Chapter 9
Partial Differential Equations

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9.2 First-Order Equations

9.2.1 Let $s = x + 2y$, $t = 2x - y$, then

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial s} + 2\frac{\partial\varphi}{\partial t}$$

$$\frac{\partial\varphi}{\partial y} = 2\frac{\partial\varphi}{\partial s} - \frac{\partial\varphi}{\partial t}$$

so the equation becomes

$$5\frac{\partial\varphi}{\partial s} + t\varphi = 0$$

$$\frac{1}{\varphi}d\varphi = -\frac{t}{5}ds$$

$$\ln\varphi = -\frac{ts}{5} + C(t)$$

$$\varphi = e^{-\frac{1}{5}(2x^2-2y^2+3xy)}f(2x-y)$$

Or note that $s + 2t = 5x$, so the solution can be transformed into

$$\varphi = e^{-\frac{st}{5}}e^{-\frac{2t^2}{5}}e^{\frac{2t^2}{5}}f(t) = e^{-\frac{5x\cdot t}{5}}g(t)$$

$$= e^{-2x^2+xy}g(2x-y)$$

9.2.2 Let $s = x - 2y$, $t = 2x + y$, then

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial s} + 2\frac{\partial\varphi}{\partial t}$$

$$\frac{\partial\varphi}{\partial y} = -2\frac{\partial\varphi}{\partial s} + \frac{\partial\varphi}{\partial t}$$

so the equation becomes

$$5\frac{\partial\varphi}{\partial s} + \frac{3t-s}{5} = 0$$

$$\frac{\partial\varphi}{\partial s} = \frac{s-3t}{25}$$

$$\varphi = \frac{s^2-6st}{50} + f(t)$$

$$= \frac{s^2-6st}{50} + \frac{9t^2}{50} - \frac{9t^2}{50} + f(t)$$

$$= \frac{(s-3t)^2}{50} + g(t)$$

$$= \frac{(x+y)^2}{2} + g(2x+y)$$

9.2.3 Let $s = x + y - z,\ t = x - y,\ u = x + y + 2z$, then

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial s} + \frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial u}$$
$$\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial s} - \frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial u}$$
$$\frac{\partial\varphi}{\partial x} = -\frac{\partial\varphi}{\partial s} + 2\frac{\partial\varphi}{\partial u}$$

so the equation becomes

$$3\frac{\partial\varphi}{\partial s} = 0$$
$$\varphi = f(t, u) = f(x - y, x + y + 2z)$$

or note that $t + u = 2(x + z)$, so the equation can be transformed into

$$\varphi = f(t, 2(x + z) - t) = g(t, x + z) = g(x - y, x + z)$$

9.2.4 Let $s = x + y + z,\ t = x - y,\ u = x + y - 2z$, then

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial s} + \frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial u}$$
$$\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial s} - \frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial u}$$
$$\frac{\partial\varphi}{\partial z} = \frac{\partial\varphi}{\partial s} - 2\frac{\partial\varphi}{\partial u}$$

so the equation becomes

$$3\frac{\partial\varphi}{\partial s} = t$$
$$\begin{aligned}\varphi &= \frac{st}{3} + f(t, u)\\ &= \frac{st}{3} - \frac{ut}{3} + \frac{ut}{3} + f(t, u)\\ &= \frac{3z \cdot t}{3} + g(t, u)\\ &= (x - y)z + g(x - y, x + y - 2z)\end{aligned}$$

9.2.5 (a)

$$\frac{\partial\varphi}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial\varphi}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial\varphi}{\partial v} = y\frac{\partial\varphi}{\partial u} + 2x\frac{\partial\varphi}{\partial v}$$
$$\frac{\partial\varphi}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial\varphi}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial\varphi}{\partial v} = x\frac{\partial\varphi}{\partial u} - 2y\frac{\partial\varphi}{\partial v}$$

so the equation becomes

$$(x^2 + y^2)\frac{\partial\varphi}{\partial u} = 0$$
$$\varphi = f(v) = f(x^2 - y^2)$$

(b) The characteristics are $x^2 - y^2 = \textit{constant}$, which are hyperbolas centered at $(0, 0)$ and with $x = y,\ x = -y$ as asymptotes.

9.2.6 Let $u = x^2 - y^2,\ v = xy$, then

$$\frac{\partial\varphi}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial\varphi}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial\varphi}{\partial v} = 2x\frac{\partial\varphi}{\partial u} + y\frac{\partial\varphi}{\partial v}$$
$$\frac{\partial\varphi}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial\varphi}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial\varphi}{\partial v} = -2y\frac{\partial\varphi}{\partial u} + x\frac{\partial\varphi}{\partial v}$$

so the equation beocomes

$$2(x^2 + y^2)\frac{\partial\varphi}{\partial u} = 0$$
$$\varphi = f(v) = f(xy)$$

9.3 Second-Order Equations

9.3.1

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\varphi}{\partial\eta}\frac{\partial\eta}{\partial x} = c^{\frac{1}{2}}\frac{\partial\varphi}{\partial\xi}$$

$$\frac{\partial^2\varphi}{\partial x^2} = c^{\frac{1}{2}}\left(\frac{\partial^2\varphi}{\partial\xi^2}\frac{\partial\xi}{\partial x} + \frac{\partial^2\varphi}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial x}\right) = c\frac{\partial^2\varphi}{\partial\xi^2}$$

$$\frac{\partial^2\varphi}{\partial x\partial y} = c^{\frac{1}{2}}\left(\frac{\partial^2\varphi}{\partial\xi^2}\frac{\partial\xi}{\partial y} + \frac{\partial^2\varphi}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial y}\right) = -b\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\xi\partial\eta}$$

$$\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{\partial\varphi}{\partial\eta}\frac{\partial\eta}{\partial y} = -c^{-\frac{1}{2}}b\frac{\partial\varphi}{\partial\xi} + c^{-\frac{1}{2}}\frac{\partial\varphi}{\partial\eta}$$

$$\frac{\partial^2\varphi}{\partial y^2} = -c^{-\frac{1}{2}}b\left(\frac{\partial^2\varphi}{\partial\xi^2}\frac{\partial\xi}{\partial y} + \frac{\partial^2\varphi}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial y}\right) + c^{-\frac{1}{2}}\left(\frac{\partial^2\varphi}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial y} + \frac{\partial^2\varphi}{\partial\eta^2}\frac{\partial\eta}{\partial y}\right) = c^{-1}b^2\frac{\partial^2\varphi}{\partial\xi^2} - 2c^{-1}b\frac{\partial^2\varphi}{\partial\xi\partial\eta} + c^{-1}\frac{\partial^2\varphi}{\partial\eta^2}$$

Substituting, we get

$$\begin{aligned}\mathcal{L} &= a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x\partial y} + c\frac{\partial^2}{\partial y^2} \\ &= (ac-b^2)\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\end{aligned}$$

9.4 Separation of Variables

9.4.1

$$(\nabla^2+k^2)(a_1\varphi_1+a_2\varphi_2) = a_1\nabla^2\varphi_1 + a_1k^2\varphi_1 + a_2\nabla^2\varphi_2 + a_2k^2\varphi_2 = a_1(\nabla^2+k^2)\varphi_1 + a_2(\nabla^2+k^2)\varphi_2$$

9.4.2 Let $\varphi(\rho,\varphi,z) = P(\rho)\Phi(\varphi)Z(z)$ and substitute:

$$\frac{\Phi Z}{\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \frac{PZ}{\rho^2}\frac{d^2\Phi}{d\varphi^2} + P\Phi\frac{d^2Z}{dz^2} + \left[k^2 + f(\rho) + \frac{1}{\rho^2}g(\varphi) + h(z)\right]P\Phi Z = 0$$

The equation can be seperated into

$$\frac{1}{Z}\frac{d^2Z}{dz^2} + h(z) = l^2$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} + g(\varphi) = -m^2$$

$$\frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \left[f(\rho)+l^2+k^2\right]\rho^2 - m^2 = 0$$

9.4.3 Let $\psi(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\phi)$ and substitute, we have

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = -k^2$$

Ir can be rearranged into

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + r^2k^2 = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\varphi^2}$$

By equating each side to λ we can separate R. The rest of the equation can be rearranged into

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \lambda\sin^2\theta$$

By equating each side to $-m^2$ we can separate Φ and Θ.

So the equation can be separated into

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+r^2k^2=\lambda$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=-m^2$$

$$-\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\lambda\sin^2\theta=-m^2$$

which are the same as equation (9.74), (9.77), and (9.78).

9.4.4 Let $\psi(r,\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi)$. Substitute and divide by $R\Theta\Phi$, we get

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+f(r)+\frac{1}{r^2}\left[\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+g(\theta)\right]+\frac{1}{r^2\sin^2\theta}\left[\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}+h(\varphi)\right]+k^2=0$$

It can be seperated into

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}+h(\varphi)=-m^2$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+r^2f(r)+r^2k^2=\lambda$$

$$-\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-g(\theta)+\frac{m^2}{\sin^2\theta}=\lambda$$

9.4.5 Let $\psi(x,y,z)=X(x)Y(y)Z(z)$ and substitute, we have

$$YZ\frac{d^2X}{dx^2}+XZ\frac{d^2Y}{dy^2}+XY\frac{d^2Z}{dz^2}+\frac{2mE}{\hbar^2}XYZ=0$$

It can be separated into

$$\frac{1}{X}\frac{d^2X}{dx^2}=-l^2\qquad\frac{1}{Y}\frac{d^2Y}{dy^2}=-m^2\qquad\frac{1}{Z}\frac{d^2Z}{dz^2}=-n^2$$

where $l^2+m^2+n^2=\frac{2mE}{\hbar^2}$.

The solution of X is $X=A\sin lx+B\cos lx$. When the boundary conditions $X(0)=X(a)=0$ ia applied, we must require $B=0$ and $la=\lambda\pi$, where λ is a positive integer. Similarly $mb=\mu\pi$, $nc=\nu\pi$, where μ,ν are positive integers. So

$$E=\frac{\hbar^2}{2m}(l^2+m^2+n^2)=\frac{\pi^2\hbar^2}{2m}\left(\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}\right)$$

and the minimum of E is

$$E_{min}=\frac{\pi^2\hbar^2}{2m}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

in which case $\lambda=\mu=\nu=1$.

9.4.6 From Exercise 3.10.32 (c), we have

$$\mathbf{L}^2=-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

Let $\psi(r,\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi)$. Substitute into the equation and divide by $R\Theta\Phi$, we have

$$-\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)-\frac{1}{\Phi\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}=l(l+1)$$

which can be separated into

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = -m^2$$

and

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + l(l+1)\Theta = 0$$

Let $t = \cos\theta$ and $\Theta(\theta) = P(\cos\theta) = P(t)$, the Θ equation becomes

$$(1-t^2)P''(t) - 2tP'(t) - \frac{m^2}{1-t^2}P(t) + l(l+1)P(t) = 0$$

which is the associated Legendre equation.

9.4.7 (a) Multiply the equation by $-\frac{2}{\hbar}\left(\frac{m}{k}\right)^{1/2}$ and use the definitions of a and λ, we have

$$\frac{1}{a^2}\frac{d^2\psi}{dx^2} - a^2x^2\psi + \lambda\psi = 0$$

Note that $\frac{d^2\psi}{dx^2} = \frac{d^2\psi}{d\xi^2}\left(\frac{d\xi}{dx}\right)^2 = a^2\frac{d^2\psi}{d\xi^2}$, and $a^2x^2 = \xi^2$, we have

$$\frac{d^2\psi}{d\xi^2} + (\lambda - \xi^2)\psi = 0$$

(b)

$$\frac{d\psi}{d\xi} = \left[y'(\xi) - \xi y(\xi)\right]e^{-\frac{\xi^2}{2}}$$

$$\frac{d^2\psi}{d\xi^2} = \left[y''(\xi) - 2\xi y'(\xi) + (\xi^2-1)y(\xi)\right]e^{-\frac{\xi^2}{2}}$$

Substitute and eliminate $e^{-\frac{\xi^2}{2}}$, we have

$$y''(\xi) - 2\xi y'(\xi) + (\lambda - 1)y(\xi) = 0$$

which is the Hermite differential equation.

9.5 Laplace and Poisson Equations

9.5.1 (a) Using Equation (3.158),

$$\nabla^2\varphi_1 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial(\frac{1}{r})}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(-1) = 0$$

(b) Substitute $r\cos\theta$ for z, then

$$\varphi_2 = \frac{1}{2r}\ln\frac{1+\cos\theta}{1-\cos\theta}$$

$$\frac{\partial\varphi_2}{\partial r} = -\frac{1}{2r^2}\ln\frac{1+\cos\theta}{1-\cos\theta}$$

$$\frac{\partial\varphi_2}{\partial\theta} = -\frac{1}{r\sin\theta}$$

Using Equation (3.158),

$$\nabla^2\varphi_2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi_2}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi_2}{\partial\theta}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(-\frac{1}{2}\ln\frac{1+\cos\theta}{1-\cos\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{1}{r}\right) = 0$$

9.5.2

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} = 0$$

so

$$\begin{aligned}\nabla^2\left(\frac{\partial\Psi}{\partial z}\right) &= \frac{\partial^3\Psi}{\partial x^2\partial z} + \frac{\partial^3\Psi}{\partial y^2\partial z} + \frac{\partial^3\Psi}{\partial z^3}\\ &= \frac{\partial}{\partial z}\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right) = 0\end{aligned}$$

which means $\frac{\partial\Psi}{\partial z}$ is also a solution of Laplace's equation.

9.5.3 Suppose ψ_1 and ψ_2 are distinct solutions to the Laplace or Poisson equation for the same Dirichlet boundary conditions, then $\psi = \psi_1 - \psi_2$ will also be a solution to the Laplace equation with a zero Dirichlet boundary conditions. From Eq. (9.88),

$$\int\limits_S \psi\frac{\partial\psi}{\partial\mathbf{n}}dS = \int\limits_V \psi\nabla^2\psi\, d\tau + \int\limits_V \nabla\psi\cdot\nabla\psi\, d\tau$$

$\int_S \psi\frac{\partial\psi}{\partial\mathbf{n}}dS$ vanishes because ψ vanishes on the boundary. $\int_V \psi\nabla^2\psi\, d\tau$ vanishes because ψ is a solution to the Laplace equation. Therefore $\int_V \nabla\psi\cdot\nabla\psi\, d\tau$ must vanish, which means $\nabla\psi = 0$ everywhere, so $\psi = \textit{constant} = 0$ because it is zero on the boundary. So $\psi_1 = \psi_2$, which means the solution is unique.

9.6 Wave Equation

9.6.1 Using the d'Alembert's solution:

$$\begin{aligned}\psi(x,t) &= \frac{1}{2}\left[\psi(x+ct,0) + \psi(x-ct,0)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\frac{\partial\psi(x,0)}{\partial t}dx\\ &= \frac{1}{2}\left[\sin(x+ct) + \sin(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\cos x\, dx\\ &= \sin x\cos ct + \frac{1}{c}\cos x\sin ct\end{aligned}$$

9.6.2

$$\begin{aligned}\psi(x,t) &= \frac{1}{2}\left[\psi(x+ct,0) + \psi(x-ct,0)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\frac{\partial\psi(x,0)}{\partial t}dx\\ &= \frac{1}{2}\left[\delta(x+ct) + \delta(x-ct)\right]\end{aligned}$$

9.6.3

$$\begin{aligned}\psi(x,t) &= \frac{1}{2}\left[\psi(x+ct,0) + \psi(x-ct,0)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\frac{\partial\psi(x,0)}{\partial t}dx\\ &= \frac{1}{2}\left[\psi_0(x+ct) + \psi_0(x-ct)\right]\end{aligned}$$

where ψ_0 is the given square-wave pulse.

9.6.4

$$\psi(x,t) = \frac{1}{2}\left[\psi(x+ct,0) + \psi(x-ct,0)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\frac{\partial\psi(x,0)}{\partial t}dx$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct}\sin x\, dx = \frac{1}{2c}\left[\cos(x-ct) - \cos(x+ct)\right] = \frac{1}{c}\sin x\cos ct$$

9.7 Heat-Flow, or Diffusion PDE

9.7.1 Substitute $T(r,t) = R(r)T(t)$ into the equation, we have

$$R\frac{\partial T}{\partial t} = KT\nabla^2 R = KT\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)$$

which can be separated into

$$\frac{1}{KT}\frac{\partial T}{\partial t} = \frac{1}{Rr^2}\left(r^2\frac{\partial^2 R}{\partial r^2} + 2r\frac{\partial R}{\partial r}\right) = -\alpha^2$$

so the R equation is

$$r^2\frac{d^2 R}{dr^2} + 2r\frac{dR}{dr} + \alpha^2 r^2 R = 0$$

For $R = \frac{\sin \alpha r}{r}$:

$$\begin{aligned} r^2\frac{\partial^2 R}{\partial r^2} &+ 2r\frac{\partial R}{\partial r} + \alpha^2 r^2 R \\ &= \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \alpha^2 r^2 R \end{aligned}$$

$$= \frac{d}{dr}\left(\alpha r\cos \alpha r - \sin \alpha r\right) + \alpha^2 r \sin \alpha r$$

$$= \alpha \cos \alpha r - \alpha^2 r \sin \alpha r - \alpha \cos \alpha r + \alpha^2 r \sin \alpha r = 0$$

For $R = \frac{\cos \alpha r}{r}$:

$$\begin{aligned} r^2\frac{\partial^2 R}{\partial r^2} &+ 2r\frac{\partial R}{\partial r} + \alpha^2 r^2 R \\ &= \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \alpha^2 r^2 R \end{aligned}$$

$$\frac{d}{dr}\left(-\alpha r\sin \alpha r - \cos \alpha r\right) + \alpha^2 r \cos \alpha r$$

$$= -\alpha \sin \alpha r - \alpha^2 r \cos \alpha r + \alpha \sin \alpha r + \alpha^2 r \cos \alpha r = 0$$

so $\frac{\sin \alpha r}{r}$ and $\frac{\cos \alpha r}{r}$ are solutions to the equation.

9.7.2 Substitute $T(\rho,t) = P(\rho)T(t)$ into the equation, we have

$$P\frac{\partial T}{\partial t} = KT\nabla^2 P = KT\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial P}{\partial \rho}\right)$$

which can be separated into

$$\frac{1}{KT}\frac{\partial T}{\partial t} = \frac{1}{P\rho}\left(\rho\frac{\partial^2 P}{\partial \rho^2} + \frac{\partial P}{\partial \rho}\right) = -\alpha^2$$

so

$$\frac{dT}{dt} + \alpha^2 KT = 0$$

$$\rho\frac{d^2 P}{d\rho^2} + \frac{dP}{d\rho} + \alpha^2 \rho P = 0$$

9.7.3 Use Equation 9.114:

$$\begin{aligned} \psi(x,t) &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} A\delta(x - 2a\xi\sqrt{t})e^{-\xi^2}d\xi \\ &= \frac{A}{\sqrt{\pi}}\int_{-\infty}^{\infty} \delta(x-y)e^{-\frac{y^2}{4a^2 t}}\frac{dy}{2a\sqrt{t}} \\ &= \frac{A}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2 t}} \end{aligned}$$

9.7.4 From Equation 9.101, the solutions have the form

$$\psi(x,t) = (A\cos\omega x + B\sin\omega x)e^{-\omega^2 a^2 t} + C_0' x + C_0$$

Using the boundary conditions:

$$\begin{aligned}
&\psi(0,\infty) = C_0 = 1 && C_0 = 1 \\
&\psi(L,\infty) = C_0' L + C_0 = 0 && C_0' = -\frac{1}{L} \\
&\psi(0,t) = Ae^{-\omega^2 a^2 t} + 1 = 1 && A = 0 \\
&\psi(L,t) = B\sin(\omega L)\, e^{-\omega^2 a^2 t} = 0 && \omega L = n\pi \qquad \textit{n is a positive integer}
\end{aligned}$$

So

$$\psi(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2 a^2}{L^2} t} - \frac{x}{L} + 1$$

To determine a_n, use $\psi(x,0) = 0$ and the orthogonality $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}\delta_{nm}$:

$$\int_0^L \psi(x,0)\sin\left(\frac{n\pi x}{L}\right) dx = \frac{L}{2}a_n + \frac{L}{n\pi} = 0$$

so $a_n = -\frac{2}{n\pi}$, and the overall solution is

$$\psi(x,t) = -\sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2 a^2}{L^2} t} - \frac{x}{L} + 1$$

It can be verified that the solution satisfies the boundary, initial conditions and the PDE.