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Generalized Digital Second Order Systems Beyond Nyquist Frequency

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ABSTRACT

The Bilinear transform is an established design method for producing digital representations of analog filters. But due to frequency warping inherent in the transform, adequate representation in the digital domain is limited for analog filters designed near or above the Nyquist frequency. Previous work has demonstrated a method for surpassing this design limitation in the case of parametric filters. This paper is an extension of that previous work, and proposes a design methodology for digitally representing any second order system with a center frequency near or above the Nyquist frequency. By repositioning the system poles, this method compensates for the effects of frequency warping, and allows for a more exact digital replication of an analog system's magnitude response.

1 Introduction

The bilinear transform (BLT) is a prevalent technique for the digitization of analog filters. The BLT maps the highest analog frequency ($s = \infty$) to the highest digital frequency ($z = -1$) and the analog DC ($s = 0$) to digital DC ($z = 1$), and avoids any aliasing that occurs in normal sampling [1]. A limitation of the BLT is that mapping from an infinite analog frequency space to a periodic discrete frequency space results in frequency warping that is especially pronounced near the Nyquist frequency. Sierra [2] proposes a method to correct for the effects of frequency warping in parametric filters with center frequencies above the Nyquist frequency.

This paper proposes a general method for digitizing any second-order systems with analog center frequencies near or above the Nyquist frequency through repositioning the system poles. This allows us to make use of the damping ratio which characterizes all second-order systems and correct for the frequency warping inherent in the BLT.

The motivation for this work was the modeling filters prior to an analog-to-digital converter (ADC) and following a digital-to-analog converter (DAC). The system being modeled has a sampling rate of 44.1 kHz and uses filters to condition the signal for the ADC and DAC around the system's Nyquist frequency of 22.05 kHz. While modeling these filters and accurately digitizing their responses is possible at higher sampling frequencies, due to frequency warping, the analog filter behavior was not accurately replicated at 44.1 kHz and 48 kHz. A direct solution would be upsampling the signal processing for these operating frequencies, but this was deemed undesirable due to increased computational complexity and the inelegance of implementing separate algorithms for different sampling frequencies. By utilizing the proposed method, the same implementation for a filter can be used across different sampling frequencies, and the analog behavior is well replicated across all implementations.

Section 2 will cover prior work, reviewing the BLT and Sierra's methodology. In Section 3, we will demonstrate how Sierra's method can be generalized for any second-order system. Section 4 will provide a design example for digitizing a pre-emphasis filter. Section 5 offers conclusions and suggestions for future work.

2 Prior Work

The methodology outlined here is used by Sierra to match the behavior of digital parametric filters with center frequencies near and beyond the Nyquist frequency to their analog responses. A review of the BLT and Sierra's work is given.

2.1 Bilinear Transform

The BLT can be viewed as a substitution from the Laplace domain to the z-domain.

$$s = c\frac{1-z^{-1}}{1+z^{-1}} \tag{1}$$

$$z^{-1} = \frac{c-s}{c+s} \tag{2}$$

Where c is the Bilinear constant.

$$c = \frac{2}{T} \tag{3}$$

T is the inverse of the digital sampling frequency. If a perfect mapping from a singular analog frequency to a digital frequency is desired, then c is modified to such,

$$c = \frac{\omega}{\tan(\omega T/2)} \tag{4}$$

in a technique known as frequency pre-warping.

2.2 Sierra's Method

Sierra deals with parametric filters of the form:

$$H(s) = \frac{s^2 + g\frac{\omega}{Q}s + \omega^2}{s^2 + \frac{\omega}{Q}s + \omega^2} \tag{5}$$

where g is the parametric filter gain, ω is the center frequency, and Q is the quality factor. To guarantee stability for the Bilinear transform above the Nyquist frequency, Sierra proposed the following definition for the Bilinear constant:

$$c = 2f_s e^{-(\frac{\omega}{\pi f_s})^2} \tag{6}$$

Sierra's Bilinear constant is stable for frequencies $> f_s/2$ as it is always greater than 0.

Due to frequency warping inherent in the BLT, the gain at the digital center frequency, ω_d, will differ significantly from the gain at the parametric filter's center frequency when $\frac{\omega}{2\pi}$ approaches $f_s/2$. This effect can be compensated for by scaling Q by a correction factor denoted as r.

The location of the digital center frequency is found using Sierra's Bilinear constant.

$$\omega_d = 2f_s \tan^{-1}(\omega/c) \tag{7}$$

Then the digital center frequency gain is found by solving Eq. 5 at ω_d and taking the magnitude of the result.

$$g_d = ||H(j\omega_d)|| \tag{8}$$

The correction factor r is dependent on the ratio of Sierra's Bilinear constant and the sampling frequency, and a ratio of the digital center frequency gain g_d and the parametric filter gain g.

$$r = \begin{cases} \frac{c}{2f_s} \cdot \frac{g}{g_d} & g > g_d \\ \frac{c}{2f_s} \cdot \frac{g_d}{g} & g < g_d \end{cases} \tag{9}$$

The two conditions account for maintaining boost-cut symmetry in the parametric filters. The corrected quality factor Q' is the parametric parameter Q scaled by r.

$$Q' = rQ \tag{10}$$

Q' along with g_d are substituted into Eq. 5 to derive the corrected analog parametric filter transfer function

$$H_{corr}(s) = \frac{s^2 + g_d\frac{\omega}{Q'}s + \omega^2}{s^2 + \frac{\omega}{Q'}s + \omega^2} \tag{11}$$

And this transfer function can then be digitized using the BLT with Sierra's Bilinear constant.

3 Proposed Generalized Method

Sierra's method adjusts the Q of the parametric filter to compensate for the frequency warping at the parametric filter's center frequency. We can realize that a parametric filter is a specific case of a second-order system where the quality factor and center frequency are directly accessible in the system's formulation. Sierra's compensation should be possible for any second-order system if Q is adjusted through direct manipulation of the system's poles.

3.1 Generalization

The linear constant-coefficient differential equation for a second-order system is a general equation for any second order system [3].

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n{}^2 y(t) = \omega_n{}^2 x(t) \tag{12}$$

The frequency response of this second order system in the Laplace domain is then:

$$H(s) = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2} \tag{13}$$

And is characterized by the undamped natural frequency, ω_n, and the damping ratio, ζ. ω_n is in most cases approximately equal to the generalized analog center frequency, ω_a, which is used here to draw distinction from the parametric filter parameter ω. The damping ratio is related to Q by:

$$\zeta = \frac{1}{2Q} \tag{14}$$

Using Eq. 14 and the quadratic equation we can derive our poles $p_{1,2}$ in terms of Q and ω_n.

$$p_{1,2} = \omega_n\left(-\frac{1}{2Q} \pm \sqrt{\left(\frac{1}{2Q}\right)^2 - 1}\right) \tag{15}$$

We can also derive the backwards relation for ω_n and Q in terms of our poles.

$$\omega_n = \sqrt{p_1 p_2} \tag{16}$$

$$Q = \frac{\sqrt{p_1 p_2}}{-(p_1 + p_2)} \tag{17}$$

Equations 15, 16, 17 allow us to directly access and manipulate the Q for any second-order system as Q and ω_n characterize all second-order systems.

Finding our corrected quality factor Q' is similar to Sierra's method, but now our digital center frequency, ω_d, is dependent on our analog center frequency, ω_a, instead of the parametric filter parameter ω.

$$\omega_d = 2f_s tan^{-1}(\omega_a) \tag{18}$$

Our correction factor r is now dependent on the system's analog center frequency gain, g_a, instead of the parametric filter gain parameter g.

$$g_a = ||H(j\omega_a)|| \tag{19}$$

And, in the general case we no longer worry about boost-cut symmetry. Our correction factor is thus:

$$r = \frac{c}{2f_s} \cdot \frac{g_d}{g_a} \tag{20}$$

As our correction factor is only dependent on values related to the position of the system's poles, we view the act of correcting a second-order system's response as an adjustment of the system's pole positions. It is useful then, to convert our transfer function model to the zero-pole-gain model, shown in Eq. 21, when correcting our analog filter. While Eq. 21 has two zeros, our second-order system can have one to two zeros but will always has two poles. Here $z_{1,2}$ are the zeros and k the scalar gain of our system.

$$H(s) = k\frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)} \tag{21}$$

3.2 True peak-gain matching

For some systems, our previous approximation $\omega_n \approx \omega_a$ does not hold and the true peak-gain of the filter is not located at the natural frequency, and this is especially true in systems where the poles are located near each other [4]. It is necessary for our correction factor to reference the true peak-gain to properly account for the effects of frequency warping. This can be done by numerically computing the magnitude response of the system and searching for the true peak-gain and peak-gain frequency using a `max()` or `min()` function, in the case we have a cut type filter. The peak-gain value becomes our analog center frequency gain, g_a, and the peak-gain frequency our analog center frequency, ω_a. It is important to note that as our analog center frequency and natural frequency are now distinct, and ω_a is not interchangeable with ω_n when we reposition the poles of our system.

3.3 Proposed method

Concretely, our proposed method for digitizing a general second-order systems with a center frequency near or above the Nyquist frequency is:

1. Convert the system from the transfer function model to the zero-pole-gain model
2. Use poles to derive ω_n and Q using Eq. 16 and 17
3. Assuming $\omega_a \approx \omega_n$, derive the Sierra Bilinear constant using Eq. 6
4. Evaluate Eqs. 8 and 19 to find g_d and g_a
5. Find r using Eq. 20 and the corrected quality factor Q' using Eq. 10
6. Derive new pole positions for Q' using Eq. 15
7. Swap the original poles with the new poles, keeping original zeros and scalar gain and convert system back to the transfer function model
8. Perform the BLT on the corrected analog transfer function using Sierra's Bilinear constant to derive the corrected digital response
9. *Optional:* If ω_a does not correspond well to ω_n, revaluate 3 – 8 using a search for the true peak-gain and peak-gain frequency

4 Example

As an example, we will apply the proposed method to a pre-emphasis filter, shown below in Figure 1.

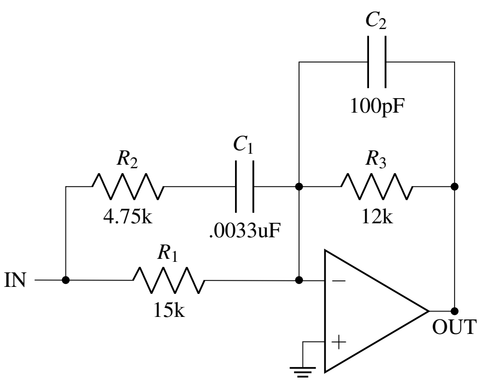

Fig. 1: The pre-emphasis filter

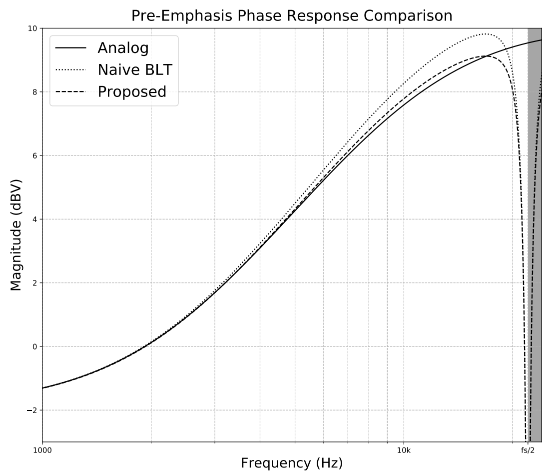

Fig. 2: Comparison of the naive BLT and the proposed method digital magnitude response to the analog magnitude response of the pre-emphasis filter

The pre-emphasis filter proceeds the ADC in the system we are modeling, and we are attempting to digitize the response at 44.1 kHz. The analog transfer function of the filter can be found through KCL and KVL methods in the Laplace domain, resulting in the second order function shown in Eq. 22.

$$\frac{C_1(R_1R_3+R_2R_3)s+R_3}{(R_1R_2R_3C_1C_2)s^2+R_1(R_2C_1+R_3C_2)s+R_1} \tag{22}$$

Solving for the poles, we find the system has poles at 10.1 kHz and 132.6 kHz. And, by applying Eq. 16 we find the filter has an analog center frequency of approximately 36.6kHz, which exceeds the Nyquist frequency for our implementation. As this pre-emphasis filter is paired with a de-emphasis filter following the DAC, it is important for the digital magnitude response to closely match the analog magnitude response.

Applying the proposed method from the previous section, we can correct the naive BLT digital response to more closely match the analog response, as shown in the Proposed response in Figure 2. Both the naive BLT and the proposed method responses still dip near the Nyquist frequency, due to the periodicity of the z-plane. In this example case, there was no need to evaluate the location and value of the true peak-gain, because the poles are distant enough from each other that the approximation $\omega_a \approx \omega_n$ still holds.

5 Conclusion

By manipulating the system's poles to adjust the system's quality factor, the proposed method allows us to closely match the digital response of an analog filter designed near or beyond the digital system's Nyquist frequency. We are also able to replicate the behavior of an analog filter similarly across many different sampling frequencies. This method doesn't replace the BLT, but allows us added flexibility when it comes to digitally representing filter in the high to very high frequency range.

Further work to improve the proposed method would be to apply it to higher order systems, most likely by breaking down the filter into second order sections and correcting each section individually. Corollary to this, is deriving a method to correct first-order systems. As first-order systems do not have damping, it is not possible to correct a first-order system by adjusting the nonexistent quality factor. In turn this affects the correction of higher order odd systems. To this end a possible approach might incorporate the Butterworth transform in [5] which provides a method for raising the filter order for first and second order systems while maintaining dominant filter characteristics such as resonant gain amount and frequency.

References

[1] Smith, J. O., *Physical Audio Signal Processing*, http://ccrma.stanford.edu/ jos/pasp/, 2010.

[2] Sierra, J., "Digital Parametric Filters Beyond Nyquist Frequency," in *Audio Engineering Society 147th Convention*, 2019.

[3] Oppenheim, A. V., Willsky, A. S., and Young, I. T., *Signals and Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1983.

[4] Smith, J. O., *Introduction to Digital Filters with Audio Applications*, http://ccrma.stanford.edu/ jos/-filters/, 2007.

[5] Zavalishin, V., "The Art of VA Filter Design," 2018.