SUSY Lectures in 2017 Summer

Kwang Sik Jeong [1]

Noted by Y.J. Park, T.G. Kim

Contents

1 Introduction **2**

1 [L1] Quantum Field Theory 2

1-1 S(action) 2

1-2 Regularization & Renormalization 3

2 [L2] Effective Field Theory 5

3 [L2] Poincaré Symmetry 6

3-1 Group 6

3-2 Representation 6

3-3 Poincaré group 6

2 Supersymmetry **8**

1 Supersymmetry 8

1-1 Super-Poincaré algebra ($N = 1$) 9

1-2 Superspace Formalism 10

[1] Dept. of Physics, Pusan National University

1 Introduction

1 [L1] Quantum Field Theory

: basic framework to study elementary particles and their interactions.

"special relativity + Quantum mechanics"

$$\text{On special relativity} \quad \begin{cases} \text{spacetime : } (ct, \vec{x}) \\ \text{force: mediated by fields. (e.g. EM)} \end{cases}$$

S: action
$\delta S = 0$: classical action(least action)

$$\langle f|i\rangle \sim \sum_{\text{path}} e^{iS/\hbar} \quad ; \text{path integral}$$

$$\begin{cases} \Delta S \gg \hbar & ; \text{Classical} \\ \Delta S \sim \hbar & ; \text{Quantum} \end{cases}$$

We use the natural unit, $\hbar = c = 1$.

$$\text{Theory} \quad \begin{cases} \text{action}(S) \\ \quad + \\ \text{regularization} \\ \quad + \\ \text{renormalization} \end{cases}$$

1-1 S(action)

$$S = \int \underbrace{d^4x}_{dtdxdydz} \mathcal{L}(\phi, \partial_\mu \phi)$$

where ϕ is a field, $\phi(x)$. Lagrange has a symmetry.

i) symmetry
- Important role to better understand natural.
- Defining an elementary particle according to the behavior of the corresponding field with respect to symmetries.
- Determining interactions among particles.
(can be hidden; spontaneous breaking)

Noether Theorem(classical field theory)
: continuous symmetry $\rightarrow$ conserved quantity

Symmetry

1. Spacetime Symmetry ($x \to x'$)
   (a) Poincaré Transformation: transformation leaving the line element invariant.
   $\to ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$
      i. Lorentz transformation; action conservation
      ii. Translations; energy-momentum conservation
   (b) General coordinate Transformation
   $\to$ general relativity
2. Internal Symmetry
$$\Phi^a(x) \to \Phi'^a(x) = M^a{}_b \Phi^b(x)$$
where $\Phi^a(x)$ is general field and a, b are internal indices.
$$M^a{}_b \begin{cases} \text{global symmetry: } M^a{}_b \text{ is spacetime-independent} \\ \text{local(gauge) symmetry: } M^a{}_b(x) \text{ is spacetime-dependent} \end{cases}$$

QFT: the most general symmetry of the S-matrix(Scattering matrix).

Note 1.1 *Coleman-Mandula(1967)*

$$\text{Poincaré symmetry} \bigotimes \text{Internal symmetry}$$

where $\bigotimes$ is a direct product.

- assumption on QFT (local, relativity, $4D$, ...)
  & scattering interacions.
- considered only bosonic generator.

Extended to include spinor generators(spin: $\frac{1}{2}$, $\frac{3}{2}$),

Note 1.2 *Haag-Lopuszański-Sohnius(1975)*

$$\text{Super-Poincaré symmetry} \bigotimes \text{Internal symmetry}$$

1-2 Regularization & Renormalization

1. Regularization
   (a) UV divergence
   : we need to regulate the theory.
   (Infinities $\to$ absorbed by appropriate counter terms.)
   e.g. cutoff regulation, dimensional regularization

2. Renormalization
   : determine how to absorb the infinities into counter terms.
   (arbitraty renormalization point μ(unphysical))

   (a) Perturbation Theory
       : expansion w.r.t $\underbrace{\text{renormalized parameters.}}_{\mu-\text{dep.}}$ $\rightarrow$ large logs: $\ln(\frac{E}{\mu}), \ln(\frac{M}{\mu})$
       $\rightarrow$ physical amplitude(independent of μ) $\rightarrow$ "set $\mu \sim E$"

   (b) RGE(Renormalization Group Equation)

2 [L2] Effective Field Theory

[Need Figure!] To make massive particle, we need high energy. So in low energy scale, we can see only small mass particles. We can't & don't need to see higher mass particles.

– Full Theory (UV)

$$\mathcal{L}_{full} = \mathcal{L}_H(\phi_H, \phi_L) + \mathcal{L}(\phi_L)$$
$$\text{where } \begin{cases} \phi_H : \text{Fields describing particles with masses } > \Lambda \\ \phi_L : \text{Fields describing particles with masses } < \Lambda \end{cases}$$

– Effective theory of light fields ϕ_L

: Integrating out ϕ_H in E.O.M. (Effects of all Heavy particle are in C_i)

$$\mathcal{L}_{eff} = \mathcal{L}(\phi_L) + \sum_k \frac{C_i}{\Lambda^{k-4}} \mathcal{O}_i^{(k)}(\phi_L)$$
$$\text{where } \begin{cases} C_i : \text{Wilson Coefficient (dim=0)} \\ \Lambda : \text{Cut-off sclae of EFT} \end{cases}$$

(1) Contribution of $\mathcal{O}_i^{(k)}$ to a process at energy scale $E \gg \Lambda$

$$C_i \left(\frac{E}{\Lambda}\right)^{k-4} \begin{cases} k = 4 : \text{Marginal} \\ k < 4 : \text{Relavent} \\ k > 4 : \text{Irrelavnt - (Almost non-renormalizable)} \end{cases}$$

(2) Matching (Around Λ) : Same physical predictions at low energy.

$$\lambda_{eff}(\mu) = \lambda_{full}(\mu) + (\text{threshold corrections})$$

e.g. [Need Figure!]

3 [L2] Poincaré Symmetry

– Symmetry of the SM

$$\begin{cases} \text{Poincaré} \begin{cases} \text{Lorentz Inv.} \\ \text{Translation Sym.} \end{cases} \\ \\ \text{Internal} \begin{cases} \text{Global: Baryon number, Lepton number, Flavor Sym.} \\ \text{Gauge/Local: } SU(3)_c \times SU(2)_c \times U(1)_Y \end{cases} \end{cases}$$

3-1 Group

Definition 3.1 (Group) *A set of elements $G = \{g_1, g_2, \cdots\}$ with product operator $*$ s.t*

1. *Closure:* $\forall g_i, g_j \in G,\ g_i * g_j \in G$
2. *Unit element :* $\exists e \in G,\ e * g = g * e = g \quad \forall g \in G$
3. *Inverse element :* $\forall g \in G,\ \exists g^{-1} \in G \quad s.t \quad g * g^{-1} = g^{-1} * g = e$
4. *Associativity :* $\forall g_i, g_j, g_k \in G,\ g_i * (g_j * g_k) = (g_i * g_j) * g_k$

3-2 Representation

Definition 3.2 (Representation) *A map $G \to R$ where $R = \{D(g)\}$ such that*

1. $D(g_1)D(g_2) = D(g_1 * g_2)$
2. $D(e) = 1$
3. $D(g^{-1}) = D(g)^{-1}$

$D(g)$ is a linear operator acting on a vector space $V = \{v_1, v_2, \cdots\}$.

3-3 Poincaré group

– Coordinate trsf leaving $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ is invariant.

$$x' = \Lambda x + a \quad \begin{cases} a : \text{Translations} \\ \Lambda : \text{Lorentz trsf (rotations + boosts)} \end{cases}$$

– **Inifinitesimal transformation :** (Λ, a)

$$\delta x^\mu = \epsilon^\mu + \omega^\mu{}_\nu x^\nu \ (\omega^{\mu\nu} = -\omega^{\nu\mu};\ \text{anti-sym})$$
$$\Rightarrow \delta X^\mu = -i\left(\epsilon_\sigma P^\sigma + \frac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma}\right)^\mu{}_\nu X^\nu \quad (X \text{ in Vector space})$$

In this representations, $\begin{cases} P_\mu = i\partial_\mu \\ M_{\mu\nu} = i(X_\mu\partial_\nu - X_\nu\partial_\mu) \end{cases}$ $\rightarrow$ the form of P and M depends on the reps.

– **Poincaré algebra**

From the composition rule : $(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$, we can derive below algebra.

1. $[P^\mu, P^\nu] = 0$
2. $[M^{\mu\nu}, P^\sigma] = i(P^\mu\eta^{\nu\sigma} - P^\nu\eta^{\mu\sigma})$
3. $[M^{\mu\nu}, M^{\rho\sigma}] = i(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho})$

– **Representation on fields : ($x' = \Lambda x + a$)**

- As a vector, $\phi^a(x) \rightarrow \phi'^a(x') = R^a{}_b(\Lambda)\phi^b(x)$ (ϕ is regarded as scalar for translations.)
- As a field(quantum) operator, $\hat{\phi}^a(x)$
  : Consider transformation of state vector $|\psi\rangle \rightarrow |\psi'\rangle = \hat{U}(\Lambda, a)|\psi\rangle$ ($\hat{U}(\Lambda, a)$ is linear & unitary.)

$$\hat{U}(\Lambda, a) = 1 - i\epsilon_\mu \hat{P}^\mu - \frac{i}{2}\omega_{\mu\nu}\hat{M}^{\mu\nu} + \cdots$$

  Now consider field operator, we can get below equations.

$$\begin{cases} \varphi^a(x) = \langle f|\hat{\phi}^a(x)|g\rangle \quad \textit{then} \quad \varphi'^a(x) = \langle f'|\hat{\phi}^a(x)|g'\rangle = \langle f|\hat{U}^\dagger\hat{\phi}^a(x)\hat{U}|g\rangle \\ \\ \varphi'^a(x') = R^a{}_b(\Lambda)\varphi^b(x) = R^a{}_b\langle f|\hat{\phi}^b(x)|g\rangle \end{cases}$$

$$\therefore\ \hat{\phi}'^a(x') = R^a{}_b\hat{\phi}^b(x) = \hat{U}^\dagger\hat{\phi}^a(x')\hat{U}$$

(Caution : It's not unitary equivalence!)

2 Supersymmetry

1 Supersymmetry

Symmetry to cancle Λ^2 - contributions? $\lambda = y^2$? **[Supersymmetry]**

− **Supersymmetry** : Boson $\leftrightarrow$ Fermion

$$\bar{Q}\,|B\rangle = |F\rangle \quad (\bar{Q} = Q^\dagger)$$
$$Q\,|F\rangle = |B\rangle$$

* SUSY operator : Anti-commuting spinor (spin 1/2) $\Rightarrow$ Spacetime Symmetry

Note 1.1 (SUSY) SUSY is the unique extension of Poincaré algebra within the QM framework. (Under reasonable physical assumptions)

− **Supermultiplet :** B, F **(Superpartner)**

(i) Equal number of fermion and boson D.O.F.

$$n_B(P_\mu) - n_F(P_\mu) \to \sum_i \langle i|(-1)^{2s}P^\mu|i\rangle$$

Since $Q \in (\frac{1}{2}, 0)$, $\bar{Q} \in (0, \frac{1}{2})$, then $\{Q, \bar{Q}\} \in (\frac{1}{2}, \frac{1}{2})$, $\{Q, \bar{Q}\} \sim P^\mu$.

$$\begin{aligned}
\sum_i \langle i|(-1)^{2s}P^\mu|i\rangle &= \sum_i \langle i|(-1)^{2s}QQ^\dagger|i\rangle + \sum_i \langle i|(-1)^{2s}Q^\dagger Q|i\rangle \\
&= \sum_i \langle i|(-1)^{2s}QQ^\dagger|i\rangle + \sum_{ij} \langle i|(-1)^{2s}Q^\dagger|j\rangle\,\langle j|Q|i\rangle \\
&= \sum_i \langle i|(-1)^{2s}QQ^\dagger|i\rangle + \sum_{ij} \langle j|Q|i\rangle\,\langle i|(-1)^{2s}Q^\dagger|j\rangle \\
&= \sum_i \langle i|(-1)^{2s}QQ^\dagger|i\rangle + \sum_{j} \langle j|Q(-1)^{2s}Q^\dagger|j\rangle \\
&= \sum_i \langle i|(-1)^{2s}QQ^\dagger|i\rangle - \sum_{j} \langle j|(-1)^{2s}QQ^\dagger|j\rangle = 0
\end{aligned}$$

$$\therefore n_B - n_F = 0 \quad for\ P^\mu \neq 0$$

(ii) Internal Symmetry

$$[Q, T_i] = 0 \quad \text{where } T_i \text{ is generator of internal symmetry group}$$
$$\Rightarrow\ B\&F \text{ in the supermultiplet : Same internal symmetry charge.}$$

(iii) Exception : R-symmetry

$$\text{Global} \quad U(1)_R \begin{cases} \text{As a vector: } Q \to e^{i\lambda}Q,\, \bar{Q} \to e^{-i\lambda}\bar{Q} \\ \text{As an operator: } \hat{Q} \to e^{-i\lambda\hat{R}}\hat{Q}e^{i\lambda\hat{R}} \end{cases}$$

$$\Rightarrow \begin{cases} [Q,R] = +Q \\ [\bar{Q},R] = -Q \end{cases}$$

1-1 Super-Poincaré algebra ($N = 1$)

− **SUSY generators** : $Q_\alpha \sim$ Spinor (spin 1/2)

$$\text{Lorentz Transform} \begin{cases} \text{As a vector: } Q_\alpha \to Q'_\alpha = \exp\left[-\frac{1}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right]_\alpha^{\ \beta} Q_\beta \\ \\ \text{As an operator: } \hat{Q}_\alpha \to \hat{Q}'_\alpha = \hat{U}^\dagger \hat{Q}_\alpha \hat{U} \end{cases}$$

For some calculations, we can derive next algebra :

(i) $[P^\mu, P^\nu] = 0$

(ii) $[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)$

(iii) $[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho})$

(iv) $[P_\mu, Q_\sigma] = \left[P_\mu, \bar{Q}^{\dot{\alpha}}\right] = 0$

(v) $[M_{\mu\nu}, Q_\alpha] = -i(\sigma_{\mu\nu})_\alpha^{\ \beta} Q_\beta$

(vi) $\left[M_{\mu\nu}, \bar{Q}_{\dot{\beta}}\right] = -i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}}\bar{Q}^{\dot{\beta}}$

(vii) $\{Q_\alpha, Q_\beta\} = 0$

(viii) $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu$

Now let's find representations.

1-2 Superspace Formalism

− **Superspace** $(\chi^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$; Corresponding to $(P^\mu, Q^\alpha, \bar{Q}_{\dot{\alpha}})$

* SUSY : Translation in the Grassmann Coordinates.

$$\delta \begin{pmatrix} \chi^\mu \\ \theta^\alpha \\ \bar{\theta}_{\dot{\alpha}} \end{pmatrix} \equiv -i(a_\rho P^\rho + \frac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma} - \xi Q - \bar{\xi}\bar{Q}) \begin{pmatrix} \chi^\mu \\ \theta^\alpha \\ \bar{\theta}_{\dot{\alpha}} \end{pmatrix}$$

$$\begin{aligned} \delta\chi^\mu &= a^\mu + \omega^\mu{}_\nu \chi^\nu - ic(\xi\sigma^\mu\bar{\theta}) + ic^*(\theta\sigma^\mu\bar{\xi}) \\ \delta\theta^\alpha &= \frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})^\alpha{}_\beta\theta^\beta + \xi^\alpha \\ \delta\bar{\theta}_{\dot{\alpha}} &= \frac{i}{2}\omega_{\mu\nu}(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}}\bar{\theta}_{\dot{\beta}} + \bar{\xi}_{\dot{\alpha}} \end{aligned}$$

Thus, one finds,

$$\begin{aligned} P_\mu &= i\partial_\mu \\ Q_\alpha &= -i\frac{\partial}{\partial\theta_\alpha} - c(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\frac{\partial}{\partial x^\mu} \\ \bar{Q}_{\dot{\alpha}} &= i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + c^*\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\frac{\partial}{\partial x^\mu} \end{aligned}$$

where c is determined by the commutation relation.

$$\begin{aligned} &\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu \\ &\Rightarrow Re(c) = 1 \quad \text{(Convenient to set } c = 1) \end{aligned}$$