Chapter 3
Vector Analysis

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3.2 Vectors in 3-D Space

3.2.1 $\mathbf{P} \times \mathbf{Q} = \sum_i \hat{\mathbf{e}}_i \sum_{jk} \varepsilon_{ijk} P_j Q_k$. $P_z = Q_z = 0$, so $\varepsilon_{ijk} P_j Q_k \neq 0$ only when $i = z$. So $\sum_i \hat{\mathbf{e}}_i \sum_{jk} \varepsilon_{ijk} P_j Q_k = \hat{\mathbf{e}}_z (P_x Q_y - P_y Q_x) \neq 0$ because $\mathbf{P}$ and $\mathbf{Q}$ are nonparallel.

3.2.2 $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = (A_x B_y - A_y B_x)^2 + (A_x B_z - A_z B_x)^2 + (A_y B_z - A_z B_y)^2 = A_x^2 B_y^2 + A_x^2 B_z^2 + A_y^2 B_x^2 + A_y^2 B_z^2 + A_z^2 B_x^2 + A_z^2 B_y^2 - 2A_x B_x A_y B_y - 2A_x B_x A_z B_z - 2A_y B_y A_z B_z$. $(AB)^2 - (\mathbf{A} \cdot \mathbf{B})^2 = (A_x^2 + A_y^2 + A_z^2)(B_x^2 + B_y^2 + B_z^2) - (A_x B_x + A_y B_y + A_z B_z)^2 = A_x^2 B_y^2 + A_x^2 B_z^2 + A_y^2 B_x^2 + A_y^2 B_z^2 + A_z^2 B_x^2 + A_z^2 B_y^2 - 2A_x B_x A_y B_y - 2A_x B_x A_z B_z - 2A_y B_y A_z B_z$, so $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = (AB)^2 - (\mathbf{A} \cdot \mathbf{B})^2$.

3.2.3 $\sin(\theta + \psi) = \frac{|\mathbf{P} \times \mathbf{Q}|}{|\mathbf{P}||\mathbf{Q}|} = |(-\sin\theta\cos\psi - \cos\theta\sin\psi)\hat{e_z}| = \sin\theta\cos\psi + \cos\theta\sin\psi$.
$\cos(\theta + \psi) = \frac{\mathbf{P} \cdot \mathbf{Q}}{|\mathbf{P}||\mathbf{Q}|} = \cos\theta\cos\psi - \sin\theta\sin\psi$.

3.2.4 (a) $\mathbf{U} \times \mathbf{V} = -3\hat{\mathbf{e}}_y - 3\hat{\mathbf{e}}_z$ is perpendicular with $\mathbf{U}$ and $\mathbf{V}$.
(b) $\frac{\mathbf{U} \times \mathbf{V}}{|\mathbf{U} \times \mathbf{V}|} = \frac{1}{\sqrt{2}}\hat{\mathbf{e}}_y + \frac{1}{\sqrt{2}}\hat{\mathbf{e}}_z$

3.2.5 All the four vectors are in the same plane, so both $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$ are perpendicular to the plane, so $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$ are parallel, so $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = 0$.

3.2.6 The area of the triangle $= \frac{1}{2}|\mathbf{B}||\mathbf{C}|\sin\alpha = \frac{1}{2}|\mathbf{A}||\mathbf{C}|\sin\beta = \frac{1}{2}|\mathbf{A}||\mathbf{B}|\sin\gamma$. Devided by $|\mathbf{A}||\mathbf{B}||\mathbf{C}|/2$, we get $\frac{\sin\alpha}{|\mathbf{A}|} = \frac{\sin\beta}{|\mathbf{B}|} = \frac{\sin\gamma}{|\mathbf{C}|}$.

3.2.7 $\hat{\mathbf{e}}_x \times \mathbf{B} = 2\hat{\mathbf{e}}_z - 4\hat{\mathbf{e}}_y$, $\hat{\mathbf{e}}_y \times \mathbf{B} = 4\hat{\mathbf{e}}_x - \hat{\mathbf{e}}_z$, $\hat{\mathbf{e}}_z \times \mathbf{B} = \hat{\mathbf{e}}_y - 2\hat{\mathbf{e}}_x$ by the experiments. $\hat{\mathbf{e}}_x \cdot (\hat{\mathbf{e}}_y \times \mathbf{B}) = \mathbf{B} \cdot (\hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_y) = \mathbf{B} \cdot (\hat{\mathbf{e}}_z) = \mathbf{B}_z = 4$, $\hat{\mathbf{e}}_y \cdot (\hat{\mathbf{e}}_z \times \mathbf{B}) = \mathbf{B} \cdot (\hat{\mathbf{e}}_y \times \hat{\mathbf{e}}_z) = \mathbf{B} \cdot (\hat{\mathbf{e}}_x) = \mathbf{B}_x = 1$, $\hat{\mathbf{e}}_z \cdot (\hat{\mathbf{e}}_x \times \mathbf{B}) = \mathbf{B} \cdot (\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_x) = \mathbf{B} \cdot (\hat{\mathbf{e}}_y) = \mathbf{B}_y = 2$. So $\mathbf{B} = \hat{\mathbf{e}}_x + 2\hat{\mathbf{e}}_y + 4\hat{\mathbf{e}}_z$.

3.2.8 (a) $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = (\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y) \cdot (-\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y - \hat{\mathbf{e}}_z) = 0$. It is true because $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{B} \times \mathbf{C}$ are in the same plane, so the volume of the parallelepiped is zero.
(b) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y) \times (-\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y - \hat{\mathbf{e}}_z) = -\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y + 2\hat{\mathbf{e}}_z$.

3.2.9 $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) + \mathbf{c}(\mathbf{b} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + \mathbf{a}(\mathbf{c} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) = 0$.

3.2.10 (a) $\mathbf{A}_r = \hat{\mathbf{r}}(\mathbf{A} \cdot \hat{\mathbf{r}})$ is quite obvious by the definition.
(b) $\mathbf{A}_t = \mathbf{A} - \mathbf{A}_r = \mathbf{A}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) - \hat{\mathbf{r}}(\mathbf{A} \cdot \hat{\mathbf{r}}) = -\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{A})$.

3.2.11 If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are coplanar, then $\mathbf{B} \times \mathbf{C}$ is perpendicular to $\mathbf{B}$, $\mathbf{C}$ and therefore perpendicular to $\mathbf{A}$, so $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = 0$. If $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = 0$, then $\mathbf{A}$ is perpendicular to $\mathbf{B} \times \mathbf{C}$, but $\mathbf{B} \times \mathbf{C}$ is perpendicular to the plane of $\mathbf{B}$ and $\mathbf{C}$, so $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are coplanar.

3.2.12 $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = -120$, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = -60\hat{\mathbf{e}}_x - 40\hat{\mathbf{e}}_y + 50\hat{\mathbf{e}}_z$, $\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 24\hat{\mathbf{e}}_x + 88\hat{\mathbf{e}}_y - 62\hat{\mathbf{e}}_z$, $\mathbf{B} \times (\mathbf{C} \times \mathbf{A}) = 36\hat{\mathbf{e}}_x - 48\hat{\mathbf{e}}_y + 12\hat{\mathbf{e}}_z$.

3.2.13 $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{B}\times(\mathbf{C}\times\mathbf{D}))\cdot\mathbf{A} = (\mathbf{C}(\mathbf{B}\cdot\mathbf{D})-\mathbf{D}(\mathbf{B}\cdot\mathbf{C}))\cdot\mathbf{A} = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$.

3.2.14 $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{C}(\mathbf{A}\times\mathbf{B}\cdot\mathbf{D}) - \mathbf{D}(\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}) = (\mathbf{A}\cdot\mathbf{B}\times\mathbf{D})\mathbf{C} - (\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})\mathbf{D}$.

3.2.15 (a) $\mathbf{F}_2 = q_2\mathbf{v}_2\times\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q_1q_2}{r^2}\mathbf{v}_2\times(\mathbf{v}_1\times\hat{\mathbf{r}})$.

(b) With $\mathbf{v}_1$ replaced by $\mathbf{v}_2$, $\mathbf{v}_2$ replaced by $\mathbf{v}_1$, $\hat{\mathbf{r}}$ replaced by $-\hat{\mathbf{r}}$, $\mathbf{F}_1 = -\frac{\mu_0}{4\pi}\frac{q_1q_2}{r^2}\mathbf{v}_1\times(\mathbf{v}_2\times\hat{\mathbf{r}})$.

(c) $\hat{\mathbf{r}}$ are perpendicular to $\mathbf{v}_2, \mathbf{v}_1$, and $\mathbf{v}_2, \mathbf{v}_1$ are parallel, so $\mathbf{F}_2 = -\frac{\mu_0}{4\pi}\frac{q_1q_2}{r^2}v_2v_1\hat{\mathbf{r}} = -\mathbf{F}_1$.

3.3 Coordinate Transformations

3.3.1

$$\begin{pmatrix}\cos(\varphi_1+\varphi_2) & \sin(\varphi_1+\varphi_2)\\ -\sin(\varphi_1+\varphi_2) & \cos(\varphi_1+\varphi_2)\end{pmatrix} = \begin{pmatrix}\cos\varphi_2 & \sin\varphi_2\\ -\sin\varphi_2 & \cos\varphi_2\end{pmatrix}\begin{pmatrix}\cos\varphi_1 & \sin\varphi_1\\ -\sin\varphi_1 & \cos\varphi_1\end{pmatrix}$$

3.3.2 Let the three reflecting surfaces be parallel to xy, xz, yz surfaces. Let the direction vector of the incident light be (k_1, k_2, k_3). Then after each reflection, one of the coordinate changes sign, so the direction vector of reflected light is $(-k_1, -k_2, -k_3)$, parallel to the incident light.

3.3.3 $(\mathbf{x}')^T\mathbf{y}' = \mathbf{x}^T\mathrm{S}^T\mathrm{S}\mathbf{y} = \mathbf{x}^T\mathbf{y}$ because S is orthogonal.

3.3.4 (a) $\det\mathrm{S} = 1$

(b) $\mathbf{a}\cdot\mathbf{b} = -1$, $\mathrm{S}\mathbf{a}\cdot\mathrm{S}\mathbf{b} = (0.8, 0.12, 1.16)\cdot(1.2, 0.68, -1.76) = -1$

(c) $\mathbf{a}\times\mathbf{b} = (-2, 1, 2)$, $\mathrm{S}(\mathbf{a}\times\mathbf{b}) = (-1, 2.8, 0.4)$, $\mathrm{S}\mathbf{a}\times\mathrm{S}\mathbf{b} = (-1, 2.8, 0.4)$. It is as expected because pseudovectors transform as vectors when the orthogonal transformation is not a reflection ($\det(\mathrm{S}) = -1$).

3.3.5 (a) $\det(\mathrm{S}) = -1$

(b) $(\mathrm{S}\mathbf{a})\times(\mathrm{S}\mathbf{b}) = (-0.4, -1.64, -2.48)$; $\mathrm{S}(\mathbf{a}\times\mathbf{b}) = (0.4, 1.64, 2.48)$. The sign changes.

(c) $(\mathrm{S}\mathbf{a}\times\mathrm{S}\mathbf{b})\cdot\mathrm{S}\mathbf{c} = -3$; $\mathrm{S}((\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = 3$. The sign changes.

(d) $\mathrm{S}\mathbf{a}\times(\mathrm{S}\mathbf{b}\times\mathrm{S}\mathbf{c}) = (-0.4, -8.84, 7.12)$; $\mathrm{S}(\mathbf{a}\times(\mathbf{b}\times\mathbf{c})) = (-0.4, -8.84, 7.12)$. The sign does not change.

(e) $\mathbf{a}\times\mathbf{b}$ is a pseudovector, $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ is a pseudoscalar, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is a vector.

3.4 Rotations in $\mathbb{R}^3$

3.4.1 The corresponding transformation matrix is

$$\begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}$$

Make the substitution by $\cos\varphi = -\sin\alpha$, $\sin\varphi = \cos\alpha$, $\cos\theta = \cos\beta$, $\sin\theta = \sin\beta$, $\cos\psi = \sin\gamma$, $\sin\psi = -\cos\gamma$. The matrix becomes

$$\begin{pmatrix}\sin\gamma & -\cos\gamma & 0\\ \cos\gamma & \sin\gamma & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\beta & \sin\beta\\ 0 & -\sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}-\sin\alpha & \cos\alpha & 0\\ -\cos\alpha & -\sin\alpha & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}\sin\gamma & -\cos\gamma & 0\\ \cos\gamma & \sin\gamma & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-\sin\alpha & \cos\alpha & 0\\ -\cos\beta\cos\alpha & -\cos\beta\sin\alpha & \sin\beta\\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta\end{pmatrix}$$

$$= \begin{pmatrix}-\sin\gamma\sin\alpha + \cos\gamma\cos\beta\cos\alpha & \sin\gamma\cos\alpha + \cos\gamma\cos\beta\sin\alpha & -\cos\gamma\sin\beta\\ -\cos\gamma\sin\alpha - \sin\gamma\cos\beta\cos\alpha & \cos\gamma\cos\alpha - \sin\gamma\cos\beta\sin\alpha & \sin\gamma\sin\beta\\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta\end{pmatrix}$$

which is the same as the transformation matrix in Eq. (3.37).

3.4.2 (In the original system, the North Pole is in the x_3-axis direction, and the middle point of Prime Meridian$(0°, 0°)$ is in the x_2-axis direction) Rotate $\alpha = 70°$ around x_3-axis to align x_1-axis with $20°$ west, and rotate $\beta = 60°$ around x_2-axis to align the North Pole with $30°$ north, and Rotate $\gamma = -80°$ around x_3-axis to align the $10°$ west Meridian with the Meridian in new system). Calculating from Eq. (3.37), the transformation matrix is

$$\begin{pmatrix} 0.9551 & -0.2552 & -0.1504 \\ 0.0052 & 0.5221 & -0.8529 \\ 0.2962 & 0.8138 & 0.5000 \end{pmatrix}$$

3.4.3 All the trigonometric function except $\cos\beta$ change sign. Substituting, we found that the rotation matrix S remains unchanged.

3.4.4 $\mathrm{S}(\alpha,\beta,\gamma) = \mathrm{S}_3(\gamma)\mathrm{S}_2(\beta)\mathrm{S}_1(\alpha)$. Note that $\mathrm{S}_i^{-1}(x) = \tilde{\mathrm{S}}_i(x)$ (orthogonality) and $\mathrm{S}_i(x)^{-1} = \mathrm{S}_i(-x)$ (property of rotation matrix). So

$$\mathrm{S}^{-1}(\alpha,\beta,\gamma) = \mathrm{S}_1^{-1}(\alpha)\mathrm{S}_2^{-1}(\beta)\mathrm{S}_3^{-1}(\gamma) = \tilde{\mathrm{S}}_1(\alpha)\tilde{\mathrm{S}}_2(\beta)\tilde{\mathrm{S}}_3(\gamma) = (\mathrm{S}_3(\gamma)\mathrm{S}_2\tilde{(}\beta)\mathrm{S}_1(\alpha)) = \tilde{\mathrm{S}}(\alpha,\beta,\gamma)$$

$$\mathrm{S}^{-1}(\alpha,\beta,\gamma) = \mathrm{S}_1^{-1}(\alpha)\mathrm{S}_2^{-1}(\beta)\mathrm{S}_3^{-1}(\gamma) = \mathrm{S}_1(-\alpha)\mathrm{S}_2(-\beta)\mathrm{S}_3(-\gamma) = \mathrm{S}_3(-\alpha)\mathrm{S}_2(-\beta)\mathrm{S}_1(-\gamma) = \mathrm{S}(-\gamma,-\beta,-\alpha)$$

3.4.5 (a) Decompose $\mathbf{r}$ into $\mathbf{r}_{\parallel}$ parallel to $\hat{\mathbf{n}}$ and $\mathbf{r}_{\perp}$ perpendicular to $\hat{\mathbf{n}}$. $\mathbf{r}_{\parallel} = \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})$, $\mathbf{r}_{\perp} = \mathbf{r} - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})$. After rotation, $\mathbf{r}_{\parallel}$ remains unchanged, and $\mathbf{r}_{\perp}$ becomes $\frac{\mathbf{r}_{\perp}}{|\mathbf{r}_{\perp}|} \times \hat{\mathbf{n}}(|\mathbf{r}_{\perp}|\sin\boldsymbol{\Phi})$ in the $\hat{\mathbf{e}}_{\boldsymbol{\Phi}}$ direction and $\frac{\mathbf{r}_{\perp}}{|\mathbf{r}_{\perp}|}|\mathbf{r}_{\perp}|\cos\boldsymbol{\Phi}$ in the $\hat{\mathbf{e}}_{\mathbf{r}_{\perp}}$ direction. So

$$\begin{aligned}\mathbf{r}' = \mathbf{r}_{\parallel} + \mathbf{r}_{\perp}\times\hat{\mathbf{n}}\sin\boldsymbol{\Phi} + \mathbf{r}_{\perp}\cos\boldsymbol{\Phi} &= \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r}) + (\mathbf{r} - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r}))\times\hat{\mathbf{n}}\sin\boldsymbol{\Phi} + (\mathbf{r} - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r}))\cos\boldsymbol{\Phi} \\ &= \mathbf{r}\cos\boldsymbol{\Phi} + \mathbf{r}\times\hat{\mathbf{n}}\sin\boldsymbol{\Phi} + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})(1-\cos\boldsymbol{\Phi})\end{aligned}$$

(b) Let $\mathbf{r} = (x, y, z)$, then

$$\mathbf{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x\cos\boldsymbol{\Phi} \\ y\cos\boldsymbol{\Phi} \\ z\cos\boldsymbol{\Phi} \end{pmatrix} + \begin{pmatrix} y\sin\boldsymbol{\Phi} \\ -z\sin\boldsymbol{\Phi} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z(1-\cos\boldsymbol{\Phi}) \end{pmatrix} = \begin{pmatrix} x\cos\boldsymbol{\Phi} + y\sin\boldsymbol{\Phi} \\ -z\sin\boldsymbol{\Phi} + y\cos\boldsymbol{\Phi} \\ z \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\boldsymbol{\Phi} & \sin\boldsymbol{\Phi} & 0 \\ -\sin\boldsymbol{\Phi} & \cos\boldsymbol{\Phi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

which is identical with Eq. (3.55).

(c) $\mathbf{r}\cdot(\mathbf{r}\times\hat{\mathbf{n}}) = 0$, $(\mathbf{r}\times\hat{\mathbf{n}})\cdot\hat{\mathbf{n}} = 0$ because they are perpendicular. Let θ be the angle between $\mathbf{r}$ and $\hat{\mathbf{n}}$, then $|\mathbf{r}\times\hat{\mathbf{n}}| = r\sin\theta$, $|\hat{\mathbf{n}}\cdot\mathbf{r}| = r\cos\theta$.

$$\begin{aligned} r'^2 = \mathbf{r}'\cdot\mathbf{r}' &= r^2\cos^2\Phi + (\mathbf{r}\times\hat{\mathbf{n}})^2\sin^2\Phi + (\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)^2 + 2(\hat{\mathbf{n}}\cdot\mathbf{r})^2(\cos\Phi)(1-\cos\Phi) \\ &= r^2\cos^2\Phi + r^2\sin^2\theta\sin^2\Phi + r^2\cos^2\theta(1-\cos\Phi)^2 + 2r^2\cos^2\theta(\cos\Phi)(1-\cos\Phi) \\ &= r^2\cos^2\Phi + r^2\sin^2\theta\sin^2\Phi + r^2\cos^2\theta(1-\cos^2\Phi) \\ &= r^2\cos^2\Phi + r^2\sin^2\Phi \\ &= r^2 \end{aligned}$$

3.5 Differential Vector Operators

3.5.1 (a) $\boldsymbol{\nabla}\mathbf{S} = -3(x^2+y^2+z^2)^{-5/2}(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z) = -3(14)^{-5/2}(1\hat{\mathbf{e}}_x + 2\hat{\mathbf{e}}_y + 3\hat{\mathbf{e}}_z)$ at point $(1,2,3)$.

(b) $\boldsymbol{\nabla}\mathbf{S} = 3\cdot 14^{-5/2}\cdot 14^{1/2} = \frac{3}{196}$

(c) $(\frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}})$

3.5.2 (a) $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})\cdot(dx, dy, dz) = df = 0$ when $f(x,y,z) = constant$, so $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ is perpendicular to the equipotential surfaces. $\boldsymbol{\nabla}(x^2+y^2+z^2) = 2x\hat{\mathbf{e}}_x + 2y\hat{\mathbf{e}}_y + 2z\hat{\mathbf{e}}_z = 2\hat{\mathbf{e}}_x + 2\hat{\mathbf{e}}_y + 2\hat{\mathbf{e}}_z$ at point $(1,1,1)$, and its unit vector is $\frac{\hat{\mathbf{e}}_x+\hat{\mathbf{e}}_y+\hat{\mathbf{e}}_z}{\sqrt{3}}$.

(b) $(x-1, y-1, z-1)\cdot(1,1,1) = 0$, so the tangent surface is $x+y+z = 3$.

3.5.3 $r_{12} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$, so $\boldsymbol{\nabla}_1 r_{12} = \frac{\partial r_{12}}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial r_{12}}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial r_{12}}{\partial z}\hat{\mathbf{e}}_z = \frac{x_1 - x_2}{r_{12}}\hat{\mathbf{e}}_x + \frac{y_1 - y_2}{r_{12}}\hat{\mathbf{e}}_y + \frac{z_1 - z_2}{r_{12}}\hat{\mathbf{e}}_z = \frac{\mathbf{r_{12}}}{r_{12}}$ which is the unit vector in the direction of $\mathbf{r_{12}}$.

3.5.4

$$\mathbf{F} = F_x\hat{\mathbf{e}}_x + F_y\hat{\mathbf{e}}_y + F_z\hat{\mathbf{e}}_z$$

$$d\mathbf{F} = (\frac{\partial F_x}{\partial x}dx + \frac{\partial F_x}{\partial y}dy + \frac{\partial F_x}{\partial z}dz + \frac{\partial F_x}{\partial t}dt)\hat{\mathbf{e}}_x + (\frac{\partial F_y}{\partial x}dx + \frac{\partial F_y}{\partial y}dy + \frac{\partial F_y}{\partial z}dz + \frac{\partial F_y}{\partial t}dt)\hat{\mathbf{e}}_y + (\frac{\partial F_z}{\partial x}dx + \frac{\partial F_z}{\partial y}dy + \frac{\partial F_z}{\partial z}dz + \frac{\partial F_z}{\partial t}dt)\hat{\mathbf{e}}_z$$

$$= (dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z})F_x\hat{\mathbf{e}}_x + (dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z})F_y\hat{\mathbf{e}}_y + (dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z})F_z\hat{\mathbf{e}}_z + (\frac{\partial F_x}{\partial t}\hat{\mathbf{e}}_x + \frac{\partial F_y}{\partial t}\hat{\mathbf{e}}_y + \frac{\partial F_z}{\partial t}\hat{\mathbf{e}}_z)dt$$

$$= (dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z})(F_x\hat{\mathbf{e}}_x + F_y\hat{\mathbf{e}}_y + F_z\hat{\mathbf{e}}_z) + \frac{\partial \mathbf{F}}{\partial t}dt$$

$$= (d\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{F} + \frac{\partial \mathbf{F}}{\partial t}dt$$

3.5.5 $\boldsymbol{\nabla}(uv) = \frac{\partial uv}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial uv}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial uv}{\partial z}\hat{\mathbf{e}}_z = (\frac{\partial u}{\partial x}v + u\frac{\partial v}{\partial x})\hat{\mathbf{e}}_x + (\frac{\partial u}{\partial y}v + u\frac{\partial v}{\partial y})\hat{\mathbf{e}}_y + (\frac{\partial u}{\partial z}v + u\frac{\partial v}{\partial z})\hat{\mathbf{e}}_z$
$= (\frac{\partial u}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial u}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial u}{\partial z}\hat{\mathbf{e}}_z)v + u(\frac{\partial v}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial v}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial v}{\partial z}\hat{\mathbf{e}}_z) = v\boldsymbol{\nabla}u + u\boldsymbol{\nabla}v$

3.5.6 (a) $\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = -r\omega\sin\omega t\,\hat{\mathbf{e}}_x + r\omega\cos\omega t\,\hat{\mathbf{e}}_y$, so

$\mathbf{r}\times\dot{\mathbf{r}} = (r\cos\omega t\,\hat{\mathbf{e}}_x + r\sin\omega t\,\hat{\mathbf{e}}_t)\times(-r\omega\sin\omega t\,\hat{\mathbf{e}}_x + r\omega\cos\omega t\,\hat{\mathbf{e}}_y) = r\omega^2(\cos^2\omega t + \sin^2\omega t)\hat{\mathbf{e}}_z = r\omega^2\hat{\mathbf{e}}_z$

(b) $\ddot{\mathbf{r}} = -r\omega^2\cos\omega t\,\hat{\mathbf{e}}_x - r\omega^2\sin\omega t\,\hat{\mathbf{e}}_y$, so $\ddot{\mathbf{r}} + \omega^2\mathbf{r} = 0$.

3.5.7 $\mathbf{A}' = \mathrm{S}\mathbf{A}$, which means $A_i' = \sum_j \mathrm{S}_{ij}A_j$. Because S_{ij} is independent of t, so $\frac{dA_i'}{dt} = \sum_j \mathrm{S}_{ij}\frac{dA_j}{dt}$, which means $\frac{d\mathbf{A}'}{dt} = \mathrm{S}\frac{d\mathbf{A}}{dt}$.

3.5.8 (a) $\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \frac{d}{dt}(A_xB_x + A_yB_y + A_zB_z) = \frac{dA_x}{dt}B_x + A_x\frac{dB_x}{dt} + \frac{dA_y}{dt}B_y + A_y\frac{dB_y}{dt} + \frac{dA_z}{dt}B_z + A_z\frac{dB_z}{dt}$
$= \frac{d\mathbf{A}}{dt}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{dt}$

(b)

$$\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \frac{d}{dt}\Big((A_yB_z - A_zB_y)\hat{\mathbf{e}}_x + (A_zB_x - A_xB_z)\hat{\mathbf{e}}_y + (A_xB_y - A_yB_x)\hat{\mathbf{e}}_z\Big)$$

$$=\Big(\tfrac{dA_y}{dt}B_z + A_y\tfrac{dB_z}{dt} - \tfrac{dA_z}{dt}B_y - A_z\tfrac{dB_y}{dt}\Big)\hat{\mathbf{e}}_x + \Big(\tfrac{dA_z}{dt}B_x + A_z\tfrac{dB_x}{dt} - \tfrac{dA_x}{dt}B_z - A_x\tfrac{dB_z}{dt}\Big)\hat{\mathbf{e}}_y + \Big(\tfrac{dA_x}{dt}B_y + A_x\tfrac{dB_y}{dt} - \tfrac{dA_y}{dt}B_x - A_y\tfrac{dB_x}{dt}\Big)\hat{\mathbf{e}}_z$$

$$=\Big(\tfrac{dA_y}{dt}B_z - \tfrac{dA_z}{dt}B_y\Big)\hat{\mathbf{e}}_x + \Big(\tfrac{dA_z}{dt}B_x - \tfrac{dA_x}{dt}B_z\Big)\hat{\mathbf{e}}_y + \Big(\tfrac{dA_x}{dt}B_y - \tfrac{dA_y}{dt}B_x\Big)\hat{\mathbf{e}}_z + \Big(A_y\tfrac{dB_z}{dt} - A_z\tfrac{dB_y}{dt}\Big)\hat{\mathbf{e}}_x + \Big(A_z\tfrac{dB_x}{dt} - A_x\tfrac{dB_z}{dt}\Big)\hat{\mathbf{e}}_y + \Big(A_x\tfrac{dB_y}{dt} - A_y\tfrac{dB_x}{dt}\Big)\hat{\mathbf{e}}_z$$

$$= \frac{d\mathbf{A}}{dt}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{dt}$$

3.5.9 $\frac{d}{dx}a_ib_j = \frac{da_i}{dx}b_j + a_i\frac{db_j}{dx}$, so we can decompose $\boldsymbol{\nabla}\cdot(\mathbf{a}\times\mathbf{b})$ into $\boldsymbol{\nabla}_a\cdot(\mathbf{a}\times\mathbf{b}) + \boldsymbol{\nabla}_b\cdot(\mathbf{a}\times\mathbf{b})$, with $\boldsymbol{\nabla}_a$ operating on $\mathbf{a}$ only, and $\boldsymbol{\nabla}_b$ operating on $\mathbf{b}$ only. Then by the symmetry of scalar triple product, $\boldsymbol{\nabla}_a\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\boldsymbol{\nabla}_a\times\mathbf{a}) = \mathbf{b}\cdot(\boldsymbol{\nabla}\times\mathbf{a})$, and $\boldsymbol{\nabla}_b\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\cdot(\mathbf{b}\times\boldsymbol{\nabla}_b) = -\mathbf{a}\cdot(\boldsymbol{\nabla}_b\times\mathbf{b}) = -\mathbf{a}\cdot(\boldsymbol{\nabla}\times\mathbf{b})$. So $\boldsymbol{\nabla}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\boldsymbol{\nabla}\times\mathbf{a}) - \mathbf{a}\cdot(\boldsymbol{\nabla}\times\mathbf{b})$.

3.5.10 $\mathbf{L} = \mathbf{r}\times(-i\boldsymbol{\nabla}) = (x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z)\times(-i\frac{\partial}{\partial x}\hat{\mathbf{e}}_x - i\frac{\partial}{\partial y}\hat{\mathbf{e}}_y - i\frac{\partial}{\partial z}\hat{\mathbf{e}}_z) =$
$-i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\hat{\mathbf{e}}_x - i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\hat{\mathbf{e}}_y - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\hat{\mathbf{e}}_z$

3.5.11

$$L_xL_y - L_yL_x$$

$$= -\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) + \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$= -\left(y\frac{\partial}{\partial x} + yz\frac{\partial^2}{\partial z\partial x} - yx\frac{\partial^2}{\partial z^2} - z^2\frac{\partial^2}{\partial y\partial x} + xz\frac{\partial^2}{\partial y\partial z}\right) + \left(zy\frac{\partial^2}{\partial x\partial z} - z^2\frac{\partial^2}{\partial x\partial y} - xy\frac{\partial^2}{\partial z^2} + x\frac{\partial}{\partial y} + xz\frac{\partial^2}{\partial z\partial y}\right)$$

$$= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = iL_z$$

3.5.12 The problem is ill-defined because the way of vector multiplication has not been specified(scalar product, cross product or tensor product). If $[\mathbf{a}, \mathbf{b}] = 0$, $[\mathbf{a}, \mathbf{L}] = 0$, $[\mathbf{b}, \mathbf{L}] = 0$ means $[a_i, b_j] = 0$, $[a_i, L_j] = 0$, $[b_i, L_j] = 0$ for all $i, j \in \{x, y, z\}$, then a and b, a and L, b and L commute, but L and L do not commute ($[L_i, L_j] = iL_k$). So

$$[\mathbf{a} \cdot \mathbf{L}, \mathbf{b} \cdot \mathbf{L}]$$

$$= [a_x L_x + a_y L_y + a_z L_z, b_x L_x + b_y L_y + b_z L_z]$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} [a_i L_i, b_j L_j]$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} (a_i L_i b_j L_j - b_j L_j a_i L_i)$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} (a_i b_j L_i L_j - b_j a_i L_j L_i)$$

$$= a_x b_y (L_x L_y - L_y L_x) + a_x b_z (L_x L_z - L_z L_x) + a_y b_x (L_y L_x - L_x L_y) + a_y b_z (L_y L_z - L_z L_y) + a_z b_x (L_z L_x - L_x L_z) + a_z b_y (L_z L_y - L_y L_z)$$

$$= a_x b_y i L_z - a_x b_z i L_y - a_y b_x i L_z + a_y b_z i L_x + a_z b_x i L_y - a_z b_y i L_x$$

$$= i(a_y b_z - a_z b_y) L_x + i(a_z b_x - a_x b_z) L_y + i(a_x b_y - a_y b_x) L_z$$

$$= i(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{L}$$

3.5.13 A stream line of a vector field should be parallel to the vector at every point in the space. So $\frac{dy}{dx} = \frac{b_y}{b_x} = \frac{x}{-y}$, where $y = y(x)$ is a stream line. Solving the differential equation, $xdx + ydy = 0$, $\frac{x^2}{2} + \frac{y^2}{2} = k'$, $x^2 + y^2 = k$, which is a circle. The direction of the stream line at $(1, 1)$ is $-\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y$, which is counterclockwise relative $(0, 0)$, the center of the circle.

3.6 Differential Vector Operators: Further Properties

3.6.1 By the identity in Exercise 3.5.9, $\boldsymbol{\nabla} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\boldsymbol{\nabla} \times \mathbf{u}) - \mathbf{u} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) = 0$ because $\boldsymbol{\nabla} \times \mathbf{u} = 0$ and $\boldsymbol{\nabla} \times \mathbf{v} = 0$ (irrotational).

3.6.2 $\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{r}) = \mathbf{r} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{r}) = 0$ because both $\mathbf{A}$ and the position vector $\mathbf{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$ are irrotational, so their curl vanish.

3.6.3 The linear velocity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, so $\boldsymbol{\nabla} \cdot \mathbf{v} = \boldsymbol{\nabla} \cdot (\boldsymbol{\omega} \times \mathbf{r}) = \mathbf{r} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\omega}) - \boldsymbol{\omega} \cdot (\boldsymbol{\nabla} \times \mathbf{r}) = 0$ because the curl of constant vector $\boldsymbol{\omega}$ and position vector $\mathbf{r}$ are zero.

3.6.4 $\boldsymbol{\nabla} \times \mathbf{V} \neq 0$ but $\boldsymbol{\nabla} \times (g\mathbf{V}) = g(\boldsymbol{\nabla} \times \mathbf{V}) + (\boldsymbol{\nabla} g) \times \mathbf{V} = 0$. So $\mathbf{V} \cdot \big(g(\boldsymbol{\nabla} \times \mathbf{V}) + (\boldsymbol{\nabla} g) \times \mathbf{V}\big) = g\mathbf{V} \cdot (\boldsymbol{\nabla} \times \mathbf{V}) + \mathbf{V} \cdot ((\boldsymbol{\nabla} g) \times \mathbf{V}) = 0$. But $\mathbf{V} \cdot ((\boldsymbol{\nabla} g) \times \mathbf{V}) = 0$ (perpendicular) and $g \neq 0$, so $\mathbf{V} \cdot (\boldsymbol{\nabla} \times \mathbf{V}) = 0$.

3.6.5 All the terms of $\mathbf{A} \times \mathbf{B}$ have the form $A_i B_j \hat{\mathbf{e}}_k$, and $\frac{\partial}{\partial x}(A_i B_j) = \frac{\partial A_i}{\partial x} B_j + A_i \frac{\partial B_j}{\partial x}$. So we can separate $\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B})$ into $\boldsymbol{\nabla}_A \times (\mathbf{A} \times \mathbf{B})$ and $\boldsymbol{\nabla}_B \times (\mathbf{A} \times \mathbf{B})$, with $\boldsymbol{\nabla}_A$ acting only on $\mathbf{A}$, and $\boldsymbol{\nabla}_B$ acting only on $\mathbf{B}$. Using the BAC-CAB rule, but noting that $\boldsymbol{\nabla}_A$ must go before $\mathbf{A}$ and after $\mathbf{B}$, $\boldsymbol{\nabla}_B$ go before $\mathbf{B}$ and after $\mathbf{A}$, we get

$$\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = \boldsymbol{\nabla}_A \times (\mathbf{A} \times \mathbf{B}) + \boldsymbol{\nabla}_B \times (\mathbf{A} \times \mathbf{B})$$

$$= (\mathbf{B} \cdot \boldsymbol{\nabla}_A)\mathbf{A} - \mathbf{B}(\boldsymbol{\nabla}_A \cdot \mathbf{A}) + \mathbf{A}(\boldsymbol{\nabla}_B \cdot \mathbf{B}) - (\mathbf{A} \cdot \boldsymbol{\nabla}_B)\mathbf{B}$$

$$= (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} - \mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{A}) + \mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{B}) - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B}$$

3.6.6

$$[(\mathbf{A}\times\boldsymbol{\nabla})\times\mathbf{B}]_x = A_z\frac{\partial B_z}{\partial x} - A_x\frac{\partial B_z}{\partial z} - A_x\frac{\partial B_y}{\partial y} + A_y\frac{\partial B_y}{\partial x}$$

$$[(\mathbf{B}\times\boldsymbol{\nabla})\times\mathbf{A}]_x = B_z\frac{\partial A_z}{\partial x} - B_x\frac{\partial A_z}{\partial z} - B_x\frac{\partial A_y}{\partial y} + B_y\frac{\partial A_y}{\partial x}$$

$$[\mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B})]_x = A_x\frac{\partial B_x}{\partial x} + A_x\frac{\partial B_y}{\partial y} + A_x\frac{\partial B_z}{\partial z}$$

$$[\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A})]_x = B_x\frac{\partial A_x}{\partial x} + B_x\frac{\partial A_y}{\partial y} + B_x\frac{\partial A_z}{\partial z}$$

Summing together we get

$$A_x\frac{\partial B_x}{\partial x} + B_x\frac{\partial A_x}{\partial x} + A_y\frac{\partial B_y}{\partial x} + B_y\frac{\partial A_y}{\partial x} + A_z\frac{\partial B_z}{\partial x} + B_z\frac{\partial A_z}{\partial x}$$

which is $\frac{\partial(\mathbf{A}\cdot\mathbf{B})}{\partial x} = [\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B})]_x$. The same is for y and z components. Therefore,

$$\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\times\boldsymbol{\nabla})\times\mathbf{B} + (\mathbf{B}\times\boldsymbol{\nabla})\times\mathbf{A} + \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) + \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A})$$

3.6.7 To distinguish between the two $\mathbf{A}$, let the first $\mathbf{A}$ be $\mathbf{A}_1$ and the second be $\mathbf{A}_2$. Applying BAC-CAB rule, $\mathbf{A}_1\times(\boldsymbol{\nabla}\times\mathbf{A}_2) = \boldsymbol{\nabla}_2(\mathbf{A}_1\cdot\mathbf{A}_2) - (\mathbf{A}_1\cdot\boldsymbol{\nabla})\mathbf{A}_2$ with $\boldsymbol{\nabla}_2$ acting only on $\mathbf{A}_2$. Noting that $\boldsymbol{\nabla}(\mathbf{A}_1\cdot\mathbf{A}_2) = \boldsymbol{\nabla}_1(\mathbf{A}_1\cdot\mathbf{A}_2) + \boldsymbol{\nabla}_2(\mathbf{A}_1\cdot\mathbf{A}_2)$ and $\boldsymbol{\nabla}_1(\mathbf{A}_1\cdot\mathbf{A}_2) = \boldsymbol{\nabla}_2(\mathbf{A}_1\cdot\mathbf{A}_2)$, we have $\boldsymbol{\nabla}_2(\mathbf{A}_1\cdot\mathbf{A}_2) = \frac{\boldsymbol{\nabla}(\mathbf{A}_1\cdot\mathbf{A}_2)}{2}$, so

$$\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \frac{1}{2}\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{A}) - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{A}$$

3.6.8

$$\begin{aligned}\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{r}) &= \boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{A}\times\mathbf{B})\\ &= \boldsymbol{\nabla}(x[\mathbf{A}\times\mathbf{B}]_x + y[\mathbf{A}\times\mathbf{B}]_y + z[\mathbf{A}\times\mathbf{B}]_z)\\ &= [\mathbf{A}\times\mathbf{B}]_x\hat{\mathbf{e}}_x + [\mathbf{A}\times\mathbf{B}]_y\hat{\mathbf{e}}_y + [\mathbf{A}\times\mathbf{B}]_z\hat{\mathbf{e}}_z\\ &= \mathbf{A}\times\mathbf{B}\end{aligned}$$

because $\mathbf{A}\times\mathbf{B}$ is constant, so $\frac{\partial[\mathbf{A}\times\mathbf{B}]_j}{\partial x_i} = 0$

3.6.9

$$[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{V})]_x = \frac{\partial^2 V_y}{\partial y\partial x} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial^2 V_z}{\partial z\partial x}$$

$$[\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{V})]_x = \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial x\partial y} + \frac{\partial^2 V_z}{\partial x\partial z}$$

$$-[\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\mathbf{V}]_x = -\frac{\partial^2 V_x}{\partial x^2} - \frac{\partial^2 V_y}{\partial y^2} - \frac{\partial^2 V_z}{\partial z^2}$$

So $[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{V})]_x = [\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{V})]_x - [\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\mathbf{V}]_x$, as well as the y and z components. Therefore

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{V}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{V}) - \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\mathbf{V}$$

3.6.10

$$\begin{aligned}[\boldsymbol{\nabla}\times(\varphi\boldsymbol{\nabla}\varphi)]_x &= \frac{\partial}{\partial y}\left(\varphi\frac{\partial\varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\varphi\frac{\partial\varphi}{\partial y}\right)\\ &= \frac{\partial\varphi}{\partial y}\frac{\partial\varphi}{\partial z} + \varphi\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial\varphi}{\partial z}\frac{\partial\varphi}{\partial y} - \varphi\frac{\partial^2\varphi}{\partial z\partial y} = 0\end{aligned}$$

The same is for the y and z components. Therefore, $\boldsymbol{\nabla}\times(\varphi\boldsymbol{\nabla}\varphi) = 0$

3.6.11 (a) If $\mathbf{F} = \mathbf{G} + k$, k is a constant, then $\boldsymbol{\nabla}\times\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{G}$ beacuse $\frac{\partial k}{\partial x_i} = 0$

(b) If $\mathbf{F} = \mathbf{G} + \boldsymbol{\nabla}\varphi$, then $\boldsymbol{\nabla}\times\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{G} + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\varphi) = \boldsymbol{\nabla}\times\mathbf{G}$ because $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\varphi) = 0$.

3.6.12 From Exercise 3.6.7, $\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v})=\frac{1}{2}\boldsymbol{\nabla}(v^2)-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$, so

$$-\boldsymbol{\nabla}\times\big(\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v})\big)=-\frac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}(v^2)+\boldsymbol{\nabla}\times\big((\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\big)=\boldsymbol{\nabla}\times\big((\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\big)$$

because $\boldsymbol{\nabla}\times\boldsymbol{\nabla}(v^2)=0$.

3.6.13 From Exercise 3.5.9,

$$\boldsymbol{\nabla}\cdot\big((\boldsymbol{\nabla}u)\times(\boldsymbol{\nabla}v)\big)=(\boldsymbol{\nabla}v)\cdot\big(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}u)\big)-(\boldsymbol{\nabla}u)\cdot\big(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}v)\big)=0$$

because $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}u)=0$ and $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}v)=0$

3.6.14 $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\varphi=\boldsymbol{\nabla}^2\varphi=0$, and $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\varphi=0$ for any φ, so $\boldsymbol{\nabla}\varphi$ is both solenoidal and irrotational.

3.6.15 By Equation (3.70), $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})-(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\mathbf{A}$, so the equation becomes $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})-(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\mathbf{A}-k^2\mathbf{A}=0$. Let $\boldsymbol{\nabla}\cdot$ operate on both side of the equation, and note that $\boldsymbol{\nabla}\cdot\big((\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\mathbf{A}\big)=(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})(\boldsymbol{\nabla}\cdot\mathbf{A})$ because $\frac{\partial}{\partial x_i}\left(\frac{\partial^2 A_k}{\partial x_j^2}\right)=\frac{\partial^2}{\partial x_j^2}\left(\frac{\partial A_k}{\partial x_i}\right)$. So the equation becomes $(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})(\boldsymbol{\nabla}\cdot\mathbf{A})-(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})(\boldsymbol{\nabla}\cdot\mathbf{A})-k^2(\boldsymbol{\nabla}\cdot\mathbf{A})=-k^2(\boldsymbol{\nabla}\cdot\mathbf{A})=0$, so $\boldsymbol{\nabla}\cdot\mathbf{A}=0$. Substituting back to the second equation, we get $(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\mathbf{A}+k^2\mathbf{A}=\boldsymbol{\nabla}^2\mathbf{A}+k^2\mathbf{A}=0$

3.6.16 Let $\Psi=\frac{k}{2}\Phi^2$, then

$$\begin{aligned}\boldsymbol{\nabla}^2\Psi&=\frac{k}{2}\left(\frac{\partial^2\Phi^2}{\partial x^2}+\frac{\partial^2\Phi^2}{\partial y^2}+\frac{\partial^2\Phi^2}{\partial z^2}\right)\\&=\frac{k}{2}\left(\frac{\partial}{\partial x}\left(2\Phi\frac{\partial\Phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(2\Phi\frac{\partial\Phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(2\Phi\frac{\partial\Phi}{\partial z}\right)\right)\\&=\frac{k}{2}\left(2\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial x}+2\Phi\frac{\partial^2\Phi}{\partial x^2}+2\frac{\partial\Phi}{\partial y}\frac{\partial\Phi}{\partial y}+2\Phi\frac{\partial^2\Phi}{\partial y^2}+2\frac{\partial\Phi}{\partial z}\frac{\partial\Phi}{\partial z}+2\Phi\frac{\partial^2\Phi}{\partial z^2}\right)\\&=k\left(\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right)=k|\boldsymbol{\nabla}\Phi|^2\end{aligned}$$

because $\boldsymbol{\nabla}^2\Phi=\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0$. So $\Psi=\frac{k}{2}\Phi^2$ is a solution of the equation.

3.6.17 Substituting, we get

$$\begin{pmatrix}\frac{1}{c}\frac{\partial}{\partial t} & -i\frac{\partial}{\partial z} & i\frac{\partial}{\partial y}\\ i\frac{\partial}{\partial z} & \frac{1}{c}\frac{\partial}{\partial t} & -i\frac{\partial}{\partial x}\\ -i\frac{\partial}{\partial y} & i\frac{\partial}{\partial x} & \frac{1}{c}\frac{\partial}{\partial t}\end{pmatrix}\begin{pmatrix}B_x-i\frac{E_x}{c}\\ B_y-i\frac{E_y}{c}\\ B_z-i\frac{E_z}{c}\end{pmatrix}$$

$$=\begin{pmatrix}\left(\frac{1}{c}\frac{\partial B_x}{\partial t}-\frac{1}{c}\frac{\partial E_y}{\partial z}+\frac{1}{c}\frac{\partial E_z}{\partial y}\right)+i\left(-\frac{1}{c^2}\frac{\partial E_x}{\partial t}-\frac{\partial B_y}{\partial z}+\frac{\partial B_z}{\partial y}\right)\\ \left(\frac{1}{c}\frac{\partial B_y}{\partial t}-\frac{1}{c}\frac{\partial E_z}{\partial x}+\frac{1}{c}\frac{\partial E_x}{\partial z}\right)+i\left(-\frac{1}{c^2}\frac{\partial E_y}{\partial t}-\frac{\partial B_z}{\partial x}+\frac{\partial B_x}{\partial z}\right)\\ \left(\frac{1}{c}\frac{\partial B_z}{\partial t}-\frac{1}{c}\frac{\partial E_x}{\partial y}+\frac{1}{c}\frac{\partial E_y}{\partial x}\right)+i\left(-\frac{1}{c^2}\frac{\partial E_z}{\partial t}-\frac{\partial B_x}{\partial y}+\frac{\partial B_y}{\partial x}\right)\end{pmatrix}=0$$

The real and imaginary part must to be zero, respectively. So the three equations from the real part form $\boldsymbol{\nabla}\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$, and the three equations from the imaginary part form $\boldsymbol{\nabla}\times\mathbf{B}=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$.

3.6.18 Note that $\boldsymbol{\sigma}_i^2=\mathbf{1}_2$, $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j=i\boldsymbol{\sigma}_k$. So

$$(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b})$$

$$\begin{aligned}&=(a_x\boldsymbol{\sigma}_1+a_y\boldsymbol{\sigma}_2+a_z\boldsymbol{\sigma}_3)(b_x\boldsymbol{\sigma}_1+b_y\boldsymbol{\sigma}_2+b_z\boldsymbol{\sigma}_3)\\&=(a_xb_x+a_yb_y+a_zb_z)\mathbf{1}_2+(a_xb_y-a_yb_x)i\boldsymbol{\sigma}_3+(a_yb_z-a_zb_y)i\boldsymbol{\sigma}_1+(a_zb_x-a_xb_z)i\boldsymbol{\sigma}_2\\&=(\mathbf{a}\cdot\mathbf{b})\mathbf{1}_2+i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})\end{aligned}$$

3.7 Vector Integration

3.7.1 The total vector area is

$$\int d\boldsymbol{\sigma} = \frac{1}{2}\mathbf{B}\times\mathbf{A} + \frac{1}{2}\mathbf{C}\times\mathbf{B} + \frac{1}{2}\mathbf{A}\times\mathbf{C} + \frac{1}{2}(\mathbf{C}-\mathbf{B})\times(\mathbf{A}-\mathbf{B}) = 0$$

3.7.2 (a) $x^2+y^2=1$, so $y=\sqrt{1-x^2}$; $2xdx+2ydy=0$, so $dy=-\frac{x}{y}dx=-\frac{x}{\sqrt{1-x^2}}dx$. So

$$w = \int (-\mathbf{F})\cdot d\mathbf{r}$$

$$= \int \frac{y}{x^2+y^2}dx + \int \frac{-x}{x^2+y^2}dy$$

$$= \int_1^{-1} \sqrt{1-x^2}\,dx + \int_1^{-1} \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$$= \int_1^{-1} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = -\pi$$

(b) $x^2+y^2=1$, so $y=-\sqrt{1-x^2}$; $2xdx+2ydy=0$, so $dy=-\frac{x}{y}dx=\frac{x}{\sqrt{1-x^2}}dx$. So

$$w = \int (-\mathbf{F})\cdot d\mathbf{r}$$

$$= \int \frac{y}{x^2+y^2}dx + \int \frac{-x}{x^2+y^2}dy$$

$$= -\int_1^{-1} \sqrt{1-x^2}\,dx - \int_1^{-1} \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$$= \int_1^{-1} \frac{-1}{\sqrt{1-x^2}}\,dx$$

$$= \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{-1}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \pi$$

3.7.3 Choose the path $(1,1)\to(1,3)\to(3,3)$. Then

$$w = \int_1^3 \mathbf{F}(x,1)\cdot(dx\hat{\mathbf{e}}_x) + \int_1^3 \mathbf{F}(3,y)\cdot(dy\hat{\mathbf{e}}_y)$$

$$= \int_1^3 (x-1)\,dx + \int_1^3 (3+y)\,dy$$

$$= 2+10 = 12$$

3.7.4 $\oint \mathbf{r}\cdot d\mathbf{r} = \oint(xdx+ydy+zdz) = (\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2})\big|_{\mathbf{a}}^{\mathbf{a}} = 0$ where $\mathbf{a}$ is the starting point.

3.7.5 For the surfaces parallel to yz surface,

$$\int_{S_{yz}} \mathbf{r}\cdot d\boldsymbol{\sigma} = \int_{S_{yz}} (x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z)\cdot(d\sigma\hat{\mathbf{e}}_x) = \int_{S_{yz}} (xd\sigma) = x$$

equals to 1 at $x=1$ and 0 at $x=0$. The same is for y and z, so

$$\frac{1}{3}\int_S \mathbf{r}\cdot d\boldsymbol{\sigma} = \frac{1}{3}(1+1+1) = 1$$

3.8 Integral Theorems

3.8.1 Let $\mathbf{a}$ be a constant vector, then

$$\mathbf{a} \cdot \oint_{\partial V} d\boldsymbol{\sigma} = \oint_{\partial V} \mathbf{a} \cdot d\boldsymbol{\sigma} = \int_V (\boldsymbol{\nabla} \cdot \mathbf{a}) d\tau = 0$$

Because $\mathbf{a}$ can be in arbitrary direction, $\oint_{\delta V} d\boldsymbol{\sigma}$ must be zero.

3.8.2

$$\frac{1}{3} \oint_S \mathbf{r} \cdot d\boldsymbol{\sigma} = \frac{1}{3} \int_V (\boldsymbol{\nabla} \cdot \mathbf{r}) d\tau = \frac{1}{3} \int_V 3 d\tau = V$$

3.8.3

$$\oint_S \mathbf{B} \cdot d\boldsymbol{\sigma} = \oint_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\boldsymbol{\sigma} = \int_V \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) d\tau = 0$$

because the divergence of a curl vanishes.

3.8.4 $\boldsymbol{\nabla} \cdot (\varphi \mathbf{E}) = (\boldsymbol{\nabla} \varphi) \cdot \mathbf{E} + \varphi(\boldsymbol{\nabla} \cdot \mathbf{E})$, so $\rho\varphi = \varepsilon_0 (\boldsymbol{\nabla} \cdot \mathbf{E})\varphi = \varepsilon_0 \boldsymbol{\nabla} \cdot (\varphi \mathbf{E}) - \varepsilon_0 (\boldsymbol{\nabla}\varphi) \cdot \mathbf{E}$, and

$$\int \rho\varphi \, d\tau = \varepsilon_0 \int \boldsymbol{\nabla} \cdot (\varphi \mathbf{E}) \, d\tau - \varepsilon_0 \int (\boldsymbol{\nabla}\varphi) \cdot \mathbf{E} \, d\tau$$

$$= \varepsilon_0 \oint \varphi \mathbf{E} \cdot d\boldsymbol{\sigma} + \varepsilon_0 \int E^2 d\tau$$

φ vanishes at least as fast as r^{-1}, so $\mathbf{E} = -\boldsymbol{\nabla}\varphi$ vanishes at least as fast as r^{-2}, and $\varphi\mathbf{E}$ vanishes at least as fast as r^{-3}. But $d\boldsymbol{\sigma}$ is in the order of r^2, so $\oint \varphi \mathbf{E} \cdot d\boldsymbol{\sigma}$ vanishes at large r. Therefore,

$$\int \rho\varphi \, d\tau = \varepsilon_0 \int E^2 d\tau$$

3.8.5 $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$ because it is steady-state current distribution. So $\boldsymbol{\nabla} \cdot (x_i \mathbf{J}) = (\boldsymbol{\nabla} x_i) \cdot \mathbf{J} + x_i (\boldsymbol{\nabla} \cdot \mathbf{J}) = (\boldsymbol{\nabla} x_i) \cdot \mathbf{J} = \hat{\mathbf{e}}_i \cdot \mathbf{J} = J_i$. So

$$\int J_i d\tau = \int \boldsymbol{\nabla} \cdot (x_i \mathbf{J}) d\tau = \oint x_i \mathbf{J} \cdot d\boldsymbol{\sigma} = 0$$

because $\mathbf{J}$ vanishes on the surface. So

$$\int \mathbf{J} d\tau = \sum \int J_i d\tau \, \hat{\mathbf{e}}_i = 0$$

3.8.6

$$\frac{1}{2} \oint \mathbf{t} \cdot d\boldsymbol{\lambda} = \frac{1}{2} \int (\boldsymbol{\nabla} \times \mathbf{t}) \cdot d\boldsymbol{\sigma} = \frac{1}{2} \int 2\hat{\mathbf{e}}_z \cdot d\boldsymbol{\sigma} = A$$

3.8.7 (a) $\oint \mathbf{r} \times d\mathbf{r} = \oint (x dy - y dx) \hat{\mathbf{e}}_z = 2A\hat{\mathbf{e}}_z$ from Exercise 3.8.6 .

(b) $\mathbf{r} = a\cos\theta \hat{\mathbf{e}}_x + b \sin\theta \hat{\mathbf{e}}_y$, so $d\mathbf{r} = -a \sin\theta d\theta \hat{\mathbf{e}}_x + b \cos\theta d\theta \hat{\mathbf{e}}_y$, and

$$\oint \mathbf{r} \times d\mathbf{r} = \int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta) d\theta = 2\pi ab = 2A$$

so the area of the ellipse is πab.

3.8.8

$$\oint \mathbf{r}\times d\mathbf{r} = -\oint d\mathbf{r}\times\mathbf{r}$$
$$= -\int\limits_S (d\boldsymbol{\sigma}\times\boldsymbol{\nabla})\times\mathbf{r}$$
$$= -\int\limits_S \left((dxdy\hat{\mathbf{e}}_z)\times(\frac{\partial}{\partial x}\hat{\mathbf{e}}_x+\frac{\partial}{\partial y}\hat{\mathbf{e}}_y+\frac{\partial}{\partial z}\hat{\mathbf{e}}_z)\right)\times\mathbf{r}$$
$$= -\int\limits_S (-dxdy\frac{\partial}{\partial y}\hat{\mathbf{e}}_x + dxdy\frac{\partial}{\partial x}\hat{\mathbf{e}}_y)\times(x\hat{\mathbf{e}}_x+y\hat{\mathbf{e}}_y)$$
$$= -\int\limits_S -2dxdy = 2A$$

3.8.9

$$\oint u\boldsymbol{\nabla}v\cdot d\boldsymbol{\lambda} + \oint v\boldsymbol{\nabla}u\cdot d\boldsymbol{\lambda} = \oint \boldsymbol{\nabla}(uv)\cdot d\boldsymbol{\lambda} = \int \boldsymbol{\nabla}\times(\boldsymbol{\nabla}uv)\cdot d\boldsymbol{\sigma} = 0$$

so $\oint u\boldsymbol{\nabla}v\cdot d\boldsymbol{\lambda} = -\oint v\boldsymbol{\nabla}u\cdot d\boldsymbol{\lambda}$.

3.8.10 $\boldsymbol{\nabla}\times(f\mathbf{V}) = (\boldsymbol{\nabla}f)\times\mathbf{V} + f(\boldsymbol{\nabla}\times\mathbf{V})$ from Eq. 3.73 . So

$$\oint u\boldsymbol{\nabla}v\cdot d\boldsymbol{\lambda} = \int\limits_S \boldsymbol{\nabla}\times(u\boldsymbol{\nabla}v)\cdot d\boldsymbol{\sigma}$$
$$= \int\limits_S (\boldsymbol{\nabla}u)\times(\boldsymbol{\nabla}v)\cdot d\boldsymbol{\sigma} + \int\limits_S u(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}v))\cdot d\boldsymbol{\sigma}$$
$$= \int\limits_S (\boldsymbol{\nabla}u)\times(\boldsymbol{\nabla}v)\cdot d\boldsymbol{\sigma}$$

because $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}v) = 0$.

3.8.11 Let $\mathbf{a}$ be a constant vector, then

$$\int\limits_V \boldsymbol{\nabla}\cdot(\mathbf{a}\times\mathbf{P})d\tau = \int\limits_V (\boldsymbol{\nabla}\times\mathbf{a})\cdot\mathbf{P}d\tau - \int\limits_V \mathbf{a}\cdot(\boldsymbol{\nabla}\times\mathbf{P})d\tau = -\mathbf{a}\cdot\int\limits_V \boldsymbol{\nabla}\times\mathbf{P}d\tau$$

Also,

$$\int\limits_V \boldsymbol{\nabla}\cdot(\mathbf{a}\times\mathbf{P})d\tau = \oint\limits_{\partial V} \mathbf{a}\times\mathbf{P}\cdot d\boldsymbol{\sigma} = \oint\limits_{\partial V} \mathbf{P}\times d\boldsymbol{\sigma}\cdot\mathbf{a} = \mathbf{a}\cdot\oint\limits_{\partial V} \mathbf{P}\times d\boldsymbol{\sigma}$$

So $\mathbf{a}\cdot\left(\oint_{\partial V}\mathbf{P}\times d\boldsymbol{\sigma} + \int_V \boldsymbol{\nabla}\times\mathbf{P}d\tau\right) = 0$. Because $\mathbf{a}$ can be in arbitrary direction, $\oint_{\partial V}\mathbf{P}\times d\boldsymbol{\sigma} + \int_V \boldsymbol{\nabla}\times\mathbf{P}d\tau$ must be zero, and therefore

$$\oint\limits_{\partial V} d\boldsymbol{\sigma}\times\mathbf{P} = \int\limits_V \boldsymbol{\nabla}\times\mathbf{P}d\tau$$

3.8.12 Let $\mathbf{a}$ be a constant vector, then

$$\oint\limits_{\partial S} (\mathbf{a}\varphi)\cdot d\mathbf{r} = \mathbf{a}\cdot\oint\limits_{\partial S}\varphi d\mathbf{r}$$

Also,

$$\oint\limits_{\partial S} (\mathbf{a}\varphi)\cdot d\mathbf{r} = \int\limits_S \boldsymbol{\nabla}\times(\varphi\mathbf{a})\cdot d\boldsymbol{\sigma} = \int\limits_S (\boldsymbol{\nabla}\varphi)\times\mathbf{a}\cdot d\boldsymbol{\sigma} = \int\limits_S d\boldsymbol{\sigma}\times(\boldsymbol{\nabla}\varphi)\cdot\mathbf{a} = \mathbf{a}\cdot\int\limits_S d\boldsymbol{\sigma}\times(\boldsymbol{\nabla}\varphi)$$

so $\mathbf{a}\cdot\left(\int_S d\boldsymbol{\sigma}\times(\boldsymbol{\nabla}\varphi) - \oint_{\partial S}\varphi d\mathbf{r}\right) = 0$. Because $\mathbf{a}$ can be in arbitrary direction, $\int_S d\boldsymbol{\sigma}\times(\boldsymbol{\nabla}\varphi) - \oint_{\partial S}\varphi d\mathbf{r}$ must be zero, and therefore

$$\int\limits_S d\boldsymbol{\sigma}\times(\boldsymbol{\nabla}\varphi) = \oint\limits_{\partial S}\varphi d\mathbf{r}$$

3.8.13 Let $\mathbf{a}$ be a constant vector, then

$$\oint_{\partial S}(\mathbf{a}\times\mathbf{P})\cdot d\mathbf{r}=\oint_{\partial S}(\mathbf{P}\times d\mathbf{r})\cdot\mathbf{a}=\mathbf{a}\cdot\oint_{\partial S}\mathbf{P}\times d\mathbf{r}$$

Because $\mathbf{a}$ is a constant, $\boldsymbol{\nabla}\times(\mathbf{a}\times\mathbf{P})=\boldsymbol{\nabla}_P\times(\mathbf{a}\times\mathbf{P})$, with $\boldsymbol{\nabla}_P$ acting only on $\mathbf{P}$. So $\boldsymbol{\nabla}_P$ needs to go before $\mathbf{P}$, but can go before and after $\mathbf{a}$. So

$$\boldsymbol{\nabla}_P\times(\mathbf{a}\times\mathbf{P})\cdot d\boldsymbol{\sigma}=d\boldsymbol{\sigma}\times\boldsymbol{\nabla}_P\cdot(\mathbf{a}\times\mathbf{P})=(d\boldsymbol{\sigma}\times\boldsymbol{\nabla}_P)\cdot\mathbf{a}\times\mathbf{P}=-\mathbf{a}\cdot(d\boldsymbol{\sigma}\times\boldsymbol{\nabla}_P)\times\mathbf{P}=-\mathbf{a}\cdot(d\boldsymbol{\sigma}\times\boldsymbol{\nabla})\times\mathbf{P}$$

and

$$\oint_S(\mathbf{a}\times\mathbf{P})\cdot d\mathbf{r}=\int_S\boldsymbol{\nabla}\times(\mathbf{a}\times\mathbf{P})\cdot d\boldsymbol{\sigma}=-\int_S\mathbf{a}\cdot(d\boldsymbol{\sigma}\times\boldsymbol{\nabla})\times\mathbf{P}=-\mathbf{a}\cdot\int_S(d\boldsymbol{\sigma}\times\boldsymbol{\nabla})\times\mathbf{P}$$

so $\mathbf{a}\cdot\left(\int_S(d\boldsymbol{\sigma}\times\boldsymbol{\nabla})\times\mathbf{P}+\oint_{\partial S}\mathbf{P}\times d\mathbf{r}\right)=0$. Because $\mathbf{a}$ can be in arbitrary direction, $\int_S(d\boldsymbol{\sigma}\times\boldsymbol{\nabla})\times\mathbf{P}+\oint_{\partial S}\mathbf{P}\times d\mathbf{r}$ must be zero, and therefore

$$\int_S(d\boldsymbol{\sigma}\times\boldsymbol{\nabla})\times\mathbf{P}=\oint_{\partial S}d\mathbf{r}\times\mathbf{P}$$

3.9 Potential Theory

3.9.1 $\mathbf{F}=r^{2n}\mathbf{r}$

(a) $\boldsymbol{\nabla}\cdot\mathbf{F}=(\boldsymbol{\nabla}r^{2n})\cdot\mathbf{r}+r^{2n}(\boldsymbol{\nabla}\cdot\mathbf{r})=2nr^{2n-1}\frac{\mathbf{r}}{r}\cdot\mathbf{r}+r^{2n}(3)=(2n+3)r^{2n}$

(b) $\boldsymbol{\nabla}\times\mathbf{F}=(\boldsymbol{\nabla}r^{2n})\times\mathbf{r}+r^{2n}(\boldsymbol{\nabla}\times\mathbf{r})=2nr^{2n-1}\frac{\mathbf{r}}{r}\times\mathbf{r}+0=0$

(c) $\boldsymbol{\nabla}\times\mathbf{F}=0$, so the scalar potential exists. $\int_a^b\mathbf{F}\cdot d\mathbf{r}=-\int_a^b\boldsymbol{\nabla}\varphi\cdot d\mathbf{r}=-\varphi\big|_a^b=\varphi(a)-\varphi(b)$. Take the path $(0,0,0)\to(x,0,0)\to(x,y,0)\to(x,y,z)$. Then

$$\int_{(0,0,0)}^{(x,y,z)}\mathbf{F}\cdot d\mathbf{r}=\int_0^x x^{2n}xdx+\int_0^y(x^2+y^2)^n ydy+\int_0^z(x^2+y^2+z^2)^n zdz$$
$$=\frac{x^{2n+2}}{2n+2}-0+\frac{(x^2+y^2)^{n+1}}{2(n+1)}-\frac{(x^2)^{n+1}}{2(n+1)}+\frac{(x^2+y^2+z^2)^{n+1}}{2(n+1)}-\frac{(x^2+y^2)^{n+1}}{2(n+1)}$$
$$=\frac{r^{2n+2}}{2n+2}=\varphi(0,0,0)-\varphi(x,y,z)$$

when $n\neq-1$. Defining $\varphi(0,0,0)$ to be zero, then $\varphi(x,y,z)=-\frac{r^{2n+2}}{2n+2}$.

(d) If $n=-1$,

$$\int_{(1,1,1)}^{(x,y,z)}\mathbf{F}\cdot d\mathbf{r}=\int_1^x\frac{1}{x}dx+\int_1^y\frac{y}{x^2+y^2}dy+\int_1^z\frac{z}{x^2+y^2+z^2}dz$$
$$=\ln|x|+\frac{1}{2}\ln|\frac{x^2+y^2}{x^2}|+\frac{1}{2}\ln|\frac{x^2+y^2+z^2}{x^2+y^2}|$$
$$=\frac{1}{2}\ln|x^2+y^2+z^2|$$
$$=\ln r=\varphi(1,1,1)-\varphi(\mathbf{r})$$

Defining $\varphi(1,1,1)=0$, then $\varphi(\mathbf{r})=-\ln r$ diverges at both the origin and infinity.

3.9.2 Applying Gauss law, at $r\leq a$, $\oint\mathbf{E}\cdot d\boldsymbol{\sigma}=E4\pi r^2=\frac{Q}{\varepsilon_0}\frac{r^3}{a^3}$, so $E=\frac{Qr}{4\pi\varepsilon_0a^3}$; at $r>a$, $\oint\mathbf{E}\cdot d\boldsymbol{\sigma}=E4\pi r^2=\frac{Q}{\varepsilon_0}$, so $E=\frac{Qr}{4\pi\varepsilon_0r^2}$. Defining the potential to be zero at $r\to\infty$, then $\int_r^\infty\mathbf{E}\cdot d\mathbf{r}=\varphi(r)-\varphi(\infty)=\varphi(r)$. At $r>a$, $\varphi(r)=\int_r^\infty\frac{Q}{4\pi\varepsilon_0r^2}dr=\frac{Q}{4\pi\varepsilon_0r}$; at $r\leq a$, $\varphi(r)=\int_r^a\frac{Qr}{4\pi\varepsilon_0a^3}dr+\int_a^\infty\frac{Q}{4\pi\varepsilon_0r^2}dr=\frac{Q}{4\pi\varepsilon_0a}\left(\frac{1}{2}-\frac{1}{2}\frac{r^2}{a^2}\right)+\frac{Q}{4\pi\varepsilon_0a}=\frac{Q}{4\pi\varepsilon_0a}\left(\frac{3}{2}-\frac{1}{2}\frac{r^2}{a^2}\right)$. So the electrostatic potential is

$$\varphi(r)=\begin{cases}\frac{Q}{4\pi\varepsilon_0a}\left(\frac{3}{2}-\frac{1}{2}\frac{r^2}{a^2}\right), & r\leq a\\ \frac{Q}{4\pi\varepsilon_0r}, & r>a\end{cases}$$

3.9.3 It can be verified that $\nabla \times \mathbf{F} = 0$, so the potential exists.

$$\int_{(0,0,0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{r} = \varphi(0,0,0) - \varphi(x,y,z)$$

$$= \frac{GMm}{R^3} \int_{(0,0,0)}^{(x,y,z)} (-xdx - ydy + 2zdz)$$

$$= \frac{GMm}{R^3} \left(-\frac{x^2}{2} - \frac{y^2}{2} + z^2 \right) = \varphi(0,0,0) - \varphi(x,y,z)$$

Define $\varphi(0,0,0)$ to be zero, then

$$\varphi(x,y,z) = -\frac{GMm}{R^3} \left(-\frac{x^2}{2} - \frac{y^2}{2} + z^2 \right)$$

3.9.4 $\nabla \cdot \mathbf{B} = \frac{\mu_0 I}{2\pi} \left(\frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} \right) = 0$, so the vector potential exists. If $\nabla \times \mathbf{A}' = \mathbf{B}$, and let $\nabla\varphi = -A'_x \hat{\mathbf{e}}_x$, then $\mathbf{A} = \mathbf{A}' + \nabla\varphi$ is also a vector potential with zero x-component because $\nabla \times \mathbf{A} = \nabla \times \mathbf{A}' + \nabla \times (\nabla\varphi) = \mathbf{B}$. Let $\mathbf{A} = A_y \hat{\mathbf{e}}_y + A_z \hat{\mathbf{e}}_z$, then

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{\mathbf{e}}_x - \frac{\partial A_z}{\partial x} \hat{\mathbf{e}}_y + \frac{\partial A_y}{\partial x} \hat{\mathbf{e}}_z$$

$$= -\frac{\mu_0 I}{2\pi} \frac{y}{x^2+y^2} \hat{\mathbf{e}}_x + \frac{\mu_0 I}{2\pi} \frac{x}{x^2+y^2} \hat{\mathbf{e}}_y$$

For the y-component, $-\frac{\partial A_z}{\partial x} = \frac{\mu_0 I}{2\pi} \frac{x}{x^2+y^2}$, so $A_z = -\frac{\mu_0 I}{4\pi} \ln(x^2+y^2) + C_1(y,z)$. For the z-component, $\frac{\partial A_y}{\partial x} = 0$, so $A_y = C_2(y,z)$. Substituting into the equation of the x-component, we get $-\frac{\mu_0 I}{2\pi} \frac{y}{x^2+y^2} + \frac{\partial C_1}{\partial y} - \frac{\partial C_2}{\partial z} = -\frac{\mu_0 I}{2\pi} \frac{y}{x^2+y^2}$, so the equation will be satisfied if we simply choose $C_1 = C_2 = 0$. Therefore,

$$\mathbf{A} = -\frac{\mu_0 I}{4\pi} \ln(x^2+y^2) \hat{\mathbf{e}}_z$$

is a vector potential of $\mathbf{B}$.

3.9.5 $\nabla \cdot \mathbf{B} = 3\frac{1}{r^3} - 3\frac{x^2+y^2+z^2}{r^5} = 0$, so the vector potential exist. As in Exercise 3.9.4, we can make one component of the vector potential be zero, and we choose the z-component. So $\mathbf{A} = A_x \hat{\mathbf{e}}_x + A_y \hat{\mathbf{e}}_y$, and

$$\nabla \times \mathbf{A} = -\frac{\partial A_y}{\partial z} \hat{\mathbf{e}}_x + \frac{\partial A_x}{\partial z} \hat{\mathbf{e}}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{\mathbf{e}}_z$$

$$= \frac{x}{r^3} \hat{\mathbf{e}}_x + \frac{y}{r^3} \hat{\mathbf{e}}_y + \frac{z}{r^3} \hat{\mathbf{e}}_z$$

so

$$-\frac{\partial A_y}{\partial z} = \frac{x}{r^3} \xrightarrow{\textit{integrating}} A_y = \frac{-xz}{(x^2+y^2)r} + C_1(x,y)$$

$$\frac{\partial A_x}{\partial z} = \frac{y}{r^3} \xrightarrow{\textit{integrating}} A_x = \frac{yz}{(x^2+y^2)r} + C_2(x,y)$$

Substituted into $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{z}{r^3}$, we get

$$\frac{z}{r^3} + \frac{\partial C_1}{\partial x} - \frac{\partial C_2}{\partial y} = \frac{z}{r^3}$$

which will be satisfied if we simply choose $C_1 = C_2 = 0$. Therefore,

$$\mathbf{A} = \frac{yz}{(x^2+y^2)r} \hat{\mathbf{e}}_x - \frac{xz}{(x^2+y^2)r} \hat{\mathbf{e}}_y$$

is a solution of $\nabla \times \mathbf{A} = \mathbf{B}$

3.9.6 If $\mathbf{B}$ is a constant vector, then

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{A} &= \frac{1}{2}\boldsymbol{\nabla}\times(\mathbf{B}\times\mathbf{r})\\ &= \frac{1}{2}\boldsymbol{\nabla}_{\mathbf{r}}\times(\mathbf{B}\times\mathbf{r})\\ &= \frac{1}{2}[\mathbf{B}(\boldsymbol{\nabla}_{\mathbf{r}}\cdot\mathbf{r})-(\mathbf{B}\cdot\boldsymbol{\nabla}_{\mathbf{r}})\mathbf{r}]\\ &= \frac{1}{2}[3\mathbf{B}-\mathbf{B}]=\mathbf{B}\end{aligned}$$

So the two equations are satisfied by any constant vector $\mathbf{B}$.

3.9.7 (a) $\boldsymbol{\nabla}\cdot\mathbf{B}=\boldsymbol{\nabla}\cdot((\boldsymbol{\nabla}u)\times(\boldsymbol{\nabla}v))=(\boldsymbol{\nabla}v)\cdot(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}u))-(\boldsymbol{\nabla}u)\cdot(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}v))=0$ because $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}u)=0$ and $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}v)=0$.

(b)

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{A} &= \frac{1}{2}\boldsymbol{\nabla}\times(u\boldsymbol{\nabla}v)-\frac{1}{2}\boldsymbol{\nabla}\times(v\boldsymbol{\nabla}u)\\ &= \frac{1}{2}(\boldsymbol{\nabla}u)\times(\boldsymbol{\nabla}v)+\frac{1}{2}u(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}v))-\frac{1}{2}(\boldsymbol{\nabla}v)\times(\boldsymbol{\nabla}u)-\frac{1}{2}v(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}u))\\ &= (\boldsymbol{\nabla}u)\times(\boldsymbol{\nabla}v)=\mathbf{B}\end{aligned}$$

3.9.8 Let $\mathbf{A}'=\mathbf{A}+\boldsymbol{\nabla}\varphi$, then $\mathbf{B}'=\boldsymbol{\nabla}\times(\mathbf{A}+\boldsymbol{\nabla}\varphi)=\boldsymbol{\nabla}\times\mathbf{A}=\mathbf{B}$ because $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\varphi)=0$, so the left side of the equation is unchanged. $\oint\mathbf{A}'\cdot d\mathbf{r}=\oint\mathbf{A}\cdot d\mathbf{r}+\oint\boldsymbol{\nabla}\varphi\cdot d\mathbf{r}=\oint\mathbf{A}\cdot d\mathbf{r}$ because $\oint\boldsymbol{\nabla}\varphi\cdot d\mathbf{r}=\oint d\varphi=0$, so the right side of the equation is unchanged.

3.9.9 Choose point P to be the origin of the coordinate system $(0,0,0)$. Let $u=\frac{1}{r}$, $v=\varphi$, and apply Green's theorem Eq. 3.85,

$$\int_V\left(\frac{1}{r}\boldsymbol{\nabla}^2\varphi-\varphi\boldsymbol{\nabla}^2(\frac{1}{r})\right)d\tau=\oint_{\partial V}\left(\frac{1}{r}\boldsymbol{\nabla}\varphi-\varphi\boldsymbol{\nabla}(\frac{1}{r})\right)\cdot d\boldsymbol{\sigma}$$

where the volume V is a sphere centered at $(0,0,0)$ with radius r. Because there are no charges on or within the sphere, we have $\boldsymbol{\nabla}^2\varphi=0$, and by Eq. 3.120, $\boldsymbol{\nabla}^2(\frac{1}{r})=-4\pi\delta(\mathbf{r})$. So the left side of the equation equals to $\int_V 4\pi\varphi\delta(\mathbf{r})d\tau=4\pi\varphi(0)$. As for the right side,

$$\oint_{\partial V}\frac{1}{r}\boldsymbol{\nabla}\varphi\cdot d\boldsymbol{\sigma}=\frac{1}{r}\int_V\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\varphi)d\tau=\frac{1}{r}\int_V\boldsymbol{\nabla}^2\varphi d\tau=0$$

$$-\oint_V\varphi\boldsymbol{\nabla}(\frac{1}{r})\cdot d\boldsymbol{\sigma}=\oint_V\varphi\frac{1}{r^2}\hat{\mathbf{r}}\cdot d\boldsymbol{\sigma}=\frac{\oint_V\varphi d\sigma}{r^2}$$

so the equation becomes $4\pi\varphi(0)=\frac{\oint_V\varphi d\sigma}{r^2}$, so $\varphi(0)=\frac{\oint_V\varphi d\sigma}{4\pi r^2}$, which means the potential at P is the average of the potential over the spherical surface centered on P with radius r.

3.9.10 $\boldsymbol{\nabla}\times\mathbf{B}=\mu\mathbf{J}$ and $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$, so $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})=\mu\mathbf{J}$. But $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})-(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\mathbf{A}=-\boldsymbol{\nabla}^2\mathbf{A}$ because $\boldsymbol{\nabla}\cdot\mathbf{A}=0$, so $\boldsymbol{\nabla}^2\mathbf{A}=-\mu\mathbf{J}$.

3.9.11 From the Maxwell's equations, $\boldsymbol{\nabla}\times\mathbf{B}=\mu_0\mathbf{J}+\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$, so

$$\begin{aligned}&\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})\\ &=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})-(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\mathbf{A}=\mu_0\mathbf{J}+\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\end{aligned}$$

Using the Lorentz gauge Eq. 3.109,

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})=\boldsymbol{\nabla}(-\frac{1}{c^2}\frac{\partial\varphi}{\partial t})=-\frac{1}{c^2}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\varphi)=\frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t})=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}$$

Substitute into the first equation, we get

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\boldsymbol{\nabla}^2\mathbf{A}=\mu_0\mathbf{J}$$

3.9.12 As in Exercise 3.9.4, we can make one component of the vector potential be zero, so we make $\mathbf{A} = (A_x, 0, A_z)$. So

$$\boldsymbol{\nabla} \times \mathbf{A} = \frac{\partial A_z}{\partial y}\hat{\mathbf{e}}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{e}}_y - \frac{\partial A_x}{\partial y}\hat{\mathbf{e}}_z = B_x\hat{\mathbf{e}}_x + B_y\hat{\mathbf{e}}_y + B_z\hat{\mathbf{e}}_z$$

For the x- and z-components,

$$\frac{\partial A_z}{\partial y} = B_x \xrightarrow{integrating} A_z = \int_{y_0}^{y} B_x(x,y,z)dy + C_1(x,y)$$

$$-\frac{\partial A_x}{\partial y} = B_z \xrightarrow{integrating} A_x = -\int_{y_0}^{y} B_z(x,y,z)dy + C_2(x,y)$$

Substitute into the equation of y-component,

$$-\frac{\partial}{\partial z}\int_{y_0}^{y} B_z dy + \frac{\partial C_2}{\partial z} - \frac{\partial}{\partial x}\int_{y_0}^{y} B_x dy - \frac{\partial C_1}{\partial x} = B_y$$

Let $y = y_0$, and let $C_2 = 0$, then

$$-\frac{\partial C_1}{\partial x} = B_y(x, y_0, z) \xrightarrow{integrating} C_1 = -\int_{x_0}^{x} B_y(x, y_0, z)dx$$

Therefore,

$$\mathbf{A} = -\hat{\mathbf{e}}_x\int_{y_0}^{y} B_z(x,y,z)dy + \hat{\mathbf{e}}_z\left[\int_{y_0}^{y} B_x(x,y,z)dy - \int_{x_0}^{x} B_y(x,y_0,z)dx\right]$$

3.10 Curvilinear Coordinates

3.10.1 (a) The surfaces of $u = constant$ are hyperbolas with $x = 0$ and $y = 0$ as asymptotes when viewing from the z-axis. The surfaces of $v = constant$ are hyperbolas with $y = x$ and $y = -x$ as asymptotes when viewing from the z-axis. The surfaces of $z = constant$ are surfaces parallel to the x-y plane.

(b)

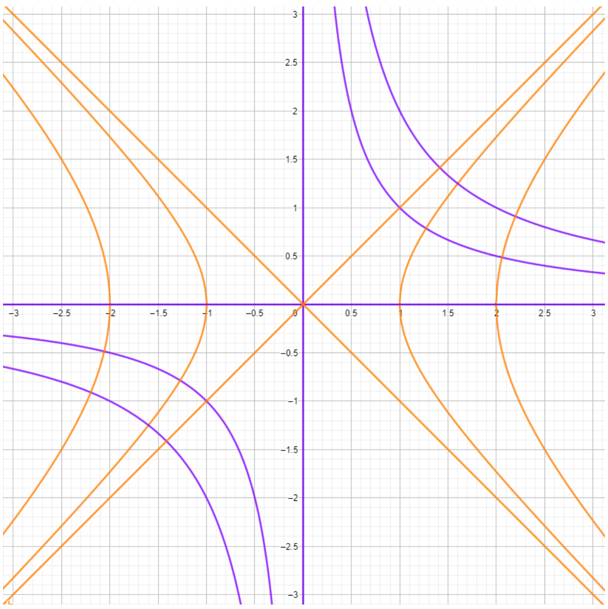

Figure 1: The purple lines are $xy = 0$, $xy = 1$, $xy = 2$
the orange lines are $x^2 - y^2 = 0$, $x^2 - y^2 = 1$, $x^2 - y^2 = 2$.

(c) Take the derivative of $xy = u$ we get $ydx + xdy = 0$. That is, $(y, x) \cdot (dx, dy) = 0$, so (y, x) is a normal vector of $xy = u$ and therefore in the direction of $\hat{\mathbf{e}}_u$. Because $\frac{\partial u}{\partial x} = y$, so $\hat{\mathbf{e}}_u$ should be in the direction of (y, x), not $(-y, -x)$. Normalizing, we get $\hat{\mathbf{e}}_u = \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_x + \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_y$. By a similar process, from $2xdx - 2ydy = 0$, we get $\hat{\mathbf{e}}_v = \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_x + \frac{-y}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_y$.

(d) $\hat{\mathbf{e}}_u \times \hat{\mathbf{e}}_v = \frac{-x^2-y^2}{x^2+y^2}\hat{\mathbf{e}}_z = -\hat{\mathbf{e}}_z$, so it is a left-handed system.

3.10.2

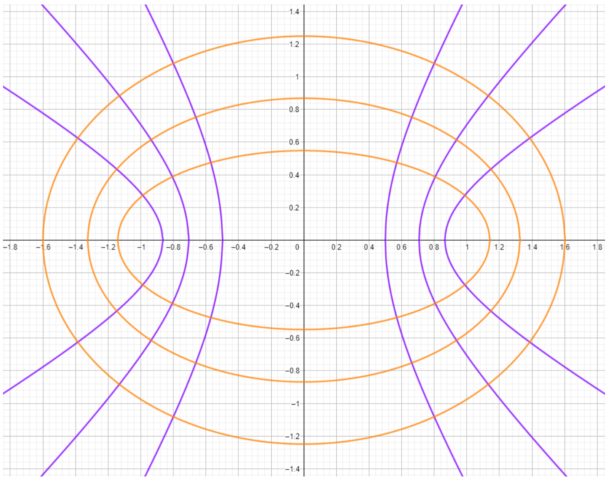

Figure 2: The orange lines are $u = \frac{\pi}{6}$, $u = \frac{\pi}{4}$, $u = \frac{\pi}{3}$
the purple lines are $v = \frac{\pi}{6}$, $v = \frac{\pi}{4}$, $v = \frac{\pi}{3}$

The unit vectors $\hat{\mathbf{e}}_u$ and $\hat{\mathbf{e}}_v$ are perpendicular to the lines, and pointing right at $x > 0$ and pointing left at $x < 0$.

3.10.3 The unit vectors of orthogonal coordinates are perpendicular to each other, so we have $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$, and $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \hat{\mathbf{e}}_k$. Therefore, $(A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3) \cdot (B_1\hat{\mathbf{e}}_1 + B_2\hat{\mathbf{e}}_2 + B_3\hat{\mathbf{e}}_3) = A_1B_1 + A_2B_2 + A_3B_3$, and

$$(A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3) \times (B_1\hat{\mathbf{e}}_1 + B_2\hat{\mathbf{e}}_2 + B_3\hat{\mathbf{e}}_3) = (A_2B_3 - A_3B_2)\hat{\mathbf{e}}_1 + (A_3B_1 - A_1B_3)\hat{\mathbf{e}}_2 + (A_1B_2 - A_2B_1)\hat{\mathbf{e}}_3$$

3.10.4 (a) $\hat{\mathbf{e}}_1 = 1\hat{\mathbf{e}}_1 + 0\hat{\mathbf{e}}_2 + 0\hat{\mathbf{e}}_3$. Using the divergence formula for curvilinear coordinates,

$$\boldsymbol{\nabla} \cdot \varphi = \frac{1}{h_1h_2h_3}\frac{\partial(h_2h_3)}{\partial q_1}$$

(b)

$$\boldsymbol{\nabla} \times \hat{\mathbf{e}}_1 = \frac{1}{h_1h_2h_3}\begin{vmatrix} \hat{\mathbf{e}}_1h_1 & \hat{\mathbf{e}}_2h_2 & \hat{\mathbf{e}}_3h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 & 0 & 0 \end{vmatrix}$$
$$= \frac{1}{h_1h_2h_3}\left[\hat{\mathbf{e}}_2h_2\frac{\partial h_1}{\partial q_3} - \hat{\mathbf{e}}_3h_3\frac{\partial h_1}{\partial q_2}\right]$$
$$= \frac{1}{h_1}\left[\hat{\mathbf{e}}_2\frac{1}{h_3}\frac{\partial h_1}{\partial q_3} - \hat{\mathbf{e}}_3\frac{1}{h_2}\frac{\partial h_1}{\partial q_2}\right]$$

***3.10.5** $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = 1 = \frac{1}{h_i^2}\frac{\partial\mathbf{r}}{\partial q_i} \cdot \frac{\partial\mathbf{r}}{\partial q_i} = \frac{1}{h_i^2}\left((\frac{\partial x}{\partial q_1})^2 + (\frac{\partial y}{\partial q_1})^2 + (\frac{\partial x}{\partial q_1})^2\right)$, so $h_i^2 = (\frac{\partial x}{\partial q_i})^2 + (\frac{\partial y}{\partial q_i})^2 + (\frac{\partial x}{\partial q_i})^2$, in agreement with Eq. 3.131.

From $h_i\hat{\mathbf{e}}_i = \frac{\partial\mathbf{r}}{\partial q_i}$, we can get

$$\frac{\partial h_i}{\partial q_j}\hat{\mathbf{e}}_i + h_i\frac{\partial\hat{\mathbf{e}}_i}{\partial q_j} = \frac{\partial(h_i\hat{\mathbf{e}}_i)}{\partial q_j} = \frac{\partial^2\mathbf{r}}{\partial q_j\partial q_i} = \frac{\partial^2\mathbf{r}}{\partial q_i\partial q_j} = \frac{\partial(h_j\hat{\mathbf{e}}_j)}{\partial q_i} = \frac{\partial h_j}{\partial q_i}\hat{\mathbf{e}}_j + h_j\frac{\partial\hat{\mathbf{e}}_j}{\partial q_i}$$

so

$$h_i \frac{\partial \hat{\mathbf{e}}_i}{\partial q_j} - \frac{\partial h_j}{\partial q_i} \hat{\mathbf{e}}_j = h_j \frac{\partial \hat{\mathbf{e}}_j}{\partial q_i} - \frac{\partial h_i}{\partial q_j} \hat{\mathbf{e}}_i = \mathbf{a}$$

If $\mathbf{a} = 0$, then $\frac{\partial \hat{\mathbf{e}}_i}{\partial q_j} = \frac{1}{h_i} \frac{\partial h_j}{\partial q_i} \hat{\mathbf{e}}_j$ can be proved. However, I don't know how to prove it. (I know $\mathbf{a} \cdot \hat{\mathbf{e}}_i = \mathbf{a} \cdot \hat{\mathbf{e}}_j = 0$, and also by taking $\mathbf{a} \cdot \mathbf{a}$, it is equivalent to prove $\frac{\partial \hat{\mathbf{e}}_i}{\partial q_j} \cdot \frac{\partial \hat{\mathbf{e}}_j}{\partial q_i} = 0$, however, I can't find a proof for this either)

From $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = 0$, we have $\frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} \cdot \hat{\mathbf{e}}_j + \hat{\mathbf{e}}_i \cdot \frac{\partial \hat{\mathbf{e}}_j}{\partial q_i} = 0$, so we have

$$\frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} \cdot \hat{\mathbf{e}}_j = -\hat{\mathbf{e}}_i \cdot \frac{\partial \hat{\mathbf{e}}_j}{\partial q_i} = -\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i \frac{1}{h_j} \frac{\partial h_i}{\partial q_j} = -\frac{1}{h_j} \frac{\partial h_i}{\partial q_j}$$

Also, $\frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} \cdot \hat{\mathbf{e}}_j = \frac{1}{2} \frac{\partial (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i)}{\partial q_i} = 0$, so

$$\frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} = \sum_{j \neq i} \hat{\mathbf{e}}_j \left(\frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} \cdot \hat{\mathbf{e}}_j \right) = -\sum_{j \neq i} \hat{\mathbf{e}}_j \frac{1}{h_j} \frac{\partial h_i}{\partial q_j}$$

3.10.6 $\mathbf{r} = \rho \cos\varphi \hat{\mathbf{e}}_x + \rho \sin\varphi \hat{\mathbf{e}}_y + z \hat{\mathbf{e}}_z$

$\frac{\partial \mathbf{r}}{\partial \rho} = \cos\varphi \hat{\mathbf{e}}_x + \sin\varphi \hat{\mathbf{e}}_y = h_\rho \hat{\mathbf{e}}_\rho$, so $h_\rho = 1$ and $\hat{\mathbf{e}}_\rho = \cos\varphi \hat{\mathbf{e}}_x + \sin\varphi \hat{\mathbf{e}}_y$
$\frac{\partial \mathbf{r}}{\partial \varphi} = -\rho \sin\varphi \hat{\mathbf{e}}_x + \rho \cos\varphi \hat{\mathbf{e}}_y = h_\varphi \hat{\mathbf{e}}_\varphi$, so $h_\varphi = \rho$ and $\hat{\mathbf{e}}_\varphi = -\sin\varphi \hat{\mathbf{e}}_x + \cos\varphi \hat{\mathbf{e}}_y$
$\frac{\partial \mathbf{r}}{\partial z} = \hat{\mathbf{e}}_z = h_z \hat{\mathbf{e}}_z$, so $h_z = 1$ and $\hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z$

3.10.7 From Exercise 3.10.6,
$\cos\varphi \hat{\mathbf{e}}_\rho - \sin\varphi \hat{\mathbf{e}}_\varphi = (\cos^2\varphi + \sin^2\varphi) \hat{\mathbf{e}}_x = \hat{\mathbf{e}}_x$, so $\hat{\mathbf{e}}_x = \cos\varphi \hat{\mathbf{e}}_\rho - \sin\varphi \hat{\mathbf{e}}_\varphi$
$\sin\varphi \hat{\mathbf{e}}_\rho + \cos\varphi \hat{\mathbf{e}}_\varphi = (\sin^2\varphi + \cos^2\varphi) \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_y$, so $\hat{\mathbf{e}}_y = \sin\varphi \hat{\mathbf{e}}_\rho + \cos\varphi \hat{\mathbf{e}}_\varphi$
$\hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z$

3.10.8 From exercise 3.10.6, $\frac{\partial \hat{\mathbf{e}}_\rho}{\partial \varphi} = -\hat{\mathbf{e}}_x \sin\varphi + \hat{\mathbf{e}}_y \cos\varphi = \hat{\mathbf{e}}_\varphi$, and $\frac{\partial \hat{\mathbf{e}}_\varphi}{\partial \varphi} = -\hat{\mathbf{e}}_x \cos\varphi - \hat{\mathbf{e}}_y \sin\varphi = -\hat{\mathbf{e}}_\rho$. All the other derivatives vanish because $\hat{\mathbf{e}}_\rho$ and $\hat{\mathbf{e}}_\varphi$ are functions of φ only, and $\hat{\mathbf{e}}_z$ is a constant vector.

3.10.9 From exercise 3.10.8, $\frac{\partial \hat{\mathbf{e}}_\rho}{\partial \varphi} = \hat{\mathbf{e}}_\varphi$ and $\frac{\partial \hat{\mathbf{e}}_\varphi}{\partial \varphi} = -\hat{\mathbf{e}}_\rho$, so

$$\boldsymbol{\nabla} \cdot \mathbf{V} = \left(\hat{\mathbf{e}}_\rho \frac{\partial}{\partial \rho} + \hat{\mathbf{e}}_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}\right) \cdot (\hat{\mathbf{e}}_\rho V_\rho + \hat{\mathbf{e}}_\varphi V_\varphi + \hat{\mathbf{e}}_z V_z)$$

$$= \hat{\mathbf{e}}_\rho \cdot \left(\hat{\mathbf{e}}_\rho \frac{\partial V_\rho}{\partial \rho} + \hat{\mathbf{e}}_\varphi \frac{\partial V_\rho}{\partial \rho} + \hat{\mathbf{e}}_z \frac{\partial V_z}{\partial \rho}\right) + \frac{1}{\rho} \hat{\mathbf{e}}_\varphi \cdot \left(\hat{\mathbf{e}}_\varphi V_\rho + \hat{\mathbf{e}}_\rho \frac{\partial V_\rho}{\partial \varphi} - \hat{\mathbf{e}}_\rho V_\varphi + \hat{\mathbf{e}}_\varphi \frac{\partial V_\varphi}{\partial \varphi} + \hat{\mathbf{e}}_z \frac{\partial V_z}{\partial \varphi}\right) + \hat{\mathbf{e}}_z \cdot \left(\hat{\mathbf{e}}_z \frac{\partial V_z}{\partial z}\right)$$

$$= \frac{\partial V_\rho}{\partial \rho} + \frac{V_\rho}{\rho} + \frac{1}{\rho} \frac{\partial V_\varphi}{\partial \varphi} + \frac{\partial V_z}{\partial z}$$

$$= \frac{1}{\rho} \frac{\partial (\rho V_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial V_\varphi}{\partial \varphi} + \frac{\partial V_z}{\partial z}$$

3.10.10 (a) From exercise 3.10.6, $\hat{\mathbf{e}}_\rho \rho + \hat{\mathbf{e}}_z z = \hat{\mathbf{e}}_x \rho \cos\varphi + \hat{\mathbf{e}}_y \rho \sin\varphi + \hat{\mathbf{e}}_z z = \hat{\mathbf{e}}_x x + \hat{\mathbf{e}}_y y + \hat{\mathbf{e}}_z z = \mathbf{r}$
(b)

$$\boldsymbol{\nabla} \cdot \mathbf{r} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho^2) + \frac{\partial V_z}{\partial z} = 2 + 1 = 3$$

$$\boldsymbol{\nabla} \times \mathbf{r} = \frac{1}{\rho} \begin{vmatrix} \hat{\mathbf{e}}_\rho & \hat{\mathbf{e}}_\varphi \rho & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ \rho & 0 & z \end{vmatrix} = 0$$

3.10.11 (a) A point $P = (\rho\cos\varphi, \rho\sin\varphi, z)$ after reflection would be $P' = (-\rho\cos\varphi, -\rho\sin\varphi, -z) = (\rho\cos(\varphi\pm\pi), \rho\sin(\varphi\pm\pi), -z)$, so it corresponds to the transformation

$$\rho \to \rho, \quad \varphi \to \varphi \pm \pi, \quad z \to -z$$

(b)

$$\begin{aligned}
\hat{\mathbf{e}}'_\rho &= \hat{\mathbf{e}}_x \cos(\varphi\pm\pi) + \hat{\mathbf{e}}_y \sin(\varphi\pm\pi) = -\hat{\mathbf{e}}_x\cos\varphi - \hat{\mathbf{e}}_y\sin\varphi = -\hat{\mathbf{e}}_\rho \\
\hat{\mathbf{e}}'_\varphi &= -\hat{\mathbf{e}}_x \sin(\varphi\pm\pi) + \hat{\mathbf{e}}_y \cos(\varphi\pm\pi) = \hat{\mathbf{e}}_x\sin\varphi - \hat{\mathbf{e}}_y\cos\varphi = -\hat{\mathbf{e}}_\varphi \\
\hat{\mathbf{e}}'_z &= \hat{\mathbf{e}}_z
\end{aligned}$$

3.10.12 (a)

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r} = (\omega\hat{\mathbf{e}}_z)\times(\rho\hat{\mathbf{e}}_\varphi + z\hat{\mathbf{e}}_z) = \omega\rho\hat{\mathbf{e}}_\varphi$$

(b)

$$\boldsymbol{\nabla}\times\mathbf{v} = \frac{1}{\rho}\begin{vmatrix} \hat{\mathbf{e}}_\rho & \hat{\mathbf{e}}_\varphi\rho & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ 0 & \omega\rho^2 & 0 \end{vmatrix} = \frac{1}{\rho}(2\omega\rho)\hat{\mathbf{e}}_z = 2\boldsymbol{\omega}$$

3.10.13

$$\begin{aligned}
\frac{d\hat{\mathbf{e}}_\rho}{dt} &= -\hat{\mathbf{e}}_x\dot\varphi\sin\varphi + \hat{\mathbf{e}}_y\dot\varphi\cos\varphi = \hat{\mathbf{e}}_\varphi\dot\varphi \\
\frac{d\hat{\mathbf{e}}_\varphi}{dt} &= -\hat{\mathbf{e}}_x\dot\varphi\cos\varphi - \hat{\mathbf{e}}_y\dot\varphi\sin\varphi = -\hat{\mathbf{e}}_\rho\dot\varphi \\
\frac{d\hat{\mathbf{e}}_z}{dt} &= 0 \\
\mathbf{r} &= \hat{\mathbf{e}}_\rho\rho + \hat{\mathbf{e}}_z z \\
\mathbf{v} = \dot{\mathbf{r}} &= \hat{\mathbf{e}}_\varphi\dot\varphi\rho + \hat{\mathbf{e}}_\rho\dot\rho + \hat{\mathbf{e}}_z\dot z \\
&= \hat{\mathbf{e}}_\rho\dot\rho + \hat{\mathbf{e}}_\varphi\rho\dot\varphi + \hat{\mathbf{e}}_z\dot z \\
\mathbf{a} = \dot{\mathbf{v}} &= \hat{\mathbf{e}}_\varphi\dot\varphi\dot\rho + \hat{\mathbf{e}}_\rho\ddot\rho - \hat{\mathbf{e}}_\rho\dot\varphi\rho\dot\varphi + \hat{\mathbf{e}}_\rho(\dot\rho\dot\varphi + \rho\ddot\varphi) + \hat{\mathbf{e}}_z\ddot z \\
&= \hat{\mathbf{e}}_\rho(\ddot\rho - \rho\dot\varphi^2) + \hat{\mathbf{e}}_\varphi(\rho\ddot\varphi + 2\dot\rho\dot\varphi) + \hat{\mathbf{e}}_z\ddot z
\end{aligned}$$

3.10.14

$$\boldsymbol{\nabla}\times\mathbf{v} = \frac{1}{\rho}\begin{vmatrix} \hat{\mathbf{e}}_\rho & \hat{\mathbf{e}}_\varphi\rho & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ V_\rho(\rho,\varphi) & \rho V_\varphi(\rho,\varphi) & 0 \end{vmatrix} = \frac{1}{\rho}\left[\hat{\mathbf{e}}_z\left(\frac{\partial(\rho V_\varphi)}{\partial\rho} - \frac{\partial V_\rho}{\partial\varphi}\right)\right]$$

3.10.15

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\mathbf{e}}_\rho & \hat{\mathbf{e}}_\varphi\rho & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\mu I}{2\pi}\ln(\frac{1}{\rho}) \end{vmatrix} = \frac{1}{\rho}\left[-\hat{\mathbf{e}}_\varphi\rho\frac{\mu I}{2\pi}\rho\frac{-1}{\rho^2}\right] = \hat{\mathbf{e}}_\varphi\frac{\mu I}{2\pi\rho}$$

3.10.16 (a) From exercise 3.10.7, we have

$$\mathbf{F} = -(\hat{\mathbf{e}}_\rho \cos\varphi - \hat{\mathbf{e}}_\varphi \sin\varphi)\frac{\rho\sin\varphi}{\rho^2} + (\hat{\mathbf{e}}_\rho \sin\varphi + \hat{\mathbf{e}}_\varphi \cos\varphi)\frac{\rho\cos\varphi}{\rho^2} = \hat{\mathbf{e}}_\rho \frac{1}{\rho}$$

(b)

$$\boldsymbol{\nabla}\times\mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\mathbf{e}}_\rho & \hat{\mathbf{e}}_\varphi\rho & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ 0 & \rho\frac{1}{\rho} & 0 \end{vmatrix} = 0$$

(c)

$$\oint(\hat{\mathbf{e}}_\varphi\frac{1}{\rho})\cdot(\hat{\mathbf{e}}_\rho d\rho + \hat{\mathbf{e}}_\varphi \rho d\varphi + \hat{\mathbf{e}}_z dz) = \oint d\varphi = 2\pi$$

(d) The range of φ of cylindrical coordinates is $0 \le \varphi < 2\pi$, so $\int_0^{2\pi} d\varphi$ is not defined.

3.10.17

$$(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B} = B_\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}(\hat{\mathbf{e}}_\varphi B_\varphi(\rho)) = \frac{B_\varphi}{\rho}[-\hat{\mathbf{e}}_\rho B_\varphi + 0] = -\hat{\mathbf{e}}_\rho\frac{B_\varphi^2}{\rho}$$

3.10.18

$$\frac{\partial\mathbf{r}}{\partial r} = \hat{\mathbf{e}}_x\sin\theta\cos\varphi + \hat{\mathbf{e}}_y\sin\theta\sin\varphi + \hat{\mathbf{e}}_z\cos\theta = h_r\hat{\mathbf{e}}_r$$

so

$$h_r = 1, \quad \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_x\sin\theta\cos\varphi + \hat{\mathbf{e}}_y\sin\theta\sin\varphi + \hat{\mathbf{e}}_z\cos\theta$$

$$\frac{\partial\mathbf{r}}{\partial\theta} = \hat{\mathbf{e}}_x r\cos\theta\cos\varphi + \hat{\mathbf{e}}_y r\cos\theta\sin\varphi - \hat{\mathbf{e}}_z r\sin\theta = h_\theta\hat{\mathbf{e}}_\theta$$

so

$$h_\theta = r, \quad \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_x\cos\theta\cos\varphi + \hat{\mathbf{e}}_y\cos\theta\sin\varphi - \hat{\mathbf{e}}_z\sin\theta$$

$$\frac{\partial\mathbf{r}}{\partial\varphi} = -\hat{\mathbf{e}}_x r\sin\theta\sin\varphi + \hat{\mathbf{e}}_y r\sin\theta\cos\varphi = h_\varphi\hat{\mathbf{e}}_\varphi$$

so

$$h_\varphi = r\sin\theta, \quad \hat{\mathbf{e}}_\varphi = -\hat{\mathbf{e}}_x\sin\varphi + \hat{\mathbf{e}}_y\cos\varphi$$

3.10.19

$$\hat{\mathbf{e}}_r = \hat{\mathbf{e}}_x\sin\theta\cos\varphi + \hat{\mathbf{e}}_y\sin\theta\sin\varphi + \hat{\mathbf{e}}_z\cos\theta \tag{1}$$

$$\hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_x\cos\theta\cos\varphi + \hat{\mathbf{e}}_y\cos\theta\sin\varphi - \hat{\mathbf{e}}_z\sin\theta \tag{2}$$

$$\hat{\mathbf{e}}_\varphi = -\hat{\mathbf{e}}_x\sin\varphi + \hat{\mathbf{e}}_y\cos\varphi \tag{3}$$

$\cos\theta\times(1) - \sin\theta\times(2)$, we get

$$\hat{\mathbf{e}}_z = \hat{\mathbf{e}}_r\cos\theta - \hat{\mathbf{e}}_\theta\sin\theta$$

$\sin\theta\times(1) + \cos\theta\times(2)$, we get

$$\hat{\mathbf{e}}_r\sin\theta + \hat{\mathbf{e}}_\theta\cos\theta = \hat{\mathbf{e}}_x\cos\varphi + \hat{\mathbf{e}}_y\sin\varphi \tag{4}$$

$\cos\varphi\times(4) - \sin\varphi\times(3)$, we get

$$\hat{\mathbf{e}}_x = \hat{\mathbf{e}}_r\sin\theta\cos\varphi + \hat{\mathbf{e}}_\theta\cos\theta\cos\varphi - \hat{\mathbf{e}}_\varphi\sin\varphi$$

$\cos\varphi\times(3) + \sin\varphi\times(4)$, we get

$$\hat{\mathbf{e}}_y = \hat{\mathbf{e}}_r\sin\theta\sin\varphi + \hat{\mathbf{e}}_\theta\cos\theta\sin\varphi + \hat{\mathbf{e}}_\varphi\cos\varphi$$

3.10.20 (a) The point $\mathbf{r} = (0,0,0)$ is related to $\mathbf{r}' = (0,\theta,\varphi)$ for any $0 \le \theta \le \pi$ and $0 \le \varphi < 2\pi$. If $\mathbf{r}' = \mathrm{B}\mathbf{r}$, then $\mathbf{r} = (0,0,0)$ can only be related to $\mathbf{r}' = (0,0,0)$, a contradiction, so the matrix B cannot exist.
(If $x, y, z \neq 0$, then simply

$$\mathrm{B} = \begin{pmatrix} \frac{r}{x} & 0 & 0 \\ 0 & \frac{\theta}{y} & 0 \\ 0 & 0 & \frac{\varphi}{z} \end{pmatrix}$$

satisfies the condition.)

(b) From exercise 3.10.19,

$$\begin{aligned}\mathbf{V} &= \hat{\mathbf{e}}_x V_x + \hat{\mathbf{e}}_y V_y + \hat{\mathbf{e}}_z V_z \\ &= (\hat{\mathbf{e}}_r \sin\theta\cos\varphi + \hat{\mathbf{e}}_\theta \cos\theta\cos\varphi - \hat{\mathbf{e}}_\varphi \sin\varphi)V_x \\ &\quad +(\hat{\mathbf{e}}_r \sin\theta\sin\varphi + \hat{\mathbf{e}}_\theta \cos\theta\sin\varphi + \hat{\mathbf{e}}_\varphi \cos\varphi)V_y \\ &\quad +(\hat{\mathbf{e}}_r \cos\theta - \hat{\mathbf{e}}_\theta \sin\theta)V_z \\ &= \hat{\mathbf{e}}_r V_r + \hat{\mathbf{e}}_\theta V_\theta + \hat{\mathbf{e}}_\varphi V_\varphi\end{aligned}$$

which is

$$\begin{pmatrix} V_r \\ V_\theta \\ V_\varphi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

Let the matrix be M, and let its transpose be $\mathbf{M}^T$, then it can be verified that $\mathrm{MM}^T = \mathbf{1}$, so it is orthogonal.

3.10.21 (Let φ of spherical coordinate be ϕ, and φ of cylindrical coordinate remain the same.) The relations between the two coordinates are $\rho = r\sin\theta$, $\varphi = \phi$, $z = r\cos\theta$.

$$\begin{aligned}\hat{\mathbf{e}}_r &= \frac{\partial\mathbf{r}}{\partial r} = \frac{\partial\mathbf{r}}{\partial\rho}\frac{\partial\rho}{\partial r} + \frac{\partial\mathbf{r}}{\partial\varphi}\frac{\partial\varphi}{\partial r} + \frac{\partial\mathbf{r}}{\partial z}\frac{\partial z}{\partial r} = \hat{\mathbf{e}}_\rho \sin\theta + \hat{\mathbf{e}}_\varphi \rho\cdot 0 + \hat{\mathbf{e}}_z \cos\theta \\ &= \hat{\mathbf{e}}_\rho \sin\theta + \hat{\mathbf{e}}_z \cos\theta \\ \hat{\mathbf{e}}_\theta &= \frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta} = \frac{1}{r}\left(\frac{\partial\mathbf{r}}{\partial\rho}\frac{\partial\rho}{\partial\theta} + \frac{\partial\mathbf{r}}{\partial\varphi}\frac{\partial\varphi}{\partial\theta} + \frac{\partial\mathbf{r}}{\partial z}\frac{\partial z}{\partial\theta}\right) = \frac{1}{r}\hat{\mathbf{e}}_\rho r\cos\theta + \frac{1}{r}\hat{\mathbf{e}}_\varphi\rho\cdot 0 + \frac{1}{r}\hat{\mathbf{e}}_z(-r\sin\theta) \\ &= \hat{\mathbf{e}}_\rho\cos\theta - \hat{\mathbf{e}}_z\sin\theta \\ \hat{\mathbf{e}}_\phi &= \frac{1}{r\sin\theta}\frac{\partial\mathbf{r}}{\partial\phi} = \frac{1}{r\sin\theta}\left(\frac{\partial\mathbf{r}}{\partial\rho}\frac{\partial\rho}{\partial\phi} + \frac{\partial\phi}{\partial\varphi}\frac{\partial\varphi}{\partial\phi} + \frac{\partial\mathbf{r}}{\partial z}\frac{\partial z}{\partial\phi}\right) = \frac{1}{r\sin\theta}\hat{\mathbf{e}}_\varphi\rho\cdot 1 \\ &= \hat{\mathbf{e}}_\varphi\end{aligned}$$

so

$$\begin{aligned}\mathbf{V} &= V_r\hat{\mathbf{e}}_r + V_\theta\hat{\mathbf{e}}_\theta + V_\phi\hat{\mathbf{e}}_\phi \\ &= V_r(\hat{\mathbf{e}}_\rho\sin\theta + \hat{\mathbf{e}}_z\cos\theta) + V_\theta(\hat{\mathbf{e}}_\rho\cos\theta - \hat{\mathbf{e}}_z\sin\theta) + V_\phi\hat{\mathbf{e}}_\varphi \\ &= V_\rho\hat{\mathbf{e}}_\rho + V_\varphi\hat{\mathbf{e}}_\varphi + V_z\hat{\mathbf{e}}_z\end{aligned}$$

which is

$$\begin{pmatrix} V_\rho \\ V_\varphi \\ V_z \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} V_r \\ V_\theta \\ V_\phi \end{pmatrix}$$

Let the matrix be M. Note that it is orthogonal, so the inverse transformation is

$$\mathrm{M}^{-1} = \mathrm{M}^T = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{pmatrix}$$

3.10.22 (a)

$$\frac{\partial \hat{\mathbf{e}}_r}{\partial r} = 0 \qquad \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \hat{\mathbf{e}}_\theta \qquad \frac{\partial \hat{\mathbf{e}}_r}{\partial \varphi} = \hat{\mathbf{e}}_\varphi \sin\theta$$

$$\frac{\partial \hat{\mathbf{e}}_\theta}{\partial r} = 0 \qquad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r \qquad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \varphi} = \hat{\mathbf{e}}_\varphi \cos\theta$$

$$\frac{\partial \hat{\mathbf{e}}_\varphi}{\partial r} = 0 \qquad \frac{\partial \hat{\mathbf{e}}_\varphi}{\partial \theta} = 0 \qquad \frac{\partial \hat{\mathbf{e}}_\varphi}{\partial \varphi} = -\hat{\mathbf{e}}_r \sin\theta - \hat{\mathbf{e}}_\theta \cos\theta$$

(b)

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\psi &= (\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi})\cdot(\hat{\mathbf{e}}_r\frac{\partial\psi}{\partial r} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\mathbf{e}}_\varphi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}) \\
&= \hat{\mathbf{e}}_r\cdot\left(\hat{\mathbf{e}}_r\frac{\partial^2\psi}{\partial r^2} + \hat{\mathbf{e}}_\theta(\cdots) + \hat{\mathbf{e}}_\varphi(\cdots)\right) + \frac{1}{r}\hat{\mathbf{e}}_\theta\cdot\left(\hat{\mathbf{e}}_\theta\frac{\partial\psi}{\partial r} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial^2\psi}{\partial\theta^2} + \hat{\mathbf{e}}_r(\cdots) + \hat{\mathbf{e}}_\varphi(\cdots)\right) \\
&\quad + \frac{1}{r\sin\theta}\hat{\mathbf{e}}_\varphi\cdot\left(\hat{\mathbf{e}}_\varphi\sin\theta\frac{\partial\psi}{\partial r} + \hat{\mathbf{e}}_\varphi\cos\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\mathbf{e}}_\varphi\frac{1}{r\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2} + \hat{\mathbf{e}}_r(\cdots) + \hat{\mathbf{e}}_\theta(\cdots)\right) \\
&= \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} \\
&= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) + \frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right]
\end{aligned}$$

3.10.23 (a) $\boldsymbol{\omega} = \hat{\mathbf{e}}_z\omega = \hat{\mathbf{e}}_r\omega\cos\theta - \hat{\mathbf{e}}_\theta\omega\sin\theta$, so

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r} = (\hat{\mathbf{e}}_r\omega\cos\theta - \hat{\mathbf{e}}_\theta\omega\sin\theta)\times(\hat{\mathbf{e}}_r r) = \hat{\mathbf{e}}_\varphi\omega r\sin\theta$$

(b)

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{v} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & \hat{\mathbf{e}}_\theta r & \hat{\mathbf{e}}_\varphi r\sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ 0 & 0 & \omega r^2\sin^2\theta \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta}(\hat{\mathbf{e}}_r 2\omega r^2\sin\theta\cos\theta - \hat{\mathbf{e}}_\theta 2\omega r^2\sin^2\theta) \\
&= 2\hat{\mathbf{e}}_r\omega\cos\theta - 2\hat{\mathbf{e}}_\theta\omega\sin\theta = 2\boldsymbol{\omega}
\end{aligned}$$

3.10.24

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{V} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & \hat{\mathbf{e}}_\theta r & \hat{\mathbf{e}}_\varphi r\sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ 0 & V_\theta & V_\varphi \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta}\left[\hat{\mathbf{e}}_r(\frac{\partial V_\varphi}{\partial\theta} - \frac{\partial V_\theta}{\partial\varphi}) - \hat{\mathbf{e}}_\theta r(\frac{\partial V_\varphi}{\partial r}) + \hat{\mathbf{e}}_\varphi r\sin\theta(\frac{\partial V_\theta}{\partial r})\right]
\end{aligned}$$

has no tangential components, so $\frac{\partial V_\varphi}{\partial r} = \frac{\partial V_\theta}{\partial r} = 0$. That is, the tangential components of $\mathbf{V}$ have no radial dependence.

3.10.25 (a) A point $P = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$ after reflection would become $P' = (-r\sin\theta\cos\varphi, -r\sin\theta\sin\varphi, -r\cos\theta) = (r\sin(\pi-\theta)\cos(\varphi\pm\pi), r\sin(\pi-\theta)\sin(\varphi\pm\pi), r\cos(\pi-\theta))$, so it corresponds to the transformation

$$r\to r, \quad \theta\to\pi-\theta, \quad \varphi\to\varphi\pm\pi$$

(b)

$$\begin{aligned}
\hat{\mathbf{e}}'_r &= \hat{\mathbf{e}}_x\sin(\pi-\theta)\cos(\varphi\pm\pi) + \hat{\mathbf{e}}_y\sin(\pi-\theta)\sin(\varphi\pm\pi) + \hat{\mathbf{e}}_z\cos(\pi-\theta) = -\hat{\mathbf{e}}_r \\
\hat{\mathbf{e}}_\theta &= \hat{\mathbf{e}}_x\cos(\pi-\theta)\cos(\varphi\pm\pi) + \hat{\mathbf{e}}_y\cos(\pi-\theta)\sin(\varphi\pm\pi) - \hat{\mathbf{e}}_z\sin(\pi-\theta) = \hat{\mathbf{e}}_\theta \\
\hat{\mathbf{e}}_\varphi &= -\hat{\mathbf{e}}_x\sin(\varphi\pm\pi) + \hat{\mathbf{e}}_y\cos(\varphi\pm\pi) = -\hat{\mathbf{e}}_\varphi
\end{aligned}$$

3.10.26 (a)

$$(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{r} = (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z)$$
$$= A_x\hat{\mathbf{e}}_x + A_y\hat{\mathbf{e}}_y + A_z\hat{\mathbf{e}}_z = \mathbf{A}$$

(b) From exercise 3.10.22

$$(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{r} = (A_r\frac{\partial}{\partial r} + A_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + A_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi})(r\hat{\mathbf{e}}_r) = A_r\hat{\mathbf{e}}_r + A_\theta\frac{1}{r}r\hat{\mathbf{e}}_\theta + A_\varphi\frac{1}{r\sin\theta}r\sin\theta\hat{\mathbf{e}}_\varphi$$
$$= A_r\hat{\mathbf{e}}_r + A_\theta\hat{\mathbf{e}}_\theta + A_\varphi\hat{\mathbf{e}}_\varphi = \mathbf{A}$$

3.10.27 From exercise 3.10.22

$$\frac{d\hat{\mathbf{e}}_r}{dt} = \frac{\partial\hat{\mathbf{e}}_r}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial\hat{\mathbf{e}}_r}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial\hat{\mathbf{e}}_r}{\partial\varphi}\frac{\partial\varphi}{\partial t} = \hat{\mathbf{e}}_\theta\dot{\theta} + \hat{\mathbf{e}}_\varphi\sin\theta\dot{\varphi}$$
$$\frac{d\hat{\mathbf{e}}_\theta}{dt} = \frac{\partial\hat{\mathbf{e}}_\theta}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial\hat{\mathbf{e}}_\theta}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial\hat{\mathbf{e}}_\theta}{\partial\varphi}\frac{\partial\varphi}{\partial t} = -\hat{\mathbf{e}}_r\dot{\theta} + \hat{\mathbf{e}}_\varphi\cos\theta\dot{\varphi}$$
$$\frac{d\hat{\mathbf{e}}_\varphi}{dt} = \frac{\partial\hat{\mathbf{e}}_\varphi}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial\hat{\mathbf{e}}_\varphi}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial\hat{\mathbf{e}}_\varphi}{\partial\varphi}\frac{\partial\varphi}{\partial t} = -\hat{\mathbf{e}}_r\sin\theta\dot{\varphi} - \hat{\mathbf{e}}_\theta\cos\theta\dot{\varphi}$$
$$\mathbf{r} = \hat{\mathbf{e}}_r r$$

so

$$\mathbf{v} = \dot{\mathbf{r}} = \hat{\mathbf{e}}_\theta\dot{\theta}r + \hat{\mathbf{e}}_\varphi\sin\theta\dot{\varphi}r + \hat{\mathbf{e}}_r\dot{r}$$
$$= \hat{\mathbf{e}}_r\dot{r} + \hat{\mathbf{e}}_\theta r\dot{\theta} + \hat{\mathbf{e}}_\varphi r\sin\theta\dot{\varphi}$$
$$\mathbf{a} = \dot{\mathbf{v}} = \hat{\mathbf{e}}_\theta\dot{r}\dot{\theta} + \hat{\mathbf{e}}_\varphi\dot{r}\sin\theta\dot{\varphi} + \hat{\mathbf{e}}_r\ddot{r} - \hat{\mathbf{e}}_r r\dot{\theta}^2 + \hat{\mathbf{e}}_\varphi r\cos\theta\dot{\theta}\dot{\varphi} + \hat{\mathbf{e}}_\theta\dot{r}\dot{\theta} + \hat{\mathbf{e}}_\theta r\ddot{\theta}$$
$$-\hat{\mathbf{e}}_r r\sin^2\theta\dot{\varphi}^2 - \hat{\mathbf{e}}_\theta r\sin\theta\cos\theta\dot{\varphi}^2 + \hat{\mathbf{e}}_\varphi\dot{r}\sin\theta\dot{\varphi} + \hat{\mathbf{e}}_\varphi r\cos\theta\dot{\theta}\dot{\varphi} + \hat{\mathbf{e}}_\varphi r\sin\theta\ddot{\varphi}$$
$$= \hat{\mathbf{e}}_r(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\varphi}^2) + \hat{\mathbf{e}}_\theta(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\varphi}^2) + \hat{\mathbf{e}}_\varphi(r\sin\theta\ddot{\varphi} + 2\dot{r}\sin\theta\dot{\varphi} + 2r\cos\theta\dot{\theta}\dot{\varphi})$$

3.10.28

$$\boldsymbol{\nabla} = \hat{\mathbf{e}}_x\frac{\partial}{\partial x} + \hat{\mathbf{e}}_y\frac{\partial}{\partial y} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}$$
$$= \hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$
$$= (\hat{\mathbf{e}}_x\sin\theta\cos\varphi + \hat{\mathbf{e}}_y\sin\theta\sin\varphi + \hat{\mathbf{e}}_z\cos\theta)\frac{\partial}{\partial r}$$
$$+(\hat{\mathbf{e}}_x\cos\theta\cos\varphi + \hat{\mathbf{e}}_y\cos\theta\sin\varphi - \hat{\mathbf{e}}_z\sin\theta)\frac{1}{r}\frac{\partial}{\partial\theta}$$
$$+(-\hat{\mathbf{e}}_x\sin\varphi + \hat{\mathbf{e}}_y\cos\varphi)\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$
$$= \hat{\mathbf{e}}_x(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi})$$
$$+\hat{\mathbf{e}}_y(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi})$$
$$+\hat{\mathbf{e}}_z(\cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial\theta})$$

equating the x, y, z components, we get

$$\frac{\partial}{\partial x} = \sin\theta\cos\varphi\frac{\partial}{\partial r} + \cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$$
$$\frac{\partial}{\partial y} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$$
$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}$$

3.10.29 $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$. Using results from exercise 3.10.28, we can have

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = r\sin\theta\cos\varphi(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi})$$

$$-r\sin\theta\sin\varphi(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}) = \frac{\partial}{\partial\varphi}$$

so

$$-i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\frac{\partial}{\partial\varphi}$$

3.10.30

$$\mathbf{L} = -i(\mathbf{r}\times\boldsymbol{\nabla}) = -i(\hat{\mathbf{e}}_r r)\times(\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi})$$

$$= \hat{\mathbf{e}}_\theta i\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - \hat{\mathbf{e}}_\varphi i\frac{\partial}{\partial\theta}$$

$$= (\hat{\mathbf{e}}_x\cos\theta\cos\varphi + \hat{\mathbf{e}}_y\cos\theta\sin\varphi - \hat{\mathbf{e}}_z\sin\theta)i\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} + (\hat{\mathbf{e}}_x\sin\varphi - \hat{\mathbf{e}}_y\cos\varphi)i\frac{\partial}{\partial\theta}$$

$$= \hat{\mathbf{e}}_x(i\sin\varphi\frac{\partial}{\partial\theta} + i\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}) + \hat{\mathbf{e}}_y(-i\cos\varphi\frac{\partial}{\partial\theta} + i\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}) + \hat{\mathbf{e}}_z(-i\frac{\partial}{\partial\varphi})$$

so

$$L_x + iL_y = i\sin\varphi\frac{\partial}{\partial\theta} + i\cot\theta\cos\varphi\frac{\partial}{\partial\varphi} + \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}$$

$$= (\cos\varphi + i\sin\varphi)(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}) = e^{i\varphi}(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi})$$

$$L_x - iL_y = i\sin\varphi\frac{\partial}{\partial\theta} + i\cot\theta\cos\varphi\frac{\partial}{\partial\varphi} - \cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}$$

$$= (-\cos\varphi + i\sin\varphi)(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}) = -e^{-i\varphi}(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi})$$

3.10.31 From exercise 3.10.30

$$\mathbf{L} = \hat{\mathbf{e}}_x L_x + \hat{\mathbf{e}}_y L_y + \hat{\mathbf{e}}_z L_z$$

$$= \hat{\mathbf{e}}_x(i\sin\varphi\frac{\partial}{\partial\theta} + i\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}) + \hat{\mathbf{e}}_y(-i\cos\varphi\frac{\partial}{\partial\theta} + i\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}) + \hat{\mathbf{e}}_z(-i\frac{\partial}{\partial\varphi})$$

so

$$\mathbf{L}\times\mathbf{L} = \hat{\mathbf{e}}_x(L_yL_z - L_zL_y) + \hat{\mathbf{e}}_y(L_zL_x - L_zL_z) + \hat{\mathbf{e}}_z(L_xL_y - L_yL_x)$$

$$= \hat{\mathbf{e}}_x(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}) + \hat{\mathbf{e}}_y(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}) + \hat{\mathbf{e}}_z(\frac{\partial}{\partial\varphi})$$

$$= \hat{\mathbf{e}}_x iL_x + \hat{\mathbf{e}}_y iL_y + \hat{\mathbf{e}}_z iL_z = i\mathbf{L}$$

3.10.32 (a)(b) It is the first half of exercise 3.10.30.

(c) The author suggest to do it in Cartesian coordinate, but I think it's easier to do in spherical coordinate, with the help of results from 3.10.22(a).

$$\mathbf{L}^2 = \mathbf{L}\cdot\mathbf{L} = -(\hat{\mathbf{e}}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - \hat{\mathbf{e}}_\varphi\frac{\partial}{\partial\theta})\cdot(\hat{\mathbf{e}}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - \hat{\mathbf{e}}_\varphi\frac{\partial}{\partial\theta})$$

$$= -\left[\hat{\mathbf{e}}_\theta\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\varphi}(\hat{\mathbf{e}}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}) + \hat{\mathbf{e}}_\varphi\cdot\frac{\partial}{\partial\theta}(\hat{\mathbf{e}}_\varphi\frac{\partial}{\partial\theta}) - \hat{\mathbf{e}}_\theta\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\varphi}(\hat{\mathbf{e}}_\varphi\frac{\partial}{\partial\theta}) - \hat{\mathbf{e}}_\varphi\cdot\frac{\partial}{\partial\theta}(\hat{\mathbf{e}}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi})\right]$$

$$= -\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right]$$

$$= -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$= -r^2\boldsymbol{\nabla}^2 + \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

(We use Eq. 3.158 in the last equality.)

3.10.33 (a)

$$\hat{\mathbf{e}}_r\frac{\partial}{\partial r}-i\frac{\mathbf{r}\times\mathbf{L}}{r^2}=\hat{\mathbf{e}}_r\frac{\partial}{\partial r}-\frac{\mathbf{r}\times(\mathbf{r}\times\boldsymbol{\nabla})}{r^2}=\hat{\mathbf{e}}_r\frac{\partial}{\partial r}-\frac{\mathbf{r}(\mathbf{r}\cdot\boldsymbol{\nabla})-(\mathbf{r}\cdot\mathbf{r})\boldsymbol{\nabla}}{r^2}=\hat{\mathbf{e}}_r\frac{\partial}{\partial r}-\frac{\hat{\mathbf{e}}_r r(r\frac{\partial}{\partial r})-r^2\boldsymbol{\nabla}}{r^2}=\boldsymbol{\nabla}$$

(b) (*There is probably a mistake:* $\boldsymbol{\nabla}(1+r\frac{\partial}{\partial r})$ *should be* $\boldsymbol{\nabla}+\boldsymbol{\nabla}(r\frac{\partial}{\partial r})$)

$r\frac{\partial}{\partial r}=\mathbf{r}\cdot\boldsymbol{\nabla}$ in the spherical coordinate, so the left side of the equation is $\mathbf{r}\boldsymbol{\nabla}^2-\boldsymbol{\nabla}-\boldsymbol{\nabla}(\mathbf{r}\cdot\boldsymbol{\nabla})$.

$$\begin{aligned}\left[\mathbf{r}\boldsymbol{\nabla}^2-\boldsymbol{\nabla}-\boldsymbol{\nabla}(\mathbf{r}\cdot\boldsymbol{\nabla})\right]_x&=x(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})-\frac{\partial}{\partial x}-\frac{\partial}{\partial x}(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})\\&=-2\frac{\partial}{\partial x}+x\frac{\partial^2}{\partial y^2}+x\frac{\partial^2}{\partial z^2}-y\frac{\partial^2}{\partial x\partial y}-z\frac{\partial^2}{\partial x\partial z}\\ [i\boldsymbol{\nabla}\times\mathbf{L}]_x=[\boldsymbol{\nabla}\times(\mathbf{r}\times\boldsymbol{\nabla})]_x&=\frac{\partial}{\partial y}(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x})-\frac{\partial}{\partial z}(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z})\\&=-2\frac{\partial}{\partial x}+x\frac{\partial^2}{\partial y^2}+x\frac{\partial^2}{\partial z^2}-y\frac{\partial^2}{\partial x\partial y}-z\frac{\partial^2}{\partial x\partial z}\end{aligned}$$

So the x-components of two side of the equation are equal. It can be verified that so are the y- and z-components. Therefore,

$$\mathbf{r}\boldsymbol{\nabla}^2-\boldsymbol{\nabla}-\boldsymbol{\nabla}(\mathbf{r}\cdot\boldsymbol{\nabla})=i\boldsymbol{\nabla}\times\mathbf{L}$$

3.10.34

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\psi}{dr}\right]=\frac{1}{r^2}(2r\frac{d\psi}{dr}+r^2\frac{d^2\psi}{dr^2})=\frac{d^2\psi}{dr^2}+\frac{2}{r}\frac{d\psi}{dr}$$

$$\frac{1}{r}\frac{d^2}{dr^2}\left[r\psi\right]=\frac{1}{r}\frac{d}{dr}(\psi+r\frac{d\psi}{dr})=\frac{1}{r}(\frac{d\psi}{dr}+\frac{d\psi}{dr}+r\frac{d^2\psi}{dr^2})=\frac{d^2\psi}{dr^2}+\frac{2}{r}\frac{d\psi}{dr}$$

so all the three form are equaivalant.

3.10.35 (a)

$$\boldsymbol{\nabla}\times\mathbf{F}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\mathbf{e}}_r & \hat{\mathbf{e}}_\theta r & \hat{\mathbf{e}}_\varphi r\sin\theta\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi}\\ \frac{2P\cos\theta}{r^3} & \frac{P}{r^2}\sin\theta & 0\end{vmatrix}=\frac{1}{r^2\sin\theta}\left(\hat{\mathbf{e}}_\varphi r\sin\theta(-2\frac{P}{r^3}\sin\theta+\frac{2P\sin\theta}{r^3})\right)=0$$

(b) $r=1$ and $\theta=\frac{\pi}{2}$, so $\mathbf{F}=\hat{\mathbf{e}}_\theta P$ and $d\mathbf{r}=\hat{\mathbf{e}}_r dr+\hat{\mathbf{e}}_\varphi d\varphi$.

$$\oint\mathbf{F}\cdot d\mathbf{r}=(\hat{\mathbf{e}}_\theta P)\cdot(\hat{\mathbf{e}}_r dr+\hat{\mathbf{e}}_\varphi d\varphi)=0$$

We cannot assert whether $\mathbf{F}$ is conservative or not unless we evaluate every integral over closed loop.

(c) $\int_a^b\mathbf{F}\cdot d\mathbf{r}=\psi(a)-\psi(b)$. Take the path $(r,\theta,\varphi)\to(\infty,\theta,\varphi)$, and define the potential at infinity $\psi(\infty)$ to be zero. Then we have

$$\psi(\mathbf{r})=\psi(\mathbf{r})-\psi(\infty)=\int_r^\infty\frac{2P\cos\theta}{r^3}dr=-\frac{P\cos\theta}{r^2}\Big|_r^\infty=\frac{P\cos\theta}{r^2}$$

3.10.36 (a)

$$\boldsymbol{\nabla}\times\mathbf{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\mathbf{e}}_r & \hat{\mathbf{e}}_\theta r & \hat{\mathbf{e}}_\varphi r\sin\theta\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi}\\ 0 & 0 & -\cos\theta\end{vmatrix}=\frac{1}{r^2\sin\theta}(\hat{\mathbf{e}}_r\sin\theta)=\frac{\hat{\mathbf{e}}_r}{r^2}$$

(b) $r=\sqrt{x^2+y^2+z^2}$, $\theta=\cos^{-1}\frac{z}{r}$, $\varphi=\tan^{-1}\frac{y}{x}$, $\hat{\mathbf{e}}_\varphi=-\hat{\mathbf{e}}_x\sin\varphi+\hat{\mathbf{e}}_y\cos\varphi$. So

$$\mathbf{A}=-(-\hat{\mathbf{e}}_x\frac{y}{\sqrt{x^2+y^2}}+\hat{\mathbf{e}}_y\frac{x}{\sqrt{x^2+y^2}})\frac{z}{\sqrt{x^2+y^2}}\frac{1}{r}=\hat{\mathbf{e}}_x\frac{yz}{r(x^2+y^2)}-\hat{\mathbf{e}}_y\frac{xz}{r(x^2+y^2)}$$

(c)

$$\boldsymbol{\nabla}\times\mathbf{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\mathbf{e}}_r & \hat{\mathbf{e}}_\theta r & \hat{\mathbf{e}}_\varphi r\sin\theta\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi}\\ 0 & -\varphi\sin\theta & 0\end{vmatrix}=\frac{1}{r^2\sin\theta}(\hat{\mathbf{e}}_r\sin\theta)=\frac{\hat{\mathbf{e}}_r}{r^2}$$

3.10.37 $\mathbf{r}=\hat{\mathbf{e}}_r r$, so from exercise 3.10.22 we have $\frac{\partial\mathbf{r}}{\partial r}=\hat{\mathbf{e}}_r$, $\frac{\partial\mathbf{r}}{\partial\theta}=\hat{\mathbf{e}}_\theta r$, $\frac{\partial\mathbf{r}}{\partial\varphi}=\hat{\mathbf{e}}_\varphi r\sin\theta$. So

$$\mathbf{E}=-\boldsymbol{\nabla}\psi=-\left[\hat{\mathbf{e}}_r\frac{\partial}{\partial r}(\frac{\mathbf{P}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3})+\hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}(\frac{\mathbf{P}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3})+\hat{\mathbf{e}}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(\frac{\mathbf{P}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3})\right]$$

$$=-\hat{\mathbf{e}}_r\left(\frac{1}{4\pi\varepsilon_0 r^3}\mathbf{P}\cdot\frac{\partial\mathbf{r}}{\partial r}+\frac{\mathbf{P}\cdot\mathbf{r}}{4\pi\varepsilon_0}\frac{\partial}{\partial r}(\frac{1}{r^3})\right)-\hat{\mathbf{e}}_\theta\frac{1}{r}\frac{1}{4\pi\varepsilon_0 r^3}\mathbf{P}\cdot\frac{\partial\mathbf{r}}{\partial\theta}-\hat{\mathbf{e}}_\varphi\frac{1}{r\sin\theta}\frac{1}{4\pi\varepsilon_0 r^3}\mathbf{P}\cdot\frac{\partial\mathbf{r}}{\partial\varphi}$$

$$=-\hat{\mathbf{e}}_r(-2)\frac{P_r}{4\pi\varepsilon_0 r^3}-\hat{\mathbf{e}}_\theta\frac{P_\theta}{4\pi\varepsilon_0 r^3}-\hat{\mathbf{e}}_\varphi\frac{P_\varphi}{4\pi\varepsilon_0 r^3}$$

$$=\frac{1}{4\pi\varepsilon_0 r^3}(3\hat{\mathbf{e}}_r P_r-\hat{\mathbf{e}}_r P_r-\hat{\mathbf{e}}_\theta P_\theta-\hat{\mathbf{e}}_\varphi P_\varphi)$$

$$=\frac{3\hat{\mathbf{r}}(\mathbf{P}\cdot\hat{\mathbf{r}})}{4\pi\varepsilon_0 r^3}$$

where $\hat{\mathbf{r}}=\hat{\mathbf{e}}_r$ is the unit vector in the $\mathbf{r}$ direction.