IMPR 2017 - Exercise #4

Task 1: Fourier transform - Theoretical part

a. $g(x) = [2, 1, 3, 1, 3, 2]$, $G(k) = \sum_{x=0}^{N-1} g(x) \cdot e^{\frac{-2\pi ikx}{N}}, k = 0, 1, \dots, N \rightarrow G(k) = \sum_{x=0}^{5} g(x) \cdot e^{\frac{-2\pi ikx}{6}}$

$$G(0) = \sum_{x=0}^{5} g(x) \cdot e^{\frac{-2\pi i0x}{6}} = 2 + 1 + 3 + 1 + 3 + 2 = 12$$

$$G(1) = \sum_{x=0}^{5} g(x) \cdot e^{\frac{-2\pi i1x}{6}} = 2 \cdot e^0 + 1 \cdot e^{\frac{-2\pi i1}{6}} + 3 \cdot e^{\frac{-2\pi i2}{6}} + 1 \cdot e^{\frac{-2\pi i3}{6}} + 3 \cdot e^{\frac{-2\pi i4}{6}} + 2 \cdot e^{\frac{-2\pi i5}{6}}$$

$$= 2 + e^{\frac{-\pi i}{3}} + 3 \cdot e^{\frac{-2\pi i}{3}} + e^{-\pi i} + 3 \cdot e^{\frac{-4\pi i}{3}} + 2 \cdot e^{\frac{-5\pi i}{3}}$$

notice rule: $e^{i\pi a} = -1^a$

$$= 2 + \frac{1}{(-1)^{\frac{1}{3}}} + \frac{3}{(-1)^{\frac{2}{3}}} - 1 - \frac{3}{(-1)^{\frac{1}{3}}} - \frac{2}{(-1)^{\frac{2}{3}}} = 1 - \frac{2}{(-1)^{\frac{1}{3}}} + \frac{1}{(-1)^{\frac{2}{3}}} = -0.5 - 0.866i$$

$$G(2) = \sum_{x=0}^{5} g(x) \cdot e^{\frac{-2\pi i2x}{6}} = \sum_{x=0}^{5} g(x) \cdot e^{\frac{-4\pi ix}{6}}$$

$$= 2 \cdot e^0 + 1 \cdot e^{\frac{-4\pi i1}{6}} + 3 \cdot e^{\frac{-4\pi i2}{6}} + 1 \cdot e^{\frac{-4\pi i3}{6}} + 3 \cdot e^{\frac{-4\pi i4}{6}} + 2 \cdot e^{\frac{-4\pi i5}{6}}$$

$$= 2 + e^{\frac{-4\pi i}{6}} + 3 \cdot e^{\frac{-8\pi i}{6}} + e^{-2\pi i} + 3 \cdot e^{\frac{-16\pi i}{6}} + 2 \cdot e^{\frac{-20\pi i}{6}}$$

$$= 2 + \left(-1^{-\frac{2}{3}}\right) + 3\left(-1^{-\frac{4}{3}}\right) + 1 + 3\left(-1^{-\frac{8}{3}}\right) + 2\left(-1^{-\frac{10}{3}}\right)$$

$$= 2 + \frac{1}{(-1)^{\frac{2}{3}}} + \frac{3}{(-1)^{\frac{1}{2}}} + 1 + \frac{3}{(-1)^{\frac{2}{3}}} + \frac{2}{(-1)^{\frac{1}{3}}} = 3 + \frac{4}{(-1)^{\frac{2}{3}}} + \frac{5}{(-1)^{\frac{1}{3}}} = -1.5 + 0.866i$$

$$G(3) = \sum_{x=0}^{5} g(x) \cdot e^{\frac{-2\pi i3x}{6}} = \sum_{x=0}^{5} g(x) \cdot e^{-\pi ix} = 2 + e^{-\pi i} + 3e^{-2\pi i} + e^{-3\pi i} + 3e^{-4\pi i} + 2e^{-5\pi i}$$

$$= 2 + (-1) + 3 + (-1) + 4 + (-2) = 4$$

$$G(4) = \sum_{x=0}^{5} g(x) \cdot e^{\frac{-2\pi i4x}{6}} = \sum_{x=0}^{5} g(x) \cdot e^{\frac{-8\pi ix}{6}} = 2 + e^{\frac{-8\pi i}{6}} + 3 \cdot e^{\frac{-16\pi i}{6}} + e^{\frac{-24\pi i}{6}} + 3 \cdot e^{\frac{-32\pi i}{6}} + 2 \cdot e^{\frac{-40\pi i}{6}}$$

$$= 2 - \frac{1}{(-1)^{\frac{1}{3}}} + \frac{3}{(-1)^{\frac{2}{3}}} + 1 - \frac{3}{(-1)^{\frac{1}{3}}} + \frac{2}{(-1)^{\frac{2}{3}}} = 3 - \frac{4}{(-1)^{\frac{1}{3}}} + \frac{5}{(-1)^{\frac{2}{3}}} = -1.5 - 0.866i$$

$$G(5) = \sum_{x=0}^{5} g(x) \cdot e^{\frac{-2\pi i5x}{6}} = \sum_{x=0}^{5} g(x) \cdot e^{\frac{-10\pi ix}{6}} = 2 + e^{\frac{-10\pi i}{6}} + 3 \cdot e^{\frac{-20\pi i}{6}} + e^{\frac{-30\pi i}{6}} + 3 \cdot e^{\frac{-40\pi i}{6}} + 2 \cdot e^{\frac{-50\pi i}{6}}$$

$$= 2 - \frac{1}{(-1)^{\frac{2}{3}}} - \frac{3}{(-1)^{\frac{1}{3}}} - 1 + \frac{3}{(-1)^{\frac{2}{3}}} + \frac{2}{(-1)^{\frac{1}{3}}} = 1 - \frac{1}{(-1)^{\frac{1}{3}}} + \frac{2}{(-1)^{\frac{2}{3}}} = -0.5 - 0.866i$$

DFT of $[2, 1, 3, 1, 3, 2]$ is $[12, -0.5 - 0.866i, -1.5 + 0.866i, 4, -1.5 - 0.866i, -0.5 - 0.866i]$

b. $G(k) = [12, -0.5 - 0.866i, 1.5 + 0.866i, 4, 1.5 - 0.866i, -0.5 + 0.866i]$,

$$g(x) = \frac{1}{N}\sum_{k=0}^{N-1} G(k) \cdot e^{\frac{2\pi ikx}{N}} \rightarrow g(x) = \frac{1}{6}\sum_{k=0}^{5} G(k) \cdot e^{\frac{2\pi ikx}{6}}$$

$$g(0) = \frac{1}{6}\sum_{k=0}^{5} G(k)\cdot e^{\frac{2\pi i k 0}{6}} = \frac{1}{6}\sum_{k=0}^{5} G(k)$$

$$= \frac{1}{6}(12 - 0.5 - 0.866i + 1.5 + 0.866i + 4 + 1.5 - 0.866i - 0.5 + 0.866i) = \frac{12}{6} = 2$$

$$g(1) = \frac{1}{6}\sum_{k=0}^{5} G(k)\cdot e^{\frac{2\pi i k}{6}}$$

$$= \frac{1}{6} \cdot \left(12 + (-0.5 - 0.866i)e^{\frac{2\pi i}{6}} + (\,1.5 + 0.866i)e^{\frac{4\pi i}{6}} + 4e^{\frac{6\pi i}{6}} + (1.5 - 0.866i)e^{\frac{8\pi i}{6}} + (\,-0.5 + 0.866i)e^{\frac{10\pi i}{6}}\right) = \frac{12 + 0.00003i}{6} = 2 + 0.000005i$$

$$g(2) = \frac{1}{6}\sum_{k=0}^{5} G(k)\cdot e^{\frac{4\pi i k}{6}}$$

$$= \frac{1}{6} \cdot \left(12 + (-0.5 - 0.866i)e^{\frac{4\pi i}{6}} + (\,1.5 + 0.866i)e^{\frac{8\pi i}{6}} + 4e^{\frac{12\pi i}{6}} + (1.5 - 0.866i)e^{\frac{16\pi i}{6}} + (\,-0.5 + 0.866i)e^{\frac{20\pi i}{6}}\right) = \frac{18 - 0.00003i}{6} = 3 - 0.000005i$$

$$g(3) = \frac{1}{6}\sum_{k=0}^{5} G(k)\cdot e^{\frac{6\pi i k}{6}} = \frac{1}{6}\sum_{k=0}^{5} G(k)\cdot e^{\pi i k}$$

$$= \frac{1}{6} \cdot (12 + (-0.5 - 0.866i)e^{\pi i} + (\,1.5 + 0.866i)e^{2\pi i} + 4e^{3\pi i} + (1.5 - 0.866i)e^{4\pi i} + (\,-0.5 + 0.866i)e^{5\pi i}) = \frac{12 + 0.00003i}{6} = 2 + 0.000005i$$

$$g(4) = \frac{1}{6}\sum_{k=0}^{5} G(k)\cdot e^{\frac{8\pi i k}{6}}$$

$$= \frac{1}{6} \cdot \left(12 + (-0.5 - 0.866i)e^{\frac{8\pi i}{6}} + (\,1.5 + 0.866i)e^{\frac{16\pi i}{6}} + 4e^{\frac{24\pi i}{6}} + (1.5 - 0.866i)e^{\frac{32\pi i}{6}} + (\,-0.5 + 0.866i)e^{\frac{40\pi i}{6}}\right) = \frac{18 - 0.00003i}{6} = 3 - 0.000005i$$

$$g(5) = \frac{1}{6}\sum_{k=0}^{5} G(k)\cdot e^{\frac{10\pi i k}{6}}$$

$$= \frac{1}{6} \cdot \left(12 + (-0.5 - 0.866i)e^{\frac{10\pi i}{6}} + (\,1.5 + 0.866i)e^{\frac{20\pi i}{6}} + 4e^{\frac{30\pi i}{6}} + (1.5 - 0.866i)e^{\frac{40\pi i}{6}} + (\,-0.5 + 0.866i)e^{\frac{50\pi i}{6}}\right) = \frac{12 + 0.00003i}{6} = 2 + 0.000005i$$

IDFT of $[12, -0.5 - 0.866i, 1.5 + 0.866i, 4, 1.5 - 0.866i, -0.5 + 0.866i]$ is $[2, 2 + 0.000005i, 3 - 0.000005i, 2 + 0.000005i, 3 - 0.000005i, 2 + 0.000005i\ \ ]$

c. The Fourier transform $G(\omega)$ of a sine function with wavelength ω_0 is: $\frac{i}{2}\left(\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right)$:

$$g(x) = \sin(2\pi\omega_0 x)$$

$$\begin{aligned}
G(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i\omega x} \cdot \left(e^{2\pi i\omega_0 x} - e^{-2\pi i\omega_0 x}\right) \cdot \frac{1}{2i}\,dx \\
= \int_{-\infty}^{\infty} e^{-2\pi i\omega x} \cdot \left(e^{2\pi i\omega_0 x} - e^{-2\pi i\omega_0 x}\right) \cdot \frac{1\cdot(-i)}{2i\cdot(-i)}\,dx \\
= \int_{-\infty}^{\infty} e^{-2\pi i\omega x} \cdot \left(e^{2\pi i\omega_0 x} - e^{-2\pi i\omega_0 x}\right) \cdot \frac{-i}{2}\,dx \\
= \int_{-\infty}^{\infty} e^{-2\pi i\omega x} \cdot \left(e^{-2\pi i\omega_0 x} - e^{2\pi i\omega_0 x}\right) \cdot \frac{i}{2}\,dx \\
= \frac{i}{2}\int_{-\infty}^{\infty} e^{-2\pi i x(\omega+\omega_0)} - e^{-2\pi i x(\omega-\omega_0)}\,dx = \frac{i}{2}[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]
\end{aligned}$$

d. DFT is cyclic:

$$F(k+N) = \sum_{n=0}^{N-1} x_n e^{\frac{-i2\pi(k+N)}{N}} = \sum_{n=0}^{N-1} x_n e^{\frac{-i2\pi k - i2\pi N}{N}}$$

$$e^{\frac{-i2\pi k - i2\pi N}{N}} = e^{\frac{-i2\pi k}{N}} \cdot e^{\frac{-i2\pi N}{N}} = e^{\frac{-i2\pi k}{N}} \cdot e^{-i2\pi}$$

$$e^{-i2\pi} = \cos(-2\pi) + i\sin(-2\pi) = 1$$

$$\Rightarrow e^{\frac{-i2\pi k}{N}} \cdot e^{-i2\pi} = e^{\frac{-i2\pi k}{N}} \cdot 1 = e^{\frac{-i2\pi k}{N}}$$

$$\Rightarrow \sum_{n=0}^{N-1} x_n e^{\frac{-i2\pi k - i2\pi N}{N}} = \sum_{n=0}^{N-1} x_n e^{\frac{-i2\pi k}{N}} = F(k)$$