Chapter 7
Ordinary Differential Equations

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7.2 First-Order Equations

7.2.1 (a)

$$\frac{1}{I}dI = -\frac{1}{RC}dt$$
$$\ln\frac{I}{I_0} = -\frac{t}{RC}$$
$$I = I_0 e^{-\frac{t}{RC}}$$

(b)

$$I_0 = \frac{V_0}{R} = 0.1\,mA$$
$$I(100) = 0.1e^{-\frac{10^2}{10^4}} = 0.099\,mA$$

7.2.2

$$\frac{1}{f}df = -\frac{s}{s^2+1}ds$$
$$\ln f = -\frac{1}{2}\ln(s^2+1) + C'$$
$$f = \frac{C}{\sqrt{s^2+1}}$$

7.2.3

$$-\frac{1}{N^2}dN = kdt$$
$$\frac{1}{N} - \frac{1}{N_0} = kt$$
$$N = N_0(1+N_0kt)^{-1} = (1+\frac{t}{\tau_0})^{-1}$$

7.2.4 (a)

$$\frac{1}{(A_0-C)(B_0-C)}dC = \alpha dt$$
$$\frac{1}{A_0-B_0}(-\frac{1}{A_0-C} + \frac{1}{B_0-C})dC = \alpha dt$$
$$\frac{1}{A_0-B_0}(\ln\frac{A_0-C}{A_0} - \ln\frac{B_0-C}{B_0}) = \alpha t$$
$$\frac{A_0-C}{B_0-C}\frac{B_0}{A_0} = e^{(A_0-B_0)\alpha t}$$
$$C = \frac{A_0B_0(e^{(A_0-B_0)\alpha t}-1)}{A_0e^{(A_0-B_0)\alpha t} - B_0}$$

(b)

$$\frac{1}{(A_0-C)^2}dC = \alpha dt$$
$$\frac{1}{A_0-C} - \frac{1}{A_0} = \alpha t$$
$$C = \frac{A_0^2\,\alpha t}{1+A_0\alpha t}$$

7.2.5

$$-v^{-n}dv = \frac{k}{m}dt$$
$$\frac{1}{n-1}(v^{-n+1} - v_0^{-n+1}) = \frac{k}{m}t$$
$$v = v_0\left(1+(n-1)\frac{k}{m}v_0^{n-1}t\right)^{1/(1-n)}$$

because $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$, so

$$-v^{1-n}dv = \frac{k}{m}dx$$
$$\frac{1}{n-2}(v^{2-n} - v_0^{2-n}) = \frac{k}{m}x$$
$$v = v_0\left(1+(n-2)\frac{k}{m}v_0^{n-2}x\right)^{1/(2-n)}$$

7.2.6 $y = ux$, $dy = udx + xdu$. Substituting,

$$\frac{udx + xdu}{dx} = g(u)$$
$$xdu = (g(u)-u)dx$$

which is a separable equation in u and x.

7.2.7 The equation being exact means that we can find a φ such that

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy = P(x,y)dx + Q(x,y)dy$$

So

$$\frac{\partial\varphi}{\partial x} = P(x,y), \qquad \varphi = \int_{x_0}^{x} P(x,y)dx + A(y)$$
$$\frac{\partial\varphi}{\partial y} = Q(x,y), \qquad \varphi = \int_{y_0}^{y} Q(x,y)dy + B(x)$$

for $x = x_0$,

$$\varphi = A(y) = \int_{y_0}^{y} Q(x_0,y)dy + B(x_0)$$

so

$$\varphi = \int_{x_0}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x_0,y)dy + B(x_0)$$

because $d\varphi$ will not change when adding a constant, so we can omit $B(x_0)$, and φ becomes

$$\varphi = \int_{x_0}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x_0,y)dy = constant$$

7.2.8 The equation being exact implies $\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$. So

$$\frac{\partial \varphi}{\partial x} = P(x,y) + \int_{y_0}^{y} \frac{\partial Q(x_0,y)}{\partial x} dy = P(x,y)$$

$$\begin{aligned}\frac{\partial \varphi}{\partial y} &= \int_{x_0}^{x} \frac{\partial P(x,y)}{\partial y} dx + Q(x_0,y) \\ &= \int_{x_0}^{x} \frac{\partial Q(x,y)}{\partial x} dx + Q(x_0,y) \\ &= [Q(x,y) - Q(x_0,y)] + Q(x_0,y) = Q(x,y)\end{aligned}$$

7.2.9 Eq. 7.12 can be rearranged to

$$[\alpha(x)p(x)y - \alpha(x)q(x)]dx + \alpha(x)dy = 0 = P(x,y)dx + Q(x,y)dy$$

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial[\alpha(x)p(x)y - \alpha(x)q(x)]}{\partial y} = \alpha(x)p(x)$$

$$\frac{\partial Q(x,y)}{\partial x} = \frac{\partial \alpha(x)}{\partial x} = \alpha(x)p(x)$$

so $\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$, which means the equation is exact.

7.2.10 The necessary and sufficient condition for the equation to be exact is $\frac{\partial f(x)}{\partial y} = \frac{\partial (g(x)h(y))}{\partial x}$. But $\frac{\partial f(x)}{\partial y} = 0$, so $\frac{\partial (g(x)h(y))}{\partial x} = \frac{\partial g(x)}{\partial x} h(y) = 0$. Because $h(y) \neq 0$, so it must be $\frac{\partial g(x)}{\partial x} = 0$, which means $g(x) = constant$.

7.2.11

$$\begin{aligned}\frac{dy}{dx} &= e^{-\int^x p(t)dt}(-p(x))\left(\int^x e^{\int^s p(t)dt} q(s)ds + c\right) + e^{-\int^x p(t)dt}\left(e^{\int^x p(t)dt} q(x)\right) \\ &= -p(x)y + q(x)\end{aligned}$$

so

$$\frac{dy}{dx} + p(x)y = q(x)$$

7.2.12

$$\frac{m}{mg - bv} dv = dt$$

$$-\frac{m}{b} \ln \frac{mg - bv}{mg - bv_0} = t$$

$$mg - bv = (mg - bv_0)e^{-\frac{b}{m}t}$$

$$v = \frac{mg}{b} - (\frac{mg}{b} - v_0)e^{-\frac{b}{m}t}$$

If $v_0 = 0$, the equation become

$$v = \frac{mg}{b}(1 - e^{-\frac{b}{m}t})$$

7.2.13

$$\frac{1}{N_1}dN_1 = \lambda_1 dt$$
$$\ln\frac{N_1}{N_0} = -\lambda_1 t$$
$$N_1 = N_0 e^{-\lambda_1 t}$$

so

$$\frac{dN_2}{dt} = \lambda_1 N_0 e^{-\lambda_1 t} - \lambda_2 N_2$$
$$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_0 e^{-\lambda_1 t}$$

Let the integrating factor α be

$$\alpha = e^{\int \lambda_2 dt} = e^{\lambda_2 t}$$

Multiplying, the equation becomes

$$\frac{d}{dt}(e^{\lambda_2 t}N_2) = \lambda_1 N_0 e^{(\lambda_2-\lambda_1)t}$$
$$e^{\lambda_2 t}(N_2 - 0) = \frac{\lambda_1}{\lambda_2-\lambda_1}N_0\left(e^{(\lambda_2-\lambda_1)t} - 1\right)$$

So $N_2(t)$ is

$$N_2 = \frac{\lambda_1}{\lambda_2-\lambda_1}N_0(e^{-\lambda_1 t} - e^{-\lambda_2 t})$$

7.2.14 Let V be the volume, and r be the radius of the drop. $V \propto r^3$, so $\frac{dV}{dr} \propto r^2$, and $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} \propto r^2\frac{dr}{dt}$. Also $\frac{dV}{dt} \propto -r^2$ by the model, so $r^2\frac{dr}{dt} = -kr^2$, $\frac{dr}{dt} = -k$, $r = r_0 - kt$.

7.2.15 (a) $\frac{1}{v}dv = -adt$, $\ln\frac{v}{v_0} = -at$, $v = v_0 e^{-at}$.
(b) $\frac{1}{v}dv + adt = 0$, $\varphi = \int\frac{1}{v}dv + \int adt = constant$, $\ln v + at = \ln v_0$, $v = v_0\, e^{-at}$.
(c) $v(t) = \frac{C}{\alpha(t)}$, $\alpha(t) = e^{\int adt} = e^{at}$, $v = Ce^{-at}$, $v = v_0 e^{-at}$.

7.2.16 (a) $m\dot{v} = mg - bv^2$. Let $v_0 = \sqrt{\frac{mg}{b}}$ and rearrange:

$$\frac{1}{v_0^2 - v^2}dv = \frac{b}{m}dt$$
$$\frac{1}{2v_0}(\frac{1}{v+v_0} - \frac{1}{v-v_0})dv = \frac{b}{m}dt$$
$$\frac{1}{2v_0}\ln\frac{v+v_0}{v_i+v_0}\frac{v_i-v_0}{v-v_0} = \frac{b}{m}t$$

Let $T = \sqrt{\frac{m}{gb}}$,

$$\frac{v+v_0}{v_i+v_0}\frac{v_i-v_0}{v-v_0} = e^{\frac{2t}{T}}$$
$$(v_i - v_0)e^{-\frac{t}{T}}v + (v_i - v_0)e^{-\frac{t}{T}}v_0 = (v_i+v_0)e^{\frac{t}{T}}v - (v_i+v_0)e^{\frac{t}{T}}v_0$$
$$(v_i\sinh\frac{t}{T} + v_0\cosh\frac{t}{T})v = (v_i\cosh\frac{t}{T} + v_0\sinh\frac{t}{T})v_0$$
$$v = v_0\frac{v_i + v_0\tanh\frac{t}{T}}{v_i\tanh\frac{t}{T} + v_0}$$

(b)

$$v_0 = \sqrt{\frac{mg}{b}} = 52\ m/s$$

7.2.17 Let $u = xy,\ du = ydx + xdy = ydx + \frac{u}{y}dy$. Substitute u for x,

$$(uy - y)\frac{du - \frac{u}{y}dy}{y} + \frac{u}{y}dy = 0$$

$$\frac{u-1}{u^2 - 2u}du = \frac{1}{y}dy$$

$$\frac{1}{2}(\frac{1}{u} + \frac{1}{u-2})du = \frac{1}{y}dy$$

$$\frac{1}{2}\ln u(u-2) = \ln y + C'$$

$$u(u-2) = Cy^2$$

$$\frac{x^2y - 2x}{y} = C$$

7.2.18 Let $y = ux$, $dy = udx + xdu$ and substitute u for y:

$$(x^2 - u^2x^2e^u)dx + (x^2 + ux^2)e^u(udx + xdu) = 0$$

$$\frac{1}{x}dx = -\frac{(1+u)e^u}{1+ue^u}du$$

$$\ln x = -\ln(1 + ue^u) + C'$$

$$x(1 + ue^u) = C$$

$$x + ye^{\frac{y}{x}} = C$$

7.3 ODEs with Constant Coefficients

7.3.1 Let $y = e^{mx}$ and substitute:

$$m^3 - 2m^2 - m + 2 = 0$$

$$(m-2)(m-1)(m+1) = 0$$

$$y = C_1e^{2x} + C_2e^x + C_3e^{-x}$$

7.3.2 Let $y = e^{mx}$ and substitute:

$$m^3 - 2m^2 + m - 2 = 0$$

$$(m-2)(m-i)(m+i) = 0$$

$$y = C_1e^{2x} + C_2'e^{ix} + C_3'e^{-ix}$$

$$= C_1e^{2x} + C_2\cos x + C_3\sin x$$

7.3.3 Let $y = e^{mx}$ and substitute:

$$m^3 - 3m + 2 = 0$$

$$(m-1)^2(m+2) = 0$$

$$y = C_1e^x + C_2xe^x + C_3e^{-2x}$$

7.3.4 Let $y = e^{mx}$ and substitute:

$$m^2 + 2m + 2 = 0$$

$$m = -1 + i,\ -1 - i$$

$$y = C_1'e^{(-1+i)x} + C_2'e^{(-1-i)x}$$

$$= e^{-x}(C_1'e^{ix} + C_2'e^{-ix})$$

$$= e^{-x}(C_1\cos x + C_2\sin x)$$

7.4 Second-Order Linear ODEs

7.4.1

$$y'' + \frac{-2x}{1-x^2}y' + \frac{l(l+1)}{1-x^2}y = 0$$

so $P(x) = \frac{-2x}{1-x^2},\ Q(x) = \frac{l(l+1)}{1-x^2}$.

$$\lim_{x\to -1} P(x) = \infty, \qquad \lim_{x\to -1}(x+1)P(x) = 1, \qquad \lim_{x\to -1}(x+1)^2 Q(x) = 0$$

so $x = -1$ is a regular singularity.

$$\lim_{x\to 1} P(x) = \infty, \qquad \lim_{x\to 1}(x-1)P(x) = 1, \qquad \lim_{x\to 1}(x-1)^2 Q(x) = 0$$

so $x = 1$ is a regular singularity.

$$\frac{2z - P(z^{-1})}{z^2} = \frac{2z}{z^2-1}, \qquad \frac{Q(z^{-1})}{z^4} = \frac{l(l+1)}{z^2(z^2-1)}$$

$$\lim_{z\to 0} \frac{Q(z^{-1})}{z^4} = \infty, \qquad \lim_{z\to 0} z\cdot\frac{2z - P(z^{-1})}{z^2} = 0 \qquad \lim_{z\to 0} z^2\cdot\frac{Q(z^{-1})}{z^4} = -l(l+1)$$

so $x = \infty$ is a regular singularity.

7.4.2

$$y'' + \frac{1-x}{x}y' + \frac{a}{x}y = 0$$

so $P(x) = \frac{1-x}{x},\ Q(x) = \frac{a}{x}$.

$$\lim_{x\to 0} P(x) = \infty, \qquad \lim_{x\to 0} xP(x) = 1, \qquad \lim_{x\to 0} x^2 Q(x) = 0$$

so $x = 0$ is a regular singularity.

$$\frac{2z - P(z^{-1})}{z^2} = \frac{z+1}{z^2}, \qquad \frac{Q(z^{-1})}{z^4} = \frac{a}{z^3}$$

$$\lim_{z\to 0} \frac{2z - P(z^{-1})}{z^2} = \infty, \qquad \lim_{z\to 0} z\cdot\frac{2z - P(z^{-1})}{z^2} = \infty$$

so $x = \infty$ is an irregular singularity.

7.4.3

$$y'' + \frac{-x}{1-x^2}y' + \frac{n^2}{1-x^2}y = 0$$

so $P(x) = \frac{-x}{1-x^2},\ Q(x) = \frac{n^2}{1-x^2}$.

$$\lim_{x\to -1} P(x) = \infty, \qquad \lim_{x\to -1}(x+1)P(x) = \frac{1}{2}, \qquad \lim_{x\to -1}(x+1)^2 Q(x) = 0$$

so $x = -1$ is a regular singularity.

$$\lim_{x\to 1} P(x) = \infty, \qquad \lim_{x\to 1}(x-1)P(x) = \frac{1}{2}, \qquad \lim_{x\to 1}(x-1)^2 Q(x) = 0$$

so $x = 1$ is a regular singularity.

$$\frac{2z - P(z^{-1})}{z^2} = \frac{2z^2-1}{z(z^2-1)}, \qquad \frac{Q(z^{-1})}{z^4} = \frac{n^2}{z^2(z^2-1)}$$

$$\lim_{z\to 0} \frac{2z - P(z^{-1})}{z^2} = \infty, \qquad \lim_{z\to 0} z\cdot\frac{2z - P(z^{-1})}{z^2} = 1 \qquad \lim_{z\to 0} z^2\cdot\frac{Q(z^{-1})}{z^4} = -n^2$$

so $x = \infty$ is a regular singularity.

7.4.4

$$y'' - 2xy' + 2\alpha y = 0$$

so $P(x) = -2x$, $Q(x) = 2\alpha$. $P(x)$ and $Q(x)$ will not diverge at any finite x, so the only possible singularity is at $x = \infty$.

$$\frac{2z - P(z^{-1})}{z^2} = \frac{2z^2+2}{z^3}, \qquad \frac{Q(z^{-1})}{z^4} = \frac{2\alpha}{z^4}$$

$$\lim_{z\to 0} \frac{2z - P(z^{-1})}{z^2} = \infty, \qquad \lim_{z\to 0} z \cdot \frac{2z - P(z^{-1})}{z^2} = \infty$$

so $x = \infty$ is an irregular singularity.

7.4.5 (The $+c$ in the hypergeometric function in Table 7.1 should be $-c$.)

Note that $y'' = \frac{d^2y}{dx^2}$, so y'' should be transformed to $\frac{1}{(-\frac{1}{2})^2}y'' = 4y''$. Similarly y' should be transformed to $\frac{1}{-\frac{1}{2}}y' = -2y'$. Substitute x, y'', y', a, b, c into the hypergeometric function:

$$\frac{1-x}{2}(\frac{1-x}{2} - 1)4y'' + \left[(1-l+l+1)\frac{1-x}{2} - 1\right](-2y') - l(l+1)y = 0$$

Rearrange and multiply by -1, we get

$$(1-x^2)y'' - 2xy' + l(l+1) = 0$$

which is the Legendre's equation.

7.5 Series Solutions—Frobenius' Method

7.5.1 (Let the equation be $y''+P(x)y'+Q(x)y = \mathcal{L}y = 0$. There should be more conditions for theorem to hold, for example, $P(x)$ and $Q(x)$ being continuous or analytic at $x = x_0$, or x_0 is a regular singular point. We prove for the "analytic" case, while I don't know the correctness or the proof for other cases. $P(x)$ being analytic at x_0 means that it is infinitely differentiable at x_0, that is, $P^{(n)}(x_0)$ is finite for every n.)

If y_1 and y_2 are two functions satisfying $\mathcal{L}y = 0$ and $y(x_0) = y_0$, $y'(x_0) = y_0'$, then $\varphi = y_1 - y_2$ satisfies $\mathcal{L}\varphi = 0$ and $\varphi(x_0) = 0$, $\varphi'(x_0) = 0$. Let $x = x_0$, we have

$$\varphi''(x_0) + P(x_0)\varphi'(x_0) + Q(x_0)\varphi(x_0) = 0$$

The second and third terms vanish, so $\varphi''(x_0) = 0$. Differentiate the equation and let $x = x_0$, we have

$$\varphi'''(x_0) + P(x_0)\varphi''(x_0) + \left[P'(x_0) + Q(x_0)\right]\varphi'(x_0) + Q'(x_0)\varphi(x_0) = 0$$

All except the first term vanish, so $\varphi'''(x_0) = 0$. Continue the process by differentiate the equation n times and let $x = x_0$, we will get the equation of the form

$$\varphi^{(n)}(x_0) + \sum_{j=1}^{n-1} a_j \varphi^{(j)}(x_0) = 0$$

Because $\varphi(x_0) = \varphi'(x_0) = \cdots = \varphi^{(n-1)}(x_0) = 0$, so $\sum_{j=1}^{n-1} a_j\varphi^{(j)}(x_0) = 0$, which means $\varphi^{(n)}(x_0) = 0$. So we know that $\varphi^{(n)}(x_0) = 0$ for every n, while $\varphi(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!}\varphi^{(n)}(x_0)$, so $\varphi(x) = 0$, which means $y_1 - y_2 = 0$, y_1 is unique.

7.5.2 Let $y(x) = \sum_{j=0}^{\infty} a_j(x-x_0)^{s+j}$ and substitute into the equation:

$$\sum_{j=0}^{\infty} a_j(s+j)(s+j-1)(x-x_0)^{s+j-2} + \sum_{j=0}^{\infty} P(x)a_j(s+j)(x-x_0)^{s+j-1} + \sum_{j=0}^{\infty} Q(x)a_j(x-x_0)^{s+j} = 0$$

Because x_0 is an ordinary point, which means $P(x_0)$ and $Q(x_0)$ are finite, so we can expand $P(x)$ and $Q(x)$ at x_0, that is, $P(x) = \sum_{n=0}^{\infty} p_n(x-x_0)^n$, $Q(x) = \sum_{m=0}^{\infty} q_m(x-x_0)^m$. Substitute $P(x)$ and $Q(x)$ into the above equation,

$$\sum_{j=0}^{\infty} a_j(s+j)(s+j-1)(x-x_0)^{s+j-2} + \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} p_n a_j(s+j)(x-x_0)^{s+j-1+n} + \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} q_m a_j(x-x_0)^{s+j+m} = 0$$

We found that term of the lowest order is $a_0 s(s-1)(x-x_0)^{s-2}$, whose coefficient must equal zero, so

$$s(s-1) = 0, \qquad s = 0, 1$$

7.5.3 Substitute $y = \sum_{j=0}^{\infty} a_j x^{s+j}$ into $y'' + \omega^2 y = 0$, we get

$$\sum_{j=0}^{\infty} a_j(s+j)(s+j-1)x^{s+j-2} + \sum_{j=0}^{\infty} \omega^2 a_j x^{s+j} = 0$$

The coefficient of x^{s-1} is $a_1(s+1)s$, which should equal zero, so when

$$\begin{array}{lll} s = 0: & (s+1)s = 0, & a_1 = arbitrary \\ s = 1: & (s+1)s = 2 \neq 0, & a_1 = 0 \end{array}$$

7.5.4 Substitute $y = \sum_{j=0}^{\infty} a_j x^{s+j}$ into the equations:

$$Legendre: \quad (1-x^2)\sum_{j=0}^{\infty} a_j(s+j)(s+j-1)x^{s+j-2} - 2x\sum_{j=0}^{\infty} a_j(s+j)x^{s+j-1} + l(l+1)\sum_{j=0}^{\infty} a_j x^{s+j} = 0$$

$$Chebyshev: \quad (1-x^2)\sum_{j=0}^{\infty} a_j(s+j)(s+j-1)x^{s+j-2} - x\sum_{j=0}^{\infty} a_j(s+j)x^{s+j-1} + n^2\sum_{j=0}^{\infty} a_j x^{s+j} = 0$$

$$Hermite: \quad \sum_{j=0}^{\infty} a_j(s+j)(s+j-1)x^{s+j-2} - 2x\sum_{j=0}^{\infty} a_j(s+j)x^{s+j-1} + 2\alpha\sum_{j=0}^{\infty} a_j x^{s+j} = 0$$

All the three equations have indicial roots $s = 0, 1$, and the coefficient of x^{s-1} are all $a_1(s+1)s$, which should be zero. So for $s = 0$, a_1 may be any finite value; for $s = 1$, a_1 must be set equal to zero.

$$Bessel: \quad x^2\sum_{j=0}^{\infty} a_j(s+j)(s+j-1)x^{s+j-2} + x\sum_{j=0}^{\infty} a_j(s+j)x^{s+j-1} + (x^2-n^2)\sum_{j=0}^{\infty} a_j x^{s+j} = 0$$

The indicial roots are $s = \pm n$, and the coefficient of x^{s+1} term is $a_1(s+1+n)(s+1-n)$, which should be zero. $(s+1+n)(s+1-n) = \pm 2n + 1 \neq 0$ when $n \neq \pm\frac{1}{2}$, so a_1 must be set equal to zero.

7.5.5 Substitute $y = \sum_{j=0}^{\infty} a_j x^{s+j}$ into the equation:

$$x(x-1)\sum_{j=0}^{\infty} a_j(s+j)(s+j-1)x^{s+j-2} + \left[(1+a+b)x - c\right]\sum_{j=0}^{\infty} a_j(s+j)x^{s+j-1} + ab\sum_{j=0}^{\infty} a_j x^{s+j} = 0$$

The coefficient of x^{s-1} is $s(s-1+c) = 0$, which means $s = 0, 1-c$. The coefficient of x^{s+j} with $j \geq 0$ is

$$a_j(s+j)(s+j-1) - a_{j+1}(s+j+1)(s+j) + (1+a+b)a_j(s+j) - c\,a_{j+1}(s+j+1) + ab\,a_j = 0$$

Choose $s = 0$ for simplicity and rearrange,

$$a_{j+1} = a_j\frac{(j+a)(j+b)}{(j+1)(j+c)}$$

So

$$a_j = a_0\frac{a(a+1)\cdots(a+j-1)b(b+1)\cdots(b+j-1)}{j!\,c(c+1)\cdots(c+j-1)} = a_0\frac{(a)_j(b)_j}{j!(c)_j}$$

where we define $(a)_j = a(a+1)\cdots(a+j-1)$, and so do $(b)_j$ and $(c)_j$. Let $a_0 = 1$, then the solution is

$$y = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{j!(c)_j} x^j = 1 + \frac{ab}{1!\,c} x + \frac{a(a+1)b(b+1)}{2!\,c(c+1)} x^2 + \cdots$$

The inverse of radius of convergence is

$$R^{-1} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+a)(n+b)}{(n+1)(n+c)} = 1$$

so the series converges for $-1 < x < 1$.

At $x = 1$, we use the Gauss' test:

$$\frac{a_n}{a_{n+1}} = \frac{n^2 + (1+c)n + c}{n^2 + (a+b)n + ab} = 1 + \frac{1+c-a-b}{n} + \frac{B(n)}{n^2}$$

so the series converges for $1+c-a-b > 1$, which is $c > a+b$, and diverges for $1+c-a-b \leq 1$, which is $c \leq a+b$.

At $x = -1$, the series is an alternating series, so the series converges if the coefficient monotonically decreases to 0 for $n \to \infty$. The condition is probably satisfied if $a+b < 1+c$, while I don't know how to prove it.

7.5.6 Substitute $y = \sum_{j=0}^{\infty} a_j x^{s+j}$ into the equation:

$$x \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j-2} + (c-x) \sum_{j=0}^{\infty} a_j (s+j) x^{s+j-1} - a \sum_{j=0}^{\infty} a_j x^{s+j} = 0$$

The coefficient of x^{s-1} is $s(s-1+c) = 0$, which means $s = 0, 1-c$. The coefficient of x^{s+j} with $j \geq 0$ is

$$a_{j+1}(s+j+1)(s+j) + c\,a_{j+1}(s+j+1) - a_j(s+j) - a\,a_j = 0$$

Choose $s = 0$ for simplicity and rearrange,

$$a_{j+1} = a_j \frac{(j+a)}{(j+1)(j+c)}$$

So

$$a_j = a_0 \frac{a(a+1)\cdots(a+j-1)}{j!\,c(c+1)\cdots(c+j-1)} = a_0 \frac{(a)_j}{j!(c)_j}$$

Let $a_0 = 1$, then the solution is

$$y = \sum_{j=0}^{\infty} \frac{(a)_j}{j!(c)_j} x^j = 1 + \frac{a}{1!\,c} x + \frac{a(a+1)}{2!\,c(c+1)} x^2 + \cdots$$

The inverse of radius of convergence is

$$R^{-1} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+a)}{(n+1)(n+c)} = 0$$

so $R = \infty$, which means the series converges for every x.

7.5.7 Let $u = \sum_{j=0}^{\infty} a_j \xi^{s+j}$ and substitute:

$$\sum_{j=0}^{\infty} a_j (s+j)^2 \xi^{s+j-1} + (\frac{1}{2}E\xi + \alpha - \frac{m^2}{4\xi} - \frac{1}{4}F\xi^2) \sum_{j=0}^{\infty} a_j \xi^{s+j} = 0$$

The coefficients of each order should be zero, so

$$\begin{aligned}
&\xi^{s-1}: && a_0 s^2 - \frac{m^2}{4} a_0 = 0 && s = \pm\frac{m}{2} = \frac{m}{2} \\
&\xi^{s}: && a_1(\frac{m}{2}+1)^2 + \alpha a_0 - \frac{m^2}{4} a_1 = 0 && a_1 = a_0 \frac{-\alpha}{1+m} \\
&\xi^{s+1}: && a_2(\frac{m}{2}+2)^2 + \frac{E}{2} a_0 + \alpha a_1 - \frac{m^2}{4} a_2 = 0 && a_2 = a_o \left[\frac{\alpha^2}{2(m+1)(m+2)} - \frac{E}{4(m+2)} \right]
\end{aligned}$$

so

$$y = a_0 \xi^{\frac{m}{2}} \left\{ 1 - \frac{\alpha}{m+1}\xi + \left[\frac{\alpha^2}{2(m+1)(m+2)} - \frac{E}{4(m+2)} \right] + \cdots \right\}$$

7.5.8 Let $u = \sum_{j=0}^{\infty} a_j \eta^{s+j}$ and substitute:

$$\sum_{j=0}^{\infty} a_j (s+j)(s+j-1)\eta^{s+j-2} - \sum_{j=0}^{\infty} a_j (s+j)(s+j+1)\eta^{s+j} + \alpha \sum_{j=0}^{\infty} a_j \eta^{s+j} + \beta\eta^2 \sum_{j=0}^{\infty} a_j \eta^{s+j} = 0$$

The coefficients of each order should be zero, so

$$\begin{aligned}
&\eta^{s-2}: && a_0 s(s-1) = 0, && s = 0, 1 \quad \textit{(choose } s = 1\textit{)} \\
&\eta^{s-1}: && a_1 (s+1)s = 0, && a_1 = 0 \\
&\eta^{s}: && a_2 (s+2)(s+1) - a_0 s(s+1) + \alpha a_0 = 0, && a_2 = a_0 \frac{2-\alpha}{6} \\
&\eta^{s+1}: && a_3 (s+3)(s+2) - a_1 (s+1)(s+2) + \alpha a_1 = 0, && a_3 = 0 \\
&\eta^{s+2}: && a_4 (s+4)(s+3) - a_2 (s+2)(s+3) + \alpha a_2 + \beta a_0 = 0, && a_4 = a_0 \left[\frac{(12-\alpha)(2-\alpha)}{120} - \frac{\beta}{20} \right]
\end{aligned}$$

so

$$y = a_0 \eta \left\{ 1 + \frac{2-\alpha}{6}\eta^2 + \left[\frac{(12-\alpha)(2-\alpha)}{120} - \frac{\beta}{20} \right] \eta^4 + \cdots \right\}$$

7.5.9 Let $\varphi = \sum_{j=0}^{\infty} a_j x^{s+j}$ and substitute (expand e^{-ax} for small x):

$$\sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j-2} + E' \sum_{j=0}^{\infty} a_j x^{s+j} - \frac{A'(1 - ax + \frac{a^2x^2}{2} - \cdots)}{x} \sum_{j=0}^{\infty} a_j x^{s+j}$$

The coefficients of each order should be zero, so

$$\begin{aligned}
&x^{s-2}: && a_0 s(s-1) = 0, && s = 0, 1 \quad \textit{(choose } s = 1\textit{)} \\
&x^{s-1}: && a_1 (s+1)s - A' a_0 - 0, && a_1 = a_0 \frac{A'}{2} \\
&x^{s}: && a_2 (s+2)(s+1) + E' a_0 - A' a_1 + A' a\, a_0 = 0, && a_2 = \frac{a_0}{6}\left(\frac{A'^2}{2} - E' - aA'\right)
\end{aligned}$$

so

$$\varphi = a_0 \left\{ x + \frac{A'}{2}x^2 + \frac{1}{6}\left(\frac{A'^2}{2} - E' - aA'\right)x^3 + \cdots \right\}$$

7.5.10 Substitute $y = \sum_{j=0}^{\infty} a_j x^{s+j}$ into the equation:

$$\sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j-2} + \frac{1}{x^2} \sum_{j=0}^{\infty} a_j (s+j) x^{s+j-1} - \frac{2}{x^2} \sum_{j=0}^{\infty} a_j x^{s+j} = 0$$

The coefficient of x^{s-3} is $a_0 s = 0$, which means $s = 0$. The coefficient of x^{s+j-2} with $j \geq 0$ is

$$a_j (s+j)(s+j-1) + a_{j+1}(s+j+1) - 2a_j = 0$$

$$a_{j+1} = a_j \frac{2 - j(j-1)}{j+1}$$

So $a_1 = a_0 \frac{2}{1} = 2a_0,\ a_2 = a_1 \frac{2}{2} = 2a_0,\ a_3 = a_2 \frac{2-2}{3} = 0,\ a_j = 0$ for $j \geq 4$. Therefore, $y = a_0(1 + 2x + 2x^2)$, substitute into the equation,

$$4 + \frac{1}{x^2}(4x+2) - \frac{2}{x^2}(1 + 2x + 2x^2) = 0$$

so it is indeed a solution.

7.5.11 Let $I_0(x) = \frac{e^x}{\sqrt{2\pi x}} f(x)$, then

$$\frac{d}{dx} I_0(x) = \frac{e^x}{\sqrt{2\pi x}} \left[(1 - \frac{1}{2}x^{-1}) f(x) + f'(x) \right]$$
$$\frac{d^2}{dx^2} I_0(x) = \frac{e^x}{\sqrt{2\pi x}} \left[(1 - x^{-1} + \frac{3}{4}x^{-2}) f(x) + (2 - x^{-1}) f'(x) + f''(x) \right]$$

Substitute into the equation and eliminate $\frac{e^x}{\sqrt{2\pi x}}$, we get

$$x^2 f''(x) + 2x^2 f'(x) + \frac{1}{4} f(x) = 0$$

Let $f(x) = \sum_j b_j x^{-j}$ and substitute, we get

$$x^2 \sum_j b_j j(j+1) x^{-j-2} + 2x^2 \sum_j b_j(-j) x^{-j-1} + \frac{1}{4} \sum_j b_j x^{-j} = 0$$

The coefficient of x^{-j} must vanish, so

$$b_j j(j+1) - 2b_{j+1}(j+1) + \frac{1}{4} b_j = 0$$

$$b_{j+1} = b_j \frac{j(j+1) + \frac{1}{4}}{2(j+1)}$$

Let $b_0 = 1$, then $b_1 = \frac{1}{8} b_0 = \frac{1}{8}$, $b_2 = \frac{9}{16} b_1 = \frac{9}{128}$.

7.5.12 Use the recursive relation in Exercise 8.3.1 and write a program for calculation, we have

sum to x^k	$x = 0.95$	$x = 0.99$	$x = 1.00$
x^{200}	0.21543	− 0.255451	− 0.650013
x^{400}	0.215429	− 0.268429	− 0.85409
x^{600}	0.215429	− 0.269403	− 0.973583
x^{800}	0.215429	− 0.269494	− 1.0584
x^{1000}	0.215429	− 0.269504	− 1.1242
x^{1200}	0.215429	− 0.269505	− 1.17797
x^{1400}	0.215429	− 0.269505	− 1.22343
x^{1600}	0.215429	− 0.269505	− 1.26282
x^{1800}	0.215429	− 0.269505	− 1.29756
x^{2000}	0.215429	− 0.269505	− 1.32864

7.5.13 (a) Use the recursive relation in Exercise 8.3.3

$$a_{j+2} = 2a_j \frac{j - \alpha}{(j+1)(j+2)} \qquad (j \ odd)$$

Let $\alpha = 0$ and $a_1 = 1$, write a program to calculate $\sum_{j\ odd} a_j x^j$ (for simplicity, we cut the calculation when the last term is less than 10^{-6}, which is a stricter condition than that in the problem). Note that

$$\frac{a_{j+2}\, x^{j+2}}{a_j\, x^j} = \frac{2jx^2}{(j+1)(j+2)} < \frac{2x^2}{j}$$

so if a_k is the first term after truncation, then the sum of remaining terms is less than the infinite geometric series with a_k as the first term and $\frac{2x^2}{j}$ as the ratio, which is

$$\frac{a_k}{1 - \frac{2x^2}{j}}$$

This can set the upper bound of the error of the summation.

x	*partial sum*	*truncate at*	*error*	*upper bound*
1	1.4626516	$a_{19}x^{19} = 1.45 \times 10^{-7}$	2×10^{-7}	1.4626518
2	16.4526271	$a_{37}x^{37} = 5.80 \times 10^{-7}$	8×10^{-7}	16.4526279
3	1444.5451221	$a_{65}x^{65} = 6.02 \times 10^{-7}$	9×10^{-7}	1444.5451230

(b) Expand e^{x^2} and integrate:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\int_0^x e^{x^2} dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)} = \sum_{j\ odd} a_j x^j$$

where $j = 2n+1$, so $a_j = \frac{1}{(\frac{j-1}{2})!j}$. Therefore, $a_1 = 1$, and

$$\frac{a_{j+2}}{a_j} = \frac{(\frac{j-1}{2})!j}{(\frac{j+1}{2})!(j+2)} = \frac{2j}{(j+1)(j+2)}$$

which is the recursive relation of Hermite series $y_{odd}(\alpha = 0)$, so the series is equal to $\int_0^x e^{x^2} dx$.

(c)

$$\int_0^1 e^{x^2} dx = 1.46265175$$

$$\int_0^2 e^{x^2} dx = 16.45262777$$

$$\int_0^3 e^{x^2} dx = 1444.54512289$$

7.6 Other Solutions

7.6.1 If $\mathbf{A} = a\hat{\mathbf{e}}_x + b\hat{\mathbf{e}}_y + c\hat{\mathbf{e}}_z = 0$, then $\mathbf{A} \cdot \mathbf{A} = a^2 + b^2 + c^2 = 0$, so $a = b = c = 0$, which means $\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$ are linearly independent.

7.6.2

$$a\mathbf{A} + b\mathbf{B} + c\mathbf{C} = \begin{pmatrix} aA_1 + bB_1 + cC_1 \\ aA_1 + bB_1 + cC_1 \\ aA_3 + bB_3 + cC_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so

$$\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let the square matrix be M. a, b, c have non-trivial solution if and only if $\det(\mathrm{M}) = 0$, so the sufficient and necessary condition for the vectors to be linearly independent is $\det(\mathrm{M}) \neq 0$. Note that $\det(\mathrm{M}) = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$, so the two criterions is equivalent.

7.6.3 $\frac{d}{dx}(\frac{x^n}{n!}) = \frac{x^{n-1}}{(n-1)!}$, so the Wronskian is

$$W = \begin{vmatrix} 1 & \frac{x^1}{1!} & \frac{x^2}{2!} & & \frac{x^N}{N!} \\ 0 & 1 & \frac{x^1}{1!} & \cdots & \frac{x^{N-1}}{(N-1)!} \\ 0 & 0 & 1 & & \frac{x^{N-2}}{(N-2)!} \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = 1 \neq 0$$

which means the functions are linearly independent.

7.6.4

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = 0$$

so $\frac{y_1'}{y_1} = \frac{y_2'}{y_2}$, $\ln y_1 = \ln y_2 + \ln c$, $y_1 = c y_2$.

7.6.5 $W(x_0) = W(x_0 + \varepsilon) = 0$, so

$$W'(x_0) = \lim_{\varepsilon \to 0} \frac{W(x_0+\varepsilon) - W(x_0)}{\varepsilon} = 0$$
$$W(x_0) = \mathit{constant} = 0$$

so the Wronskian is zero for all x, and the functions are linearly dependent.

7.6.6

$$W = \begin{vmatrix} \sin x & e^x & e^{-x} \\ \cos x & e^x & -e^{-x} \\ -\sin x & e^x & e^{-x} \end{vmatrix} = 4\sin x = 0 \qquad \mathit{for}\ x = \pm n\pi,\ n = 0, 1, 2, \cdots$$

7.6.7 The functions must be differentiable for the Wronskian to be valid, but $|x|$ is not differentiable at $x = 0$.

7.6.8 $\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}(e^x + \frac{1}{e^x})$, so between $\cosh x$ and e^x there is a dependence which is not linear.

7.6.9 Let $W = P_n(x)Q_n'(x) - P_n'(x)Q_n(x) = 0$, then by Eq. 7.60 we have

$$W' = -\frac{-2x}{1-x^2}W$$
$$\frac{1}{W}dW = \frac{2x}{1-x^2}dx$$
$$\ln W = -\ln(1-x^2) + \ln A_n$$
$$W = \frac{A_n}{1-x^2}$$

7.6.10 Let y_1, y_2, y_3 be three solutions of the equation, so $y_i'' + P(x)y_i' + Q(x)y_i = 0$, $i = 1, 2, 3$. The wronskian is

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -P(x)y_1' - Q(x)y_1 & -P(x)y_2' - Q(x)y_2 & -P(x)y_3' - Q(x)y_3 \end{vmatrix}$$
$$= -P(x)\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \end{vmatrix} - Q(x)\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1 & y_2 & y_3 \end{vmatrix} = 0$$

so the Wronskian is zero for all x and the three solutions are linearly dependent.

7.6.11 (a) The equation is self-adjoint, so $p'(x) = q(x)$ (Eq. 8.9). Therefore,

$$\begin{aligned} W'(x) &= -\frac{q(x)}{p(x)}W(x) = -\frac{p'(x)}{p(x)}W(x) \\ \frac{W'(x)}{W(x)} &= -\frac{p'(x)}{p(x)} \\ \ln W(x) &= -\ln p(x) + \ln C \\ W(x) &= \frac{C}{p(x)} \end{aligned}$$

(b)

$$\begin{aligned} y_1^2 \frac{d}{dx}(\frac{y_2}{y_1}) &= y_1 y_2' - y_1' y_2 = W(x) = \frac{C}{p(x)} \\ \frac{d}{dx}(\frac{y_2}{y_1}) &= \frac{C}{p(x)[y_1(x)]^2} \\ y_2(x) &= C y_1(x) \int^x \frac{dt}{p(t)[y_1(t)]^2} \end{aligned}$$

7.6.12

$$\begin{aligned} y &= e^{-\frac{1}{2}\int^x P(t)dt} z \\ y' &= e^{-\frac{1}{2}\int^x P(t)dt}\Big(z' - \frac{1}{2}zP(x)\Big) \\ y'' &= e^{-\frac{1}{2}\int^x P(t)dt}\left[z'' - P(x)z' + \Big(\frac{1}{4}P^2(x) - \frac{1}{2}P'(x)\Big)z\right] \end{aligned}$$

Substitute into $y'' + P(x)y' + Q(x)y = 0$ and eliminate $e^{-\frac{1}{2}\int^x P(t)dt}$, we get

$$z'' + \left[Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}P^2(x)\right] z = 0$$

7.6.13 The Laplacian in spherical polar coordinates of $\varphi(r)$ is $\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{d\varphi(r)}{dr}\right] = \varphi''(r) + \frac{2}{r}\varphi'(r)$. Use the substitution in Exercise 7.6.12:

$$\varphi(r) = ze^{-\frac{1}{2}\int^r \frac{2}{t}dt} = ze^{-\ln r} = \frac{z}{r}$$

So $z = r\varphi(r)$, and the Laplacian becomes

$$\frac{1}{r}\left[z'' + \Big(0 - \frac{1}{2}(-\frac{2}{r^2}) - \frac{1}{4}\frac{4}{r^2}\Big)z\right] = \frac{1}{r}z'' = \frac{1}{r}\frac{d^2}{dr^2}[r\varphi(r)]$$

which corresponds to the results from Exercise 3.10.34.

7.6.14

$$\begin{aligned} y_2(x) &= y_1(x)\int^x \frac{e^{-\int^s P(t)dt}}{[y_1(s)]^2}ds \\ y_2'(x) &= y_1'(x)\int^x \frac{e^{-\int^s P(t)dt}}{[y_1(s)]^2}ds + y_1(x)\frac{e^{-\int^x P(t)dt}}{[y_1(x)]^2} \\ y_2''(x) &= y_1''(x)\int^x \frac{e^{-\int^s P(t)dt}}{[y_1(s)]^2}ds + 2y_1'(x)\frac{e^{-\int^x P(t)dt}}{[y_1(x)]^2} + y_1(x)\frac{e^{-\int^x P(t)dt}\big[-P(x)[y_1(x)]^2 - 2y_1(x)y_1'(x)\big]}{[y_1(x)]^4} \\ &= y_1''(x)\int^x \frac{e^{-\int^s P(t)dt}}{[y_1(s)]^2}ds - \frac{e^{-\int^x P(t)dt}P(x)}{y_1(x)} \end{aligned}$$

so

$$\begin{aligned} & y_2''(x) + P(x)y_2'(x) + Q(x)y_2(x) \\ &= y_1''(x)\int^x \frac{e^{-\int^s P(t)dt}}{[y_1(s)]^2}ds + P(x)y_1'(x)\int^x \frac{e^{-\int^s P(t)dt}}{[y_1(s)]^2}ds + Q(x)y_1(x)\int^x \frac{e^{-\int^s P(t)dt}}{[y_1(s)]^2}ds \\ &= \big[y_1''(x) + P(x)y_1'(x) + Q(x)y_1(x)\big]\int^x \frac{e^{-\int^s P(t)dt}}{[y_1(s)]^2}ds = 0 \end{aligned}$$

7.6.15 Inclusion of lower limits will introduce a constant to the integral, so the function become

$$y_3(x) = y_1(x)\left[\int^x \frac{e^{-\int^s P(t)dt+a}}{[y_1(s)]^2}ds + b\right]$$

$$= e^a y_1(x)\int^x \frac{e^{-\int^s P(t)dt}}{[y_1(s)]^2}ds + by_1(x)$$

$$= a' y_2(x) + b' y_1(x)$$

which is a linear combination of y_1 and y_2, so no new independent solution is generated.

7.6.16 From Eq. 7.67,

$$y_2 = r^m \int^r \frac{e^{-\int^s \frac{1}{t}dt}}{(s^m)^2}ds = r^m \int^r s^{-1-2m}ds = r^m \frac{r^{-2m}}{-2m} \propto r^{-m}$$

so r^{-m} is the second solution.

7.6.17 Use equation 7.67,

$$y_2(x) = y_1(x)\int^x \frac{e^{-\int^{x_2} 0\cdot dx_1}}{[y_1(x_2)]^2}dx_2$$

$$= y_1(x)\int^x (x_2 - \frac{x_2^3}{3!} + \frac{x_2^5}{5!} - \cdots)^{-2}dx_2$$

$$= y_1(x)\int^x x_2^{-2}(1 - \frac{x_2^2}{3!} + \frac{x_2^4}{5!} - \cdots)^{-2}dx_2$$

$$= y_1(x)\int^x x_2^{-2}(1 + c_2x_2^2 + c_4x_2^4 + \cdots)dx_2$$

so there is no x_2^{-1} term in the integral, and therefore $c_n = 0$.

7.6.18 From Eq. 7.49, the first solution of Bessel's equation is

$$y_1 = \sum_{j=0}^{\infty}(-1)^j \frac{1}{j!(n+j)!}(\frac{x}{2})^{n+2j} = x^n(b_0 + b_2x^2 + b_4x^4 + \cdots)$$

so by using Eq. 7.67, the second solution is

$$y_2(x) = y_1(x)\int^x \frac{e^{-\int^{x_2} \frac{1}{x_1}\cdot dx_1}}{[y_1(x_2)]^2}dx_2$$

$$= y_1(x)\int^x x_2^{-1-2n}(b_0 + b_2x^2 + b_4x^4 + \cdots)^{-2}dx_2$$

$$= y_1(x)\int^x x_2^{-1-2n}(c_0 + c_2x^2 + c_4x^4 + \cdots)dx_2$$

All the terms in the integral have the form $x_2^{-1-2n+2k}$, where k is an integer, but n is not an integer, so $x_2^{-1-2n+2k} \neq x^{-1}$, which means there is no x_2^{-1} term in the integral, and therefore y_2 does not contain a logarithmic term.

7.6.19 (a)

$$y_2(x) = y_1(x)\int^x \frac{e^{-\int^{x_2} -2x_1 dx_1}}{[y_1(x_2)]^2}dx_2 = \int^x e^{x_2^2}dx_2$$
$$= \int^x \left(\sum_{n=0}^{\infty} \frac{x_2^{2n}}{n!}\right) dx_2 = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}$$
$$= x \sum_{j\ even} \frac{x^j}{(j+1)(\frac{j}{2})!} = x \sum_{j\ even} a_j x^j$$

where we use the substitution $j = 2n$. So

$$\frac{a_{j+2}}{a_j} = \frac{1}{\frac{(j+3)}{(j+1)}\frac{(j+2)}{2}} = \frac{2(j+1)}{(j+2)(j+3)}$$

which is the recursive relation of y_{odd} (Exercise 8.3.3), so $y_2 = y_{odd}$.

(b)

$$y_2(x) = y_1(x)\int^x \frac{e^{-\int^{x_2} -2x_1 dx_1}}{[y_1(x_2)]^2}dx_2 = x\int^x x_2^{-2}e^{x_2^2}dx_2$$
$$= x\int^x \left(\sum_{n=0}^{\infty} \frac{x_2^{2n-2}}{n!}\right) dx_2 = x\sum_{n=0}^{\infty} \frac{x^{2n-1}}{(2n-1)n!}$$
$$= \sum_{j\ even} \frac{x^j}{(j-1)(\frac{j}{2})!} = \sum_{j\ even} a_j x^j$$

where we use the substitution $j = 2n$. So

$$\frac{a_{j+2}}{a_j} = \frac{1}{\frac{(j+1)}{(j-1)}\frac{(j+2)}{2}} = \frac{2(j-1)}{(j+1)(j+2)}$$

which is the recursive relation of y_{even} (Exercise 8.3.3), so $y_2 = y_{even}$.

7.6.20

$$y_2(x) = y_1(x)\int^x \frac{e^{-\int^s \frac{1-t}{t}dt}}{[y_1(s)]^2}ds$$
$$= \int^x e^{-(\ln s - s)}ds = \int^x \frac{e^s}{s}ds$$
$$= \int^x \left(\sum_{n=0}^{\infty} \frac{s^{n-1}}{n!}\right) ds = \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$

7.6.21 (a)

$$y_2(x) = \int^x \left(\sum_{n=0}^{\infty} \frac{s^{n-1}}{n!}\right) ds = \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$

(b)

$$y_2' = \frac{e^x}{x}$$
$$y_2'' = \frac{e^x}{x}(1 - \frac{1}{x})$$
$$xy_2'' + (1-x)y_2' = \frac{e^x}{x}(x - 1 + 1 - x) = 0$$

(c)

$$y_2' = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

$$y_2'' = -\frac{1}{x^2} + \sum_{n=2}^{\infty} \frac{(n-1)x^{n-2}}{n!}$$

$$xy_2'' + (1-x)y_2' = -\frac{1}{x} + \sum_{n=2}^{\infty} \frac{(n-1)x^{n-1}}{n!} + \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} - 1 - \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{nx^n}{(n+1)!} + \sum_{n=1}^{\infty} \frac{x^n}{(n+1)!} + 1 - 1 - \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$= \sum_{n=1}^{\infty} \left(\frac{n}{(n+1)!} + \frac{1}{(n+1)!} - \frac{1}{n!} \right) x^n = 0$$

7.6.22 (a)

$$y_2(x) = y_1(x) \int^x \frac{e^{-\int^s \frac{-t}{1-t^2} dt}}{[y_1(s)]^2} ds$$

$$= \int^x e^{-\frac{1}{2}\ln(1-s^2)} ds = \int^x \frac{1}{\sqrt{1-s^2}} ds = \sin^{-1} x$$

(b)

$$(1-x^2)\frac{dy'}{dx} = xy'$$

$$\frac{1}{y'} dy' = \frac{x}{1-x^2} dx$$

$$\ln y' = -\frac{1}{2}\ln(1-x^2)$$

$$y' = \frac{1}{\sqrt{1-x^2}}$$

$$y = \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x$$

7.6.23

$$y_2(x) = y_1(x) \int^x \frac{e^{-\int^s \frac{-t}{1-t^2} dt}}{[y_1(s)]^2} ds$$

$$= x \int^x \frac{e^{-\frac{1}{2}\ln(1-s^2)}}{s^2} ds = x \int^x \frac{1}{s^2\sqrt{1-s^2}} ds = x \frac{-\sqrt{1-x^2}}{x} = -\sqrt{1-x^2}$$

7.6.24 (a) Let $y_1(r) = \sum_{j=0}^{\infty} a_j r^{s+j}$ and substitute:

$$-\frac{\hbar^2}{2m} \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) r^{s+j-2} + l(l+1)\frac{\hbar^2}{2m} r^{-2} \sum_{j=0}^{\infty} a_j r^{s+j} + \left(\frac{b_{-1}}{r} + b_0 + b_1 r + \cdots - E\right) \sum_{j=0}^{\infty} a_j r^{s+j} = 0$$

The coefficient of r^{s-2} is

$$-\frac{\hbar^2}{2m} a_0 s(s-1) + l(l+1)\frac{\hbar^2}{2m} a_0 = 0$$

$$s^2 - s - l(l+1) = 0$$

$$s = l+1, -l$$

so

$$y_1(x) = a_0 r^{l+1} + a_1 r^{l+2} + \cdots$$

(b)

$$y_2(x) = y_1(x) \int^x \frac{e^{-\int^s 0 dt}}{[y_1(s)]^2} ds$$

$$= (a_0 r^{l+1} + a_1 r^{l+2} + \cdots) \int^x (a_0 s^{l+1} + a_1 s^{l+2} + \cdots)^{-2} ds$$

$$
\begin{aligned}
&= (a_0 r^{l+1} + a_1 r^{l+2} + \cdots) \int^x a_0^{-2} s^{-2l-2}(1 + c_1 s + c_2 s^2 + \cdots) ds \\
&= (a_0 r^{l+1} + a_1 r^{l+2} + \cdots)(k_0 r^{-2l-1} + k_1 r^{-2l} + \cdots) \\
&= p_0 r^{-l} + p_1 r^{-l+1} + \cdots
\end{aligned}
$$

where $p_0 = -\frac{1}{a_0(2l+1)} \neq 0$, so $y_2(x)$ diverges at the origin as r^{-l}.

7.6.25

$$
\begin{aligned}
y_2'(x) &= y_1'(x) f(x) + y_1(x) f'(x) \\
y_2''(x) &= y_1''(x) f(x) + 2y_1'(x) f'(x) + y_1(x) f''(x)
\end{aligned}
$$

Substitute into $y_2'' + P(x) y_2' + Q(x) y_2 = 0$ and rearrange:

$$
y_1(x) f''(x) + \big[2y_1'(x) + P(x) y_1(x)\big] f'(x) + \big[y_1''(x) + P(x) y_1'(x) + Q(x) y_1(x)\big] f(x) = 0
$$

Note that $y_1''(x) + P(x) y_1'(x) + Q(x) y_1(x) = 0$, so the equation becomes

$$
\begin{aligned}
\frac{1}{f'(x)} df'(x) &= -\left(\frac{2y_1'(x)}{y_1(x)} + P(x)\right) dx \\
\ln f'(x) &= -\ln[y_1(x)]^2 - \int^x P(t) dt \\
f'(x) &= \frac{e^{-\int^x P(t)dt}}{[y_1(x)]^2} \\
f(x) &= \int^x \frac{e^{-\int^s P(t)dt}}{[y_1(s)]^2} ds
\end{aligned}
$$

7.6.26 (a) Substitute y_1, y_2 into the equation:

$$
a_0 \frac{1 \pm \alpha}{2} \frac{-1 \pm \alpha}{2} x^{\frac{-3 \pm \alpha}{2}} + \frac{1-\alpha^2}{4} x^{-2} a_0 x^{\frac{1 \pm \alpha}{2}} = 0
$$

so y_1, y_2 are indeed solutions.

(b)

$$
y_2(x) = y_1(x) \int^x \frac{e^{-\int^s 0 dt}}{[y_1(s)]^2} ds = a_0 x^{\frac{1}{2}} \int^x a_0^{-2} s^{-1} ds = a_0^{-1} x^{\frac{1}{2}} \ln x \propto x^{\frac{1}{2}} \ln x
$$

(c)

$$
\begin{aligned}
\lim_{\alpha \to 0} \left(\frac{y_1 - y_2}{\alpha}\right) &= \lim_{\alpha \to 0} \left(\frac{a_0 x^{\frac{1}{2}} (x^{\frac{\alpha}{2}} - x^{-\frac{\alpha}{2}})}{\alpha}\right) = a_0 x^{\frac{1}{2}} \lim_{\alpha \to 0} \frac{e^{\frac{\alpha}{2} \ln x} - e^{-\frac{\alpha}{2} \ln x}}{\alpha} \\
&= a_0 x^{\frac{1}{2}} \lim_{\alpha \to 0} \left[e^{\frac{\alpha}{2} \ln x} \frac{\ln x}{2} - e^{-\frac{\alpha}{2} \ln x} (-\frac{\ln x}{2})\right] = a_0 x^{\frac{1}{2}} \ln x \propto x^{\frac{1}{2}} \ln x
\end{aligned}
$$

where we use the L'Hôpital's rule to obtain the limit.

7.7 Inhomogeneous Linear ODEs

7.7.1 Let $y_p = u_1 y_1 + u_2 y_2$, then from Eq 7.98 we have

$$
\begin{aligned}
y_1 u_1' + y_2 u_2' &= 0 \\
y_1' u_1' + y_2' u_2' &= F(x)
\end{aligned}
$$

Using the Cramer's rules and integrating:

$$
u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ F & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 F}{W[y_1, y_2]} \qquad u_1 = \int^x \frac{-y_2(s) F(s)}{W[y_1(s), y_2(s)]} ds
$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & F \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 F}{W[y_1, y_2]} \qquad u_2 = \int^x \frac{y_1(s)F(s)}{W[y_1(s), y_2(s)]} ds$$

so

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= -y_1(x) \int^x \frac{y_2(s)F(s)}{W[y_1(s), y_2(s)]} ds + y_2(x) \int^x \frac{y_1(s)F(s)}{W[y_1(s), y_2(s)]} ds \end{aligned}$$

7.7.2 For the homogeneous equation, let $y = e^{mx}$ and substitue:

$$m^2 + 1 = 0 \qquad m = \pm i$$

$$y = C_1' e^{ix} + C_2' e^{-ix} = C_1 \cos x + C_2 \sin x$$

For the particular solution, we can found $y_p = 1$ by observation, or by Eq 7.98

$$\begin{aligned} \cos x \, u_1' + \sin x \, u_2' &= 0 \\ -\sin x \, u_1' + \cos x \, u_2' &= 1 \end{aligned}$$

$$\begin{aligned} u_1' &= -\sin x \qquad & u_1 &= \cos x \\ u_2' &= \cos x \qquad & u_2 &= \sin x \end{aligned}$$

$$y_p = u_1 y_1 + u_2 y_2 = \cos^2 x + \sin^2 x = 1$$

so

$$y = C_1 \cos x + C_2 \sin x + 1$$

7.7.3 For the homogeneous equation, let $y = e^{mx}$ and substitue:

$$m^2 + 4 = 0 \qquad m = \pm 2i$$

$$y = C_1 e^{2ix} + C_2 e^{-2ix}$$

For the particular solution, from Eq 7.98,

$$\begin{aligned} e^{2ix} u_1' + e^{-2ix} u_2' &= 0 \\ 2ie^{2ix} u_1' - 2ie^{-2ix} u_2' &= e^x \end{aligned}$$

$$\begin{aligned} u_1' &= \frac{e^x}{4ie^{2ix}} = \frac{1}{4i} e^{(1-2i)x} \qquad & u_1 &= \frac{1}{4i(1-2i)} e^{(1-2i)x} \\ u_2' &= \frac{-e^x}{4ie^{-2ix}} = \frac{-1}{4i} e^{(1+2i)x} \qquad & u_2 &= \frac{-1}{4i(1+2i)} e^{(1+2i)x} \end{aligned}$$

$$y_p = u_1 y_1 + u_2 y_2 = e^x \left[\frac{1}{4i(1-2i)} - \frac{1}{4i(1+2i)} \right] = \frac{1}{5} e^x$$

so

$$y = C_1 e^{2ix} + C_2 e^{-2ix} + \frac{1}{5} e^x$$

7.7.4 For the homogeneous equation, let $y = e^{mx}$ and substitue:

$$m^2 - 3m + 2 = 0 \qquad m = 1, 2$$

$$y = C_1 e^x + C_2 e^{2x}$$

For the particular solution, from Eq 7.98,

$$\begin{aligned} e^x u_1' + e^{2x} u_2' &= 0 \\ e^x u_1' + 2e^{2x} u_2' &= \sin x \end{aligned}$$

$$\begin{aligned} u_1' &= -e^{-x} \sin x \qquad & u_1 &= \frac{1}{2} e^{-x} (\cos x + \sin x) \\ u_2' &= e^{-2x} \sin x \qquad & u_2 &= -\frac{1}{5} e^{-2x} (\cos x + 2 \sin x) \end{aligned}$$

$$y_p = u_1 y_1 + u_2 y_2 = \frac{3}{10} \cos x + \frac{1}{10} \sin x$$

so

$$y = C_1 e^x + C_2 e^{2x} + \frac{3}{10} \cos x + \frac{1}{10} \sin x$$

7.7.5 Let $y = \sum_{j=0}^{\infty} a_j x^{s+j}$ and substitute:

$$x \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j-2} - (1+x) \sum_{j=0}^{\infty} a_j (s+j) x^{s+j-1} + \sum_{j=0}^{\infty} a_j x^{s+j} = 0$$

The coefficient of x^{s-1} is $a_0 s(s-2) = 0$, which means $s = 0, 2$. The coefficient of x^{s+j} with $j \geq 0$ is

$$a_{j+1}(s+j+1)(s+j) - a_{j+1}(s+j+1) - a_j(s+j) + a_j = 0$$

Choose $s = 0$ and rearrange:

$$a_{j+1} = a_j \frac{1}{j+1}$$

$$a_j = a_0 \frac{1}{j!}$$

so

$$y_1 = \sum_{j=0}^{\infty} a_j x^{s+j} = \sum_{j=0}^{\infty} a_0 \frac{x^j}{j!} = a_0 e^x \propto e^x$$

Using Eq 7.67,

$$y_2(x) = y_1(x) \int^x \frac{e^{\int^s (\frac{1}{t}+1) dt}}{[y_1(s)]^2} ds$$

$$= e^x \int^x s e^s e^{-2s} ds = e^x(-x e^{-x} - e^{-x}) = -1 - x \propto 1 + x$$

Let $y_p = u_1 y_1 + u_2 y_2$ and use Eq 7.98,

$$\begin{aligned} e^x u_1' + (1+x) u_2' &= 0 \\ e^x u_1' + \qquad\quad u_2' &= x \end{aligned}$$

$$\begin{aligned} u_1' &= e^{-x}(1+x) \qquad & u_1 &= -e^{-x}(x+2) \\ u_2' &= -1 \qquad & u_2 &= -x \end{aligned}$$

$$y_p = u_1 y_1 + u_2 y_2 = -(x+2) - x(1+x) = -x^2 - 2x - 2$$

where $-2x - 2 = -2(x+1) = -2y_2$ and therefore can be omitted, so $y_p = -x^2$, and

$$y = C_1 e^x + C_2 (1+x) - x^2$$

7.8 Nonlinear Differential Equations

7.8.1 Let $y = 2 + u$ and substitute into the equation:

$$u' = (2+u)^2 - (2+u) - 2 = u^2 + 3u$$

which is a Bernoulli equation, so let $v = u^{1-2} = u^{-1}$ and substitute:

$$v' = -u^{-2}u' = -(1+3v)$$

$$\frac{1}{3v+1}dv = -dx$$

$$\frac{1}{3}\ln(3v+1) = -x + C'$$

$$v = \frac{Ce^{-3x} - 1}{3}$$

$$u = v^{-1} = \frac{3}{Ce^{-3x} - 1}$$

so

$$y = 2 + \frac{3}{Ce^{-3x} - 1}$$

7.8.2 Let $y = x^2 + u$ and substitute into the equation:

$$2x + u' = \frac{(x^2+u)^2}{x^3} - \frac{(x^2+u)}{x} + 2x$$

$$u' = \frac{1}{x}u + \frac{1}{x^3}u^2$$

which is a Bernoulli equation, so let $v = u^{1-2} = u^{-1}$ and substitute:

$$v' = -u^{-2}u' = -(\frac{1}{x}v + \frac{1}{x^3})$$

$$v' + \frac{1}{x}v = -\frac{1}{x^3}$$

which is a linear first-order equation. The integrating factor is

$$\alpha = e^{\int \frac{1}{x}dx} = x$$

Multiplying,

$$\frac{d}{dx}(xv) = xv' + v = -\frac{1}{x^2}$$

$$xv = x^{-1} + C$$

$$v = \frac{1 + Cx}{x^2}$$

$$u = v^{-1} = \frac{x^2}{1 + Cx}$$

so

$$y = x^2 + \frac{x^2}{1 + Cx}$$

7.8.3 Let $u = y^{1-3} = y^{-2}$ and substitute:

$$u' = -2y^{-3}y' = -2(-xu + x) = 2x(u-1)$$

$$\frac{1}{u-1}du = 2xdx$$

$$\ln(u-1) = x^2 + C'$$

$$u = Ce^{x^2} + 1$$

$$y = u^{-\frac{1}{2}} = \frac{1}{\sqrt{Ce^{x^2} + 1}}$$

7.8.4 (a)

$$y'' = 0 \qquad y' = a \qquad y = ax + b$$

Substitute into the equation:

$$ax + b = xa + a^2$$

so $b = a^2$, and therefore

$$y = ax + a^2$$

(b)

$$f'(y') = 2y' = -x \qquad y = -\frac{x^2}{4} + c$$

Substitute into the equation:

$$-\frac{x^2}{4} + c = -\frac{x^2}{2} + \frac{x^2}{4}$$

so $c = 0$, and therefore

$$y = -\frac{x^2}{4}$$

The envelope of a family of curves $f(x, y, a)$ can be obtained by solving

$$\begin{cases} f(x, y, a) = 0 \\ f_a(x, y, a) = 0 \end{cases}$$

where f_a is the derivative of f regarding a. So

$$\begin{cases} y = ax + a^2 \\ 0 = x + 2a \end{cases}$$

which means $a = -\frac{x}{2}$ and $y = -\frac{x^2}{4}$.