Linear Algebra

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1 Basic notation

I use the following notation:

- $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ denotes a matrix with m rows and n columns.
- $\boldsymbol{x} \in \mathbb{R}^n$ denotes a *vector* with n entries. By default, it is a **column vector**. If I want to explicitly represent a **row vector**, I use $\boldsymbol{x}^T$ (transpose of $\boldsymbol{x}$).
- The ith element of $\boldsymbol{x}$ is denoted by x_i

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

- I use the notation a_{ij} (or $\boldsymbol{A}_{\text{ij}}$, $\boldsymbol{A}_{\text{i,j}}$) for the element at coordination (i, j) of the matrix.

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

- jth column of $\boldsymbol{A}$ is denoted by a_j or $\boldsymbol{A}_{:,j}$

$$\boldsymbol{A} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_3 \\ | & | & & | \end{bmatrix}$$

- ith row of $\boldsymbol{A}$ is denoted by a_i or $\boldsymbol{A}_{i,:}$

$$A = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \cdots & \\ — & a_3^T & — \end{bmatrix}.$$

2 Dot Product

Definition 2.1. (Dot Product): *The dot product between to vector $\boldsymbol{u}$ and $\boldsymbol{v}$, denoted by $\boldsymbol{u} \cdot \boldsymbol{v}$ and is defined as a scalar value*

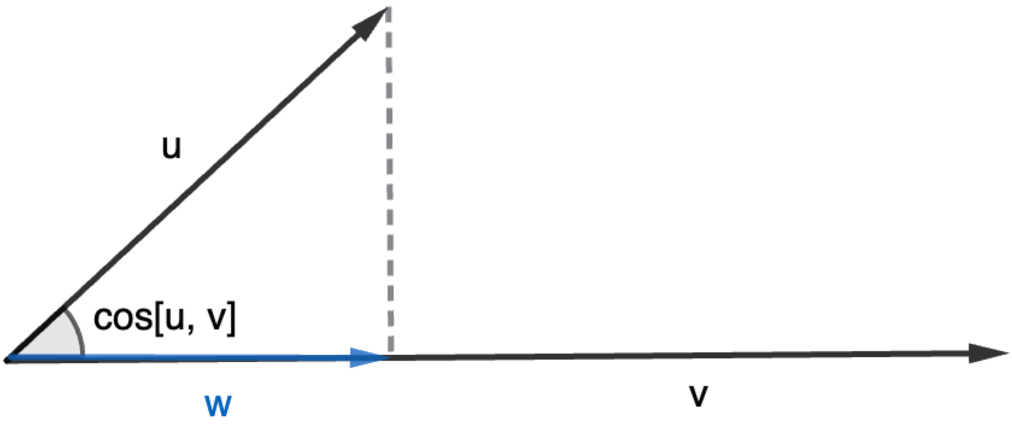
$$\boldsymbol{u} \cdot \boldsymbol{v} = \begin{cases} \|\boldsymbol{u}\| \; \|\boldsymbol{v}\| \cos[\boldsymbol{u}, \boldsymbol{v}] \text{ if } \boldsymbol{u} \neq 0 \text{ and } \boldsymbol{v} \neq 0 \\ 0 \text{ if } \boldsymbol{u} = 0 \text{ or } \boldsymbol{v} = 0 \end{cases} \tag{2.1}$$

2.1 Unit vectors and Normalization

A unit vector is a vector whose length is 1, i.e., $\boldsymbol{v}$ is a unit vector if $\|\boldsymbol{v}\| = 1$. A **normalized vector** $\boldsymbol{n}$ is created from v by dividing $\boldsymbol{v}$ by its length, $\|\boldsymbol{v}\|$, i.e.,

$$\boldsymbol{n} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v} \tag{2.2}$$

2.2 Projection



Definition 2.2. (Orthgonal projection): *If $\boldsymbol{v}$ is a non-zero vector, then the orthogonal projection of $\boldsymbol{u}$ onto v is denoted $P_v\boldsymbol{u}$, and is defined by*

$$P_v\boldsymbol{u} = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{v}\|^2}\boldsymbol{v}$$

Proof. Assume one vector $\boldsymbol{u}$, shall be projected orthogonally onto another vector, let's say $\boldsymbol{v}$, creating a new vector, $\boldsymbol{w}$. Since $\boldsymbol{u}$ and $\boldsymbol{v}$ makes up a triangle, the following must hold:

$$\cos[\boldsymbol{u}, \boldsymbol{v}] = \frac{\|\boldsymbol{w}\|}{\|\boldsymbol{u}\|}$$

This means that $\|\boldsymbol{w}\| = \|\boldsymbol{u}\| \cos[\boldsymbol{u}, \boldsymbol{v}]$. If v has length 1, i.e., $\|\boldsymbol{v}\| = 1$, then the projected vector can be computed as

$$\boldsymbol{w} = \|\boldsymbol{w}\| \ \boldsymbol{v} = \|\boldsymbol{u}\| \cos[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{v}$$

In case of $\boldsymbol{v}$ does not have length 1, we can normalize it using (2.2). If $\boldsymbol{v}$ is replaced by $\frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}$ in equation above, the following is obtained

$$\boldsymbol{w} = \frac{\|\boldsymbol{u}\| \cos[\boldsymbol{u}, \boldsymbol{v}]}{\|\boldsymbol{v}\|}\boldsymbol{v}$$

Next, multiply both the numerator and denominator by $\|\boldsymbol{v}\|$

$$\boldsymbol{w} = \frac{\|\boldsymbol{u}\| \ \|\boldsymbol{v}\| \cos[\boldsymbol{u}, \boldsymbol{v}]}{\|\boldsymbol{v}\|^2}\boldsymbol{v}$$

The numerator is Definition 2.1, now we can write shorter

$$\boldsymbol{w} = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{v}\|^2}\boldsymbol{v}$$

□

2.3 Rules and Properties

Theorem 2.3. (Dot product rules):

(i) $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$

(ii) $k(\boldsymbol{u} \cdot \boldsymbol{v}) = (k\boldsymbol{u}) \cdot \boldsymbol{v}$

(iii) $\boldsymbol{v} \cdot (\boldsymbol{u} + \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{u} + \boldsymbol{v} \cdot \boldsymbol{w}$

(iv) $\boldsymbol{v} \cdot \boldsymbol{v} = \|v\|^2 \geq 0$, *with equality only when* $\boldsymbol{v} = 0$

2.4 Orthonomal Basis

Definition 2.4. (Orthonormal basis): *For an n-dimensional orthonormal basis, consisting of the set of basis vectors,* $\{\boldsymbol{e}_1, ..., \boldsymbol{e}_n\}$*, the following must hold*

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases}$$

This means that basis vectors are of unit length, and pairwise orthogonal.

Definition 2.5. (Dot Product calculation in Orthonormal basis): *In any orthonormal basis, the dot product between two n-dimensional vectors,* $\boldsymbol{u}$ *and* $\boldsymbol{v}$*, can be calculated as*

$$\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i=1}^{n} u_i v_i$$

2.4.1 Vector length in orthonormal basis

Assume a n-dimensional vector $\boldsymbol{v}$, its length in an orthonormal basis can be calculated as

$$\|\boldsymbol{v}\| = \sqrt{\sum_{i=1}^{n} v_i^2}$$

3 Linear Dependence and Independence

Definition 3.1. (Linear combination): *The vector,* $\boldsymbol{u}$*, is a linear combination of the vectors,* $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ *when* $\boldsymbol{u}$ *is expressed as*

$$\boldsymbol{u} = k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + ... + k_n\boldsymbol{v}_n = \sum_{i=1}^{n} k_i v_i$$

where $k_1, k_2, ..., k_n$ *are scalar values.*

For example, $\boldsymbol{w} = w_x\boldsymbol{e}_1 + w_y\boldsymbol{e}_2 + w_z\boldsymbol{e}_3$, that is, 3-dimensional vector $\boldsymbol{w}$ is a *linear combination* of the basis vectors, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, and w_x, w_y, w_z are the scalar values.

Definition 3.2. (Linear independence and dependence): *The set of vectors, $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$, is said to be linearly independent, if the equation*

$$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + ... + k_n\boldsymbol{v}_n = 0$$

***only** has a single solution which is*

$$k_1 = k_2 = ... = k_n = 0$$

If there is at least one other solution, then the set of vectors is linearly dependent

4 Spanning

Definition 4.1. (Span): *The set of vectors $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_q\}$ in $\mathbb{R}^n$ is said to span $\mathbb{R}^n$, if the equation*

$$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + ... + k_n\boldsymbol{v}_n = \boldsymbol{u}$$

has at least one solution, for every vector $\boldsymbol{u}$.

5 The Matrix

Definition 5.1. (Matrix transpose): *The transpose of an $r \times c$ matrix, $\boldsymbol{A}$, is denoted by $\boldsymbol{A}^T$ of size $c \times r$ and is formed by making the columns of $\boldsymbol{A}$ into rows in $\boldsymbol{A}^T$.*

***Example* 5.2.** Assume we have the following matrices,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 6 & 5 \\ 6 & 2 & 4 \\ 5 & 4 & 3 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Their corresponding transposes are

$$\boldsymbol{A}^T = \begin{bmatrix} 1 & 6 & 5 \\ 6 & 2 & 4 \\ 5 & 4 & 3 \end{bmatrix}, \boldsymbol{B}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \boldsymbol{C}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Definition 5.3. (Symmetric matrix): *A square matrix is called symmetric if $\boldsymbol{A} = \boldsymbol{A}^T$*

5.1 Matrix Operations

5.1.1 Matrix Multiplication by a Scalar

Definition 5.4. (Matrix Multiplication by a Scalar): *A matrix $\boldsymbol{A}$ can be multiplied by a scalar k to form a new matrix $\boldsymbol{S} = k\boldsymbol{A}$, which is of the same size as A*

$$\boldsymbol{S} = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

***Example* 5.5.** A 2×2 matrix $\boldsymbol{A}$ is

$$\boldsymbol{A} = \begin{bmatrix} 5 & -2 \\ 3 & 8 \end{bmatrix}$$

multiply it by $k = 4$, we get

$$k\boldsymbol{A} = 4 \begin{bmatrix} 5 & -2 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 4 \cdot 5 & 4 \cdot (-2) \\ 4 \cdot 3 & 4 \cdot 8 \end{bmatrix} = \begin{bmatrix} 20 & -8 \\ 12 & 32 \end{bmatrix}$$

5.1.2 Matrix addition

Definition 5.6. (Matrix Addition): *If 2 matrices* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *have the same size, then the 2 matrices can be added to create a new matrix,* $\boldsymbol{S} = \boldsymbol{A} + \boldsymbol{B}$ *of the same size, where each element* s_{ij} *is the sum of the elements in the same position in matrix* $\boldsymbol{A}$ *and* $\boldsymbol{B}$

$$\boldsymbol{S} = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

***Example* 5.7.** Assume we have two 2×2 matrices $\boldsymbol{A}$ and $\boldsymbol{B}$

$$\boldsymbol{A} = \begin{bmatrix} 5 & -2 \\ 3 & 8 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} -1 & 2 \\ 4 & -6 \end{bmatrix}$$

The matrix addition, $\boldsymbol{S} = \boldsymbol{A} + \boldsymbol{B}$, is

$$\boldsymbol{S} = \boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 5 & -2 \\ 3 & 8 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 4 & -6 \end{bmatrix} = \begin{bmatrix} 5 - 1 & -2 + 2 \\ 3 + 4 & 8 - 6 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 7 & 2 \end{bmatrix}$$

5.1.3 Matrix-Matrix Multiplication