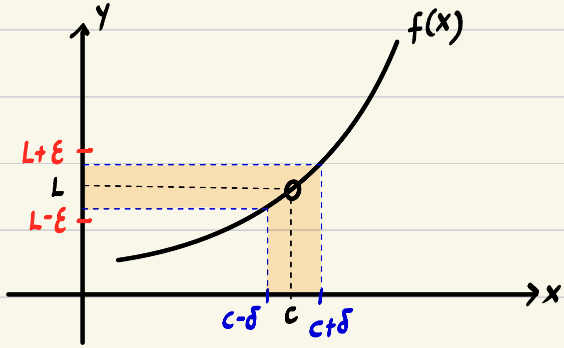
Formal Definition of Limit

Definition. We say that $\lim_{x \to c} f(x) = L$ if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$



1. f is linear.

Let $f(x) = 2x + 3$. Show that $\lim_{x \to 2} f(x) = 7$.

<u>proof</u>.

Let $\varepsilon > 0$ be given. We want to find a $\delta(\varepsilon) > 0$ such that

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 7| < \varepsilon.$$

Since $|f(x) - 7| = |(2x+3) - 7| = |2x - 4| = 2|x - 2|$, choose $\delta = \frac{\varepsilon}{2}$.

Now, check if $0 < |x - 2| < \delta \Rightarrow |f(x) - 7| < \varepsilon$.

$$|f(x) - 7| = |(2x+3) - 7| = 2 \cdot |x - 2| < 2\delta = \cancel{2} \cdot \frac{\varepsilon}{\cancel{2}} = \varepsilon \quad \square$$

2. f is quadratic

Let $f(x) = x^2 - x - 2$. Show that $\lim_{x \to 1} f(x) = -2$.

<u>proof</u>. Let $\varepsilon > 0$ be given. We want to find a $\delta(\varepsilon) > 0$ such that

$$0 < |x - 1| < \delta \Rightarrow |f(x) - (-2)| < \varepsilon.$$

Then $|f(x) - (-2)| = |x^2 - x - 2 + 2| = |x^2 - x| = |x(x-1)| = |x| \cdot |x-1|$.

To estimate $|f(x) - (-2)| = |x| . |x-1|$ by ε, we need to bound $|x|$ and $|x-1|$. We will bound $|x-1|$ by choosing δ.

So, it remains to bound $|x|$.

As $x \to 1$, x is close enough to 1. Assume $|x-1| < 1$. Then $-1 < x - 1 < 1 \Rightarrow 0 < x < 2 \Rightarrow |x| < 2$.

Now, choose $\delta = \min\left\{1, \frac{\varepsilon}{2}\right\}$ and check if

$$0 < |x - 1| < \delta \Rightarrow |f(x) - (-2)| < \varepsilon.$$

If $\frac{\varepsilon}{2} < 1$, then $\delta = \frac{\varepsilon}{2}$ and $|f(x) - (-2)| = |x| . |x-1| < 2 . \frac{\varepsilon}{2} = \varepsilon$

If $\frac{\varepsilon}{2} > 1$, then $\delta = 1$ and $|f(x) - (-2)| = |x| \cdot |x-1| < 2.1 = 2 < \varepsilon$ ∎

3. f is square-root

Let $f(x)=\sqrt{x^2+3}$. Show that $\lim_{x\to 1}\sqrt{x^2+3}=2$.

proof. Let $\varepsilon>0$ be given. We want to find a $\delta(\varepsilon)>0$ such that

$$0<|x-1|<\delta \Rightarrow |f(x)-2|<\varepsilon.$$

Then $|f(x)-2| = |\sqrt{x^2+3}-2| = \left|\frac{(\sqrt{x^2+3}-2)(\sqrt{x^2+3}+2)}{\sqrt{x^2+3}+2}\right|$

$$=\left|\frac{x^2+3-4}{\sqrt{x^2+3}+2}\right| = \left|\frac{x^2-1}{\sqrt{x^2+3}+2}\right|$$

$$=|x-1|\cdot\left|\frac{x+1}{\sqrt{x^2+3}+2}\right|$$

We need to bound $\left|\frac{x+1}{\sqrt{x^2+3}+2}\right|$.

As $x\to 1$, x is close enough to 1. Assume $|x-1|<1$.

Then $-1<x-1<1 \Rightarrow 0<x<2 \Rightarrow 1<x+1<3 \Rightarrow |x+1|<3$

and $0<x^2<4 \Rightarrow 3<x^2+3<7 \Rightarrow \sqrt{3}<\sqrt{x^2+3}<\sqrt{7}$

$$\Rightarrow \sqrt{3}+2<\sqrt{x^2+3}+2<\sqrt{7}+2$$

$$\Rightarrow \frac{1}{\sqrt{7}+2}<\frac{1}{\sqrt{x^2+3}+2}<\frac{1}{\sqrt{3}+2}$$

$$\Rightarrow \left|\frac{1}{\sqrt{x^2+3}+2}\right|<\frac{1}{\sqrt{3}+2}$$

Choose $\delta = \min\left\{1, \frac{(\sqrt{3}+2)}{3}\varepsilon\right\}$. Check if

$$0 < |x-1| < \delta \Rightarrow |f(x)-2| < \varepsilon.$$

Then,

- $\frac{(\sqrt{3}+2)}{3}\varepsilon < 1 \Rightarrow \delta = \frac{(\sqrt{3}+2)}{3}\varepsilon \Rightarrow |f(x)-2| = |x-1| \cdot \frac{|x+1|}{|\sqrt{x^2+3}+2|} < \frac{(\sqrt{3}+2)}{3}\varepsilon \cdot \frac{3}{\sqrt{3}+2}$
  $= \varepsilon$

- $\frac{(\sqrt{3}+2)}{3}\varepsilon > 1 \Rightarrow \delta = 1 \Rightarrow |f(x)-2| = |x-1| \cdot \frac{|x+1|}{|\sqrt{x^2+3}+2|} < 1 \cdot \frac{3}{\sqrt{3}+2} < \varepsilon.$