Machine Learning from Data – IDC

Question 1

a. $f(x,y) = 2e^{xy}$; constraint: $2x^2 + y^2 = 16$
constraint: $2x^2 + y^2 - 16 = 0$

$$L = 2e^{xy} - \lambda(2x^2 + y^2 - 16)$$
$$\nabla L = 0:$$
$$\nabla x = 2ye^{xy} - 4\lambda x = 0$$
$$\nabla y = 2xe^{xy} - 2\lambda y = 0$$
$$\nabla \lambda = -2x^2 - y^2 + 16 = 0$$

$$\nabla x = ye^{xy} - 2\lambda x = 0$$
$$\nabla y = xe^{xy} - \lambda y = 0$$
$$\nabla \lambda = -2x^2 - y^2 + 16 = 0$$

$$II: xe^{xy} = 2\lambda y$$
$$II: \lambda = \frac{xe^{xy}}{2y}$$

$check\ y = 0:$

$$I:\ -2\lambda x = 0 \to\ x = 0\ \ or\ \ \lambda = 0$$

$x = 0:$

$$III:\ 13 = 0 \to not\ possible$$

$\lambda = 0:$

$$II: xe^0 = 0 \to x = 0 \to\ \ III: 16 = 0 \to not\ possible$$

$\lambda = \frac{xe^{xy}}{2y}$:

$$I:\ ye^{xy} - 2x \cdot \frac{xe^{xy}}{y} = 0\ \ | \cdot y$$
$$I:\ y^2e^{xy} - 2x^2e^{xy} = 0\ |: e^{xy}$$
$$I:\ y^2 = 2x^2$$
$$III:\ y^2 = 16 - 2x^2$$
$$16 - 2x^2 = 2x^2$$
$$4x^2 = 16$$
$$x^2 = 4 \to x = \pm 2$$

$X = 2:$

$$I:\ y^2 = 2 \cdot 2^2 = 8 \to y = \pm\sqrt{8}$$

$y = \sqrt{8}:$

$$II:\ \lambda = \frac{2e^{2\cdot\sqrt{8}}}{\sqrt{8}} = \frac{e^{4\sqrt{2}}}{\sqrt{2}}$$

$y = -\sqrt{8}:$

$$II:\ \lambda = \frac{2e^{-2\cdot\sqrt{8}}}{-\sqrt{8}} = \frac{e^{-4\sqrt{2}}}{-\sqrt{2}}$$

$X = -2:$

$$I: y^2 = 2 \cdot (-2)^2 = 8 \to same\ as\ with\ x = 2$$

values are: $\left\{\left(x = \pm 2, y = \sqrt{8},\ \lambda = \frac{e^{4\sqrt{2}}}{\sqrt{2}}\right),\ \left(x = \pm 2, y = -\sqrt{8},\ \lambda = \frac{e^{-4\sqrt{2}}}{-\sqrt{2}}\right)\right\}$

b. $f(x, y) = 3\pi x y^2$; constraint: $6 + 6\pi xy + 3\pi y^2 = 12$
constraint: $6\pi xy + 3\pi y^2 - 6 = 0$

$$L = 3\pi x y^2 - \lambda(6\pi xy + 3\pi y^2 - 6)$$
$$\nabla L = 0 :$$
$$\nabla x = 3\pi y^2 - 6\lambda\pi y = 0$$
$$\nabla y = 6\pi xy - 6\lambda\pi x - 6\lambda\pi y = 0$$
$$\nabla \lambda = -6\pi xy - 3\pi y^2 + 6 = 0$$

$$\begin{cases} y^2 - 2\lambda y = 0 \\ xy - \lambda x - \lambda y = 0 \\ -2\pi xy - \pi y^2 + 2 = 0 \end{cases}$$
$$I: y(y - 2\lambda) = 0 \quad \rightarrow y_1 = 2\lambda, \qquad y_2 = 0$$

$y_2 = 0$:

$$III: 2 = 0 \rightarrow not\ possible$$

$y_1 = 2\lambda$:

$$II:\ 2\lambda x - \lambda x - 2\lambda^2 = 0$$
$$II:\ -4\pi\lambda x - 4\pi\lambda^2 + 2 = 0$$
$$II:\ \lambda(x - 2\lambda) = 0$$
$$II:\ -2\pi\lambda x - 2\pi\lambda^2 + 1 = 0$$
$$II:\ x = 2\lambda = y\ \ or\ \lambda = 0$$

$\lambda = 0$:

$$III: 1 = 0 \rightarrow not\ possible$$

$x_1 = y_1 = 2\lambda$:

$$III:\ -2\pi\lambda \cdot 2\lambda - 2\pi\lambda^2 + 1 = -4\pi\lambda^2 - 2\pi\lambda^2 + 1 = -6\pi\lambda^2 + 1 = 0$$
$$\lambda = \pm\sqrt{\frac{1}{6\pi}}$$
$$x_1 = y_1 = \pm 2 \cdot \sqrt{\frac{1}{6\pi}}$$

values are: $\left\{x = \pm 2 \cdot \sqrt{\frac{1}{6\pi}},\ y = \pm 2 \cdot \sqrt{\frac{1}{6\pi}},\ \lambda = \pm\sqrt{\frac{1}{6\pi}}\right\}$

c. $f(x, y, z) = 2x - 4y - z$; constraints: $x + y - z = 0$; $x^2 + 2y^2 = 1$

$$L = 2x - 4y - z - \lambda_1(x + y - z) - \lambda_2(x^2 + 2y^2 - 1)$$
$$\nabla L = 0 :$$
$$\nabla x = 2 - \lambda_1 - 2\lambda_2 x = 0$$
$$\nabla y = -4 - \lambda_1 - 4\lambda_2 y = 0$$
$$\nabla z = -1 + \lambda_1 = 0$$
$$\nabla \lambda_1 = -x - y + z = 0$$
$$\nabla \lambda_2 = -x^2 - 2y^2 + 1 = 0$$

$$III: 1 = \lambda_1$$

$$I:\ 1-2\lambda_2 x=0$$
$$II:\ -5-4\lambda_2 y=0$$

$$I-II:\ 1-2\lambda_2 x+5+4\lambda_2 y=0$$
$$6-2\lambda_2(x-2y)=0$$
$$6=2\lambda_2(x-2y)$$
$$3=\lambda_2(x-2y)$$
$$\lambda_2=\frac{3}{x-2y}$$

$check\ x=2y:$

$$V: -4y^2-2y^2+1=0$$
$$6y^2=1$$
$$y=\pm\sqrt{\frac{1}{6}}$$

$y=\sqrt{\frac{1}{6}}:$

$$II:\ -5-4\lambda_2\cdot\sqrt{\frac{1}{6}}=0$$
$$4\sqrt{\frac{1}{6}}\lambda_2=-5$$
$$\lambda_2=-\frac{5}{4\sqrt{\frac{1}{6}}}=-\frac{5\sqrt{6}}{4}$$
$$I: 1-2\left(-\frac{5\sqrt{6}}{4}\right)\cdot 2\sqrt{\frac{1}{6}}=0$$
$$1+\frac{20\sqrt{6}}{4}\cdot\sqrt{\frac{1}{6}}=0$$
$$6=0\rightarrow not\ possible$$

$y=-\sqrt{\frac{1}{6}}:$

$$II:\ -5-4\lambda_2\cdot\left(-\sqrt{\frac{1}{6}}\right)=0$$
$$4\sqrt{\frac{1}{6}}\lambda_2=5$$
$$\lambda_2=\frac{5}{4\sqrt{\frac{1}{6}}}=\frac{5\sqrt{6}}{4}$$
$$I: 1-2\left(\frac{5\sqrt{6}}{4}\right)\cdot 2\sqrt{\frac{1}{6}}=0$$
$$1-\frac{20\sqrt{6}}{4}\cdot\sqrt{\frac{1}{6}}=0$$
$$-5=0\rightarrow not\ possible$$

$$\lambda_2=\frac{3}{x-2y}$$

$$I: 1 = 2\lambda_2 x$$
$$\lambda_2 = \frac{1}{2x}$$

$check\ x = 0:$

$$I: 1 - 2\lambda_2 \cdot 0 = 0$$
$$1 = 0 \rightarrow not\ possible$$

$$\frac{3}{x-2y} = \frac{1}{2x}$$
$$3 \cdot 2x = 1 \cdot (x - 2y)$$
$$6x = x - 2y$$
$$5x = -2y$$
$$x = -\frac{2y}{5}$$
$$V:\ -\left(-\frac{2y}{5}\right)^2 - 2y^2 + 1 = 0$$
$$-\frac{4y^2}{25} - 2y^2 + 1 = 0 \mid \cdot 25$$
$$-4y^2 - 50y^2 + 25 = 0$$
$$25 = 54y^2$$
$$y = \pm\frac{5\sqrt{6}}{18}$$

$y = \frac{5\sqrt{6}}{18}:$

$$x = -\frac{2\left(\frac{5\sqrt{6}}{18}\right)}{5} = -2\frac{\sqrt{6}}{18}$$
$$x = -\frac{\sqrt{6}}{9}$$
$$\lambda_2 = \frac{1}{2\left(-\frac{\sqrt{6}}{9}\right)}$$
$$\lambda_2 = -\frac{3\sqrt{6}}{4}$$
$$IIII:\ -\left(-\frac{\sqrt{6}}{9}\right) - \left(\frac{5\sqrt{6}}{18}\right) + z = 0$$
$$z = \frac{\sqrt{6}}{6}$$

$y = -\frac{5\sqrt{6}}{18}:$

$$x = -\frac{2\left(-\frac{5\sqrt{6}}{18}\right)}{5}$$
$$x = \frac{\sqrt{6}}{9}$$

$$\lambda_2 = \frac{1}{2\left(\frac{\sqrt{6}}{9}\right)}$$

$$\lambda_2 = \frac{3\sqrt{6}}{4}$$

$$IIII: -\left(\frac{\sqrt{6}}{9}\right) - \left(-\frac{5\sqrt{6}}{18}\right) + z = 0$$

$$z = \frac{\sqrt{6}}{6}$$

$$\text{values are: } \left\{ \begin{matrix} \left(x = -\frac{\sqrt{6}}{9},\ y = \frac{5\sqrt{6}}{18},\ z = \frac{\sqrt{6}}{6},\ \lambda_1 = 1,\ \lambda_2 = -\frac{3\sqrt{6}}{4}\right), \\ \left(x = \frac{\sqrt{6}}{9},\ y = -\frac{5\sqrt{6}}{18},\ z = \frac{\sqrt{6}}{6},\ \lambda_1 = 1,\ \lambda_2 = \frac{3\sqrt{6}}{4}\right) \end{matrix} \right\}$$

Question 2

We will show that $K(x_i, x_j) = K(x_j, x_i)$:

$$K(x_i, x_j) = \varphi(x_i) \cdot \varphi(x_j) = \varphi(x_j) \cdot \varphi(x_i) = K(x_j, x_i)$$

Question 3

a. $\varphi(x) = \left(x_1^3, x_2^3, \sqrt{3}x_1^2x_2, \sqrt{3}x_1x_2^2, \sqrt{3}x_1^2, \sqrt{3}x_2^2, \sqrt{6}x_1x_2, \sqrt{3}x_1, \sqrt{3}x_2, 1\right)$

$$x = (x_1, x_2)$$

$$\begin{aligned}
&\varphi(x) \cdot \varphi(y) = \\
&x_1^3y_1^3 + x_2^3y_2^3 \\
&+3x_1^2x_2y_1^2y_2 + 3x_1x_2^2y_1y_2^2 \\
&+3x_1^2y_1^2 + 3x_2^2y_2^2 \\
&+6x_1x_2y_1y_2 \\
&+3x_1y_1 + 3x_2y_2 \\
&+1 \\
&= \sum_{i=1}^{2} x_i^3y_i^3 + \sum_{i=1}^{3}\sum_{j\neq I} 3x_i^2x_jy_i^2y_j + \sum_{i=1}^{2} 3x_i^2y_i^2 + 1
\end{aligned}$$

this looks like the function $(x \cdot y + 1)^3 = x^3y^3 + 3x^2y^2 + 3xy + 1$

$$= \left(\sum_{i=1}^{2} x_iy_i\right)^3 + 3\left(\sum_{i=1}^{3} x_iy_i\right)^2 + 3\left(\sum_{i=1}^{3} x_iy_i\right) + 1$$

$$\left(\sum_{i=1}^{2} x_iy_i\right)^3 = x_1^3y_1^3 + x_2^3y_2^3 + 3x_1^2x_2y_1^2y_2 + 3x_1x_2^2y_1y_2^2 = \sum_{i=1}^{2} x_i^3y_i^3 + \sum_{i=1}^{3}\sum_{j\neq I} 3x_i^2x_jy_i^2y_j$$

$$3\left(\sum_{i=1}^{2} x_iy_i\right)^2 = 3(x_1^2y_1^2 + x_2^2y_2^2 + 2x_1x_2y_1y_2) = 3x_1^2y_1^2 + 3x_2^2y_2^2 + 6x_1x_2y_1y_2$$

$$= \sum_{i=1}^{2} 3x_i^2y_i^2 + 6x_1x_2y_1y_2$$

$$(x \cdot y + 1)^3 = \sum_{i=1}^{2} x_i^3 y_i^3 + \sum_{i=1}^{3}\sum_{j \neq I} 3x_i^2 x_j y_i^2 y_j + \sum_{i=1}^{2} 3x_i^2 y_i^2 + 6x_1x_2y_1y_2 + 1$$

therefore, for the mapping

$\varphi(x) = \left(x_1^3, x_2^3, \sqrt{3}x_1^2x_2, \sqrt{3}x_1x_2^2, \sqrt{3}x_1^2, \sqrt{3}x_2^2, \sqrt{6}x_1x_2, \sqrt{3}x_1, \sqrt{3}x_2, 1\right)$

the kernel function is $(x \cdot y + 1)^3$

b. $\varphi(x) = (x_1^2, x_2^2, x_3^2, \sqrt{2}x_1x_2, \sqrt{2}x_1x_3, \sqrt{2}x_2x_3, 2x_1, 2x_2, 2x_3, 2)$

$x = (x_1, x_2, x_3)$

$$\begin{aligned}
\varphi(x) \cdot \varphi(y) = {} & \\
& x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2 \\
& +2x_1x_2y_1y_2 + 2x_1x_3y_1y_3 + 2x_2x_3y_2y_3 \\
& +4x_1y_1 + 4x_2y_2 + 4x_3y_3 \\
& +4 \\
& = \sum_{i=1}^{3} x_i^2y_i^2 + \sum_{i=1}^{3}\sum_{j=i+1}^{3} 2x_ix_jy_iy_j + \sum_{i=1}^{3} 4x_iy_i + 4
\end{aligned}$$

this looks like the function $(x \cdot y + 2)^2 = x^2y^2 + 4xy + 4$

$$= \left(\sum_{i=1}^{3} x_iy_i\right)^2 + \sum_{i=1}^{3} 4x_iy_i + 4$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} x_ix_jy_iy_j + \sum_{i=1}^{3} 4x_iy_i + 4$$

$$= \sum_{i=1}^{3}\sum_{j=i+1}^{3} 2x_ix_jy_iy_j + \sum_{i=1}^{3} x_i^2y_i^2 + \sum_{i=1}^{3} 4x_iy_i + 4$$

$$= \sum_{i=1}^{3} x_i^2y_i^2 + \sum_{i=1}^{3}\sum_{j=i+1}^{3} 2x_ix_jy_iy_j + \sum_{i=1}^{3} 4x_iy_i + 4$$

therefore, for the mapping

$\varphi(x) = (x_1^2, x_2^2, x_3^2, \sqrt{2}x_1x_2, \sqrt{2}x_1x_3, \sqrt{2}x_2x_3, 2x_1, 2x_2, 2x_3, 2)$

the kernel function is $(x \cdot y + 2)^2$