# Chapter 2
# Determinants and Matrices

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July 2021

## 2.1 Determinants

**2.1.1** (a) $1 \times (-1 \times 1) = -1$
(b) $1 \times (1 \times 1 - 2 \times 3) - 2 \times (3 \times 1 - 2 \times 0) = -11$
(c) $\frac{1}{\sqrt{2}}(-\sqrt{3}) \times \sqrt{3} \times (-\sqrt{3} \times \sqrt{3}) = \frac{9}{\sqrt{2}}$

**2.1.2** $\begin{vmatrix} 1 & 3 & 3 \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 2$ , So the homogeneous linear independent equations have no nontrivial solutions.

**2.1.3** (a) $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$
(b) $\begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix} = 0$
(c) $(1,1)$, $(2,2)$

**2.1.4** (a) $|A| = \sum_{ij\ldots} \varepsilon_{ij\ldots} a_{1i} a_{2j} \cdots$, which is the sum of all products formed by choosing one entry in each row that they are all in different columns (call it a valid combination), multiplying them together, and multiplying $+1$ or $-1$ depending on the parity of permutation $c_1 c_2 \cdots$, where $c_i$ is the column of the entry chosen in row i. $a_{ji} C_{ji} = a_{ji} M_{ji} (-1)^{j+i}$, is the sum of all products of valid combinations in $M_{ji}$, multiplying $a_{j+i}$, multiplying $(-1)^{j+i}$. If it takes n steps for a permutation in $M_{ji}$ to return to reference order, then it will take $n + |j - i|$ steps for the permutation appended $a_{ji}$ to return to reference order, so $a_{ij} M_{ij} (-1)^{|j-i|} = a_{ij} M_{ij} (-1)^{j+i}$ will contribute to the sum of products of valid combinations in A that contains $a_{ji}$, so $\sum_i a_{ji} C_{ji}$ contains all products of valid combinations, and is therefore equal to $|A|$.

Example:

$$\begin{vmatrix} & o & & \\ o & & & \\ & & & o \\ & & o & \end{vmatrix} \xrightarrow{2\,steps} \begin{vmatrix} o & & & \\ & o & & \\ & & o & \\ & & & o \end{vmatrix}$$

$$\begin{vmatrix} & o & & & \\ & & & a_{24} & \\ o & & & & \\ & & & & o \\ & & o & & \end{vmatrix} \xrightarrow{2\,steps} \begin{vmatrix} o & & & & \\ & & & a_{24} & \\ & o & & & \\ & & o & & \\ & & & & o \end{vmatrix} \xrightarrow{|4-2|\,steps} \begin{vmatrix} o & & & & \\ & a_{24} & & & \\ & & o & & \\ & & & o & \\ & & & & o \end{vmatrix}$$

(b) If $A'$ is the matrix whose $k^{th}$ column is the $j^{th}$ column of $A$ and all the other columns is the same with $A$, then $\sum_i a_{ij} C_{ik} = \sum_i a'_{ik} C'_{ik}$ is the determinant of $A'$, but $A'$ has two equal rows ($j^{th}$ and $k^{th}$ rows), so it equals to zero.

**2.1.5** (a) $\det(H_1) = 1$, $\det(H_2) = 8.3333 \times 10^{-2}$, $\det(H_3) = 4.62963 \times 10^{-4}$
(b) for $\det(H_4)$:
Subtract the last row from each row above it:

$$\begin{vmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{vmatrix} = \begin{vmatrix} \frac{3}{(1)(4)} & \frac{3}{(2)(5)} & \frac{3}{(3)(6)} & \frac{3}{(4)(7)} \\ \frac{2}{(2)(4)} & \frac{2}{(3)(5)} & \frac{2}{(4)(6)} & \frac{2}{(5)(7)} \\ \frac{1}{(3)(4)} & \frac{1}{(4)(5)} & \frac{1}{(5)(6)} & \frac{1}{(6)(7)} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{vmatrix} = \frac{(1)(2)(3)}{(4)(5)(6)(7)} \begin{vmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ 1 & 1 & 1 & 1 \end{vmatrix} = \frac{(3!)^2}{7!} \begin{vmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

Subtract the last column from each column precedes it:

$$\frac{(3!)^2}{7!} \begin{vmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ 1 & 1 & 1 & 1 \end{vmatrix} = \frac{(3!)^2}{7!} \begin{vmatrix} \frac{3}{(1)(4)} & \frac{2}{(2)(4)} & \frac{1}{(3)(4)} & \frac{1}{4} \\ \frac{3}{(2)(5)} & \frac{2}{(3)(5)} & \frac{1}{(4)(5)} & \frac{1}{5} \\ \frac{3}{(3)(6)} & \frac{2}{(4)(6)} & \frac{1}{(5)(6)} & \frac{1}{6} \\ 0 & 0 & 0 & 1 \end{vmatrix} = \frac{(3!)^2}{7!} \frac{(3!)^2}{6!} \begin{vmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ 0 & 0 & 0 & 1 \end{vmatrix} = \frac{(3!)^4}{(7!)(6!)} \det(H_3)$$

By this procedure, we found that

$$\det(H_n) = \frac{(n-1)!^4}{(2n-1)!(2n-1)!} \det(H_{n-1})$$

So $\det(H_4) = \frac{3!^4}{7!6!} \det(H_3) = 1.65344 \times 10^{-7}$, $\det(H_5) = \frac{4!^4}{9!8!} \det(H_4) = 3.74930 \times 10^{-12}$, $\det(H_6) = \frac{5!^4}{11!10!} = 5.36730 \times 10^{-18}$

**2.1.6** Linear dependence implies one row (or column) can be expressed by linear combination of other rows (columns), so $A_{ni} = a_1 A_{1i} + a_2 A_{2i} + \cdots$. Add $-a_j$ times the $j^{th}$ row to the $n^{th}$ row, then the determinant remains the same, but all entries in the $n^{th}$ row becomes 0, so the determinant equals to zero.

**2.1.7** By Gauss's elimination, $x_1 = 1.88282$, $x_2 = -0.36179$, $-0.96889$, $0.44221$, $0.41022$, $0.39219$

**2.1.8** (a) $\sum_i \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$
(b) $\sum_{ij} \delta_{ij}\varepsilon_{ijk} = \sum_i \delta_{ii}\varepsilon_{iik} = 0$
(c) If $i \neq j$, then at least two of $1, j, p, q$ is the same, and $\varepsilon_{ipq}\varepsilon_{jpq} = 0$. If $i = j = 1$, then $\sum_{pq} \varepsilon_{ipq}\varepsilon_{jpq} = \varepsilon_{123}\varepsilon_{123} + \varepsilon_{132}\varepsilon_{132} = 2$, and the case is similar when $i = j = 2$ and $i = j = 3$. Therefore, $\sum_{pq} \varepsilon_{ipq}\varepsilon_{jpq} = 2\delta_{ij}$
(d)$\sum_{ijk} \varepsilon_{ijk}\varepsilon_{ijk} = (-1)^2 \times 6 = 6$

**2.1.9** The only case that $\varepsilon_{ijk}\varepsilon_{pqk} \neq 0$ is : k is one of $(1,2,3)$ and $(i,j), (p,q)$ are the other two of $(1,2,3)$, respectively. So $i = p, j = q$ or $i = q, j = p$. For the former case, $\varepsilon_{ijk}\varepsilon_{pqk} = (\pm 1)^2 = 1 = \delta_{ip}\delta_{jq}$, and for the latter case, $\varepsilon_{ijk}\varepsilon_{pqk} = (1)(-1) = -1 = -\delta_{iq}\delta_{jp}$. Therefore, $\sum_k \varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$

## 2.2 Matrices

**2.2.1**

$$((AB)C)_{il} = \sum_m (AB)_{im} C_{ml} = \sum_m \sum_k A_{ik} B_{km} C_{ml} = \sum_k \sum_m A_{ik} B_{km} C_{ml} = \sum_k A_{ik} (BC)_{kl} = (A(BC))_{il}$$

**2.2.2** If $(A+B)(A-B) = A^2 - B^2$, than $A^2 + BA - AB - B^2 = A^2 - B^2$, so $AB - BA = [A, B] = 0$. If $[A, B] = 0$, then $(A+B)(A-B) = A^2 - B^2 - (AB - BA) = A^2 - B^2$

**2.2.3** (a)

$$(a+ib)+(c+id) \longleftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ -(b+d) & a+c \end{pmatrix} \longleftrightarrow (a+c)+i(b+d)$$

$$(a+ib)(c+id) \longleftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{pmatrix} \longleftrightarrow (ac-bd)+i(ad+bc)$$

(b)

$$(a+ib)^{-1} = \frac{1}{a^2+b^2}(a-ib) \longleftrightarrow \frac{1}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

**2.2.4** Multiply each row of $A$ by $-1$ will turn $A$ into $-A$, so $\det(-A) = (-1)^n \det(A)$.

**2.2.5** (a) If $A^2 = 0$ and $A = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$

$$A^2 = \begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} = 0$$

Then $x^2+yz=0$, $t^2+yz=0$, $y(x+t)=0$, $z(x+t)=0$. Let $y=b^2$, $z=-a^2$, then $x=\pm ab$, $t=\pm ab$. Without less of generality let $x=ab$ because the sign of $a$ and $b$ is arbitrary. If $y \neq 0$, then $t=-x=-ab$; if $y=0$, then $t=x=ab=0$ so $t=-ab$. Therefore, in all cases we can find $a,b$ such that $\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}$

(b) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, then $\det C = \det \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$ but $\det A + \det B = 1+1 = 2$

**2.2.6** $K = \begin{pmatrix} 0 & 0 & i \\ -i & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$, $K^2 = \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 1 \\ i & 0 & 0 \end{pmatrix}$, $K^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -I$, $K^4 = -K$, $K^5 = -K^2$, $K^6 = I$. So if $n = 6k$, k is positive integer, then $K^n = I$

**2.2.7**

$$[A,[B,C]] = [A, BC-CB] = (ABC-ACB)-(BCA-CBA) = ABC-ACB-BCA+CBA$$

$$[B,[A,C]]-[C,[B,A]] = [B, AC-CA]-[C, AB-BA]$$

$= (BAC-BCA)-(ACB-CAB)-[(CAB-CBA)-(ABC-BAC)] = ABC-ACB-BCA+CBA$

So $[A,[B,C]] = [B,[A,C]]-[C,[B,A]]$

**2.2.8** Use the definition of commutator and carry out the corresponding matrix multiplication, then all the three relations are trivially satisfied.

**2.2.9** Carry out the corresponding matrix multiplication, then all the relations are trivially satisfied.

**2.2.10** If $A$ and $B$ are upper right triangular matrices, then $A_{ij}=0$ when $j<i$, $B_{ij}=0$ when $j<i$. $(AB)_{ij} = \sum_k A_{ik}B_{kj}$, when $j<i$: if $k>j$, then $B_{kj}=0$, if $k \le j$, then $k<i$ and $A_{ik}=0$. So in all case $(AB)_{ij} = \sum_k A_{ik}B_{kj} = 0$ when $j<i$, so $AB$ is also an upper right triangular matrix.

**2.2.11** (a)(b) By matrix multiplication the relations hold trivially.

(c) When $i=j$, $\sigma_i\sigma_j + \sigma_j\sigma_i = 2(\sigma_i)^2 = 2I_2 = 2\delta_{ij}I_2$. When $i \neq j$: by (a) we know the inverse matrix of $\sigma_i$ is itself, and by (b) we have $\sigma_i\sigma_j = i\sigma_k$, so $(\sigma_i\sigma_j)^{-1} = \sigma_j^{-1}\sigma_i^{-1} = \sigma_j\sigma_i = (i\sigma_k)^{-1} = -i\sigma_k$, so $\sigma_i\sigma_j + \sigma_j\sigma_i = i\sigma_k - i\sigma_k = 0 = 2\delta_{ij}I_2$. So $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}I_2$ holds for all cases.

**2.2.12** (a)(b) By definition of commutator and matrix multiplication, the relations can be easily verified.

(c) $[M^2, M_i] = 2IM_i - M_i 2I = 0$; $[M_z, L^+] = [M_z, M_x] + i[M_z, M_y] = iM_y + i(-i)M_x = M_x + iM_y$; $[L^+, L^-] = [M_x + iM_y, M_x - iM_y] = i[M_y . M_x] - i[M_x, M_y] = 2M_z$

**2.2.13** It is similar with Exercise 2.2.12.

**2.2.14** If the $i^{th}$ diagonal entries of $A$ is $a_i$, then $(AB)_{ij} = a_i B_{ij}$, and $(BA)_{ij} = B_{ij} a_j$. So if $i \neq j$, then by $a_i B_{ij} = a_j B_{ij}$ and $a_i \neq a_j$, we have $B_{ij} = 0$, which means $B$ is a diagonal matrix.

**2.2.15** $(AB)_{ij} = \sum_k A_{ik} B_{kj} = \sum_k a_i \delta_{ik} b_j \delta_{kj} = a_i b_j \delta_{ij} = a_i b_i \delta_{ij}$. $(BA)_{ij} = \sum_k B_{ik} A_{kj} = \sum_k b_i \delta_{ik} a_j \delta_{kj} = a_j b_i \delta_{ij} = a_i b_i \delta_{ij}$. So $(AB)_{ij} = (BA)_{ij}$, and $A$ and $B$ commute.

**2.2.16** For any two matrices $X, Y$ we have $\text{trace}(XY) = \text{trace}(YX)$. If $A, B$ commute, $\text{trace}(ABC) = \text{trace}(BAC) = \text{trace}(CBA)$; if $B, C$ commute, $\text{trace}(ABC) = \text{trace}(ACB) = \text{trace}(CBA)$; if $A, C$ commute, $\text{trace}(ABC) = \text{trace}(CAB) = \text{trace}(ACB) = \text{trace}(CBA)$.

**2.2.17** $\text{trace}([M_j, M_k]) = \text{trace}(M_j M_k - M_k M_j) = \text{trace}(M_j M_k) - \text{trace}(M_k M_j) = 0 = \text{trace}(iM_l) = i\text{trace}(M_l)$, so $\text{trace}(M_l) = 0$, ans so as $M_j$ and $M_k$ because the commutation relation is cyclic.

**2.2.18** $\text{trace}(A) = \text{trace}(ABB) = \text{trace}(BAB) = \text{trace}(-ABB) = \text{trace}(-A) = -\text{trace}(A)$, so $\text{trace}(A) = 0$. The same is for $\text{trace}(B)$.

**2.2.19** (a) If $AB = -BA$ and both are non-singular (so the matrix inverses exist): $\text{trace}(A) = \text{trace}(ABB^{-1}) = \text{trace}(B^{-1}AB) = \text{trace}(-B^{-1}BA) = \text{trace}(-A) = -\text{trace}(A)$, so $\text{trace}(A) = 0$. The same is for $\text{trace}(B)$.

(b) $A, B$ being non-singular means $\det(A), \det(B) \neq 0$. Suppose they are anti-commuting and n is odd, then $\det(A)\det(B) = \det(AB) = \det(-BA) = \det(-B)\det(A) = (-1)^n \det(B)\det(A) = -\det(A)\det(B)$, so $\det(A)\det(B) = 0$. So either $\det(A)$ or $\det(B)$ equals to zero, contradicting to $\det(A), \det(B) \neq 0$.

**2.2.20** $(A^{-1}A)_{ik} = \sum_j (A^{-1})_{ij} A_{jk} = \sum_j \frac{(-1)^{i+j} M_{ji}}{\det(A)} A_j k = \frac{1}{\det(A)} \sum_j (-1)^{i+j} M_{ji} A_{jk}$. If $i = k$, notice that $\det(A) = \det(A^T) = \sum_j (-1)^{i+j} M_{ji} A_{ji}$, so $(A^{-1}A)_{ik} = \frac{\det(A)}{\det(A)} = 1$; if $i \neq k$, then $(A^{-1}A)_{ik} = \frac{1}{\det(A)} \sum_j (-1)^{i+j} M_{ji} A_{jk} = \frac{1}{\det(A)} A_{jk} C_{ji} = 0$ by Exercise 2.1.4(b) (it is obvious by noticing that $\sum_j (-1)^{i+j} M_{ji} A_{jk}$ is the determinant of A whose $k^{th}$ column is replaced by $j^{th}$ column, and the determinant of matrix with two same column is zero). Therefore, $(A^{-1}A)_{ik} = \delta_{ik}$, so $A^{-1}A = I$.

**2.2.21** (a) The unit matrix with $M_{ii}$ replaced by $k$.

(b) The unit matrix with $M_{im}$ replaced by $-K$.

(c) The unit matrix with $M_{ii}, M_{mm}$ replaced by 0 and $M_{im}, M_{mi}$ replaced by 1.

**2.2.22** (a) The unit matrix with $M_{ii}$ replaced by $k$.

(b) The unit matrix with $M_{mi}$ replaced by $-k$.

(c) The unit matrix with $M_{ii}, M_{mm}$ replaced by 0 and $M_{im}, M_{mi}$ replaced by 1.

**2.2.23** By Gauss-Jordan matrix inversion, $A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & \frac{11}{7} & -\frac{1}{7} \\ 0 & -\frac{1}{7} & \frac{2}{7} \end{pmatrix}$

**2.2.24** (a) $\sum_{i=1}^n T_{ij}$ is the sum of fraction of population of $j$th area having moved to other(including $j$) areas, so the sum add up to 1.

(b) $\sum_{i=1}^n Q_i = \sum_{i=1}^n (TP)_i = \sum_{i=1}^n \sum_{j=1}^n T_{ij} P_j = \sum_{j=1}^n \left(\sum_{i=1}^n T_{ij}\right) P_j = \sum_{j=1}^n P_j = 1$

**2.2.25**

$$\left(\begin{array}{cccccc|cccccc} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{32} & \frac{1}{16} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_i - \frac{1}{2}R_{i-1} \to R_i \; i=6\sim 2}$$

$$\left(\begin{array}{cccccc|cccccc} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{3}{8} & \frac{3}{16} & \frac{3}{32} & \frac{3}{64} & -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & \frac{3}{8} & \frac{3}{16} & \frac{3}{32} & 0 & -\frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & \frac{3}{8} & \frac{3}{16} & 0 & 0 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{3}{8} & 0 & 0 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 \end{array}\right) \xrightarrow[R_i - \frac{1}{2}R_{i+1} \to R_i \; i=1\sim 5]{\frac{3}{4}R_1 \to R_1}$$

$$\left(\begin{array}{cccccc|cccccc} \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{5}{4} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{5}{4} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{5}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{5}{4} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 \end{array}\right) \xrightarrow{\frac{4}{3}R_i \to R_i \; i=1\sim 6}$$

$$\left(\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & \frac{4}{3} & -\frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\frac{2}{3} & \frac{5}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{2}{3} & \frac{5}{3} & -\frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{2}{3} & \frac{5}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{2}{3} & \frac{5}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{2}{3} & \frac{4}{3} \end{array}\right)$$

**2.2.26** If $A, B$ are orthogonal, $AA^T = I$ and $BB^T = I$, then $(AB)(AB)^T = ABB^TA^T = AA^T = I$, so $AB$ is also orthogonal.

**2.2.27** $\det(AA^T) = \det(A)\det(A^T) = (\det(A))^2 = \det(I) = 1$, so $\det(A) = \pm 1$

**2.2.28** If $A = A^T$ and $B = -B^T$, then $\text{trace}(AB) = \text{trace}(BA) = \text{trace}(-B^TA^T) = \text{trace}(-(AB)^T) = -\text{trace}(AB)$, so $\text{trace}(AB) = 0$.

**2.2.29** $AA^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so $a^2 + b^2 = 1$, $c^2 + d^2 = 1$, $ac + bd = 0$. Let $\theta = \tan^{-1}\frac{b}{a}$, then $a = \cos\theta$, $b = \sin\theta$, $\frac{c}{d} = -\tan\theta$, $c = -\sin\theta$, $d = \cos\theta$. So the most general form is $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

**2.2.30** $\det(A^*) = \sum_{ij\cdots} \varepsilon_{ij\cdots} a_{1i}^* a_{2j}^* \cdots = (\sum_{ij\cdots} \varepsilon_{ij\cdots} a_{1i} a_{2j} \cdots)^* = (\det A)^*$
$\det(A^*) = \det((A^*)^T) = A^\dagger$

**2.2.31** If two of the matrices are real, then their commutator is real, so $i$ multiply the third matrix must be real, so the third matrix must be pure imaginary.

**2.2.32** $(AB)^\dagger = ((AB)^T)^* = (B^TA^T)^* = ((B)^T)^*((A)^T)^* = B^\dagger A^\dagger$

**2.2.33** $S^\dagger_{ij} = S^*_{ji}$, so trace$(S^\dagger S) = \sum_i (S^\dagger S)_{ii} = \sum_i\sum_j S^\dagger_{ij}S_{ji} = \sum_i\sum_j |S_{ji}|^2 > 0$ when $S$ is not null matrix.

**2.2.34** $A^\dagger = A$, $B^\dagger = B$. $(AB+BA)^\dagger = B^\dagger A^\dagger + A^\dagger B^\dagger = BA + AB = AB + BA$; $[i(AB-BA)]^\dagger = -i(B^\dagger A^\dagger - A^\dagger B^\dagger) = -i(BA-AB) = i(AB-BA)$. So both $(AB-BA)$ and $i(AB-BA)$ are Hermitian.

**2.2.35** $(C+C^\dagger)^\dagger = C^\dagger + C = C + C^\dagger$; $[i(C-C^\dagger)]^\dagger = -i(C^\dagger - C) = i(C-C^\dagger)$. So both matrices are Hermitian,

**2.2.36** $C = -i(AB-BA)$, $C^\dagger = i(B^\dagger A^\dagger - A^\dagger B^\dagger) = i(BA-AB) = -i(AB-BA) = C$ so $C$ is Hermitian.

**2.2.37** If $AB = BA$, then $(AB)^\dagger = B^\dagger A^\dagger = BA = AB$ so $AB$ is Hermitian; if $AB$ is Hermitian, then $AB = (AB)^\dagger = B^\dagger A^\dagger = BA$. Therefore, $AB = BA$ is a necessary and sufficient condition for $AB$ to be Hermitian.

**2.2.38** $UU^\dagger = I$, $U^\dagger = U^{-1}$, $U = (U^{-1})^\dagger$, $U^{-1}U = I = U^{-1}(U^{-1})^\dagger$, so $U^{-1}$ is unitary.

**2.2.39** It is obvious that $(A\otimes B)^T = A^T\otimes B^T$, so $(A\otimes B)^\dagger = A^\dagger\otimes B^\dagger$. If $A$ and $B$ are unitary, then $(A\otimes B)(A\otimes B)^\dagger = (A\otimes B)(A^\dagger\otimes B^\dagger) = (AA^\dagger\otimes BB^\dagger) = I_1\otimes I_2 = I$

**2.2.40** $\boldsymbol{\sigma}\cdot\mathbf{p} = \sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3 = \begin{pmatrix} 0 & p_1 \\ p_1 & 0\end{pmatrix} + \begin{pmatrix} 0 & -ip_2 \\ ip_2 & 0\end{pmatrix} + \begin{pmatrix} p_3 & 0 \\ 0 & -p_3\end{pmatrix} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3\end{pmatrix}$, so $(\boldsymbol{\sigma}\cdot\mathbf{p})^2 = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3\end{pmatrix}\begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3\end{pmatrix} = \begin{pmatrix} \mathbf{p}^2 & 0 \\ 0 & \mathbf{p}^2\end{pmatrix} = \mathbf{p}^2\mathbf{1}_2$

**2.2.41** $(\gamma^0)^2 = (\sigma_3)^2\otimes(1_2)^2 = 1_2\otimes 1_2 = 1_4$. $(\gamma^i)^2 = \gamma^2\otimes(\sigma_i)^2 = (-1_2)\otimes 1_2 = -1_4$. When $\mu\neq 0$, $\gamma^\mu\gamma^i + \gamma^i\gamma^\mu = \gamma^2\otimes(\sigma_\mu\sigma_i) + \gamma^2\otimes(\sigma_i\sigma_\mu) = \gamma^2\otimes(\sigma_\mu\sigma_i + \sigma_i\sigma_\mu) = \gamma^2\otimes 0 = 0$; when $\mu = 0$, $\gamma^0\gamma^i + \gamma^i\gamma^0 = (\sigma_3\gamma)\otimes(\sigma_i) + (\gamma\sigma_3)\otimes(\sigma_i) = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\otimes(\sigma_i) + \begin{pmatrix} 0 & -1 \\ -1 & 0\end{pmatrix}\otimes(\sigma_i) = 0$

**2.2.42** $\gamma^5\gamma^\mu = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu$. Switch $\gamma^\mu$ with $\gamma^i$ left to it: if $i = \mu$, it won't cange; if $i\neq\mu$, it will be multiplied $(-1)$ by the anti-commuting properties. So after switching four times to move $\gamma^\mu$ to the left-most side, three $(-1)$ have been multiplied, so $\gamma^5\gamma^\mu = -i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^\mu\gamma^5$, so $\gamma^5$ anti-commutes with all four $\gamma^\mu$.

**2.2.43** ($\gamma_\mu = \sum g_{\nu\mu}\gamma^\mu$ = should be $\gamma_\nu = \sum g_{\nu\mu}\gamma^\mu$) $\gamma_0 = \gamma^0$ and $\gamma_i = -\gamma^i$, $i = 1,2,3$. Along with $(\gamma^0)^2 = 1$ and $(\gamma^i)^2 = -1$, we have $\gamma_\mu\gamma^\mu = 1$, $\mu = 0,1,2,3$.

(a) If $\mu = \alpha$, $\gamma_\mu\gamma^\alpha\gamma^\mu = \gamma_\mu\gamma^\mu\gamma^\alpha = \gamma^\alpha$; if $\mu\neq\alpha$, $\gamma_\mu\gamma^\alpha\gamma^\mu = -\gamma_\mu\gamma^\mu\gamma^\alpha = -\gamma^\alpha$. So $\sum\gamma_\mu\gamma^\alpha\gamma^\mu = (1-3)\gamma^\alpha = -2\gamma^\alpha$

(b) If $\alpha = \beta$, $\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu = \gamma^\alpha\gamma^\beta = (\gamma^\alpha)^2 = g^{\alpha\alpha} = g^{\alpha\beta}$ for all $\mu = 0,1,2,3$, so $\sum\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu = 4g^{\alpha\beta}$. If $\alpha\neq\beta$, for $\mu = \alpha$ or $\mu = \beta$, $\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu = -\gamma^\alpha\gamma^\beta$, and for $\mu\neq\alpha$ and $\mu\neq\beta$, $\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu = \gamma^\alpha\gamma^\beta$, so $\sum\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu = (-2+2)\gamma^\alpha\gamma^\beta = 0 = g^{\alpha\beta}$. So $\sum\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu = 4g^{\alpha\beta}$.

(c) If $\alpha,\beta,\nu$ are different with each other, then $\sum\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^\mu = (3-1)\gamma^\alpha\gamma^\beta\gamma^\nu = 2(-1)^3\gamma^\nu\gamma^\beta\gamma^\alpha = -2\gamma^\nu\gamma^\beta\gamma^\alpha$. If only two of $\alpha,\beta,\nu$ are the same, then $\sum\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^\mu = (-3+1)\gamma^\alpha\gamma^\beta\gamma^\nu = -2(1)(-1)^2\gamma^\nu\gamma^\beta\gamma^\alpha$ $= -2\gamma^\nu\gamma^\beta\gamma^\alpha$. If $\alpha,\beta,\nu$ are all the same, then $\sum\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^\mu = (-3+1)\gamma^\alpha\gamma^\beta\gamma^\nu = -2\gamma^\nu\gamma^\beta\gamma^\alpha$. Therefore, in all cases, $\sum\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^\mu = -2\gamma^\nu\gamma^\beta\gamma^\alpha$.

**2.2.44** $(\gamma^5)^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -(-1)^{3+2+1}(\gamma^0)^2(\gamma^1)^2(\gamma^2)^2(\gamma^3)^2 = 1$, so
$M^2 = \frac{1}{4}(1 + 2\gamma^5 + (\gamma^5)^2) = \frac{1}{4}(2 + 2\gamma^5) = \frac{1}{2}(1+\gamma^5) = M$.

**2.2.45** By evaluation, we can found that the 16 Dirac matrices is equal to $(i)^n\sigma_i \otimes \sigma_j$, $i, j = 0, 1, 2, 3$ with each Dirac matrix having different $(i, j)$, if we let $\sigma_0 = 1_2$, $n$ depend on $i, j$. Then the problem is equivalent to prove that the 16 $\sigma_i \otimes \sigma_j$ form a linearly independent set. If $a, b \neq 0$ and $(i.j) \neq (k, l)$, then $a\sigma_i \otimes \sigma_j + b\sigma_k \otimes \sigma_l \neq 0$ because the four $\sigma_\mu$ form a linearly independent set. So the 16 $\sigma_i \otimes \sigma_j$ form a linearly independent set, and the 16 Dirac matrices form a linearly independent set

**2.2.46** (The 16 Dirac matrices is defined as $E_{ij} = \sigma_i \otimes \sigma_j$, $i, j = 0, 1, 2, 3$, where $\sigma_0 = I_2$)
For $(i, j) \neq (0, 0)$, $\text{trace}(E_{ij}) = \text{trace}(\sigma_i)(\sigma_j) = 0$ because $\text{trace}(\sigma_k) = 0$ when $k = 1, 2, 3$. So $\text{trace}(E_{ij}E_{mn}) \neq 0$ only when $(\sigma_i \otimes \sigma_j)(\sigma_m \otimes \sigma_n) = \sigma_0 \otimes \sigma_0$, that is, $i = m$ and $j = n$. Let $c_i = c_{mn}$, $\Gamma_i = E_{mn}$, $\sum_{i=1}^{16} c_i\Gamma_i = \sum_{i,j=0}^{3} c_{ij}E_{ij}$, then $\text{trace}(A\Gamma_i) = \text{trace}(\sum_{i,j=0}^{3} c_{ij}E_{ij}E_{mn}) = c_{mn}\text{trace}(E_{mn}E_{mn}) = c_{mn}\text{trace}(I_4) = 4c_{mn} = 4c_i$, so $c_i = \frac{1}{4}\text{trace}(A\Gamma_i)$.

**2.2.47** Note that $(\gamma^0)^T = \gamma^0$, $(\gamma^1)^T = -\gamma^1$, $(\gamma^2)^T = \gamma^2$, $(\gamma^3)^T = -\gamma^3$. $C^{-1} = -i(\gamma^0)^{-1}(\gamma^2)^{-1} = i\gamma^0\gamma^2$, so $C\gamma^\mu C^{-1} = -\gamma^2\gamma^0\gamma^\mu\gamma^0\gamma^2$. If $\mu = 0$ or 2, $-\gamma^2\gamma^0\gamma^\mu\gamma^0\gamma^2 = -(-1)(1)(-1)\gamma^\mu = -\gamma^\mu = -(\gamma^\mu)^T$; If $\mu = 1$ or 3, $-\gamma^2\gamma^0\gamma^\mu\gamma^0\gamma^2 = -(-1)^2(1)(-1)\gamma^\mu = \gamma^\mu = -(\gamma^\mu)^T$. So in all cases $C\gamma^\mu C^{-1} = -(\gamma^\mu)^T$.

**2.2.48** (a)

$$\gamma^0 mc^2 = mc^2\sigma_3 \otimes I_2 = \begin{pmatrix} mc^2 & 0 \\ 0 & -mc^2 \end{pmatrix}$$

$$c(\alpha_1p_1 + \alpha_2p_2 + \alpha_3p_3) = c\gamma^0(\gamma^1p_1 + \gamma^2p_2 + \gamma^3p_3) = c(\sigma_3 \otimes I_2)(\gamma \otimes \sigma_1p_1 + \gamma \otimes \sigma_2p_2 + \gamma \otimes \sigma_3p_3)$$

$$= c\sigma_1 \otimes (\sigma_1p_1 + \sigma_2p_2 + \sigma_3p_3) = \begin{pmatrix} 0 & c(\sigma_1p_1 + \sigma_2p_2 + \sigma_3p_3) \\ \sigma_1p_1 + \sigma_2p_2 + \sigma_3p_3 & 0 \end{pmatrix}$$

$$-EI_4 = -EI_2 \otimes I_2 = \begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix}$$

$$\psi = \begin{pmatrix} \psi_L \\ \psi_S \end{pmatrix}$$

So $\left[\gamma^0 mc^2 + c(\alpha_1p_1 + \alpha_2p_2 + \alpha_3p_3) - E\right]\psi = 0$ becomes

$$\begin{pmatrix} mc^2 - E & c(\sigma_1p_1 + \sigma_2p_2 + \sigma_3p_3) \\ c(\sigma_1p_1 + \sigma_2p_2 + \sigma_3p_3) & -mc^2 - E \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_S \end{pmatrix} = 0$$

(b) By the indicated approximation, the equation becomes

$$\begin{pmatrix} -\varepsilon & c(\sigma_1p_1 + \sigma_2p_2 + \sigma_3p_3) \\ c(\sigma_1p_1 + \sigma_2p_2 + \sigma_3p_3) & -2mc^2 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_S \end{pmatrix} = 0$$

It can be separated to $\varepsilon\psi_L = c(\alpha_1p_1 + \alpha_2p_2 + \alpha_3p_3)\psi_S$ and $c(\alpha_1p_1 + \alpha_2p_2 + \alpha_3p_3)\psi_L = 2mc^2\psi_S$. Eliminating $\psi_S$ and we obtain $\frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right)\psi_L = \varepsilon\psi_L$

(c) From the two separated equations, we can get $(\frac{\psi_S}{\psi_L})^2 = \frac{\varepsilon}{2mc^2} \ll 1$ in the non-relativistic approximation.

**2.2.49**

$$(\gamma^0)^2 = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} = I_4$$

$$(\gamma^i)^2 = \begin{pmatrix} -\sigma_i^2 & 0 \\ 0 & -\sigma_i^2 \end{pmatrix} = \begin{pmatrix} -I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = -I_4,\ i = 1, 2, 3$$

$$\gamma^\mu\gamma^i + \gamma^i\gamma^\mu = \begin{pmatrix} -\sigma_\mu\sigma_i & 0 \\ 0 & -\sigma_\mu\sigma_i \end{pmatrix} + \begin{pmatrix} -\sigma_i\sigma_\mu & 0 \\ 0 & -\sigma_i\sigma_\mu \end{pmatrix} = 0,\ \mu \neq i$$

**2.2.50** As in Exercise 2.2.48 but with $\gamma^0$ and $\gamma^i$ defined in Exercise 2.2.49, the Dirac equation becomes

$$\begin{pmatrix} -c(\sigma_1p_1 + \sigma_2p_2 + \sigma_3p_3) - E & mc^2 \\ mc^2 & c(\sigma_1p_1 + \sigma_2p_2 + \sigma_3p_3) - E \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_S \end{pmatrix} = 0$$

In the limit that m approaches zero, the equation becomes

$$\begin{pmatrix} -c(\sigma_1p_1 + \sigma_2p_2 + \sigma_3p_3) - E & 0 \\ 0 & c(\sigma_1p_1 + \sigma_2p_2 + \sigma_3p_3) - E \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_S \end{pmatrix} = 0$$

which separates into independent $2 \times 2$ blocks.

**2.2.51** (a) $|r'|^2 = r'^\dagger r' = r^\dagger U^\dagger U r = r^\dagger r = |r|^2$
(b) $r^\dagger r = r'^\dagger r' = r^\dagger U^\dagger U r$ for any $r$, so $U^\dagger U = I$.