Chapter 4
Tensors and Differential Forms

solutions by Hikari

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4.1 Tensor Analysis

4.1.1 Let all the components of a tensor A vanish in a coordinate system K. For any coordinate system K', the components of A in K' are linear combinations of components of A in K according to the transformation laws of tensors, and is therefore zero. So in every coordinate systems, all the components of A vanish.

4.1.2

$$A_{ij} = \sum_k \sum_l \frac{\partial (x^0)^k}{\partial x^i} \frac{\partial (x^0)^l}{\partial x^j} A^0_{kl} = \sum_k \sum_l \frac{\partial (x^0)^k}{\partial x^i} \frac{\partial (x^0)^l}{\partial x^j} B^0_{kl} = B_{ij}$$

4.1.3 Let the vector be $\mathbf{A}$, and its components be A^i and $(A')^i$ in the two reference frames. For $i = 1,2,3$, $A^i = 0$ and $(A')^i = 0$. Applying the transformation law,

$$(A')^i = \sum_j \frac{\partial (x')^i}{\partial x^j} A^j = \frac{\partial (x')^i}{\partial x^0} A^0$$

For $i = 1,2,3$, $(A')^i = 0$, but at least one of $\frac{\partial (x')^i}{\partial x^0} \neq 0$, so A^0 must be zero. So all the components of $\mathbf{A}$ in the first reference frame vanish, and by exercise 4.1.1, all the components of $\mathbf{A}$ vanish in every reference frame. In particular, the zeroth component of $\mathbf{A}$ vanish in every reference frame.

4.1.4 Let A be an isotropic second-rank tensor in 3-D space. Consider the 90° rotation about x_3 axis. Then $(x')^1 = x^2$, $(x')^2 = -x^1$, $(x')^3 = x^3$. So

$$A^{11} = (A')^{11} = \sum_i \sum_j \frac{\partial (x')^1}{\partial x^i} \frac{\partial (x')^1}{\partial x^j} A^{ij} = \frac{\partial (x')^1}{\partial x^2} \frac{\partial (x')^1}{\partial x^2} A^{22} = A^{22}$$

Similarly, we can prove $A^{22} = A^{33}$, so $A^{11} = A^{22} = A^{33} = k$, k is a constant.

Consider the 180° rotation about x_3 axis. Then $(x'')^1 = -x^1$, $(x'')^2 = -x^2$, $(x'')^3 = x^3$. So

$$A^{13} = (A'')^{13} = \sum_i \sum_j \frac{\partial (x'')^1}{\partial x^i} \frac{\partial (x'')^3}{\partial x^j} A^{ij} = \frac{\partial (x')^1}{\partial x^1} \frac{\partial (x')^1}{\partial x^3} A^{13} = (-1)(1) A^{13} = -A^{13}$$

so $A^{13} = 0$. Similarly, we can prove $A^{31} = A^{12} = A^{21} = A^{23} = A^{32} = 0$. Therefore,

$$A^{ij} = \begin{cases} k, & i = j \\ 0, & i \neq j \end{cases}$$

which is $k\delta^i_j$.

4.1.5 [First raletion] If $i = k$, then $R_{iklm} = -R_{kilm} = -R_{iklm}$, so $R_{iklm} = 0$. The same is for $l = m$, so $R_{iklm} \neq 0$ only if $i \neq k$ and $l \neq m$. $R_{ik\_\_}$ will determine $R_{ki\_\_}$, and $R_{\_\_lm}$ will determine $R_{\_\_ml}$, so if we let $(i,k),(l,m) \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$, then all the other components are determined. So the number of independent components is $6 \times 6 = 36$.

[Second relation] If $(i,k) \neq (l,m)$, then R_{iklm} determines R_{lmik}. So the number of components reduced is $C_2^6 = 15$, and the number of independent components becomes $36 - 15 = 21$.

[Third relation] If one of k,l,m equals i, let it be k, then the relation becomes $R_{iilm} + R_{ilmi} + R_{imil} = 0$. Using the first two relations, it becomes $R_{imil} + R_{miil} = 0$, which is the first relation, so no new information are obtained. If two of k,l,m are equal, let it be $k = l$, then the relation become $R_{ikkm} + R_{ikmk} + R_{imkk} = 0$. Using the first relation, it becomes $R_{ikkm} + R_{ikmk} = 0$, which is the first relation, so no new information are obtained. So the relation furnishes new information only if all four indices are different. Using the first relation, $R_{iklm} + R_{ilmk} + R_{imkl} = 0$ becomes $R_{ikml} + R_{ilkm} + R_{imlk} = 0$, so the parity of the permutation of k,l,m does not matter. Using the first two relation, $R_{iklm} + R_{ilmk} + R_{imkl} = 0$ becomes $R_{klmi} + R_{kmil} + R_{kilm} = 0$, so whether the first index is $1,2,3$ or 4 does not matter. Therefore, let $= 1$, and $(k,l,m) = (2,3,4)$, then we get a new equation, so the number of independent components becomes $21 - 1 = 20$.

4.1.6 If two of i,k,l,m are equal, that it be $i = k$, then $T_{iklm} = -T_{kilm} = -T_{iklm}$, so $T_{iklm} = 0$. So $T_{iklm} \neq 0$ only if all the indices are different. But there are only three possible values (3-D space) for the four indices, so at least two of i,k,l,m are equal, so $T_{iklm} = 0$. Therefore, there are no independent components.

4.1.7 By the transformation law,

$$(T')_{\cdots i} = \sum \cdots \sum_k \cdots \frac{\partial x^k}{\partial (x')^i} T_{\cdots k}$$

Defining $\left(\frac{\partial T}{\partial x}\right)_{\cdots ij} = \frac{\partial T_{\cdots i}}{\partial x_j}$. If the transformation is linear, then $\frac{\partial^2 x^\mu}{\partial (x')^j (x')^i} = 0$ for all μ. So

$$\begin{aligned}
\left(\frac{\partial T}{\partial x}\right)'_{\cdots ij} = \frac{\partial (T')_{\cdots i}}{\partial (x')^j} &= \sum \cdots \sum_k \cdots \frac{\partial x^k}{\partial (x')^i} \sum_l \frac{\partial x^l}{\partial (x')^j} \frac{\partial T_{\cdots k}}{\partial x^l} \\
&= \sum \cdots \sum_k \sum_l \cdots \frac{\partial x^k}{\partial (x')^i} \frac{\partial x^l}{\partial (x')^j} \frac{\partial T_{\cdots k}}{\partial x^l} \\
&= \sum \cdots \sum_k \sum_l \cdots \frac{\partial x^k}{\partial (x')^i} \frac{\partial x^l}{\partial (x')^j} \left(\frac{\partial T}{\partial x}\right)_{\cdots kl}
\end{aligned}$$

which is the transformation law for tensors of rank $n+1$, so $\left(\frac{\partial T}{\partial x}\right)_{\cdots ij} = \frac{\partial T_{\cdots i}}{\partial x_j}$ is a tensor of rank $n+1$.

4.1.8 By the transformation law,

$$(T')_{ijk\cdots} = \sum_l \sum_m \sum_n \cdots \sum \frac{\partial x^l}{\partial (x')^i} \frac{\partial x^m}{\partial (x')^j} \frac{\partial x^n}{\partial (x')^k} \cdots T_{lmn\cdots}$$

Note that in Cartesian coordinates, $\frac{\partial x^m}{\partial (x')^j} = \frac{\partial (x')^j}{\partial x^m}$. So

$$\begin{aligned}
\sum_j \frac{\partial (T')_{ijk\cdots}}{\partial (x')^j} &= \sum_j \sum_l \sum_m \sum_n \cdots \sum \frac{\partial x^l}{\partial (x')^i} \frac{\partial x^m}{\partial (x')^j} \frac{\partial x^n}{\partial (x')^k} \cdots \frac{\partial T_{lmn\cdots}}{\partial (x')^j} \\
&= \sum_l \sum_m \sum_n \cdots \sum \frac{\partial x^l}{\partial (x')^i} \frac{\partial x^n}{\partial (x')^k} \cdots \sum_j \frac{\partial (x')^j}{\partial x^m} \frac{\partial T_{lmn\cdots}}{\partial (x')^j} \\
&= \sum_l \sum_m \sum_n \cdots \sum \frac{\partial x^l}{\partial (x')^i} \frac{\partial x^n}{\partial (x')^k} \cdots \frac{\partial T_{lmn\cdots}}{\partial x^m} \\
&= \sum_l \sum_n \cdots \sum \frac{\partial x^l}{\partial (x')^i} \frac{\partial x^n}{\partial (x')^k} \cdots \sum_m \frac{\partial T_{lmn\cdots}}{\partial x^m}
\end{aligned}$$

which is the transformation law for tensors of rank $n-1$, so $\sum_j \frac{\partial T_{ijk\cdots}}{\partial x^j}$ is a tensor of rank $n-1$.

4.1.9 When defining $x_4 = ict$, the Lorentz transformations take the form

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

or $\mathbf{x}' = \mathrm{U}\mathbf{x}$, where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, $\beta = \frac{v}{c}$. It can be verified that U is orthogonal, so $\mathrm{U}^{-1} = \mathrm{U}^T$, and $\mathbf{x} = \mathrm{U}^T\mathbf{x}'$. $\frac{\partial x_i'}{\partial x_j} = \mathrm{U}_{ij}$, and $\frac{\partial x_j}{\partial x_i'} = \mathrm{U}^T_{ji} = \mathrm{U}_{ij}$, so $\frac{\partial x_i'}{\partial x_j} = \frac{\partial x_j}{\partial x_i'}$. (Therefore, as long as the transformation is orthogonal, this relation holds.)

$$(\Box^2)' = \sum_i \frac{\partial^2}{\partial (x_i')^2} = \sum_i \frac{\partial}{\partial x_i'}(\frac{\partial}{\partial x_i'}) = \sum_i\sum_j \frac{\partial x_j}{\partial x_i'}\frac{\partial}{\partial x_j}(\frac{\partial}{\partial x_i'}) = \sum_i\sum_j \frac{\partial x_j}{\partial x_i'}\frac{\partial^2}{\partial x_j \partial x_i'}$$

$$= \sum_j\sum_i \frac{\partial x_i'}{\partial x_j}\frac{\partial^2}{\partial x_i' \partial x_j} = \sum_j\sum_i \frac{\partial x_i'}{\partial x_j}\frac{\partial}{\partial x_i'}(\frac{\partial}{\partial x_j}) = \sum_j \frac{\partial}{\partial x_j}(\frac{\partial}{\partial x_j}) = \sum_j \frac{\partial^2}{\partial {x_j}^2} = \Box^2$$

so the d'Alembertian is invariant under Lorentz transformation.

4.1.10 (Using the Einstein convention)

$$K_{mn}A^m B^n = K_{mn}\frac{\partial x^m}{\partial (x')^i}\frac{\partial x^n}{\partial (x')^j}(A')^i (B')^j = (K')_{ij}(A')^i(B')^j$$

so

$$\left[(K')_{ij} - K_{mn}\frac{\partial x^m}{\partial (x')^i}\frac{\partial x^n}{\partial (x')^j}\right](A')^i(B')^j = 0$$

Because A′ and B′ are arbitrary, the coefficient must vanish. (For example, to prove $(K')_{12} - K_{mn}\frac{\partial x^m}{\partial (x')^1}\frac{\partial x^n}{\partial (x')^2} = 0$, set $(A')^1 = (B')^2 = 1$, all the other components of A′ and B′ $= 0$.) Therefore,

$$(K')_{ij} = K_{mn}\frac{\partial x^m}{\partial (x')^i}\frac{\partial x^n}{\partial (x')^j}$$

which means that K_{ij} is a second-rank tensor.

4.1.11 (Using the Einstein convention)

$$\begin{aligned}(B')_i^k &= \frac{\partial (x')^k}{\partial x^p}\frac{\partial x^m}{\partial (x')^i}B_m^p = \frac{\partial (x')^k}{\partial x^p}\frac{\partial x^m}{\partial (x')^i}K_{mn}A^{np} \\ &= \frac{\partial (x')^k}{\partial x^p}\frac{\partial x^m}{\partial (x')^i}K_{mn}\frac{\partial x^n}{\partial (x')^j}\frac{\partial x^p}{\partial (x')^l}(A')^{jl} \\ &= K_{mn}\frac{\partial x^m}{\partial (x')^i}\frac{\partial x^n}{\partial (x')^j}\left(\frac{\partial (x')^k}{\partial x^p}\frac{\partial x^p}{\partial (x')^l}\right)(A')^{jl} \\ &= K_{mn}\frac{\partial x^m}{\partial (x')^i}\frac{\partial x^n}{\partial (x')^j}\delta_l^k(A')^{jl} \\ &= K_{mn}\frac{\partial x^m}{\partial (x')^i}\frac{\partial x^n}{\partial (x')^j}(A')^{jk} \\ &= (K')_{ij}(A')^{jk}\end{aligned}$$

so

$$\left[(K')_{ij} - K_{mn}\frac{\partial x^m}{\partial (x')^i}\frac{\partial x^n}{\partial (x')^j}\right](A')^{jk} = 0$$

Because A′ is arbitrary, the coefficient must vanish. Therefore,

$$(K')_{ij} = K_{mn}\frac{\partial x^m}{\partial (x')^i}\frac{\partial x^n}{\partial (x')^j}$$

which means that K is a second-rank tensor.

3.2 Pseudsotensors, Dual Tensors

4.2.1 Let the transformation matrix from $\mathbf{x}$ to $\mathbf{x}'$ be A, then

$$\mathrm{A} = \begin{pmatrix} \frac{\partial (x')^1}{\partial x^1} & \frac{\partial (x')^1}{\partial x^2} & \frac{\partial (x')^1}{\partial x^3} \\ \frac{\partial (x')^2}{\partial x^1} & \frac{\partial (x')^2}{\partial x^2} & \frac{\partial (x')^2}{\partial x^3} \\ \frac{\partial (x')^3}{\partial x^1} & \frac{\partial (x')^3}{\partial x^2} & \frac{\partial (x')^3}{\partial x^3} \end{pmatrix}$$

(take $n = 3$ for example). Then $\det(\mathrm{A}) = \varepsilon_{ljk} \frac{\partial (x')^l}{\partial x^1} \frac{\partial (x')^j}{\partial x^2} \frac{\partial (x')^k}{\partial x^3}$ (using Einstein convention)(we use l instead of i for reason that will soon be clear). If we permute $(1, 2, 3)$, then we must permute (l, j, k) in the same way to retain the formula $\frac{\partial (x')^l}{\partial x^1} \frac{\partial (x')^j}{\partial x^2} \frac{\partial (x')^k}{\partial x^3}$. Therefore,

$$\varepsilon_{ljk} \frac{\partial (x')^l}{\partial x^m} \frac{\partial (x')^j}{\partial x^n} \frac{\partial (x')^k}{\partial x^p} = \varepsilon_{mnp} \varepsilon_{ljk} \frac{\partial (x')^l}{\partial x^1} \frac{\partial (x')^j}{\partial x^2} \frac{\partial (x')^k}{\partial x^3} = \varepsilon_{mnp} \det(\mathrm{A}) \tag{1}$$

C_i is a pseudovector, so the transformation law gives

$$\begin{aligned}
(C')_i = \det(\mathrm{A}) \frac{\partial x^m}{\partial (x')^i} C_m &= \det(\mathrm{A}) \frac{\partial x^m}{\partial (x')^i} \frac{1}{2} \varepsilon_{mnp} C^{np} \quad \textit{(combine } \det(\mathrm{A}) \textit{ and } \varepsilon_{mnp}\textit{)} \\
&= \frac{1}{2} \frac{\partial x^m}{\partial (x')^i} (\varepsilon_{mnp} \det(\mathrm{A})) C^{np} \quad \textit{(use equation 1)} \\
&= \frac{1}{2} \frac{\partial x^m}{\partial (x')^i} \varepsilon_{ljk} \frac{\partial (x')^l}{\partial x^m} \frac{\partial (x')^j}{\partial x^n} \frac{\partial (x')^k}{\partial x^p} C^{np} \quad \textit{(combine } \frac{\partial x^m}{\partial (x')^i} \textit{ and } \frac{\partial (x')^l}{\partial x^m}\textit{)} \\
&= \frac{1}{2} \delta^l_i \varepsilon_{ljk} \frac{\partial (x')^j}{\partial x^n} \frac{\partial (x')^k}{\partial x^p} C^{np} \\
&= \frac{1}{2} \varepsilon_{ijk} \frac{\partial (x')^j}{\partial x^n} \frac{\partial (x')^k}{\partial x^p} C^{np} = \frac{1}{2} \varepsilon_{ijk} (C')^{jk}
\end{aligned}$$

the last equality holds because $C_i = \frac{1}{2} \varepsilon_{ijk} C^{jk}$ holds in all coordinate systems, so $(C')_i = \frac{1}{2} \varepsilon_{ijk} (C')^{jk}$.

If $j \neq k$, let i be the remaining value other than j, k, then $\varepsilon_{ijk} \neq 0$, so we have

$$(C')^{jk} = \frac{\partial (x')^j}{\partial x^n} \frac{\partial (x')^k}{\partial x^p} C^{np}$$

If $j = k$, then $\frac{\partial (x')^j}{\partial x^p} \frac{\partial (x')^k}{\partial x^n} = \frac{\partial (x')^j}{\partial x^n} \frac{\partial (x')^k}{\partial x^p}$, and because C is antisymmetric, so $(C')^{jk} = (C)^{jk} = 0$ for $j = k$. Therefore,

$$\begin{aligned}
&\frac{\partial (x')^j}{\partial x^n} \frac{\partial (x')^k}{\partial x^p} C^{np} \\
= &\frac{\partial (x')^j}{\partial x^1} \frac{\partial (x')^k}{\partial x^2} (C^{12} + C^{21}) + \frac{\partial (x')^j}{\partial x^1} \frac{\partial (x')^k}{\partial x^3} (C^{13} + C^{31}) + \frac{\partial (x')^j}{\partial x^2} \frac{\partial (x')^k}{\partial x^3} (C^{23} + C^{32}) \\
= &0 = (C')^{jk}
\end{aligned}$$

so $(C')^{jk} = \frac{\partial (x')^j}{\partial x^n} \frac{\partial (x')^k}{\partial x^p} C^{np}$ still holds. Therefore, in all cases,

$$(C')^{jk} = \frac{\partial (x')^j}{\partial x^n} \frac{\partial (x')^k}{\partial x^p} C^{np}$$

holds, which implies that C^{jk} is a tensor.

4.2.2 If there is a one-to-one correspondence between two sets, then the numbers of elements of the two sets need to be the same. If there is a one-to-one correspondence between the components of a vector C_i and the components of a tensor $(AB)^{jk}$, because the numbers of components of C_i and $(AB)^{jk}$ are different (n and n^2), it should mean that the one-to-one correspondence exists between *independent* components of C_i and $(AB)^{jk}$, so the number of *independent* components of C_i and $(AB)^{jk}$ should be the same.

By the antisymmetry property of AB^{jk}, $AB^{jj} = -AB^{jj}$, so $AB^{jj} = 0$, and $AB^{jk} = -AB^{kj}$. So the the number of *independent* components of and $(AB)^{jk} = \frac{n \times n - n}{2}$, should be equal to n, the number of *independent* components of C_i. So

$$\frac{n \times n - n}{2} = n$$

so $n = 3$.

4.2.3

$$\nabla \cdot \nabla \times \mathbf{A} = \frac{\partial}{\partial x^i}(\nabla \times \mathbf{A})_i = \frac{\partial}{\partial x^i}\varepsilon_{ijk}\frac{\partial}{\partial x^j}A^k = (\varepsilon_{ijk}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j})A^k = 0$$

because $\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}$ and $\varepsilon_{ijk} + \varepsilon_{jik} = 0$.

$$(\nabla \times \nabla\varphi)_i = \varepsilon_{ijk}\frac{\partial}{\partial x^j}(\nabla\varphi)_k = \varepsilon_{ijk}\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k}\varphi = 0$$

because $\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k} = \frac{\partial}{\partial x^k}\frac{\partial}{\partial x^j}$ and $\varepsilon_{ijk} + \varepsilon_{ikj} = 0$.

4.2.4 (a)

$$\begin{aligned}(A')^{ik}_{jl} &= \frac{\partial(x')^i}{\partial x^m}\frac{\partial x^n}{\partial(x')^j}\frac{\partial(x')^k}{\partial x^p}\frac{\partial x^q}{\partial(x')^l}\delta^m_n\delta^p_q \\ &= \frac{\partial(x')^i}{\partial x^m}\frac{\partial x^m}{\partial(x')^j}\frac{\partial(x')^k}{\partial x^p}\frac{\partial x^p}{\partial(x')^l} \\ &= \frac{\partial(x')^i}{\partial(x')^j}\frac{\partial(x')^k}{\partial(x')^l} = \delta^i_j\delta^k_l\end{aligned}$$

(b)

$$\begin{aligned}(B')^{ij}_{kl} &= \frac{\partial(x')^i}{\partial x^m}\frac{\partial x^p}{\partial(x')^k}\frac{\partial(x')^j}{\partial x^n}\frac{\partial x^q}{\partial(x')^l}(\delta^m_p\delta^n_q + \delta^m_q\delta^n_p) \\ &= \frac{\partial(x')^i}{\partial(x')^k}\frac{\partial(x')^j}{\partial(x')^l} + \frac{\partial(x')^i}{\partial(x')^l}\frac{\partial(x')^j}{\partial(x')^k} = \delta^i_k\delta^j_l + \delta^i_l\delta^j_k\end{aligned}$$

(c)

$$\begin{aligned}(C')^{ij}_{kl} &= \frac{\partial(x')^i}{\partial x^m}\frac{\partial x^p}{\partial(x')^k}\frac{\partial(x')^j}{\partial x^n}\frac{\partial x^q}{\partial(x')^l}(\delta^m_p\delta^n_q - \delta^m_q\delta^n_p) \\ &= \frac{\partial(x')^i}{\partial(x')^k}\frac{\partial(x')^j}{\partial(x')^l} - \frac{\partial(x')^i}{\partial(x')^l}\frac{\partial(x')^j}{\partial(x')^k} = \delta^i_k\delta^j_l - \delta^i_l\delta^j_k\end{aligned}$$

4.2.5 Similar with Exercise 4.2.1, let A be the transformation matrix from $\mathbf{x}$ to $\mathbf{x}'$, then $\det(\mathrm{A}) = \varepsilon_{kl}\frac{\partial(x')^k}{\partial x^1}\frac{\partial(x')^l}{\partial x^2}$, and

$$\varepsilon_{kl}\frac{\partial(x')^k}{\partial x^m}\frac{\partial(x')^l}{\partial x^n} = \varepsilon_{mn}\varepsilon_{kl}\frac{\partial(x')^k}{\partial x^1}\frac{\partial(x')^l}{\partial x^2} = \varepsilon_{mn}\det(\mathrm{A}) \tag{1}$$

ε_{ij} is defined as $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = 1$, $\varepsilon_{21} = -1$, regardless of which coordinate system it is in. So it is invariant in all coordinate systms, and $(\varepsilon')_{kl} = \varepsilon_{kl}$. (We will use it later.)

$$\begin{aligned}(\varepsilon')_{ij} = (\varepsilon')_{kl}(\delta')^k_i(\delta')^l_j &= \varepsilon_{kl}\frac{\partial(x')^k}{\partial(x')^i}\frac{\partial(x')^l}{\partial(x')^j} \\ &= \varepsilon_{kl}\frac{\partial(x')^k}{\partial x^m}\frac{\partial x^m}{\partial(x')^i}\frac{\partial(x')^l}{\partial x^n}\frac{\partial x^n}{\partial(x')^j} \quad \textit{(combining } \varepsilon_{kl}\frac{\partial(x')^k}{\partial x^m}\frac{\partial(x')^l}{\partial x^n}\textit{)} \\ &= \left(\varepsilon_{kl}\frac{\partial(x')^k}{\partial x^m}\frac{\partial(x')^l}{\partial x^n}\right)\frac{\partial x^m}{\partial(x')^i}\frac{\partial x^n}{\partial(x')^j} \quad \textit{(use equation 1)} \\ &= \varepsilon_{mn}\det(\mathrm{A})\frac{\partial x^m}{\partial(x')^i}\frac{\partial x^n}{\partial(x')^j} = \det(\mathrm{A})\frac{\partial x^m}{\partial(x')^i}\frac{\partial x^n}{\partial(x')^j}\varepsilon_{mn}\end{aligned}$$

which is the transformation equation for a second-rank pseudotensor, Therefore ε_{ij} is a second-rank pseudotensor. It does not contradict the uniqueness of δ^i_j because we proved δ^i_j is the only isotropic second rank *tensor* (with a coefficient), while ε_{ij} is an isotropic second-rank *pseudotensor* (it fails to keep the transformation law under improper rotation).

4.2.6 $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in matrix form, and let the orthogonal transformation be $\mathrm{S} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$. Then the similarity transformation is

$$\varepsilon' = \mathrm{S}\varepsilon\mathrm{S}^T = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so ε_{ij} is invariant under orthogonal similarity transformations. It corresponds to the isotropic property of ε_{ij} under proper rotation.

4.2.7 Using the result from Exercise 2.1.9, we have $\varepsilon^{mnk}\varepsilon_{ijk} = \delta_i^m\delta_j^n - \delta_j^m\delta_i^n$, so

$$\varepsilon^{mnk}A_k = \varepsilon^{mnk}\frac{1}{2}\varepsilon_{ijk}B^{ij} = \frac{1}{2}(\delta_i^m\delta_j^n - \delta_j^m\delta_i^n)B^{ij} = \frac{1}{2}(B^{mn} - B^{nm}) = B^{mn}$$

4.3 Tensors in General Coordinates

4.3.1 q^i, q^j, q^k are independent, so $\boldsymbol{\varepsilon}^i, \boldsymbol{\varepsilon}^j, \boldsymbol{\varepsilon}^k$ is linear independent. (If $\boldsymbol{\varepsilon}^i, \boldsymbol{\varepsilon}^j, \boldsymbol{\varepsilon}^k$ is linear dependent, which means $a\boldsymbol{\varepsilon}^i + b\boldsymbol{\varepsilon}^j + c\boldsymbol{\varepsilon}^k = 0$, then $\frac{\partial(aq^i+bq^j+cq^k)}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial(aq^i+bq^j+cq^k)}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial(aq^i+bq^j+cq^k)}{\partial z}\hat{\mathbf{e}}_z = 0$, so $aq^i + bq^j + cq^k = d$, which means q^i, q^j, q^k are dependent.)

Express $\frac{\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k}{\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k \cdot \boldsymbol{\varepsilon}_i}$ in the bases of $\boldsymbol{\varepsilon}^i, \boldsymbol{\varepsilon}^j, \boldsymbol{\varepsilon}^k$, and note that $\boldsymbol{\varepsilon}^p \cdot \boldsymbol{\varepsilon}_q = \delta_q^p$, no matter whether $\boldsymbol{\varepsilon}^i, \boldsymbol{\varepsilon}^j, \boldsymbol{\varepsilon}^k$ and $\boldsymbol{\varepsilon}_i, \boldsymbol{\varepsilon}_j, \boldsymbol{\varepsilon}_k$ are orthogonal. Then

$$\frac{\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k}{\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k \cdot \boldsymbol{\varepsilon}_i} = A_i\boldsymbol{\varepsilon}^i + A_j\boldsymbol{\varepsilon}^j + A_k\boldsymbol{\varepsilon}^k$$

$$\frac{\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k \cdot \boldsymbol{\varepsilon}_i}{\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k \cdot \boldsymbol{\varepsilon}_i} = 1 = A_i$$

$$\frac{\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k \cdot \boldsymbol{\varepsilon}_j}{\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k \cdot \boldsymbol{\varepsilon}_i} = 0 = A_j$$

$$\frac{\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k \cdot \boldsymbol{\varepsilon}_k}{\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k \cdot \boldsymbol{\varepsilon}_i} = 0 = A_k$$

so

$$\frac{\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k}{\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k \cdot \boldsymbol{\varepsilon}_i} = \boldsymbol{\varepsilon}^i$$

4.3.2 (a) If $i \neq j$, then $g_{ij} = \boldsymbol{\varepsilon}_i \cdot \boldsymbol{\varepsilon}_j = 0$, so g_{ij} is diagonal.

(b) $g_{ji} = 0$ when $i \neq j$, so

$$\begin{aligned} &g^{ii}g_{ii} \quad \textit{(no summation on i)} \\ &= g^{ij}g_{ji} \quad \textit{(summation on j)} \\ &= \delta_i^i \quad \textit{(by definition of } g^{ij}\textit{)} \\ &= 1 \end{aligned}$$

so

$$g^{ii} = \frac{1}{g_{ii}}$$

(c) $\boldsymbol{\varepsilon}_j \cdot \boldsymbol{\varepsilon}_i = 0$, so

$$\begin{aligned} &(\boldsymbol{\varepsilon}^i \cdot \boldsymbol{\varepsilon}^i)(\boldsymbol{\varepsilon}_i \cdot \boldsymbol{\varepsilon}_i) \quad \textit{(no summation on i)} \\ &= (\boldsymbol{\varepsilon}^i \cdot \boldsymbol{\varepsilon}^j)(\boldsymbol{\varepsilon}_j \cdot \boldsymbol{\varepsilon}_i) \quad \textit{(summation on j)} \\ &= \delta_i^i = 1 \quad \textit{(by Eq. 4.46)} \end{aligned}$$

so $|\boldsymbol{\varepsilon}^i|^2|\boldsymbol{\varepsilon}_i|^2 = 1$, which means

$$|\boldsymbol{\varepsilon}^i| = \frac{1}{|\boldsymbol{\varepsilon}_i|}$$

4.3.3
$$(\boldsymbol{\varepsilon}^i\cdot\boldsymbol{\varepsilon}^j)\cdot(\boldsymbol{\varepsilon}_j\cdot\boldsymbol{\varepsilon}_k)=(\frac{\partial q^i}{\partial x}\frac{\partial q^j}{\partial x}+\frac{\partial q^i}{\partial y}\frac{\partial q^j}{\partial y}+\frac{\partial q^i}{\partial z}\frac{\partial q^j}{\partial z})(\frac{\partial x}{\partial q^j}\frac{\partial x}{\partial q^k}+\frac{\partial y}{\partial q^j}\frac{\partial y}{\partial q^k}+\frac{\partial z}{\partial q^j}\frac{\partial z}{\partial q^k})$$

Note that j is summed, so $\frac{\partial q^j}{\partial x}\frac{\partial x}{\partial q^j}=\frac{\partial x}{\partial x}=1$, and $\frac{\partial q^j}{\partial x}\frac{\partial y}{\partial q^j}=\frac{\partial y}{\partial x}=0$. Similarly, $\frac{\partial q^j}{\partial y}\frac{\partial y}{\partial q^j}=\frac{\partial q^j}{\partial z}\frac{\partial z}{\partial q^j}=1$, and other cross terms are zero. So the equation becomes

$$\frac{\partial q^i}{\partial x}\frac{\partial x}{\partial q^k}+\frac{\partial q^i}{\partial y}\frac{\partial y}{\partial q^k}+\frac{\partial q^i}{\partial z}\frac{\partial z}{\partial q^k}=\frac{\partial q^i}{\partial q^k}=\delta^i_k$$

4.3.4
$$\Gamma^m_{jk}\boldsymbol{\varepsilon}_m=\frac{\partial\boldsymbol{\varepsilon}_k}{\partial q^j}=\frac{\partial^2 x}{\partial q^j\partial q^k}\hat{\mathbf{e}}_x+\frac{\partial^2 y}{\partial q^j\partial q^k}\hat{\mathbf{e}}_y+\frac{\partial^2 z}{\partial q^j\partial q^k}\hat{\mathbf{e}}_z$$
$$=\frac{\partial^2 x}{\partial q^k\partial q^j}\hat{\mathbf{e}}_x+\frac{\partial^2 y}{\partial q^k\partial q^j}\hat{\mathbf{e}}_y+\frac{\partial^2 z}{\partial q^k\partial q^j}\hat{\mathbf{e}}_z=\frac{\partial\boldsymbol{\varepsilon}_j}{\partial q^k}=\Gamma^m_{kj}\boldsymbol{\varepsilon}_m$$

so $(\Gamma^m_{jk}-\Gamma^m_{kj})\boldsymbol{\varepsilon}_m=0$. Because $\boldsymbol{\varepsilon}_m$ are linear independent, $\Gamma^m_{jk}-\Gamma^m_{kj}$ must be zero for every m, so

$$\Gamma^m_{jk}=\Gamma^m_{kj}$$

4.3.5 $(q^1,q^2,q^3)=(\rho,\varphi,z)$, and $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, $z=z$. So

$$\boldsymbol{\varepsilon}_1=\frac{\partial x}{\partial\rho}\hat{\mathbf{e}}_x+\frac{\partial y}{\partial\rho}\hat{\mathbf{e}}_y+\frac{\partial z}{\partial\rho}\hat{\mathbf{e}}_z=\cos\varphi\,\hat{\mathbf{e}}_x+\sin\varphi\,\hat{\mathbf{e}}_y$$

$$\boldsymbol{\varepsilon}_2=\frac{\partial x}{\partial\varphi}\hat{\mathbf{e}}_x+\frac{\partial y}{\partial\varphi}\hat{\mathbf{e}}_y+\frac{\partial z}{\partial\varphi}\hat{\mathbf{e}}_z=-\rho\sin\varphi\,\hat{\mathbf{e}}_x+\rho\cos\varphi\,\hat{\mathbf{e}}_y$$

$$\boldsymbol{\varepsilon}_3=\frac{\partial x}{\partial z}\hat{\mathbf{e}}_x+\frac{\partial y}{\partial z}\hat{\mathbf{e}}_y+\frac{\partial z}{\partial z}\hat{\mathbf{e}}_z=\hat{\mathbf{e}}_z$$

$$(g_{ij})=\begin{pmatrix}\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_1 & \boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2 & \boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_3\\ \boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}_1 & \boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}_2 & \boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}_3\\ \boldsymbol{\varepsilon}_3\cdot\boldsymbol{\varepsilon}_1 & \boldsymbol{\varepsilon}_3\cdot\boldsymbol{\varepsilon}_2 & \boldsymbol{\varepsilon}_3\cdot\boldsymbol{\varepsilon}_3\end{pmatrix}=\begin{pmatrix}1&0&0\\0&\rho^2&0\\0&0&1\end{pmatrix}$$

Because $g^{ij}g_{jk}=\delta^i_k$, the unit matrix, so $(g^{ij})=(g_{jk})^{-1}$, the matrix inverse. Therefore,

$$(g^{ij})=\begin{pmatrix}1&0&0\\0&\frac{1}{\rho^2}&0\\0&0&1\end{pmatrix}$$

4.3.6 Differentiate $\boldsymbol{\varepsilon}^i\cdot\boldsymbol{\varepsilon}_k=\delta^i_k$ by q^j, we have

$$\frac{\partial\boldsymbol{\varepsilon}^i}{\partial q^j}\cdot\boldsymbol{\varepsilon}_k+\boldsymbol{\varepsilon}^i\cdot\frac{\partial\boldsymbol{\varepsilon}_k}{\partial q^j}=0$$

$$\frac{\partial\boldsymbol{\varepsilon}^i}{\partial q^j}\cdot\boldsymbol{\varepsilon}_k=-\boldsymbol{\varepsilon}^i\cdot\frac{\partial\boldsymbol{\varepsilon}_k}{\partial q^j}=-\boldsymbol{\varepsilon}^i\cdot(\Gamma^\mu_{jk}\boldsymbol{\varepsilon}_\mu)=-\Gamma^i_{jk}$$

so
$$\frac{\partial\boldsymbol{\varepsilon}^i}{\partial q^j}=-\Gamma^i_{jk}\boldsymbol{\varepsilon}^k$$

when expanded in the contravariant basis.

$V_{i;j}$ is defined as $\frac{\partial\mathbf{V}'}{\partial q^j}=V_{i;j}\boldsymbol{\varepsilon}^i$. Expand the vector in contravariant basis $\mathbf{V}'=V_i\boldsymbol{\varepsilon}^i$ and differentiate, we have

$$\frac{\partial\mathbf{V}'}{\partial q^j}=\frac{\partial V_i}{\partial q^j}\boldsymbol{\varepsilon}^i+V_i\frac{\partial\boldsymbol{\varepsilon}^i}{\partial q^j}$$

$$=\frac{\partial V_i}{\partial q^j}\boldsymbol{\varepsilon}^i-V_i\Gamma^i_{jk}\boldsymbol{\varepsilon}^k\quad\textit{(interchange i and k in the second term)}$$

$$= (\frac{\partial V_i}{\partial q^j} - V_k\Gamma^k_{ji})\boldsymbol{\varepsilon}^i \quad (\textit{ i and j in } \Gamma^k_{ji} \textit{ can be interchanged})$$

$$= (\frac{\partial V_i}{\partial q^j} - V_k\Gamma^k_{ij})\boldsymbol{\varepsilon}^i = V_{i;j}\boldsymbol{\varepsilon}^i$$

So

$$V_{i;j} = \frac{\partial V_i}{\partial q^j} - V_k\Gamma^k_{ij}$$

because the set of $\boldsymbol{\varepsilon}^i$ are linear independent.

4.3.7

$$\frac{\partial V_i}{\partial q^j} - V_k\Gamma^k_{ij} = \frac{\partial(g_{ik}V^k)}{\partial q^j} - V_k\Gamma^k_{ij}$$

$$= g_{ik}\frac{\partial V^k}{\partial q^j} + \frac{\partial g_{ik}}{\partial q^j}V^k - V_k\Gamma^k_{ij}$$

$$= g_{ik}\frac{\partial V^k}{\partial q^j} + \frac{\partial(\boldsymbol{\varepsilon}_i\cdot\boldsymbol{\varepsilon}_k)}{\partial q^j}V^k - V_k\Gamma^k_{ij}$$

$$= g_{ik}\frac{\partial V^k}{\partial q^j} + V^k\boldsymbol{\varepsilon}_i\cdot\frac{\partial\boldsymbol{\varepsilon}_k}{\partial q^j} + V^k\boldsymbol{\varepsilon}_k\cdot\frac{\partial\boldsymbol{\varepsilon}_i}{\partial q^j} - V_k\Gamma^k_{ij}$$

$$= g_{ik}\frac{\partial V^k}{\partial q^j} + V^m\boldsymbol{\varepsilon}_i\cdot\frac{\partial\boldsymbol{\varepsilon}_m}{\partial q^j} + V_k\boldsymbol{\varepsilon}^k\cdot\frac{\partial\boldsymbol{\varepsilon}_i}{\partial q^j} - V_k\Gamma^k_{ij}$$

$$= g_{ik}\frac{\partial V^k}{\partial q^j} + V^m(g_{ik}\boldsymbol{\varepsilon}^k)\cdot\frac{\partial\boldsymbol{\varepsilon}_m}{\partial q^j} + V_k\Gamma^k_{ij} - V_k\Gamma^k_{ij}$$

$$= g_{ik}\frac{\partial V^k}{\partial q^j} + g_{ik}V^m\Gamma^k_{mj}$$

$$= g_{ik}\left[\frac{\partial V^k}{\partial q^j} + V^m\Gamma^k_{mj}\right]$$

(Or note that $\mathbf{V}' = V_i\boldsymbol{\varepsilon}^i = V^k\boldsymbol{\varepsilon}_k$, so

$$\frac{\partial\mathbf{V}'}{\partial q^j} = \left[\frac{\partial V_i}{\partial q^j} - V_k\Gamma^k_{ij}\right]\boldsymbol{\varepsilon}^i = \left[\frac{\partial V^k}{\partial q^j} + V^m\Gamma^k_{mj}\right]\boldsymbol{\varepsilon}_k$$

Take the scalar product of both sides with $\boldsymbol{\varepsilon}_i$, and note that $\boldsymbol{\varepsilon}^i\cdot\boldsymbol{\varepsilon}_i = 1$, and $\boldsymbol{\varepsilon}_k\cdot\boldsymbol{\varepsilon}_i = g_{ik}$. Therefore,

$$\frac{\partial V_i}{\partial q^j} - V_k\Gamma^k_{ij} = \left[\frac{\partial V^k}{\partial q^j} + V^m\Gamma^k_{mj}\right]g_{ik}$$

which is another verification.)

4.3.8 From Eq. 4.63, $\Gamma^n_{ij} = \frac{1}{2}g^{nk}\left[\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right]$. Because g^{ij} has only diagonal components, $g^{nk}\neq 0$ only when $n=k$, so $\Gamma^n_{ij} = \frac{1}{2}g^{nn}\left[\frac{\partial g_{in}}{\partial q^j} + \frac{\partial g_{jn}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^n}\right]$ (n is not summed). The only non-constant component of (g_{ij}) is $g_{22} = \rho = q^1$, so the only non-zero derivative of g_{ij} is $\frac{\partial g_{22}}{\partial q^1} = 2\rho$. When $n=1$, $\frac{\partial g_{in}}{\partial q^j} + \frac{\partial g_{jn}}{\partial q^i} = 0$, so i,j must be 2 to have non-zero Γ^n_{ij}. When $n=2$, $\frac{\partial g_{ij}}{\partial q^n} = 0$, so one of i,j must be 1 and the other must be 2 to make $\frac{\partial g_{in}}{\partial q^j} + \frac{\partial g_{jn}}{\partial q^i} \neq 0$. When $n=3$, none of the derivatives can be non-zero. Therefore, there are only three nonzero Γ^n_{ij}: Γ^1_{22}, Γ^2_{12}, Γ^2_{21}.

$$\Gamma^1_{22} = \frac{1}{2}(1)[-2\rho] = -\rho$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}(\frac{1}{\rho^2})[2\rho] = \frac{1}{\rho}$$

4.3.9 $V^i_{;j} = \frac{\partial V^i}{\partial q^j} + V^k\Gamma^i_{kj}$, so

$$V^1_{;2} = \frac{\partial V^1}{\partial q^2} + V^2\Gamma^1_{22} = \frac{\partial V^\rho}{\partial \varphi} - V^\varphi \rho = V^\rho_{;\varphi}$$

$$V^2_{;1} = \frac{\partial V^2}{\partial q^1} + V^2\Gamma^2_{21} = \frac{\partial V^\varphi}{\partial \rho} + V^\varphi \frac{1}{\rho} = V^\varphi_{;\rho}$$

$$V^2_{;2} = \frac{\partial V^2}{\partial q^2} + V^1\Gamma^2_{12} = \frac{\partial V^\varphi}{\partial \varphi} + V^\rho \frac{1}{\rho} = V^\varphi_{;\varphi}$$

For all the other i, j,

$$V^i_{;j} = \frac{\partial V^i}{\partial q^j}$$

4.3.10 $g_{ij;k}$ and $g^{ij}_{;k}$ are not defined in the text, but I think they are probably defined as $\frac{\partial(g_{ij}\boldsymbol{\varepsilon}^i\cdot\boldsymbol{\varepsilon}^j)}{\partial q^k} = g_{ij;k}\,\boldsymbol{\varepsilon}^i\cdot\boldsymbol{\varepsilon}^j$ and $\frac{\partial(g^{ij}\boldsymbol{\varepsilon}_i\cdot\boldsymbol{\varepsilon}_j)}{\partial q^k} = g^{ij}_{;k}\,\boldsymbol{\varepsilon}_i\cdot\boldsymbol{\varepsilon}_j$.

$$\frac{\partial(g_{ij}\boldsymbol{\varepsilon}^i\cdot\boldsymbol{\varepsilon}^j)}{\partial q^k} = \frac{\partial g_{ij}}{\partial q^k}\boldsymbol{\varepsilon}^i\cdot\boldsymbol{\varepsilon}^j + g_{ij}\frac{\partial \boldsymbol{\varepsilon}^i}{\partial q^k}\cdot\boldsymbol{\varepsilon}^j + g_{ij}\boldsymbol{\varepsilon}^i\cdot\frac{\partial \boldsymbol{\varepsilon}^j}{\partial q^k}$$

$$= \frac{\partial g_{ij}}{\partial q^k}\boldsymbol{\varepsilon}^i\cdot\boldsymbol{\varepsilon}^j + g_{ij}(-\Gamma^i_{k\alpha}\boldsymbol{\varepsilon}^\alpha)\cdot\boldsymbol{\varepsilon}^j + g_{ij}\boldsymbol{\varepsilon}^i\cdot(-\Gamma^j_{k\beta}\boldsymbol{\varepsilon}^\beta)$$

(interchange i and α in the second term, and interchange j and β in the last term)

$$= \frac{\partial g_{ij}}{\partial q^k}\boldsymbol{\varepsilon}^i\cdot\boldsymbol{\varepsilon}^j - g_{\alpha j}\Gamma^\alpha_{ki}\boldsymbol{\varepsilon}^i\cdot\boldsymbol{\varepsilon}^j - g_{i\beta}\Gamma^\beta_{kj}\boldsymbol{\varepsilon}^i\cdot\boldsymbol{\varepsilon}^j$$

$$= \left[\frac{\partial g_{ij}}{\partial q^k} - g_{j\alpha}\Gamma^\alpha_{ik} - g_{i\beta}\Gamma^\beta_{jk}\right]\boldsymbol{\varepsilon}^i\cdot\boldsymbol{\varepsilon}^j$$

so

$$g_{ij;k} = \frac{\partial g_{ij}}{\partial q^k} - g_{j\alpha}\Gamma^\alpha_{ik} - g_{i\beta}\Gamma^\beta_{jk} \quad \textit{(using Eq. 4.63)}$$

$$= \frac{\partial g_{ij}}{\partial q^k} - g_{j\alpha}\frac{1}{2}g^{\alpha m}\left[\frac{\partial g_{im}}{\partial q^k} + \frac{\partial g_{km}}{\partial q^i} - \frac{\partial g_{ik}}{\partial q^m}\right] - g_{i\beta}\frac{1}{2}g^{\beta n}\left[\frac{\partial g_{jn}}{\partial q^k} + \frac{\partial g_{kn}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^n}\right]$$

$$= \frac{\partial g_{ij}}{\partial q^k} - \frac{1}{2}\delta^m_j\left[\frac{\partial g_{im}}{\partial q^k} + \frac{\partial g_{km}}{\partial q^i} - \frac{\partial g_{ik}}{\partial q^m}\right] - \frac{1}{2}\delta^n_i\left[\frac{\partial g_{jn}}{\partial q^k} + \frac{\partial g_{kn}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^n}\right]$$

$$= \frac{\partial g_{ij}}{\partial q^k} - \frac{1}{2}\left[\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ik}}{\partial q^j}\right] - \frac{1}{2}\left[\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i}\right]$$

$$= \frac{\partial g_{ij}}{\partial q^k} - \frac{\partial g_{ij}}{\partial q^k} = 0$$

We can prove $g^{ij}_{;k} = 0$ in a similar way. Or we can note that

$$g^{lj}_{;k}\,\boldsymbol{\varepsilon}_l\cdot\boldsymbol{\varepsilon}_j = \frac{\partial(g^{lj}\boldsymbol{\varepsilon}_l\cdot\boldsymbol{\varepsilon}_j)}{\partial q^k} = \frac{\partial(g_{lj}\boldsymbol{\varepsilon}^l\cdot\boldsymbol{\varepsilon}^j)}{\partial q^k} = g_{lj;k}\,\boldsymbol{\varepsilon}^l\cdot\boldsymbol{\varepsilon}^j = 0$$

multiply both side with $\boldsymbol{\varepsilon}^j\cdot\boldsymbol{\varepsilon}^i$, and note that $(\boldsymbol{\varepsilon}_l\cdot\boldsymbol{\varepsilon}_j)(\boldsymbol{\varepsilon}^j\cdot\boldsymbol{\varepsilon}^i) = \delta^i_l$, we have

$$g^{lj}_{;k}(\boldsymbol{\varepsilon}_l\cdot\boldsymbol{\varepsilon}_j)(\boldsymbol{\varepsilon}^j\cdot\boldsymbol{\varepsilon}^i) = 0$$

$$g^{lj}_{;k}\delta^i_l = 0$$

$$g^{ij}_{;k} = 0$$

4.3.11 From Example 4.3.1, the metric tensor of spherical polar coordinates is

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

so $[\det(g)]^{1/2} = r^2\sin\theta$. Use Eq. 4.69, we have

$$\nabla\cdot\mathbf{V} = \frac{1}{[\det(g)]^{1/2}}\frac{\partial}{\partial q^k}\left([\det(g)]^{1/2}V^k\right)$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta V^r) + \frac{\partial}{\partial\theta}(r^2\sin\theta V^\theta) + \frac{\partial}{\partial\varphi}(r^2\sin\theta V^\varphi)\right]$$

Compared with the results in Chapter 3, and note that

$$\begin{aligned}
\boldsymbol{\varepsilon}_r &= \frac{\partial\mathbf{r}}{\partial r} = \hat{\mathbf{e}}_r & \boldsymbol{\varepsilon}_r V^r &= \hat{\mathbf{e}}_r(V^r) = \hat{\mathbf{e}}_r V_r & V^r &= V_r \\
\boldsymbol{\varepsilon}_\theta &= \frac{\partial\mathbf{r}}{\partial\theta} = r\hat{\mathbf{e}}_\theta & \boldsymbol{\varepsilon}_\theta V^\theta &= \hat{\mathbf{e}}_\theta(rV^\theta) = \hat{\mathbf{e}}_\theta V_\theta & V^\theta &= \frac{1}{r}V_\theta \\
\boldsymbol{\varepsilon}_\varphi &= \frac{\partial\mathbf{r}}{\partial\varphi} = r\sin\theta\hat{\mathbf{e}}_\varphi & \boldsymbol{\varepsilon}_\varphi V^\varphi &= \hat{\mathbf{e}}_\varphi(r\sin\theta V^\varphi) = \hat{\mathbf{e}}_\varphi V_\varphi & V^\varphi &= \frac{1}{r\sin\theta}V_\varphi
\end{aligned}$$

Substitute V^r, V^θ, V^φ into the above equation, we get Eq. 3.157.

4.3.12 $A_i = \frac{\partial\varphi}{\partial q^i}$ bexause it is the gradient of a scalar. From Exercise 4.3.6 we have

$$A_{i;j} = \frac{\partial A_i}{\partial q^j} - A_k\Gamma^k_{ij} = \frac{\partial^2\varphi}{\partial q^j\partial q^i} - A_k\Gamma^k_{ij}$$

$$= \frac{\partial^2\varphi}{\partial q^i\partial q^j} - A_k\Gamma^k_{ji} = \frac{\partial A_j}{\partial q^i} - A_k\Gamma^k_{ji} = A_{j;i}$$

so

$$A_{i;j} - A_{j;i} = 0$$

4.4 Jacobians

4.4.1 (a) If $f(u,v) = 0$, then by differentiating we have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} = 0$$

which means $\frac{\partial u}{\partial x} : \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} : \frac{\partial v}{\partial y} = \frac{\partial u}{\partial z} : \frac{\partial v}{\partial z} = (-\frac{\partial f}{\partial v}) : \frac{\partial f}{\partial u}$, so $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z})$ are parallel to $\nabla v = (\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z})$, and therefore $(\nabla u)\times(\nabla v) = 0$

If $(\nabla u)\times(\nabla v) = 0$, then $(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}) = a(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z})$ (or u,v being interchanged, and a can be zero for one or both of ∇u,∇v being zero). So $\frac{\partial(u-av)}{\partial x} = \frac{\partial(u-av)}{\partial y} = \frac{\partial(u-av)}{\partial z} = 0$, means that $u - av = b$, a constant, so $u - av - b = 0$ is the relation for u and v.

(b)

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left[(\frac{\partial u}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial u}{\partial y}\hat{\mathbf{e}}_y)\times(\frac{\partial v}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial v}{\partial y}\hat{\mathbf{e}}_y)\right]_z = [(\nabla u)\times(\nabla v)]_z = 0$$

4.4.2 h_1,h_2 are defined as

$$\frac{\partial \mathbf{r}}{\partial q_1} = h_1\hat{\mathbf{e}}_1 = \frac{\partial x}{\partial q_1}\hat{\mathbf{e}}_x + \frac{\partial y}{\partial q_1}\hat{\mathbf{e}}_y$$
$$\frac{\partial \mathbf{r}}{\partial q_2} = h_2\hat{\mathbf{e}}_2 = \frac{\partial x}{\partial q_2}\hat{\mathbf{e}}_x + \frac{\partial y}{\partial q_2}\hat{\mathbf{e}}_y$$

Because $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are orthogonal, the area of the parallelogram formed by $h_1\hat{\mathbf{e}}_1$ and $h_2\hat{\mathbf{e}}_2$ is h_1h_2, and also the area of the parallelogram equals to the determinant of components in Cartesian coordinate, so

$$Area = h_1h_2 = \begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial y}{\partial q_1} \\ \frac{\partial x}{\partial q_2} & \frac{\partial y}{\partial q_2} \end{vmatrix} = \frac{\partial x}{\partial q_1}\frac{\partial y}{\partial q_2} - \frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_1}$$

4.4.3 (a) Solve for x, y, we have $x = \frac{vu}{v+1}$, $y = \frac{u}{v+1}$, so

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{v}{v+1}\right)\left(\frac{-u}{(v+1)^2}\right) - \left(\frac{1}{v+1}\right)\left(\frac{u}{v+1} - \frac{vu}{(v+1)^2}\right) = \frac{-u}{(v+1)^2}$$

(b)

$$J^{-1} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = (1)(\frac{-x}{y^2}) - (\frac{1}{y})(1) = \frac{-(x+y)}{y^2} = \frac{(v+1)^2}{-u}$$

so

$$J = \frac{1}{J^{-1}} = \frac{-u}{(v+1)^2}$$

4.5 Differential Forms

4.5.1 (i, j, k denotes any cyclic permutation of 1,2,3)

$$*1 = (1)(-1)^0 dt \wedge dx_1 \wedge dx_2 \wedge dx_3 = dt \wedge dx_1 \wedge dx_2 \wedge dx_3$$
$$*dx_i = (-1)(-1)^1 dt \wedge dx_j \wedge dx_k = dt \wedge dx_j \wedge dx_k$$
$$*dt = (1)(-1)^0 dx_1 \wedge dx_2 \wedge dx_3 = dx_1 \wedge dx_2 \wedge dx_3$$
$$*(dx_j \wedge dx_k) = (1)(-1)^2 dt \wedge dx_i = dt \wedge dx_i$$
$$*(dt \wedge dx_i) = (1)(-1)^1 dx_j \wedge dx_k = -dx_j \wedge dx_k$$
$$*(dx_1 \wedge dx_2 \wedge dx_3) = (-1)(-1)^3 dt = dt$$
$$*(dt \wedge dx_i \wedge dx_j) = (1)(-1)^2 dx_k = dx_k$$
$$*(dt \wedge dx_1 \wedge dx_2 \wedge dx_3) = (1)(-1)^3 = -1$$

4.5.2 Let the force field be $\mathbf{F} = F_x\hat{\mathbf{e}}_x + F_y\hat{\mathbf{e}}_y + F_z\hat{\mathbf{e}}_z$, so the infinitely small work done is $dw = \mathbf{F}\cdot d\mathbf{r} = F_xdx + F_ydy + F_zdz$, and $w = F_x(x_2 - x_1) + F_y(y_2 - y_1) + F_z(z_2 - z_1)$. Substituting, we get $F_x = \frac{a}{3}$, $F_y = \frac{b}{2}$, $F_z = c$. So

$$dw = \frac{a}{3}dx + \frac{b}{2}dy + cdz$$

4.6 Differentiating Forms

4.6.1 (a) $d\omega_1 = dx \wedge dy + dy \wedge dx = 0$
(b) $d\omega_2 = dx \wedge dy - dy \wedge dx = 2dx \wedge dy$
(c) $d(d\omega_2) = 2d(dx) \wedge dy - 2dx \wedge d(dy) = 0$

4.6.2

$$d\omega_3 = (ydx + xdy) \wedge dz + (zdx + xdz) \wedge dy - (zdy + ydz) \wedge dx = 2zdx \wedge dy - 2ydz \wedge dx$$
$$d(d\omega_3) = 2dz \wedge dx \wedge dy - 2dy \wedge dz \wedge dx = 0$$

4.6.3 (a)

$$\omega_2 \wedge \omega_3 = (x\,dy - y\,dx) \wedge (xy\,dz + xz\,dy - yz\,dx) = x^2 y\,dy \wedge dz + xy^2\,dz \wedge dz$$

$$d(\omega_2 \wedge \omega_3) = 2xy\,dx \wedge dy \wedge dz + 2xy\,dy \wedge dz \wedge dx = 4xy\,dx \wedge dy \wedge dz$$

(b)

$$d(\omega_2 \wedge \omega_3) = (d\omega_2) \wedge \omega_3 - \omega_2 \wedge (d\omega_3) = 2xy\,dx \wedge dy \wedge dz - (-2xy)\,dy \wedge dz \wedge dx = 4xy\,dx \wedge dy \wedge dz$$

4.7 Integrating Forms

4.7.1

$$\begin{aligned} & A(x,y,z) dx \wedge dy \wedge dz \\ = & A(u,v,w)(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw) \wedge (\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw) \wedge (\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw) \\ = & A(u,v,w)\left[\frac{\partial x}{\partial u}(\frac{\partial y}{\partial v}\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\frac{\partial z}{\partial v}) - \frac{\partial x}{\partial v}(\frac{\partial y}{\partial u}\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\frac{\partial z}{\partial u}) + \frac{\partial x}{\partial w}(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u})\right] du \wedge dv \wedge dw \\ = & A(u,v,w)\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} du \wedge dv \wedge dw \\ = & A(u,v,w)\frac{\partial(x,y,z)}{\partial(u,v,w)} du \wedge dv \wedge dw \end{aligned}$$

4.7.2 $\int_S \nabla \times \mathbf{H} \cdot d\mathbf{a} = kI = k\int_S \mathbf{J} \cdot d\mathbf{a}$, where $\mathbf{J}$ is the current density. In the differential form, the equation becomes

$$\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right] dy\wedge dz + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right] dz\wedge dx + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] dx\wedge dy = kJ_x\,dy\wedge dz + kJ_y\,dz\wedge dx + kJ_z\,dx\wedge dy$$

the corresponding components of the two sides equal, respectively.

4.7.3 If $\frac{\partial f}{\partial x} = A$ and $\frac{\partial f}{\partial y} = B$, then $\frac{\partial A}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial B}{\partial x}$, so being exact implies being closed. If $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} = \varphi(x,y)$, then $A = \int_{y_0}^{y} \varphi(x,y)dy + h(x)$ and $B = \int_{x_0}^{x} \varphi(x,y)dx + g(y)$. Let $f = \int_{x_0}^{x}\int_{y_0}^{y} \varphi(x,y)dxdy + \int_{x_0}^{x} h(x)dx + \int_{y_0}^{y} g(y)dy$, then $\frac{\partial f}{\partial x} = \int_{y_0}^{y} \varphi(x,y)dy + h(x) = A$ and $\frac{\partial f}{\partial y} = \int_{x_0}^{x} \varphi(x,y)dx + g(y) = B$, so being closed implies being exact. Therefore, being closed and being exact are sufficient and necessary conditions for each other.

To find the function f for exact $Adx + Bdy$, let $f = \int_{x_0}^{x} A(x,y)dx + \int_{y_0}^{y} B(x_0,y)dy$, then $\frac{\partial f}{\partial x} = A(x,y)$, and $\frac{\partial f}{\partial y} = \int_{x_0}^{x} \frac{\partial A(x,y)}{\partial y}dx + B(x_0,y) = \int_{x_0}^{x} \frac{\partial B(x,y)}{\partial x}dx + B(x_0,y) = B(x,y) - B(x_0,y) + B(x_0,y) = B(x,y)$. So f satisfy the condition.

(1) $\frac{\partial y}{\partial y} = \frac{\partial x}{\partial x} = 1$, so $ydx + xdy$ is closed and exact. Let $x_0 = y_0 = 0$, then

$$f = \int_{x_0}^{x} ydx + \int_{y_0}^{y} x_0 dy = xy$$

(2) $\frac{\partial}{\partial y}(\frac{y}{x^2+y^2}) \neq \frac{\partial}{\partial x}(\frac{x}{x^2+y^2})$, so $\frac{ydx+xdy}{x^2+y^2}$ is neither closed nor exact.

(3) $\frac{\partial}{\partial y}[\ln(xy) + 1] = \frac{\partial}{\partial x}(\frac{x}{y}) = \frac{1}{y}$, so $[\ln(xy)+1]dx + \frac{x}{y}dy$ is closed and exact. Let $x_0 = 0$, then

$$f = \int_{x_0}^{x} [\ln(xy) + 1]dx + \int_{y_0}^{y} \frac{x_0}{y}dy = x\ln(xy)$$

(4) $\frac{\partial}{\partial y}(\frac{-y}{x^2+y^2}) = \frac{\partial}{\partial x}(\frac{x}{x^2+y^2}) = \frac{-x^2+y^2}{(x^2+y^2)^2}$, so $\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ is closed and exact. Let $x_0 = 0$, then

$$f = \int_{x_0}^{x} \frac{-y}{x^2+y^2}dx + \int_{y_0}^{y} \frac{x_0}{{x_0}^2+y^2}dy = -\tan^{-1}\frac{x}{y}$$

(5) $f(z)dx = (x+iy)dx + (-y+ix)dy$. $\frac{\partial(x+iy)}{\partial y} = \frac{\partial(-y+ix)}{\partial x} = i$, so $(x+iy)dx+(-y+ix)dy$ is closed and exact. Let $x_0 = y_0 = 0$, then

$$f = \int_{x_0}^{x} (x+iy)dx + \int_{y_0}^{y} (-y+ix_0)dy = \frac{x^2-y^2}{2} + ixy$$