Chapter 8
Sturm-Liouville Theory

solutions by Hikari

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8.2 Hermitian Operators

8.2.1 When multiplied by e^{-x}, the Laguerre's ODE becomes

$$e^{-x}xy'' + e^{-x}(1-x)y' + e^{-x}ay = 0$$

Then $\frac{d}{dx}(e^{-x}x) = e^{-x}(1-x)$, so the ODE is self-adjoint if the boundary term $e^{-x}x\big[v^*u' - (v^*)'u\big]_a^b$ vanishes. The inner product becomes

$$\langle v|u\rangle = \int_a^b v^*(x)e^{-x}\mathcal{L}(x)u(x)dx = \int_a^b \big[v^*(x)\mathcal{L}(x)u(x)\big]e^{-x}dx$$

which means the ODE will be self-adjoint if we let e^{-x} be the weighting function.

8.2.2 When multiplied by e^{-x^2}, the Hermite ODE becomes

$$e^{-x^2}y'' - 2xe^{-x^2}y' + 2\alpha e^{-x^2}y = 0$$

Then $\frac{d}{dx}(e^{-x^2}) = -2xe^{-x^2}$, so the ODE is self-adjoint if the boundary term $e^{-x^2}\big[v^*u' - (v^*)'u\big]_a^b$ vanishes. The inner product becomes

$$\langle v|u\rangle = \int_a^b v^*(x)e^{-x^2}\mathcal{L}(x)u(x)dx = \int_a^b \big[v^*(x)\mathcal{L}(x)u(x)\big]e^{-x^2}dx$$

which means the ODE will be self-adjoint if we let e^{-x^2} be the weighting function.

8.2.3 When multiplied by $\frac{1}{\sqrt{1-x^2}}$, the Chebyshev ODE becomes

$$\sqrt{1-x^2}\,y'' - \frac{x}{\sqrt{1-x^2}}y' + \frac{n^2}{\sqrt{1-x^2}}y = 0$$

Then $\frac{d}{dx}(\sqrt{1-x^2}) = -\frac{x}{\sqrt{1-x^2}}$, so the ODE is self-adjoint if the boundary term $\sqrt{1-x^2}\big[v^*u' - (v^*)'u\big]_a^b$ vanishes. The inner product becomes

$$\langle v|u\rangle = \int_a^b v^*(x)\frac{1}{\sqrt{1-x^2}}\mathcal{L}(x)u(x)dx = \int_a^b \big[v^*(x)\mathcal{L}(x)u(x)\big]\frac{1}{\sqrt{1-x^2}}dx$$

which means the ODE will be self-adjoint if we let $\frac{1}{\sqrt{1-x^2}}$ be the weighting function.

8.2.4 The boundary condition for $p_0(x)y'' + p_1(x)y' + p_2(x) = 0$ to be self-adjoint is

$$w(x)p_0(x)\big[v^*u' - (v^*)'u\big]_a^b = 0$$

for every u, v, with $w(x)$ being the weighting function.

Legendre:	$(1-x^2)\big[v^*u' - (v^*)'u\big]_{-1}^{1} = 0$	*because* $1-x^2=0$ *at* $x=-1,1$
Chebyshev:	$\sqrt{1-x^2}\big[v^*u' - (v^*)'u\big]_{-1}^{1} = 0$	*because* $\sqrt{1-x^2}=0$ *at* $x=-1,1$
Hermite:	$e^{-x^2}\big[v^*u' - (v^*)'u\big]_{-\infty}^{\infty} = 0$	*because* e^{-x^2} *approach to zero faster than polynomial* $v^*u'-(v^*)'u$ *at* $-\infty,\infty$
Laguerre:	$e^{-x}x\big[v^*u' - (v^*)'u\big]_{0}^{\infty} = 0$	*because* $x=0$, *and* e^{-x} *approach to zero faster than polynomial* $x\big[v^*u'-(v^*)'u\big]$ *at* ∞

8.2.5 The eigenvectors of an Hermitian operator with different eigenvalues are orthogonal (chapter 6) and therefore linearly independent (because if $au_1+bu_2=0$, then $\langle au_1+bu_2|au_1+bu_2\rangle = |a|^2\langle u_1|u_1\rangle + |b|^2\langle u_2|u_2\rangle = 0$, which means $a=b=0$).

8.2.6 (a) Let $u=\frac{1+x}{1-x}$, so $x=\frac{u-1}{u+1}$, $dx=\frac{2}{(u+1)^2}du$, and $u=0,\infty$ when $x=-1,1$. The integral becomes

$$\int_0^\infty \frac{1}{2}\frac{u-1}{u+1}(\ln u)\frac{2}{(u+1)^2}du = \int_0^\infty \frac{u-1}{(u+1)^3}\ln u\,du$$

$$= \frac{-u}{(u+1)^2}\ln u\Big|_0^\infty - \int_0^\infty \frac{-u}{(u+1)^2}\frac{1}{u}du = \frac{-u}{(u+1)^2}\ln u\Big|_0^\infty - \frac{1}{u+1}\Big|_0^\infty = (0-0)-(0-1) = 1$$

(b) The boundary term

$$(1-x^2)\left[x\frac{1}{1-x^2} - \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right]_{-1}^{1}$$

does not vanish (diverges), so the proof of the ODE being self-adjoint failed, and therefore the eigenfunctions are not necessarily orthogonal.

8.2.7 The boundary term

$$\sqrt{1-x^2}\left[\frac{-x}{\sqrt{1-x^2}} - 0\right]_{-1}^{1} = -2$$

does not vanish, so the proof of the ODE being self-adjoint failed, and therefore the eigenfunctions are not necessarily orthogonal.

8.2.8 (We prove the case when $w(x)=1$ (or similarly a constant), while I'm not sure the correctness when $w(x)$ is a function of x. But if $w(x)$ is the weighting function, which means $\langle u_m|u_n\rangle = \int_a^b u_m^* u_n w\,dx$, then there is no problem in the following proof.)

We have $(pu_n')' = -\lambda_n w u_n$ from the equation, and $\langle u_m|u_n\rangle = \int_a^b u_m^* u_n dx = \delta_{mn}$ by the orthogonality. Integrating $\langle u_m'|u_n'\rangle$ by parts:

$$\begin{aligned}\langle u_m'|u_n'\rangle &= \int_a^b (u_m')^* p u_n' dx \\ &= u_m^* p u_n'\Big|_a^b - \int_a^b u_m^*(pu_n')'dx \\ &= u_m^* p u_n'\Big|_a^b + \int_a^b u_m^*\lambda_n w u_n dx \\ &= u_m^* p u_n'\Big|_a^b + \lambda_n\langle u_m|u_n\rangle = \lambda_n\delta_{mn}\end{aligned}$$

which means u_m', u_n' are orthogonal, as long as the boundary condition $u_m^* p u_n'\Big|_a^b = 0$ is satisfied.
(Or equivalently $(u_m^*)' p u_n\Big|_a^b = 0$, because we have $\big[u_m^* p u_n' - (u_m^*)' p u_n\big]_a^b = 0$ from the boundary conditions making u_m, u_n orthogonal.)

8.2.9 Assume linear dependence, so $\varphi_n = \sum_{i=1}^{n-1} a_i \varphi_i$, then

$$A\varphi_n = \lambda_n \varphi_n = \sum_{i=1}^{n-1} \lambda_n a_i \varphi_i$$

and also

$$A\varphi_n = A(\sum_{i=1}^{n-1} a_i \varphi_i) = \sum_{i=1}^{n-1} a_i \lambda_i \varphi_i$$

so

$$\sum_{i=1}^{n-1} a_i (\lambda_n - \lambda_i) \varphi_i = 0$$

Note that $\lambda_n - \lambda_1 \neq 0$, so the equation implies $\varphi_1, \cdots, \varphi_{n-1}$ are linearly dependent. Therefore, the linear dependence between $\varphi_1, \cdots, \varphi_n$ implies the linear dependence between $\varphi_1, \cdots, \varphi_{n-1}$, and by repeating the process, we will find φ_1, φ_2 being linearly dependent, which is impossible because it implies $\lambda_1 = \lambda_2$ ($\lambda_2\varphi_2 = A\varphi_2 = Ak\varphi_1 = \lambda_1 k\varphi_1 = \lambda_1\varphi_2$). So $\varphi_1, \cdots, \varphi_n$ must be linearly independent.

8.2.10 (a) Using Eq 8.15, the ODE will be self-adjoint if multiplied by

$$\frac{1}{1-x^2} e^{\int \frac{-(2\alpha+1)x}{1-x^2} dx} = \frac{1}{1-x^2} e^{(\alpha+\frac{1}{2})\ln(1-x^2)} = (1-x^2)^{\alpha-\frac{1}{2}}$$

so the ODE becomes

$$\left\{(1-x^2)^{\alpha+\frac{1}{2}} \frac{d^2}{dx^2} - (2\alpha+1)x(1-x^2)^{\alpha-\frac{1}{2}} \frac{d}{dx} + n(n+2\alpha)(1-x^2)^{\alpha-\frac{1}{2}}\right\} C_n^{(\alpha)}(x) = 0$$

which is self-adjoint because

$$\frac{d}{dx}(1-x^2)^{\alpha+\frac{1}{2}} = -(2\alpha+1)x(1-x^2)^{\alpha-\frac{1}{2}}$$

(b) The boundary conditions will be satisfied if we choose the interval to be $[-1, 1]$:

$$(1-x^2)^{\alpha+\frac{1}{2}} \left[v^* u' - (v^*)' u\right]_{-1}^{1} = 0$$

so the eigenfunctions of different eigenvalues will be orthogonal if the inner product is defined as

$$\langle C_m | C_n \rangle = \int_{-1}^{1} C_m^* C_n (1-x^2)^{\alpha-\frac{1}{2}} \, dx$$

8.3 ODE Eigenvalue Problems

8.3.1 Let $y = \sum_{j=0}^{\infty} a_j x^{s+j}$ and substitute:

$$(1-x^2)\sum_{j=0}^{\infty} a_j (s+j)(s+j-1)x^{s+j-2} - 2x\sum_{j=0}^{\infty} a_j (s+j)x^{s+j-1} + n(n+1)\sum_{j=0}^{\infty} a_j x^{s+j} = 0$$

The coefficients of each order must be zero, so

$$\begin{aligned}
x^{s-2}: &\quad a_0 s(s-1) = 0 \\
x^{s-1}: &\quad a_1 (s+1)s = 0 \\
x^{s+j}: &\quad a_{j+2}(s+j+2)(s+j+1) - a_j(s+j)(s+j-1) - 2a_j(s+j) + n(n+1)a_j = 0 \\
&\quad a_{j+2} = \frac{(s+j)(s+j+1) - n(n+1)}{(s+j+1)(s+j+2)} a_j
\end{aligned}$$

(a) $a_0 \neq 0$, so $s(s-1) = 0$.

(b) Let $s=0$ and $a_1=0$ (so $a_3, a_5, \cdots$ vanish), then

$$a_{j+2} = \frac{j(j+1)-n(n+1)}{(j+1)(j+2)} a_j = \frac{(j-n)(j+n+1)}{(j+1)(j+2)} a_j$$

$$y_{even} = \sum_{j\ even} a_j x^j = a_0 \left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \cdots\right]$$

(c) Let $s=1$, then $a_1(s+1)s=0$ implies $a_1=0$, and

$$a_{j+2} = \frac{(j+1)(j+2)-n(n+1)}{(j+2)(j+3)} a_j = \frac{(j+1-n)(j+2+n)}{(j+2)(j+3)} a_j$$

$$y_{odd} = \sum_{j\ even} a_j x^{1+j} = a_0 \left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \cdots\right]$$

(d) For y_{even}, let $u_k = a_{2k}$, so

$$u_{k+1} = \frac{2k(2k+1)-n(n+1)}{(2k+1)(2k+2)} u_k$$

then the series at $x=\pm 1$ becomes

$$y_{even} = \sum_{j\ even} a_j x^j = \sum_{k=0}^{\infty} u_k (x^2)^k = \sum_{k=0}^{\infty} u_k$$

and

$$\frac{u_k}{u_{k+1}} = \frac{(2k+1)(2k+2)}{2k(2k+1)-n(n+1)} = 1 + \frac{1}{k} + \frac{B(k)}{k^2}$$

where $B(k)$ is bounded for large k, so by Gauss' test the series diverge.

Similarly, for y_{odd}, let $u_k = a_{2k}$, so

$$u_{k+1} = \frac{(2k+1)(2k+2)-n(n+1)}{(2k+2)(2k+3)} u_k$$

then the series at $x=\pm 1$ becomes

$$y_{odd} = \sum_{j\ even} a_j x^{1+j} = \sum_{k=0}^{\infty} x u_k (x^2)^k = \pm \sum_{k=0}^{\infty} u_k$$

and

$$\frac{u_k}{u_{k+1}} = \frac{(2k+2)(2k+3)}{(2k+1)(2k+2)-n(n+1)} = 1 + \frac{1}{k} + \frac{B(k)}{k^2}$$

where $B(k)$ is bounded for large k, so by Gauss' test the series diverge.

(e) From the recurrence relations of y_{even} and y_{odd}, if n is a non-negative even integer, then y_{even} will terminate at $a_n x^n$; if n is a non-negative odd integer, then y_{odd} will terminate at $a_{n-1}x^n$. So the series are converted into finite polynomials.

8.3.2 When multiplied by e^{-x^2}, the equation becomes

$$e^{-x^2} y'' - 2xe^{-x^2} y' + 2\alpha e^{-x^2} y = 0$$

so

$$\frac{d}{dx}\left[e^{-x^2}\right] = -2xe^{-x^2}$$

and

$$e^{-x^2}\left[v^* u' - (v^*)' u\right]_{-\infty}^{\infty} = 0$$

which means the ODE is self-adjoint (Hermitian).

8.3.3 Let $y = \sum_{j=0}^{\infty} a_j x^{s+j}$ and substitute:

$$\sum_{j=0}^{\infty} a_j(s+j)(s+j-1)x^{s+j-2} - 2x\sum_{j=0}^{\infty} a_j(s+j)x^{s+j-1} + 2\alpha\sum_{j=0}^{\infty} a_j x^{s+j} = 0$$

The coefficients of each order must be zero, so

$$\begin{aligned}
x^{s-2}: &\quad a_0 s(s-1) = 0\\
x^{s-1}: &\quad a_1(s+1)s = 0\\
x^{s+j}: &\quad a_{j+2}(s+j+2)(s+j+1) - 2a_j(s+j) + 2\alpha a_j = 0\\
&\quad a_{j+2} = a_j\frac{2(s+j-\alpha)}{(s+j+1)(s+j+2)}
\end{aligned}$$

(a) $s = 0$:

$$a_{j+2} = a_j\frac{2(j-\alpha)}{(j+1)(j+2)}$$

$$y_{even} = \sum_{j\ even} a_j x^j = a_0\left[1 + \frac{2(-\alpha)}{2!}x^2 + \frac{2^2(-\alpha)(2-\alpha)}{4!}x^4 + \cdots\right]$$

$s = 1$:

$$a_{j+2} = a_j\frac{2(j+1-\alpha)}{(j+2)(j+3)}$$

$$y_{odd} = \sum_{j\ even} a_j x^{1+j} = a_0\left[x + \frac{2(1-\alpha)}{3!}x^3 + \frac{2^2(1-\alpha)(3-\alpha)}{5!}x^5 + \cdots\right]$$

(b) For y_{even}, let $u_k = a_{2k}$, then

$$u_{k+1} = u_k\frac{2(2k-\alpha)}{(2k+1)(2k+2)}$$

$$\lim_{k\to\infty}\frac{a_{2k+2}x^{2k+2}}{a_{2k}x^{2k}} = \lim_{k\to\infty}\frac{u_{k+1}}{u_k}x^2 = \lim_{k\to\infty}\frac{4x^2k - 2x^2\alpha}{4k^2+6k+2} = \lim_{k\to\infty}\frac{x^2}{k} = 0$$

so by ratio test the series converge.

Similarly, for y_{odd}, let $u_k = a_{2k}$, then

$$u_{k+1} = u_k\frac{2(2k+1-\alpha)}{(2k+2)(2k+3)}$$

$$\lim_{k\to\infty}\frac{a_{2k+2}x^{2k+3}}{a_{2k}x^{2k+1}} = \lim_{k\to\infty}\frac{u_{k+1}}{u_k}x^2 = \lim_{k\to\infty}\frac{4x^2k + 2x^2(1-\alpha)}{4k^2+10k+6} = \lim_{k\to\infty}\frac{x^2}{k} = 0$$

so by ratio test the series converge.

$e^{x^2} = \sum_{k=0}^{\infty}\frac{x^{2k}}{k!}$, so the ratio of seccessive terms is

$$\lim_{k\to\infty}\frac{x^{2(k+1)}}{(k+1)!}\frac{k!}{x^{2k}} = \lim_{k\to\infty}\frac{x^2}{k+1} = \lim_{k\to\infty}\frac{x^2}{k}$$

which is the same as the ratios of seccessive terms of y_{even} and y_{odd}.

(c) From the recurrence relations of y_{even} and y_{odd}, if α is a non-negative even integer, then y_{even} will terminate at $a_\alpha x^\alpha$; if n is a non-negative odd integer, then y_{odd} will terminate at $a_{\alpha-1}x^\alpha$. So the series are converted into finite polynomials.

8.3.4 Let $L_n(x) = \sum_{j=0}^{\infty} a_j x^{s+j}$ and substitute:

$$x\sum_{j=0}^{\infty} a_j(s+j)(s+j-1)x^{s+j-2} + (1-x)\sum_{j=0}^{\infty} a_j(s+j)x^{s+j-1} + n\sum_{j=0}^{\infty} a_j x^{s+j} = 0$$

The coefficients of each order must be zero, so

$$x^{s-1}: \quad a_0 s(s-1) + a_0 s = a_0 s^2 = 0 \qquad s = 0$$

$$x^{s+j}: \quad a_{j+1}(j+1)j + a_{j+1}(j+1) - a_j j + n a_j = 0 \qquad a_{j+1} = a_j \frac{j-n}{(j+1)^2}$$

so

$$y = \sum_{j=0}^{\infty} a_j x^j = a_0\left[1 + \frac{(-n)}{1^2}x + \frac{(-n)(1-n)}{1^2\cdot 2^2}x^2 + \cdots\right]$$

From the recurrence relation of y, if n is a non-negative integer, then y will terminate at $a_n x^n$, and the series will be converted into a finite polynomial.

8.3.5 Let $T_n = \sum_{j=0}^{\infty} a_j x^{s+j}$ and substitute:

$$(1-x^2)\sum_{j=0}^{\infty} a_j(s+j)(s+j-1)x^{s+j-2} - x\sum_{j=0}^{\infty} a_j(s+j)x^{s+j-1} + n^2\sum_{j=0}^{\infty} a_j x^{s+j} = 0$$

The coefficients of each order must be zero, so

$$x^{s-2}: \quad a_0 s(s-1) = 0$$

$$x^{s-1}: \quad a_1(s+1)s = 0$$

$$x^{s+j}: \quad a_{j+2}(s+j+2)(s+j+1) - a_j(s+j)(s+j-1) - a_j(s+j) + n^2 a_j = 0$$

$$a_{j+2} = a_j\frac{(s+j-n)(s+j+n)}{(s+j+1)(s+j+2)}$$

For $s = 0$, let $a_1 = 0$, then

$$a_{j+2} = a_j\frac{(j-n)(j+n)}{(j+1)(j+2)}$$

$$y_{even} = \sum_{j\ even} a_j x^j = a_0\left[1 + \frac{(-n)n}{2!}x^2 + \frac{(2-n)(-n)n(2+n)}{4!}x^4 + \cdots\right]$$

For $s = 1$, $a_1 = 0$, and

$$a_{j+2} = a_j\frac{(j+1-n)(j+1+n)}{(j+2)(j+3)}$$

$$y_{odd} = \sum_{j\ even} a_j x^{1+j} = a_0\left[x + \frac{(1-n)(1+n)}{3!}x^3 + \frac{(3-n)(1-n)(1+n)(3+n)}{5!}x^5 + \cdots\right]$$

To test the convergence at $x = \pm 1$, let $u_k = a_{2k}$:

y_{even} :

$$u_{k+1} = u_k\frac{(2k-n)(2k+n)}{(2k+1)(2k+2)}$$

$$y_{even} = \sum_{j\ even} a_j x^j = \sum_{k=0}^{\infty} u_k (x^2)^k = \sum_{k=0}^{\infty} u_k$$

$$\frac{u_k}{u_{k+1}} = \frac{4k^2+6k+2}{4k^2-n^2} = 1 + \frac{3}{2k} + \frac{B(k)}{k^2}$$

so y_{even} converges at $x = \pm 1$ by Gauss' test.

y_{odd} :

$$u_{k+1} = u_k\frac{(2k+1-n)(2k+1+n)}{(2k+2)(2k+3)}$$

$$y_{odd} = \sum_{j\ even} a_j x^{1+j} = \sum_{k=0}^{\infty} x u_k (x^2)^k = \pm \sum_{k=0}^{\infty} u_k$$

$$\frac{u_k}{u_{k+1}} = \frac{4k^2+10k+6}{4k^2+4k+1-n^2} = 1 + \frac{3}{2k} + \frac{B(k)}{k^2}$$

so y_{odd} converges at $x = \pm 1$ by Gauss' test.

8.3.6 Let $U_n = \sum_{j=0}^{\infty} a_j x^{s+j}$ and substitute:

$$(1-x^2)\sum_{j=0}^{\infty} a_j(s+j)(s+j-1)x^{s+j-2} - 3x\sum_{j=0}^{\infty} a_j(s+j)x^{s+j-1} + n(n+2)\sum_{j=0}^{\infty} a_j x^{s+j} = 0$$

The coefficients of each order must be zero, so

$$\begin{aligned}
x^{s-2}: &\quad a_0 s(s-1) = 0 \\
x^{s-1}: &\quad a_1(s+1)s = 0 \\
x^{s+j}: &\quad a_{j+2}(s+j+2)(s+j+1) - a_j(s+j)(s+j-1) - 3a_j(s+j) + n(n+2)a_j = 0 \\
&\quad a_{j+2} = a_j \frac{(s+j)(s+j+2) - n(n+2)}{(s+j+1)(s+j+2)} = a_j \frac{(s+j-n)(s+j+n+2)}{(s+j+1)(s+j+2)}
\end{aligned}$$

Choose $s = 1$, then $a_1 = 0$, and

$$y_{odd} = \sum_{j\ even} a_j x^{1+j} = a_0\left[x + \frac{(1-n)(3+n)}{3!}x^3 + \frac{(3-n)(1-n)(3+n)(5+n)}{5!}x^5 + \cdots\right]$$

From the recurrence relation, if n is a positive odd integer, then y_{odd} will terminate at $a_{n-1}x^n$, and the series will be converted into a finite polynomial.

8.4 Variation Method

8.4.1 (a)

$$\langle\varphi|\varphi\rangle = \int_0^{\infty} 4\alpha^3 x^2 e^{-2\alpha x} dx = 4\alpha^3\left[2\frac{e^{-2\alpha x}}{(-2\alpha)^3}\right]_0^{\infty} = 1$$

(b)

$$\langle x^{-1}\rangle = \int_0^{\infty} 4\alpha^3 x e^{-2\alpha x} dx = 4\alpha^3\left[-\frac{e^{-2\alpha x}}{(-2\alpha)^2}\right]_0^{\infty} = \alpha$$

(c)

$$\begin{aligned}
\langle\frac{d^2}{dx^2}\rangle &= \int_0^{\infty} 4\alpha^3 x e^{-\alpha x}\frac{d^2}{dx^2}(xe^{-\alpha x})dx \\
&= 4\alpha^5\int_0^{\infty} x^2 e^{-2\alpha x}dx - 8\alpha^4\int_0^{\infty} xe^{-2\alpha x}dx \\
&= 4\alpha^5\left[2\frac{e^{-2\alpha x}}{(-2\alpha)^3}\right]_0^{\infty} - 8\alpha^4\left[-\frac{e^{-2\alpha x}}{(-2\alpha)^2}\right]_0^{\infty} \\
&= \alpha^2 - 2\alpha^2 = -\alpha^2
\end{aligned}$$

(d)

$$\left\langle \varphi \left| -\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{x} \right| \varphi \right\rangle = \frac{\alpha^2}{2} - \alpha$$

$$\frac{d}{d\alpha}\left[\frac{\alpha^2}{2} - \alpha\right] = \alpha - 1 = 0$$

so

$$\alpha = 1$$

and the minimum value of the expectation value is

$$\left[\frac{\alpha^2}{2} - \alpha\right]_{\alpha=1} = -\frac{1}{2}$$