Chapter 10
Green's Functions

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10.1 One-Dimensional Problems

10.1.1 The solution given by the Green's function is

$$y(x) = \int_0^x t \cdot f(t)\,dt + x \int_x^1 f(t)\,dt$$

then

$$\begin{aligned}
y'(x) &= xf(x) + \int_x^1 f(t)\,dt - xf(x) = \int_x^1 f(t)\,dt \\
y''(x) &= -f(x) \\
y(0) &= \int_0^0 t \cdot f(t)\,dt + 0 \int_x^1 f(t)\,dt = 0 \\
y'(1) &= \int_1^1 f(t)\,dt = 0
\end{aligned}$$

so the equation $\mathcal{L}y = -y''(x) = f(x)$ is satisfied, and the boundary conditions $y(0) = 0$ and $y'(1) = 0$ are also satisfied.

10.1.2 (a) $\sin x$ satisfies the homogeneous equation and $y(0) = 0$, and $\cos(x-1)$ satisfies the homogeneous equation and $y'(1) = 0$, so the Green's function has the form

$$G(x,t) = \begin{cases} h_1(t)\sin x, & 0 \le x < t \\ h_2(t)\cos(x-1), & t < x \le 1 \end{cases}$$

Using the general properties of Green's function:

$$\begin{aligned}
G(t_+,t) &= G(t_-,t) && \longrightarrow \quad h_2(t)\cos(t-1) = h_1(t)\sin t \\
\frac{\partial G}{\partial x}(t_+,t) - \frac{\partial G}{\partial x}(t_-,t) &= \frac{1}{p(t)} && \longrightarrow \quad -h_2(t)\sin(t-1) - h_1(t)\cos t = 1
\end{aligned}$$

so $h_1(t) = -\frac{\cos(t-1)}{\cos(1)}$, and $h_2(t) = -\frac{\sin t}{\cos(1)}$. Therefore the Green's function is

$$G(x,t) = \begin{cases} -\frac{\sin x \cos(t-1)}{\cos(1)}, & 0 \le x < t \\ -\frac{\cos(x-1)\sin t}{\cos(1)}, & t < x \le 1 \end{cases}$$

(b) e^x satisfies the homogeneous equation and $y(-\infty) = 0$, and e^{-x} satisfies the homogeneous equation and $y(\infty) = 0$, so the Green's function has the form

$$G(x,t) = \begin{cases} e^x h_1(t), & -\infty < x < t \\ e^{-x} h_2(t), & t < x < \infty \end{cases}$$

Using the general properties of Green's function:

$$
\begin{aligned}
&G(t_+,t) = G(t_-,t) && \longrightarrow \quad e^{-t}h_2(t) = e^t h_1(t) \\
&\frac{\partial G}{\partial x}(t_+,t) - \frac{\partial G}{\partial x}(t_-,t) = \frac{1}{p(t)} && \longrightarrow \quad -e^{-t}h_2(t) - e^t h_1(t) = 1
\end{aligned}
$$

so $h_1(t) = -\frac{e^{-t}}{2}$, and $h_2(t) = -\frac{e^t}{2}$. Therefore the Green's function is

$$
G(x,t) = \begin{cases} -\frac{e^{x-t}}{2}, & -\infty < x < t \\ -\frac{e^{t-x}}{2}, & t < x < \infty \end{cases}
$$

10.1.3 The solution is

$$
y(x) = \int_0^x \sin(x-t) f(t) dt
$$

By Leibniz integral rule,

$$
\begin{aligned}
y'(x) &= \sin(x-x)f(x) + \int_0^x \frac{\partial \sin(x-t)}{\partial x} f(t)\,dt = \int_0^x \cos(x-t)f(t)\,dt \\
y''(x) &= \cos(x-x)f(x) + \int_0^x \frac{\partial \cos(x-t)}{\partial x} f(t)\,dt = f(x) - y(x) \\
y(0) &= \int_0^0 \sin(x-t)f(t)\,dt = 0 \\
y'(0) &= \int_0^0 \cos(x-t)f(t)\,dt = 0
\end{aligned}
$$

so the equation $y'' + y = f(x)$ is satisfied, and the initial conditions $y(0) = y'(0) = 0$ are also satisfied.

10.1.4 $\sin(\frac{x}{2})$ satisfies the homogeneous equation and $y(0) = 0$, and $\cos(\frac{x}{2})$ satisfies the homogeneous equation and $y(\pi) = 0$, so the Green's function has the form

$$
G(x,t) = \begin{cases} h_1(t)\sin(\frac{x}{2}), & 0 \le x < t \\ h_2(t)\cos(\frac{x}{2}), & t < x \le \pi \end{cases}
$$

Using the general properties of Green's function:

$$
\begin{aligned}
&G(t_+,t) = G(t_-,t) && \longrightarrow \quad h_2(t)\cos(\frac{t}{2}) = h_1(t)\sin(\frac{t}{2}) \\
&\frac{\partial G}{\partial x}(t_+,t) - \frac{\partial G}{\partial x}(t_-,t) = \frac{1}{p(t)} && \longrightarrow \quad -\frac{1}{2}h_2(t)\sin(\frac{t}{2}) - \frac{1}{2}h_1(t)\cos(\frac{t}{2}) = -1
\end{aligned}
$$

so $h_1(t) = 2\cos(\frac{t}{2})$, and $h_2(t) = 2\sin(\frac{t}{2})$. Therefore the Green's function is

$$
G(x,t) = \begin{cases} 2\sin(\frac{x}{2})\cos(\frac{t}{2}), & 0 \le x < t \\ 2\cos(\frac{x}{2})\sin(\frac{t}{2}), & t < x \le \pi \end{cases}
$$

10.1.5 Let $u = kx$, then the equation becomes Bessel's equation with order $n = 1$, so the solution is $J_1(kx)$ and $Y_1(kx)$. $J_1(kx)$ satisfies $y(0) = 0$, and $Y_1(k)J_1(kx) - J_1(k)Y_1(kx)$ satisfies $y(1) = 0$. To find the Green's function, we must put it into the self-sdjoint form:

$$
x\frac{d^2y}{dx^2} + \frac{dy}{dx} + (k^2x - \frac{1}{x})y = \frac{f(x)}{x}
$$

Then the Green's function has the form (we use g instead of G to remind that it is the Green's function of the self-adjoint equation, not the original equation)

$$
g(x,t) = \begin{cases} h_1(t)J_1(kx), & 0 \le x < t \\ h_2(t)\left[Y_1(k)J_1(kx) - J_1(k)Y_1(kx)\right], & t < x \le 1 \end{cases}
$$

Using the general properties of Green's function:

$$\begin{aligned}
&G(t_+,t) = G(t_-,t) &&\longrightarrow \quad h_2(t)\left[Y_1(k)J_(kt) - J_1(k)Y_1(kt)\right] = h_1(t)J_1(kt)\\
&\frac{\partial G}{\partial x}(t_+,t) - \frac{\partial G}{\partial x}(t_-,t) = \frac{1}{p(t)} &&\longrightarrow \quad h_2(t)\left[Y_1(k)J_1'(kt) - J_1(k)Y_1'(kt)\right] - h_1(t)J_1'(kt) = \frac{1}{t}
\end{aligned}$$

so $h_1(t) = -\frac{\pi}{2J_1(k)}\left[Y_1(k)J_1(kt) - J_1(k)Y_1(kt)\right]$, and $h_2(t) = -\frac{\pi}{2J_1(k)}J_1(kt)$. Therefore the Green's function is

$$g(x,t) = \begin{cases} -\frac{\pi}{2J_1(k)}J_1(kx)\left[Y_1(k)J_1(kt) - J_1(k)Y_1(kt)\right], & 0 \le x < t\\ -\frac{\pi}{2J_1(k)}\left[Y_1(k)J_1(kx) - J_1(k)Y_1(kx)\right]J_1(kt), & t < x \le 1\end{cases}$$

The solution is given by $y(x) = \int_0^1 g(x,t)\frac{f(x)}{x}dx$, which means the Green's function of the original equagtion is

$$G(x,t) = \frac{g(x,t)}{x} = \begin{cases} \frac{\pi}{2x}J_1(kx)\left[Y_1(kt) - \frac{Y_1(k)}{J_1(k)}J_1(kt)\right], & 0 \le x < t\\ \frac{\pi}{2x}\left[Y_1(kx) - \frac{Y_1(k)}{J_1(k)}J_1(kx)\right]J_1(kt), & t < x \le 1\end{cases}$$

10.1.6 The equation is Legendre's differential equation of order $n = 0$, so the solution is $P_0(x) = 1$ and $Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$. $Q_0(x)$ is infinite at $x = \pm 1$, so the function for $x < t$ and $x > t$ that satisfies the boundary conditions will both be a multiple of $P_0(x)$, which results in the absence of the discontinuity in $\left.\frac{dG(x,t)}{dx}\right|_{x=t}$. So no Green's function can be constructed.

10.1.7 The solution of the homogeneous equation has the form $c_1e^{-kt} + c_2$. The only solution that satisfies $\psi(0) = \psi'(0) = 0$ is the trivial solution $\psi(t) = 0$, while there is no other boundary condition. To find the Green's function, we must put the equation into self-adjoint form:

$$e^{kt}\frac{d^2\psi}{dt^2} + ke^{kt}\frac{d\psi}{dt} = e^{kt}f(t)$$

Then the Green's function has the form

$$g(t,u) = \begin{cases} 0, & 0 \le t < u\\ h_1(u)e^{-kt} + h_2(u), & u < t\end{cases}$$

Using the general properties of Green's function:

$$\begin{aligned}
&G(u_+,u) = G(u_-,u) &&\longrightarrow \quad h_1(u)e^{-kt} + h_2(u) = 0\\
&\frac{\partial G}{\partial t}(u_+,u) - \frac{\partial G}{\partial t}(u_-,u) = \frac{1}{p(u)} &&\longrightarrow \quad -kh_1(u)e^{-ku} = e^{-ku}
\end{aligned}$$

so $h_1(u) = -\frac{1}{k}$, and $h_2(u) = \frac{e^{-ku}}{k}$. Therefore the Green's function of the self-adjoint equation is

$$g(t,u) = \begin{cases} 0, & 0 \le t < u\\ \frac{1}{k}\left(-e^{-kt} + e^{-ku}\right), & u < t\end{cases}$$

and the Green's function of the original equation is

$$G(t,u) = g(t,u)e^{ku} = \begin{cases} 0, & 0 \le t < u\\ \frac{1}{k}\left(1 - e^{k(u-t)}\right), & u < t\end{cases}$$

If $f(t) = e^{-t}$, then the solution is given by

$$y(t) = \int_0^t \frac{1}{k}\left(1 - e^{k(u-t)}\right)e^{-u}du = \frac{(k-1) - ke^{-t} + e^{-kt}}{k(k-1)}$$

10.1.8 From the Green's function, the solution is given by

$$\psi(x) = \int_{-\infty}^{x} -\frac{i}{2k} e^{ik(x-x')} g(x')dx' + \int_{x}^{\infty} -\frac{i}{2k} e^{ik(x'-x)} g(x')dx'$$

Using the Leibniz integral rule:

$$\begin{aligned}
\frac{d\psi(x)}{dx} &= -\frac{i}{2k} e^{ik(x-x)} g(x) + \int_{-\infty}^{x} -\frac{i}{2k}(ik) e^{ik(x-x')} g(x')dx' + \frac{i}{2k} e^{ik(x-x)} g(x) + \int_{x}^{\infty} -\frac{i}{2k}(-ik) e^{ik(x'-x)} g(x')dx' \\
&= \int_{-\infty}^{x} \frac{1}{2} e^{ik(x-x')} g(x')dx' + \int_{x}^{\infty} -\frac{1}{2} e^{ik(x'-x)} g(x')dx' \\
\frac{d^2\psi(x)}{dx^2} &= \frac{1}{2} e^{ik(x-x)} g(x) + \int_{-\infty}^{x} \frac{1}{2}(ik) e^{ik(x-x')} g(x')dx' + \frac{1}{2} e^{ik(x-x)} g(x) + \int_{x}^{\infty} -\frac{1}{2}(-ik) e^{ik(x'-x)} g(x')dx' \\
&= g(x) - k^2\psi(x)
\end{aligned}$$

so the solution satisfies the equation $\frac{d^2\psi}{dx^2} + k^2\psi = g(x)$

10.1.9 e^{kx} satisfies the homogeneous equation and $y(-\infty) = 0$, and e^{-kx} satisfies the homogeneous equation and $y(\infty) = 0$, so the Green's function has the form

$$G(x,t) = \begin{cases} h_1(t)e^{kx}, & -\infty < x < t \\ h_2(t)e^{-kx}, & t < x < \infty \end{cases}$$

Using the general properties of Green's function:

$$\begin{aligned}
&G(t_+,t) = G(t_-,t) && \longrightarrow \quad h_2(t)e^{-kt} = h_1(t)e^{kt} \\
&\frac{\partial G}{\partial x}(t_+,t) - \frac{\partial G}{\partial x}(t_-,t) = \frac{1}{p(t)} && \longrightarrow \quad -kh_2(t)e^{-kt} - kh_1(t)e^{kt} = 1
\end{aligned}$$

so $h_1(t) = -\frac{1}{2k}e^{-kt}$, and $h_2(t) = -\frac{1}{2k}e^{kt}$. Therefore the Green's function is

$$G(x,t) = \begin{cases} -\frac{1}{2k}e^{k(x-t)}, & -\infty < x < t \\ -\frac{1}{2k}e^{k(t-x)}, & t < x < \infty \end{cases} = -\frac{1}{2k}e^{-k|x-t|}$$

10.1.10 (a) From Example 10.1.1,

$$G(x,t) = \begin{cases} x(1-t), & 0 \le x < t \\ t(1-x), & t < x \le 1 \end{cases}$$

is the Green's function of the equation $-y'' = f(x)$, with boundary conditions $y(0) = y(1) = 0$. The operator $\mathcal{L} = -\frac{d^2}{dx^2}$ and boundary conditions $y(0) = y(1) = 0$ have the orthonormal eigenfunctions $\varphi_n = \sqrt{2}\sin n\pi x$ with eigenvalues $\lambda_n = n^2\pi^2$, so by Equation 10.14, the Green's function is given by

$$G(x,t) = \sum_n \frac{\varphi_n^*(t)\varphi_n(x)}{\lambda_n} = \sum_{n=1}^{\infty} \frac{2\sin n\pi x\, \sin n\pi t}{n^2\pi^2}$$

(b) From Exercise 10.1.1,

$$G(x,t) = \begin{cases} x, & 0 \le x < t \\ t, & t < x \le 1 \end{cases}$$

is the Green's function of the equation $-y'' = f(x)$, with boundary conditions $y(0) = 0$, $y'(1) = 0$. The operator $\mathcal{L} = -\frac{d^2}{dx^2}$ and boundary conditions $y(0) = 0$, $y'(1) = 0$ have the orthonormal eigenfunctions $\varphi_n = \sqrt{2}\sin(n+\frac{1}{2})\pi x$ with eigenvalues $\lambda_n = (n+\frac{1}{2})^2\pi^2$, so by Equation 10.14, the Green's function is given by

$$G(x,t) = \sum_n \frac{\varphi_n^*(t)\varphi_n(x)}{\lambda_n} = \sum_{n=1}^{\infty} \frac{2\sin(n+\frac{1}{2})\pi x\, \sin(n+\frac{1}{2})\pi t}{(n+\frac{1}{2})^2\pi^2}$$

10.1.11 (a)

$$y''(x) = y(x)$$
$$y'(x) - y'(-1) = \int_{-1}^{x} y(t)dt$$
$$y'(x) = c + \int_{-1}^{x} y(t)dt$$
$$y(x) - y(-1) = c\int_{-1}^{x} dx + \int_{-1}^{x} ds \int_{-1}^{s} y(t)dt$$
$$= c(x+1) + \int_{-1}^{x} y(t)dt \int_{t}^{x} ds$$
$$= c(x+1) + \int_{-1}^{x} y(t)(x-t)dt$$

where we change the order of integration in the last three equation (the area to be integrated is an upper triangle in the t-s surface). Substitute $y(1) = 1$ and $y(-1) = 1$:

$$y(1) - y(-1) = 2c + \int_{-1}^{1} y(t)(1-t)dt = 0$$

$$c = -\frac{1}{2}\int_{-1}^{1} y(t)(1-t)dt$$

so

$$y(x) = y(-1) + c(x+1) + \int_{-1}^{x} y(t)(x-t)dt$$
$$= 1 - \frac{1}{2}\int_{-1}^{1}(x+1)y(t)(1-t)dt + \int_{-1}^{x} y(t)(x-t)dt$$
$$= 1 - \int_{-1}^{x}\frac{1}{2}(1-x)(t+1)y(t)dt - \int_{x}^{1}\frac{1}{2}(1-t)(x+1)y(t)dt$$
$$= 1 - \int_{-1}^{1} K(x,t)\,y(t)\,dt$$

where

$$K(x,t) = \begin{cases} \frac{1}{2}(1-x)(t+1), & x > t \\ \frac{1}{2}(1-t)(x+1), & x < t \end{cases}$$

(b) Let $u(x) = y(x) - 1$, then the equation becomes $u''(x) = u(x) + 1$, and the boundary conditions becomes $u(1) = u(-1) = 0$. $u(x) = x + 1$ satisfies the homogeneous equation and $u(-1) = 0$, and $u(x) = x - 1$ satisfies the homogeneous equation and $u(1) = 0$. So the Green's function has the form

$$G(x,t) = \begin{cases} h_1(t)(x+1), & x < t \\ h_2(t)(x-1), & x > t \end{cases}$$

Using the general properties of Green's function:

$$\begin{aligned} &G(t_+,t) = G(t_-,t) && \longrightarrow \quad h_2(t)(t-1) = h_1(t)(t+1) \\ &\frac{\partial G}{\partial x}(t_+,t) - \frac{\partial G}{\partial x}(t_-,t) = \frac{1}{p(t)} && \longrightarrow \quad h_2(t) - h_1(t) = 1 \end{aligned}$$

so $h_1(t) = \frac{1}{2}(t-1)$, and $h_2(t) = \frac{1}{2}(t+1)$. Therefore the Green's function is

$$G(x,t) = \begin{cases} \frac{1}{2}(x+1)(t-1), & x < t \\ \frac{1}{2}(x-1)(t+1), & x > t \end{cases}$$

To match the notation with the book, define

$$K(x,t) = -G(x,t) = \begin{cases} \frac{1}{2}(1-x)(t+1), & x > t \\ \frac{1}{2}(1-t)(x+1), & x < t \end{cases}$$

then the solution (integral equation) is given by

$$y(x) = 1 + u(x) = 1 + \int_{-1}^{1} G(x,t)\,[u(t)+1]\,dt = 1 - \int_{-1}^{1} K(x,t)\,y(t)\,dt$$

10.1.12

$$y'(x) - y'(0) + a_1 y(x) - a_1 y(0) + a_2 \int_0^x y(t)dt = 0$$

$$y(x) - y(0) - y'(0)(x-0) + a_1 \int_0^x y(t)dt - a_1 y(0)(x-0) + a_2 \int_0^x ds \int_0^s y(t)dt = 0$$

$$y(x) - y_0' x + a_1 \int_0^x y(t)dt + a_2 \int_0^x y(t)(x-t)dt = 0$$

Substitute $y(1) = 0$:

$$-y_0' + a_1 \int_0^1 y(t)dt + a_2 \int_0^1 y(t)(1-t)dt = 0$$

$$y_0' = a_1 \int_0^1 y(t)dt + a_2 \int_0^1 y(t)(1-t)dt$$

$$y(x) = y_0' x - a_1 \int_0^x y(t)dt - a_2 \int_0^x y(t)(x-t)dt$$

$$= a_1 \int_0^1 xy(t)dt + a_2 \int_0^1 x(1-t)y(t)dt - a_1 \int_0^x y(t)dt - a_2 \int_0^x (x-t)y(t)dt$$

$$= \int_0^x \big[a_2 t(1-x) + a_1(x-1)\big] y(t)dt + \int_x^1 \big[a_2 x(1-t) + a_1 x\big] y(t)dt$$

$$= \int_0^1 K(x,t)y(t)dt$$

where

$$K(x,t) = \begin{cases} a_2 t(1-x) + a_1(x-1), & t < x \\ a_2 x(1-t) + a_1 x, & x < t \end{cases}$$

If $a_1 = 0$, then the equation is self-adjoint, so $K(x,t)$ is the Green's function of the equation, and therefore has the properties of Green's function (symmetry, continuity, etc.)

10.1.13 Regard $V_0 \frac{e^{-r}}{r} y(r)$ as the inhomogeneous term:

$$\frac{d^2 y(r)}{dr^2} - k^2 y(r) = -V_0 \frac{e^{-r}}{r} y(r)$$

The solution of the homogeneous equation has the form $c_1 e^{kr} + c_2 e^{-kr}$. $\sinh kr = \frac{1}{2}(e^{kr} - e^{-kr})$ satisfies the homogeneous equation and y(0)=0, and e^{-kr} satisfies the homogeneous equation and $y(\infty) = 0$. So the Green's function has the form

$$G(r,t) = \begin{cases} h_1(t)\sinh kr, & 0 \le r < t \\ h_2(t) e^{-kr}, & t < r < \infty \end{cases}$$

Using the general properties of Green's function:

$$\begin{aligned} &G(t_+, t) = G(t_-, t) && \longrightarrow \quad h_2(t)e^{-kt} = h_1(t)\sinh ht \\ &\frac{\partial G}{\partial r}(t_+, t) - \frac{\partial G}{\partial r}(t_-, t) = \frac{1}{p(t)} && \longrightarrow \quad -kh_2(t)e^{-kt} - kh_1(t)\cosh kt = 1 \end{aligned}$$

so $h_1(t) = -\frac{1}{k}e^{-kt}$, and $h_2(t) = -\frac{1}{k}\sinh kt$. Therefore the Green's function is

$$G(r,t) = \begin{cases} -\frac{1}{k}e^{-kt}\sinh kr, & 0 \le r < t \\ -\frac{1}{k}e^{-kr}\sinh kt, & t < r < \infty \end{cases}$$

and the solution (integral equation) is given by

$$y(r) = \int_0^\infty G(r,t)f(t)dt = -V_0\int_0^\infty G(r,t)\frac{e^{-t}}{t}y(t)\,dt$$

10.2 Problems in Two and Three Dimensions

10.2.1

$$\mathcal{L}\int_a^b \Big[G(x_1,x_2)+\varphi(x_1)\Big]f(x_2)dx_2 = \int_a^b \Big[\mathcal{L}G(x_1,x_2)+\mathcal{L}\varphi(x_1)\Big]f(x_2)dx_2 = \mathcal{L}\int_a^b G(x_1,x_2)f(x_2)dx_2$$

if $\mathcal{L}\varphi(x_1) = 0$. That means the Green's function will still give the correct solution if added a solution of the homogeneous equation. $\frac{1}{2}|x_1 - x_2|$ is the Green's function of Laplace equation, and $-\frac{1}{2}x_1 - \frac{1}{2}x_2$ is a solution of the homogeneous solution, so

$$\frac{1}{2}|x_1 - x_2| - \frac{1}{2}x_1 - \frac{1}{2}x_2 = \begin{cases} -x_1, & 0 \le x_1 < x_2 \\ -x_2, & x_2 < x_1 \le 1 \end{cases}$$

is also a Green's function of Laplace equation, which is consistent with the one found in Example 10.1.1 (negative signs arise because the operator is defined as $\mathcal{L} = -\frac{d^2}{dx^2}$ in Example 10.1.1).

10.2.2

$$\mathcal{L}\psi(\mathbf{r}) = \boldsymbol{\nabla}\cdot\big[p(\mathbf{r})\boldsymbol{\nabla}\psi(\mathbf{r})\big] + q(\mathbf{r})\psi(\mathbf{r})$$

$$\langle\chi|\mathcal{L}\psi\rangle = \int\limits_V \chi^*\mathcal{L}\psi\,d\tau = \int\limits_V \chi^*\boldsymbol{\nabla}\cdot\big[p\boldsymbol{\nabla}\psi\big]\,d\tau + \int\limits_V \chi^* q\psi\,d\tau$$

$$= \int\limits_V \boldsymbol{\nabla}\cdot(\chi^* p\boldsymbol{\nabla}\psi)\,d\tau - \int\limits_V (\boldsymbol{\nabla}\chi^*)\cdot(p\boldsymbol{\nabla}\psi)\,d\tau + \int\limits_V \chi^* q\psi\,d\tau$$

$$= \oint\limits_A \chi^* p(\boldsymbol{\nabla}\psi)\cdot d\boldsymbol{\sigma} - \int\limits_V \boldsymbol{\nabla}\cdot(\psi p\boldsymbol{\nabla}\chi^*)\,d\tau + \int\limits_V \psi\boldsymbol{\nabla}\cdot(p\boldsymbol{\nabla}\chi^*)\,d\tau + \int\limits_V \chi^* q\psi\,d\tau$$

$$= \oint\limits_A \chi^* p(\boldsymbol{\nabla}\psi)\cdot d\boldsymbol{\sigma} - \int\limits_A \psi p(\boldsymbol{\nabla}\chi^*)\cdot d\boldsymbol{\sigma} + \int\limits_V \psi\big[\boldsymbol{\nabla}\cdot(p\boldsymbol{\nabla}\chi^*) + q\chi^*\big]\,d\tau$$

$$= \oint\limits_A (\chi^* p\boldsymbol{\nabla}\psi - \psi p\boldsymbol{\nabla}\chi^*)\cdot d\boldsymbol{\sigma} + \int\limits_V \psi\mathcal{L}\chi^*\,d\tau$$

$$= \langle\mathcal{L}\chi|\psi\rangle$$

which implies $\mathcal{L}$ is Hermitian.
(The surface integral vanishes by the Dirichlet boundary conditions.)
($\boldsymbol{\nabla}\cdot(f\mathbf{V}) = (\boldsymbol{\nabla}f)\cdot\mathbf{V} + f\boldsymbol{\nabla}\cdot\mathbf{V}$ and $\int_V \boldsymbol{\nabla}\cdot\mathbf{V}\,d\tau = \oint_A \mathbf{V}\cdot d\boldsymbol{\sigma}$ have been used several times.)

10.2.3 (The space to be integrated given in the book is not quite clear, and the result we get is different from the book.)

$$\lim_{|\mathbf{r}_1-\mathbf{r}_2|\to 0}\int k^2 G(\mathbf{r}_1,\mathbf{r}_2)d^3r_2$$

$$= \lim_{a\to 0}\int\limits_{|\mathbf{r}_1-\mathbf{r}_2|<a} -k^2\,\frac{e^{ik|\mathbf{r}_1-\mathbf{r}_2|}}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}\,d^3r_2$$

$$= -\lim_{a\to 0}\int_0^a\int_0^\pi\int_0^{2\pi} k^2\,\frac{e^{ikr}}{4\pi r}\,r^2\sin\theta\,d\theta\,d\varphi$$

$$= -\lim_{a\to 0}k^2\int_0^a re^{ikr}dr = \lim_{a\to 0}(ika\,e^{ika} - e^{ika} + 1) = 0$$

Substitute k with ik, the case becomes modified Helmholtz operator, and the result is the same.

10.2.4 For

$$G(\mathbf{r}_1,\mathbf{r}_2) = -\frac{e^{ik|\mathbf{r}_1-\mathbf{r}_2|}}{4\pi|\mathbf{r}_1-\mathbf{r}_2|} = -\frac{e^{ikr_{12}}}{4\pi r_{12}}$$

We want to show that $(\boldsymbol{\nabla}^2 + k^2)G(\mathbf{r}_1,\mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2)$.

For $\mathbf{r}_1 \neq \mathbf{r}_2$,

$$(\boldsymbol{\nabla}^2 + k^2)G(\mathbf{r_1},\mathbf{r}_2) = -\boldsymbol{\nabla}^2\frac{e^{ikr}}{4\pi r} - k^2\frac{e^{ikr}}{4\pi r}$$

$$= -\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}(\frac{e^{ikr}}{4\pi r})\right] - \frac{k^2e^{ikr}}{4\pi r}$$

$$= -\frac{1}{r^2}\frac{d}{dr}\left[\frac{ikr\,e^{ikr}}{4\pi} - \frac{e^{ikr}}{4\pi}\right] - \frac{k^2e^{ikr}}{4\pi r}$$

$$= -\frac{ike^{ikr}}{4\pi r^2} + \frac{k^2e^{ikr}}{4\pi r} + \frac{ike^{ikr}}{4\pi r^2} - \frac{k^2e^{ikr}}{4\pi r} = 0$$

For $\mathbf{r}_1 = \mathbf{r}_2$, the function diverges, but for every $a > 0$,

$$\int\limits_{r_{12}<a}(\boldsymbol{\nabla}^2 + k^2)G(\mathbf{r}_1,\mathbf{r}_2)d^3r_1$$

$$= -\int\limits_{r_{12}<a}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\frac{e^{ikr_{12}}}{4\pi r_{12}}d^3r_{12} - \int\limits_{r_{12}<a}k^2\frac{e^{ikr_{12}}}{4\pi r_{12}}d^3r_{12}$$

$$= -\oint\limits_{r_{12}=a}\boldsymbol{\nabla}\frac{e^{ikr_{12}}}{4\pi r_{12}}\cdot d\boldsymbol{\sigma}_{12} - \int_0^a\int_0^\pi\int_0^{2\pi}k^2\frac{e^{ikr}}{4\pi r}r^2\sin\theta\,d\theta\,d\varphi$$

$$= -\oint\limits_{r_{12}=a}\left(\frac{ike^{ikr_{12}}}{4\pi r_{12}} - \frac{e^{ikr_{12}}}{4\pi r_{12}^2}\right)\hat{\mathbf{r}}\cdot d\boldsymbol{\sigma}_{12} - k^2\int_0^a re^{ikr}dr$$

$$= -\left(\frac{ike^{ika}}{4\pi a} - \frac{e^{ika}}{4\pi a^2}\right)4\pi a^2 - k^2\left(\frac{ae^{ika}}{ik} + \frac{e^{ika}-1}{k^2}\right)$$

$$= -ika\,e^{ika} + e^{ika} + ika\,e^{ika} - e^{ika} + 1 = 1$$

Therefore, it must be

$$(\boldsymbol{\nabla}^2 + k^2)G(\mathbf{r_1},\mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2)$$

10.2.5

$$-\frac{e^{ik|\mathbf{r}_1-\mathbf{r}_2|}}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}$$

is the fundamental Green's function of the Helmholtz equation, and

$$\frac{i\sin k|\mathbf{r}_1-\mathbf{r}_2|}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}$$

is a solution of the Helmholtz equation, so

$$-\frac{e^{ik|\mathbf{r}_1-\mathbf{r}_2|}}{4\pi|\mathbf{r}_1-\mathbf{r}_2|} + \frac{i\sin k|\mathbf{r}_1-\mathbf{r}_2|}{4\pi|\mathbf{r}_1-\mathbf{r}_2|} = \frac{-\cos k|\mathbf{r}_1-\mathbf{r}_2|}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}$$

is also a Green's function of the Helmholtz equation, and the asymptotic r dependence is $\cos kr$, which is a standing wave.

10.2.6 In Exercise 10.2.4, substitute k with ik, then the Helmholtz equation $(\boldsymbol{\nabla}^2 + k^2)\psi = 0$ becomes $(\boldsymbol{\nabla}^2 - k^2)\psi = 0$, which is the modified Helmholtz equation, and the Green's function becomes

$$G(\mathbf{r}_1, \mathbf{r}_2) = -\frac{e^{-k|\mathbf{r}_1-\mathbf{r}_2|}}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|}$$

which is the fundamental Green's function of the modified Helmholtz equation, and

$$\lim_{|\mathbf{r}_1-\mathbf{r}_2|\to\infty} -\frac{e^{-k|\mathbf{r}_1-\mathbf{r}_2|}}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|} = 0$$

10.2.7 Using the Poisson's equation for electrostatics:

$$\boldsymbol{\nabla}^2\varphi = -\frac{\rho}{\varepsilon_0}$$

For $\mathbf{r} \neq 0$,

$$\begin{aligned}
\rho(\mathbf{r}) &= -\varepsilon_0 \boldsymbol{\nabla}^2\varphi(\mathbf{r}) \\
&= -\varepsilon_0 \frac{Z}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}(\frac{e^{-ar}}{r})\right] \\
&= -\frac{Z}{4\pi}\frac{1}{r^2}\frac{d}{dr}\left[-ar\,e^{-ar} - e^{-ar}\right] \\
&= -\frac{Za^2}{4\pi}\frac{e^{-ar}}{r}
\end{aligned}$$

For $\mathbf{r} = 0$, the function diverges, but for every $R > 0$,

$$\begin{aligned}
\int\limits_{r<R} \rho(\mathbf{r})d^3r &= \int\limits_{r<R} -\varepsilon_0\boldsymbol{\nabla}^2\varphi(\mathbf{r})d^3r \\
&= -\frac{Z}{4\pi}\int\limits_{r<R} \boldsymbol{\nabla}^2(\frac{e^{-ar}}{r})d^3r \\
&= -\frac{Z}{4\pi}\oint\limits_{r=R} \boldsymbol{\nabla}(\frac{e^{-ar}}{r})\cdot d\boldsymbol{\sigma} \\
&= ZaR\,e^{-aR} + Ze^{-aR}
\end{aligned}$$

Note that

$$\int\limits_{r<R}\frac{Za^2}{4\pi}\frac{e^{-ar}}{r}d^3r = \int_0^R \frac{Za^2}{4\pi}\frac{e^{-ar}}{r}4\pi r^2\,dr = Za^2\int_0^R re^{-ar}dr = -ZaR\,e^{-aR} - Ze^{-aR} + Z$$

so for every R,

$$\int\limits_{r<R}\left[\rho(\mathbf{r}) + \frac{Za^2}{4\pi}\frac{e^{-ar}}{r}\right]d^3r = Z$$

but

$$\rho(\mathbf{r}) + \frac{Za^2}{4\pi}\frac{e^{-ar}}{r} = 0, \quad \mathit{for}\ \mathbf{r} \neq 0$$

which means

$$\rho(\mathbf{r}) + \frac{Za^2}{4\pi}\frac{e^{-ar}}{r} = Z\delta(r)$$

Therefore,

$$\rho(\mathbf{r}) = Z\delta(r) - \frac{Za^2}{4\pi}\frac{e^{-ar}}{r}$$