# Chapter 1
# Mathematical Preliminaries

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## 1.1 Infinite Series

**1.1.1** (a) If $A > 0$, $\lim_{n\to\infty} n^p u_n = A$, so $|n^p u_n - A| < \frac{A}{2}$ when $n \geq N$ for some N. Then $0 < \frac{A}{2}\frac{1}{n^p} < u_n < \frac{3A}{2}\frac{1}{n^p}$. $\sum_{n=1}^{\infty} \frac{3A}{2}\frac{1}{n^p}$ converges when $p > 1$ by Cauchy integral test, so $\sum_{n=1}^{\infty} u_n$ converges by comparison test.

If $A < 0$, then $\lim_{n\to\infty} n^p(-u_n) = -A$, $-A > 0$. From above $\sum_1^{\infty}(-u_n)$ converges, so $\sum_1^{\infty}(u_n)$ converges.

If $A = 0$, $|n^p u_n| < 1$ when $n \geq N$ for some N. Then $-\frac{1}{n^p} < u_n < \frac{1}{n^p}$, so $|u_n| < \frac{1}{n^p}$ for sufficiently large n. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges by Cauchy integral test, so $\sum_{n=1}^{\infty} u_n$ converges by comparison test.

(b) $\lim_{n\to\infty} n u_n = A$, so $A - \frac{A}{2} < n u_n < A + \frac{A}{2}$ when $n \geq N$ for some N. So $u_n > \frac{A}{2}\frac{1}{n}$ for sufficiently large n. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so $\sum_{n=1}^{\infty} u_n$ diverges.

**1.1.2** Let $b'_n = \frac{b_n}{2K}$, then $\lim_{n\to\infty} \frac{b'_n}{a_n} = \frac{1}{2}$, so for sufficiently large n, $\frac{1}{2} - \frac{1}{2} = 0 < \frac{b'_n}{a_n} < 1 = \frac{1}{2} + \frac{1}{2}$. Then $0 < b'_n < a_n$ or $0 > b'_n > a_n$, so $\sum a_n$ converges implies $\sum b'_n$ converges by comparison test[1], and therefore $\sum b_n$ converges.

Let $b''_n = \frac{2b_n}{KK}$, then $\lim_{n\to\infty} \frac{b''_n}{a_n} = 2$, so for sufficiently large n $2 + 1 = 3 > \frac{b''_n}{a_n} > 1 = 2 - 1$.Then $3a_n > b''_n > a_n$ or $3a_n < b''_n < a_n$, so $\sum a_n$ diverges implies $\sum b''_n$ diverges by comparison test[1], and therefore $\sum b_n$ diverges.

**1.1.3** $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = -\frac{1}{\ln x}\Big|_2^{\infty} = \frac{1}{\ln 2}$, so by Cauchy integral test $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

**1.1.4**

$$\frac{u_n}{u_{n+1}} = 1 + \frac{(a_1 - b_1)n + (a_0 - b_0))}{n^2 + b_1 n + b_0} = 1 + \frac{a_1 - b_1}{n} + \frac{B(n)}{n^2}$$

where $B(n)$ is bounded for large n (It can be verified by binomial theorem, that each term in $B(n)$ has negative or zero power of n. By Gauss' test, $\sum_{n=1}^{\infty} u_n$ converges if $a_1 - b_1 > 1$ and diverges if $a_1 - b_1 \leq 1$.

**1.1.5** (a) $\ln n < n$, so $\frac{1}{\ln n} > \frac{1}{n} > 0$ for all positive integers n, and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges implies $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

(b) $\frac{(n+1)!}{10^{n+1}}/\frac{n!}{10^n} = \frac{n+1}{10} \geq 1$ for $n \geq 9$, so $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ diverges by ratio test.

(c) $2n(2n+1) > (2n)^2$, so $0 < \frac{1}{2n(2n+1)} < \frac{1}{(2n)^2}$. $\sum_{n=1}^{\infty} \frac{1}{4n^2}$ converges by integral test, so $\sum_{n=1}^{\infty} \frac{1}{2n(2n+1)}$ converges by comparison test.

(d) $\frac{1}{\sqrt{n(n+1)}} > \frac{1}{\sqrt{(n+1)^2}} = \frac{1}{n+1} > 0$. $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges by integral test, so $\sum_{n=1}^{\infty}[n(n+1)]^{-\frac{1}{2}}$ diverges.

(e) $\int_0^{\infty} \frac{1}{2x+1} = \frac{1}{2}\ln(2n+1)\Big|_0^{\infty}$ is infinite, so $\sum_{n=0}^{\infty} \frac{1}{2n+1}$ diverges by integral test.

[1] It's different with the comparison test in the text, but I guess it is right

**1.1.6** (a) $0 < \frac{1}{n(n+1)} < \frac{1}{n^2}$. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by integral test, so $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ converges by comparison test.

(b) $\int_2^{\infty}\frac{1}{n\ln n}\,dx = \ln\ln n\big|_2^{\infty}$ is infinite, so $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ converges by integral test.

(c) $\frac{1}{(n+1)2^{n+1}}/\frac{1}{n2^n} = \frac{n}{n+1}\frac{1}{2} \le \frac{1}{2}$ for all n, so $\sum_{n=1}^{\infty}\frac{1}{n2^n}$ converges by retio test.

(d) $\ln\frac{n+1}{n} = \ln(n+1) - \ln n$, so $\sum_{n=1}^{\infty}\ln(1+\frac{1}{n}) = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots = \lim_{n\to\infty}\ln n$ is infinite, so $\sum_{n=1}^{\infty}\ln(1+\frac{1}{n})$ diverges.

(e) $n^{\frac{1}{n}} > 1$. Let $x_n = n^{\frac{1}{n}} - 1$, then $(1+x_n)^n = n$, and $\frac{n(n+1)}{2}x_n^2 \le n$ by binomial theorem, so $0 \le x_n \le \sqrt{\frac{2}{n-1}}$, and $\lim_{n\to\infty}\sqrt{\frac{2}{n-1}} = 0$ implies $\lim_{n\to\infty}x_n = 0$, so $\lim_{n\to\infty}n^{\frac{1}{n}} = 1$.

$\lim_{n\to\infty}n^{\frac{1}{n}} = 1$ implies for sufficiently large n, $0 = 1 - 1 < n^{\frac{1}{n}} < 1 + 1 = 2$, so $\frac{1}{n^{\frac{1}{n}}} > \frac{1}{2}$, and $\frac{1}{n\cdot n^{\frac{1}{n}}} > \frac{1}{2n}$. $\sum_{n=1}^{\infty}\frac{1}{2n}$ diverges by integral test, so $\sum_{n=1}^{\infty}\frac{1}{n\cdot n^{\frac{1}{n}}}$ diverges by comparison test.

**1.1.7** If $a_1 \ge a_2 \ge a_3 \ge \cdots \ge 0$, then $\sum_{n=1}^{\infty}a_n$ converges if and only if $\sum_{k=1}^{\infty}2^k a_{2k}$ converges[2].

$\sum_{k=1}^{\infty}2^k\frac{1}{(2^k)^p(\ln 2^k)^q} = 2^{(1-p)k}\frac{1}{k^q}\frac{1}{(\ln 2)^q}$. If $p > 1$, then $\sum_{k=1}^{\infty}\frac{1}{(2^{p-1})^k}\frac{1}{(\ln 2)^q}$ converges by ratio test, so $\sum_{k=1}^{\infty}2^k\frac{1}{(2^k)^p(\ln 2^k)^q}$ converges, and $\sum_{n=2}^{\infty}\frac{1}{n^p(\ln n)^q}$ converges.

If $p = 1$, $\sum_{k=1}^{\infty}2^k\frac{1}{(2^k)^p(\ln 2^k)^q} = \frac{1}{k^q}\frac{1}{(\ln 2)^q}$, converges if $q > 1$.

Therefore, if $p > 1$, or $p = 1$ and $q > 1$, then $\sum_{n=2}^{\infty}\frac{1}{n^p(\ln n)^q}$ converges.

**1.1.8**

$$\gamma = \lim_{n\to\infty}(\sum_{m=1}^{n}\frac{1}{m} - \ln n) = \sum_{m=1}^{1000}\frac{1}{m} + \lim_{n\to\infty}(\sum_{m=1001}^{n}\frac{1}{m} - \ln n)$$

$$\lim_{n\to\infty} n = \sum_{m=1001}^{\infty}\ln\frac{m}{m-1} - \ln 1000$$

$$\gamma = \sum_{m=1}^{1000}\frac{1}{m} - \ln 1000 + \sum_{m=1001}^{\infty}(\frac{1}{m} - \ln\frac{m}{m-1})$$

$$\int_{1001}^{\infty}(\frac{1}{x} - \ln\frac{x}{x-1})\,dx \ge \sum_{m=1001}^{\infty}(\frac{1}{m} - \ln\frac{m}{m-1}) \ge \int_{1001}^{\infty}(\frac{1}{x} - \ln\frac{x}{x-1})\,dx + (\frac{1}{1001} - \ln\frac{1001}{1000})$$

$$\int_{1001}^{\infty}(\frac{1}{x} - \ln\frac{x}{x-1})\,dx = \ln\frac{x}{x-1} + x\ln\frac{x-1}{x}\Big|_{1001}^{\infty} = -0.000\,499\,667$$

$$\frac{1}{1001} - \ln\frac{1001}{1000} = -0.000\,000\,499$$

$$\sum_{m=1}^{1000}\frac{1}{m} - \ln 1000 = 0.577\,714\,7$$

$$0.577\,715 - 0.000\,499 > \gamma > 0.577\,714 - 0.000\,500 - 0.000\,001$$

$$0.577\,213 < \gamma < 0.577\,216$$

**1.1.9** The number in each shell is proportional to $r^2$, but the solid angle of each shell is proportional to $\frac{1}{r^2}$, so the total solid angle occupied by stars in each shell is the same. Let it be $\omega_0$.

If the solid angle occupied by stars from shell 1 to n is $a_n$, then $a_{n+1} = a_n + a - a_n\cdot\frac{a}{4\pi}$, where $a_n\cdot\frac{a}{4\pi}$ is the solid angle of stars in shell $n+1$ that is blocked by stars in shell 1 to n. So $a_{n+1} - 4\pi = (a_n - 4\pi)(1 - \frac{a}{4\pi})$, and $a_n = (a - 4\pi)(1 - \frac{a}{4\pi})^{n-1} + 4\pi$. $1 - \frac{a}{4\pi} < 1$, so $\lim_{n\to\infty}a_n = 4\pi$.

[2]See Walter Rudin, *Principles of Mathmatical Analysis*, Chapter 3, theorem 3.27

**1.1.10**

$$\frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^2 = 1 + \frac{4n+3}{4n^2+4n+1} = 1 + \frac{1}{n} + \frac{B(n)}{n^2}$$

where B(n) is bound for large n. By Gauss' test,

$$\sum_{n=1}^{\infty} \left[\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right]^2$$

converges.

**1.1.11** (a) $\frac{\ln n}{n}$ is monotonically decreasing, and $\lim_{n\to\infty} \frac{\ln n}{n} = 0$, so the series converges by Leibniz criterion. $\frac{\ln n}{n} \geq \frac{1}{n}$ when $n \geq 3$, and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by integral test, so $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by comparison test, so the series is not absolutely convergent.

(b) The series $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \cdots$ converges by Leibniz criterion, and so does the series $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \cdots$. So the series $\frac{1}{1} + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \cdots$ is the sum of two convergent series, and is therefore convergent. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by integral test, so the series is not absolutely convergent.

(c) Combine adjacent terms with same sign to form a new series $\sum_{n=1}^{\infty} a_n$

$$(1) - (\frac{1}{2} + \frac{1}{3}) + (\frac{1}{4} + \frac{1}{5} + \frac{1}{6}) - (\frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}) + \cdots$$

where

$$a_n = \frac{1}{\frac{n^2-n}{2}+1} + \frac{1}{\frac{n^2-n}{2}+2} + \cdots + \frac{1}{\frac{n^2-n}{2}+n}$$

$$a_{n+1} = \frac{1}{\frac{n^2+n}{2}+1} + \frac{1}{\frac{n^2+n}{2}+2} + \cdots + \frac{1}{\frac{n^2+n}{2}+n} + \frac{1}{\frac{n^2+n}{2}+n+1}$$

so $a_n$ has n terms and $a_{n+1}$ has $n+1$ terms.

$$a_n - a_{n+1} = (\frac{1}{\frac{n^2-n}{2}+1} - \frac{1}{\frac{n^2+n}{2}+1}) + (\frac{1}{\frac{n^2-n}{2}+2} - \frac{1}{\frac{n^2+n}{2}+2}) + \cdots + (\frac{1}{\frac{n^2-n}{2}+n} - \frac{1}{\frac{n^2+n}{2}+n}) - \frac{2}{n^2+3n+2}$$

The $k^{st}$ term (except for the last term) of $a_n - a_{n+1}$ is

$$\frac{1}{\frac{n^2-n}{2}+k} - \frac{1}{\frac{n^2+n}{2}+k} \geq \frac{1}{\frac{n^2-n}{2}+n} - \frac{1}{\frac{n^2+n}{2}+n} = \frac{4}{n^3+4n^2+3n}$$

so

$$a_n - a_{n+1} \geq \frac{4}{n^3+4n^2+3n} \cdot n - \frac{2}{n^2+3n+2} = \frac{4}{n^2+4n+3} \cdot n - \frac{2}{n^2+3n+2} \geq 0$$

which means the sequence $a_n$ is monotonically decreasing.

$$0 \leq a_n \leq \frac{1}{\frac{n^2-n}{2}} \cdot n = \frac{2}{n-1}$$

so $\lim_{n\to\infty} a_n = 0$ because $\lim_{n\to\infty} \frac{2}{n-1} = 0$. Therefore the series converges by Leibniz criterion. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by integral test, so the series is not absolutely convergent.

**1.1.12**

$$\beta(2) = 1 - \sum_{k=1}^{\infty} \frac{16k}{(16k^2-1)^2} = 1 - \sum_{k=1}^{40} \frac{16k}{(16k^2-1)^2} - \sum_{k=41}^{\infty} \frac{16k}{(16k^2-1)^2}$$

$$\int_{41}^{\infty} \frac{16x}{(16x^2-1)^2}\,dx \leq \sum_{k=41}^{\infty} \frac{16k}{(16k^2-1)^2} \leq \int_{41}^{\infty} \frac{16x}{(16x^2-1)^2}\,dx + \frac{16 \cdot 41}{(16 \cdot 41^2 - 1)^2}$$

$$1 - \sum_{k=1}^{40} \frac{16k}{(16k^2-1)^2} = 0.915\,984\,644$$

$$\int_{41}^{\infty} \frac{16x}{(16x^2-1)^2}\,dx = \frac{-1}{2(16x^2-1)}\Big|_{41}^{\infty} = 0.000\,018\,591$$

$$\frac{16 \cdot 41}{(16 \cdot 41^2 - 1)^2} = 0.000\,000\,907$$

$$0.915965146 \le \beta(2) \le 0.915966053$$

So $\beta(2) = 0.915966$ (to six-digit accuracy).

**1.1.13**

$$\zeta(2) + a\alpha_1 + b\alpha_2 = \sum_{n=1}^{\infty}\left(\frac{1}{n^2} + \frac{a}{n(n+1)} + \frac{b}{n(n+1)(n+2)}\right) = \frac{(1+a)n^2 + (3+2a+b)n + 2}{n^2(n+1)(n+2)}$$

$$\zeta(2) - \alpha_1 - \alpha_2 = \frac{2}{n^2(n+1)(n+2)}$$

**1.1.14**

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} + \sum_{n=1}^{\infty} \frac{1}{(2n)^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

$$\lambda(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{\infty} \frac{1}{(2n)^3} = \frac{7}{8}\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{7}{8}\zeta(3)$$

$$\lambda(3) = 1.051800$$

to six decimal place.

**1.1.15** (a)

$$\sum_{n=2}^{\infty}[\zeta(n) - 1] = \sum_{n=2}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^n} - 1\right) = \sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{k^n} = \sum_{k=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{k^n} = \sum_{k=2}^{\infty}\frac{\frac{1}{k^2}}{1-\frac{1}{k}} = \sum_{k=2}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k}\right) = 1$$

(b)

$$\sum_{n=2}^{\infty}(-1)^n[\zeta(n) - 1] = \sum_{n=2}^{\infty}(-1)^n\left(\sum_{k=1}^{\infty}\frac{1}{k^n} - 1\right) = \sum_{n=2}^{\infty}(-1)^n\sum_{k=2}^{\infty}\frac{1}{k^n} = \sum_{k=2}^{\infty}\sum_{n=2}^{\infty}(-1)^n\frac{1}{k^n}$$

$$= \sum_{k=2}^{\infty}\frac{\frac{1}{k^2}}{1-(-\frac{1}{k})} = \sum_{k=2}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{2}$$

**1.1.16** (a)

$$\zeta(3) - \alpha_2' = 1 + \sum_{n=1}^{\infty}\left(\frac{1}{n^3} - \frac{1}{(n-1)n(n+1)}\right) = 1 + \sum_{n=2}^{\infty}\frac{-1}{(n-1)n^3(n+1)}$$

$$\zeta(3) = \frac{5}{4} - \sum_{n=2}^{\infty}\frac{1}{n^3(n^2-1)}$$

(b)

$$\zeta(3) + \alpha_4' = \frac{5}{4} - \sum_{n=2}^{\infty}\frac{1}{(n-1)n^3(n+1)} + \sum_{n=3}^{\infty}\frac{1}{(n-2)(n-1)n(n+1)(n+2)}$$

$$= \frac{29}{24} + \sum_{n=3}^{\infty}\frac{4}{(n-2)(n-1)n^3(n+1)(n+2)}$$

$$\zeta(3) = \frac{115}{96} + \sum_{n=3}^{\infty}\frac{4}{(n-2)(n-1)n^3(n+1)(n+2)}$$

(c)
$$\int_N^\infty \frac{1}{x^3}\,dx \le \sum_{n=N}^\infty \frac{1}{n^3} \le \int_N^\infty \frac{1}{x^3}\,dx + \frac{1}{N^3}$$
$\frac{1}{126^3} = 4.999\times 10^{-7} < 5\times 10^{-7}$, so 125 terms are required.
$$\int_N^\infty \frac{1}{x^3(x^2-1)}\,dx \le \sum_{n=N}^\infty \frac{1}{n^3(n^2-1)} \le \int_N^\infty \frac{1}{x^3(x^2-1)}\,dx + \frac{1}{N^3(N^2-1)}$$
$\frac{1}{19^3(19^2-1)} = 4\times 10^{-7} < 5\times 10^{-7}$, so 17 terms ($n=2$ to $n=18$) are required.
$$\int_N^\infty \frac{4}{x^3(x^2-1)(x^2-4)}\,dx \le \sum_{n=N}^\infty \frac{1}{n^3(n^2-1)(n^2-4)} \le \int_N^\infty \frac{4}{x^3(x^2-1)(x^2-4)}\,dx + \frac{4}{N^3(N^2-1)(N^2-4)}$$
$\frac{4}{10^3(10^2-1)(10^2-4)} = 4.2\times 10^{-7} < 5\times 10^{-7}$, so 7 terms ($n=3$ to $n=9$) are required.

## 1.2 Series of Functions

**1.2.1** (a) When $x>0$, $\lim_{n\to\infty}\frac{(-1)^{n-1}}{n^x}=0$, and the sequence $\frac{1}{n^x}$ is monotonically decreasing, so by Leibniz criterion, $\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^x}$ converges. When $x\le 0$, $\lim_{n\to\infty}\frac{(-1)^{n-1}}{n^x}\ne 0$, so $\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^x}$ diverges. Therefore, the series converges for $x>0$, and if the series uniformly converges in the interval $[a,b]$, then $0<a<b$.

For $x\in[a,b]$, $0<a<b$,
$$|S(x)-s_n(x)| = |\sum_{k=n+1}^\infty \frac{(-1)^{k-1}}{k^x}| \le \frac{1}{(n+1)^x} < \frac{1}{n^x} < \frac{1}{n^a}$$
For any $\epsilon>0$, let $N=(\frac{1}{\epsilon})^{\frac{1}{a}}+1$, then
$$|S(x)-s_n(x)| < \frac{1}{n^a} < \frac{1}{N^a} = \epsilon$$
for all $n\ge N$, so the series is uniformly convergent in the range $[a,b]$ if $0<a<b$.

(b) The series converges when $x>1$ and diverges when $x\le 1$ by integral test, so if it uniformly converges in interval $[a,b]$, then $1<a<b$. For $x\in[a,b]$ and $1<a<b$, $|\frac{1}{n^x}|\le\frac{1}{n^a}$. $\sum_n \frac{1}{n^a}$ is a convergent series, so by Weierstrass M test, $\sum_{n=1}^\infty\frac{1}{n^x}$ is uniformly convergent in the range $[a,b]$ if $1<a<b$.

**1.2.2** The series converges when $|x|<1$ and diverges when $|x|\ge 1$. If it uniformly converges in the interval $[a,b]$, then $-1<a<b<1$. For $x\in[a,b]$ and $-1<a<b<1$, let $c=\max(|a|,|b|)$, then $|x^n|\le x^c$. Because $c<1$, so $\sum_{n=0}^\infty x^c$ converges, so by Weietress M test, $\sum_{n=0}^\infty x^n$ uniformly converges in the range $[a,b]$ if $-1<a<b<1$.

**1.2.3** (a) When $0<x\le 1$, $\lim_{n\to\infty}\frac{1}{1+x^n}\ne 0$, so the series diverges. When $x>1$, $0<\frac{1}{1+x^n}<\frac{1}{x^n}$. The geometry series $\frac{1}{x^n}$ converges, so $\sum_{n=0}^\infty\frac{1}{1+x^n}$ converges by comparison test.

(b) If the series is uniformly convergent in the interval $[a,b]$, then $1<a<b$. For $x\in[a,b]$ and $1<a<b$, $\frac{1}{1+x^n}<\frac{1}{x^n}\le\frac{1}{a^n}$. The geometry series $\sum_{n=0}^\infty\frac{1}{a^n}$ converges, so by Weierstrass M test, $\sum_{n=0}^\infty\frac{1}{1+x^n}$ is uniformly convergent in the range $[a,b]$ if $1<a<b$.

**1.2.4** For $x\in[a,b]$, $|a_n\cos nx+b_n\sin nx|\le|a_n|+|b_n|$. $\sum|a_n|$ and $\sum|b_n|$ converges, so $\sum(|a_n|+|b_n|)$ converges, so $\sum(a_n\cos nx+b_n\sin nx)$ is uniformly convergent by Weierstrass M test in the range $[a,b]$ for any $a<b$.

**1.2.5** Let $j=2n$, $a_n(x)=u_{2n}(x)$, then $a_{n+1}(x)=\frac{(2n+1)(2n+2)-l(l+1)}{(2n+2)(2n+3)}x^2a_n(x)$. $\lim_{n\to\infty}\frac{a_{n+1}(x)}{a_n(x)}=x^2>1$ when $|x|>1$ and $\lim_{n\to\infty}\frac{a_{n+1}(x)}{a_n(x)}<1$ when $|x|<1$, so by ratio test the series diverges when $|x|>1$ and converges when $|x|<1$. When $x=1$, $\frac{a_n}{a_{n+1}}=\frac{4n^2+10n+6}{4n^2+6n+2-l(l+1)}=1+\frac{4n+4+l(l+1)}{4n^2+6n+2-l(l+1)}=1+\frac{1}{n}+\frac{B(n)}{n^2}$, where $B(n)$ is bound for large n. By Gauss' test, the series diverges when $x=1$, so the range of convergence is $-1<x<1$.

**1.2.6** Let $j = 2m$, $a_m(x) = u_{2m}(x)$. when $x = \pm 1$, $\frac{a_m(x)}{a_{m+1}(x)} = 1 + \frac{3}{2m} + \frac{B(m)}{m^2}$, where $B(m)$ is bound for large m. By Gauss' test, the series converges at $x = \pm 1$.

**1.2.7** Let $j = 2m$, $u_m = a_{2m}$, then $\frac{u_m}{u_{m+1}} = \frac{4m^2+(4k+6)m+(k+1)(k+2)}{4m^2+(4k+4\alpha)m+k(k+2\alpha)-n(n+2\alpha)} = 1 + \frac{6-4\alpha}{4m} + \frac{B(m)}{m^2}$, where $B(m)$ is bound for large m. By Gauss' test, the series converges $\alpha < \frac{1}{2}$ and diverges when $\alpha \geq \frac{1}{2}$.

**1.2.8**

$$\sin x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left( \frac{d^n \sin x}{dx^n}\Big|_{x=0} \right) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}(-1)^k \sin 0 + \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}(-1)^k \cos 0 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left( \frac{d^n \cos x}{dx^n}\Big|_{x=0} \right) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}(-1)^k \cos 0 + \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}(-1)^{k+1} \sin 0 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

**1.2.9** expand $\sin x$ and $\cos x$ for 4 terms, and perform long division.

$$\begin{array}{ccccc|cccccc}
 & & & & & \frac{1}{x} & -\frac{x}{3} & -\frac{x^3}{45} & -\frac{2x^5}{945} & \cdots \\
\hline
x & -\frac{x^3}{6} & +\frac{x^5}{120} & -\frac{x^7}{5040} & \surd & 1 & -\frac{x^2}{2} & +\frac{x^4}{24} & -\frac{x^6}{720} & \cdots \\
 & & & & & 1 & -\frac{x^2}{6} & +\frac{x^4}{120} & -\frac{x^6}{5040} & \cdots \\
\hline
 & & & & & & -\frac{x^2}{3} & +\frac{x^4}{30} & -\frac{x^6}{840} & \cdots \\
 & & & & & & -\frac{x^2}{3} & +\frac{x^4}{18} & -\frac{x^6}{360} & \cdots \\
\hline
 & & & & & & & -\frac{x^4}{45} & +\frac{x^6}{630} & \cdots \\
 & & & & & & & -\frac{x^4}{45} & +\frac{x^6}{270} & \cdots \\
\hline
 & & & & & & & & -\frac{2x^6}{945} & \cdots
\end{array}$$

So $\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} + \cdots$.

**1.2.10**

$$\frac{d}{dx} \coth^{-1} x = \frac{1}{1-x^2} = \frac{d}{dx}\left(\frac{1}{2}\ln\frac{x+1}{x-1}\right) = \frac{1}{1-x^2}$$

So $\frac{d^n}{dx^n} \coth^{-1} x = \frac{d^n}{dx^n}\left(\frac{1}{2}\ln\frac{x+1}{x-1}\right)$, and $\sum_{n=0}^{\infty} \frac{x^n}{n!}\frac{d^n}{dx^n} \coth^{-1} x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\frac{d^n}{dx^n}\left(\frac{1}{2}\ln\frac{x+1}{x-1}\right)$.
Therefore, $\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1}$.

**1.2.11** (a) $\frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}}$ is undefined at $x = 0$, so Maclaurin expansion does not exist.
(b)

$$x^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!}\frac{d^n}{dx^n}x^{\frac{1}{2}} = \sum_{n=0}^{\infty} u_n(x)$$

$$\lim_{n\to\infty} \frac{u_{n+1}(x)}{u_n(x)} = \lim_{n\to\infty} \left| \frac{(x-x_0)^{n+1}}{(n+1)!}\frac{d^{n+1}}{dx^{n+1}}x^{\frac{1}{2}} \Big/ \frac{(x-x_0)^n}{n!}\frac{d^n}{dx^n}x^{\frac{1}{2}} \right| = \lim_{n\to\infty}\left| \frac{x-x_0}{x_0}\frac{2n-1}{2n+2} \right| < 1$$

when $|x - x_0| < x_0$, so the series converges when $|x - x_0| < x_0$. When $x = 0$, all the terms are less than 0, and

$$\frac{u_n(x)}{u_{n+1}(x)} = \frac{2n+2}{2n-1} = 1 + \frac{2}{2n-1} = 1 + \frac{1}{n} + \frac{B(n)}{n^2}$$

where B(n) is bound for large n, so the series diverges by Gauss' test. When $x = 2x_0$, $\sum_{n=0}^{\infty} u_n(x)$ is an alternating series, and $\lim_{n\to\infty} u_n(x) = 0$, so the series converges by Leibniz criterion. Therefore, the range of convergence is $0 < x \leq 2x_0$.

**1.2.12**
$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f(x_0)+(x-x_0)f'(x_0)+\sum_{n=2}^{\infty}a_n(x-x_0)^n}{g(x_0)+(x-x_0)g'(x_0)+\sum_{n=2}^{\infty}b_n(x-x_0)^n}$$
When $f(x_0)=g(x_0)=0$,
$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{(x-x_0)f'(x_0)+\sum_{n=2}^{\infty}a_n(x-x_0)^n}{(x-x_0)g'(x_0)+\sum_{n=2}^{\infty}b_n(x-x_0)^n}=\lim_{x\to x_0}\frac{f'(x_0)+\sum_{n=1}^{\infty}c_n(x-x_0)^n}{g'(x_0)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$
by binomial theorem[3].

**1.2.13** (a)
$$\frac{1}{n}-\ln\frac{n}{n-1}=\frac{1}{n}+\ln\frac{n-1}{n}=\frac{1}{n}+\ln(1-\frac{1}{n})=\frac{1}{n}-\sum_{k=1}^{\infty}(\frac{1}{n})^k=-\sum_{k=2}^{\infty}(\frac{1}{n})^k<0$$
(b)
$$\frac{1}{n}-\ln\frac{n+1}{n}=\frac{1}{n}-\ln(1+\frac{1}{n})=\frac{1}{n}-\sum_{k=1}^{\infty}(-1)^{k+1}(\frac{1}{n})^k=\sum_{k=2}^{\infty}(-1)^k(\frac{1}{n})^k>0$$
Euler-Mascheroni constant:
$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)=1+\sum_{n=2}^{\infty}(\frac{1}{n}-\ln\frac{n}{n-1})<1$$
$$\gamma=1+\sum_{n=2}^{\infty}(\frac{1}{n}-\ln\frac{n}{n-1})=1+\sum_{n=1}^{\infty}(\frac{1}{n+1}-\ln\frac{n+1}{n})=1-1+\sum_{n=1}^{\infty}(\frac{1}{n}-\ln\frac{n+1}{n})+\lim_{n\to\infty}\frac{1}{n+1}>0$$
So the Euler-Mascheroni constant $\gamma$ is bound by 0 and 1, and is therefore finite.

**1.2.14**
$$\psi(x\pm h)=\psi(x)\pm h\psi'(x)+\frac{h^2}{2}\psi''(x)\pm\frac{h^3}{6}\psi^{(3)}(x)+\frac{h^4}{24}\psi^{(4)}(x)+\cdots$$
$$\frac{1}{h^2}\left[\psi(x+h)-2\psi(x)+\psi(x-h)\right]=\psi''(x)+\frac{h^2}{12}\psi^{(4)}(x)+\cdots$$
So the error is $\frac{h^2}{12}\psi^{(4)}(x)$.

**1.2.15** $\sin x=x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\cdots$, $\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{7x^7}{315}+\cdots$ (expansion of $\tan x$ can be obtained by long division of expansions of $\sin x$ and $\cos x$). The term of highest order in $(\sin(\tan x)-\tan(\sin x))$ is $x^7$, the coefficient is
$\left\{\left[\frac{7}{315}-\frac{1}{6}\left(3\times\frac{2}{15}+3\times\frac{1}{3^2}\right)+\frac{1}{120}\left(5\times\frac{1}{3}\right)-\frac{1}{5040}\right]-\left[-\frac{1}{5040}+\frac{1}{3}\left(3\times\frac{1}{120}+3\times\frac{1}{36}\right)+\frac{2}{15}\left(5\times\frac{-1}{6}+\frac{7}{315}\right)\right]\right\}x^7$
$$\lim_{x\to\infty}\left[\frac{\sin(\tan x)-\tan(\sin x)}{x^7}\right]=-\frac{1}{30}$$

**1.2.16** If the convergence ranges of the series $\sum_{n=0}^{\infty}a_nx^n$ is $-R<x<R$, then $\lim_{n\to\infty}\left|\frac{a_n+1}{a_n}R\right|=1$. The differentiated series is $\sum_{n=1}^{\infty}na_nx^{n-1}$, $\lim_{n\to\infty}\left|\frac{n+1}{n}\frac{a_n+1}{a_n}x\right|=\lim_{n\to\infty}\left|\frac{a_n+1}{a_n}x\right|<1$ when $-R<x<R$ (ignoring the end points). The integrated series is $\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$, $\lim_{n\to\infty}\left|\frac{n+1}{n+2}\frac{a_n+1}{a_n}x\right|=\lim_{n\to\infty}\left|\frac{a_n+1}{a_n}x\right|<1$ when $-R<x<R$ (ignoring the end points). So all the series have the same converging range (ignoring the end points).

[3]This is not a formal proof of l'Hôpital's rule

## 1.3 Binomial Theorem

**1.3.1**

$$P(x) = c\left(\frac{\cosh x}{\sinh(x)} - \frac{1}{x}\right) = c\frac{(1+\frac{x^2}{2}+\frac{x^4}{24}+\cdots)-(1+\frac{x^3}{6}+\frac{x^4}{120})}{x+\frac{x^3}{6}+\frac{x^5}{120}+\cdots} = c\frac{\frac{x}{3}+\frac{x^3}{30}+\cdots}{1+\frac{x^2}{6}+\cdots}$$

$$= c(\frac{x}{3}+\frac{x^3}{30}+\cdots)(1-\frac{x^2}{6}+\cdots) = c\left(\frac{x}{3}-\frac{x^3}{45}+\cdots\right)$$

**1.3.2** By binomial theorem, $\frac{1}{1-x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$, $\int_0^1 \frac{1}{1+x^2}dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$. At $x=1$, $\lim_{n\to\infty}(-1)^n x^{2n} \neq 0$, so the series diverges; $\lim_{n\to\infty}\frac{x^{2n+1}}{2n+1}=0$ and the sequence $\frac{x^{2n+1}}{2n+1}$ is monotonically decreasing, so by Leibniz criterion the series converges.

**1.3.3** $e^{-t}t^n = \sum_{p=0}^{\infty}\frac{(-t)^p}{p!}t^n$. When $-a \le t \le a$ $(a>0)$, $\left|\frac{(-t)^p}{p!}t^n\right| \le \frac{a^{p+n}}{p!}$ and $\sum_{p=0}^{\infty}\frac{a^{p+n}}{p!}$ is convergent, so by Weierstrass M test $\sum_{p=0}^{\infty}\frac{(-t)^p}{p!}t^n$ is uniformly convergent in $-a \le t \le a$ for every $a>0$, and the series can be integrated by terms.

$$\int_0^x e^{-t}t^n dt = \int_0^x \sum_{p=0}^{\infty}\frac{(-t)^p}{p!}t^n = \sum_{p=0}^{\infty}\int_0^x (-1)^p\frac{t^{n+p}}{p!}dt = \sum_{p=0}^{\infty}(-1)^p\frac{x^{n+p+1}}{p!(n+p+1)}$$

For $-a \le x \le a\,(a>0)$, the integrated series converges by Leibniz criterion, so the range of convergence is $-a \le t \le a$ for every $a>0$.

**1.3.4** (a) $x = \sinh y = \frac{e^y-e^{-y}}{2} = y + \frac{y^3}{6}+\frac{y^5}{120}+\cdots$. Because x in terms of y has only terms of odd orders, y in terms of x can only has terms of odd orders too. Let $y = a_1x + a_3x^3 + a_5x^5 + \cdots$, substituting into $x = y + \frac{y^3}{6}+\frac{y^5}{120}+\cdots$, obtaining $x = a_1x + (a_3+\frac{a_1^3}{6})x^3 + (a_5 + \frac{a_1^2a_3}{2}+\frac{a_1^5}{120})x^5+\cdots$, so $a_1=1$, $a_3 = -\frac{1}{6}$, $a_5 = \frac{3}{40}$. Therefore $\sinh^{-1}x = x - \frac{x^3}{6}+\frac{3x^5}{40}+\cdots$.

(b) $\frac{d}{dx}\sinh^{-1}x = (x^2+1)^{-1/2}$, $\frac{d^2}{dx^2}\sinh^{-1}x = -x(x^2+1)^{-3/2}$, $\frac{d^3}{dx^3}\sinh^{-1}x = -(x^2+1)^{-3/2} + 3x^2(x^2+1)^{-5/2}$, $\frac{d^4}{dx^4}\sinh^{-1}x = 9x(x^2+1)^{-5/2} - 15x^3(x^2+1)^{-7/2}$, $\frac{d^5}{dx^5}\sinh^{-1}x = 9(x^2+1)^{-5/2} - 90x^2(x^2+1)^{-7/2} + 105x^4(x^2+1)^{-9/2}$. So $\sinh^{-1}x = \sum_{n=0}^{\infty}\frac{x^n}{n!}\frac{d^n}{dx^n}\sinh^{-1}x = x - \frac{x^3}{6}+\frac{3x^5}{40}+\cdots$.

**1.3.5**

$$\frac{1}{(1-x)^{n+1}} = \sum_{m=0}^{\infty}\binom{-(n+1)}{m}(-x)^m = \sum_{m=0}^{\infty}\frac{(n+m)!}{m!n!}x^m = \sum_{m'=n}^{\infty}\frac{m'!}{(m'-n)!n!}x^{m'-n} = \sum_{m=n}^{\infty}\binom{m}{n}x^{m-n}$$

In the third equal sign we use a substitution $m' = m+n$.

**1.3.6**

$$(1+x)^{-\frac{m}{2}} = \sum_{n=0}^{\infty}\binom{-\frac{m}{2}}{n}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n\frac{(m+2n-2)!!}{2^n(m-2)!!}}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{(m+2n-2)!!}{2^n n!(m-2)!!}$$

**1.3.7** (a) $\nu' = \nu(1 \pm \frac{v}{c} + (\frac{v}{c})^2+\cdots)$
(b) $\nu' = \nu(1\pm\frac{v}{c})$
(c) $\nu' = \nu(1\pm\frac{v}{c}+\frac{1}{2}(\frac{v}{c})^2)+\cdots$

**1.3.8** (a) $v_1 = c(\delta + \frac{1}{2}\delta^2)$
(b) $v_2 = c\delta(1+\frac{1}{2}\delta)(1-2\delta+\cdots) = c(\delta - \frac{3}{2}\delta^2+\cdots)$
(c) Solve for $v_3$, obtaining $v_3 = c\frac{\delta^2+2\delta}{2+2\delta+\delta^2} = c\delta(1+\frac{\delta}{2})(1-\delta\cdots) = c(\delta-\frac{1}{2}\delta^2+\cdots)$

**1.3.9**

$$\frac{w}{c} = \frac{2(1-\alpha)}{1+(1-\alpha^2)}\frac{1-\alpha}{1-\alpha+\frac{\alpha^2}{2}} = (1-\alpha)\left[1-(-\alpha+\frac{\alpha^2}{2})+(-\alpha+\frac{\alpha^2}{2})^2-(-\alpha+\frac{\alpha^2}{2})^3+\cdots\right]$$

$$= (1-\alpha)(1+\alpha+\frac{\alpha^2}{2}+\cdots) = 1-\frac{\alpha^2}{2}-\frac{\alpha^3}{2}+\cdots$$

**1.3.10**

$$x = \frac{c^2}{g}\left\{1 + \frac{1}{2}\left(g\frac{t}{c}\right) - \frac{1}{8}\left(g\frac{t}{c}\right)^4 + \frac{1}{16}\left(g\frac{t}{c}\right)^6 - \cdots\right\} = \frac{1}{2}gt^2 - \frac{1}{8}\frac{g^3t^4}{c^2} + \frac{1}{16}\frac{g^5t^6}{c^4} - \cdots$$

**1.3.11**

$$\frac{\gamma^2}{(s+n-|k|^2)^2} = \frac{\gamma^2}{\left(n+|k|(1-\frac{1}{2}\frac{\gamma^2}{|k|^2}+\cdots-1)\right)^2} = \frac{\gamma^2}{n^2(1-\frac{\gamma^2}{2n|k|})^2+\cdots} = \frac{\gamma^2}{n^2}(1+\frac{\gamma^2}{n|k|}+\cdots) = \frac{\gamma^2}{n^2}+\frac{\gamma^4}{n^3|k|}+\cdots$$

$$E = mc^2\left[1 - \frac{1}{2}\left(\frac{\gamma^2}{n^2}+\frac{\gamma^4}{n^3|k|}+\cdots\right) + \frac{3}{8}\left(\frac{\gamma^2}{n^2}+\frac{\gamma^4}{n^3|k|}+\cdots\right)^2 + \cdots\right]$$

$$= mc^2\left[1 - \frac{1}{2}\frac{\gamma^2}{n^2} + (\frac{3}{8} - \frac{n}{2|k|})\frac{\gamma^4}{n^4} + \cdots\right]$$

**1.3.12** (a)

$$R = \frac{2mc^2(1+\frac{E_k}{2mc^2})^{1/2} - 2mc^2}{E_k} = \frac{\frac{1}{2}E_k + \cdots}{E_k} \approx \frac{1}{2}$$

(b)

$$R = \sqrt{\frac{2mc^2(E_k+2mc^2)}{E_k^2}} - \frac{2mc^2}{E_k} \approx 0$$

**1.3.13** The first series converges for $|x| < 1$, and the second series converges for $|x^{-1}| < 1$, that is, $|x| > 1$. The ranges of convergence of the two series do not overlap, so when adding together to get $\sum_{n=-\infty}^{\infty} x^n$, nowhere can the series converge.

**1.3.14** (a) By binomial theorem, when $|x| < 1$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}\binom{-1}{n}(-x)^n = \sum_{n=0}^{\infty} x^n$$

$$\frac{x}{(1-x)^2} = x\sum_{n=0}^{\infty}\binom{-2}{n}(-x)^n = x\sum_{n=0}^{\infty}(n+1)x^n = \sum_{n=1}^{\infty} nx^n$$

$$\langle\varepsilon\rangle = \frac{\varepsilon_0 e^{\frac{-\varepsilon_0}{kT}}}{(1-\frac{-\varepsilon_0}{kT})^2}\Big/\frac{1}{1-\frac{-\varepsilon_0}{kT}} == \frac{\varepsilon_0}{e^{\frac{\varepsilon_0}{kT}}-1}$$

(b)

$$\langle\varepsilon\rangle = \frac{\varepsilon_0}{1+\frac{\varepsilon_0}{kT}+\cdots-1} \approx kT$$

**1.3.15**

$$\tan^{-1}x = \int_0^x \frac{dt}{1+t^2} = \int_0^\infty \sum_{n=0}^{\infty}(-1)^n t^{2n} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

When $-a < t < a$ and $0 < a < 1$, the series $\sum_{n=0}^{\infty}(-1)^n t^{2n}$ is uniformly convergent by Weierstrass M test, so the series can be integrated by term.

**1.3.16**

$$\frac{2+2\varepsilon}{1+2\varepsilon} = 2(1+\varepsilon)(1-2\varepsilon+4\varepsilon^2-\cdots) = 2-2\varepsilon+4\varepsilon^2-\cdots$$

$$\frac{\ln(1+2\varepsilon)}{\varepsilon} = \frac{2\varepsilon-\frac{4\varepsilon^2}{2}+\frac{8\varepsilon^3}{3}-\cdots}{\varepsilon} = 2-2\varepsilon+\frac{8}{3}\varepsilon^2-\cdots$$

$$\lim_{n\to\infty} f(\varepsilon) = \lim_{n\to\infty}\frac{(1+\varepsilon)}{\varepsilon^2}\left(\frac{4}{3}\varepsilon^2+\cdots\right) = \frac{4}{3}$$

**1.3.17**

$$\xi_1 = 1+\frac{A^2(1-2A^{-1}+A^{-2})}{2A}\left[-(A^{-1}+\frac{A^{-2}}{2}+\frac{A^{-3}}{3}+\frac{A^{-4}}{4}+\cdots)-(A^{-1}-\frac{A^{-2}}{2}+\frac{A^{-3}}{3}-\frac{A^{-4}}{4}+\cdots)\right]$$

$$= 1+\frac{A}{2}(1-2A^{-1}+A^{-2})(-2A^{-1}-\frac{2}{3}A^{-3}-\cdots) = 2A^{-1}-\frac{4}{3}A^{-2}+\frac{2}{3}A^{-3}+\cdots$$

$$\xi_2 = \frac{2}{A}(1+\frac{2}{3}A^{-1})^{-1} = 2A^{-1}(1-\frac{2}{3}A^{-1}+\frac{4}{9}A^{-2}-\cdots) = 2A^{-1}-\frac{4}{3}A^{-2}+\frac{8}{9}A^{-3}+\cdots$$

The difference in the coefficients of the $A^{-3}$ term is $\frac{2}{9}$.

**1.3.18** (a)

$$\arctan x = \int_0^\infty \frac{1}{1+t^2}dt = \int_0^\infty \sum_{k=0}^{\infty}(-1)^k t^{2k}dt = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$$

(In exercise 1.3.15 we have verified that this series expansion is valid in $-a \le x \le a$ for $0 < a < 1$).

$$\int_0^1 \arctan t\frac{dt}{t} = \int_0^1 \sum_{n=0}^{\infty}\frac{t^{2k}}{2k+1}dt = \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)^2}$$

which is the Catalan's constant.

(b) Integrating by parts,

$$-\int_0^1 \ln x\frac{dx}{1+x^2} = -\left[(\ln x)\arctan x\Big|_0^1 - \int_0^1 \frac{1}{x}\arctan x dx\right] = \int_0^1 \arctan x\frac{dx}{x}$$

which is the same with (a).

## 1.4 Mathematical Induction

**1.4.1** If the equation holds for $n = k-1$, then $\sum_{j=1}^{k-1} j^4 = \frac{k-1}{30}(2k-1)(k)(3k^2-3k-1) = \frac{k^5}{5}-\frac{k^4}{2}+\frac{k^3}{3}-\frac{k}{30}$, and $\sum_{j=1}^{k} j^4 = \sum_{j=1}^{k-1} j^4 + k^4 = \frac{k^5}{5}+\frac{k^4}{2}+\frac{k^3}{3}-\frac{k}{30} = \frac{k}{30}(2k+1)(k+1)(3k^2+3k-1)$. $\sum_{j=1}^{1} j^4 = 1 = \frac{1}{30}(2+1)(1+1)(3+3-1)$. The equation holds for $n = 1$, and if it holds for $n = k-1$ then it will holds for $n = k$, so the equation holds for all positive integers by mathematical induction.

**1.4.2** If $(\frac{d}{dx})^n[f(x)g(x)] = \sum_{j=0}^{n}\binom{n}{j}\left[(\frac{d}{dx})^j f(x)\right]\left[(\frac{d}{dx})^{n-j} g(x)\right]$, then

$$\left(\frac{d}{dx}\right)^{n+1}[f(x)g(x)] = \sum_{j=0}^{n}\binom{n}{j}\left[\left(\frac{d}{dx}\right)^{j+1} f(x)\right]\left[\left(\frac{d}{dx}\right)^{n-j} g(x)\right]+\sum_{j=0}^{n}\binom{n}{j}\left[\left(\frac{d}{dx}\right)^{j} f(x)\right]\left[\left(\frac{d}{dx}\right)^{n-j+1} g(x)\right]$$

$$= \sum_{j=1}^{n+1}\binom{n}{j-1}\left[\left(\frac{d}{dx}\right)^{j} f(x)\right]\left[\left(\frac{d}{dx}\right)^{n+1-j} g(x)\right]+\sum_{j=0}^{n}\binom{n}{j}\left[\left(\frac{d}{dx}\right)^{j} f(x)\right]\left[\left(\frac{d}{dx}\right)^{n+1-j} g(x)\right]$$

$$= \sum_{j=1}^{n}\binom{n+1}{j}\left[\left(\frac{d}{dx}\right)^{j} f(x)\right]\left[\left(\frac{d}{dx}\right)^{n+1-j} g(x)\right]+\left[\left(\frac{d}{dx}\right)^{n+1} f(x)\right]\left[\left(\frac{d}{dx}\right) g(x)\right]+\left[\left(\frac{d}{dx}\right) f(x)\right]\left[\left(\frac{d}{dx}\right)^{n+1} g(x)\right]$$

$$= \sum_{j=0}^{n+1}\binom{n+1}{j}\left[\left(\frac{d}{dx}\right)^{j} f(x)\right]\left[\left(\frac{d}{dx}\right)^{n+1-j} g(x)\right]$$

so the equation holds for all positive integers by mathematical induction.

## 1.5 Operations On Series Expansions Of Functions

**1.5.1**

$$\int_{-x}^{x}\frac{dt}{1-t^{\,2}} = \frac{1}{2}\int_{-x}^{x}\left(\frac{1}{1-t}+\frac{1}{1+t}\right)dt = \frac{1}{2}\left[-\ln\frac{1-x}{1+x}+\ln\frac{1+x}{1-x}\right] = \ln\frac{1+x}{1-x}$$

**1.5.2** If the equation holds for p

$$\frac{1}{n(n+1)\cdots(n+p)} = \frac{1}{p!}\sum_{j=0}^{p}(-1)^j\binom{p}{j}\frac{1}{n+j}$$

$$\begin{aligned}
\frac{1}{n(n+1)\cdots(n+p)(n+p+1)} &= \frac{1}{p!}\sum_{j=0}^{p}(-1)^j\binom{p}{j}\frac{1}{n+j}\frac{1}{n+p+1} \\
&= \frac{1}{p!}\sum_{j=0}^{p}(-1)^j\binom{p}{j}\left(\frac{1}{n+j}-\frac{1}{n+p+1}\right)\frac{1}{p+1-j} \\
&= \frac{1}{(p+1)!}\sum_{j=0}^{p}(-1)^j\frac{p+1}{p+1-j}\binom{p}{j}\left(\frac{1}{n+j}-\frac{1}{n+p+1}\right) \\
&= \frac{1}{(p+1)!}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}\frac{1}{n+j} - \frac{1}{(p+1)!}\frac{1}{n+p+1}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j} \\
&= \frac{1}{(p+1)!}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}\frac{1}{n+j} - \frac{1}{(p+1)!}\frac{1}{n+p+1}(-1)^{p+1}\sum_{j=1}^{p+1}(-1)^j\binom{p+1}{j} \\
&= \frac{1}{(p+1)!}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}\frac{1}{n+j} + \frac{1}{(p+1)!}\frac{1}{n+p+1}(-1)^{p+1}\sum_{j=1}^{p+1}(-1)^{j-1}\binom{p+1}{j} \\
&= \frac{1}{(p+1)!}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}\frac{1}{n+j} + \frac{1}{(p+1)!}\frac{1}{n+p+1}(-1)^{p+1} \\
&= \frac{1}{(p+1)!}\sum_{j=0}^{p+1}(-1)^j\binom{p+1}{j}\frac{1}{n+j}
\end{aligned}$$

So the equation holds for p+1. The equation hold for 1. Therefore, it holds for all positive integers by mathematical induction.

**1.5.3**

$$u_n(p) = \frac{1}{(n+1)\cdots(n+p-1)}\frac{1}{p}\left[\frac{1}{n}-\frac{1}{n+p}\right] = \frac{1}{p}\left[u_n(p-1)-u_{n+1}(p-1)\right]$$

$$\sum_{n=1}^{\infty}u_n(p) = \frac{u_1(p-1)}{p} = \frac{1}{p\cdot p!}$$

**1.5.4** Substituting Eq. (1.88) into Eq. (1.87):

$$f(x) = \sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}\frac{x^n}{(1+x)^{n+1}}c_{n-j}$$

Rearrange the series by $n' = n-j$

$$f(x) = \sum_{n'=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{n'}\binom{n'+j}{j}\frac{x^{n'+j}}{(1+x)^{n'+1+j}}c_{n'} = \sum_{n'=0}^{\infty}(-1)^{n'}c_{n'}\frac{x^{n'}}{(1+x)^{n'+1}}\sum_{j=0}^{\infty}\binom{n'+j}{j}\left(\frac{x}{1+x}\right)^j$$

$$= \sum_{n'=0}^{\infty}(-1)^{n'}c_{n'}\frac{x^{n'}}{(1+x)^{n'+1}}\sum_{j=0}^{\infty}\binom{-n'-1}{j}\left(-\frac{x}{1+x}\right)^j = \sum_{n'=0}^{\infty}(-1)^{n'}c_{n'}\frac{x^{n'}}{(1+x)^{n'+1}}\left(1-\frac{x}{1+x}\right)^{-n'-1}$$

$$= \sum_{n'=0}^{\infty}(-1)^{n'}c_{n'}\frac{x^{n'}}{(1+x)^{n'+1}}(1+x)^{n'+1} = \sum_{n'=0}^{\infty}(-1)^{n'}c_{n'}x^{n'} = \sum_{n=0}^{\infty}(-1)^n c_n x^n$$

**1.5.5** Let $\arctan x = \sum_{n=0}^{\infty}(-1)^n c_n x^n$, then from $n = 0$, $c_n = 0, -\frac{1}{1}, 0, \frac{1}{3}, 0, -\frac{1}{5}, \cdots$, which means there are only odd terms. $a_n = \sum_{j=0}^{n}(-1)^j \binom{n}{j} c_{n-j}$, when n is odd, $a_n = \sum_{j=0,2,4,\cdots,n-1} \binom{n}{j} c_{n-j} = \sum_{k=1,3,5,\cdots,n} \binom{n}{k} c_k = -\binom{n}{1}\frac{1}{1} + \binom{n}{3}\frac{1}{3} - \binom{n}{5}\frac{1}{5} + \cdots$; when n is even, $a_n = \sum_{j=1,3,5,\cdots,n-1}(-1)\binom{n}{j} c_{n-j} = \sum_{k=1,3,5,\cdots,n-1}(-1)\binom{n}{k} c_k = \binom{n}{1}\frac{1}{1} - \binom{n}{3}\frac{1}{3} + \binom{n}{5}\frac{1}{5} - \cdots$. The numerical verification of $\arctan(1)$ and $\arctan(3^{-1/2})$ is straightforward by using Eq. (1.87).

## 1.6 Some Important Series

**1.6.1** $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$, $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$, so $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$.

## 1.7 Vectors

**1.7.1** $3(A_x)^2 = 1.732^2$, so $A_x = A_y = A_z \approx 1.000$

**1.7.2** $(\mathbf{B} - \mathbf{A}) + (\mathbf{C} - \mathbf{B}) + (\mathbf{A} - \mathbf{C}) = 0$

**1.7.3** (a) $(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = 0$
(b) $\mathbf{r} = \mathbf{r} + \mathbf{a}$

**1.7.4** Let $\mathbf{v}_i'$ and $\mathbf{r}_i'$ be the velocity and position of the $i$th galaxy viewed from $\mathbf{r}_1$. Then $\mathbf{v}_i' = \mathbf{v}_i - \mathbf{v}_1 = H_0\mathbf{r}_i - H_0\mathbf{r}_1 = H_0\mathbf{r}_i'$

**1.7.5** Diagonals: $\pm(1,1,1), \pm(-1,1,1), \pm(1,-1,1), \pm(1,1,-1)$
Diagonals of faces: $\pm(1,-1,0), \pm(1,0,-1), \pm(0,1,-1)$

**1.7.6** (a) $(x - a_x)a_x + (y - a_y)a_y + (z - a_z)a_z = 0$. It is the surface containing the tip of $\mathbf{a}$ and perpendicular to $\mathbf{a}$.

(b)$(x - a_x)x + (y - a_y)a_y + (z - a_z)a_z = 0$, so $(x - \frac{a_x}{2})^2 + (y - \frac{a_y}{2})^2 + (z - \frac{a_z}{2})^2 = \frac{a_x^2 + a_y^2 + a_z^2}{4}$. It is a sphere cantered at the tip of $\frac{1}{2}\mathbf{a}$ with radius $\frac{1}{2}|\mathbf{a}|$.

**1.7.7** $(1,0,1) \cdot (0,1,-1)/(\sqrt{2} \cdot \sqrt{2}) = -1/2 = \cos\theta$, so $\theta = 120°$

**1.7.8** $(1 + t - 2, 1 + 2t - 1, 1 + 3t - 3) \cdot (1,2,3) = 0$, so $t = \frac{1}{2}$, and the nearest point is $(\frac{3}{2}, 2, \frac{5}{2})$. So the nearest distance with $(2,1,3)$ is $\sqrt{\frac{3}{2}}$.

**1.7.9** Let the position vector of three vertices of the triangle be $\mathbf{A}, \mathbf{B}, \mathbf{C}$. Let the median from $\mathbf{C}$ and the median from $\mathbf{B}$ intersect at point $\mathbf{X}$. Then $\mathbf{X} - \mathbf{B} = m(\frac{\mathbf{A}}{2} + \frac{\mathbf{C}}{2} - \mathbf{B})$, $\mathbf{X} - \mathbf{C} = m(\frac{\mathbf{A}}{2} + \frac{\mathbf{B}}{2} - \mathbf{C})$. Solve for $m, n$ obtaining $m = n = \frac{2}{3}$, and $\mathbf{X} = \frac{\mathbf{A}}{3} + \frac{\mathbf{B}}{3} + \frac{\mathbf{C}}{3}$. $\mathbf{X} - \mathbf{A} = \frac{-2\mathbf{A}}{3} + \frac{\mathbf{B}}{3} + \frac{\mathbf{C}}{3} = \frac{2}{3}(-\mathbf{A} + \frac{\mathbf{B}}{2} + \frac{\mathbf{C}}{2})$. Therefore, the three medians intersect at one point, and the distances from this point to three vertices are 2/3 of the median's length.

**1.7.10** $|\mathbf{A}|^2 = |\mathbf{B}|^2 + |\mathbf{C}|^2 - 2\mathbf{B} \cdot \mathbf{C} = |\mathbf{B}|^2 + |\mathbf{C}|^2 - 2|\mathbf{B}||\mathbf{C}|$

**1.7.11** $\mathbf{Q} = -2\mathbf{P}$, so $\mathbf{Q}$ and $\mathbf{P}$ are anti-parallel. $\mathbf{P} \cdot \mathbf{R} = 0$ and $\mathbf{Q} \cdot \mathbf{R} = 0$, so $\mathbf{R}$ is perpendicular to $\mathbf{P}$ and $\mathbf{Q}$.

## 1.8 Complex Numbers And Functions

**1.8.1** $x + iy = re^{i\theta}$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$. $(x + iy)^{-1} = r^{-1}e^{-i\theta} = \frac{1}{\sqrt{x^2+y^2}}(\cos(-\theta) + i\sin(-\theta)) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

**1.8.2** $(re^{i\theta})^{1/2} = r^{1/2}e^{i\theta/2}$ is a complex number. $i^{1/2} = (e^{i\pi/2})^{1/2} = e^{i\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

**1.8.3** (a)

$$\cos n\theta = \Re\big[e^{in\theta}\big] = \Re\big[(e^{i\theta})^n\big] = \Re\left[\sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}\theta\sin^k\theta(i)^k\right] =$$

$$\sum_{k=0,2,4,\cdots}\binom{n}{k}\cos^{n-k}\theta\sin^k\theta(-1)^{k/2} = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^n\theta - \cdots$$

(b)

$$\sin n\theta = \Im\big[e^{in\theta}\big] = \Im\big[(e^{i\theta})^n\big] = \Im\left[\sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}\theta\sin^k\theta(i)^k\right] =$$

$$\sum_{k=1,3,5,\cdots}\binom{n}{k}\cos^{n-k}\theta\sin^k\theta(-1)^{(k-1)/2} = \binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \cdots$$

**1.8.4** (a)

$$\sum_{n=0}^{N-1}\cos nx = \Re\left[\sum_{n=0}^{N-1}e^{inx}\right] = \Re\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] = \Re\left[\frac{e^{iNx/2}}{e^{ix/2}}\left(\frac{e^{iNx/2}-e^{-iNx/2}}{e^{ix/2}-e^{-ix/2}}\right)\right] = \left(\cos\frac{(N-1)x}{2}\right)\frac{\sin(Nx/2)}{\sin(x/2)}$$

(b)

$$\sum_{n=0}^{N-1}\sin nx = \Im\left[\sum_{n=0}^{N-1}e^{inx}\right] = \Im\left[\frac{1-e^{iNx}}{1-e^{ix}}\right] = \Im\left[\frac{e^{iNx/2}}{e^{ix/2}}\left(\frac{e^{iNx/2}-e^{-iNx/2}}{e^{ix/2}-e^{-ix/2}}\right)\right] = \left(\sin\frac{(N-1)x}{2}\right)\frac{\sin(Nx/2)}{\sin(x/2)}$$

**1.8.5** $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$, $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$, $\sinh z = \frac{e^z - e^{-z}}{2} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$, $\cosh z = \frac{e^x + e^{-x}}{2} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$. Substituting $iz$ for $z$, then we obtain all the four equations.

**1.8.6** (a)

$$\sin(x+iy) = \frac{e^{-y+ix}-e^{y-ix}}{2i} = \frac{e^{-y}\cos x - e^{y}\cos x}{2i} + \frac{ie^{-y}\sin x + ie^{y}\sin x}{2i} = \sin x\cosh y + i\cos x\sinh y$$

$$\cos(x+iy) = \frac{e^{-y+ix}+e^{y-ix}}{2} = \frac{e^{-y}\cos x + e^{y}\cos x}{2} + \frac{ie^{-y}\sin x - ie^{y}\sin x}{2} = \cos x\cosh y - i\sin x\sinh y$$

(b)

$$|\sin z|^2 = \sin^2 x(1+\sinh^2 y) + (1-\sin^2 x)\sinh^2 y = \sin^2 x + \sinh^2 y$$
$$|\cos z|^2 = \cos^2 x(1+\sinh^2 y) + (1-\cos^2 x)\sinh^2 y = \cos^2 x + \sinh^2 y$$

**1.8.7** (a)

$$\sinh(x+iy) = i\sin(y-ix) = i(\sin y\cosh x - i\cos y\sinh x) = \sinh x\cos y + i\cosh x\sin y$$

$$\cosh x + iy = \cos y - ix = \cos y\cosh x + i\sin y\sinh x = \cosh x\cos y + i\sinh x\sin y$$

(b)

$$|\sinh(x+iy)|^2 = |\sin(y-ix)|^2 = \sin^2 y + \sinh^2(-x) = \sinh^2 x + \sin^2 y$$
$$|\cosh(x+iy)|^2 = |\cos(y-ix)|^2 = \cos^2 y + \sinh^2(-x) = 1 - \sin^2 y + \cosh^2 x - 1 = \cosh^2 x - \sin^2 y$$

**1.8.8** (a)

$$\frac{\sinh\left(\frac{z}{2}\right)}{\cosh\left(\frac{z}{2}\right)} = \frac{\sinh\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right) + i\cosh\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)}{\cosh\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right) + i\sinh\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)} = \frac{\cosh(\frac{x}{2})\sinh(\frac{x}{2}) + i\cos(\frac{y}{2})\sin(\frac{y}{2})}{\cosh^2(\frac{x}{2})\cos^2(\frac{y}{2}) + \sinh^2(\frac{x}{2})\sin^2(\frac{y}{2})}$$

$$= \frac{\frac{1}{2}\sinh x + i\frac{1}{2}\sin y}{\frac{\cosh x+1}{2}\frac{1+\cos y}{2} + \frac{\cosh x-1}{2}\frac{1-\cos y}{2}} = \frac{\sinh x + i\sin y}{\cosh x + \cos y}$$

(b)

$$\coth(\frac{z}{2}) = \frac{1}{\tanh(\frac{z}{2})} = \frac{\cosh x + \cos y}{\sinh x + i\sin y} = \frac{(\cosh x + \cos y)(\sinh x - i\sin y)}{\sinh^2 x + \sin^2 y}$$

$$= \frac{(\cosh x + \cos y)(\sinh x - i\sin y)}{\cosh^2 x - \cos^2 y} = \frac{\sinh x - i\sin y}{\cosh x - \cos y}$$

**1.8.9**

$$\tan^{-1} x = \int_0^x \frac{1}{1+x^2}dx = \int_0^x (1 - x^2 + x^4 - \cdots)dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

$$\ln(1 - ix) = -ix - \frac{(ix)^2}{2} - \frac{(ix)^3}{3} - \frac{(ix)^4}{4} - \frac{(ix)^5}{5} - \cdots$$

$$\ln(1 + ix) = ix - \frac{(ix)^2}{2} + \frac{(ix)^3}{3} - \frac{(ix)^4}{4} + \frac{(ix)^5}{5} - \cdots$$

$$\frac{i}{2}\ln\left(\frac{1-ix}{1+ix}\right) = \frac{i}{2}\left(\ln(1-ix) - \ln(1+ix)\right) = \frac{i}{2}(-2)\left((ix) + \frac{(ix)^3}{3} + \frac{(ix)^5}{5} + \cdots\right) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

**1.8.10** (a)

$$(-8)^{1/3} = \left(8e^{i(1+2n)\pi}\right)^{1/3} = 2e^{i\frac{\pi}{3}}, 2e^{i\pi}, 2e^{-i\frac{\pi}{3}} = 1 + \sqrt{3}i, -2, 1 - \sqrt{3}i$$

(b)

$$i^{1/4} = (e^{i(\frac{1}{2}+2n)\pi})^{1/4} = e^{i\frac{\pi}{8}}, e^{i\frac{5\pi}{8}}, e^{i\frac{9\pi}{8}}, e^{i\frac{13\pi}{8}}$$

(c)

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

**1.8.11** (a)

$$(1+i)^3 = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^3 = 2^{3/2}e^{i3\pi/4}$$

(b)

$$\left(e^{i(-1+2n)\pi}\right)^{1/5} = e^{ik\pi/5}$$

where $k = 1, 3, 5, 7, 9$

# 1.9 Derivatives and Extrema

**1.9.1**

$$f(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\frac{\partial^n f}{\partial x^n}\Big|_{0,y} = \sum_{n=0}^{\infty} \frac{x^n}{n!}\sum_{m=0}^{\infty}\frac{y^m}{m!}\frac{\partial^{m+n} f}{\partial y^m \partial x^n}\Big|_{0,0}$$

Rearrange the double series by $p = m + n$ and $q = n$,

$$f(x,y) = \sum_{p=0}^{\infty}\sum_{q=0}^{p} \frac{x^{p-q}}{(p-q)!}\frac{y^q}{q!}\frac{\partial^{m+n} f}{\partial y^m \partial x^n}\Big|_{0,0} = \sum_{p=0}^{\infty}\sum_{q=0}^{p}\frac{1}{p!}\binom{p}{q}x^{p-q}y^q\frac{\partial^{m+n} f}{\partial y^m \partial x^n}\Big|_{0,0}$$

**1.9.2** Expand the function in every variable as in Exercise 1.9.1, then

$$f(x_1, \cdots, x_m) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_m^{n_m}}{n_m!} \frac{\partial^{n_1+\cdots+n_m}}{\partial x_1^{n_1} \cdots \partial x_m^{n_m}} f(0, \cdots, 0)$$

Let $n = n_1 + \cdots + n_m$, substitute $\alpha_i t$ for $x_i$, and apply the generalized form of binomial theorem (Eq 1.80)

$$f(x_1, \cdots, x_m) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} \frac{1}{n!} \frac{n!}{n_1! \cdots n_m!} t^{n_1 \cdots + n_m} \left(\alpha_1 \frac{\partial}{\partial x_1}\right)^{n_1} \cdots \left(\alpha_m \frac{\partial}{\partial x_m}\right)^{n_m} f(0, \cdots, 0)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\alpha_1 \frac{\partial}{\partial x_1} + \cdots + \alpha_m \frac{\partial}{\partial x_m}\right)^n f(0, \cdots, 0) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial x_i}\right)^n f(0, \cdots, 0)$$

## 1.10 Evaluation of Integrals

**1.10.1**

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt = -t^{n-1} e^{-t}\Big|_0^\infty + \int_0^\infty (n-1) t^{n-2} e^{-t} dt = (n-1)\Gamma(n-1)$$

We have integrated by parts and applied the definition of $\Gamma(n-1)$. Keep reducing n by this method until $n = 1$, and note that $\Gamma(1) = 1$, we obtain

$$\Gamma(n) = (n-1)!$$

**1.10.2** Let $J(a) = \int_o^\infty e^{-ax} \frac{\sin x}{x} dx$, then $\int_0^\infty \frac{\sin x}{x} dx = J(0)$.

$$\frac{dJ(a)}{da} = -\int_0^\infty e^{-ax} \sin x dx = -\Im\left[\int_0^\infty e^{(-a+i)x}\right] dx = -\Im\left[\frac{e^{(-a+i)x}}{-a+i}\Big|_0^\infty\right] = \frac{-1}{a^2+1}$$

$$J(a) = \int \frac{-1}{a^2+1} da = -\tan^{-1} a + c = -\tan^{-1} a + \frac{\pi}{2}$$

where $c$ is determined by $J(\infty) = 0$.

$$\int_0^\infty \frac{\sin x}{x} dx = J(0) = \frac{\pi}{2}$$

**1.10.3**

$$\int_0^\infty \frac{dx}{\cosh x} = \int_0^\infty \frac{2}{e^x + e^{-x}} dx = \int_0^\infty \frac{2e^{-x}}{1 + e^{-2x}} = \int_0^\infty 2e^{-x}(1 - e^{-2x} + e^{4x} - \cdots) dx = 2(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \cdots)$$

where the expansion is valid for all positive x because $e^{-2x} < 1$ for all positive x. $\tan^{-1} 1 = \frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2} dx = \int_0^1 (1 - x^2 + x^4 - \cdots) dx = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$, so

$$\int_0^\infty \frac{dx}{\cosh x} = 2\tan^{-1} 1 = \frac{\pi}{2}$$

**1.10.4**

$$\int_0^\infty \frac{dx}{e^{ax} + 1} = \int_0^\infty \frac{e^{-ax}}{1 + e^{-ax}} = \int_0^\infty e^{-ax}(1 - e^{-ax} + e^{-2ax} - \cdots) dx = \frac{1}{a}(\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots) = \frac{1}{a} \ln(1+1) = \frac{\ln 2}{a}$$

**1.10.5** Integrating by parts

$$\int_\pi^\infty \frac{\sin x}{x^2} dx = \frac{-1}{x} \sin x\Big|_\pi^\infty - \int_\pi^\infty -\frac{\cos x}{x} dx = -\mathrm{Ci}(\pi)$$

where $\mathrm{Ci}(x)$ is the cosine integral (see Table 1.2)

**1.10.6** Let $J(a) = \int_0^\infty e^{-ax}\frac{\sin x}{x}dx$, which is the same as Exercise 1.10.2, then $\int_0^\infty e^{-x}\frac{\sin x}{x}dx = J(1)$.

$$\int_0^\infty e^{-x}\frac{\sin x}{x}dx = -\tan^{-1}1 + \frac{\pi}{2} = \frac{\pi}{4}$$

**1.10.7** Integrating by parts

$$\int_0^x \operatorname{erf}(t)dt = \operatorname{erf}(t)\cdot t\Big|_0^x - \int_0^x d\operatorname{erf}(t)\cdot t = x\operatorname{erf}(x) - \int_0^x \frac{2}{\sqrt{\pi}}e^{-t^2}dt\cdot t = x\operatorname{erf}(x) + \frac{e^{-x^2}-1}{\sqrt{\pi}}$$

**1.10.8** Integrating by parts

$$\int_1^x \mathrm{E}_1(t)\,dt = \mathrm{E}_1(t)\,t\Big|_1^x - \int_1^x d\,\mathrm{E}_1(t)\cdot t = \mathrm{E}_1(t)\,t\Big|_1^x - \int_1^x -t^{-1}e^{-t}\,dt\cdot t = \mathrm{E}_1(t)\,t\Big|_1^x - e^{-t}\Big|_1^x$$

$$= x\,\mathrm{E}_1(x) - \mathrm{E}_1(1) + e^{-1} - e^{-x}$$

**1.10.9** Let $y = x + 1$

$$\int_0^\infty \frac{e^{-x}}{x+1}dx = \int_1^\infty \frac{e^{-y+1}}{y}dy = e\,\mathrm{E}_1(1)$$

**1.10.10** Integrating by parts

$$I = \int_0^\infty (\tan^{-1}x)^2\frac{1}{x^2}dx = -(\tan^{-1}x)^2\frac{1}{x}\Big|_0^\infty - \int_0^\infty 2\tan^{-1}x\frac{1}{1+x^2}\frac{-1}{x}dx = \int_0^\infty \frac{2\tan^{-1}x}{(1+x^2)x}dx$$

Let $J(a) = \int_0^\infty \frac{2\tan^{-1}ax}{(1+x^2)x}dx$, then $I = J(1)$

$$\frac{dJ(a)}{da} = \int_0^\infty \frac{2}{(1+x^2)x}\frac{x}{1+a^2x^2} = \frac{2}{1-a^2}\int_0^\infty \left(\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right)dx$$

$$= \frac{2}{1-a^2}\left[\tan^{-1}x\Big|_0^\infty - a\tan^{-1}ax\Big|_0^\infty\right] = \frac{2}{1-a^2}\left[(1-a)\frac{\pi}{2}\right] = \frac{\pi}{1+a}$$

$$J(a) = \int \frac{\pi}{1+a}da = \pi\ln|1+a| + c = \pi\ln|1+a|$$

where $c = 0$ is determined by $J(0) = 0$. So

$$\int_0^\infty \left(\frac{\tan^{-1}x}{x}\right)^2 dx = J(1) = \pi\ln 2$$

**1.10.11**

$$A = 4\int_0^a dx\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy = 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}dx$$

Let $\frac{x}{a} = \sin\theta$, then $\sqrt{1-\frac{x^2}{a^2}} = \cos\theta$, $dx = a\cos\theta d\theta$, and the range of integration becomes 0 to $\frac{\pi}{2}$

$$A = 4\int_0^{\frac{\pi}{2}} ab\cos^2\theta d\theta = 4ab\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{\frac{\pi}{2}} = \pi ab$$

**1.10.12**

$$A = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\theta\int_{\frac{1}{2}\sec\theta}^1 r\,dr = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\left(1 - \frac{\sec^2\theta}{4}\right)d\theta = \left(\frac{\theta}{2} - \frac{\tan\theta}{8}\right)\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

Which is the area of the circular sector minus the area of triangle.

## 1.11 Dirac Delta Function

**1.11.1**

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\delta_n(x)dx = \lim_{n\to\infty}\int_{-\frac{1}{2n}}^{\frac{1}{2n}} f(x)n\,dx = \lim_{n\to\infty} f(\xi_n)\frac{1}{n}n = f(0)$$

where $-\frac{1}{2n} \le \xi_n \le \frac{1}{2n}$ (by mean value theorem).

**1.11.2** Let $nx = \tan\theta$, then $1 + n^2x^2 = \sec^2\theta$, and $ndx = \sec^2\theta d\theta$. The range of integration becomes $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$$\int_{-\infty}^{\infty} \frac{n}{\pi}\frac{1}{1+n^2x^2}dx == \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{n}{\pi}\frac{1}{\sec^2\theta}\frac{1}{n}\sec^2\theta\,d\theta = \frac{\theta}{\pi}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1$$

***1.11.3** The proof probably requires contour integral of complex analysis, while I have no idea whether there is any elementary proof.

**1.11.4** Let $a(x - x_1) = y$, then $x = \frac{y}{a} + x_1$, $adx = dy$

$$\int_{-\infty}^{\infty} f(x)\delta[a(x-x_1)]dx = \frac{1}{a}\int_{-\infty}^{\infty} f(\frac{y}{a}+x_1)\delta(y)dy = \frac{1}{a}f(\frac{0}{a}+x_1) = \frac{1}{a}f(x_1) = \int_{-\infty}^{\infty} f(x)\frac{1}{a}\delta(x-x_1)dx$$

so $\delta[a(x-x_1)] = \frac{1}{a}\delta(x-x_1)$.

**1.11.5** When $(x-x_1)(x-x_2) \neq 0$, $\delta[(x-x_1)(x-x_2)] = 0$, so

$$\int_{-\infty}^{\infty} f(x)\delta[(x-x_1)(x-x_2)]dx = \int_{x_1-\varepsilon}^{x_1+\varepsilon} f(x)\delta[(x-x_1)(x-x_2)]dx + \int_{x_2-\varepsilon}^{x_2+\varepsilon} f(x)\delta[(x-x_1)(x-x_2)]dx$$

for arbitrarily small positive $\varepsilon$. Then

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{x_1-\varepsilon}^{x_1+\varepsilon} f(x)\delta[(x-x_1)(x-x_2)]dx + \int_{x_2-\varepsilon}^{x_2+\varepsilon} f(x)\delta[(x-x_1)(x-x_2)]dx \\
&= \int_{x_1-\varepsilon}^{x_1+\varepsilon} f(x)\delta[(x-x_1)(x_1-x_2)]dx + \int_{x_2-\varepsilon}^{x_2+\varepsilon} f(x)\delta[(x_2-x_1)(x-x_2)]dx \\
&= \int_{x_1-\varepsilon}^{x_1+\varepsilon} f(x)\frac{1}{|x_1-x_2|}\delta(x-x_1)dx + \int_{x_2-\varepsilon}^{x_2+\varepsilon} f(x)\frac{1}{|x_1-x_2|}\delta(x-x_2)dx \\
&= \int_{-\infty}^{\infty} f(x)\frac{\delta(x-x_1)+\delta(x-x_2)}{|x_1-x_2|}dx
\end{aligned}$$

Therefore,

$$\delta[(x-x_1)(x-x_2)] = \frac{\delta(x-x_1)+\delta(x-x_2)}{|x_1-x_2|}$$

**1.11.6** Integrating by parts

$$\int_{-\infty}^{\infty} f(x)x\frac{d\delta(x)}{dx}dx = f(x)x\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}[f'(x)x + f(x)]\delta(x)dx = 0 - f(0) = -\int_{-\infty}^{\infty} f(x)\delta(x)dx$$

Therefore,

$$x\frac{d\delta(x)}{dx} = -\delta(x)$$

**1.11.7** Integrating by parts

$$\int_{-\infty}^{\infty} \frac{d\delta(x)}{dx}f(x)dx = \delta(x)f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\delta(x)f'(x)dx = -f'(0)$$

**1.11.8** (The equation holds only when $f(x)$ has only one zero point) When $f(x) \neq 0$, $\delta(f(x)) = 0$, so

$$\int_{-\infty}^{\infty} h(x)\delta(f(x))dx = \int_{x_0-\varepsilon}^{x_0+\varepsilon} h(x)\delta\left((x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2}f''(x_0) + \cdots\right)dx$$

for arbitrary small positive $\varepsilon$. Then

$$\lim_{\varepsilon\to 0}\int_{x_0-\varepsilon}^{x_0+\varepsilon} h(x)\delta(f(x))dx = \lim_{\varepsilon\to 0}\int_{x_0-\varepsilon}^{x_0+\varepsilon} h(x)\delta\left((x-x_0)f'(x_0)\right)dx = \int_{-\infty}^{\infty} h(x)\frac{1}{|f'(x_0)|}\delta(x-x_0)dx$$

by Exercise 1.11.4. Therefore

$$\delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x-x_0)$$

**1.11.9** (a)

$$\int_{-\infty}^{\infty} \frac{n}{2\cosh^2 nx}dx = \frac{1}{2}\tanh nx\Big|_{-\infty}^{\infty} = 1$$

(b)

$$\int_{-\infty}^{x} \frac{n}{2\cosh^2 nx}dx = \frac{1}{2}(\tanh nx + 1) = u_n(x)$$

When $n \to \infty$ and $x < 0$, $u_n(x) = \frac{1}{2}(-1+1) = 0$; when $n \to \infty$ and $x > 0$, $u_n(x) = \frac{1}{2}(1+1) = 1$.