Chapter 5
Differentiation

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1.

$$\phi(y) = \left|\frac{f(x) - f(y)}{x - y}\right| \leq |x - y|$$

$|x - y| \to 0$ as $y \to x$, so $\phi(y) \to 0$ as $y \to x$, which means $f'(x) = 0$. By Theorem 5.11, f is constant.

2. For $x_1, x_2 \in (a, b)$ and $x_2 > x_1$, there exists $x \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = (x_2 - x_1)f'(x)$, and since $f'(x) > 0$, $f(x_2) - f(x_1) > 0$. So f is strictly increasing in (a, b), and therefore the inverse function g exists.

Let $x, t \in (a, b)$, and $y = f(x)$, $u = f(t)$. Then

$$\phi(u) = \frac{g(u) - g(y)}{u - y} = \frac{t - x}{f(t) - f(x)} = \frac{1}{\frac{f(t)-f(x)}{t-x}} \to \frac{1}{f'(x)}$$

when $t \to x$, which means for every $\varepsilon > 0$ there is $\delta' > 0$ such that $|t - x| < \delta'$ implies $|\phi(u) - \frac{1}{f'(x)}| < \varepsilon$. f is a continuous 1-1 mapping of the compact metric space $[a', b']$ where $a < a' < b' < b$, so $g = f^{-1}$ is a continuous mapping by Theorem 4.17. So for $\delta' > 0$ there is $\delta > 0$ such that $|u - y| < \delta$ implies $|g(u) - g(y)| = |t - x| < \delta'$ and therefore $|\phi(u) - \frac{1}{f'(x)}| < \varepsilon$, so

$$g'(f(x)) = g'(y) = \lim_{u \to y} \phi(u) = \frac{1}{f'(x)}$$

3. Let $\varepsilon < \frac{1}{M}$, then $f'(x) = 1 + \varepsilon g'(x) > 1 - \frac{1}{M} \cdot M = 0$, so by Exercise 2, f is strictly increasing and therefore one-to-one.

4. Consider the function

$$f(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1}$$

which is differential. $f(0) = f(1) = 0$, so by Theorem 5.10, there is a $x \in (0, 1)$ such that

$$f'(x) = C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$

5. For every $\varepsilon > 0$, there is a x_0 such that $x > x_0$ implies $|f'(x)| < \varepsilon$, then for $x > x_0$, there is a $x_1 \in (x, x + 1)$ such that

$$|g(x)| = \left|\frac{f(x + 1) - f(x)}{x + 1 - x}\right| = |f'(x_1)| < \varepsilon$$

so $g(x) \to 0$ as $x \to +\infty$.

6.

$$g'(x) = \frac{f'(x)}{x} - \frac{f(x)}{x^2} = \frac{1}{x}\left(f'(x) - \frac{f(x) - f(0)}{x - 0}\right) = \frac{1}{x}\left(f'(x) - f'(x_1)\right) > 0$$

where $0 < x_1 < x$, so $f'(x) > f'(x_1)$. By Theorem 5.11, g is monotonically increasing.

7.

$$\lim_{t\to x}\frac{f(t)}{g(t)}=\lim_{t\to x}\frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}}=\frac{\lim_{t\to x}\frac{f(t)-f(x)}{t-x}}{\lim_{t\to x}\frac{g(t)-g(x)}{t-x}}=\frac{f'(x)}{g'(x)}$$

8. f' is continuous on the compact metric space $[a,b]$, so it is uniformly continuous by Theorem 4.19. For every $\varepsilon>0$, there is $\delta>0$ such that $x,y\in[a,b]$ and $|y-x|<\delta$ implies $|f'(y)-f'(x)|<\varepsilon$, then for $x,t\in[a,b]$ and $0<|t-x|<\delta$, there is a p between x and t such that

$$\left|\frac{f(t)-f(x)}{t-x}-f'(x)\right|=|f'(p)-f'(x)|<\varepsilon$$

since $|p-x|<|t-x|<\delta$. The results hold for vector-valued functions with arbitrary dimension k since for the i-th component, we can find δ_i such that

$$\left|\frac{f_i(t)-f_i(x)}{t-x}-f_i'(x)\right|<\frac{\varepsilon}{\sqrt{k}}$$

when $0<|t-x|<\delta_i$, then for $|t-x|<\delta=\min(\delta_1,\delta_2,\cdots,\delta_k)$,

$$\left|\frac{\mathbf{f}(t)-\mathbf{f}(x)}{t-x}-\mathbf{f}'(x)\right|<\sqrt{\left(\frac{\varepsilon}{\sqrt{k}}\right)^2\cdot k}=\varepsilon$$

9. For every t there is a $u\in(0,t)$ or $(t,0)$ such that

$$\frac{f(t)-f(0)}{t-0}=f'(u)$$

Then

$$\lim_{t\to 0}\frac{f(t)-f(0)}{t-0}=\lim_{t\to 0}f'(u)=3$$

since $u\to 0$ as $t\to 0$. So $f'(0)=3$.

10. Let $f(x)=f_1(x)+if_2(x)$, then by Theorem 5.13,

$$\lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0}\frac{f_1(x)}{x}+\lim_{x\to 0}\frac{if_2(x)}{x}=\lim_{x\to 0}f_1'(x)+\lim_{x\to 0}if_2'(x)=\lim_{x\to 0}f(x)=A$$

Similarly $\lim_{x\to 0}\frac{g(x)}{x}=B$. Therefore,

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\left\{\lim_{x\to 0}\frac{f(x)}{x}-A\right\}\cdot\lim_{x\to 0}\frac{x}{g(x)}+A\cdot\lim_{x\to 0}\frac{x}{g(x)}=\{A-A\}\cdot\frac{1}{B}+A\cdot\frac{1}{B}=\frac{A}{B}$$

11.

$$f''(x)=\frac{f''(x)}{2}+\frac{f''(x)}{2}=\frac{1}{2}\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}+\frac{1}{2}\lim_{h\to 0}\frac{f'(x-h)-f'(x)}{-h}=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}$$

$f''(x)$ exists, so $f'(x)$ exists in a neighborhood of x, which means $f(x)$ is differentiable in the neighborhood. Let $A(h)=f(x+h)+f(x-h)-2f(x),\ B(h)=h^2$, then $A(h)$ is differentiable in a neighborhood of $h=0$ with $A'(h)=f'(x+h)-f'(x-h)$, and $B(h)$ is differentiable in a neighborhood of $h=0$ with $B'(h)=2h$. As $h\to 0,\ A(h)\to 0$ and $B(h)\to 0$, so by Theorem 5.13,

$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=\lim_{h\to 0}\frac{A(h)}{B(h)}=\lim_{h\to 0}\frac{A'(h)}{B'(h)}=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}=f''(x)$$

Let f be such that

$$f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$$

then

$$\lim_{h\to 0}\frac{f(0+h)+f(0-h)-2f(0)}{h^2}=0$$

while $f'(0)$ and $f''(0)$ do not exist.

12. For $x > 0$, $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$, $f^{(3)}(x) = 6$. For $x < 0$, $f(x) = -x^3$, $f'(x) = -3x^2$, $f''(x) = -6x$, $f^{(3)}(x) = -6$. For $x = 0$,

$$f'(0) = \lim_{t\to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t\to 0} f'(p) = \lim_{p\to 0} \pm 3p^2 = 0$$

where $p \in (0, t)$ or $(t, 0)$.

$$f''(0) = \lim_{t\to 0} \frac{f'(t) - f'(0)}{t - 0} = \lim_{t\to 0} f''(q) = \lim_{q\to 0} \pm 6q = 0$$

where $q \in (0, t)$ or $(t, 0)$.

$$f^{(3)}(0+) = 6, \qquad f^{(3)}(0-) = -6$$

so $f^{(3)}(0)$ does not exist.

13. f is a complex function since x^a may be complex for $x < 0$. Assume the derivative laws of real functions holds for complex functions.

(a) f is continuous for all $x \neq 0$. At $x = 0$, f is continuous if and only if $\lim_{t\to 0} f(t) = f(0)$. If $a \leq 0$, $\lim_{t\to 0} f(t)$ does not exist. If $a > 0$, $\lim_{t\to 0} f(t) = 0 = f(0)$.

(b)

$$f'(0) = \lim_{t\to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t\to 0} t^{a-1} \sin(|x|^{-c})$$

If $a \leq 1$, the limit does not exist. If $a > 1$, the limit exists and is equal to 0.

(c) For $x \neq 0$,

$$f'(x) = ax^{a-1} \sin(|x|^{-c}) - cx^{a+1}|x|^{-c-2} \cos(|x|^{-c})$$

If $a < 1 + c$, consider $x > 0$ and $|x|^{-c} = 2n\pi$ where n is an integer, then

$$f'(x) = -cx^{a-c-1} \to \infty$$

as $x \to 0$, so f' is not bounded.

If $a \geq 1 + c$, then

$$|f'(x)| \leq |a||x|^{a-1} + c|x|^{a-c-1} \leq |a| + c$$

so f' is bounded.

(d) f' is continuous for all $x \neq 0$. At $x = 0$, f' is continuous if and only if $\lim_{t\to 0} f'(t) = f'(0) = 0$. If $a \leq 1 + c$, consider $t > 0$ and $|t|^{-c} = 2n\pi$ where n is an integer, then

$$\lim_{t\to 0} f'(t) = \lim_{t\to 0} -c\, t^{a-c-1} = \begin{cases} \infty & \textit{if } a - c - 1 < 0 \\ -c \neq 0 & \textit{if } a - c - 1 = 0 \end{cases}$$

so f' is not continuous.

If $a > 1 + c$,

$$\lim_{t\to 0} |f'(t)| \leq \lim_{t\to 0} |a||t|^{a-1} + \lim_{t\to 0} c|t|^{a-c-1} = 0$$

so $\lim_{t\to 0} f'(t) = 0 = f'(0)$.

(e)

$$f''(0) = \lim_{t\to 0} \frac{f'(t) - f'(0)}{t - 0} = \lim_{t\to 0} a\, t^{a-2} \sin(|t|^{-c}) - \lim_{t\to 0} c\, t^{a}|t|^{-c-2} \cos(|t|^{-c})$$

If $a \leq 2 + c$, consider $t > 0$, then

$$f''(0) = \lim_{t\to 0} t^{a-c-2} \left(a\, t^{c} \sin(t^{-c}) - c \cos(t^{-c})\right) = \lim_{t\to 0} -c\, t^{a-c-2} \cos(t^{-c})$$

which does not exist.

If $a > 2 + c$, then

$$|f''(0)| \leq \lim_{t\to 0} |a||t|^{a-2} + \lim_{t\to 0} c|t|^{a-c-2} = 0$$

so $f''(0) = 0$.

(f) For $x \neq 0$,

$$\begin{aligned}f''(x) &= \left(a(a-1)x^{a-2} - c^2 x^{a+2}|x|^{-2c-4}\right)\sin(|x|^{-c}) \\ &\quad + \left(-acx^a|x|^{-c-2} - c(a+1)x^a|x|^{-c-2} - c(-c-2)x^{a+2}|x|^{-c-4}\right)\cos(|x|^{-c})\end{aligned}$$

If $a < 2+2c$, consider $x > 0$ and $|x|^{-c} = (2n+\frac{1}{2})\pi$ where n is an integer, then

$$f''(x) = x^{a-2c-2}\left(a(a-1)x^{2c} - c^2\right) \to \infty$$

as $x \to 0$, so f'' is not bounded.

If $a \geq 2+2c$, then

$$|f''(x)| \leq |a(a-1)| + |c^2| + |ac| + |c(a+1)| + |c(c+2)|$$

so f'' is bounded.

(g) f'' is continuous for all $x \neq 0$. At $x = 0$, f'' is continuous if and only if $\lim_{t\to 0} f''(t) = f''(0) = 0$. If $a \leq 2+2c$, consider $t > 0$ and $|t|^{-c} = (2n+\frac{1}{2})\pi$ where n is an integer, then

$$\lim_{t\to 0} f''(t) = \lim_{t\to 0} t^{a-2c-2}\left(a(a-1)t^{2c} - c^2\right) = \begin{cases}\infty & \textit{if } a-2c-2<0 \\ -c^2 \neq 0 & \textit{if } a-2c-2=0\end{cases}$$

so f'' is not continuous.

If $a > 2+2c$,

$$\lim_{t\to 0}|f''(t)| \leq \lim_{t\to 0}\left(|a(a-1)||t|^{a-2} + |c^2+c-2ac||t|^{a-c-2} + |c^2||t|^{a-2c-2}\right) = 0$$

so $\lim_{t\to 0} f''(t) = 0 = f''(0)$.

14. If f is convex, let x_1, x_2 be such that $a < x_1 < x_2 < b$, then by Exercise 4.23,

$$f'(x_1) = \lim_{t\to x_1}\frac{f(t)-f(x_1)}{t-x_1} \leq \frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \lim_{t\to x_2}\frac{f(t)-f(x_2)}{t-x_2} = f'(x_2)$$

which means f is monotonically increasing.

If f is monotonically increasing, let $x, y \in (a,b)$, $0 < \lambda < 1$, and $z = \lambda x + (1-\lambda)y$. Then by Theorem 5.10,

$$\frac{f(z)-f(x)}{z-x} = f(w_1) \leq f(w_2) = \frac{f(y)-f(z)}{y-z}$$

where $w_1 \in (x,z)$ and $w_2 \in (z,y)$. Rearranging,

$$(y-x)f(z) \leq (y-z)f(x) + (z-x)f(y)$$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

which means f is convex.

$f''(x) \geq 0$ if and only if $f'(x)$ is monotonically increasing, if and only if f is convex.

15. For every $x \in (a,\infty)$ and $h > 0$, by Taylor's theorem,

$$f(x+2h) = f(x) + f'(x)\cdot 2h + \frac{f''(\xi)}{2}(2h)^2$$

where $\xi \in (x, x+2h)$, so

$$f'(x) = \frac{1}{2h}\left[f(x+2h) - f(x)\right] - hf''(\xi)$$

$$|f'(x)| \leq \frac{1}{2h}\left[M_0 - (-M_0)\right] + hM_2 = \frac{M_0}{h} + hM_2$$

If $M_0 = 0$, then $M_1 = 0$, so $M_1^2 \leq 4M_0M_2$ holds. If $M_2 = 0$, then M_1 is a constant, which means f is a linear function, which is bounded only if f is a constant function, which means $f'(x) = 0$ and $M_1 = 0$, so $M_1^2 \le 4M_0M_2$ holds. So assume $M_0, M_2 \ne 0$, and let $h = \sqrt{\frac{M_0}{M_2}}$, then $|f'(x)| \le 2\sqrt{M_0M_2}$ for every x, so $M_1 \le 2\sqrt{M_0M_2},\ M_1^2 \le 4M_0M_2$.

Let f be such that

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0) \\ \frac{x^2-1}{x^2+1} & (0 \le x < \infty) \end{cases}$$

then

$$f'(x) = \begin{cases} 4x & (-1 < x < 0) \\ \frac{4x}{(x^2+1)^2} & (0 < x < \infty) \end{cases}$$

$$f''(x) = \begin{cases} 4 & (-1 < x < 0) \\ \frac{4(1-3x^2)}{(x^2+1)^3} & (0 < x < \infty) \end{cases}$$

$$f^{(3)}(x) = \begin{cases} 0 & (-1 < x < 0) \\ \frac{48x(x^2-1)}{(x^2+1)^4} & (0 < x < \infty) \end{cases}$$

$f'(0+) = f'(0-) = \lim_{t\to 0} f'(t) = 0$, so $f'(0) = 0$ by Exercise 9. $f''(0+) = f''(0-) = \lim_{t\to 0} f''(t) = 4$, so $f''(0) = 4$ by Exercise 9. So f is twice-differentiable. $f'(x)$ has a root at $x = 0$ only, so

$$M_0 = \max\left(|f(-1)|, |f(0)|, |f(\infty)|\right) = 1$$

$f''(x)$ has a root at $x = \frac{1}{\sqrt{3}}$ only, so

$$M_1 = \max\left(|f'(-1)|, |f'(0)|, |f'(\frac{1}{\sqrt{3}})|, |f'(\infty)|\right) = 4$$

$f^{(3)}(x)$ has roots at $-1 < x \le 0$ and $x = 1$, so

$$M_2 = \max\left(4, |f''(1)|, |f''(\infty)|\right) = 4$$

Therefore, $M_1^2 = 4M_0M_2$

Let $\mathbf{f}$ be a twice-differentiable vector-valued function on (a, ∞), and let M_0, M_1, M_2 be the least upper bound of $|\mathbf{f}(x)|, |\mathbf{f}'(x)|, |\mathbf{f}''(x)|$. For every $0 < \alpha < M_1$, there is a x_0 such that $\alpha < |\mathbf{f}'(x_0)| < M_1$.Let $\mathbf{u} = \frac{\mathbf{f}'(x_0)}{|\mathbf{f}'(x_0)|}$ and $\phi(x) = \mathbf{f}'(x) \cdot \mathbf{u}$, and let N_0, N_1, N_2 be the least upper bound of $|\phi(x)|, |\phi'(x)|, |\phi''(x)|$. We have

$$\alpha < |\mathbf{f}'(x_0)| = |\phi(x_0)| \le N_1$$

$\phi(x)$ is a twice differentiable real function, so by the above results,

$$N_1^2 \le 4N_0N_2$$

Since $|\phi(x)| = |\mathbf{f}(x)\cdot\mathbf{u}| \le |\mathbf{f}(x)|\cdot|\mathbf{u}| = |\mathbf{f}(x)| \le M_0$, and $|\phi''(x)| = |\mathbf{f}''(x)\cdot\mathbf{u}| \le |\mathbf{f}''(x)|\cdot|\mathbf{u}| = |\mathbf{f}''(x)| \le M_2$, we have

$$N_0 \le M_0 \qquad N_2 \le M_2$$

Summarizing, we have

$$\alpha^2 \le N_1^2 \le 4N_0N_2 \le 4M_0M_2$$

for every $0 < \alpha < M_1$, so

$$M_1^2 \le 4M_0M_2$$

16. Let M_0, M_1, M_2 be the upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$ on (a, ∞). Then by Exercise 15,

$$\left(\lim_{a\to\infty} M_1\right)^2 \le 4\left(\lim_{a\to\infty} M_0\right)\left(\lim_{a\to\infty} M_2\right) = 0$$

so $\lim_{a\to\infty} M_1 = 0$, which means $f'(x) \to 0$ as $x \to \infty$.

17. By Taylor's Theorem,

$$\begin{aligned} f(1) &= f(0) + f'(0) + \frac{f''(0)}{2} + \frac{f^{(3)}(s)}{6} \\ f(-1) &= f(0) - f'(0) + \frac{f''(0)}{2} - \frac{f^{(3)}(t)}{6} \end{aligned}$$

for some $s \in (0,1),\ t \in (-1,0)$. Subtracting the two equations, we have

$$f^{(3)}(s) + f^{(3)}(t) = 6$$

so $f^{(3)}(s) \geq 3$ or $f^{(3)}(t) \geq 3$.

18. The relation $f^{(k)}(t) = (t-\beta)Q^{(k)}(t) + kQ^{(k-1)}(t)$ holds for $k = 1$. If it holds for $k = n$, so $f^{(n)}(t) = (t-\beta)Q^{(n)}(t) + nQ^{(n-1)}(t)$, then

$$f^{(n+1)}(t) = (t-\beta)Q^{(n+1)}(t) + Q^{(n)}(t) + nQ^{(n)}(t) = (t-\beta)Q^{(n+1)}(t) + (n+1)Q^{(n)}(t)$$

so it holds for $k = n+1$. By induction, the relation holds for all $k \geq 1$. Therefore,

$$\begin{aligned} P(\beta) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k \\ &= f(\alpha) - \sum_{k=1}^{n-1} \frac{Q^{(k)}(\alpha)}{k!}(\beta-\alpha)^{k+1} + \sum_{k=1}^{n-1} \frac{Q^{(k-1)}(t)}{(k-1)!}(\beta-\alpha)^k \\ &= f(\beta) - (\beta-\alpha)Q(t) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n + Q(t)(\beta-\alpha) \\ &= f(\beta) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n \end{aligned}$$

Therefore,

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n$$

19. (a)(b)

$$\begin{aligned} D_n &= \frac{f(\beta_n) - f(0)}{\beta_n - \alpha_n} - \frac{f(\alpha_n) - f(0)}{\beta_n - \alpha_n} \\ &= \frac{f(\beta_n) - f(0)}{\beta_n - 0}\,\frac{\beta_n}{\beta_n - \alpha_n} + \frac{f(\alpha_n) - f(0)}{\alpha_n - 0}\,\frac{-\alpha_n}{\beta_n - \alpha_n} \\ &= \left(\frac{f(\beta_n) - f(0)}{\beta_n - 0} - \frac{f(\alpha_n) - f(0)}{\alpha_n - 0}\right)\frac{\beta_n}{\beta_n - \alpha_n} + \frac{f(\alpha_n) - f(0)}{\alpha_n - 0} \end{aligned}$$

$\frac{\beta_n}{\beta_n - \alpha_n}$ is bounded by 1 in (a), and is assumed to be bounded in (b), so

$$\begin{aligned} \lim_{n\to\infty} D_n &= \left(\lim_{\beta_n \to 0} \frac{f(\beta_n) - f(0)}{\beta_n - 0} - \lim_{\alpha_n \to 0} \frac{f(\alpha_n) - f(0)}{\alpha_n - 0}\right) \lim_{n\to\infty} \frac{\beta_n}{\beta_n - \alpha_n} + \lim_{\alpha_n \to 0} \frac{f(\alpha_n) - f(0)}{\alpha_n - 0} \\ &= (f'(0) - f'(0)) \cdot \lim_{n\to\infty} \frac{\beta_n}{\beta_n - \alpha_n} + f'(0) \\ &= f'(0) \end{aligned}$$

(c) f is differentiable in $(-1,1)$, so by Theorem 5.10, there is $\gamma_n \in (\alpha_n, \beta_n)$ such that

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = f'(\gamma_n)$$

Therefore,

$$\lim_{n\to\infty} D_n = \lim_{\gamma_n \to 0} f'(\gamma_n) = f'(0)$$

since $\gamma_n \to 0$ as $n \to \infty$, and f' is continuous.

Let f be such that

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

$f(x)$ is differentiable at $x \neq 0$, and at $x = 0$,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x \sin \frac{1}{x} = 0$$

so $f'(0)$ exists. Let $\beta_n = \frac{1}{2\pi(n - \frac{1}{4})}$ and $\alpha_n = \frac{1}{2\pi n}$, then

$$\lim_{n \to \infty} D_n = \lim_{n \to \infty} \frac{-\beta_n^2 - 0}{\beta_n - \alpha_n} = \lim_{n \to \infty} -\frac{2}{\pi} \frac{n}{n - \frac{1}{4}} = -\frac{2}{\pi} \neq f'(0)$$

20. Let $\mathbf{f}$ be a vector-valued function, and let all the other definitions be the same as in Theorem 5.15. Let $\mathbf{u}$ be a constant vector with $|\mathbf{u}| = 1$, then $\mathbf{u} \cdot \mathbf{f}$ is a real function on which Theorem 5.15 can apply, so there exists a point x between α and β such that

$$|\mathbf{u} \cdot \mathbf{f}(\beta) - \mathbf{u} \cdot \mathbf{P}(\beta)| = \left| \frac{\mathbf{u} \cdot \mathbf{f}^{(n)}(x)}{n!} (\beta - \alpha)^n \right| \leq \left| \frac{\mathbf{f}^{(n)}(x)}{n!} \right| (\beta - \alpha)^n$$

Let $\mathbf{u} = \frac{\mathbf{f}(\beta) - \mathbf{P}(\beta)}{|\mathbf{f}(\beta) - \mathbf{P}(\beta)|}$, then $|\mathbf{u} \cdot \mathbf{f}(\beta) - \mathbf{u} \cdot \mathbf{P}(\beta)| = |\mathbf{f}(\beta) - \mathbf{P}(\beta)|$, so

$$|\mathbf{f}(\beta) - \mathbf{P}(\beta)| \leq \left| \frac{\mathbf{f}^{(n)}(x)}{n!} \right| (\beta - \alpha)^n$$

which is the required inequality.

21. Let E be a closed subset of R^1, then by Exercise 2.29, $E^c = \bigcup_k (a_k, b_k)$ where a_k and b_k can be possibly infinite. Define the function f as

$$f(x) = \begin{cases} e^{-\frac{1}{(x-a_k)^2 (x-b_k)^2}} & x \in (a_k, b_k) \subset E^c,\ a_k \neq -\infty,\ b_k \neq \infty \\ e^{-\frac{1}{(x-a_k)^2}} & x \in (a_k, \infty) \subset E^c \\ e^{-\frac{1}{(x-b_k)^2}} & x \in (-\infty, b_k) \subset E^c \\ 0 & x \in E \end{cases}$$

The zero set of f is E. It is differentiable of all orders at $x \neq a_k, b_k$, and at $x = a_k$,

$$f'(a_k+) = \lim_{x \to a_k} \frac{e^{-\frac{1}{(x-a_k)^2 (x-b_k)^2}}}{x - a_k} = 0 = f'(a_k-)$$

so $f'(a_k)$ exists. Continue the process, $f^{(n)}(a_k)$ exists for every n. The facts hold similarly for $x = b_k$. Therefore, f has derivatives of all orders on R^1.

22. If there are two different points x_1, x_2 such that $f(x_1) = x_1$ and $f(x_2) = x_2$, then by Theorem 5.10, there is a t between x_1, x_2 such that

$$f'(t) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{x_1 - x_2}{x_1 - x_2} = 1$$

a contradiction, so f has at most one fixed point.

(b) $(1 + e^{-t})^{-1} \neq 0$, so $f(t) = t + (1 + e^{-t})^{-1} \neq t$, which means f has no fixed point.

$$f'(t) = 1 - \frac{e^t}{(1 + e^t)^2}$$

which lies between $(0, 1)$ for all finite t.

(c) For $k \geq 1$,

$$\frac{|x_{k+2} - x_{k+1}|}{|x_{k+1} - x_k|} = \left| \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} \right| = |f'(t_k)| \leq A$$

where t_k is between x_{k+1} and x_k. Therefore,

$$|x_{n+1} - x_n| \leq |x_n - x_{n-1}|A \leq \cdots \leq |x_2 - x_1|A^{n-1}$$

Let N be a positive integer, then for $n, m > N$ and $n \geq m$,

$$|x_n - x_m| \leq |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m| \leq |x_2 - x_1|(A^{n-2} + \cdots + A^{m-1}) \leq |x_2 - x_1| \sum_{k=N}^{\infty} A^k = |x_2 - x_1| \frac{A^N}{1-A}$$

which tends to 0 as $N \to \infty$, so $\{x_n\}$ is a Cauchy sequence, and $x = \lim_{n\to\infty} x_n$ exists.

$$f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x$$

so x is a fixed point of f.

(d) The zig-zag path consists of vertical segments moving from (x_n, x_n) on $y = x$ to $(x_n . x_{n+1})$ on $y = f(x)$, and horizontal segments moving from (x_n, x_{n+1}) on $y = f(x)$ to (x_{n+1}, x_{n+1}) on $y = x$.

23. $f(x) = x$ has at most three roots, which are $x = \alpha, \beta, \gamma$, and $f(x) < x$ for $x < \alpha$, $f(x) > x$ for $\alpha < x < \beta$, $f(x) < x$ for $\beta < x < \gamma$, $f(x) > x$ for $x > \gamma$. $f'(x) = x^2 \geq 0$, so $f(x)$ is monotonically increasing, and is strictly monotonically increasing at $x \neq 0$.

(a) If $x_n < \alpha$, then $x_{n+1} = f(x_n) < x_n$, which means $\{x_n\}$ is monotonically decreasing if $x_1 < \alpha$. If $\{x_n\}$ is bounded and therefore $\alpha' = \lim_{n\to\infty} x_n$ exists, then

$$f(\alpha') = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = \alpha'$$

which means $\alpha' < \alpha$ is also a root of $f(x) = x$, a contradiction. Therefore, $x_n \to -\infty$ as $n \to \infty$.

(b) If $\alpha < x_n < \beta$, then $x_{n+1} = f(x_n) > x_n$, and since $f(x)$ is monotonically increasing, $x_{n+1} = f(x_n) < f(\beta) = \beta$. So if $\alpha < x_1 < \beta$, then $\{x_n\}$ is a monotonically increasing sequence bounded above by β, which means the limit $\beta' = \lim_{n\to\infty} x_n$ exists, and since β' is a root of $f(x) = x$, $\beta' = \beta$. Similarly, if $\beta < x_1 < \gamma$, then $\{x_n\}$ is a monotonically decreasing sequence bounded below by β, so $\lim_{n\to\infty} x_n = \beta$.

(c) Similarly with (a), if $x_1 > \gamma$, then $\{x_n\}$ is a non-bounded monotonically increasing sequence, so $x_n \to \infty$ as $n \to \infty$.

24.

$$f: \quad x_{n+1} - \sqrt{\alpha} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} = \frac{1}{2}\left(1 - \frac{\sqrt{\alpha}}{x_n}\right)(x_n - \sqrt{\alpha})$$

$$g: \quad x_{n+1} - \sqrt{\alpha} = \frac{\alpha + x_n}{1 + x_n} - \sqrt{\alpha} = \frac{1 - \sqrt{\alpha}}{1 + x_n}(x_n - \sqrt{\alpha})$$

$\lim_{x_n\to\sqrt{\alpha}} \frac{1}{2}\left(1 - \frac{\sqrt{\alpha}}{x_n}\right) = 0$, while $\lim_{x_n\to\sqrt{\alpha}} \frac{1-\sqrt{\alpha}}{1+x_n} = \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}} \neq 0$, so $\{x_n\}$ obtained by $x_{n+1} = f(x_n)$ converges to $\sqrt{\alpha}$ faster than $\{x_n\}$ obtained by $x_{n+1} = g(x_n)$.

The zig-zag paths of f and g when $\alpha = 2$ are shown in Figure 1.

25. (a) The tangent to the graph of f at $(x_n, f(x_n))$ is $y = f'(x_n)(x - x_n) + f(x_n)$, which intersects with x-axis at $\left(x_n - \frac{f(x_n)}{f'(x_n)}\, , 0\right) = (x_{n+1}\, , 0)$.

(b) $x_1 > \xi$. If $x_n > \xi$, then $f(x_n) > 0$, so

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} < x_n$$

Also, by Theorem 5.10, there is $t \in (\xi, x_n)$ such that

$$\frac{f(x_n) - f(\xi)}{x_n - \xi} = f'(t) < f'(x_n)$$

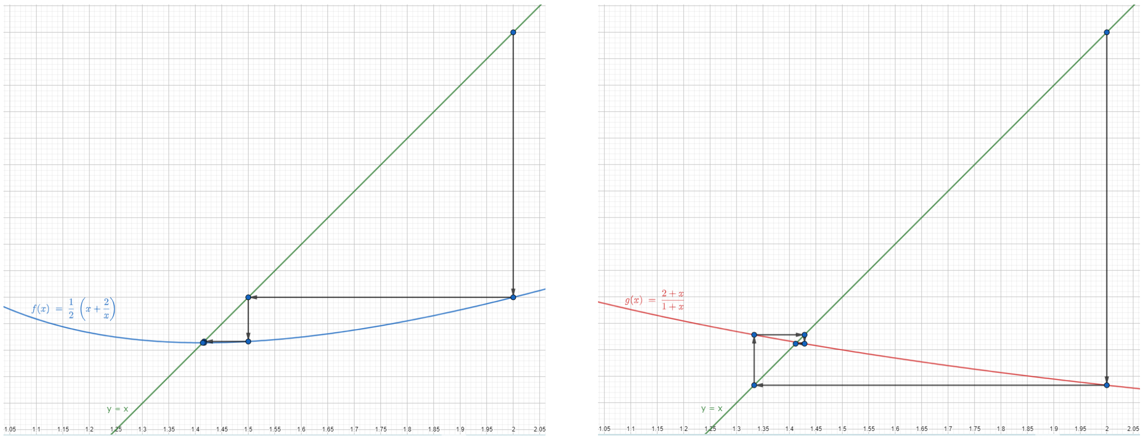

Figure 1: The zig-zag paths of f and g in Exercise 24.

since f' is monotonically increasing by $f''(x) > 0$. Rearranging,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} > \xi$$

Therefore, $\{x_n\}$ is monotonically decreasing and bounded below by ξ. Let the limit be $\xi' = \lim_{n\to\infty} x_n$, then since $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = \xi'$,

$$\xi' = \xi' - \frac{f(\xi')}{f'(\xi')}$$

so $f(\xi') = 0$, which means $\xi' = \xi$.

(c) By Taylor's Theorem, there is $t_n \in (\xi, x_n)$ such that

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2$$

Rearranging,

$$x_{n+1} - \xi = x_n - \frac{f(x_n)}{f'(x_n)} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi^2)$$

(d)

$$x_{n+1} - \xi \leq \frac{1}{A}\left[A(x_n - \xi)\right]^2$$

The inequality holds for $n = 1$. If it holds for $n = k$, so

$$0 \leq x_{k+1} - \xi \leq \frac{1}{A}\left[A(x_1 - \xi)\right]^{2^n}$$

then $x_{k+2} - \xi \geq 0$ by (b), and

$$x_{k+2} - \xi \leq A(x_{k+1} - \xi)^2 \leq \frac{1}{A}\left[A(x_1 - \xi)\right]^{2^{k+1}}$$

so by induction, the inequality holds for every n.

The algorithms in Exercise 3.16 and 3.18 are Newton's methods applied on $x_n^2 - \alpha$ and $x_n^p - \alpha$.

(e) $g(x) = x$ is equivalent to $f(x) = 0$, so finding a fixed point of g is finding the root of $f(x)$.

$$g'(x) = 1 - \frac{\left[f'(x)\right]^2 - f(x)f''(x)}{\left[f'(x)\right]^2} = \frac{f(x)f''(x)}{\left[f'(x)\right]^2} \leq \frac{f(x)M}{\delta^2} \to 0$$

as $x \to \xi$.

(f)

$$x_{n+1} = x_n - \frac{x_n^{\frac{1}{3}}}{\frac{1}{3}x_n^{-\frac{2}{3}}} = -2x_n$$

so $\{x_n\}$ is an alternating sequence such that $\{|x_n|\} \to \infty$ as $n \to \infty$.

26. Let $x_0 = a + \frac{1}{2A}$. For $a \le x \le x_0$,

$$\left|\frac{f(x) - f(a)}{x - a}\right| = |f'(t)| \le M_1$$

where $t \in (a, x) \in (a, x_0)$. Rearranging,

$$|f(x)| \le M_1(x - a) \le M_1(x_0 - a) \le AM_0(x_0 - a) = \frac{M_0}{2}$$

since M_0 is the least upper bound of $|f(x)|$, we have $M_0 \le \frac{M_0}{2}$, which means $M_0 = 0$, so $f(x) = 0$ on $[a, a + \frac{1}{2A}]$. Repeat the process for finite steps, we have $f(x) = 0$ on $[a, b]$.

27. Let $f(x) = y_2(x) - y_1(x)$, then f is differentiable on $[a, b]$, and $f(a) = c - c = 0$. If the inequality holds, then

$$|f'(x)| = |y_2' - y_1'| = |\phi(x, y_2) - \phi(x, y_1)| \le A|y_2 - y_1| = A|f(x)|$$

so by Exercise 26, $f(x) = 0$, which means $y_2 = y_1$, the solution is unique.

Consider $f(x)$ which is the solution of $y' = \sqrt{y}$ and $y(0) = 0$. Since $y' = \sqrt{y} \ge 0$, y is monotonically increasing. Let a be the real number such that $f(x) = 0$ for $0 \le x \le a$ and $f(x) > 0$ for $x > a$. Let $F(x) = \sqrt{f(x)}$, then $F'(x) = \frac{f'(x)}{2\sqrt{f(x)}} = \frac{1}{2}$, so $F(x) = \frac{1}{2}(x + c)$, and $f(x) = \frac{(x+c)^2}{4}$. Since $f(a) = 0$, we have $c = -a$. So the solution is

$$f(x) = \begin{cases} 0 & 0 \le x \le a \\ \frac{(x-a)^2}{4} & x > a \end{cases}$$

where a can be arbitrary.

28. *statement:* Let $\boldsymbol{\phi}$ be a vector-valued function in R^k defined on a $(k + 1)$-cell, given by $a \le x \le b$, $\alpha_j \le y_j \le \beta_j$. If there is a constant A such that

$$|\boldsymbol{\phi}(x, \mathbf{y}_2) - \boldsymbol{\phi}(x, \mathbf{y}_1)| \le A|\mathbf{y}_2 - \mathbf{y}_1|$$

then the initial-value problem

$$\mathbf{y}' = \boldsymbol{\phi}(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c}$$

has at most one solution.

proof: The result in Exercise 26 holds for vector-valued function since by Theorem 5.19,

$$|\mathbf{f}(x) - \mathbf{f}(a)| \le |\mathbf{f}'(t)|(x - a) \le M_1(x_0 - a) \le AM_0(x_0 - a)$$

and the other parts of the proof remain the same as Exercise 26.

Let $\mathbf{f} = \mathbf{y}_2(x) - \mathbf{y}_1(x)$ where $\mathbf{y}_2$, $\mathbf{y}_1$ are the solutions of the initial-value problem, then the proof is the same as Exercise 27, except all the functions are vector-valued. So $\mathbf{y}_2 = \mathbf{y}_1$, the solution is unique.

29. Let $\mathbf{y} = (y_1, \cdots, y_k)$ and $\mathbf{g} = (g_1, \cdots, g_k)$ be vectors in R^k. Let $\boldsymbol{\phi}$ be a function form R^{k+1} to R^k defined as

$$\boldsymbol{\phi}(x, \mathbf{y}) = (y_2, \cdots, y_k, f(x) - \mathbf{g} \cdot \mathbf{y})$$

If $y_j = y^{(j-1)}$ for $1 \le j \le k$, then

$$y_j' = y^{(j)} = y_{j+1} \quad \textit{for } 1 \le j \le k - 1$$

$$y_k' = y^{(k)} = f(x) - \sum_{j=1}^{k} g_j(x) y^{j-1} = f(x) - \sum_{j=1}^{k} g_j(x) y_j = f(x) - \mathbf{g} \cdot \mathbf{y}$$

$$y^{(j-1)}(a) = y_j(a) = c_j \quad \textit{for } 1 \le j \le k$$

so the initial-value problem given is equivalent to

$$\mathbf{y}' = \boldsymbol{\phi}(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c}$$

$g_1, \cdots, g_k$ are continuous real function on the compact space $[a, b]$, so $|g_j|$ is bounded, which means $|\mathbf{g}|$ is bounded. Let $|\mathbf{g}| \leq M$, and let $A = \sqrt{1 + M^2}$, then

$$\begin{aligned}
|\boldsymbol{\phi}(x, \mathbf{y}_b) - \boldsymbol{\phi}(x, \mathbf{y}_a)| &= \sqrt{\sum_{j=2}^{k} (y_{bj} - y_{aj})^2 + |\mathbf{g} \cdot (\mathbf{y}_b - \mathbf{y}_a)|^2} \\
&\leq \sqrt{|\mathbf{y}_b - \mathbf{y}_a|^2 + |\mathbf{g}|^2 |\mathbf{y}_b - \mathbf{y}_a|^2} \\
&= \sqrt{1 + |\mathbf{g}|^2} |\mathbf{y}_b - \mathbf{y}_a| \\
&\leq A |\mathbf{y}_b - \mathbf{y}_a|
\end{aligned}$$

so by Exercise 28, the solution is unique.