical axis plots the density as the loss is continuous. (See also Exhibit 9.2.)

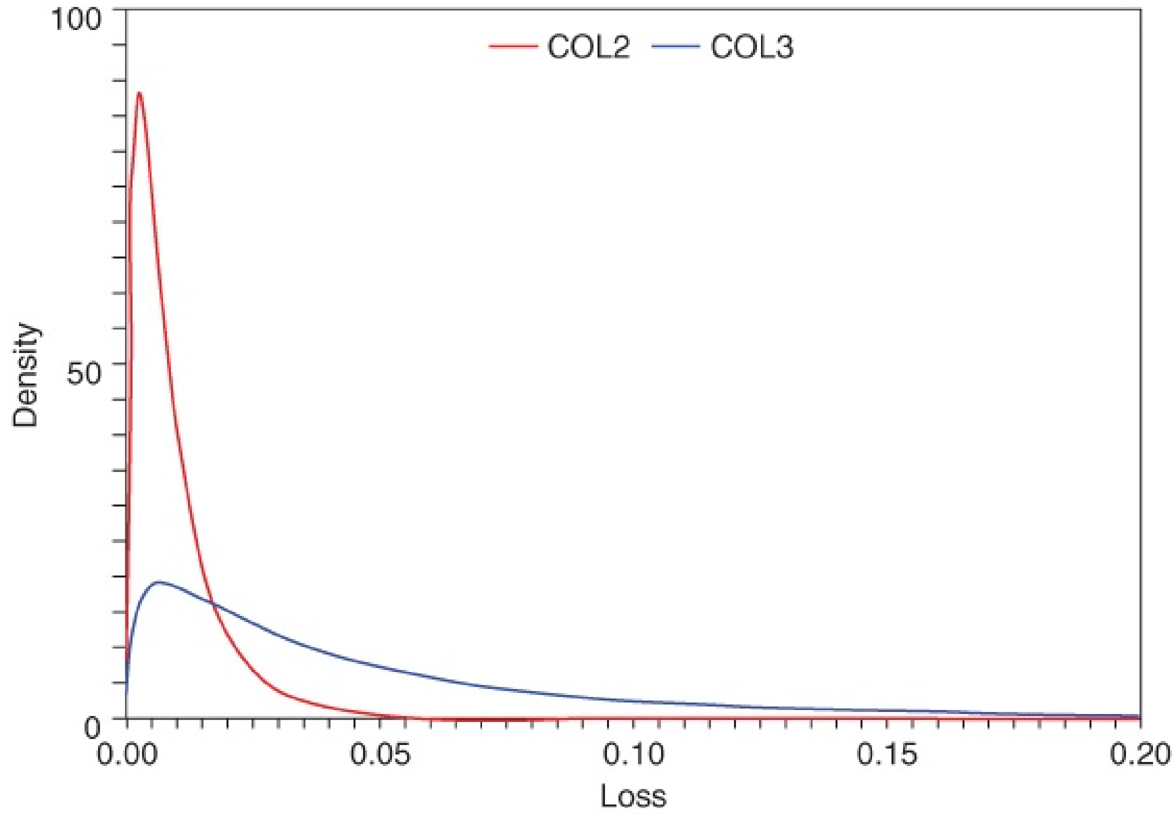

Exhibit 9.2 Analytical Loss Distributions

```
ODS GRAPHICS ON;
```

```
GOPTIONS RESET=GLOBAL GUNIT=PCT NOBORDER CBACK=WHITE
COLORS=(BLACK BLUE GREEN RED)
FTITLE=SWISSB FTEXT=SWISS HTITLE=3 HTEXT=3;
SYMBOL1 COLOR=RED INTERPOL=JOIN WIDTH=3 VALUE=NONE HEIGHT=0;
SYMBOL2 VALUE=NONE COLOR=BLUE INTERPOL=JOIN WIDTH=3 HEIGHT=0;
AXIS1 ORDER=(0 TO 0.2 BY 0.05) OFFSET=(0,0)
LABEL=('Loss')
MAJOR=(HEIGHT=1) MINOR=(HEIGHT=1)
WIDTH=3;
AXIS2 ORDER=(0 TO 100 BY 50) OFFSET=(0,0)
LABEL=('Density')
MAJOR=(HEIGHT=1) MINOR=(HEIGHT=1)
WIDTH=3;
LEGEND1 LABEL=NONE
SHAPE=SYMBOL(4,2)
POSITION=(TOP CENTER INSIDE)
MODE=SHARE;
PROC GPLOT DATA=analytic;
PLOT PD_1*loss PD_5*loss / OVERLAY LEGEND=LEGEND1
HAXIS=AXIS1
VAXIS=AXIS2;
RUN;
ODS GRAPHICS OFF;
```

Numerical Solution

In a portfolio with a finite number of borrowers, idiosyncratic risk is not canceled out, which means it must be taken into account when the loss distribution is computed. We condition again on the systematic factor and assume homogeneous borrowers (i.e., $PD_i = PD, \forall i$). Note that the remaining stochastic variables ϵ_i are independent. Assume further that defaults are independent for a given realization of the systematic factor $X = x$. The number of defaults $D = \sum_{i=1}^{n} D_i$ is conditionally binomially distributed:

$$P(D = d|x) = \binom{n}{d} CPD(x)^d (1 - CPD(x))^{n-d}, d = 0, \dots, n$$

The final unconditional distribution is then found by mixing (integrating) over the distribution of the random factor X as

$$P(D = d) = \int_{-\infty}^{\infty} \binom{n}{d} CPD(x)^d (1 - CPD(x))^{n-d} \phi(x) dx, d = 0, \dots, n$$

The integral can be calculated using IML, as shown in the following code using CALL QUAD within PROC IML. CALL QUAD solves integrals that are included in a separate module in PROC IML numerically. CALL QUAD has a number of inputs, of which we use the first four in the statement “CALL QUAD(z,“fun”,a,eps);”:

- z is a numeric vector containing the results of the integration.
- “fun” specifies the name of an IML module that contains the integral to be solved.

- "a" contains the limits of the integral. We specify "a = .M .P;" which implies integration from $-\infty$ to ∞.
- eps is a specification of accuracy of the numerical process, and we specify: "eps = 1.34E-15;" The choice of eps is optional and involves a trade-off between computing time and accuracy: The smaller this number, the longer the numerical solution takes and the more accurate the result will be.

More details on numerical integration using SAS can be found in the SAS manual; see SAS Institute Inc. (2015).

```
ODS GRAPHICS ON;
PROC IML SYMSIZE=10000000 WORKSIZE = 10000000;
/*Set asset correlation*/
rho = 0.2;
/*Set number of borrowers*/
N = 100;
/*Set PD*/
p = 0.05;
/*Define output matrix*/
out = J(1,7,0);
/*Create module fun - name is arbitrary*/
START fun(x) GLOBAL(k,p,N,rho);
pi = CONSTANT('PI');
/*Compute the CPD and multiply with the standard normal pdf */
CPD = PROBNORM((1/SQRT(1-rho)) * (PROBIT(p)-SQRT(rho)*x));
v   = PROBBNML(CPD,n,k) * (EXP(-(x*x)/2))/(SQRT(2*pi));
RETURN(v);
FINISH fun;
/*Increase k up to desired percentile*/
k=0;
DO UNTIL(z>0.9999);
/*Call QUAD to compute the integral */
a   = {.M .P};
eps = 1.34E-15;
CALL QUAD(z,"fun",a,eps);
quantile=k;
default_rate= quantile/N;
IF k = 0 THEN DO;
prob = z;
out = out//(N||rho||p||quantile||default_rate||z||prob);
END;
IF k> 0 THEN DO;
prob = z-out[k+1,6];
out = out//(N||rho||p||quantile||default_rate||z||prob);
END;
k=k+1;
END;
out = out[2:NROW(out),];
/*Export output data set*/
CREATE  numeric FROM out;
APPEND FROM out;
QUIT;
```

The following code generates the histogram for the loss distribution. (See Exhibit 9.3.) The vertical axis plots probabilities, as we have discrete loss observations.

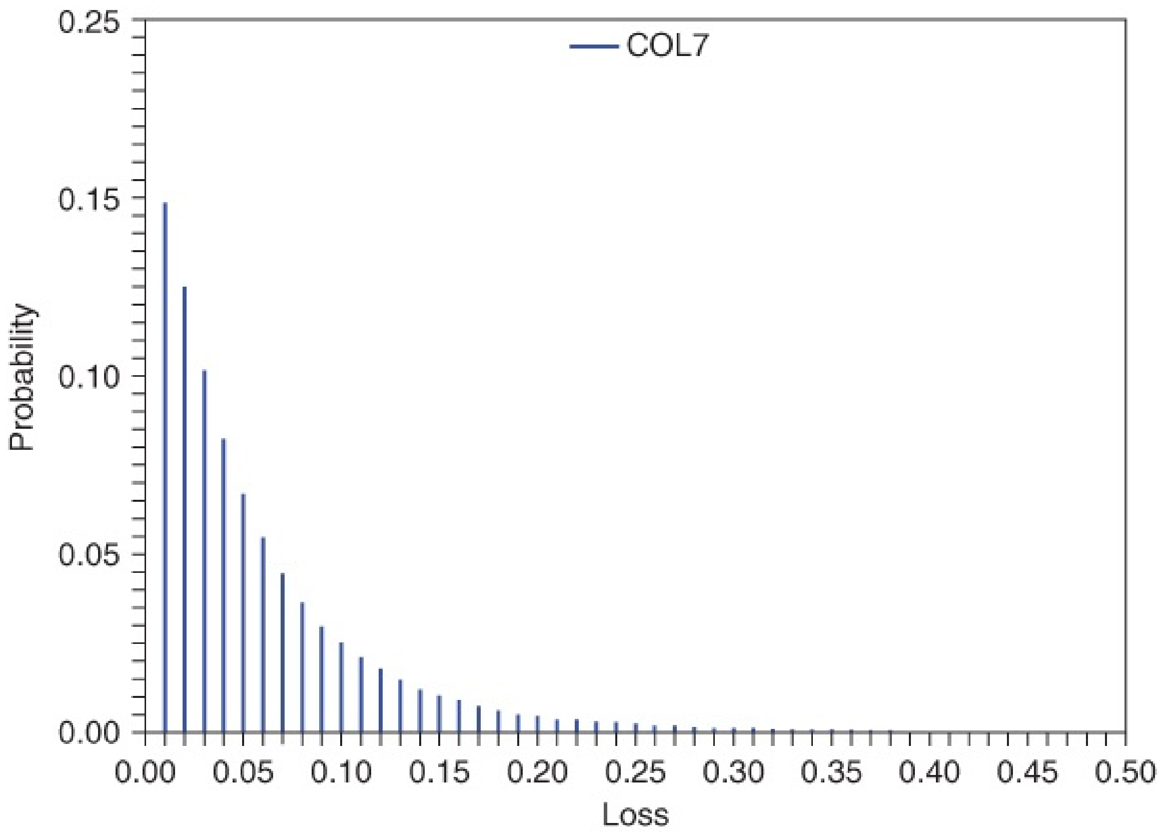

Exhibit 9.3 Numerically Computed Loss Distribution

```
GOPTIONS RESET=GLOBAL GUNIT=PCT NOBORDER CBACK=WHITE
COLORS=(BLACK BLUE GREEN RED)
FTITLE=SWISSB FTEXT=SWISS HTITLE=3 HTEXT=3;
SYMBOL1 COLOR=BLUE INTERPOL=NEEDLE VALUE=NONE HEIGHT=0 WIDTH=3;
AXIS1 ORDER=(0 TO 0.5 BY 0.05) OFFSET=(0,0)
LABEL=('Loss') MAJOR=(HEIGHT=1) MINOR=(HEIGHT=1) WIDTH=3;
AXIS2 ORDER=(0 TO 0.2 BY 0.05) OFFSET=(0,0)
LABEL=('Probability') MAJOR=(HEIGHT=1) MINOR=(HEIGHT=1) WIDTH=3;
ODS GRAPHICS ON;
PROC GPLOT DATA=numeric;
PLOT COL7*COL5 /
HAXIS=AXIS1
VAXIS=AXIS2;
RUN;
ODS GRAPHICS OFF;
```

Monte Carlo Simulation

The most flexible way of deriving loss distributions is via Monte Carlo simulations (e.g., using PROC IML). Here, the random variables of the model are drawn by a random number generator according to the specified distributions. In the preceding model, the model parameters have to be given, and standard normally distributed random variables are simulated from which the variable R_i is computed (denoted as “lnV” in the code). R_i is then compared to the default threshold c_i and a default variable is set to 1 if $R_i < c_i$ and 0 otherwise. The data are adequately stored and can be analyzed for instance by PROC UNIVARIATE, as shown in the following code.

```
PROC IML SYMSIZE=10000000 WORKSIZE = 10000000;
/*Set number of simulations*/
N_sim = 100000;
/*Set number of borrowers*/
N  = 10000;
/*Set homogeneous PDs*/
p = 0.05;
/*Create vector of PDs - may easily be extended to individual PDs*/
PD = J(N,1,p);
/*Set asset correlations*/
w_2 = 0.2 ;
w = SQRT(w_2);
ksi = SQRT(1-w_2);
/*Beginning of simulation loop*/
DO sim = 1 TO N_sim;
x1  = J(1,1,0);
x2  = J(N,1,0);
/*Simulation standard normal random numbers*/
z   = RANNOR (x1);
eps = RANNOR (x2);
/*Generation of asset returns*/
lnV = J(N,1,w) * z  + J(N, 1,ksi) #  eps;
threshold    =   PROBIT(PD);
defaults = lnv<threshold;
default_rate    = defaults[+]/N;
out= out//(sim||N||defaults[+]||default_rate);
END;
/*Name variables*/
x={'sim', 'N', 'D', 'Loss'};
/*Export output data*/
CREATE sim FROM out [colname=x];
APPEND FROM out;
QUIT;
```

We obtain 100,000 realizations of the default rate. (See Exhibit 9.4.) The default rate can be interpreted as a loss rate under the assumption of an EAD and a LGD of unity. We generate a histogram (Exhibit 9.5) using PROC UNIVARIATE for the rather large number of observations in our output data set:

The UNIVARIATE Procedure

Variable: Loss

Moments			
N	100000	Sum Weights	100000
Mean	0.05001205	Sum Observations	5001.2053
Std Deviation	0.05223872	Variance	0.00272888
Skewness	2.30169689	Kurtosis	7.77479974
Uncorrected SS	523.006189	Corrected SS	272.885645
Coeff Variation	104.452258	Std Error Mean	0.00016519

Basic Statistical Measures			
Location		Variability	
Mean	0.050012	Std Deviation	0.05224
Median	0.033100	Variance	0.00273
Mode	0.007500	Range	0.62720
		Interquartile Range	0.05180

Quantiles (Definition 5)	
Level	Quantile
100% Max	0.62720
99%	0.24905
95%	0.15530
90%	0.11560
75% Q3	0.06670
50% Median	0.03310
25% Q1	0.01490
10%	0.00650
5%	0.00390
1%	0.00130
0% Min	0.00000

Exhibit 9.4 Monte Carlo Simulation

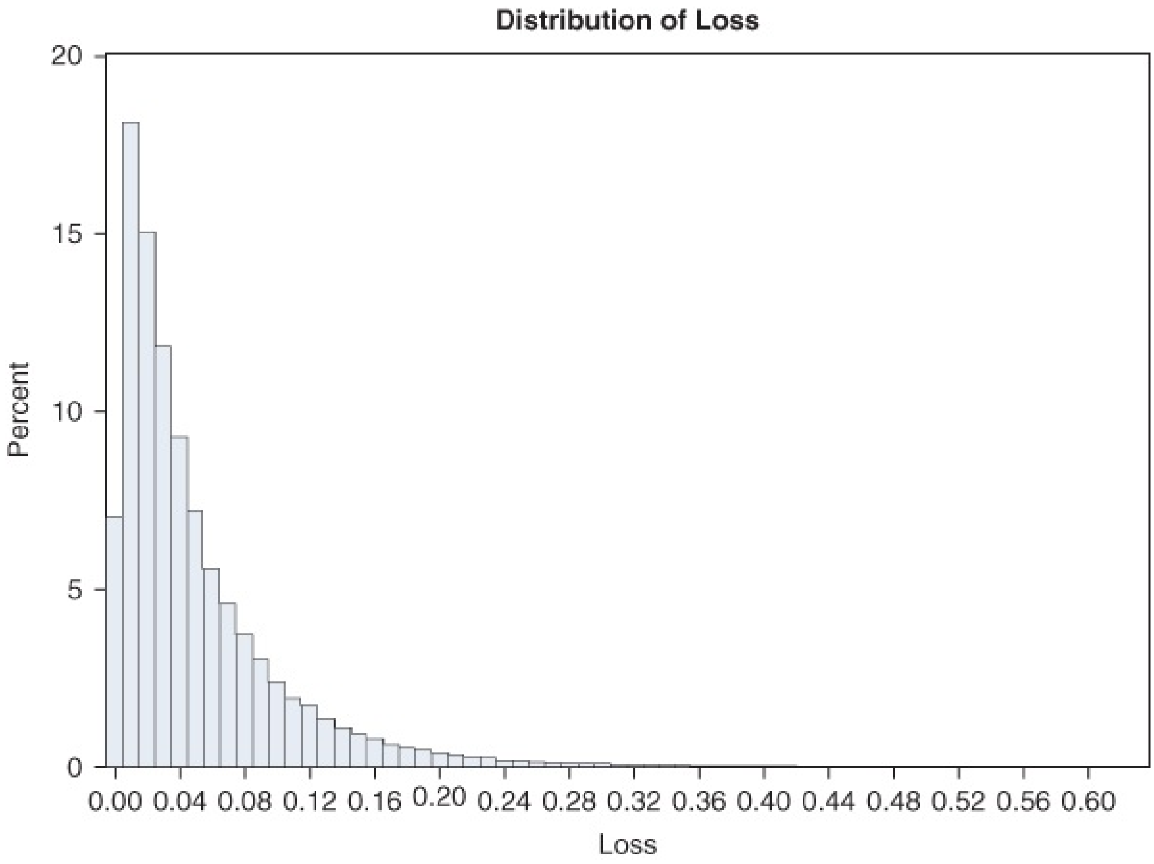

Exhibit 9.5 Simulated Loss Distribution

```
ODS GRAPHICS ON;
PROC UNIVARIATE DATA = sim ;
VAR Loss;
OUTPUT OUT=sim_univariate PCTLPRE=P_
pctlpts=50, 95, 97.5, 99, 99.5, 99.9, 100;
HISTOGRAM / MIDPOINTS=0 TO 0.2 BY 0.01 HOFFSET=0 NOFRAME
CBARLINE=CYAN CFILL=CYAN
OUTHISTOGRAM = sim_histogram;
RUN;
ODS GRAPHICS OFF;
```

ESTIMATING CORRELATIONS

For parameter estimation, we consider some of the most popular estimation techniques such as the method of moments (MM), maximum likelihood (ML), and probit-linear regressions. The last technique will be divided into a special case for the ASRF model where closed formulas exist, and more general cases with finite numbers of borrowers where we consider various extensions, dependent upon the data and the objective.

Method of Moments

The method of moments technique replaces population moments by their empirical counterparts. In credit risk, this technique has been derived and applied by Gordy (2000). It requires a time series of defaults in a homogeneous segment where

$d_t = \sum_{i=1}^{N_t} d_{it}\ (i = 1, \ldots, n_t; t = 1, \ldots, T)$ is the number of defaults and n_t is the number of borrowers observed in year t. The estimator for the default probability is simply given by the average yearly default rate as

$$\widehat{PD} = \frac{1}{T} \sum_{t=1}^{T} \frac{d_t}{n_t}$$

and the variance of the default rate is estimated by the historical mean squared error (MSE) as

$$\hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^{T} \left(\frac{d_t}{n_t} - \widehat{PD} \right)^2$$

An estimator for the variance of the CPD is given by

$$\widehat{Var}(CPD) = \frac{\hat{\sigma}^2 - \frac{1}{T} \sum_{t=1}^{T} \frac{1}{n_t} \widehat{PD}(1 - \widehat{PD})}{1 - \frac{1}{T} \sum_{t=1}^{T} \frac{1}{n_t}}$$

Finally, the estimator $\hat{\rho}$ for the asset correlation is obtained as a (numerical) solution for

$$\widehat{Var}(CPD) = \Phi_2(\Phi^{-1}(\widehat{PD}), \Phi^{-1}(\widehat{PD}), \hat{\rho}) - \widehat{PD}^2$$

where $\Phi_2(.,.,\hat{\rho})$ denotes the CDF of the bivariate standardized normal distribution with correlation $\hat{\rho}$.

This can be implemented in SAS as follows. First, we generate the mean gross domestic product (GDP) growth rate and default rate by time by sorting the data using PROC SORT and computing the averages of GDP growth rate and default indicator using PROC MEANS by time. Then, we compute the lagged and probit transformed default rate in a data step:

```
DATA tmp;
SET data.mortgage;
RUN;
PROC SORT DATA = tmp;
BY time;
RUN;
PROC MEANS DATA = tmp;
VAR default_time gdp_time;
BY time;
OUTPUT OUT = means;
RUN;
DATA tmp2;
SET means;
IF _STAT_ = “MEAN”;
n_default = default_time * _FREQ_;
default_time_1 = LAG(default_time);
Probit_dr = PROBIT(default_time);
```

```
RUN;
```

Note that the resulting data set tmp2 is the basis for this and the following two sections (Maximum-Likelihood ASRF and Probit-Linear Regression).

```
PROC IML;
USE tmp2;
READ ALL Var _NUM_ INTO data;
default_rate = data[,4];
/* Average default rate */
lambda_hat = default_rate[:];
/* Average default rate squared*/
lambda_hat2 = lambda_hat**2;
/* 1 divided by N_t*/
N_inverse = 1/data[,3];
/* Probit of the default rate*/
threshold = PROBIT(lambda_hat);
/* Historical mean squared error*/
MSE = (default_rate-lambda_hat)‘*(default_rate-lambda_hat)
/ NROW(data);
/* Variance of the CPD*/
Var_lambda_hat = (MSE - N_inverse‘ * J(Nrow(data),1,1)
*lambda_hat * (1-lambda_hat) / NROW(data))
/ (1- N_inverse‘ * J(Nrow(data),1,1)/NROW(data));
PRINT lambda_hat MSE Var_lambda_hat threshold lambda_hat2;
out =   lambda_hat||threshold||Var_lambda_hat;
CREATE output_MM FROM out;
APPEND FROM out;
QUIT;
```

The following code uses CALL NLPNRA to perform a nonlinear optimization by the Newton-Raphson method. (See Exhibit 9.6.) A more detailed description of this routine can be found in the SAS manual; see SAS Institute Inc. (2015).

Optimization Start					
Parameter Estimates					
N	**Parameter**	**Estimate**	**Gradient Objective Function**	**Lower Bound Constraint**	**Upper Bound Constraint**
1	**X1**	0.990000	0.142121	0.000000100	1.000000
Value of Objective Function = 0.0177221014					

Newton-Raphson Optimization with Line Search	
Without Parameter Scaling	
Gradient Computed by Finite Differences	
CRP Jacobian Computed by Finite Differences	
Parameter Estimates	1
Lower Bounds	1
Upper Bounds	1

Optimization Start			
Active Constraints	0	**Objective Function**	0.0177221014
Max Abs Gradient Element	0.1421210742		

Iteration		Restarts	Function Calls	Active Constraints		Objective Function	Objective Function Change	Max Abs Gradient Element	Step Size	Slope Searc Direc
1		0	3	0		0.01568	0.00204	0.0812	1.000	−0.00
2		0	4	0		0.01235	0.00333	0.0454	1.000	−0.00
3		0	5	0		0.00752	0.00483	0.0238	1.000	−0.00
4		0	6	0		0.00250	0.00503	0.0103	1.000	−0.00
5		0	7	0		0.0000171	0.00248	0.00315	1.000	−0.00
6		0	10	0		1.63156E-6	0.000015	0.00308	0.0228	−0.00
7	*	0	16	0		1.56846E-7	1.475E-6	0.00309	0.00006	−167
8	*	0	21	0		2.1182E-11	1.568E-7	0.000368	0.00098	−0.00
9	*	0	33	0		2.1182E-11	0	0.00368	1E-9	−14E

Optimization Results			
Iterations	9	Function Calls	34
Hessian Calls	10	Active Constraints	0
Objective Function	2.118166E-11	Max Abs Gradient Element	0.0003680843
Slope of Search Direction	−1.39024E-10	Ridge	0.1899715969

Optimization Results			
Parameter Estimates			
N	**Parameter**	**Estimate**	**Gradient Objective Function**
1	**X1**	0.045432	0.000368
Value of Objective Function = 2.118166E-11			

Exhibit 9.6 Parameter Estimates

```
PROC IML;
USE output_MM ;
READ ALL Var _NUM_ INTO data;
/* Read the data required for backing out the asset correlation*/
threshold = data[1,2];
lambda_hat = data[1,1];
Var = data[1,3];
START optim(x) GLOBAL(threshold, lambda_hat, Var);
/* Formula for the variance of the CPD
with argument x as asset correlation*/
/* Difference between left-hand and right-hand side
of the formula is denoted by f
and should be close to zero*/
f = ABS((PROBBNRM(threshold, threshold, x) - lambda_hat**2 - Var));
RETURN (f);
FINISH optim;
x = 0.99;
optn = {0, 2};
con = {0.0000001 , 0.9999999};
CALL NLPNRA(rc,xres,“optim”,x,optn, con) ;
QUIT;
```

The output shows the starting value of 0.99 together with the value of the objective function. Then the algorithm iterates a number of times and improves the value of the objective functions. It stops after further improvement is impossible and returns the value for the asset correlation of 4.5 percent.

Maximum-Likelihood ASRF

Another widely used estimation technique is the maximum likelihood method. A special simple case of the likelihood is obtained under the assumption that the model has exactly one systematic risk factor and is homogeneous and asymptotic (i.e., has infinitely many identical borrowers). This is called the asymptotic single risk factor (ASRF) model. Under these assumptions, the CPD becomes the default rate of the asymptotic portfolio, which stochastically depends on only the systematic random factor and has two parameters: the PD and the correlation. Düellmann, Klaus, Trapp, and Monika (2004) show that the ML estimator for the PD and the asset correlation are given by

$$\widehat{PD} = \Phi\left(\frac{\hat{c}}{\sqrt{1+\widehat{Var}(\hat{c}_t)}}\right) \tag{9.1}$$

and

$$\hat{\rho} = \frac{\widehat{Var}(\hat{c}_t)}{1+\widehat{Var}(\hat{c}_t)} \tag{9.2}$$

where $\hat{c} = \frac{1}{T}\sum_{t=1}^{T}\hat{c}_t$, $\widehat{Var}(\hat{c}_t) = \frac{1}{T}\sum_{t=1}^{T}\hat{c}_t^{\,2} - \hat{c}^2$, and $\hat{c}_t = \Phi^{-1}\left(\frac{d_t}{n_t}\right)$.

This can be implemented in SAS as follows:

```
ODS GRAPHICS ON;
PROC IML;
USE tmp2;
READ all Var _NUM_ INTO data;
/* Probit of the yearly default rate*/
c_t  = PROBIT(data[,4]);
/* Average of theses Probits */
c_bar = c_t[:];
/* Variance of these Probits */
Var_c = c_t‘*c_t / NROW(data) - c_bar**2;
/* Estimate for the PD */
PD_ML = PROBNORM(c_bar / (SQRT(1+Var_c)));
rho_ML = Var_c / (1+Var_c);
/* Estimate for the asset correlation */
PRINT c_bar Var_c rho_ML PD_ML;
RUN;
ODS GRAPHICS OFF;
```

The output returns the computed quantities (Exhibit 9.7); in the particular case, the asset correlation is estimated as 5.4 percent and the PD is estimated as 2.11 percent.

c_bar	Var_c	rho_ML	PD_ML
−2.086967	0.057137	0.0540488	0.0211892

Exhibit 9.7 ASRF Maximum Likelihood Method

Probit-Linear Regression

Probit-Linear Regression without Covariates

The ASRF model can also be estimated in a simple way using a probit-linear regression model. In the ASRF model, it holds that the default rate $dr_t = \frac{d_t}{n_t}$ in time t equals the CPD. Then after transformation we have the regression model:

$$\begin{aligned}\Phi^{-1}(dr_t) &= \frac{c-\sqrt{\rho}X}{\sqrt{1-\rho}} \\ &= \frac{c}{\sqrt{1-\rho}} - \frac{\sqrt{\rho}}{\sqrt{1-\rho}}X \\ &= a + \sigma \quad \epsilon\end{aligned}$$

where $a = \frac{c}{\sqrt{1-\rho}}$, $\sigma = -\frac{\sqrt{\rho}}{\sqrt{1-\rho}}$, and $X = \epsilon \sim N(0,1)$ i.i.d.

Note that we assume an infinitely granular portfolio (i.e., the assumption underlying the ASRF), which implies an infinitely large number of exposures with infinitely small exposure amounts. The result of this is that all idiosyncratic risk is diversified and that the portfolio default rates are exposed only to systematic risk, which is modeled by X in the ASRF and constitutes the error in a linear model. A linear regression model plots the MSE, which can be transformed into an estimate for the asset correlation (see later discussion).

The following code shows the implementation via PROC MODEL in SAS (see Exhibits 9.8 and 9.9).

The MODEL Procedure

Nonlinear OLS Summary of Residual Errors							
Equation	**DF Model**	**DF Error**	**SSE**	**MSE**	**Root MSE**	***R*-Squared**	**Adj *R*-Sq**
Probit_dr	1	59	3.4282	0.0581	0.2411	−0.0000	−0.0000

Nonlinear OLS Parameter Estimates				
Parameter	**Estimate**	**Approx Std Err**	***t* Value**	**Approx Pr > |*t*|**
a	−2.08697	0.0311	−67.06	<.0001

Number of Observations		Statistics for System	
Used	60	**Objective**	0.0571
Missing	0	**Objective*N**	3.4282

Exhibit 9.8 Probit-Linear Regression

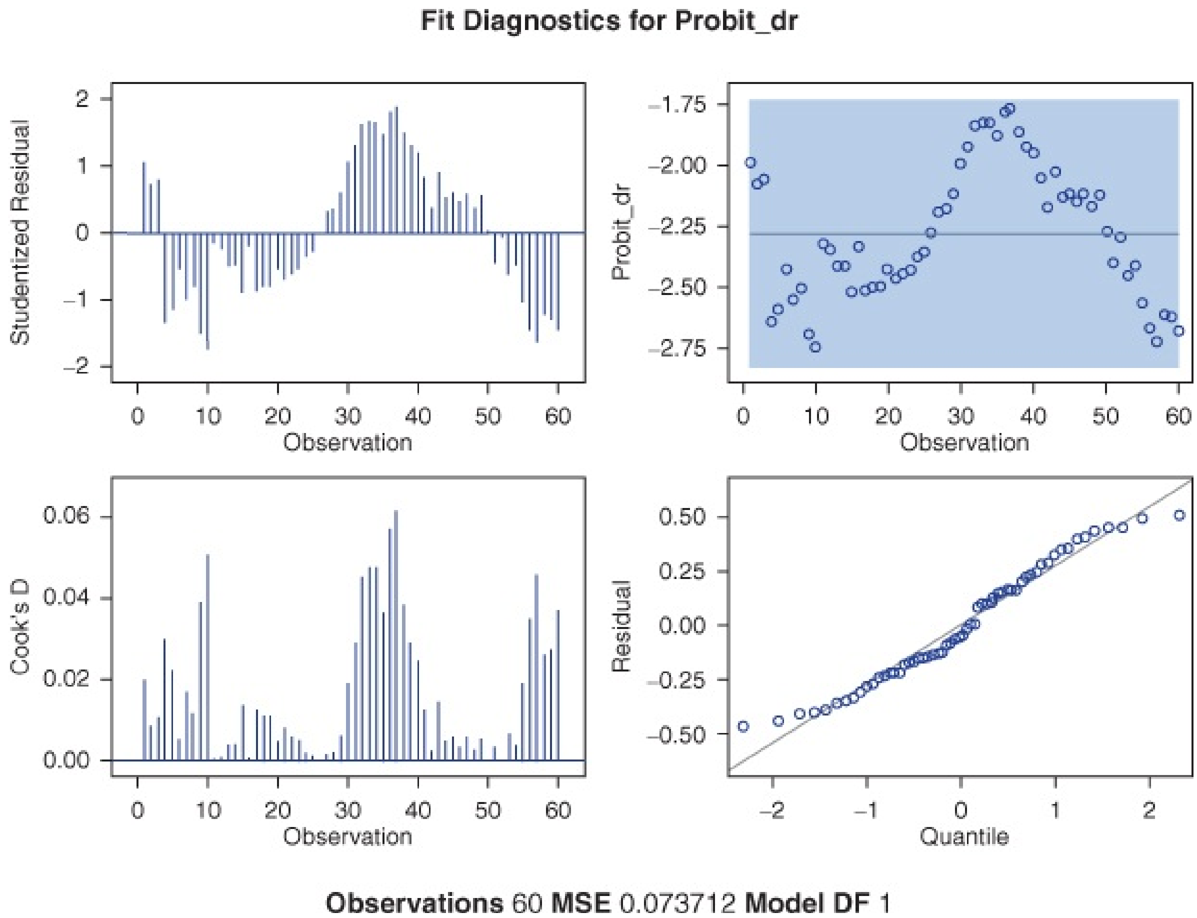

Exhibit 9.9 Probit-Linear Regression

```
/*Probit-Linear Regression*/
ODS GRAPHICS ON;
PROC MODEL DATA = tmp2;
Probit_dr = a;
FIT Probit_dr /  OUTEST=outest;
RUN;
ODS GRAPHICS OFF;
```

The code estimates two parameters, namely the constant a and the variance of the residual b^2, which is the mean squared error (MSE). From $\sigma^2 = \frac{\rho}{1-\rho}$ it follows that $\hat{\rho} = \frac{\sigma^2}{1+\sigma^2}$. Thus, the estimate for the asset correlation is $\hat{\rho} = \frac{0.0581}{1+0.0581} = 0.055 = 5.5\%$. The estimate for the default probability is obtained by noting that $c = a\sqrt{1-\rho}$ and $PD = \Phi(c)$. Thus, $\hat{c} = -2.0870 \cdot \sqrt{0.945} = -2.0288$, and therefore $\widehat{PD} = 0.0212 = 2.12\%$.

The procedure also returns some fit diagnostics for the regression model. The figures show that the residuals, which are the realizations of the systematic effects over time, exhibit a clear autocorrelation pattern. This is because the model does not include any explanatory, exogenous variables, such as macroeconomic variables, and therefore the time variation of default rates is fully captured by the residual. In the next model, we show that it is quite easy to account for this and include time-varying covariates.

Probit-Linear Regression with Covariates

An important extension of the probit-linear regression model and an advantage over the analytical solutions is that covariates, particularly autoregressive (AR) effects and/or macroeconomic covariates, can be included. As an example, we include the realized one-year lagged default rate in the regression model (see Exhibits 9.10 and 9.11).

The MODEL Procedure

Nonlinear OLS Summary of Residual Errors							
Equation	**DF Model**	**DF Error**	**SSE**	**MSE**	**Root MSE**	***R*-Squared**	**Adj *R*-Sq**
Probit_dr	2	57	0.7687	0.0135	0.1161	0.7726	0.7686

Nonlinear OLS Parameter Estimates				
Parameter	**Estimate**	**Approx Std Err**	***t* Value**	**Approx Pr > |*t*|**
a	−0.2078	0.1356	−1.53	0.1310
b	0.902061	0.0648	13.92	<.0001

Number of Observations		Statistics for System	
Used	59	**Objective**	0.0130
Missing	1	**Objective*N**	0.7687

Exhibit 9.10 Probit-Linear Regression

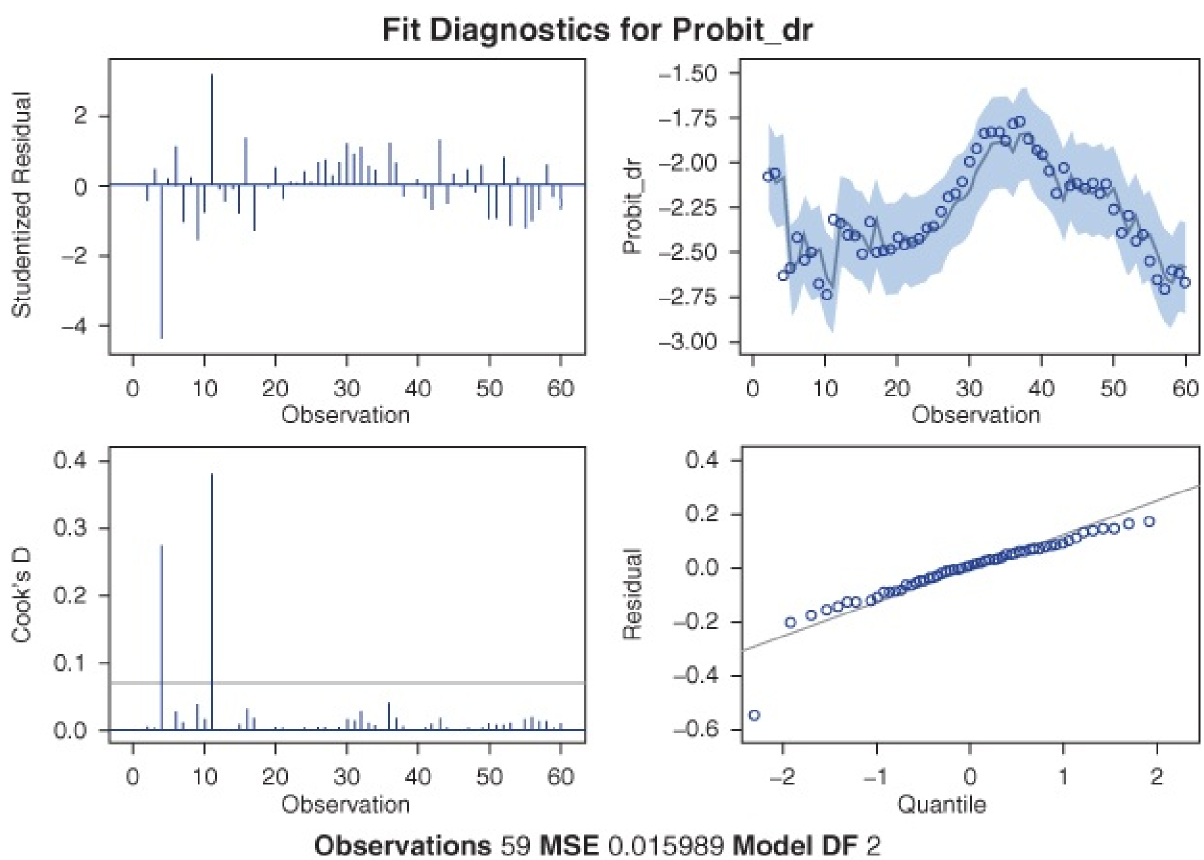

Exhibit 9.11 Probit-Linear Regression with Lagged Default Rate

```
/*Probit-Linear Regression with covariates*/
ODS GRAPHICS ON;
PROC MODEL DATA = tmp2;
Probit_dr = a + b * PROBIT(default_time_1);
FIT Probit_dr / OUTEST=outest;
RUN;
ODS GRAPHICS OFF;
```

The model now has an additional coefficient b, which measures the impact of the lagged default rate on the current default risk. It is significantly positive and captures some of the autocorrelation of the default rates. The PD is now time-dependent and the residual is the difference between the time-dependent PD and the realized default rate, and is the remainder of the unexplained systematic risk. Therefore, the remaining asset correlation after including autocorrelated PDs is only $\hat{\rho} = \frac{0.0135}{1+0.0135} = 0.0133 = 1.33\%$. The regression diagnostics show that the fit is much better than without covariates. Instead, or in addition to the lagged transformed default rate, other (particularly macroeconomic) covariates can also be included. In the next example, shown in Exhibits 9.12 and 9.13, we include GDP. However, as most of the time variation is already captured by the lagged effect, the GDP is not significant for this data set.

The MODEL Procedure

Nonlinear OLS Summary of Residual Errors							
Equation	DF Model	DF Error	SSE	MSE	Root MSE	*R*-Squared	Adj *R*-Sq
Probit_dr	3	56	0.7395	0.0132	0.1149	0.7812	0.7734

Nonlinear OLS Parameter Estimates				
Parameter	Estimate	Approx Std Err	*t* Value	Approx Pr > |*t*|
a	−0.36139	0.1693	−2.13	0.0372
b	0.813225	0.0877	9.28	<.0001
c	−0.01739	0.0117	−1.49	0.1426

Number of Observations		Statistics for System	
Used	59	Objective	0.0125
Missing	1	Objective**N*	0.7395

Exhibit 9.12 Probit-Linear Regression with Macroeconomic Variable

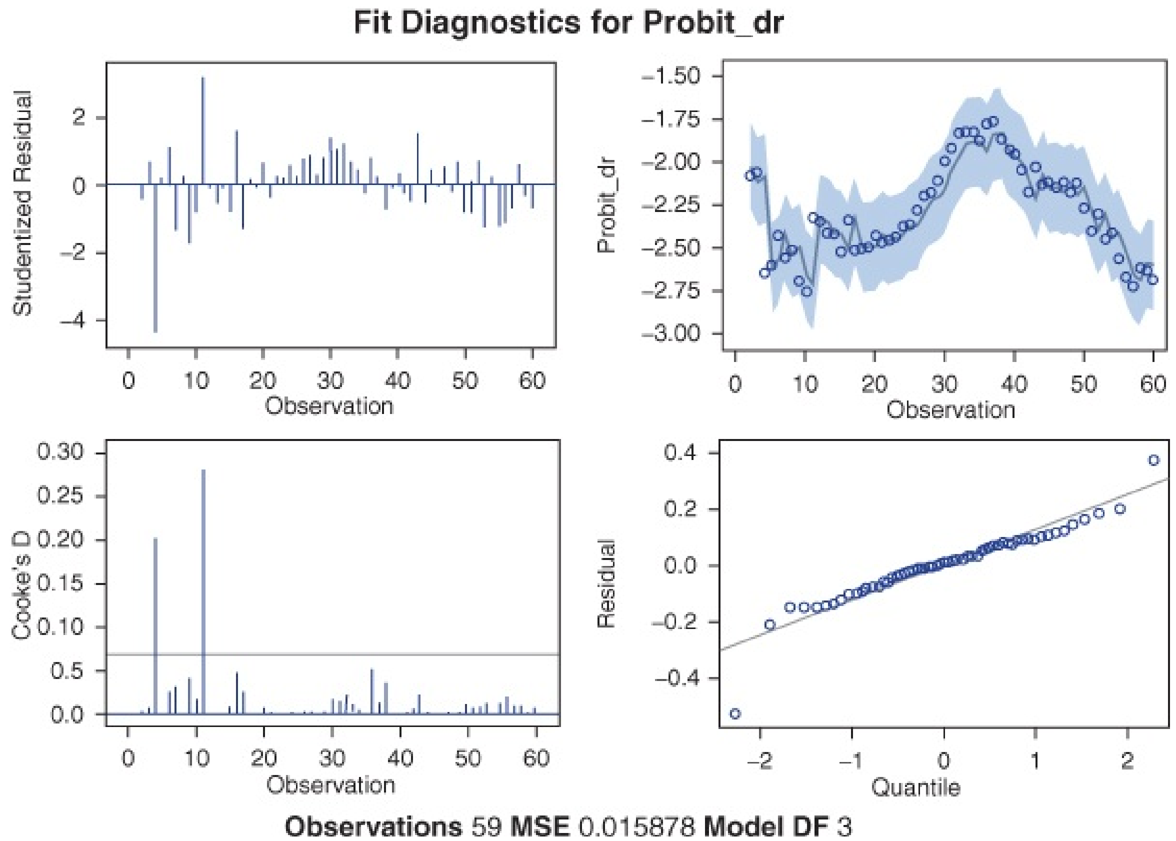

Exhibit 9.13 Probit-Linear Regression with Macroeconomic Variable

```
/*Probit-Linear Regression with covariates*/
ODS GRAPHICS ON;
PROC MODEL DATA = tmp2;
Probit_dr = a + b * PROBIT(default_time_1) + c * gdp_time;
FIT Probit_dr / /*out = resid outresid*/ OUTEST=outest;
RUN;
ODS GRAPHICS OFF;
```

As an alternative to including the lagged default rate as covariate, one might also directly estimate an autoregressive (AR) model using PROC AUTOREG. In the example shown in Exhibits 9.14 and 9.15, we include an AR(3) effect (i.e., with three lags) and GDP. The output shows that only the first lag is significant. Without AR effect, GDP is significant. After including the AR effect, GDP is still significant. The AR models can easily be extended in PROC AUTOREG to various GARCH family types. We refer to the SAS manual for a detailed description and leave it up to the interested reader to specify more advanced econometric models. For a general introduction and overview of time series models, see, for example, Box (2015).

The AUTOREG Procedure

Ordinary Least Squares Estimates			
SSE	1.88259464	DFE	58
MSE	0.03246	Root MSE	0.18016
SBC	−29.240303	AIC	−33.428993
MAE	0.14427436	AICC	−33.218466
MAPE	6.91479828	HQC	−31.790566
Durbin-Watson	0.4941	Total *R*-Squared	0.4509

Parameter Estimates					
Variable	DF	Estimate	Standard Error	*t* Value	Approx Pr > |*t*|
Intercept	1	−1.9211	0.0334	−57.45	<.0001
gdp_time	1	−0.0899	0.0130	−6.90	<.0001

<table>
<tr><th colspan="4">Estimates of Autocorrelations</th></tr>
<tr><th>Lag</th><th>Covariance</th><th>Correlation</th><th>-1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1</th></tr>
<tr><td>0</td><td>0.0314</td><td>1.000000</td><td>| |********************|</td></tr>
<tr><td>1</td><td>0.0224</td><td>0.714006</td><td>| |************** |</td></tr>
<tr><td>2</td><td>0.0182</td><td>0.578784</td><td>| |************ |</td></tr>
<tr><td>3</td><td>0.0152</td><td>0.483873</td><td>| |********** |</td></tr>
</table>

Estimates of Autoregressive Parameters			
Lag	Coefficient	Standard Error	*t* Value
1	−0.605246	0.134606	−4.50
2	−0.104585	0.156781	−0.67
3	−0.058893	0.134606	−0.44

Maximum Likelihood Estimates			
SSE	0.64109687	DFE	55
MSE	0.01166	Root MSE	0.10796
SBC	−79.932239	AIC	−90.403962
MAE	0.08140146	AICC	−89.292851
MAPE	3.89550756	HQC	−86.307895
Log Likelihood	50.2019809	Transformed Regression *R*-Squared	0.1325
Durbin-Watson	1.9190	Total *R*-Squared	0.8130
		Observations	60

Exhibit 9.14 AR Model with Macroeconomic Variable

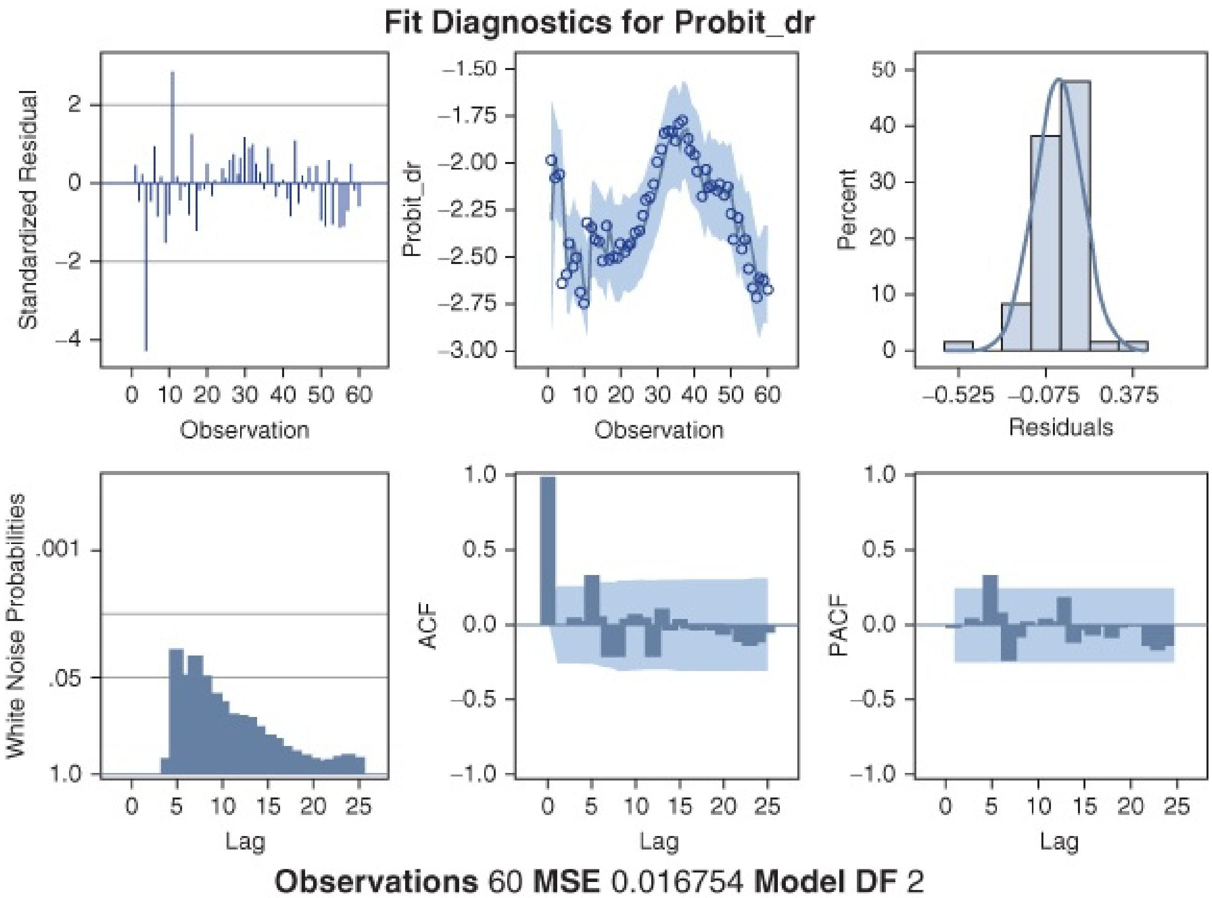

Exhibit 9.15 AR Model with Macroeconomic Variable

```
/*AR model*/
ODS GRAPHICS ON;
PROC AUTOREG DATA=tmp2 ;
ar_1 :   MODEL Probit_dr = gdp_time /  NLAG=3 METHOD=ML;
RUN;
ODS GRAPHICS OFF;
```

EXTENSIONS

Model Specifications Other Than Gaussian

The models in this chapter might seem rather restrictive at first sight because the basic framework introduced normally distributed variables. In risk management, however, particularly in market risk, the assumption of a normal distribution is often not reasonable. Asset returns usually have fatter tails than the normal distribution and exhibit asymmetry. The question then becomes how alternative distributional assumptions can be implemented in SAS and how they affect the shape of the loss distribution. In this subsection, we show how other distributions can be simulated. For exposition purposes, we compare the Gaussian model with a Clayton copula model and a simplified version of the CreditRisk+ (CR+) model. To compute the parameters for the non-Gaussian models, we use a moments-matching approach as in Gordy

(2000) such that mean and variance of the distribution are the same. Given the PD and the asset correlation ρ in the Gaussian model, the variance is obtained as $Var = \Phi_2(c, c, \rho) - PD^2$ where $c = \Phi^{-1}(PD)$ and $\Phi_2(.)$ is the CDF of the bivarate normal distribution. The parameters for the Gamma distribution in CR+ are obtained as $\alpha = PD^2 / Var$ and $\beta = 1/\alpha$. For the Clayton copula with dispersion parameter θ, the variance is $Var_C = \phi^{-1}(2\phi(PD)) - PD^2$ with $\phi(t) = t^{-\theta} - 1$ and $\phi^{-1}(s) = (1+s)^{-1/\theta}$. From this θ can be backed out.

The Monte Carlo simulation for the Gaussian model is straightforward, as shown earlier. In the CR+ and the Clayton model, the systematic factor is gamma distributed. Given the systematic factor, the conditional PD is computed and defaults are binomial or Poisson distributed, respectively (see, e.g., Das and Geng 2003). The IML code generates 100,000 Monte Carlo simulation runs with 1,000 borrowers for each model. The output shows the three simulated loss distributions, and we see in Exhibit 9.16 that the differences between the distributions are almost negligible. Similar results have been obtained by other studies and other models; see, for example Hamerle and Rösch (2005) and Schloegl and O'Kane (2005) for comparisons of Gaussian and Student's t copula models.

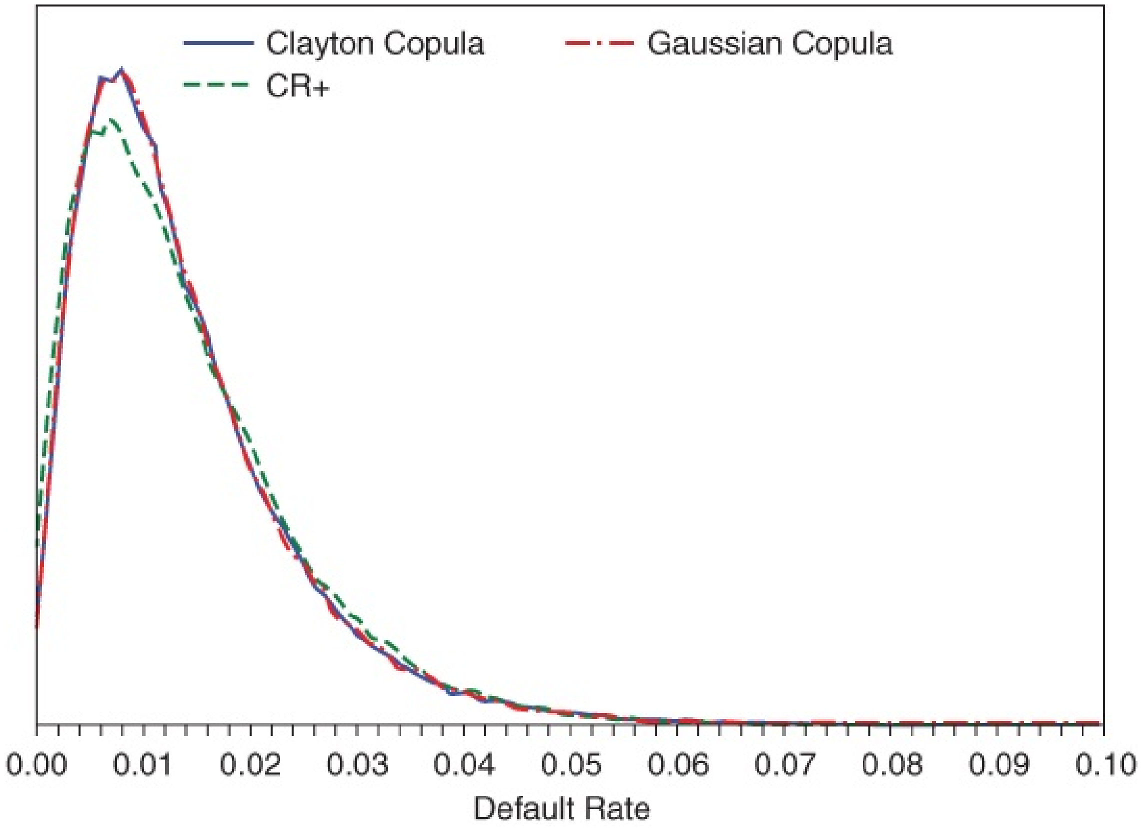

Exhibit 9.16 Comparison of Loss Distributions

```
/*Reparameterizing Gaussian, Clayton and CR+ Model and Simulation*/
PROC IML;
/*Set parameters*/
 Nsim = 100000;
 N = 1000;
/*Parameters for the Gaussian model*/
 pd_N = 0.0212;
 c = PROBIT(PD_N);
```

```
  rho_N = 0.055;
  ksi_N = SQRT(1-rho_N);
  Var_CPD = PROBBNRM(c,c,rho_N) - PD_N**2;
 /*Parameters for the CR+ model*/
  pd_C = pd_N;
  alpha_C = (PD_C**2)/Var_CPD;
  beta_C = 1/alpha_C;
  /*Parameters for the Clayton Copula model*/
  pd_T = PD_N;
  theta = 0.020401;
  /*Generation of the default distribution for Gaussian model*/
  x1_N = J(Nsim,1,0);
  z_N = RANNOR(x1_N);
  CPD_N = PROBNORM((PROBIT(pd_n)- SQRT(rho_N) * z_N)/ksi_N);
  out_N = RANBIN(0,N,CPD_N)/N;
  /*Generation of the default distribution for CR+ Model*/
  x1_C = J(Nsim,1,0);
  z_C = beta_C * RANGAM (x1_C,alpha_C);
  CPD_C = N * PD_c * z_C ;
  out_C = RANPOI(0,CPD_C)/N;
  /*Generation of the default distribution for Clayton Copula Model*/
  x1_T = J(Nsim,1,0);
  z_T = RANGAM (x1_T,1/theta);
  CPD_T = exp(-(PD_T**(-theta)-1)*z_T);
  out_T = RANBIN(0,N,CPD_T)/N;
  /*Export output data*/
  CREATE Svsim_n FROM out_N;
  APPEND FROM out_N;
  CREATE  Svsim_T FROM out_T;
  APPEND FROM out_T ;
  CREATE  Svsim_CR FROM out_C;
  APPEND FROM out_C ;
  QUIT;
  PROC UNIVARIATE DATA = svsim_t;
  VAR COL1;
  OUTPUT OUT=temp_pctl_t pctlpre=P_
  pctlpts=50, 95, 99, 99.5, 99.9, 100;
  HISTOGRAM / MIDPOINTS=0 TO 0.1 BY 0.001
  OUTHISTOGRAM = temp_T NOPLOT;
  RUN;
  PROC UNIVARIATE DATA = svsim_n;
  VAR COL1;
  OUTPUT OUT=temp_pctl_g pctlpre=P_
  pctlpts=50, 95, 99, 99.5, 99.9, 100;
  HISTOGRAM / MIDPOINTS=0 TO 0.1 BY 0.001
  OUTHISTOGRAM = temp_G NOPLOT;
  RUN;
  PROC UNIVARIATE DATA = svsim_CR;
  VAR COL1;
  OUTPUT OUT=temp_pctl_c pctlpre=P_
  pctlpts=50, 95, 99, 99.5, 99.9, 100;
  HISTOGRAM / MIDPOINTS=0 TO 0.1 BY 0.001
  OUTHISTOGRAM = temp_C NOPLOT;
  RUN;
  DATA temp_t1;
```

```
 SET temp_T (KEEP=_MIDPT_ _OBSPCT_
 RENAME=(_MIDPT_=default_rate  _OBSPCT_=T)) ;
 RUN;
 DATA temp_g1;
 SET temp_g (KEEP=_MIDPT_ _OBSPCT_
 RENAME=(_MIDPT_=default_rate  _OBSPCT_=g)) ;
 RUN;
 DATA temp_c1;
 SET temp_c (KEEP=_MIDPT_ _OBSPCT_
 RENAME=(_MIDPT_=default_rate  _OBSPCT_=c)) ;
 RUN;
 DATA temp;
 MERGE temp_T1 temp_G1 temp_C1;
 BY default_rate;
 ATTRIB default_Rate LABEL = 'Default Rate' T
 LABEL = 'Clayton Copula' G LABEL = 'Gaussian Copula'
 C LABEL = 'CR+';
 RUN;
```

The loss distributions are then plotted using PROC GPLOT:

```
ODS GRAPHICS ON / WIDTH=10IN IMAGEMAP=ON;
GOPTIONS RESET=GLOBAL GUNIT=PCT NOBORDER CBACK=WHITE
COLORS=(BLACK BLUE GREEN RED)
FTITLE=SWISSB FTEXT=SWISS HTITLE=5 HTEXT=4.5;
SYMBOL1 COLOR=BLUE INTERPOL=SM LINE=1 WIDTH=3 HEIGHT=1;
SYMBOL2 FONT=MARKER VALUE=NONE COLOR=RED LINE=21 WIDTH=3
INTERPOL=SM HEIGHT=0;
SYMBOL3 FONT=MARKER COLOR=GREEN INTERPOL=SM LINE=21 WIDTH=3
HEIGHT=1;
AXIS1 ORDER=(0 TO 0.12 BY 0.02) OFFSET=(0,0) LABEL=('Default rate')
MAJOR=(HEIGHT=1) MINOR=(HEIGHT=1) WIDTH=3;
AXIS2 ORDER=(0 TO 6 BY 2) OFFSET=(0,0) LABEL=('Percent')
MAJOR=(HEIGHT=1) MINOR=NONE  WIDTH=3;
LEGEND1 LABEL=NONE SHAPE=SYMBOL(4,2)
POSITION=(TOP RIGHT INSIDE) MODE=SHARE;
PROC GPLOT DATA=temp;
PLOT T*default_rate G*default_rate  C*default_rate
 / OVERLAY LEGEND=LEGEND1
HAXIS=AXIS1
VAXIS=AXIS2;
RUN;
ODS GRAPHICS OFF;
```

Models with Idiosyncratic Effects

The earlier models for estimating correlations used aggregated data (default rates). This is sufficient if the portfolio or segment is large and idiosyncratic risk is diversified. For smaller segments, the models can be extended to include idiosyncratic effects. Then we actually obtain a nonlinear panel regression model with mixed effects; see Rösch and Scheule (2004), Rösch (2005), and Hamerle and Rösch (2006).

A general feature of these models, relative to the method of moments and the maximum-likelihood ASRF, is that the correlations are computed conditional on the covariates. Furthermore, the models include idiosyncratic as well as systematic information, whereas the probit-linear regression model only controls for systematic (i.e., macroeconomic) covariates. The effect of controlling for time-varying (systematic or idiosyncratic) information is that the resulting asset correlations (i.e., the residual systematic risk exposure as we have demonstrated for probit-linear regression models) are smaller than for unconditional models that do not control for such information. As a result, the loss distribution that is based on these PD and asset correlation measures becomes narrower and the tails are much lower.

The interpretation of these results is:

- Banks can lower their tail risk estimates and hence economic capital by providing for time-varying information that adds accuracy to their PD estimates.
- Banks that rely on unconditional models overestimate unexplained stochastic systematic risk, as they obtain spurious correlations that can be explained by macroeconomic risk factors.
- Asset correlation models that control for systematic information are more accurate with regard to the unknown data-generating process.

These approaches are quite advanced; they usually require programming the likelihood, solving multidimensional integrals, and using an efficient optimizer and are beyond the scope of this book. For further information, refer to the referenced literature (e.g., Hamerle and Rösch 2006).

Model and Estimation Risk

Usually, banks have yearly data available for estimation, and time series are hardly longer than 20 or 30 years. Thus, we have only about 20 to 30 observations for estimating correlations, which imposes sparse data and nonstationarity issues, as correlations are estimated using the time-series dimension rather than the cross-section dimension of the data. Even if we assume stationarity and no structural breaks, the standard errors of the correlation estimates might be quite high and asset correlation estimates might be downward biased; see Gordy and Heitfield (2010). As loss distributions and risk measures derived thereof react very sensitively to changes in correlations, it is very important to consider estimation errors when computing risk measures. The good news is that the model specification is not such a big issue (within certain limits), but the bad news is that measurement and estimation error can be a problem, as risk measures might be under- or overstated. An analysis of value-at-risk volatility is given in Hamerle and Rösch (2005) and in Rösch and Scheule (2013). Moreover, the models in this section could be extended to deal with time-varying correlations and structural breaks, which are beyond the scope of this book.

PRACTICE QUESTIONS

1. Compute loss distributions analytically and numerically for PDs of 1 percent, 2 percent, 3

percent, and 5 percent, and correlations of 0.01, 0.05, 0.1, and 0.3. Let the number of borrowers in the numerical solution vary between 10, 100, and 1,000. Interpret the results.

2. Generate the loss distribution via Monte Carlo simulation using the same parameter settings and compare the results. What are the advantages and the disadvantages of the Monte Carlo simulations?
3. Discuss how the inclusion of macroeconomic variables into a model for correlation estimation might affect the estimate for the correlation. Interpret the results in terms of the earlier discussion about point-in-time (PIT) versus through-the-cycle (TTC).
4. In the regulatory framework of Basel, the correlation is an important parameter for the internal ratings based approach (IRBA). Discuss how it affects regulatory capital.
5. Categorize the FICO score into three categories using PROCRANK. Compute and plot the default rates for the three categories using PROC GPLOT. Estimate the PD and asset correlation of the three classes using the method of moments. Use data set mortgage.
6. Estimate the PD and asset correlation of the three FICO classes from the previous question using maximum-likelihood ASRF and a probit linear regression approach. Use data set mortgage.

References

Basel Committee on Banking Supervision. 2006. "International Convergence of Capital Measurement and Capital Standards: A Revised Framework, Comprehensive Version."

Basel Committee on Banking Supervision. 2011. "Basel III: A Global Regulatory Framework for More Resilient Banks and Banking Systems."

Box, G. 2015. *Time Series Analysis: Forecasting and Control*. Hoboken, NJ: John Wiley & Sons.

Das, S., and G. Geng. 2003. "Simulating Correlated Default Processes Using Copulas: A Criterion-Based Approach." Working paper, Santa Clara University.

Düellmann, Klaus, Trapp, and Monika. 2004. "Systematic Risk in Recovery Rates—An Empirical Analysis of US Corporate Credit Exposures." Deutsche Bundesbank Discussion Paper, Series 2: Banking and Financial Supervision, No 02/2004.

European Union. 2013. "Regulation (EU) no 575/2013 of the European Parliament and of the Council of 26 June 2013."

Gordy, M. 2000. "A Comparative Anatomy of Credit Risk Models." *Journal of Banking and Finance* 24: 119–149.

Gordy, M. 2003. "A Risk-Factor Model Foundation for Ratings-Based Bank Capital Rules." *Journal of Financial Intermediation* 12: 199–232.

Gordy, M., and E. Heitfield. 2010. “Small-Sample Estimation of Models of Portfolio Credit Risk.” In *Recent Advances in Financial Engineering: Proceedings of the KIER-TMU International Workshop on Financial Engineering 2009*, edited by Masaaki Kijima et al. Singapore: World Scientific Publishing.

Hamerle, A., and D. Rösch. 2005. “Misspecified Copulas in Credit Risk Models: How Good Is Gaussian?” *Journal of Risk* 8: 41–58.

Hamerle, A., and D. Rösch. 2006. “Parameterizing Credit Risk Models.” *Journal of Credit Risk* 3: 101–122.

Koyluoglu, H. U., and A. Hickman. 1998. “A Generalized Framework for Credit Risk Portfolio Models.” Manuscript, Oliver Wyman & Company.

Rösch, D. 2005. “An Empirical Comparison of Default Risk Forecasts from Alternative Credit Rating Philosophies.” *International Journal of Forecasting* 25(1): 37–51.

Rösch, D., and H. Scheule. 2004. “Forecasting Retail Portfolio Credit Risk.” *Journal of Risk Finance* 5(2): 16–32.

Rösch, D., and H. Scheule. 2013. “Forecasting Mortgage Securitization Risk under Systematic Risk and Parameter Uncertainty.” *Journal of Risk and Insurance* 81(3): 563–586.

SAS Institute Inc. 2015. *SAS/IML 14.1 User's Guide: Technical Report*. Cary, NC: SAS Institute.

Schloegl, E., and D. O'Kane. 2005. “A Note on the Large Homogeneous Portfolio Approximation with the Student-t Copula.” *Finance and Stochastics* 9: 577–584.

Vasicek, O. 1987. “Probability of Loss on Loan Portfolio.” Working paper, KMV Corporation.

Vasicek, O. 1991. “Limiting Loan Loss Probability Distribution.” Working paper, KMV Corporation.