Midterm Answer

Variance, Covariance, Derivative Softmax function

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Prove that $V[x] = E[x^2] - E[x]^2$ where PDF of x is continuous

Solution

$$\begin{aligned}
V[x] &= E[(x-\mu)^2] \\
&= \int (x-\mu)^2 f(x)dx \\
&= \int x^2 f(x)dx - 2\mu \underbrace{\int xf(x)dx}_{\mu} + \mu^2 \underbrace{\int f(x)dx}_{1} \\
&= E[x^2] - \mu^2 \\
&= E[x^2] - E^2[x]
\end{aligned}$$

Prove that $Cov(x, y) = E[xy] - E[x]E[y]$ where PDF of x,y are continuous

Solution

$$\begin{aligned}
Cov(x,y) &= E[(x-\mu_x)(y-\mu_y)] \\
&= \int (x-\mu_x)(y-\mu_y)h(x,y)dxdy \\
&= \int xyh(x,y)dxdy - \mu_x \int yh(x,y)dxdy \\
&\qquad - \mu_y \int xh(x,y)dxdy + \mu_x\mu_y \underbrace{\int h(x,y)dxdy}_{1} \\
&= E[xy] - \mu_x \underbrace{\int yg(y)dy}_{\mu_y} - \mu_y \underbrace{\int xf(x)dx}_{\mu_x} + \mu_x\mu_y \\
&= E[xy] - 2\mu_x\mu_y + \mu_x\mu_y \\
&= E[xy] - E[x]E[y]
\end{aligned}$$

Find derivatives of $softmax(x_k) = \frac{e^{x_k}}{\sum\limits_k e^{x_k}}$

Solution

Let the function put $\sigma(x_k)$,

$$
\begin{aligned}
\frac{d}{dx_i}\sigma(x_k) &= \frac{(\delta_{ki}e^{x_k})\sum\limits_k e^{x_k} - e^{x_k}(e^{x_i})}{\left(\sum\limits_k e^{x_k}\right)^2} \\
&= \frac{e^{x_k}}{\sum\limits_k e^{x_k}}\left(\delta_{ki}\cancelto{1}{\frac{\sum\limits_k e^{x_k}}{\sum\limits_k e^{x_k}}} - \frac{e^{k_i}}{\sum\limits_k e^{x_k}}\right) \\
&= \sigma(x_k)(\delta_{ki} - \sigma(x_i))
\end{aligned}
$$