Expected Utility in Insurance Selection

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March 18, 2020

In Section 1, I will introduce one of the literature that attempted to employ the idea of Game Theory into the insurance selection. Then, I will extend the author conversation on different premium pricing and a discount factor. In Section 2, I will solve the questions from Chapter 4 in the book "Playing for Real" by Binmore et al. (2015) and show how risk preferences take role in the expected utilities and insurance selections.

1 Game Theory and Fire Insurance

1.1 Full Insurance or No Insurance

Williams (1960) initiates one of the applications of quantitative analysis to real-world decision making. Williams (1960) sets up a problem as follows:

- There is a person who is deciding to buy fire insurance or not
- The decision is either buying full insurance or not buying insurance at all
- There is only one insurance option, meaning there are no alternative insurances

Table 1: Loss in Utility

Decision	Total Loss	No Loss
No Insurance	L+N+W	W
Fire Insurance	P+N	P

In this table,

L = loss in utility for which the insured will be reimbursed if he has insurance;

N = loss in utility for which insurance affords no protection (inconvenience, for example);

P = loss in utility if the premium is paid;

W = loss in utility due to worry if the person decides not to buy insurance.

Note. Adapted from "Game-Theory and Insurance Consumption." by C. A. Williams.1960, *The Journal of Insurance* , 27(4), 47. Copyright 1960 by JSTOR.

In Table 1, the author identifies the values of the expected utility losses for the cases where fire occurs with insurance, where fire occurs without insurance, where fire does not occur with insurance, and where fire does not occur without insurance (Williams, 1960).

The author identifies that the insurance consumers can make the most rational and typical choice by minimizing the expected loss in utility and by minimizing the expected regret (Williams, 1960).
That is, the consumers buy fire insurance when

$$P \leq (w_1 p_1 + w_2 p_2 + ... + w_k p_k)L + W$$

where $p_1, p_2, ..., p_k$ are the probabilities that fire will occur, $w_1, w_2, ..., w_k$ are weights for the probabilities, and the sum of the weights is unity (Williams, 1960).

1.2 Multiple Insurance Options

In Table 2, the author presents consumers' decision making factors when there are multiple premium options.

Table 2: Loss in Utility

Amount of Insurance	Insurable Loss in Dollars				
	\$0	\$1000	\$2000	\$3000	\$4000
\$0	W_0	$W_0 + L_2$	$W_0 + L_2$	$W_0 + L_3$	$W_0 + L_4$
\$1000	$P_1 + W_1$	$P_1 + W_1$	$P_1 + W_1 + L_1$	$P_1 + W_1 + L_2$	$P_1 + W_1 + L_3$
\$2000	$P_2 + W_2$	$P_2 + W_2$	$P_2 + W_2$	$P_2 + W_2 + L_1$	$P_2 + W_2 + L_2$
\$3000	$P_3 + W_3$	$P_3 + W_3$	$P_3 + W_3$	$P_3 + W_3$	$P_3 + W_3 + L_1$
\$4000	P_4	P_4	P_4	P_4	P_4

In this table,

$L_1, L_2, L_3, L_4 =$ losses in utility for dollar insurable losses of \$1000, \$2000, \$3000, \$4000;

$P_1, P_2, P_3, P_4 =$ losses in utility when premiums are paid for \$1000, \$2000, \$3000, \$4000 of insurance;

$W_0, W_1, W_2, W_3 =$ worry factors corresponding to insurance amounts of \$0, \$1000, \$2000, \$3000.

Note. Adapted from "Game-Theory and Insurance Consumption." by C. A. Williams.1960, *The Journal of Insurance* , 27(4), 47. Copyright 1960 by JSTOR.

By specifying the probabilities of each insurance loss, I compute the loss in expected utility. Assuming the probabilities of insurable losses for \$0, \$1000, \$2000, \$3000, \$4000 $p_0, p_{1000}, p_{2000}, p_{3000}, 1 - p_{1000} - p_{2000} - p_{3000}$ respectively, I can express the expected loss in utility for each amount of insurance as follows:

$$\begin{aligned}
EU_0 &= p_0 W_0 + p_{1000}(W_0 + L_1) + p_{2000}(W_0 + L_2) + p_{3000}(W_0 + L_3) + \\
&\quad (1 - p_{1000} - p_{2000} - p_{3000})(W_0 + L_4) \\
EU_{1000} &= p_0(P_1 + W_1) + p_{1000}(P_1 + W_1) + p_{2000}(W_1 + L_1 + P_1) + \\
&\quad p_{3000}(W_1 + L_2 + P_1) + (1 - p_0 - p_{1000} - p_{2000} - p_{3000})(W_1 + L_3 + P_1) \\
EU_{2000} &= p_0(P_2 + W_2) + p_{1000}(P_2 + W_2) + p_{2000}(W_2 + P_2) + p_{3000}(W_2 + L_1 + P_2) + \\
&\quad (1 - p_0 - p_{1000} - p_{2000} - p_{3000})(W_2 + L_2 + P_2) \\
EU_{3000} &= p_0(P_3 + W_3) + p_{1000}(P_3 + W_3) + p_{2000}(W_3 + P_3) + p_{3000}(W_3 + P_3) + \\
&\quad (1 - p_0 - p_{1000} - p_{2000} - p_{3000})(W_3 + L_1 + P_3) \\
EU_{4000} &= p_0 P_4 + p_{1000} P_4 + p_{1000} P_4 + p_{2000} P_4 + p_{3000} P_4 + \\
&\quad (1 - p_0 - p_{1000} - p_{2000} - p_{3000}) P_4
\end{aligned}$$

The consumers will choose the price with the least expected loss in utility. As the author does not specify the equations of the utility functions for the losses, I am unable to find the exact numbers of losses computationally.

1.3 A Discount Factor

Williams (1960) suggests to include a discount factor into insurance pricing for future research.
I introduce a discount factor d ($0 \leq d \leq 1$) for P. When the person buys insurance in two consecutive years, their loss in utility when the premium is paid is Pd. When they purchase the insurance a year later as well, their loss in utility when the premium is paid is Pd^2.
I assume that the person has been purchasing the fire insurance for K-1 times continuously and that the same person is deciding to purchase the fire insurance at time K. I modify the Table 1 by including the discount factor. The result is shown in Table 3.

Table 3: Loss in Utility at time K

Amount of Insurance	Total Loss	No Loss
No Insurance	$L + N + W$	W
Fire Insurance	$Pd^k + N$	Pd^k

When the probability of fire is p, the probability of not having fire is $1 - p$. I can find the expected loss in utility for each case where they buy insurance and where they do not.

$$\begin{aligned}
EU_{NoInsurance} &= p(L + N + W) + (1 - p)W \\
EU_{FireInsurance} &= p(Pd^K + N) + Pd^K(1 - p)
\end{aligned}$$

They are going to purchase the fire insurance at time K when

$$EU_{FireInsurance} \leq EU_{NoInsurance}$$

That is,

$$\begin{aligned} p(Pd^K + N) + Pd^K(1-p) &\leq p(L + N + W) + (1-p)W \\ Pd^K &\leq pL + W \end{aligned}$$

2 Expected Utility

2.1 Risk Aversion

Figure 1

A charity is to sponsor a garden party to raise money, but the organizer is worried about the possibility of rain, which will occur on the day chosen for the event with probability p. She therefore considers insuring against rain. Her Von Neumann and Morgenstern utility for money $u : \mathbb{R} \to \mathbb{R}$ satisfies $u'(x) > 0$ and $u''(x) < 0$ for all x. Why does she like more money rather than less? Why is she strictly risk averse? Why is the function u' strictly decreasing?

If it is sunny on the day of the event, the charity will make \$$y$. If it rains, the charity will make only \$$z$. The insurance company offers full insurance against the potential loss of \$$(y - z)$ from rain at a premium of \$$M$, but the organizer may decide against full coverage by paying only a fraction f of the full premium. This means that she pays \$$Mf$ before the event, and the insurance company repays \$0 if it is sunny and \$$(y - z)f$ if it rains. (Keep things simple by *not* making the realistic assumption that f is restricted to the range $0 \leq f \leq 1$.)

a. What is the insurance company's dollar expectation if she buys full insurance? Why does it make sense to call the insurance contract fair if $M = p(y - z)$?

b. Why does the organizer choose f to maximize $(1 - p)u(y - Mf) + pu(z + (y - z)f - Mf)$? What do you get when this expression is differentiated with respect to f?

c. Show that the organizer buys full insurance ($f = 1$) if the insurance contract is fair.

d. Show that the insurance contract is fair if the organizer buys full insurance.

e. If the insurance contract is unfair, with $M > p(y - z)$, show that the organizer definitely buys less than full insurance ($f < 1$).

f. How would the organizer feel about taking out a fair insurance contract if she were risk neutral?

Note. Adapted from "Playing for real: a text on game theory" by K. Binmore. 2015, *Oxford University Press.* Copyright 2015 by Oxford University Press.

Now, I solve questions from Question 21, Chapter 4 (Figure 1) and apply the idea of expected utility to risk preference.

a. The insurance company's dollar expectation when the consumer buys the insurance is

$$M - ((1-p)0 + p(y-z)) = M - py + pz$$

The insurance contract is fair when $M - py + pz = 0$, that is when $M = py + pz$. It is fair because the price they ask to the consumer and the amount of money they reimburse is the same.

b. f is the only value that the organizer can have a control. The expected utility for the consumer is as follows.

$$EU_{consumer} = (1-p)u(y - fM) + pu(z + f(y-z) - fM)$$

The derivative of this expected utility is as follows.

$$\frac{\partial EU_{consumer}}{\partial f} = -M(1-p)u'(y - fM) + ((y-z) - M)pu'(z + f(y-z) - fM)$$

c. The consumer's expected utility is highest when the derivative of the expected utility is 0. I show that I can get the derivative of the expected utility 0 when $f = 1$.

$$\begin{aligned}
\frac{\partial EU_{consumer}}{\partial f} &= -M(1-p)u'(y - fM) + ((y-z) - M)pu'(z + f(y-z) - fM) \\
&= -M(1-p)u'(y - M) + (y - z - M)pu'(z + y - z - M) \\
&= -M(1-p)u'(y - M) + (y - z - M)pu'(y - M) \\
&= u'(y - M)(-M + pM + yp - zp - pM) \\
&= u'(y - M)(-M + yp - zp)
\end{aligned}$$

The fair insurance contract means $M = p(y - z)$.

$$\begin{aligned}
\frac{\partial EU_{consumer}}{\partial f} &= u'(y - M)(-M + yp - zp) \\
&= u'(y - M)(-(p(y - z) + yp - zp) \\
&= u'(y - M)(-(py - pz) + yp - zp) \\
&= u'(y - M)(-py + pz + yp - zp) \\
&= u'(y - M)0 = 0
\end{aligned}$$

When the insurance contract is fair, I can maximize the consumer's utility with $f = 1$.

d. I show what condition can lead the derivative of expected utility to be 0 when $f = 1$.

$$\begin{aligned}
\frac{\partial EU_{consumer}}{\partial f} &= -M(1-p)u'(y-fM) + (y-z-M)pu'(z+f(y-z)-fM) \\
&= -M(1-p)u'(y-M) + (y-z-M)pu'(z+y-z-M) \\
&= -M(1-p)u'(y-M) + (y-z-M)pu'(y-M) \\
&= u'(y-M)(-M(1-p) + (y-z-M)p) \\
&= u'(y-M)(-M + pM + yp - zp - pM) \\
&= u'(y-M)(-M + yp - zp)
\end{aligned}$$

Since $u'(x)>0$ for any x, $-M + yp - zp$ must be equal to 0 to maximize the $\frac{\partial EU_{consumer}}{\partial f}$.

$$\begin{aligned}
-M + yp - zp &= 0 \\
M &= yp - zp \\
&= p(y-z)
\end{aligned}$$

This is the definition of the fair contract.

e. When $M>p(y-z)$,

$$\begin{aligned}
p(y-z) &< M \\
-M + py - pz &< 0
\end{aligned}$$

That is,

$$\frac{\partial EU_{consumer}}{\partial f} = u'(y-M)(-M + yp - zp) < 0$$

As the derivative is negative, the expected utility is not at its maximum. The consumer can increase the expected utility by reducing f. With $f < 1$, the consumer will buy less than full insurance.

f. When the consumer's risk preference is neutral, their expected utility function is linear. That is,

$$\begin{aligned}
EU_{consumer} &= (1-p)u(y-fM) + pu(z+f(y-z)-fM) \\
&= (1-p)(u(y) - Mu(f)) + p(u(z) + (y-z)u(f) - Mu(f)) \\
&= (1-p)u(y) - M(1-p)u(f) + pu(z) + (y-z)pu(f) - Mpu(f) \\
&= (1-p)u(y) - Mu(f) + pMu(f) + pu(z) + (y-z)pu(f) - pMu(f) \\
&= (1-p)u(y) - pu(z) + (-M + pM + py - pz - pM)u(f) \\
&= (1-p)u(y) + pu(z) + (-M + py - pz)u(f)
\end{aligned}$$

When the insurance is fair,

$$-M + py - pz = 0$$
$$and$$
$$EU_{consumer} = (1-p)u(y) + pu(z)$$

This expected utility is independent of the value of f. Therefore, $EU_{consumer}$ is constant. That is, the consumer is indifferent whether they buy insurance or not.

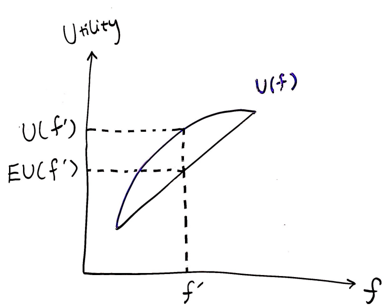
2.2 Other Risk Preferences

So far, I have seen two risk preferences; risk averse and risk neutral.
As discussed in the questions, the utility function for risk averse individual can be represented with a concave curve (Figure 2). For a risk averse individual, the utility of the expected f' is smaller than the expected utility of f' and therefore they always prefers to buy an insurance.

$$EU(f') \leq U(f')$$

Risk neutral individual's utility curve is linear (Figure 3). For a risk neutral

Figure 2: Risk Averse



individual, the utility of the expected f' is equal to the expected utility of f' and therefore they are indifferent to buy an insurance.

$$EU(f') = U(f')$$

Risk loving individual's utility curve is convex (Figure 4). For a risk loving individual, the utility of the expected f' is greater than the expected utility of f' and therefore they prefers not to buy an insurance.

$$EU(f') \geq U(f')$$

Figure 3: Risk Neutral

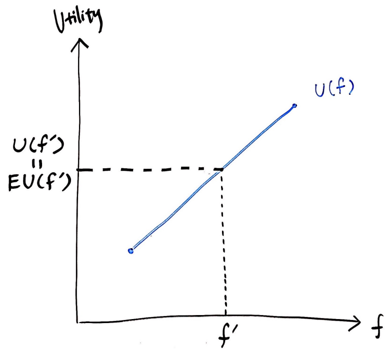
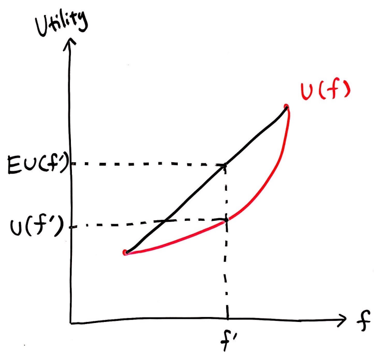


Figure 4: Risk Loving



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