Chapter 1
The Real and Complex Number Systems

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June 2021

1. If $r+x$ is rational, than $x=(r+x)-r$ is rational, a contradiction, so $r+x$ is irrational.
If rx is rational, than $x=(rx)\frac{1}{r}$ is rational, a contradiction, so rx is irrational.

2. If there is a rational number $\frac{p}{q}$ (p,q are intergers and $gcd(p,q)=1$), whose square is 12. Then $p^2=12q^2$. $3|12$, so $3|p^2$, $3|p$. So $9|p^2$, $9|12q^2$, $3|4q^2$, $3|q$. So $3|p$ and $3|q$, contrary to $gcd(p,q)=1$.

3. (a) $x\neq 0$, so $\frac{1}{x}$ exists. $y=y\cdot x\cdot\frac{1}{x}=z\cdot x\cdot\frac{1}{x}=z$
(b) Take $z=1$ in (a).
(c) Take $z=\frac{1}{x}$ in (a).
(d) $\frac{1}{x}\cdot x=1$, so $x=\frac{1}{\frac{1}{x}}$ from (c).

4. For an element x of E, $\alpha\leq x$ and $x\leq\beta$, so $\alpha\leq\beta$.

5. For every $x\in A$, $x\geq\inf A$, so $-x\leq-\inf A$, so $-\inf A$ is an upper bound of -A. If $\gamma<-\inf A$, $-\gamma>\inf A$, so $-\gamma$ is not a lower bound of A, then there is a element $x<-\gamma$ in A. Then there is an element $-x>\gamma$ in -A, which means γ is not an upper bound of -A. So $-\inf A=\sup(-A)$, and $\inf A=-\sup(-A)$.

6.* (a) $((b^m)^{\frac{1}{n}})^{nq}=b^{mq}=b^{np}=((b^p)^{\frac{1}{q}})^{nq}=k$. By Theorem 1.21, $k^{\frac{1}{nq}}$ is unique, so $(b^m)^{\frac{1}{n}}=(b^p)^{\frac{1}{q}}$.

(b) Let $r=\frac{m}{n}$, $s=\frac{p}{q}$. $(b^{r+s})^{nq}=(b^{\frac{mq+np}{nq}})^{nq}=b^{mq+np}=((b^m)^{\frac{1}{n}}(b^p)^{\frac{1}{q}})^{nq}=(b^rb^s)^{nq}$. By Theorem 1.21, $b^{r+s}=b^rb^s$.

(c) If $t<r$, then $r-t>0$, so $b^{r-t}>1$, $b^r>b^t$. So $b^r\geq b^t$ for all $b^t\in B(r)$, so b^r is an upper bound of $B(r)$. But $b^r\in B(r)$, so every $\gamma<b^r$ is not an upper bound of $B(r)$, so $b^r=\sup B(r)$.

(d) If $b^{x+y}<b^xb^y$, then $\frac{b^{x+y}}{b^y}<b^x$, and there is an rational $t\leq x$ that $\frac{b^{x+y}}{b^y}<b^t$. Then $\frac{b^{x+y}}{b^t}<b^y$, and there is an rational $s\leq y$ that $\frac{b^{x+y}}{b^t}<b^s$. So $b^{s+t}>b^{x+y}$, but $s+t\leq x+y$, contradicting to the definition of b^{x+y}.
If $b^{x+y}>b^xb^y$, then there is a rational $r<x+y$ that $b^r>b^xb^y$ *, which means $b^r>b^pb^q$ for all rational $p\leq x$ and $q\leq y$. Let ε be an positive real number such that $r<x+y-\varepsilon$. By Theorem 1.20(b), there are rational p and q such that $x-\frac{\varepsilon}{2}<p<x$ and $y-\frac{\varepsilon}{2}<q<y$. So $r<x+y-\varepsilon<p+q$, and $b^r<b^{p+q}=b^pb^q$, contradicting with $b^r>b^pb^q$.
Therefore, only $b^{x+y}=b^xb^y$ can be the case.

7. (a) $b^n-1=(b-1)(b^{n-1}+b^{n-2}+\cdots+1)\geq(b-1)(1+1+\cdots+1)=(b-1)n$

(b) $b^{\frac{1}{n}}>1$, so substitute b with $b^{\frac{1}{n}}$ in (a) we obtain (b).

(c) $b-1\geq n(b^{\frac{1}{n}}-1)>\frac{b-1}{t-1}(b^{\frac{1}{n}}-1)$, so $(b^{\frac{1}{n}}-1)<(t-1)$ since $b-1>0$ and $t-1>0$. So $b^{\frac{1}{n}}<t$.

(d) $b^w<y$, so $y\cdot b^{-w}>1$. So from (c), $b^{\frac{1}{n}}<y\cdot b^{-w}$ when $n>\frac{b-1}{y\cdot b^{-w}-1}$. So $b^{w+\frac{1}{n}}<y$ for sufficiently large n.

(e) $b^w>y$, so $\frac{b^w}{y}>1$. So from (c), $b^{\frac{1}{n}}<\frac{b^w}{y}$ when $n>\frac{b-1}{\frac{b^w}{y}-1}$. Since $b^{\frac{1}{n}},y>0$, $b^{w-\frac{1}{n}}>y$ for sufficiently large n.

(f) If $b^x < y$, from (d) $b^{x+\frac{1}{n}} < y$ for some n. Then $x + \frac{1}{n} \in A$, but $x + \frac{1}{n} > x$, contrary to the fact that x in an upper bound of A.

If $b^x > y$, form (e) $b^{x-\frac{1}{n}} > y$ for some n. So $b^{x-\frac{1}{n}} > b^w$, and $x - \frac{1}{n} > w$ for all w in A since $b > 1$. So $x - \frac{1}{n}$ is an upper bound of A, but $x - \frac{1}{n} < x$, contrary to $x = \sup A$.

Therefore, only $b^x = y$ can be the case.

(g) If $x < x'$, let q be a rational that $x < q < x'$(Theorem 1.20(b)). By definition of real power in Exercise 6, $b^x < b^q \leq b^{x'}$, so $b^x < b^{x'}$. Similarly, $x > x'$ implies $b^x > b^{x'}$. Therefore, if $b^x = b^{x'}$, then $x = x'$ and x is unique

8. If complex field is an ordered field, then $1 > 0$ from proposition 1.18(d), $-1 < 0$ from proposition 1.18(a). But $-1 = i^2 > 0$ from proposition 1.18(d), a contradiction, so complex field cannot be an ordered field.

9. When $a < c$, $z < w$. When $a > c$, $z > w$. When $a = c$ and $b < d$, $z < w$. When $a = c$ and $b > d$, $z > w$. When $a = c$ and $b = d$, $z = w$. So definition 1.5(i) is satisfied.

If $z < w$ and $w < r$ ($r = e + fi$), then $a < c$ or $a = c$, $c < e$ or $c = e$. If $a < c$, then $a < e$, so $z < r$. If $a = c$ and $c < e$, then $a < e$, so $z < r$. If $a = c$ and $c = e$, then $b < d$ and $d < f$, so $b < f$, so $z < r$. Therefore, for all cases $z < r$ when $z < w$ and $w < r$, satisfying definition 1.5(ii).

Assume that complex number under this definition has least-upper-bound property. Consider the set $S = \{xi | x \in \mathbb{R}\}$. 1 is greater than all the $0+xi$, so S is bound above. Let $a+bi = \sup S$, $a \geq 0$. If $a > 0$, then $\frac{a}{2} > 0$ is also an upper bound, a contradiction. So $a = 0$, but then $(b+1)i > a+bi$ and $(b+1)i \in S$, a contradiction. Therefore, complex number under this definition cannot have least-upper-bound property.

10. $z^2 = (a^2 - b^2) + 2abi$, $a^2 - b^2 = u$, $2ab = (|w|^2 - u^2)^{\frac{1}{2}} = |v|$, so when $v \geq 0$, $z^2 = w$, when $v \leq 0$, $\overline{z}^2 = u - 2abi = u + vi = w$. When $w \neq 0$, either a or $b \neq 0$, and $z \neq -z$, so $\pm z^2 = w$ when $v \geq 0$ and $\pm \overline{z}^2 = w$ when $v \leq 0$.

11. If $z = 0$, then $r = 0$ and w can be any complex number that $|w| = 1$. w and r are not uniquely determined by z.

If $z \neq 0$, let $z = a + bi$. Let $r = \sqrt{a^2 + b^2}$ and $w = \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}i$. Then $|w| = \frac{a^2+b^2}{a^2+b^2} = 1$, and $z = rw$. If $z = rw$, then $|z| = |r||w| = |r| = r$, and $w = \frac{z}{r}$, so r and w are uniquely determined by z.

12. If $|z_1 + z_2 + \cdots + z_{n-1}| \leq |z_1| + |z_2| + \cdots + |z_{n-1}|$, then by Theorem 1.33(e), $|z_1 + z_2 + \cdots + z_n| \leq |z_1 + z_2 + \cdots + z_{n-1}| + |z_n| \leq |z_1| + |z_2| + \cdots + |z_{n-1}| + |z_n|$. $|z_1| \leq |z_1|$, so the proof completes by induction.

13. $|x| = |x - y + y| \leq |x - y| + |y|$, so $|x| - |y| \leq |x - y|$; $|y| = |y - x + x| \leq |y - x| + |x|$, so $|y| - |x| \leq |y - x| = |x - y|$. So $||x| - |y|| \leq |x - y|$.

14. $|1 + z|^2 + |1 - z|^2 = (1 + z)(1 + \bar{z}) + (1 - z)(1 - \bar{z}) = 1 + z + \bar{z} + 1 + 1 - z - \bar{z} + 1 = 4.$

15. According to the proof of Theorem 1.35, when the equality holds, $\sum |Ba_j - Cb_j|^2 = 0$, so $Ba_j - Cb_j = 0$ for all j, and $a_j = \frac{C}{B}b_j = kb_j$ where k is a constant independent of j. If $a_j = kb_j$ for all j, $C = \sum kb_j\bar{b_j} = kB$, and $Ba_j - Cb_j = Bkb_j - kBb_j = 0$, the equality holds. So $a_j = kb_j$ with k being a constant independent of j is a sufficient and necessary condition for the equality to hold.

16. (a) If $\mathbf{z} = \frac{\mathbf{x}+\mathbf{y}}{2} + \mathbf{w}$, $\mathbf{w} \cdot (\mathbf{x} - \mathbf{y}) = 0$, and $|\mathbf{w}| = \sqrt{r^2 - \frac{d^2}{4}}$, then $|\mathbf{z} - \mathbf{x}| = | - \frac{\mathbf{x}-\mathbf{y}}{2} + \mathbf{w}| = \sqrt{|\mathbf{w}|^2 + |\frac{\mathbf{x}-\mathbf{y}}{2}|^2 - 2\mathbf{w} \cdot \frac{\mathbf{x}-\mathbf{y}}{2}} = \sqrt{r^2 - \frac{d^2}{4} + \frac{d^2}{4}} = r$, and similarly $|\mathbf{z} - \mathbf{y}| = r$. So we want to prove that there are infinitely many $\mathbf{w}$ that satisfy $\mathbf{w} \cdot (\mathbf{x} - \mathbf{y}) = 0$ and $|\mathbf{w}| = \sqrt{r^2 - \frac{d^2}{4}}$. Let $\mathbf{x} - \mathbf{y} = \mathbf{u} \neq 0$, $\sqrt{r^2 - \frac{d^2}{4}} = s > 0$. Let u_i be the coordinate that $u_i \neq 0$, and choose other two coordinates u_j, u_k(they exist because the dimension $k \geq 3$). Let $w_i = au_j + bu_k$, $w_j = -au_i$, $w_k = -bu_i$, and all the other coordinates $= 0$. Then $\mathbf{w} \cdot \mathbf{u} = 0$, and we want to prove that there are infinitely many a, b that satisfy $|\mathbf{w}| = s$. Substituting, we get

$$(au_j + bu_k)^2 + (-au_i)^2 + (-bu_i)^2 = s^2$$

$$(u_i^2 + u_k^2)b^2 + 2u_j u_k a \cdot b + a^2(u_i^2 + u_j^2) - s^2 = 0$$

To solve the quadratic equation for b, the discriminant $\Delta = 4u_j^2u_k^2a^2 - 4(u_i^2 + u_k^2)\left[a^2(u_i^2 + u_j^2) - k^2\right]$. When $a^2 < \frac{k^2}{u_i^2 + u_j^2}$, $\Delta > 0$, the solution of b exists, so there are infinitely many a and b satisfying $|\mathbf{w}| = s$. So there are infinitely many $\mathbf{w}$ satisfying $\mathbf{w} \cdot (\mathbf{x} - \mathbf{y}) = 0$ and $|\mathbf{w}| = \sqrt{r^2 - \frac{d^2}{4}}$, so there are infinitely many $\mathbf{z}$ satisfying $|\mathbf{z} - \mathbf{x}| = r$ and $|\mathbf{z} - \mathbf{y}| = r$.

(b) Let $\mathbf{z} = \frac{\mathbf{x}+\mathbf{y}}{2}$, then $|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = |\frac{\mathbf{x}-\mathbf{y}}{2}| = \frac{d}{2} = r$. If $\mathbf{z}' \neq \mathbf{z}$, then $|\mathbf{z}' - \mathbf{z}| = \varepsilon > 0$, and $|\mathbf{z}' - \mathbf{x}| \geq |\mathbf{z}' - \mathbf{z}| + |\mathbf{z} - \mathbf{x}| = \varepsilon + r > r$, so $\mathbf{z}'$ cannot satisfy the condition, and $\mathbf{z}$ is unique.

(c) If such $\mathbf{z}$ exists, then $2r = |\mathbf{x} - \mathbf{z}| + |\mathbf{z} - \mathbf{y}| \geq |\mathbf{x} - \mathbf{y}| = d$, contradicting with $2r < d$, so the $\mathbf{z}$ cannot exist.

17. $|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = \sum(x_i + y_i)^2 + \sum(x_i - y_i)^2 = 2\sum x_i^2 + 2\sum y_i^2 + \sum 2x_iy_i + \sum(-2)x_iy_i = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$. It can be geometrically interpreted as the sum of squares of two diagonals of parallelograms being equal to the sum of squares of four sides of parallelograms.

18. If $x_1 = x_2 = 0$, let $y_1 = y_2 = 1$, $y_k = 0$ for $k > 2$, then $\mathbf{y} \neq 0$ and $\mathbf{x} \cdot \mathbf{y} = 0$. If one of $x_1, x_2 \neq 0$, let $y_1 = x_2, y_2 = -x_1$, $y_k = 0$ for $k > 2$, then $\mathbf{y} \neq 0$ and $\mathbf{x} \cdot \mathbf{y} = 0$.

When $k = 1$, if $\mathbf{x} \neq 0$ and $\mathbf{y} \neq 0$, then $\mathbf{x} \cdot \mathbf{y} = x_1y_1 \neq 0$, so the $\mathbf{y}$ satisfying the condition does not necessary exist.

19.

$$\sum(x_i - a_i)^2 = 4\sum(x_i - b_i)^2$$

$$\sum 3x_i^2 - (8b_i - 2a_i)x_i + (4b_i^2 - a_i^2) = 0$$

$$\sum 3(x_i - \frac{4b_i - a_i}{3})^2 + (4b_i^2 - a_i^2) - \frac{(4b_i - a_i)^2}{3} = 0$$

$$\sum 3(x_i - \frac{4b_i - a_i}{3})^2 = \sum \frac{4a_i^2 - 8a_ib_i + 4b_i}{3}$$

$$\sum(x_i - \frac{4b_i - a_i}{3})^2 = \sum \frac{4}{9}(b_i - a_i)^2 = \sum(\frac{2}{3}(b_i - a_i))^2$$

So if $\mathbf{c} = \frac{4\mathbf{b}-\mathbf{a}}{3}$ and $r = \frac{2}{3}|\mathbf{b} - \mathbf{a}|$, then the two equations are sufficient and necessary conditions for each other.

20. The proof of least-upper-bound property and axioms (A1) to (A3) did not use property (III) and therefore still hold. Let 0^* be $\{x | x \in \mathbb{Q}, x \leq 0\}$. For every $r \in \alpha$ and $s \in 0^*$, $r + s \leq r$, hence $r + s \in \alpha$, so $\alpha + 0^* \subset \alpha$. For every $r \in \alpha$, $r = r + 0 \in \alpha + 0^*$, so $\alpha \subset \alpha + 0^*$. So $\alpha + 0^* = \alpha$, and (A4) holds. If (A5) holds, let $\alpha = \{x | x \in \mathbb{Q}, x < 1\}$, then there is an α' such that $\alpha + \alpha' = 0^*$. So there are $r \in \alpha$ and $s \in \alpha'$ that $r + s = 0$, and because $r < 1$, $s > -1$. Let $s = -1 + \varepsilon$, $\varepsilon > 0$, and let $t = 1 - \frac{\varepsilon}{2}$, then $t \in \alpha$. $s + t \in \alpha' + \alpha = 0^*$, so $s + t \leq 0$, but $s + t = -1 + \varepsilon + 1 - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} > 0$, a contradiction, so (A5) cannot hold.