Nim game

1 Problem setting

You are playing a Nim game with your friend:

- Initially, there is a heap of stones on the table
- You and your friend will alternate taking turns, and **you go first**.
- On each turn, the person whose turn it is will remove 1 to 3 stones from the heap.
- The one who removes the last stone is the winner.

Given n, he number of stones in the heap, return *true* if you can win the game assuming both you and your friend play optimally, otherwise return *false*.

Constraints $1 \leq n \leq 2^{31} - 1$

2 How to solve this problem

For those who don't know what a *Nim game* is, you can take a look at Nim

Basically, The game nim is played with heaps (or piles) of chips (or counters, beans, pebbles, matches). Players alternate in making a move, by removing some chips from one of the heaps (at least one chip, possibly the entire heap). The first player who cannot move any more loses the game (Stengel 2021)

Let's now call first player as A and second player as B

Let's look at some first cases of n:

- $n = 1$: A removes 1 stones and B don't have any move => A wins
- $n = 2$: A removes 2 stones and B don't have any move => A wins
- $n = 3$: A removes 3 stones and B don't have any move => A wins
- $n = 4$: with any number $1 \leq k \leq 3$ A removes, B always has some stones left to moves => B win. We can say this is a *losing position* for A.

You may see that $n = 4$ is the first case where A loses. Let's consider some more cases:

- $n = 5$: A removes 1 stones and B has 4 stones left, as in case $n = 4$ above, B is in a *losing condition* => A wins
- $n = 6$: A removes 2 stones and B has 4 stones left => A wins
- $n = 7$: A removes 3 stones and B has 4 stones left => A wins
- $n = 8$: for any number A moves, B still can give the deadly number 4 to A => B wins

There is a pattern right here: a player who is in position that for every number $n \geq 0$ satisfies $n \equiv 0 (\operatorname{mod} 4)$ is *losing*, else *winning*. Is that true?

Proof.

- Base case $n = 0$, $n \equiv 0 (\operatorname{mod} 4)$, the player loses as required.
- Induction step:

Assume for all $m \leq n$ the rule holds:

- if $m \equiv 0 (\operatorname{mod} 4)$, it is a losing position
- if $m \equiv 1, 2, 3 (\operatorname{mod} 4)$, it is a winning position

Now consider $n + 1$:

- if $n + 1 \equiv 0 (\operatorname{mod} 4)$, then by rule, it is a losing position
- if $n + 1 \equiv 1, 2, 3 (\operatorname{mod} 4)$:
  - the player can move by removing 1, 2, 3

  - in each case, he can choose a move to reach a number divisible by 4 (losing position for the opponent):
    - if $n + 1 \equiv 1 (\operatorname{mod} 4)$, removes 1 -> $n \equiv 0 (\operatorname{mod} 4)$
    - if $n + 1 \equiv 2 (\operatorname{mod} 4)$, removes 2 -> $n \equiv 0 (\operatorname{mod} 4)$
    - if $n + 1 \equiv 3 (\operatorname{mod} 4)$, removes 3 -> $n \equiv 0 (\operatorname{mod} 4)$

Thus, the player can always win from $n + 1 \equiv 1, 2, 3 (\operatorname{mod} 4)$ and loses at $n \equiv 0 (\operatorname{mod} 4)$ By induction, the theorem holds for all $n \geq 0$ ■

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class Solution:
  def canWinNim(self, n: int) -> bool:
    return n % 4 != 0
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3 Extension: Nim game with two heaps

Suppose the game is played with only two heaps and the players can remove any number of stones from a heap they choose (not restricted to the initial setup in the Leetcode question). The numbers of stones in the two heaps are called m and n, respectively. In which positions (m, n) can player A win?

Let's observe a few cases:
- case $(1, 1)$: player A removes 1 and B removes 1. Thus A loses.
- case $1, 2$:
  - player A remove 1 from the second heap, game is back to case $(1, 1)$ -> A is losing
  - player A remove 1 from the second heap -> game is back to case $1, 1$ but for B -> A is winning

So if the two heaps have equal size, e.g. (1, 1), then the first player to move loses (so this is a losing position) because player B can *copy* player A's move by equalising the two heaps. If the two heaps has different size, then player A can equalize them by removing an approriate number of stones from the larger heap, putting B in a losing position.

Lemma 3.1. *The nim position m, n is winning if and only if $m \neq n$, otherwise losing, for all $m, n \geq 0$*

This lemma is a direct consequence of the **Sprague–Grundy theorem** for impartial games, and specifically for normal play Nim (Berlekamp, Conway, and Guy 2001).

Bibliography

Berlekamp, Elwyn R., John H. Conway, and Richard K. Guy. 2001. *Winning Ways for Your Mathematical Plays, Volume 1*. 2nd ed. A. K. Peters

Stengel, Bernhard von. 2021. *Game Theory Basics*. Cambridge: Cambridge University Press