Chapter 3
Numerical Sequences and Series

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November 2021

1. Let $\{s_n\}$ converges to s, so for every $\varepsilon > 0$, there is an integer N such that $n \geq N$ implies $|s_n - s| < \varepsilon$. By Problem 1.13, $\big||s_n| - |s|\big| < |s_n - s| < \varepsilon$, so for every $\varepsilon > 0$, there is an integer N such that $n \geq N$ implies $\big||s_n| - |s|\big| < \varepsilon$, which means $\{|s_n|\}$ converges to $|s|$.

The converse is not true. Let $s_n = (-1)^n$, then $\{|s_n|\}$ converges to 1, but $\{s_n\}$ diverges.

2.

$$\lim_{n\to\infty}(\sqrt{n^2+n} - n) = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$$

3. $s_1 < 2$. If $s_k < 2$, then $s_{k+1} = \sqrt{2+\sqrt{s_k}} < \sqrt{2+\sqrt{2}} < 2$, so $s_n < 2$ for every n by induction. $s_2 > s_1$. If $s_k > s_{k-1}$, then $\sqrt{2+\sqrt{s_k}} > \sqrt{2+\sqrt{s_{k-1}}}$, which means $s_{k+1} > s_k$. s o $\{s_n\}$ is monotonically increasing by induction. By Theorem 3.14, $\{s_n\}$ being monotonic and bounded implies convergence.

4. $s_{2m+1} = 1 - \frac{1}{2^m}$ holds for $m = 0$. If it holds for $m = k$, so $s_{2k+1} = 1 - \frac{1}{2^k}$, then $s_{2(k+1)+1} = \frac{1}{2} + s_{2k+2} = \frac{1}{2} + \frac{s_{2k+1}}{2} = 1 - \frac{1}{2^{k+1}}$, which means it also holds for $m = k+1$. So $s_{2m+1} = 1 - \frac{1}{2^m}$ holds for every m by induction.

$s_{2m} = \frac{1}{2} - \frac{1}{2^m}$ holds for $m = 1$. If it holds for $m = k$, so $s_{2k} = \frac{1}{2} - \frac{1}{2^k}$, then $s_{2(k+1)} = \frac{s_{2k+1}}{2} = \frac{1}{4} + \frac{s_{2k}}{2} = \frac{1}{2} - \frac{1}{2^{k+1}}$, which means it also holds for $m = k+1$. So $s_{2m} = \frac{1}{2} - \frac{1}{2^m}$ holds for every m by induction.

$\{s_{2m+1}\} = \{1 - \frac{1}{2^m}\}$ converges to 1, so 1 is a subsequential limit of $\{s_n\}$. If $x > 1$, then for every n we have $s_n < 1 < x$. By Theorem 3.17, 1 is the upper limit of $\{s_n\}$.

$\{s_{2m}\} = \{\frac{1}{2} - \frac{1}{2^m}\}$ converges to $\frac{1}{2}$, so $\frac{1}{2}$ is a subsequential limit of $\{s_n\}$. If $x < \frac{1}{2}$, let $x = \frac{1}{2} - \varepsilon$ where $\varepsilon > 0$, and let N be an integer such that $N > -\log_2 \varepsilon$, then $s_n \geq \frac{1}{2} - \frac{1}{2^n} > \frac{1}{2} - \varepsilon = x$ for all $n \geq N$. By Theorem 3.17, $\frac{1}{2}$ is the lower limit of $\{s_n\}$.

5. If at least one of $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$ equals to $+\infty$, and the other does not equal to $-\infty$, then

$$\limsup_{n\to\infty}(a_n + b_n) = \infty = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

the inequality holds.

For the rest of the case, we have $\limsup_{n\to\infty} a_n < +\infty$ and $\limsup_{n\to\infty} < +\infty$. By Theorem 3.17, $\limsup_{n\to\infty}(a_n + b_n)$ is also a subsequential limit of $\{a_n + b_n\}$, so there is a subsequence $\{a_{n_k} + b_{n_k}\}$ such that

$$\limsup_{n\to\infty}(a_n + b_n) = \lim_{k\to\infty}(a_{n_k} + b_{n_k}) \tag{1}$$

$\limsup_{k\to\infty} a_{n_k}$, the upper limit of $\{a_{n_k}\}$, is also a subsequential limit of $\{a_{n_k}\}$, so there is a subsequence $\{a_{n_{k_j}}\}$ such that

$$\lim_{j\to\infty} a_{n_{k_j}} = \limsup_{k\to\infty} a_{n_k}$$

$\{a_{n_k} + b_{n_k}\}$ converges, so every subsequential limit equals to $\lim_{k\to\infty}(a_{n_k} + b_{n_k})$, which means

$$\lim_{k\to\infty}(a_{n_k} + b_{n_k}) = \lim_{j\to\infty}(a_{n_{k_j}} + b_{n_{k_j}}) \tag{2}$$

Both $\{a_{n_{k_j}} + b_{n_{k_j}}\}$ and $\{a_{n_{k_j}}\}$ converge, so $\{b_{n_{k_j}}\}$ also converges, and we have

$$\lim_{j\to\infty}(a_{n_{k_j}} + b_{n_{k_j}}) = \lim_{j\to\infty} a_{n_{k_j}} + \lim_{j\to\infty} b_{n_{k_j}} \tag{3}$$

$\lim_{j\to\infty} b_{n_{k_j}}$ is a subsequential limit of $\{b_{n_k}\}$, so we have

$$\lim_{j\to\infty} b_{n_{k_j}} \le \limsup_{k\to\infty} b_{n_k}$$

Therefore, we have

$$\lim_{j\to\infty} a_{n_{k_j}} + \lim_{j\to\infty} b_{n_{k_j}} \le \limsup_{k\to\infty} a_{n_k} + \limsup_{k\to\infty} b_{n_k} \tag{4}$$

It is obvious that

$$\limsup_{k\to\infty} a_{n_k} \le \limsup_{n\to\infty} a_n$$

and also

$$\limsup_{k\to\infty} b_{n_k} \le \limsup_{n\to\infty} b_n$$

so we have

$$\limsup_{k\to\infty} a_{n_k} + \limsup_{k\to\infty} b_{n_k} \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \tag{5}$$

Combining equation (1) to (5), we have

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

6. (a)

$$a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{2\sqrt{n+1}}$$

$\sum \frac{1}{2(n+1)^{1/2}}$ diverges by Theorem 3.28, so $\sum a_n$ diverges by comparison test.

(b)

$$a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} < \frac{1}{n\cdot 2\sqrt{n}}$$

$\sum \frac{1}{2n^{3/2}}$ converges by Theorem 3.28, so $\sum a_n$ converges by comparison test.

(c)

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty}(\sqrt[n]{n} - 1) = 0$$

so $\sum a_n$ converges by root test.

(d) For $|z| \le 1$,

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{|1+z^n|} \ge \lim_{n\to\infty}\frac{1}{1+|z|^n} = \begin{cases} 1, & |z| < 1 \\ \frac{1}{2}, & |z| = 1 \end{cases}$$

which is not zero, so $\sum a_n$ diverges by Theorem 3.23.

For $|z| > 1$,

$$\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty}\frac{|1+z^n|}{|1+z^{n+1}|} \le \limsup_{n\to\infty}\frac{|z|^n+1}{|z|^{n+1}-1} = \frac{1}{|z|} < 1$$

so $\sum a_n$ converges by ratio test

7.

$$(\sqrt{a_n} - \frac{1}{n})^2 = a_n + \frac{1}{n^2} - \frac{2\sqrt{a_n}}{n} \ge 0$$

$$\frac{\sqrt{a_n}}{n} \le \frac{a_n}{2} + \frac{1}{2n^2}$$

$\sum \frac{a_n}{2}$ and $\sum \frac{1}{2n^2}$ converges, so $\sum \frac{\sqrt{a_n}}{2}$ converges by comparison test.

8. Being monotonic and bounded implies convergence, so let $\lim_{n\to\infty} b_n = b$. If $\{b_n\}$ is increasing, let $d_n = b - b_n$, then $d_0 \geq d_1 \geq d_2 \geq \cdots$, and $\lim_{n\to\infty} d_n = 0$.

$$\sum a_n b_n = \sum a_n b - \sum a_n d_n$$

$\sum a_n b$ converges because $\sum a_n$ converges. $\sum a_n d_n$ converges by Theorem 3.42 because $\sum a_n$ is bounded and d_n satisfy the conditions. Therefore, $\sum a_n b_n$ converges.

If $\{b_n\}$ is decreasing, let $d_n = b_n - b$, then the convergence can be proved similarly.

9. (a) The series converges when

$$\limsup_{n\to\infty} \left| \frac{(n+1)^3 z^{n+1}}{n^3 z^n} \right| = |z| \limsup_{n\to\infty} \left| \frac{(n+1)^3}{n^3} \right| = |z| < 1$$

so the radius of convergence $R = 1$.

(b) The series converges when

$$\limsup_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!} z^{n+1}}{\frac{2^n}{n!} z^n} \right| = |z| \limsup_{n\to\infty} \left| \frac{2}{n+1} \right| = 0 < 1$$

so the radius of convergence $R = \infty$.

(c) The series converges when

$$\limsup_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)^2} z^{n+1}}{\frac{2^n}{n^2} z^n} \right| = |z| \limsup_{n\to\infty} \left| \frac{2n^2}{(n+1)^2} \right| = 2|z| < 1$$

so the radius of convergence $R = \frac{1}{2}$.

(d) The series converges when

$$\limsup_{n\to\infty} \left| \frac{\frac{(n+1)^3}{3^{n+1}} z^{n+1}}{\frac{n^3}{3^n} z^n} \right| = |z| \limsup_{n\to\infty} \left| \frac{(n+1)^3}{3n^3} \right| = \frac{|z|}{3} < 1$$

so the radius of convergence $R = 3$.

10. For every positive integer N, there is an integer n such that $a_n \neq 0$. Because $|a_n| \geq 1$, we have $\sqrt[n]{|a_n|} \geq 1$. Therefore,

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \geq 1$$

$$R = \frac{1}{\alpha} \leq 1$$

11. (a) If $\{a_n\}$ is not bounded, then $\lim_{n\to\infty} a_n = \infty \ or\ -\infty$, so

$$\lim_{n\to\infty} \frac{a_n}{1+a_n} = 1 \neq 0$$

which means $\sum \frac{a_n}{1+a_n}$ diverges by Theorem 3.23.

If $\{a_n\}$ is bounded, so there is a M such that $|a_n| < M$ for every n, then

$$\frac{a_n}{1+a_n} \geq \frac{a_n}{1+M}$$

$\sum \frac{a_n}{1+M}$ diverges because $\sum a_n$ diverges, so $\sum \frac{a_n}{1+a_n}$ diverges by comparison test.

(b) $a_n > 0$, so s_n is monotonically increasing, and $s_{N+1}, s_{N+2}, \cdots \leq s_{N+k}$. Therefore,

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}$$

s_n is monotonically increasing but diverges, which implies that it is not bounded, so $\lim_{n\to\infty} s_n = \infty \, or - \infty$. For every integer N,

$$\lim_{k\to\infty}\left|\sum_{n=N+1}^{N+k}\frac{a_n}{s_n}\right| \geq \lim_{k\to\infty}\left|1-\frac{s_N}{s_{N+k}}\right| = 1$$

so for $\varepsilon < 1$, the Cauchy criterion (Theorem 3.11) is not satisfied, which means $\sum \frac{a_n}{s_n}$ diverges.

(c)

$$\frac{a_n}{s_n^2} = \frac{s_n - s_{n-1}}{s_n^2} \leq \frac{s_n - s_{n-1}}{s_n \cdot s_{n-1}} = \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

Because $\lim_{n\to\infty} s_n = \infty \, or - \infty$, so

$$\sum_{n=2}^{\infty}\left(\frac{1}{s_{n-1}} - \frac{1}{s_n}\right) = \frac{1}{s_1} - \lim_{n\to\infty}\frac{1}{s_n} = \frac{1}{s_1}$$

which is convergent, so $\frac{a_n}{s_n^2}$ converges by comparison test.

(d) $\sum \frac{a_n}{1+a_n}$ may be convergent or divergent. For example, let $a_n = 1$ for every n, then

$$\sum \frac{a_n}{1+na_n} = \sum \frac{1}{n+1}$$

which is divergent. On the other hand, let

$$a_n = \begin{cases} 1, & n = m^2, \quad m \in \mathbb{N} \\ \frac{1}{n^2}, & otherwise \end{cases}$$

Then

$$\sum_{n=1}^{N}\frac{a_n}{1+na_n} = \sum_{m=1}^{\infty}\frac{1}{1+m^2} + \sum_{\substack{n=1\\ n\neq m^2}}^{\infty}\frac{\frac{1}{n^2}}{1+n\cdot\frac{1}{n^2}}$$

The first series converges because

$$\frac{1}{1+m^2} < \frac{1}{m^2}$$

and $\sum \frac{1}{m^2}$ converges. The second series converges because

$$\frac{\frac{1}{n^2}}{1+n\cdot\frac{1}{n^2}} = \frac{1}{n(n+1)} < \frac{1}{n^2}$$

and $\sum \frac{1}{n^2}$ converges. Therefore, $\sum \frac{a_n}{1+na_n}$ converges.

12. (a) $a_n > 0$, so $s_n = \sum_{m=1}^{n} a_m$ is monotonically increasing, and $r_n = \sum_{m=n}^{\infty} a_m$ is monotonically decreasing, which means $r_m > r_{m+1}, \cdots, r_n$. Therefore,

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_m} = 1 - \frac{r_{n+1}}{r_m} > 1 - \frac{r_n}{r_m}$$

By Cauchy criterion, $\lim_{n\to\infty} r_n = 0$. For every positive integer N,

$$\lim_{k\to\infty}\left|\sum_{n=N}^{k}\frac{a_n}{r_n}\right| > \lim_{k\to\infty}\left|1-\frac{r_k}{r_N}\right| = 1$$

so the Cauchy criterion (Theorem 3.22) cannot be satisfied, which means $\sum \frac{a_n}{r_n}$ diverges.

(b) $r_n \neq r_{n+1}$, so

$$(\sqrt{r_n} - \sqrt{r_{n+1}})^2 > 0$$

$$r_n - 2\sqrt{r_n}\sqrt{r_{n+1}} + r_{n+1} > 0$$

$$2r_n - 2\sqrt{r_n}\sqrt{r_{n+1}} > r_n - r_{n+1}$$

$$2(\sqrt{r_n} - \sqrt{r_{n+1}}) > \frac{r_n - r_{n+1}}{\sqrt{r_n}} = \frac{a_n}{\sqrt{r_n}}$$

We have

$$\sum_{n=1}^{\infty} 2(\sqrt{r_n} - \sqrt{r_{n+1}}) = 2\sqrt{r_1} - \lim_{n\to\infty} 2\sqrt{r_n} = 2\sqrt{r_1}$$

which is convergent, so $\sum \frac{a_n}{\sqrt{r_n}}$ converges by comparison test.

13. Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series, and let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Let d_n be the Cauchy product of $\sum |a_n|$ and $\sum |b_n|$, which is

$$d_n = \sum_{k=0}^{n} |a_k||b_{n-k}| = \sum_{k=0}^{n} |a_k b_{n-k}|$$

Both $\sum |a_n|$ and $\sum |b_n|$ converges absolutely, so $\sum d_n$ converges by Theorem 3.50.

$$|c_n| = \left|\sum_{k=0}^{n} a_k b_{n-k}\right| \le \sum_{k=0}^{n} |a_k b_{n-k}| = d_n$$

$\sum d_n$ converges, so $\sum |c_n|$ converges by comparison test, which means $\sum c_n$ converges absolutely.

14. (a) For every $\varepsilon > 0$, let N be a positive integer such that $|s_n - s| < \frac{\varepsilon}{2}$ for $n > N$. Let M be a positive integer such that $\frac{(s_0-s)+\cdots+(s_N-s)}{M} < \frac{\varepsilon}{2}$. Then for $n > \max(N, M)$, we have

$$\sigma_n - s = \frac{(s_0 - s) + \cdots + (s_n - s)}{n+1} < \frac{(s_0 - s) + \cdots + (s_N - s)}{M} + \frac{(s_{N+1} - s) + \cdots + (s_n - s)}{n - N} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which means $\lim \sigma_n = s$.

(b) Let $s_n = (-1)^n$, then $\{s_n\}$ diverges, but $\lim \sigma_n = 0$ because $(s_0 + \cdots + s_n) \le 1$ and $\lim \frac{1}{n+1} = 0$.

(c) Let s_n be

$$s_n = \begin{cases} m, & n = m^3,\ m \in \mathbb{N} \\ \frac{1}{2^n}, & otherwise \end{cases}$$

then $\limsup s_n = \infty$ because it is not bounded.

$$\sigma_n \le \frac{\frac{1}{2^0} + \cdots + \frac{1}{2^n}}{n+1} + \frac{1 + \cdots + M}{n+1} \le \frac{2}{n+1} + \frac{M^2}{n+1} \le \frac{2}{n+1} + n^{-\frac{1}{3}}$$

where M is the largest positive integer such that $M^3 \le n$. We have $\lim \frac{2}{n+1} = \lim n^{-\frac{1}{3}} = 0$, and $\sigma_n \ge 0$, so $\lim \sigma_n = 0$.

(d)

$$s_n - \sigma_n = \frac{ns_n - \sum_{k=0}^{n-1} s_k}{n+1} = \frac{\sum_{k=1}^{n} ks_k - \sum_{k=0}^{n-1}(k+1)s_k}{n+1} = \frac{\sum_{k=1}^{n}(ks_k - ks_{k-1})}{n+1} = \frac{\sum_{k=1}^{n} ka_k}{n+1}$$

Note that $\frac{\sum_{k=0}^{n} ka_k}{n+1}$ is the arithmetic means of the sequence $\{na_n\}$, so by (a), $\lim(na_n) = 0$ implies $\lim(s_n - \sigma_n) = 0$, which is $\lim s_n = \lim \sigma_n$.

(e) For $m < n$,

$$\begin{aligned}
&\frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{\sum_{i=m+1}^{n}(s_n - s_i)}{n-m} \\
&= \frac{m+1}{n-m}\sigma_n - \frac{\sum_{i=0}^{m} s_i}{n-m} + \frac{(n-m)s_n}{n-m} - \frac{\sum_{i=m+1}^{n} s_i}{n-m} \\
&= \frac{(n-m)s_n}{n-m} + \frac{m+1}{n-m}\sigma_n - \frac{\sum_{i=0}^{n} s_i}{n-m} \\
&= s_n + \frac{m+1}{n-m}\sigma_n - \frac{n+1}{n-m}\sigma_n \\
&= s_n - \sigma_n
\end{aligned}$$

For $i \geq m+1$,

$$|s_n - s_i| = \left| \sum_{k=i+1}^{n} (s_k - s_{k-1}) \right| \leq \sum_{k=i+1}^{n} |s_k - s_{k-1}| = \sum_{k=i+1}^{n} |a_k|$$

$$\sum_{k=i+1}^{n} |a_k| \leq \sum_{k=i+1}^{n} \frac{M}{k} \leq \sum_{k=i+1}^{n} \frac{M}{i+1} = \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}$$

For $\varepsilon > 0$, let m be the integer such that

$$m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1$$

form which we have

$$\frac{m+1}{n-m} \leq \frac{1}{\varepsilon} \qquad \frac{n-m-1}{m+2} < \varepsilon$$

We have $\lim_{n\to\infty} \frac{m+1}{n-m}(\sigma_n - \sigma_m) = 0$, and

$$\frac{1}{n-m} \sum_{i=m+1}^{n} (s_n - s_i) \leq \frac{(n-m-1)M}{m+2} < M\varepsilon$$

Therefore,

$$\limsup_{n\to\infty} |s_n - \sigma| < M\varepsilon$$

Since ε is arbitrary, $\lim s_n = \sigma$.

15.

- *Theorem 3.22* From Theorem 3.11, in R^k, a sequence converges if and only if it is a Cauchy sequence. Therefore, $s_n = \sum a_n$ converges if and only if for every $\varepsilon > 0$, there is an integer $N = N' + 1$ such that

$$|s_m - s_{n-1}| = \left| \sum_{k=n}^{m} a_k \right| \leq \varepsilon$$

  if $m \geq n \geq N' + 1 = N$.

- *Theorem 3.23* By taking $m = n$, we have

$$|a_n| \leq \varepsilon \qquad (n \geq N)$$

  which means $\lim |a_n| = \lim a_n = 0$.

- *Theorem 3.25* The proof is identical with that in the text because $|\sum_{k=n}^{m} a_k| \leq \sum_{k=m}^{n} |a_k|$ holds in every R^k.

- *Theorem 3.33* The proof is identical with that in the text because the comparison text (Theorem 3.25) and Theorem 3.23 holds in every R^k.

- *Theorem 3.34* The proof is identical with that in the text as Theorem 3.33.

- *Theorem 3.42* The proof is identical with that in the text because the Cauchy criterion (Theorem 3.22) holds in every R^k.

- *Theorem 3.45* The proof is identical with that in the text because the equality and Cauchy criterion hold in every R^k.

- *Theorem 3.47* The proof is identical with that in the text because the proof of the limit rules (Theorem 3.3) holds in every R^k.

- *Theorem 3.55* The proof is identical with that in the text because the Cauchy criterion holds in every R^k.

16. (a) $x_1 > \sqrt{\alpha}$. If $x_n > \sqrt{\alpha}$, then $x_{n+1} > \sqrt{\alpha}$ because

$$(x_n - \sqrt{\alpha})^2 > 0$$

$$\frac{x_n^2 + \alpha}{2x_n} > \frac{2x_n\sqrt{\alpha}}{2x_n}$$

$$x_{n+1} > \sqrt{\alpha}$$

So by induction, $x_n > \sqrt{\alpha}$ for every n. From $x_n^2 > \alpha$, we have

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) < \frac{1}{2}(x_n + x_n) = x_n$$

So $\{x_n\}$ decreases monotonically. Along with the fact that $\{x_n\}$ is bounded below, $\{x_n\}$ converges. Let $\lim x_n = x$, then

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) = \frac{x}{2} + \frac{\alpha}{2x} = \lim_{n\to\infty} x_n = x$$

Solving for x, we have

$$\lim_{n\to\infty} x_n = \sqrt{\alpha}$$

(b)

$$\varepsilon_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} = \frac{x_n^2 - 2x_n\sqrt{\alpha} + \alpha}{2x_n} = \frac{(x_n - \sqrt{\alpha})^2}{2x_n} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

Setting $\beta = 2\sqrt{\alpha}$, then $\varepsilon_{n+1} < \frac{\varepsilon_n^2}{\beta}$ for every n. Consider the inequality

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$$

It holds for $n = 1$:

$$\varepsilon_2 < \frac{\varepsilon_1^2}{\beta} = \beta\left(\frac{\varepsilon_1}{\beta}\right)^2$$

If it holds for $n = k$:

$$\varepsilon_{k+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^k}$$

then it also holds for $n = k + 1$:

$$\varepsilon_{k+2} < \frac{\varepsilon_{k+1}^2}{\beta} < \frac{1}{\beta}\left[\beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^k}\right]^2 = \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^{k+1}}$$

So the inequality holds for every n by induction.

(c) From $25 < 27,\ 5 < 3\sqrt{3},\ 20 < 12\sqrt{3},\ 10(2 - \sqrt{3}) < 2\sqrt{3}$, we have

$$\frac{\varepsilon_1}{\beta} = \frac{2 - \sqrt{3}}{2\sqrt{3}} < \frac{1}{10}$$

From $2\sqrt{3} = \sqrt{12} < \sqrt{16} = 4$, we have

$$\beta = 2\sqrt{3} < 4$$

Therefore, from the inequality in (b), we have

$$\varepsilon_5 < 4\left(\frac{1}{10}\right)^{2^4} = 4 \cdot 10^{-16}$$

$$\varepsilon_6 < 4\left(\frac{1}{10}\right)^{2^5} = 4 \cdot 10^{-32}$$

17. (a)(b)

$$x_{n+1} - \sqrt{\alpha} = \frac{\alpha + x_n}{1 + x_n} - \sqrt{\alpha} = -\frac{(\sqrt{\alpha} - 1)(x_n - \sqrt{\alpha})}{1 + x_n}$$

So if $x_n - \sqrt{\alpha} < 0$, $x_{n+1} - \sqrt{\alpha} > 0$, and if $x_n - \sqrt{\alpha} > 0$, $x_{n+1} - \sqrt{\alpha} < 0$. Because $x_1 > \sqrt{\alpha}$, we have $x_1, x_3, x_5, \cdots > \sqrt{\alpha}$, and $x_2, x_4, x_6, \cdots < \sqrt{\alpha}$.

$$x_{n+2} = \frac{\alpha + x_{n+1}}{1 + x_{n+1}} = \frac{\alpha + \frac{\alpha + x_n}{1 + x_n}}{1 + \frac{\alpha + x_n}{1 + x_n}} = \frac{(\alpha + 1)x_n + 2\alpha}{2x_n + \alpha + 1} = x_n + \frac{-2(x_n^2 - \alpha)}{2x_n + \alpha + 1}$$

For the odd terms, $x_n^2 - \alpha > 0$, so $x_{n+2} < x_n$, which is $x_1 > x_3 > x_5 > \cdots$.

For the even terms, $x_n^2 - \alpha < 0$, so $x_{n+2} > x_n$, which is $x_2 < x_4 < x_6 < \cdots$.

(c) The sequence of odd terms is monotonically decreasing and bounded below by $\sqrt{\alpha}$, so it converges. Let the limit be x:

$$\lim_{n \to \infty} x_{n+2} = \frac{(\alpha + 1)x + 2\alpha}{2x + \alpha + 1} = \lim_{n \to \infty} x_n = x$$

Solving for x, we have $x = \sqrt{\alpha}$. Similarly, the sequence of even terms being monotonically increasing and bounded above by $\sqrt{\alpha}$ implies convergence, and the limit is also $\sqrt{\alpha}$. Therefore, both the sequences of odd and even terms converging to $\sqrt{\alpha}$ implies that

$$\lim_{n \to \infty} x_n = \sqrt{\alpha}$$

(d) Let $\delta_n = |x_n - \sqrt{\alpha}|$, then

$$\delta_{n+1} = \frac{(\sqrt{\alpha} - 1)\delta_n}{1 + x_n} > \frac{(\sqrt{\alpha} - 1)\delta_n}{1 + x_1} = \gamma\delta_n$$

where $\gamma < 1$. By induction we have

$$\delta_{n+1} > \gamma^n \delta_1$$

Compared with the sequence in Exercise 16:

$$\varepsilon_{n+1} < \beta \left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$$

The sequence a^{2^n} for $a < 1$ converges faster than b^n for $b < 1$, so for $n > N$ where N is large enough, we have

$$\varepsilon_{n+1} < \beta \left(\frac{\varepsilon_1}{\beta}\right)^{2^n} < \gamma^n \delta_1 < \delta_{n+1}$$

which means the sequence in Exercise 16 converges faster than that in Exercise 17.

18. For $0 < a < 1$, we have

$$1 - a^p = (1 - a)(1 + a + \cdots + a^{p-1}) < (1 - a)(1 + 1 + \cdots + 1) = p(1 - a)$$

$x_1 > \sqrt[p]{\alpha}$. If $x_n > \sqrt[p]{\alpha}$, so $0 < \frac{\sqrt[p]{\alpha}}{x_n} < 1$, then

$$1 - \left(\frac{\sqrt[p]{\alpha}}{x_n}\right)^p < p\left(1 - \frac{\sqrt[p]{\alpha}}{x_n}\right)$$

$$\frac{x_n}{p} - \frac{\alpha}{p} x_n^{-p+1} < x_n - \sqrt[p]{\alpha}$$

$$x_{n+1} = \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1} > \sqrt[p]{\alpha}$$

So by induction, $x_n > \sqrt[p]{\alpha}$ for every n. From $x_n^p > \alpha$, or $x_n > \alpha x_n^{-p+1}$, we have

$$x_{n+1} = \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1} < \frac{p-1}{p} x_n + \frac{1}{p} x_n = x_n$$

So $\{x_n\}$ decreases monotonically. Along with the fact that $\{x_n\}$ is bounded below, $\{x_n\}$ converges. Let $\lim x_n = x$, then

$$\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \left(\frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1}\right) = \frac{p-1}{p} x + \frac{\alpha}{p} x^{-p+1} = \lim_{n \to \infty} x_n = x$$

solving for x, we have

$$\lim_{n \to \infty} x_n = \sqrt[p]{\alpha}$$

19. Let the Cantor set be denoted as P.

If $x \in P$, then $0 \le x \le 1$, and $3x$ satisfies either the two inequalities:

$$0 \le 3x \le 1$$

$$2 \le 3x \le 3$$

Let α_1 be 0 or 2, respectively, and let $\beta_1 = 3x - \alpha_1$. Then $0 \le \beta_1 \le 1$, and $3\beta_1$ satisfies either the two inequalities:

$$0 \le 3\beta_1 \le 1$$

$$2 \le 3\beta_1 \le 3$$

Let α_2 be 0 or 2, respectively, and let $\beta_2 = 3\beta_1 - \alpha_2$. Continue the process to obtain the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, where α_n is 0 or 2, and $0 \le \beta_n \le 1$ for every n. Then we have

$$x = \sum_{n=1}^{N} \frac{\alpha_n}{3} + \frac{\beta_N}{3}$$

Since $\frac{\beta_N}{3^N}$ can be arbitrarily small for large enough N, we have

$$x = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\alpha_n}{3^n} = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}$$

which means that $x \in X(a)$. Therefore, $P \subset X(a)$.

If $x \notin P$, then for some N we have $1 < 3\beta_n < 2$, so

$$x = \sum_{n=1}^{N} \frac{\alpha_n}{3^n} + \frac{\beta_N}{3^N}$$

$$\sum_{n=1}^{N} \frac{\alpha_n}{3^n} + \frac{1}{3^{N+1}} < x < \sum_{n=1}^{N} \frac{\alpha_n}{3^n} + \frac{2}{3^{N+1}}$$

so if $x = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}$, then α_{N+1} cannot be 0 or 2, which means $x \notin X(a)$. Therefore, $X(a) \subset P$.

From $P \subset X(a)$ and $X(a) \subset P$, we have $X(a) = P$.

20. For every $\varepsilon > 0$, let N_1 be the positive integer such that for $n, m \ge N_1$,

$$d(p_n, p_m) < \frac{\varepsilon}{2}$$

Let N_2 be the positive integer such that for $n_i \ge N_2$,

$$d(p_{n_i}, p) < \frac{\varepsilon}{2}$$

Let $N = \max(N_1, N_2)$. Then for $n \ge N$, choose $n_i \ge N$, and we have

$$d(p_n, p) \le d(p_n, p_{n_i}) + d(p_{n_i}, p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which implies $\{p_n\}$ converges to p.

21. Construct a sequence $\{x_n\}$ such that $x_n \in E_n$ for each n. Because $E_n \supset E_{n+1}$, E_n contains the points $x_n, x_{n+1}, x_{n+2}, \cdots$, and therefore $\lim \operatorname{diam} E_n = 0$ implies that $\{x_n\}$ is a Cauchy sequence. Since the metric space is complete, $\{x_n\}$ converges to a point x. Since x is a limit point of the set of $\{x_n\}$, it is a limit point of every E_n, and since E_n is closed, $x \in E_n$ for every n. Therefore,

$$x \in \bigcap_{n=1}^{\infty} E_n$$

If $E = \bigcap_1^\infty E_n$ consists of more that one point, then $\operatorname{diam} E > 0$. But for each n, $\operatorname{diam} E_n \ge \operatorname{diam} E$. This contradicts the assumption that $\operatorname{diam} E_n \to 0$. Therefore, $\bigcap_1^\infty E_n$ consists of exactly one point.

22. Find a neighborhood E_1 with diameter less than 1 such that $\overline{E_1} \subset G_1$ (it exists because G_1 is open). Since G_2 is dense, every neighborhood centered at any point contains a point of G_2, so $E_1 \cap G_2$ is nonempty, and because both E_1 and G_2 are open, $E_1 \cap G_2$ is open. Find a neighborhood E_2 with diameter less than $\frac{1}{2}$ such that $\overline{E_2} \subset E_1 \cap G_2$. Continue the process to construct a sequence of sets $\{\overline{E_n}\}$, where $\overline{E_n} \supset \overline{E_{n+1}}$, and $\overline{E_n} \subset G_n$. Every $\overline{E_n}$ is closed, and because $0 \leq \operatorname{diam} \overline{E_n} \leq \frac{1}{n}$,

$$\lim_{n \to \infty} \operatorname{diam} \overline{E_n} = 0$$

so by Exercise 21, there is a point x contained in every $\overline{E_n}$, so x is contained in every G_n, which means $\bigcap_1^\infty G_n$ is not empty.

23. $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences, so for every $\varepsilon > 0$, there is a positive integer N_1 such that $d(p_n, p_m) < \frac{\varepsilon}{2}$ for $n, m \geq N_1$, and a positive integer N_2 such that $d(q_n, q_m) < \frac{\varepsilon}{2}$ for $n, m \geq N_2$. Let $N = \max(N_1, N_2)$. For $n, m \geq N$, we have

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_n, q_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which implies $\{d(p_n, q_n)\}$ is a Cauchy sequence. Since every Cauchy sequence in R converges, $\{d(p_n, q_n)\}$ converges.

24. (a)

- *reflexive:*

$$\lim_{n \to \infty} d(p_n, p_n) = 0$$

  so $\{p_n\} \sim \{p_n\}$.

- *symmetric:* If $\{p_n\} \sim \{q_n\}$, so $\lim d(p_n, q_n) = 0$, then

$$\lim_{n \to \infty} d(q_n, p_n) = \lim_{n \to \infty} d(p_n, q_n) = 0$$

  so $\{q_n\} \sim \{p_n\}$.

- *transitive* If $\{p_n\} \sim \{q_n\}$ and $\{q_n\} \sim \{r_n\}$, so $\lim d(p_n, q_n) = \lim d(q_n, r_n) = 0$, then

$$d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n)$$

$$\lim_{n \to \infty} d(p_n, r_n) = 0$$

  so $\{p_n\} \sim \{r_n\}$.

(b) If $\{p_n\}, \{p'_n\} \in P$, and $\{q_n\}, \{q'_n\} \in Q$, then

$$d(p'_n, q'_n) \leq d(p'_n, p_n) + d(p_n, q_n) + d(q_n, q'_n)$$

$$|d(p'_n, q'_n) - d(p_n, q_n)| \leq d(p'_n, p_n) + d(q_n, q'_n)$$

Since $\lim d(p'_n, p_n) = \lim d(q_n, q'_n) = 0$, we have

$$\lim_{n \to \infty} d(p'_n, q'_n) = \lim_{n \to \infty} d(p_n, q_n)$$

So $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences.
Verify the three definition of distance function:

- *Definition (a)*

$$\Delta(P, Q) = \lim_{n \to \infty} d(p_n, q_n) \begin{cases} > 0 & \textit{if } P \neq Q \\ = 0 & \textit{if } P = Q \end{cases}$$

- *Definition (b)*

$$\Delta(P, Q) = \lim_{n \to \infty} d(p_n, q_n) = \lim_{n \to \infty} d(q_n, p_n) = \Delta(Q, P)$$

- *Definition (c)*

$$\Delta(P,Q) = \lim_{n\to\infty} d(p_n,q_n) \le \lim_{n\to\infty} d(p_n,r_n) + \lim_{n\to\infty} d(r_n,q_n) = \Delta(P,R) + \Delta(R,Q)$$

So Δ is a distance function in X^*.

(c) Let $\{P_n\}$ be a Cauchy sequence in X^*, so every P_n is an equivalence class. Let $p_n = \{p_{n,m}\}$ be an element of P_n. so p_n is a Cauchy sequence in X. By the definition of Cauchy sequence, there is a positive integer N_n such that for $s,t \ge N_n$,

$$d(p_{n,s}, p_{n,t}) < \frac{1}{2^n}$$

Let q be a sequence in X such that $q = \{q_n\} = \{p_{n,N_n}\}$.

For $\varepsilon > 0$, there is a positive integer M_1 such that for $m,n \ge M_1$,

$$\Delta(P_m,P_n) < \frac{\varepsilon}{6}$$

There is a positive integer M_2 such that

$$\frac{1}{2^{M_2}} < \frac{\varepsilon}{3}$$

Let $M = \max(M_1,M_2)$. For $m,n \ge M$, since

$$\Delta(P_m,P_n) = \lim_{k\to\infty} d(p_{m,k},p_{n,k})$$

there is a positive integer N_0 such that for $k \ge N_0$,

$$|d(p_{m,k},p_{n,k}) - \Delta(P_m,P_n)| < \frac{\varepsilon}{6}$$

$$d(p_{m,k},p_{n,k}) < \frac{\varepsilon}{6} + \Delta(P_m,P_n) < \frac{\varepsilon}{3}$$

Let $N = \max(N_0,N_m,N_n)$. Let k be a positive integer such that $k \ge N$. Then

$$\begin{aligned} d(q_m,q_n) &= d(p_{m,N_m},p_{n,N_n}) \\ &\le d(p_{m,N_m},p_{m,k}) + d(p_{m,k},p_{n,k}) + d(p_{n,k},p_{n,N_n}) \\ &< \frac{1}{2^m} + \frac{\varepsilon}{3} + \frac{1}{2^n} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

Therefore, for $\varepsilon > 0$, there is a positive integer M such that for $m,n \ge M$, $d(q_m,q_n) < \varepsilon$, which means that $q = \{q_n\}$ is a Cauchy sequence. Let $Q \in X^*$ be the equivalence class containing q.

$$\lim_{n\to\infty} \Delta(P_n,Q) = \lim_{n\to\infty}\lim_{m\to\infty} d(p_{n,m},q_n)$$

For $m \ge N_n$, $d(p_{n,m},q_n) < \frac{1}{2^n}$, so

$$\Delta(P_n,Q) = \lim_{m\to\infty} d(p_{n,m},q_n) < \frac{1}{2^n}$$

Because $\lim_{n\to\infty} \frac{1}{2^n} = 0$,

$$\lim_{n\to\infty} \Delta(P_n,Q) = 0$$

which means that $\{P_n\}$ converges to Q. So every Cauchy sequence in X^* converges, which means that X^* is complete.

(d)

$$\Delta(P_p,P_q) = \lim_{n\to\infty} d(p,q) = d(p,q)$$

(e) Let P be an element of X^*, which contains a Cauchy sequence $p = \{p_n\}$. For $\varepsilon > 0$, there is a positive integer N such that for $n, m \geq N$,

$$d(p_n, p_m) < \varepsilon$$

Consider $p_N \in X$, and $\varphi(p_N) \in \varphi(X)$.

$$\Delta(P, \varphi(p_N)) = \lim_{n \to \infty} d(p_n, p_N) < \varepsilon$$

which means $\varphi(p_N)$ is contained in the neighborhood of P with radius ε. Therefore, every neighborhood centered at any point of X^* contains an element of $\varphi(X)$, which means $\varphi(X)$ is dense in X^*.

If X is complete, then $\{p_n\}$ converges. Let $\lim_{n \to \infty} p_n = a$, then

$$\Delta(P, \varphi(a)) = \lim_{n \to \infty} d(p_n, a) = 0$$

so $P = \varphi(a) \in \varphi(X)$. Therefore, $X^* \subset \varphi(X)$, and obviously $\varphi(X) \subset X^*$, so $\varphi(X) = X^*$.

25. The completion of X must be a complete metric space, and contains a dense subset isometric with X. If X is the rational numbers with the metric $d(x, y) = |x - y|$, then the real numbers with the same metric satisfies the conditions. So the completion of Q is R.