Chapter 4
Continuity

solutions by Hikari

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1. f satisfying the condition is not necessary continuous. Consider the function

$$f(x) = \begin{cases} 1, & \textit{if x=0} \\ 0, & \textit{otherwise} \end{cases}$$

then $\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$ for every x, while f is not continuous at $x = 0$.

2. Let $q \in f(\overline{E})$, so there is a $p \in \overline{E}$ such that $f(p) = q$. Consider the neighborhood of q:

$$N_\varepsilon = \{y \in Y \mid d_Y(y, q) < \varepsilon\}$$

Since f is continuous, there is a $\delta > 0$ such that

$$x \in N_\delta = \{x \in X \mid d_X(x, p) < \delta\}$$

implies $f(x) \in N_\varepsilon$. Since p is a point or a limit point of E, there is a point of E contained in N_δ, and therefore there is a point of $f(E)$ contained in N_ε. So every neighborhood of q contains a point of $f(E)$, which means $q \in \overline{f(E)}$. Therefore, $f(\overline{E}) \subset \overline{f(E)}$.

Let f be a mapping from $X = (0,1)$ to $Y = R$, such that

$$f(x) = x$$

Let $E = X$. Since $\overline{E} = E$ in X, we have

$$f(\overline{E}) = f(E) = (0,1)$$

$$\overline{f(E)} = \overline{(0,1)} = [0,1]$$

so $f(\overline{E})$ is a proper subset of $\overline{f(E)}$.

3. $\{0\}$ is closed in R, so by Corollary of Theorem 4.8, $Z(f) = f^{-1}(\{0\})$ is closed in X, since f is continuous.

4. E is dense in X, so $\overline{E} = X$. From Exercise 2, $f(X) = f(\overline{E}) \subset \overline{f(E)}$, so every point of $f(X)$ is a point or a limit point of $f(E)$, which means $f(E)$ is dense in $f(X)$.

Let $p \in X$. Since E is dense in X, there is a sequence $\{p_n\} \to p$ where $p_n \in E$ for each n, so $g(p_n) = f(p_n)$. Since g and f are continuous,

$$g(p) = \lim_{n \to \infty} g(p_n) = \lim_{n \to \infty} f(p_n) = f(p)$$

so $g(p) = f(p)$ for all $p \in X$.

5. The complement of E is open, so by Exercise 2.29, it is the union of an at most countable collection of disjoint segments. Let $E^c = \bigcup_i (a_i, b_i)$. Let g_i be a real function on (a_i, b_i). If $a_i = \infty$ or $-\infty$, let $g_i(x) = f(b_i)$. If $b_i = \infty$ or $-\infty$, let $g_i(x) = f(a_i)$. If both a_i and b_i are finite, let g_i be that

$$g_i(x) = \frac{f(b_i) - f(a_i)}{b_i - a_i}(x - a_i) + f(a_i)$$

so $g_i(x)$ is a linear function $g_i(x) = m_i x + k_i$ where $g_i(a_i) = f(a_i)$ and $g_i(b_i) = f(b_i)$. Let g be a real function on R^1 such that

$$g(x) = \begin{cases} f(x), & \textit{if } x \in E \\ g_i(x), & \textit{if } x \in (a_i, b_i) \in E^c \end{cases}$$

g is continuous at every point except those a_i and b_i since $f(x)$ and $g_i(x)$ are continuous. At a_i, for every $\varepsilon > 0$, there is a $\delta_1 > 0$ such that $|x - a_i| < \delta_1$ implies $|f(x_i) - f(a_i)| < \varepsilon$. Let $\delta_2 = 0$ if $g_i(x) = f(a_i)$, or $\delta_2 = \frac{\varepsilon}{|m|}$ otherwise. Let $\delta = \max(\delta_1, \delta_2)$. For $|x - a_i| < \delta$,

$$|g(x) - g(a_i)| < \varepsilon$$

so g is continuous at a_i. Similarly g is continuous at b_i. So g is a continuous real function on R^1 such that $g(x) = f(x)$ for all $x \in E$.

Let f be a real continuous function defined on $E = (0,1)$ such that $f(x) = \frac{1}{x}$. Note that E is open. The extension g of f cannot be continuous at 0 whatever value $g(0)$ is assigned since $\lim_{x\to 0} f(x) = \infty \neq g(0)$.

Let $\mathbf{f} = (f_1, f_2, \cdot, f_n)$ be a vector-valued continuous function defined on a closed set $E \subset R^1$. By Theorem 4.10, each f_k is continuous, so there exist continuous function g_k on R^1 such that $g_k(x) = f_k(x)$ for all $x \in E$. Let $\mathbf{g} = (g_1, g_2, \cdot, g_n)$, then by Theorem 4.10, $\mathbf{g}$ is a continuous function on R^1 such that $\mathbf{g}(x) = \mathbf{f}(x)$ for all $x \in E$.

6.* If f is continuous on E, let ϕ be a function from $E \in R^1$ to R^2 such that

$$\phi(x) = (x, f(x))$$

then the graph of f is $\phi(E)$. Since both the functions $\phi_1(x) = x$ and $\phi_2(x) = f(x)$ are continuous, ϕ is continuous by Theorem 4.10. Since E is compact, $\phi(E)$ is compact by Theorem 4.14.

If f is not continuous at some point p, then there is a sequence $\{p_n\}$ in E such that $\{p_n\}$ converges to p, but $\{f(p_n)\}$ does not converge to $f(p)$. Consider the sequence $\{q_n\}$ in $\phi(E)$ such that $q_n = (p_n, f(p_n))$. If a subsequence $\{q_{n_k}\}$ converges to (a, b), then $\{p_{n_k}\} \to a$ and $\{f(p_{n_k})\} \to b$. Since $\{p_n\}$ converges to p, $a = p$, but since $\{f(p_n)\}$ does not converge to $f(p)$, $b \neq f(p)$,* so $(a, b) \notin \phi(E)$. Therefore, there is no subsequence of $\{q_n\}$ in $\phi(E)$ converges to a point of $\phi(E)$, which means $\phi(E)$ is not compact by Theorem 3.6.

7. $(|x| - y^2)^2 \geq 0$, so $x^2 + y^4 \geq 2|x|y^2$, and

$$|f(x,y)| = \frac{|xy^2|}{|x^2 + y^4|} \leq \frac{|xy^2|}{2|xy^2|} = \frac{1}{2}$$

so $f(x, y)$ is bounded by $\frac{1}{2}$.

Let $x = y^3$, then

$$g(x,y) = \frac{y^3 \cdot y^2}{y^6 + y^6} = \frac{1}{2y}$$

since $\frac{1}{2y} \to \infty$ as $y \to 0$, $g(x, y)$ is unbounded in every neighborhood of $(0, 0)$.

Let $x = y^2$, then

$$f(x,y) = \frac{y^2 \cdot y^2}{y^4 + y^4} = \frac{1}{2}$$

For $\varepsilon < \frac{1}{2}$, in every neighborhood $|(x, y) - (0, 0)| < \delta$ there is a point (a^2, a) such that $|f(a^2, a) - f(0, 0)| = \frac{1}{2} > \varepsilon$, so f is not continuous at $(0, 0)$.

If $f(x,y)$ and $g(x,y)$ are restricted to a straight line not passing $(0,0)$, then the denominators of $f(x,y)$ and $g(x,y)$ are not zero, which implies that they are continuous by Theorem 4.9. If f(x,y) and $g(x,y)$ are restricted to a straight line $y = mx$ passing $(0,0)$, then

$$f(x,y) = \frac{x \cdot m^2x^2}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}$$

$$g(x,y) = \frac{x \cdot m^2x^2}{x^2 + m^6x^6} = \frac{m^2x}{1 + m^6x^4}$$

both the denominators of which are not zero, so they are continuous by Theorem 4.9.

8. Let x be a limit point of E, so there is a sequence $\{x_n\}$ in E such that $\{x_n\} \to x$. For $\varepsilon > 0$, there is a $\delta > 0$ such that $|p - q| < \delta$ implies $|f(p) - f(q)| < \varepsilon$, since f is uniformly continuous; for $\delta > 0$, there is a positive integer N such that $m, n \geq N$ implies $|x_n - x_m| < \delta$, since $\{x_n\}$ converges and is therefore a Cauchy sequence. Therefore, for $\varepsilon > 0$ there is a positive integer N such that $m, n \geq N$ implies $|f(x_n) - f(x_m)| < \varepsilon$, which means $\{f(x_n)\}$ is a Cauchy sequence. Since every Cauchy sequence in R^1 converges, $\{f(x_n)\}$ converges to a point y, where $y = y(x)$ depends on x. Let F be a function on $\overline{E}$ to R^1 such that

$$F(x) = \begin{cases} f(x), & \textit{if } x \in E \\ y(x), & \textit{if } x \in \overline{E} \setminus E \end{cases}$$

By the construction of y, F is continuous. Since $\overline{E}$ is closed and bounded, it is compact, so $F(\overline{E})$ is compact and therefore bounded by Theorem 4.14. Note that $f(E) = F(E) \subset F(\overline{E})$, so $f(E)$ is bounded.

Let $f(x) = x$ be a function from $E = R^1$ to R^1. It is uniformly continuous if we let $\delta = \varepsilon$, since $|f(x) - f(y)| = |x - y| < \delta = \varepsilon$. But E is not bounded. $f(E) = f(R^1) = R^1$ is not bounded.

9. If f is a uniformly continuous function from X to Y, then for every $\varepsilon > 0$, there is a $\delta > 0$ such that $d_X(p,q) < \delta$ implies $d_Y(f(p), f(q)) < \varepsilon$. If $E \subset X$ and $\operatorname{diam} E < \delta$, then for $p, q \in E$, we have $d_X(p,q) \leq \operatorname{diam} E < \delta$, and therefore $d_Y(f(p), f(q)) < \varepsilon$. Since $\operatorname{diam} f(E) = \sup d_Y(f(p), f(q))$ for $f(p), f(q) \in E$, we have $\operatorname{diam} f(E) < \varepsilon$.

If for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\operatorname{diam} E < \delta$ implies $\operatorname{diam} f(E) < \varepsilon$, then for every $d_X(p,q) < \delta$, let $E = \{p,q\}$, then $\operatorname{diam} E = d_X(p,q) < \delta$, and therefore $d_Y(f(p), f(q)) = \operatorname{diam} f(E) < \varepsilon$, which means f is uniformly continuous.

10. Let f be a continuous mapping of a compact metric space X into a metric space Y. Suppose f is not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}, \{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \varepsilon$ for every n. By Theorem 3.6, a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ converges to a point $p \in X$, and a subsequence $\{q_{n_{i_j}}\} = \{q_{n_k}\}$ of $\{q_{n_i}\}$ converges to a point $q \in X$. So $\{p_{n_k}\} \to p$ and $\{q_{n_k}\} \to q$, but note that $d_X(p_n, q_n) \to 0$, so $p = q$. Since f is continuous, we have $\{f(p_{n_k})\} \to f(p)$ and $\{f(q_{n_k})\} \to f(q)$, while $f(p) = f(q)$, so $d_Y(f(p_{n_k}), f(q_{n_k})) \to 0$, contradicting with $d_Y(f(p_n), f(q_n)) > \varepsilon$. So f is uniformly continuous.

11. Suppose f is a uniformly continuous mapping from X to Y, and $\{x_n\}$ is a Cauchy sequence in X. For every $\varepsilon > 0$, there is a $\delta > 0$ such that $d_X(p,q) < \delta$ implies $d_Y(f(p), f(q)) < \varepsilon$; for $\delta > 0$, there is a positive integer N such that $m, n \geq N$ implies $d_X(x_m, x_n) < \delta$, and therefore $d_Y(f(x_m), f(x_n)) < \varepsilon$. So $\{f(x_n)\}$ is a Cauchy sequence in Y.

(Exercise 13) For each $x \in X$, let $\{x_n\}$ be a sequence in E such that $\{x_n\} \to x$ (the sequence exists since E is dense in X). Since $\{x_n\}$ is a Cauchy sequence, $\{f(x_n)\}$ is also a Cauchy sequence and therefore converges, since $\{f(x_n)\}$ is in R^1. Let the limit be $F(x)$.

F is a function defined on X, and $F(x) = f(x)$ for $x \in E$ since f is continuous on E. So F is an extension of f from E to X. For every $\varepsilon > 0$, there is a $\delta > 0$ such that $p, q \in E$ and $d(p,q) < \delta$ implies $d(f(p), f(q)) < \frac{\varepsilon}{3}$. For $x, y \in X$ and $d(x,y) < \delta$, let N be a positive integer such that $d(f(x_N), F(x)) < \frac{\varepsilon}{3}$ and $d(x_N, x) < \frac{\delta - d(x,y)}{2}$; let M be a positive integer such that $d(f(y_M), F(y)) < \frac{\varepsilon}{3}$ and $d(y_N, y) < \frac{\delta - d(x,y)}{2}$. Let $p = x_N$ and $q = y_M$. Then we have

$$d(p,q) \leq d(p,x) + d(x,y) + d(y,q) < \frac{\delta - d(x,y)}{2} + d(x,y) + \frac{\delta - d(x,y)}{2} = \delta$$

and therefore

$$d(F(x), F(y)) \leq d(F(x), F(p)) + d(F(p), F(q)) + d(F(q), F(y)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Therefore, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $x, y \in X$ and $d(x, y) < \delta$ implies $d(F(x), F(y)) < \varepsilon$, which means F is a uniformly continuous extension of f from E to X.

12. *restate:* If f is a uniformly continuous function from X to Y, and g is a uniformly function form $f(X) \subset Y$ to Z, then the composition $h = g \circ f$ is a uniformly continuous function from X to Z.

proof: For $\varepsilon > 0$, there is a $\eta > 0$ such that $d_Y(x, y) < \eta$ implies $d_Z(g(x), g(y)) < \varepsilon$; for $\eta > 0$, there is a $\delta > 0$ such that $d_X(p, q) < \delta$ implies $d_Y(f(p), f(q)) < \eta$. Therefore, for $\varepsilon > 0$, there is a $\delta > 0$ such that $d_X(p, q) < \delta$ implies $d_Z(g(f(p)), g(f(q))) < \varepsilon$, which means $g \circ f$ is uniformly continuous.

13. For each $p \in X$ and each positive integer n, let $V_n(p)$ be the set of all $q \in E$ with $d(p, q) < \frac{1}{n}$. For every n, $V_n(p)$ is nonempty since E is dense in X, so $f(V_n(p))$ is nonepmty for every n. For $\varepsilon > 0$, there is a $\delta > 0$ such that $E \subset X$ with $\operatorname{diam} E < \delta$ implies $\operatorname{diam} f(E) < \varepsilon$; for $\delta > 0$, let N be the positive integer such that $\frac{1}{N} < \delta$, then $\operatorname{diam} V_n(p) < \delta$ for $n \geq N$. Therefore, for $\varepsilon > 0$ there is a positive integer N such that $n \geq N$ implies $\operatorname{diam} f(V_n(p)) < \varepsilon$, which means $\operatorname{diam} \overline{f(V_n(p))} \to 0$. Let M be the positive integer such that $\operatorname{diam} f(V_n(p)) < 1$ for $n \geq M$. Then $\overline{f(V_M(p))}, \overline{f(V_{M+1}(p))}, \overline{f(V_{M+2}(p))}, \cdots$ are closed and bounded, so they are compact. From Theorem 3.10, $\overline{f(V_M(p))} \supset \overline{f(V_{M+1}(p))} \supset \cdots$ and $\operatorname{diam} \overline{f(V_n(p))} \to 0$ implies there is exactly one point contained in every $\overline{f(V_n(p))}$, let it be $g(p)$.

g is a function defined on X, and for $x \in E$, $g(x) = f(x)$, so g is an extension of f from E to X. For every $\varepsilon > 0$, there is a $\delta > 0$ such that $p, q \in E$ and $d(p, q) < \delta$ implies $d(f(p), f(q)) < \frac{\varepsilon}{3}$. For $x, y \in X$ such that $d(x, y) < \delta$, let N, M be positive integers such that $\overline{f(V_N(x))} < \frac{\varepsilon}{3}$ and $\overline{f(V_M(y))} < \frac{\varepsilon}{3}$. Let p be an element of E such that $d(p, x) < \min(\frac{1}{N}, \frac{\delta - d(x,y)}{2})$, and let q be an element of E such that $d(p, y) < \min(\frac{1}{M}, \frac{\delta - d(x,y)}{2})$ (p, q exist because E is dense in X). Then we have

$$d(p, q) \leq d(p, x) + d(x, y) + d(y, q) < \frac{\delta - d(x, y)}{2} + d(x, y) + \frac{\delta - d(x, y)}{2} = \delta$$

and therefore

$$d(g(x), g(y)) \leq d(g(x), g(p)) + d(g(p), g(q)) + d(g(q), g(y)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Therefore, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $x, y \in X$ and $d(x, y) < \delta$ implies $d(g(x), g(y)) < \varepsilon$, so g is a uniformly continuous extension of f from E to X.

The range space can be replaced by R^k, compact spaces or complete metric spaces, since by Exercise 3.21, the existence and uniqueness of $g(p)$ so constructed does not change, and the rest of the proof holds in every metric space.

The consequence may not hold if the range space is incomplete. For example, let $E = Q$, the rational numbers which is dense in R, and let $f(x) = x$ be a function from Q to Q, which is uniformly continuous. If f has a continuous extension from Q to R, then the extension F_1 is a function from R to Q such that $F_1(x) = x$ for $x \in Q$. Let F_2 be the identity function $F_2(x) = x$ from R to R. We have $F_1(p) = F_2(p)$ for $p \in Q$, while if $q \in R \setminus Q$, then $F_1(q) \in Q$ but $F_2(q) \in R \setminus Q$, so $F_1(q) \neq F_2(q)$, a contradiction to Exercise 4. Therefore, f does not have a continuous extension from Q to R.

14. If $f(0) = 0$ or $f(1) = 1$, then we are done, so supposed $f(0) > 0$ and $f(1) < 1$. Let $g(x) = f(x) - x$, which is continuous since $f(x)$ is continuous. $g(0) = f(0) - 0 > 0$, and $g(1) = f(1) - 1 < 0$, so by Theorem 4.23, there is a point $p \in (0, 1)$ such that $g(p) = 0$. Then $f(p) = p$.

15. Let f be a continuous mapping of R^1 into R^1. If it is not monotonic, then there are $a < b < c$ such that $f(a) < f(b)$ and $f(b) > f(c)$, or $f(a) > f(b)$ and $f(b) < f(c)$. Consider the first case only, since the second case can be proved similarly. The interval $I = [a, c]$ is compact, so $f(I)$ is compact, and therefore $\sup f(I)$ exists and $\sup f(I) \in f(I)$. Let u be the point in $[a, c]$ such that $f(u) = \sup f(I)$. Since $f(a), f(c) < f(b) \leq f(u)$, $u \neq a, b$, so $u \in (a, b)$. Consider the segment $S = (a, b)$ which is open. $f(u) \in f(S)$ while it is not an interior point of $f(S)$, since every neighborhood $N_\varepsilon(f(u))$ contains a point p such that $f(u) < p < f(u) + \varepsilon$, which is not in $f(S)$. So $f(S)$ is not open, which means f is not an open mapping. Therefore, every continuous open mapping of R^1 into R^1 is monotonic.

16. Let $f(x) = [x]$ and $g(x) = (x)$. Both the functions are continuous on segments $(n, n+1)$, where n is an integer. At $x = n$, $f(n-) = n - 1$, $f(n+) = n$, $g(n-) = 1$, $g(n+) = 0$, so both the functions have simple discontinuity at integer points.

17. For $f(x-) < f(x+)$, a rational number p such that $f(x-) < p < f(x+)$ exists by Theorem 1.20. If for every $\delta > 0$, there is a t such that $x - \delta < t < x$ and $f(t) \geq p$, let t_n be that t when $\delta = \frac{1}{n}$, then $\{t_n\} \to x$, but $\{f(t_n)\}$ either diverges or converges to $q \geq p$, a contradiction to the definition of $f(x-) < p$. Therefore, there is a $\delta > 0$ such that $x - \delta < t < x$ implies $f(t) < p$. Let q be a rational number in (a, b) such that $x - \delta < q < x$, then $q < t < x$ implies $f(t) < p$. Similarly, we can find a rational number r in (a, b) such that $x < t < r$ implies $f(t) > p$.

If there are two points $x_1 < x_2$ of E associated to the same (p, q, r), let t be a point such that $x_1 < t < x_2$, then $x_1 < t < r$ so $f(t) > p$, but $q < t < x_2$ so $f(t) < p$, a contradiction, which means at most one point of E is associated with each triple (p, q, r). So the set of simple discontinuity such that $f(x-) < f(x+)$ is countable. Similar is for the case of $f(x-) > f(x+)$.

Let p be the set of points on which $f(x-) = f(x+) < f(x)$. With each point x of E, associate a triple (p, q, r) of rational numbers such that $f(x-) = f(x+) < p < f(x)$, and $a < q < t < x$ or $x < t < r < b$ implies $f(t) < p$. If there are two points $x_1 < x_2$ of P associated to the same (p, q, r), then $q < x_1 < x_2$ implies $f(x_1) < p$, but $f(x_1) > p$, a contradiction, which means at most one point of P is associated with each triple (p, q, r). So the set of simple discontinuity such that $f(x-) = f(x+) < f(x)$ is countable. Similar is for the case of $f(x-) = f(x+) > f(x)$.

Therefore, the set of points at which f has a simple discontinuity is at most countable.

18. Let x be a point in R^1. For every $\varepsilon > 0$, let N be the positive integer such that $N \leq \frac{1}{\varepsilon} < N + 1$. For $1 \leq n \leq N$, let M_n be the integer such that $\frac{M_n}{n} \leq x \leq \frac{M_n+1}{n}$. If $x = \frac{M_n}{n}$, let $\delta_n = \frac{1}{n}$, otherwise let $\delta_n = \min(x - \frac{M_n}{n}, \frac{M_n+1}{n} - x)$. Let $\delta = \min(\delta_1, \delta_2, \cdots, \delta_N)$, so there is no rational number $\frac{m}{n}$ other than x with $n \leq N$ be contained the neighborhood of x with radius δ. For every t such that $0 < |t - x| < \delta$, if t is irrational, then $|f(t) - 0| = |0 - 0| < \varepsilon$; if t is rational so $t = \frac{m}{n}$, then by the construction of δ we have $n > N$, so $|f(t) - 0| = |\frac{1}{n} - 0| \leq \frac{1}{N+1} < \varepsilon$. Therefore, for each $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < |t - x| < \delta$ implies $|f(t) - 0| < \varepsilon$, which means $\lim_{t \to x} f(t) = 0$. If x is irrational, then $\lim_{t \to x} f(t) = f(x) = 0$, so f is continuous at x; if x is rational, then $f(x+) = f(x-) = \lim_{t \to x} f(t) = 0 \neq f(x)$, so f has a simple discontinuity at x.

19. Suppose f is a real function with domain R^1 which has the intermediate value property. If f is not continuous at a point x_0, then there is a sequence $\{p_n\}$ such that $\{p_n\} \to x_0$ but $\{f(p_n)\}$ does not converge to $f(x_0)$. Either the set of p_n such that $f(p_n) > f(x_0)$ or the set of p_n such that $f(p_n) < f(x_0)$ are infinite. Consider the first case only since the second case can be proved similarly. Let $\{x_n\}$ be the subsequence of $\{p_n\}$ such that $f(x_n) > f(x_0)$ for every n. $\{f(x_n)\}$ does not converge to $f(x_0)$, so there is a r' such that $f(x_n) > r' > f(x_0)$ for every n. Let r be a rational number such that $r' > r > f(x_0)$. Let t_n be the point such that $x_0 < t_n < x_n$ and $f(t_n) = r$. x_0 is a limit point of the set of t_n, which is a subset of $E = \{x \,|\, f(x) = r\}$, so x_0 is a limit point of E. But $f(x_0) \neq r$, so $x_0 \notin E$, which means E is not closed. Therefore, if f is a real function with domain R^1 which has the intermediate value property, and for every rational r, the set of all x with $f(x) = r$ is closed, then f is continuous.

20. (a) $\rho_E(x) = \inf_{z \in E} d(x, z) = 0$ if and only if for every $\varepsilon > 0$ there is a $z \in E$ such that $d(x, z) < E$, which is equivalent to $x \in \overline{E}$.

(b) $\rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z)$ for all z, so $\rho_E(x) - d(x, y) \leq d(y, z)$ for all z, which means $\rho_E(x) - d(x, y) \leq \rho_E(y)$. Interchanging x and y, we have $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$. For every $\varepsilon > 0$, let $\delta = \varepsilon$, then $d(x, y) < \delta$ implies $|\rho_E(x) - \rho_E(y)| < \varepsilon$, which means ρ_E is a uniformly continuous function.

21. K is compact and ρ_F is continuous, so $\rho_F(K)$ is compact by Theorem 4.14. So $\inf \rho_F(K) \in \rho_F(K)$, which means there is a $u \in K$ such that $\rho_F(u) = \inf \rho_F(K)$. If $\rho_F(u) = 0$, then by Exercise 20, $u \in \overline{F}$ and therefore $u \in F$ since F is closed, a contradiction to K and F being disjoint. So $\inf \rho_F(K) = \rho_F(u) > 0$. Let δ be a number such that $0 < \delta < \inf \rho_F(K)$. For a point $p \in K$ and all points $q \in F$, $d(p, q) \geq \rho_F(p)$, and for all $p \in K$, $\rho_F(p) \geq \inf \rho_F(K) > \delta$, so $d(p, q) > \delta$ for all $p \in K$ and $q \in F$.

Let $A = \{n \,|\, n \in \mathbb{N}\}$ and $B = \{n + \frac{1}{n+1} | n \in \mathbb{N}\}$. A and B are disjoint, and both of which are closed but not compact. For every $\varepsilon > 0$, let N be the positive integer such that $N + 1 > \frac{1}{\delta}$, then $p = N \in A$ and $q = N + \frac{1}{N+1} \in B$ satisfy $d(p, q) = \frac{1}{N+1} < \delta$.

22. If $\rho_A(p) = \rho_B(p) = 0$, then $p \in \overline{A} = A$ and $p \in \overline{B} = B$, a contradiction to A and B being disjoint, so $\rho_A(p) + \rho_B(p) \neq 0$ for all $p \in X$. By Theorem 4.9, f is continuous. $\rho_A(p) \geq 0$ and $\rho_B(p) \geq 0$, so

$$0 = \frac{0}{\rho_A(p) + \rho_B(p)} \leq \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)} \leq \frac{\rho_A(p)}{\rho_A(p)} = 1$$

for all $p \in X$, so the range of f lies in $[0,1]$. $f(p) = 0$ if and only if $\rho_A(p) = 0$, if and only if $p \in \overline{A} = A$. $f(p) = 1$ if and only if $\rho_B(p) = 0$, if and only if $p \in \overline{B} = B$.

f is a continuous function from X to $Y = [0,1]$. $[0, \frac{1}{2})$ is open in Y, so $V = f^{-1}([0, \frac{1}{2}))$ is open in X by Theorem 4.8. Similarly, $W = f^{-1}((\frac{1}{2}, 1])$ is open in X. $f(p) = 0 \in [0, \frac{1}{2})$ for all $p \in A$, so $A \subset V$. Similarly, $B \subset W$.

23. Let x be a point in (a, b). Find p, q and η such that $a < p < p + \eta < x < q - \eta < q < b$. Let $M_1 = \max(f(p), f(q))$. For every t such that $p < t < q$, let $\lambda = \frac{q-t}{q-p}$, then

$$t = \lambda p + (1-\lambda) q$$

$$f(t) \leq \lambda f(p) + (1-\lambda) f(q) \leq \lambda M_1 + (1-\lambda) M_1 = M_1$$

So $f(t) \leq M_1$ for all t in (p, q).

For every t such that $\frac{p+q}{2} < t < q$, let $\lambda = \frac{2t-p-q}{2(t-p)}$, then

$$\frac{p+q}{2} = \lambda p + (1-\lambda) t$$

$$f(\frac{p+q}{2}) \leq \lambda f(p) + (1-\lambda) f(t)$$

$$\begin{aligned} f(t) &\geq \frac{1}{1-\lambda} f(\frac{p+q}{2}) - \frac{\lambda}{1-\lambda} f(p) \\ &= \frac{2(t-p)}{q-p} f(\frac{p+q}{2}) - \frac{2t-p-q}{q-p} f(p) \\ &\geq -2 \left| f(\frac{p+q}{2}) \right| - |f(p)| \end{aligned}$$

Let $M_2 = -2 \left| f(\frac{p+q}{2}) \right| - |f(p)|$, then $f(t) \geq M_2$ for all t in $(\frac{p+q}{2}, q)$. Similarly, we can find M_3 such that $f(t) \geq M_3$ for all t in $(p, \frac{p+q}{2})$. Let $M = \max(|M_1|, |M_2|, |M_3|, |f(p)|, |f(q)|, |f(\frac{p+q}{2})|)$, then $|f(t)| < M$ for all t in $[p, q]$.

For every $\varepsilon > 0$, let $\delta = \frac{\eta}{2M}\varepsilon$. For every y in $(x, x+\delta)$, let $\lambda = \frac{y-x}{y-p}$, then

$$x = \lambda p + (1-\lambda) y$$

$$f(x) \leq \lambda f(p) + (1-\lambda) f(y)$$

$$f(x) - f(y) \leq \lambda \left(f(p) - f(y)\right) = \frac{y-x}{y-p} \left(f(p) - f(y)\right) < \frac{\delta}{\eta} \cdot 2M = \varepsilon$$

Let $\lambda = \frac{q-y}{q-x}$, then

$$y = \lambda x + (1-\lambda) q$$

$$f(y) \leq \lambda f(x) + (1-\lambda) f(q)$$

$$f(y) - f(x) \leq (1-\lambda)(f(q) - f(x)) = \frac{y-x}{q-x}(f(q) - f(x)) < \frac{\delta}{\eta} \cdot 2M = \varepsilon$$

So $|f(y) - f(x)| < \varepsilon$ for all y in $(x, x+\delta)$. Similarly, $|f(y) - f(x)| < \varepsilon$ for all y in $(x - \delta, x)$. Therefore, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \varepsilon$, so f is continuous at x. Since x is arbitrary, f is continuous.

If g is an increasing convex function, and f is convex, then

$$g(f(\lambda x + (1-\lambda) y)) \leq g(\lambda f(x) + (1-\lambda) f(y)) \leq \lambda g(f(x)) + (1-\lambda) g(f(y))$$

so $g \circ f$ is convex.

If f is convex in (a,b) and $a<s<t<u<b$, let $\lambda=\frac{u-t}{u-s}$, then

$$t=\lambda s+(1-\lambda)u$$

$$f(t)\le \lambda f(s)+(1-\lambda)f(u)=\frac{u-t}{u-s}f(s)+\frac{t-s}{u-s}f(u)$$

Rearranging we can obtain

$$\frac{f(t)-f(s)}{t-s}\le\frac{f(u)-f(s)}{u-s}$$
$$\frac{f(u)-f(s)}{u-s}\le\frac{f(u)-f(t)}{u-t}$$

24. Consider the inequality $f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y)$ for all $x,y\in(a,b)$. If $\lambda=0$ or 1, the inequality holds. If $\lambda=\frac{1}{2}$,

$$f(\lambda x+(1-\lambda)y)=f(\frac{x+y}{2})\le\frac{f(x)+f(y)}{2}=\lambda f(x)+(1-\lambda)f(y)$$

so the inequality holds. Suppose the inequality holds for $\lambda=\frac{k}{2^n}$ where n is a positive integer and $k=0,1,\cdots,2^n$. Consider $\lambda=\frac{r}{2^{n+1}}$ where $r=0,1,\cdots,2^{n+1}$. If r is even, then the inequality holds since $\lambda=\frac{r}{2^{n+1}}=\frac{2r'}{2^{n+1}}=\frac{r'}{2^n}$ where r' is an integer. If r is odd, then $\lambda=\frac{r}{2^{n+1}}=\frac{1}{2}\left(\frac{t}{2^n}+\frac{t+1}{2^n}\right)$ for some integer t, so

$$\begin{aligned}
&f\left[\frac{r}{2^{n+1}}x+\left(1-\frac{r}{2^{n+1}}\right)y\right]\\
&=f\left[\frac{\frac{t}{2^n}x+(1-\frac{t}{2^n})y+\frac{t+1}{2^n}x+(1-\frac{t+1}{2^n})y}{2}\right]\\
&\le\frac{f\left[\frac{t}{2^n}x+(1-\frac{t}{2^n})y\right]+f\left[\frac{t+1}{2^n}x+(1-\frac{t+1}{2^n})y\right]}{2}\\
&\le\frac{\frac{t}{2^n}f(x)+(1-\frac{t}{2^n})f(y)+\frac{t+1}{2^n}f(x)+(1-\frac{t+1}{2^n})f(y)}{2}\\
&=\frac{r}{2^{n+1}}f(x)+\left(1-\frac{r}{2^{n+1}}\right)f(y)
\end{aligned}$$

so the inequality holds for $\lambda=\frac{r}{2^{n+1}}$. Therefore, by induction, the inequality holds for all $\lambda\in A$ where $A=\{\frac{k}{2^n}\mid n,k\in\mathbb{N},\ 0\le k\le 2^n\}$. For $x,y\in(a,b)$, consider the function g from $[0,1]$ to R^1 that maps λ to $\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)$. For all $\lambda\in A$, $\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)\ge 0$, so $g(\lambda)\ge0$, which means $A\subset g^{-1}([0,\infty))$. Since f is continuous, g is continuous, and since $[0,\infty)$ is closed in R^1, $g^{-1}([0,\infty))$ is closed in $[0,1]$ by Theorem 4.8. For $\lambda\in[0,1]$, every neighborhood of λ contains a point of A and therefore a point of $g^{-1}([0,\infty))$, so λ is a point or a limit point of $g^{-1}([0,\infty))$, and since $g^{-1}([0,\infty))$ is closed, $\lambda\in g^{-1}([0,\infty))$, which is $g(\lambda)\ge0$. Therefore, for every $x,y\in(a,b)$ and $\lambda\in[0,1]$, the inequality $f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y)$ holds, so f is convex.

25. (a) Take $\mathbf{z}\notin K+C$, put $F=\mathbf{z}-C$, the set of all $\mathbf{z}-\mathbf{y}$ with $\mathbf{y}\in C$. If $\mathbf{t}\in K$ and $\mathbf{t}\in F$, then $\mathbf{t}=\mathbf{x}=\mathbf{z}-\mathbf{y}$ for some $\mathbf{x}\in K$ and $\mathbf{y}\in C$, a contradiction to $\mathbf{z}\notin K+C$, so K and F are disjoint. K is compact and F is closed since C is closed, so by Exercise 21, there exists $\delta>0$ such that $|\mathbf{p}-\mathbf{q}|>\delta$ if $\mathbf{p}\in K$, $\mathbf{q}\in F$, so $|\mathbf{p}-\mathbf{q}|=|\mathbf{p}-(\mathbf{z}-\mathbf{r})|=|\mathbf{p}+\mathbf{r}-\mathbf{z}|>\delta$ for $\mathbf{p}\in K$, $\mathbf{r}\in C$, which means the open ball with center $\mathbf{z}$ and radius δ contains no element of $K+C$. So if $\mathbf{z}\notin K+C$, then $\mathbf{z}$ is not a limit point of $K+C$, which means $K+C$ is closed.

(b) C_1 and C_2 have no limit points and therefore closed. Let $\beta_k=k\alpha-[k\alpha]$, where k is a positive integer, and $[k\alpha]$ denotes the largest integer less than or equal to $k\alpha$. If $i\ne j$ but $\beta_i=\beta_j$, then $i\alpha-[i\alpha]=j\alpha-[j\alpha]$, so

$$\alpha=\frac{[i\alpha]-[j\alpha]}{i-j}$$

which means α is rational, a contradiction. Therefore, $\beta_i\ne\beta_j$ for $i\ne j$, so there are infinitely many different elements of $\{\beta_k\}$ in $(0,1)$. For every integer N, consider the disjoint sets $(\frac{0}{N},\frac{1}{N})$, $[\frac{1}{N},\frac{2}{N})$, $\cdots$, $[\frac{N-1}{N},\frac{N}{N})$ whose union is $(0,1)$. There are only N sets but infinitely many elements of $\{\beta_k\}$ in $(0,1)$, which means there are at least two elements β_i,β_j in the same set, so $0<\beta_i-\beta_j<\frac{1}{N}$. Note that

$\beta_i - \beta_j = (i+j)\alpha - ([i\alpha] + [j\alpha])$ is an element of $C_1 + C_2$, so for every positive integer N there is an element y of $C_1 + C_2$ such that $0 < y < \frac{1}{N}$. For every $x \in R^1$ and $\varepsilon > 0$, let N be a positive integer such that $N > \frac{1}{\varepsilon}$, then the segment $(\frac{k}{N}, \frac{k+1}{N})$ for some integer k is in the neighborhood of x with radius ε. Let n be the integer such that $n \leq \frac{k}{N} < n+1$, and let y be an element of $C_1 + C_2$ such that $0 < y < \frac{1}{N}$, then there is an integer m such that $\frac{k}{N} < n + my < \frac{k+1}{N}$, while $n + my \in C_1 + C_2$, so $(\frac{k}{N}, \frac{k+1}{N})$ contains a point of $C_1 + C_2$. Therefore, every neighborhood of $x \in R^1$ contains a point of $C_1 + C_2$, which means $C_1 + C_2$ is dense in R^1. $C_1 + C_2$ is countable but R^1 is uncountable, so there exists $x \in R^1$ but $x \notin C_1 + C_2$, which means x is a limit point of $C_1 + C_2$ while not in $C_1 + C_2$, so $C_1 + C_2$ is not closed.

26. g is a continuous one-to-one mapping of the compact metric space Y onto $g(Y)$, so g^{-1} is a continuous mapping of $g(Y)$ onto Y by Theorem 4.17. Since Y is compact and g is continuous, $g(Y)$ is compact by Theorem 4.14, and since g^{-1} is continuous, g^{-1} is uniformly continuous by Theorem 4.19. Therefore, if h is uniformly continuous, then $f = g^{-1} \circ h$ is uniformly continuous by Exercise 12. If h is continuous, then $f = g^{-1} \circ h$ is continuous by Theorem 4.7.

Let $X = Z = \{\mathbf{x} \in R^2 \mid |\mathbf{x}| = 1\}$ and $Y = \{x \in R^1 \mid 0 \leq x < 2\pi\}$. Let g be the function from Y onto Z such that $g(t) = (\cos t, \sin t)$. Since g is a one-to-one mapping onto Z, g^{-1} is a well-defined function from Z to Y. Let f be the same as g^{-1} with the domain replaced by X. $h = g \circ f$ is the identity function and therefore uniformly continuous, but f is not continuous at $\mathbf{x} = (0, 1)$. Note that Y is not compact while X and Z are compact.